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ON A FUNCTIONAL EQUATION CONNECTED TO GAUSS QUADRATURE RULE

Barbara Koclęga-Kulpa, Tomasz Szostok

Abstract. We consider the functional equation

$$F(y) - F(x) = (y - x)[f(\alpha x + \beta y) + f(\beta x + \alpha y)]$$

stemming from Gauss quadrature rule. In previous results equations of this type with rational only coefficients α and β were considered. In this paper we allow these numbers to be irrational. We find all solutions of this equation for functions acting on $\mathbb{R}$. However, some results are valid also on integral domains.

1. Introduction

Functional equations connected to well known quadrature rules were studied by many authors. In the monograph [7] the equation stemming from Simpson quadrature rule

$$F(y) - F(x) = (y - x)\left[\frac{1}{6}f(x) + \frac{2}{3}f\left(\frac{x+y}{2}\right) + \frac{1}{6}f(y)\right]$$

was considered for functions acting on $\mathbb{R}$. This equation, in a bit more general form, for functions transforming an integral domain into itself has been solved in [3].

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On the other hand, M. Sablik [9] during the *7th* Katowice-Debrecen Winter Seminar on Functional Equations and Inequalities presented the general solution of the equation

$$g(x) - f(y) = (x-y)[h(x) + k(sx+ty) + k(tx+sy) + h(y)] \tag{1}$$

in the case $s, t \in \mathbb{Q}$ without any regularity assumptions concerning the functions considered. Some other equations of this type were solved in [4]. However, these results are valid only for equations with rational coefficients on the right-hand side.

In the current paper we are going to consider an equation connected to the simplest quadrature rule with irrational coefficient - the Gauss quadrature rule with two nodes. Thus we shall find the solutions of

$$F(y) - F(x) = (y-x)[f(\alpha x + \beta y) + f(\beta x + \alpha y)]. \tag{2}$$

In our method we shall need a lemma, which was established by M. Sablik ([8, Lemma 2.3]) and improved by I. Pawlikowska (cf. [6]).

By a *polynomial function of order* n in this paper we mean a solution of the functional equation $\Delta_h^{n+1} f(x) = 0$, where Δ_h^n stands for the n–th iterate of the difference operator $\Delta_h f(x) = f(x+h) - f(x)$. Observe that a continuous polynomial function of order n is a polynomial of degree at most n (see [5, Theorem 4, p. 398]).

Let us now quote Sablik's result. First we need some notations. Let G, H be Abelian groups and $SA^0(G,H) := H$, $SA^1(G,H) := \mathrm{Hom}(G,H)$ (i.e. the group of all homomorphisms from G into H), and for $i \in \mathbb{N}$, $i \geq 2$, let $SA^i(G,H)$ be the group of all i–additive and symmetric mappings from G^i into H. Furthermore, let $\mathcal{P} := \{(\alpha,\beta) \in \mathrm{Hom}(G,G)^2 : \alpha(G) \subset \beta(G)\}$. Finally, for $x \in G$ let $x^i = \underbrace{(x,\dots,x)}_{i}$, $i \in \mathbb{N}$.

Lemma 1. *Fix $N \in \mathbb{N} \cup \{0\}$ and let $I_0,\dots,I_N$ be finite subsets of $\mathcal{P}$. Suppose that H is uniquely divisible by $N!$ and that the functions $\varphi_i\colon G \to SA^i(G,H)$ and $\psi_{i,(\alpha,\beta)} : G \to SA^i(G,H)$, $(\alpha,\beta) \in I_i$, $i = 0,\dots,N$, satisfy*

$$\varphi_N(x)(y^N) + \sum_{i=0}^{N-1} \varphi_i(x)(y^i) = \sum_{i=0}^{N} \sum_{(\alpha,\beta)\in I_i} \psi_{i,(\alpha,\beta)}\big(\alpha(x) + \beta(y)\big)(y^i) \tag{3}$$

for every $x, y \in G$. *Then* φ_N *is a polynomial function of order at most* $k-1$, *where*

$$k = \sum_{i=0}^{N} \operatorname{card}\Big(\bigcup_{s=i}^{N} I_s\Big).$$

2. Results

First we need some auxiliary lemma.

Lemma 2. *Let* P *be an integral domain with a unit element* 1 *such that* $2 = 1+1 \neq 0$ *and* $3 = 1+1+1 \neq 0$. *Let the functions* $f, F\colon P \to P$ *satisfy the equation*

$$F(x) - F(y) = (x-y)[b_1 f(\alpha_1 x + \beta_1 y) + \cdots + b_n f(\alpha_n x + \beta_n y)] \tag{4}$$

for all $x, y \in P$ *and some* $b_1, \dots, b_n, \alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n \in P$. *Further, let* f *be a function of the form*

$$f(x) = a_3(x) + a_2(x) + a_1(x) + a_0, \quad x \in P, \tag{5}$$

where $a_i, i \in \{1,2,3\}$ *is a diagonalization of some* $i-$*additivie and symmetric function and* $a_0 \in P$ *is a constant.*

Then all functions $a_i, i \in \{1,2,3\}$ *also satisfy* (4) *with some* F_i.

Proof. We may assume that $F(0) = 0$, so substituting $y = 0$ in (4) we obtain

$$F(x) = x[b_1 f(\alpha_1 x) + \cdots + b_n f(\alpha_n x)], \quad x \in P.$$

Using this formula in (4) we get

$$\begin{aligned} &x[b_1 f(\alpha_1 x) + \cdots + b_n f(\alpha_n x)] - y[b_1 f(\alpha_1 y) + \cdots + b_n f(\alpha_n y)] \\ &\quad = (x-y)[b_1 f(\alpha_1 x + \beta_1 y) + \cdots + b_n f(\alpha_n x + \beta_n y)], \quad x, y \in P. \end{aligned} \tag{6}$$

It is easy to see that if f, g are solutions to this equation, then also $f+g$ satisfies this equation. Moreover, if f satisfies (6) and $\alpha \in P$ is given, then also αf is a solution. On the other hand, if αf satisfies (6) and $\alpha \neq 0$, then f

is a solution of this equation. Now take $a \neq 0$ and substitute ax, ay in place of x and y, We obtain (after canceling by a) that

$$x[b_1 f(\alpha_1 ax) + \cdots + b_n f(\alpha_n ax)] - y[b_1 f(\alpha_1 ay) + \cdots + b_n f(\alpha_n ay)]$$
$$= (x-y)[b_1 f(\alpha_1 ax + \beta_1 ay) + \cdots + b_n f(\alpha_n ax + \beta_n ay)], \quad x, y \in P,$$

which means that the function $x \mapsto f(ax)$ also satisfies our equation (if f does).

Now consider f of the form (5), and put

$$g_1(x) := 8f(x) - f(2x), \quad x \in P.$$

Then g_1 is a solution of (6) and on the other hand

$$g_1(x) = 4a_2(x) + 6a_1(x) + 7a_0. \tag{7}$$

Define g_2 by the formula

$$g_2(x) := 4g_1(x) - g_1(2x) = 12a_1(x) + 21a_0.$$

Similarly as before g_2 satisfies (6), constant function $21a_0$ is clearly a solution of (6), which means that also $12a_1$ satisfies this equation thus also a_1 satisfies (6) since $2, 3 \neq 0$. Further, in view of (7) we easily obtain that also a_2 is a solution of (6). Finally, from (5) it follows that a_3 satisfies our equation. □

Remark 1. Let P be an integral domain such that $2, 3 \neq 0$. If functions $A_i, B_i \colon P \to P$, $i = 1, 2, 3$, satisfy

$$A_3(x) + A_2(x) + A_1(x) = B_3(x) + B_2(x) + B_1(x), \quad x \in P, \tag{8}$$

and A_i, B_i are i-homogeneous, then

$$A_i = B_i, \; i = 1, 2, 3.$$

Indeed, if we take $-x$ in place of x, in (8), then we obtain

$$-A_3(x) + A_2(x) - A_1(x) = -B_3(x) + B_2(x) - B_1(x), \quad x \in P. \tag{9}$$

This, together with (8) means that $A_2 = B_2$ and, consequently

$$A_3(x) + A_1(x) = B_3(x) + B_1(x), \quad x \in P. \tag{10}$$

Moreover, taking here $2x$ instead of x, we get

$$8A_3(x) + 2A_1(x) = 8B_3(x) + 2B_1(x), \quad x \in P. \tag{11}$$

Multiplying (10) by 2 and substracting it from (11) we arrive at

$$6A_3(x) = 6B_3(x), \quad x \in P,$$

which means that $A_3 = B_3$ and obviously also $A_1 = B_1$.

Now we shall prove some lemmas concerning solutions of a simplified version of equation (2). However, at this moment we are concerned only with solutions, which are diagonalizations of symmetric and $i-$additive functions.

LEMMA 3. *Let P be a integral domain with a unit element 1 such that $2, 3 \neq 0$. If functions $G, f\colon P \to P$ satisfy the equation*

$$G(y) - G(x) = (y - x)[f(x + \gamma y) + f(\gamma x + y)], \quad x, y \in P, \tag{12}$$

and $f(x) := C(x, x, x)$ for some $3-$additive and symmetric function $C\colon P \to P$, then

$$f((\gamma + 1)x) = 2cx^3$$

for some $c \in P$ and all $x \in P$.

PROOF. Let f be given by the formula $f(x) = C(x, x, x)$, $x \in P$, where C is 3-additive and symmetric.

We may assume that $G(0) = 0$, so substituting $x = 0$ in (12) we get

$$G(y) = y[f(\gamma y) + f(y)], \quad y \in P. \tag{13}$$

Using this formula in (12), we obtain for all $x, y \in P$

$$y[f(\gamma y) + f(y)] - x[f(\gamma x) + f(x)] = (y - x)[f(x + \gamma y) + f(\gamma x + y)]. \tag{14}$$

On the other hand, using here the form of f after some calculation we arrive at

$$\begin{aligned} x[f(\gamma y) + f(y)] - y[f(\gamma x) + f(x)] = 3(y - x)[&C(x, x, \gamma y) \\ &+ C(x, \gamma y, \gamma y) + C(y, y, \gamma x) + C(y, \gamma x, \gamma x)] \end{aligned} \tag{15}$$

for all $x, y \in P$.

Now we define a new function $g(x) := f(x) + f(\gamma x),\ x \in P$. Then in view of (15) we have

$$xg(y) - yg(x) = 3(y-x)[C(x,x,\gamma y) + C(x,\gamma y,\gamma y) + C(y,y,\gamma x) + C(y,\gamma x,\gamma x)]. \tag{16}$$

Further, taking here $x = 1$ we obtain the following formula

$$g(y) - yg(1) = 3(y-1)[C(1,1,\gamma y) + C(1,\gamma y,\gamma y) + C(y,y,\gamma) + C(y,\gamma,\gamma)]. \tag{17}$$

Using Remark 1, we may compare here the terms of order one and we obtain that

$$yg(1) = 3C(y,\gamma,\gamma) + 3C(1,1,\gamma y), \quad y \in P, \tag{18}$$

further comparing expressions of order two in (17) we get

$$3y[C(y,\gamma,\gamma) + C(1,1,\gamma y)] = 3[C(1,\gamma y,\gamma y) + C(y,y,\gamma)], \quad y \in P, \tag{19}$$

thus, using here (18), we may write

$$y^2 g(1) = 3[C(1,\gamma y,\gamma y) + C(y,y,\gamma)], \quad y \in P. \tag{20}$$

Finally, comparing terms of order three in (17), we obtain

$$g(y) = 3y[C(1,\gamma y,\gamma y) + C(y,y,\gamma)], \quad y \in P,$$

which, in view of (20), gives us

$$g(y) = y^3 g(1), \quad y \in P, \tag{21}$$

so we may write that

$$f(x) + f(\gamma x) = cx^3, \quad x \in P,$$

(where $c = g(1)$) and, more precisely,

$$C(x,x,x) + C(\gamma x,\gamma x,\gamma x) = cx^3, \quad x \in P. \tag{22}$$

Now, putting $2x$ in place of y in (16), we obtain

$$6cx^4 = 3x[2C(x,x,\gamma x) + 4C(x,\gamma x,\gamma x) + 4C(x,x,\gamma x) + 2C(x,\gamma x,\gamma x)],$$

thus

$$3C(x,x,\gamma x) + 3C(x,\gamma x,\gamma x) = cx^3, \quad x \in P. \tag{23}$$

Adding equations (22) and (23), we may write

$$C(x,x,x) + 3C(x,x,\gamma x) + 3C(x,\gamma x,\gamma x) + C(\gamma x,\gamma x,\gamma x) = 2cx^3, \quad x \in P,$$

i.e.

$$f((1+\gamma)x) = 2cx^3,\ x \in P. \qquad \square$$

Lemma 4. *Let P be an integral domain with a unit element 1 such that $2 \neq 0$. If functions $G, f\colon P \to P$ satisfy the equation (12) and $f(x) := B(x,x)$ for some $2-$additive and symmetric function $B\colon P^2 \to P$, then*

$$2f((\gamma+1)x) = 3bx^2$$

for some $b \in P$ and all $x \in P$.

Proof. Let $f(x) = B(x,x)$, $x \in P$, where B is biadditive and symmetric. We may assume, without loss of generality, that $G(0) = 0$ and putting $x = 0$ in (12) we obtain

$$G(y) = y\left[f(y) + f(\gamma y)\right], \quad y \in P. \tag{24}$$

Consequently, equation (12) takes form

$$y\left[f(y) + f(\gamma y)\right] - x\left[f(x) + f(\gamma x)\right] = (y-x)[f(x+\gamma y) + f(\gamma x+y)], \quad x, y \in P.$$

Further, using here the form of f, we may write for all $x, y \in P$

$$\begin{aligned} y\left[f(y) + f(\gamma y)\right] - x\left[f(x) + f(\gamma x)\right] = (y-x)\big[&f(x) + 2B(x,\gamma y) + f(\gamma y) \\ &+ f(\gamma x) + 2B(\gamma x, y) + f(y)\big]. \end{aligned}$$

Now we define a new function $g(x) := f(x) + f(\gamma x)$, $x \in P$, then we obtain

$$yg(y) - xg(x) = (y-x)[g(x) + g(y) + 2B(x,\gamma y) + 2B(\gamma x, y)], \quad x, y \in P,$$

and, after some simple calculations,

$$xg(y) - yg(x) = 2(y-x)[B(x,\gamma y) + B(\gamma x, y)], \quad x, y \in P. \tag{25}$$

Putting here $x = 1$ we get

$$g(y) - yg(1) = 2(y-1)[B(1, \gamma y) + B(\gamma, y)], \quad x, y \in P.$$

Similarly as in the proof of the previous lemma we compare now the terms of the same order. Thus we get

$$g(y) = 2y[B(1, \gamma y) + B(\gamma, y)], \quad y \in P,$$

and

$$yg(1) = 2[B(1, \gamma y) + B(\gamma, y)], \quad y \in P,$$

which means that

$$g(y) = by^2, \quad y \in P,$$

where $b := g(1)$. Further, from the obtained form of g we have

$$f(x) + f(\gamma x) = bx^2, \quad x \in P,$$

i.e.

$$2B(x, x) + 2B(\gamma x, \gamma x) = 2bx^2, \quad x \in P. \tag{26}$$

On the other hand, taking in (25) $2x$ in place of y, and using form of g, we get

$$4B(x, \gamma x) = bx^2, \quad x \in P.$$

Adding this equality to (26), we may write

$$2B(x, x) + 2B(\gamma x, \gamma x) + 4B(\gamma x, x) = 3bx^2,$$

thus

$$2f((1+\gamma)x) = 3bx^2, \quad x \in P. \qquad \square$$

Lemma 5. *Let P be an integral domain and let $f, G \colon P \to P$ be solutions of* (12) *such that f is an additive function. Then*

$$f((\gamma + 1)x) = ax$$

for some $a \in P$ and all $x \in P$.

Proof. Similarly as before we easily obtain that f satisfies (14). Since f is additive, we get

$$y[f(\gamma y)+f(y)]-x[f(\gamma x)+f(x)]=(y-x)[f(x)+f(\gamma y)+f(\gamma x)+f(y)]$$

for all $x,y\in P$ and, consequently,

$$x[f(\gamma y)+f(y)]=y[f(\gamma x)+f(x)],\quad x,y\in P.$$

Taking here $y=1$ we arrive at

$$x[f(\gamma)+f(1)]=f(\gamma x)+f(x),\quad x\in P,$$

thus putting $a:=f(\gamma)+f(1)$ we have

$$f(\gamma x)+f(x)=ax.$$

Using here the additivity of f we obtain $f((\gamma+1)x)=ax$ for all $x\in P$. □

Now we are in position to obtain the result concerning the form of solutions of equation (2) for functions acting on an integral domain.

Theorem 1. *Let P be an integral domain divisible by* 6. *If functions $f,F:P\to P$ satisfy* (12) *with some $\gamma\in P$, such that P is divisible by γ, $\gamma+1$ and $\gamma-1$, then*

$$f(x)=cx^3+bx^2+ax+d,\quad x\in P,$$

for some $a,b,c,d\in P$.

Proof. Substituting in (12) $\frac{x-y}{\gamma+1}$ in place of x and $\frac{x+\gamma y}{\gamma+1}$ in place of y, we obtain

$$G\left(\frac{x+\gamma y}{\gamma+1}\right)-G\left(\frac{x-y}{\gamma+1}\right)=y[f((x+(\gamma-1)y)+f(x)],\quad x,y\in P,$$

and, consequently,

$$yf(x)=G\left(\frac{x+\gamma y}{\gamma+1}\right)-G\left(\frac{x-y}{\gamma+1}\right)-yf((x+(\gamma-1)y),\quad x,y\in P.$$

Using here Lemma 1 with $N = 1$, $I_0 := \left\{\left(\frac{1}{\gamma+1}\mathrm{id}, \frac{\gamma}{\gamma+1}\mathrm{id}\right), \left(\frac{1}{\gamma+1}\mathrm{id}, \frac{-1}{\gamma+1}\mathrm{id}\right)\right\}$ and $I_1 = \{(\mathrm{id}, (\gamma-1)\mathrm{id})\}$ we obtain that f is a polynomial function of degree at most 3. Now, since P is divisible by 6, we have

$$f(x) = C(x,x,x) + B(x,x) + A(x) + d, \quad x \in P,$$

where $C\colon P^3 \to P, B\colon P^2 \to P$, and $A\colon P \to P$ are symmetric and $3,2,1$–additive, respectively (see for example [5]).

Now using Lemma 2 we obtain that functions $f_3(x) := C(x,x,x)$, $f_2(x) := B(x,x)$, and $f_1(x) := A(x)$ also satisfy (12), which in view of Lemmas 3, 4 and 5 means that $f_3((\gamma+1)x) = c_0x^3$, $f_2((\gamma+1)x) = b_0x^2$, and $f_1((\gamma+1)x) = a_0x$ for some $a_0, b_0, c_0 \in P$. However, P is divisible by $\gamma+1$ and, consequently, taking $c := \frac{c_0}{(\gamma+1)^3}$, $b := \frac{b_0}{(\gamma+1)^2}$ and $a := \frac{a_0}{\gamma+1}$, we arrive at

$$f(x) = cx^3 + bx^2 + ax + d, \quad x \in P. \qquad \square$$

Now we shall deal with functions defined and taking values in $\mathbb{R}$. In this case we shall obtain a general solution of equation (2).

Theorem 2. *Let functions $f, F\colon \mathbb{R} \to \mathbb{R}$ satisfy* (2) *with some constants $\alpha, \beta \in \mathbb{R}$.*

(i) *If $\alpha = \beta \neq 0$, then $f(x) = ax + d, x \in \mathbb{R}$.*
(ii) *If $\alpha = \beta = 0$, then f may be any function.*
(iii) *If $\alpha = 0, \beta \neq 0$ or $\alpha \neq 0, \beta = 0$, then $f(x) = ax + d, x \in \mathbb{R}$.*
(iv) *If $\alpha = -\beta \neq 0$, then $f(x) = h(x) + \frac{A(x)}{x}$ for all $x \neq 0$ and some functions $h, A\colon \mathbb{R} \to \mathbb{R}$ which are odd and additive, respectively.*
(v) *If $\alpha \neq -\beta$ and $\alpha^2 + \beta^2 \neq 4\alpha\beta$, then $f(x) = ax + d,\ x \in \mathbb{R}$.*
(vi) *If $\alpha \neq -\beta$ and $\alpha^2 + \beta^2 = 4\alpha\beta$, then $f(x) = cx^3 + bx^2 + ax + d,\ x \in \mathbb{R}$.*

In each case, function F is expressed by formula

$$F(x) = x[f(\alpha x) + f(\beta x)] + F(0), \quad x \in \mathbb{R}.$$

Conversely, in each case functions given by the above formulas satisfy equation (2).

Proof. Let us first consider the case $\alpha = \beta \neq 0$. Then equation (2) takes the form

$$F(y) - F(x) = 2(y-x)f(\alpha(x+y)).$$

Put $\tilde{f}(t) := 2f(\alpha t),\ t \in \mathbb{R}$. Then we get the equation solved by J. Aczél in [1] and consequently we obtain $\tilde{f}(x) = \tilde{a}x + \tilde{d}$, which means that $f(x) = ax + d$ where $a := \frac{\tilde{a}}{2\alpha}$ and $d := \frac{\tilde{d}}{2}$.

In the case (iii) we get (after a simple substitution)

$$F(y) - F(x) = (y - x)[h(x) + h(y)]$$

and we obtain our result from the paper of Sh. Haruki [2].

Now we consider the case $\alpha = -\beta \neq 0$, this means that equation (2) takes the form

$$F(y) - F(x) = (y - x)[f(\alpha(x - y)) + f(\alpha(y - x))].$$

Define a new function $f_1 \colon \mathbb{R} \to \mathbb{R}$ by

$$f_1(x) := f(\alpha x).$$

We obtain the equation

$$F(y) - F(x) = (y - x)[f_1(x - y) + f_1(y - x)], \quad x, y \in \mathbb{R}. \tag{27}$$

Take in (27) $x = 0$ to obtain

$$F(y) = y[f_1(-y) + f_1(y)] + F(0), \quad y \in \mathbb{R}, \tag{28}$$

which used in (27) gives us

$$F(y) - F(x) = F(y - x) - F(0), \quad x, y \in \mathbb{R}.$$

Substituting here $x + y$ in place of y we get

$$F(x + y) - F(x) = F(y) - F(0), \quad x, y \in \mathbb{R}.$$

This means that function $A_1 := F - F(0)$ is additive, thus from (28) we obtain

$$x[f_1(-x) + f_1(x)] = A_1(x)$$

and further

$$f_1(-x) + f_1(x) = \frac{A_1(x)}{x}, \quad x \neq 0.$$

Now we put $H(x) := \frac{f_1(x)-f_1(-x)}{2}$, which gives us

$$f_1(x) = \frac{f_1(x)-f_1(-x)}{2} + \frac{f_1(x)+f_1(-x)}{2} = H(x) + \frac{A_1(x)}{2x}$$

for all $x \in \mathbb{R} \setminus \{0\}$. And from the definition of f we get

$$f(x) = f_1\left(\frac{x}{\alpha}\right) = H\left(\frac{x}{\alpha}\right) + \frac{\alpha A_1\left(\frac{x}{\alpha}\right)}{2x}.$$

To finish this part of the proof it suffices to take $h(x) := H\left(\frac{x}{\alpha}\right)$ and $A(x) := \frac{\alpha}{2} A_1\left(\frac{x}{\alpha}\right)$.

Finally we consider the case $\alpha \neq -\beta$. Note that both numbers α, β are different from zero. Now we put in (2) $\frac{1}{\alpha}x$ and $\frac{1}{\alpha}y$ instead of x and y, respectively. Thus

$$F\left(\frac{1}{\alpha}y\right) - F\left(\frac{1}{\alpha}x\right) = \frac{1}{\alpha}(y-x)\left[f\left(x + \frac{\beta}{\alpha}y\right) + f\left(\frac{\beta}{\alpha}x + y\right)\right].$$

Multiplying this equation by α and taking $G(x) := \alpha F\left(\frac{1}{\alpha}x\right)$, $\gamma := \frac{\beta}{\alpha}$ we obtain that f satisfies

$$G(y) - G(x) = (y-x)[f(x+\gamma y) + f(\gamma x + y)].$$

Thus we have got the equation (12) which in view of Theorem 1 means that

$$f(x) = cx^3 + bx^2 + ax + d, \quad x \in \mathbb{R}.$$

To finish this part of the proof we assume that $c \neq 0$ and we are going to show that

$$\alpha^2 + \beta^2 = 4\alpha\beta.$$

Indeed, if function $f(x) = cx^3 + bx^2 + ax + d$ satisfies (2) then from Lemma 2 we know that also $f_3(x) := cx^3$ satisfies (2). Thus we have

$$G_3(y) - G_3(x) = (y-x)[f_3(\alpha x + \beta y) + f_3(\beta x + \alpha y)], \quad x, y \in \mathbb{R}$$

for some function G_3. Using the form of f and equality

$$G_3(y) = y[f_3(\beta y) + f_3(\alpha y)], \quad y \in \mathbb{R},$$

we get after some simple calculations

$$(\beta^3+\alpha^3)(y^4-x^4)=(y-x)[(\alpha^3+\beta^3)(x^3+y^3)+3\alpha\beta(\alpha+\beta)(xy^2+x^2y)],$$

i.e.

$$(\beta^3+\alpha^3)(y^3+x^3+xy^2+x^2y)=(\alpha^3+\beta^3)(x^3+y^3)+3\alpha\beta(\alpha+\beta)(xy^2+x^2y),$$

and further

$$(\beta^3+\alpha^3)=3\alpha\beta(\alpha+\beta),$$

thus

$$4\alpha\beta=\alpha^2+\beta^2.$$

Similarly, if we assume that $b\neq 0$ we obtain the same equality for α and β.

On the other hand, it is easy to show that in every of the above cases obtained functions satisfy equation (2). □

Remark 2. Let us consider the following quadrature rule with two nodes

$$\int_x^y f(t)dt\approx\frac{1}{2}(y-x)[f(\alpha x+(1-\alpha)y)+f((1-\alpha)x+\alpha y)],$$

which leads to a functional equation

$$F(y)-F(x)=(y-x)[f(\alpha x+(1-\alpha)y)+f((1-\alpha)x+\alpha y)].$$

If we addititionally assume that this equation is satisfied by $f(x)=x^3$ then

$$\alpha=\frac{3-\sqrt{3}}{6}.$$

This means that from our result we obtain that the only two-nodes quadrature rule satisfied exactly by polynomials of degree 3 (or 2) is the well known Gauss quadrature rule. One should emphasize that we assume here no regularity of functions f, F.

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