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Conjunctors and their Residual Implicators: Characterizations and Construction Methods

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Abstract. In many practical applications of fuzzy logic it seems clear that one needs more flexibility in the choice of the conjunction: in particular, the associativity and the commutativity of a conjunction may be removed. Motivated by these considerations, we present several classes of conjunctors, i.e. binary operations on $[0,1]$ that are used to extend the boolean conjunction from $\{0,1\}$ to $[0,1]$, and characterize their respective residual implicators. We establish hence a one-to-one correspondence between construction methods for conjunctors and construction methods for residual implicators. Moreover, we introduce some construction methods directly in the class of residual implicators, and, by using a deresiduation procedure, we obtain new conjunctors.

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1. Introduction

A fuzzy logic is usually considered as a many-valued propositional logic in which the class of truth values is modelled by the unit interval $[0,1]$, and which forms an extension of the classical boolean logic (see [17],[18] and [19]).

In these logics, a key role is played by the so-called *conjunctor*, i.e. a binary operation on $[0,1]$ that is used to extend the boolean conjunction from $\{0,1\}$ to $[0,1]$. Usually, one assumes that the conjunctor is monotone, associative, commutative and has neutral element 1, i.e. it is a *triangular norm*, briefly *t–norm* (see [24]). The Łukasiewicz t–norm $T_{\mathbf{L}}(x,y) = \max\{x+y-1,0\}$, the minimum

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$T_{\mathbf{M}}(x,y) = \min\{x,y\}$ and the product $T_{\mathbf{P}}(x,y) = xy$ are three examples of t–norms that generate, respectively, the Gödel, the Łukasiewicz and the product logic [19]. Starting with the conjunctor given by a triangular norm T, one also finds a truth degree function R for the implication connective, usually, via the adjointness condition

$$T(x,z) \le y \Longleftrightarrow z \le R(x,y). \tag{1.1}$$

In particular, if the triangular norm T is is left–continuous in both arguments, then R is uniquely determined by condition (1.1) (see [17]), and the algebraic structure associated with this logic, $([0,1],\cap,\cup,T,R,0,1)$, constitutes a residuated lattice [21]. For these reasons, the mapping R constructed from a left–continuous t–norm T by means of (1.1) is called a *residual implicator*.

Today, the mathematical foundation of a t–norm based logic is widely used in many applications to engineering, computer science and fuzzy systems. However, the properties of triangular norms are quite strong and, sometimes, this fact restricts their practical use. As underlined in [14], for example, "if one works with binary conjunctions and there is no need to extend them for three or more arguments, as happens e.g. in the inference pattern called *generalized modus ponens*, associativity of the conjunction is an unnecessarily restrictive condition" (see, also, [23]). In the same spirit, recent investigations have stressed the importance of giving a mathematical foundation also to fuzzy logics with a non-commutative conjunction [7],[8],[12],[20].

Starting with these motivations, in this paper we consider different conjunctors (Section 2), already known in the literature, and we characterize the corresponding residual implicators (Section 3), by obtaining, as a relevant case, the characterization of the residual implicator of a t–norm satisfying a Lipschitz property. A one-to-one correspondence is hence established between some construction methods for conjunctors and the corresponding residual implicators (Section 4). Finally, we stress that the residual implicators are important tools in the construction of special conjunctors (Section 5).

2. Definitions and first properties

In this section, we review some definitions and properties.

The *boolean conjunction* is the mapping $B : \{0,1\} \times \{0,1\} \to \{0,1\}$ given by:

$$B(0,0) = 0, \quad B(0,1) = 0, \quad B(1,0) = 0, \quad B(1,1) = 1. \tag{2.1}$$

A *conjunctor* is a mapping $C : [0,1]^2 \to [0,1]$ that is increasing in each variable and such that its restriction to $\{0,1\} \times \{0,1\}$ is a boolean conjunction [6].

A mapping $C : [0,1]^2 \to [0,1]$ is a *semi–copula* [3],[9],[11] (also called *t-seminorm* [32]), if it is a conjunctor with neutral element 1, viz. $C(x,1) = C(1,x)$ for every x in $[0,1]$.

A semi–copula C is a *pseudo triangular norm* (briefly, *pseudo t–norm*) if it is an associative operation on $[0,1]$ (see [12]).

A semi–copula C is a *triangular norm* (briefly, *t–norm*) if it is an associative and commutative operation on $[0,1]$ (see [1],[24]).

A semi–copula C is a *quasi–copula* if it is 1–*Lipschitz*, viz. it satisfies

$$|C(x_1,y_1)-C(x_2,y_2)|\le |x_1-x_2|+|y_1-y_2|$$

for every x_1,x_2,y_1,y_2 in $[0,1]$ ([16],[31]). Roughly speaking, such conjunctors are used when one wants that the aggregation process of the truth values is stable in the sense that small input errors generate small output errors.

A semi–copula C is called a *copula* if it is 2–increasing, viz.

$$C(x_1,y_1)+C(x_2,y_2)\ge C(x_1,y_2)+C(x_2,y_1)$$

for every x_1,x_2,y_1,y_2 in $[0,1]$, $x_1\le x_2$ and $y_1\le y_2$ [31].

Every copula is a quasi–copula, but the converse is not true in general. However, if we consider associative functions, we have the following result [24, Theorem 9.10].

Proposition 2.1. *For a t–norm T, T is a quasi–copula if, and only if, T is a copula.*

In the sequel, we are mainly interested in *left–continuous* semi–copulas, viz. conjunctors that are left–continuous in both variables, and, thus, jointly left–continuous [24, Proposition 1.19].

We denote by $\mathcal{L}$, $\mathcal{T}$, $\mathcal{Q}$ and $\mathcal{C}$, respectively, the class of left–continuous semi–copulas, left–continuous t–norms, quasi–copulas and copulas. We have that $\mathcal{T}$, $\mathcal{Q}$ and $\mathcal{C}$ are proper subsets of $\mathcal{L}$. Moreover $\mathcal{C}$ is a proper subset of $\mathcal{Q}$ and, as shown, $\mathcal{T}\cap\mathcal{Q}=\mathcal{T}\cap\mathcal{C}$.

Now, we recall some construction methods that can be introduced in $\mathcal{L}$ and its subsets. Let $\varphi:[0,1]\to[0,1]$ be an increasing bijection. For each $C\in\mathcal{L}$, we define the mapping $C_\varphi:[0,1]^2\to[0,1]$ via

$$C_\varphi(x,y)=\varphi^{-1}(C(\varphi(x),\varphi(y))), \tag{2.2}$$

called the *φ–transform* of C.

Proposition 2.2 ([10, 24, 25])**.** *The following statements hold:*

(i) *if C is in $\mathcal{L}$, then C_φ is in $\mathcal{L}$;*
(ii) *if C is in $\mathcal{T}$, then C_φ is in $\mathcal{T}$;*
(iii) *if C is in $\mathcal{Q}$ and φ is concave, then C_φ is in $\mathcal{Q}$;*
(iv) *if C is in $\mathcal{C}$ and φ is concave, then C_φ is in $\mathcal{C}$.*

Let $(]a_\alpha,e_\alpha[)_{\alpha\in A}$ be a family of non–empty, pairwise disjoint open subintervals of $[0,1]$ and let $(C_\alpha)_{\alpha\in A}$ be a family of left–continuous semi–copulas. We define the mapping $C:[0,1]^2\to[0,1]$ given by

$$C(x,y)=\begin{cases} a_\alpha+(e_\alpha-a_\alpha)\;C_\alpha\left(\dfrac{x-a_\alpha}{e_\alpha-a_\alpha},\dfrac{x-a_\alpha}{e_\alpha-a_\alpha}\right), & \text{if } (x,y)\in[a_\alpha,e_\alpha]^2,\\ \min\{x,y\}, & \text{otherwise;}\end{cases}$$

called the *ordinal sum* of the summands $\langle a_\alpha,e_\alpha,C_\alpha\rangle$, $\alpha\in A$, and we shall write $C=(\langle a_\alpha,e_\alpha,C_\alpha\rangle)_{\alpha\in A}$.

Proposition 2.3 ([24],[31])**.** *Let* $(]a_\alpha, e_\alpha[)_{\alpha\in A}$ *be a family of non–empty, pairwise disjoint open subintervals of* $[0,1]$ *and let* $(C_\alpha)_{\alpha\in A}$ *be a family of left–continuous semi–copulas. Consider* $C = (\langle a_\alpha, e_\alpha, C_\alpha\rangle)_{\alpha\in A}$*. If, for every* $\alpha \in A$, C_α *is in* $\mathcal{L}$ *(resp.* $\mathcal{T}$, $\mathcal{Q}$ *and* $\mathcal{C}$*), then* C *is in* $\mathcal{L}$ *(resp.* $\mathcal{T}$, $\mathcal{Q}$ *and* $\mathcal{C}$*).*

Let H be an idempotent binary aggregation operator, viz. $H : [0,1]^2 \to [0,1]$ is increasing in each variable and $H(x,x) = x$ for each $x \in [0,1]$ [4]. For all semi–copulas C_1 and C_2, we define the mapping $C : [0,1]^2 \to [0,1]$

$$C(x,y) = H(C_1(x,y), C_2(x,y)),$$

called H*–composition* of C_1 and C_2, and denoted by $C = H(C_1, C_2)$.

Proposition 2.4 ([26],[31])**.** *Let* C *be the* H*–composition of* C_1 *and* C_2*. The following statements hold:*

(i) *if* C_1 *and* C_2 *are in* $\mathcal{L}$ *and* H *is left–continuous in each variable, then* C *is in* $\mathcal{L}$*;*

(ii) *if* C_1 *and* C_2 *are in* $\mathcal{Q}$ *and* H *is a kernel aggregation operator, viz.*
$$|H(x,y) - H(x',y')| \le \max\{|x-x'|, |y-y'|\},$$
then C *is in* $\mathcal{Q}$*;*

(iii) *if* C_1 *and* C_2 *are in* $\mathcal{C}$ *and, for every* $\lambda \in [0,1]$, $H(x,y) = \lambda x + (1-\lambda)y$*, then* C *is in* $\mathcal{C}$*;*

(iv) *if* C_1 *and* C_2 *are in* $\mathcal{T}$ *and* H *is a projection, then* C *is in* $\mathcal{T}$*.*

In particular, $\mathcal{L}$, $\mathcal{Q}$ and $\mathcal{C}$, but not $\mathcal{T}$, are stable under convex combinations. Moreover, the pointwise maximum and minimum of two elements in $\mathcal{L}$ (resp. $\mathcal{Q}$) are also in $\mathcal{L}$ (resp. Q), but this is not the case for the classes $\mathcal{C}$ and $\mathcal{T}$.

3. Characterizations of residual implicators

As already mentioned in the case of t–norms, if one constructs a fuzzy logic by using a conjunctor C as the interpretation of the logical conjunction, then the implication connective $R_C : [0,1]^2 \to [0,1]$ (if no additional logical connectives are given) is usually derived from C by means of the adjointness condition (1.1). Moreover, if C is left–continuous in both variables, then R_C is uniquely determined by the expression

$$R_C(x,y) = \sup\{z \in [0,1] \mid C(x,z) \le y\}, \tag{3.1}$$

and it is called *residual implicator* of C (or C*–residuum*, shortly).

Example 3.1. The residual implicators related to the three most common t–norms, $T_{\mathbf{L}}$, $T_{\mathbf{P}}$ and $T_{\mathbf{M}}$, are given, respectively, by

$$R_{T_{\mathbf{L}}}(x,y) = \min\{1, 1-x+y\}, \qquad R_{T_{\mathbf{P}}}(x,y) = \min\{1, \tfrac{y}{x}\},$$

and

$$R_{T_{\mathbf{M}}}(x,y) = \begin{cases} 1, & \text{if } x \le y, \\ y, & \text{otherwise.} \end{cases}$$

In general, for any continuous Archimedean t–norm T with additive generator $t : [0,1] \to [0,+\infty]$, namely $T(x,y) := t^{-1}(\min\{t(0), t(x)+t(y)\})$, we have

$$R_T(x,y) = t^{-1}(\max\{0, t(y)-t(x)\}).$$

The characterization of the residual implicator of any left–continuous semi–copula is given below (and it can be derived from the results of [5]).

Theorem 3.2. *Let C be a left–continuous semi–copula. Then the C–residuum R_C satisfies the following properties:*

(R1) *$R_C(x,y) = 1$ if, and only if, $x \le y$;*
(R2) *$R_C(1,y) = y$ for all y in $[0,1]$;*
(R3) *R_C is decreasing in the first variable;*
(R4) *R_C is increasing in the second variable;*
(R5) *R_C is left–continuous in its first variable;*
(R6) *R_C is right–continuous in its second variable.*

We denote by $\mathcal{R}$ the class of all functions $R : [0,1]^2 \to [0,1]$ satisfying (R1)–(R6), which are called simply *residual implicators* (or *R–implicators*). We call *residuation* the mapping $\Psi : \mathcal{L} \to \mathcal{R}$ given, for each C in $\mathcal{L}$, by $\Psi(C) = R_C$, where R_C is defined by (3.1). The mapping $\Psi : \mathcal{L} \to \mathcal{R}$ is bijective and its inverse Ψ^{-1}, called *deresiduation*, is given, for each R in $\mathcal{R}$, by the mapping $C_R : [0,1]^2 \to [0,1]$ defined by

$$C_R(x,y) = \inf\{z \in [0,1] \mid R(x,z) \ge y\}. \tag{3.2}$$

For each $C \in \mathcal{L}$, $\Psi(C)$ is the C–residuum, and, for each $R \in \mathcal{R}$ we call $\Psi^{-1}(R)$ the *deresiduum* of R (briefly, *R–deresiduum*).

Proposition 3.3 ([5]). *If $R \in \mathcal{R}$, then the R–deresiduum C_R is a left–continuous semi–copula.*

Remark 3.4. In [12], the authors studied the algebraic structure associated to a fuzzy logic with a non–commutative conjunctor and, for any $C \in \mathcal{L}$, they consider also another kind of residual implicator, R'_C, defined by

$$R'_C(x,y) = \sup\{z \in [0,1] \mid C(z,x) \le y\}.$$

For our purpose, it is not useful to distinguish these two kinds of residual implicators. In fact, for any $C \in \mathcal{L}$, R'_C is simply the classical residual implicator of the left–continuous semi–copula C' defined by $C'(x,y) = C(y,x)$.

Now, given $\mathcal{A} \subseteq \mathcal{L}$, we wish to characterize the set $\Psi(\mathcal{A})$.

Theorem 3.5. *Let C be a left–continuous commutative semi–copula. Then the C–residuum R_C satisfies* (R1)–(R6) *and*

(R7) *for every x, y, z in $[0,1]$, $R_C(x,z) \ge y \Longleftrightarrow R_C(y,z) \ge x$.*

Conversely, if $R \in \mathcal{R}$ and satisfies (R7)*, then the R–deresiduum C_R is a left–continuous commutative semi–copula.*

Proof. Let C be a commutative semi–copula in $\mathcal{L}$. In view of Theorem 3.2 and Proposition 3.3, we only have to prove that the commutativity of C is equivalent to (R7). Since (3.2), C is commutative if, and only if, for all $(x, y) \in [0, 1]^2$

$$\{z \in [0,1] \mid R_C(x,z) \geq y\} = \{z \in [0,1] \mid R_C(y,z) \geq x\},$$

which yields $R_C(x, z) \geq y$ if, and only if, $R_C(y, z) \geq x$, viz. (R7). □

Notice that the above result can be derived also from [28, Theorem 3] (see also [27]), where commutativity is expressed in terms of the orthosymmetric property of the contour lines of a semi–copula.

Theorem 3.6. *Let C be a left–continuous pseudo t–norm. Then the C–residuum R_C satisfies* (R1)–(R6) *and*

(R8) *for every x, y, z, u in $[0, 1]$, $R_C(y, R_C(x, z)) \geq u$ if, and only if, there exists v in $[0, 1]$ such that $R_C(x, v) \geq y$ and $R_C(v, u) \geq z$.*

Conversely, if $R \in \mathcal{R}$ and satisfies (R8)*, then the R–deresiduum C_R is a pseudo t–norm.*

Proof. Let C be a pseudo t–norm in $\mathcal{L}$. In view of Theorem 3.2 and Proposition 3.3, we only have to prove that the associativity of C is equivalent to (R8). Thanks to (3.2), C is associative if, and only if, for all $(x, y, u) \in [0, 1]^3$

$$\begin{aligned}\{z \in [0,1] \mid R_C(\inf\{v \in [0,1] \mid R_C(x,v) \geq y\}, u) \geq z\}& \\ = \{z \in [0,1] \mid R_C(x,z) \geq \inf\{v \in [0,1] \mid R_C(y,v) \geq u\}\}&,\end{aligned}$$

i.e., there exists v_1 in $[0, 1]$ such that $R_C(x, v_1) \geq y$ and $R_C(v_1, u) \geq z$ if, and only if, there exists v_2 in $[0, 1]$ such that $R_C(x, z) \geq v_2$ and $R_C(y, v_2) \geq u$. Since (R4), the latter two conditions are equivalent to $R_C(y, R_C(x, z)) \geq u$, and, hence, condition (R8) holds. □

For another characterization of pseudo t–norms, see also [27] and [28]. In particular, we derive the following characterization of left–continuous t–norms.

Theorem 3.7. *Let T be a left–continuous t–norm. Then the T–residuum R_T satisfies* (R1)–(R8)*. Conversely, if $R \in \mathcal{R}$ and satisfies* (R7) *and* (R8)*, then the R–deresiduum T_R is a left–continuous t–norm.*

As a corollary, we obtain also the following (equivalent) characterization of left–continuous t–norms.

Corollary 3.8. *The mapping $T : [0, 1]^2 \to [0, 1]$ is a left–continuous t–norm if, and only if, the T–residuum R_T satisfies* (R1)–(R6) *and the exchange principle, viz. for each x, y, z in $[0, 1]$*

(EP) $R_T(x, R_T(y, z)) = R_T(y, R_T(x, z))$.

Proof. Let T be a left–continuous t–norm. In view of Theorem 3.7, R_T satisfies (R1)–(R8). In particular, as a consequence of (R8), for every x, y, z, u in $[0, 1]$, $R_T(y, R_T(x, z)) \geq u$ if, and only if, $R_T(x, v) \geq y$ and $R_T(v, u) \geq z$ for some v in

$[0, 1]$. From (R7), $R_T(x, v) \geq y$ if, and only if, $R_T(y, v) \geq x$. Thus, again by using (R8), $R_T(y, R_T(x, z)) \geq u$ if, and only if, $R_T(x, R_T(y, z)) \geq u$, viz. (EP) holds. The converse implication can be found in [30]. □

Theorem 3.9. *Let Q be a quasi–copula. Then the Q–residuum R_Q satisfies* (R1)–(R6) *and the two following properties:*

(R9) *for every $\varepsilon > 0$, $R_Q(x + \varepsilon, y) \geq R_Q(x, y - \varepsilon)$;*

(R10) *for every $\varepsilon > 0$, $R_Q(x, y) \geq R_Q(x, y - \varepsilon) + \varepsilon$.*

Conversely, if $R \in \mathcal{R}$ and satisfies conditions (R9)–(R10)*, then the R–deresiduum Q_R is a quasi–copula.*

Proof. Let Q be a quasi–copula. In view of Theorem 3.2, R_Q satisfies (R1)–(R6). Moreover, for each $\varepsilon > 0$, $Q(x + \varepsilon, z) - Q(x, z) \leq \varepsilon$, from which

$$\{z \in [0, 1] \mid Q(x + \varepsilon, z) \leq y\} \supseteq \{z \in [0, 1] \mid Q(x, z) + \varepsilon \leq y\},$$

which, in turn, is equivalent to $R_Q(x+\varepsilon, y) \geq R_Q(x, y-\varepsilon)$, viz. (R9). Analogously, for every $\varepsilon > 0$, since $Q(x, z) - Q(x, z - \varepsilon) \leq \varepsilon$, we obtain

$$\{z \in [0, 1] \mid Q(x, z) \leq y\} \supseteq \{z \in [0, 1] \mid Q(x, z - \varepsilon) \leq y - \varepsilon\},$$

which implies

$$\sup\{z \in [0, 1] \mid Q(x, z) \leq y\} \geq \sup\{z \in [0, 1] \mid Q(x, z - \varepsilon) \leq y - \varepsilon\}.$$

Thus $R_Q(x, y) \geq \varepsilon + R_Q(x, y - \varepsilon)$ and (R10) holds.

Conversely, if $R \in \mathcal{R}$, then Proposition 3.3 ensures that the R–deresiduation Q_R is a left–continuous semi–copula. In order to prove the 1–Lipschitz property for Q_R, notice that it follows from property (R9) that, for every $\varepsilon > 0$,

$$\{z \in [0, 1] \mid R(x + \varepsilon, z) \geq y\} \supseteq \{z \in [0, 1] \mid R(x, z - \varepsilon) \geq y\},$$

and, then

$$\inf\{z \in [0, 1] \mid R(x + \varepsilon, z) \geq y\} \leq \inf\{z \in [0, 1] \mid R(x, z - \varepsilon) \geq y\},$$

which implies $Q_R(x + \varepsilon, y) \leq \varepsilon + Q_R(x, y)$. Analogously, by using (R10), it follows that

$$\{z \in [0, 1] \mid R(x, z) - \varepsilon \geq y\} \supseteq \{z \in [0, 1] \mid R(x, z - \varepsilon) \geq y\},$$

which implies

$$\inf\{z \in [0, 1] \mid R(x, z) - \varepsilon \geq y\} \leq \inf\{z \in [0, 1] \mid R(x, z - \varepsilon) \geq y\},$$

and, hence, $Q_R(x, y + \varepsilon) \leq \varepsilon + Q_R(x, y)$. These two latter results imply that Q_R is 1–Lipschitz. □

Now, by using Proposition 2.1, we obtain the following result.

Theorem 3.10. *If C is an associative copula, then the C–residuum R_C satisfies* (R1)–(R10)*. Conversely, if $R \in \mathcal{R}$ and satisfies* (R1)–(R10)*, then the R–deresiduum C_R is an associative copula.*

The characterization of the residual implicator of a copula C is still an open question, and the main difficulty in this problem is to express the 2–increasing property of C in terms of some properties of the C–residuum.

4. Induced construction methods for residual implicators

In this section, we investigate whether there exists a close relationship between some construction methods in the class of left–continuous conjunctors and corresponding construction methods in the class of residual implicators.

Proposition 4.1 ([29])**.** *Let $(]a_\alpha, e_\alpha[)_{\alpha\in A}$ be a family of non–empty, pairwise disjoint open subintervals of $[0,1]$ and let $(C_\alpha)_{\alpha\in A}$ be a family of left–continuous semi–copulas. Let C be the ordinal sum $C = (\langle a_\alpha, e_\alpha, C_\alpha\rangle)_{\alpha\in A}$. Then the C–residuum is given by*

$$R_C(x,y) = \begin{cases} a_\alpha + (e_\alpha - a_\alpha) R_{C_\alpha}\left(\dfrac{x-a_\alpha}{e_\alpha - a_\alpha}, \dfrac{y-a_\alpha}{e_\alpha-a_\alpha}\right), & \text{if } a_\alpha < y < x \le e_\alpha; \\ R_{T_{\mathbf{M}}}, & \text{otherwise.} \end{cases} \tag{4.1}$$

This result admits the following converse implicator.

Proposition 4.2. *Let $(]a_\alpha, e_\alpha[)_{\alpha\in A}$ be a family of non–empty, pairwise disjoint open subintervals of $[0,1]$ and let $(R_\alpha)_{\alpha\in A}$ be a family in $\mathcal{R}$. Let $R : [0,1]^2 \to [0,1]$ be given by*

$$R(x,y) = \begin{cases} a_\alpha + (e_\alpha - a_\alpha) R_{C_\alpha}\left(\dfrac{x-a_\alpha}{e_\alpha - a_\alpha}, \dfrac{y-a_\alpha}{e_\alpha-a_\alpha}\right), & \text{if } a_\alpha < y < x \le e_\alpha; \\ R_{T_{\mathbf{M}}}, & \text{otherwise.} \end{cases} \tag{4.2}$$

Then $R \in \mathcal{R}$.

Proof. A tedious verification shows that R satisfies conditions (R1)–(R6). □

The residual implicator R given by (4.2) is called the *R-ordinal sum* of the summands $\langle a_\alpha, e_\alpha, R_\alpha\rangle$, $\alpha \in A$, and we shall write $R = (\langle a_\alpha, e_\alpha, R_\alpha\rangle)_{\alpha\in A}$. By connecting the two propositions above , we obtain the following consequence.

Corollary 4.3. *The residuation of an ordinal sum of left–continuous semi–copulas is the R-ordinal sum of the residua of the summands. The deresiduation of an ordinal sum of R–implicators is the ordinal sum of the deresidua of the summands.*

Proposition 4.4 ([2])**.** *Let C be a left–continuous semi–copula and let φ be an increasing bijection of $[0,1]$. Let C_φ be the φ-transform of C given by (2.2). Then the residual implicator of C_φ is given by*

$$R_{C_\varphi}(x,y) = (R_C)_\varphi(x,y) = \varphi^{-1}(R_C(\varphi(x), \varphi(y))). \tag{4.3}$$

Proposition 4.5. *Given $R \in \mathcal{R}$ and an increasing bijection φ of $[0,1]$, consider the φ–transform of R given by $R_\varphi(x,y) = \varphi^{-1}(R(\varphi(x),\varphi(y)))$. Then $R_\varphi \in \mathcal{R}$.*

Proof. Let R be in $\mathcal{R}$ and let φ be an increasing bijection of $[0,1]$. We have to prove that R_φ satisfies (R1)–(R6). For every x, y in $[0,1]$, we have that $R_\varphi(x,y) = 1$ if, and only if, $\varphi^{-1}(R(\varphi(x),\varphi(y))) = 1$, which is equivalent to $x \le y$, because R satisfies (R1) and φ is increasing. Moreover, since R satisfies (R2), it is immediate that $R_\varphi(x,y) = 1$. Therefore R_φ satisfies (R1) and (R2). Since R satisfies (R3) and (R4) and φ is increasing, R_φ satisfies (R3) and (R4). In the same manner, since R satisfies (R5) and (R6) and φ is a bijection, it is immediate that R_φ satisfies (R5) and (R6). □

Corollary 4.6. *The residuation of the φ–transform of a semi–copula C is the φ–transform of the C–residuum. The deresiduation of the φ–transform of a residual implicator R is the φ–transform of the R–deresiduum.*

In the case of the pointwise composition of two left–continuous semi–copulas, we have the following partial result.

Proposition 4.7. *Let C_1 and C_2 be left continuous semi–copulas and let C be the pointwise maximum of C_1 and C_2. Then the C–residuum R_C is the pointwise minimum of R_{C_1} and R_{C_2}. Conversely, given two R–implicators R_1 and R_2, let R be the pointwise minimum of R_1 and R_2. Then the R–deresiduum C_R is the pointwise maximum of C_{R_1} and C_{R_2}.*

Proof. Let C_1 and C_2 be in $\mathcal{L}$ and let C be the pointwise maximum of C_1 and C_2. Then

$$\begin{aligned} R_C(x,y) &= \sup\{z \in [0,1] \mid \max\{C_1(x,z), C_2(x,z)\} \le y\} \\ &= \sup\{z \in [0,1] \mid C_1(x,z) \le y,\, C_2(x,z) \le y\} \\ &= \min\{R_{C_1}(x,y), R_{C_2}(x,y)\}. \end{aligned}$$

Conversely, let R_1 and R_2 be in $\mathcal{R}$ and let R be the pointwise minimum of R_1 and R_2. Then

$$\begin{aligned} C_R(x,y) &= \inf\{z \in [0,1] \mid \min\{R_1(x,z), R_2(x,z)\} \ge y\} \\ &= \inf\{z \in [0,1] \mid C_1(x,z) \le y \,\mathrm{or}\, C_2(x,z) \le y\} \\ &= \max\{C_{R_1}(x,y), C_{R_2}(x,y)\}, \end{aligned}$$

which concludes the proof. □

A similar result holds by interchanging maximum and minimum.

Proposition 4.8. *Let C_1 and C_2 be left continuous semi–copulas and let C be the pointwise minimum of C_1 and C_2. Then the C–residuum R_C is the pointwise maximum of R_{C_1} and R_{C_2}. Conversely, given two R–implicators R_1 and R_2, let R be the pointwise maximum of R_1 and R_2. Then the R–deresiduum C_R is the pointwise minimum of C_{R_1} and C_{R_2}.*

In the statements above, we have thus established a one-to-one correspondence between construction methods for conjunctors and construction methods for residual implicators. However, it is not clear whether this correspondence can be extended in a natural way also to any pointwise composition of two conjunctors, which is not a lattice operation. For example, given C_1 and C_2 in $\mathcal{L}$, it is easily proved that, for any λ in $[0,1]$, $C = \lambda C_1 + (1 - \lambda C_2)$ is also in $\mathcal{L}$, but it is not clear whether R_C can be constructed directly by means of some pointwise operations involving R_{C_1} and R_{C_2}. In particular, $R = \lambda R_{C_1} + (1-\lambda) R_{C_2}$ is a residual implicator that, in general, differs from R_C.

Example 4.9. Given the t–norms $T_{\mathbf{L}}$ and $T_{\mathbf{M}}$, consider the corresponding residual implicators $R_{T_{\mathbf{L}}}$ and $R_{T_{\mathbf{M}}}$. For each $\alpha \in [0,1]$, let R_α be the residual implicator given by $R_\alpha = \alpha R_{T_{\mathbf{L}}} + (1-\alpha) R_{T_{\mathbf{M}}}$. Then the deresiduum of R_α is given by

$$C_{R_\alpha}(x,y) = \begin{cases} 0, & \text{if } y \leq -\alpha x + \alpha, \\ x, & \text{if } y \geq (1-\alpha)x + \alpha, \\ y - \alpha(1-x), & \text{otherwise,} \end{cases}$$

which is obviously different from the convex combination of $T_{\mathbf{L}}$ and $T_{\mathbf{M}}$.

This apparently “negative” result will be used in the next section.

5. Direct construction methods for residual implicators

Having at one’s disposal a large class of conjunctors is an important tool in some applications of fuzzy set theory, fuzzy preference modelling and multicriteria decision making [15]. In this section, we analyze two special constructions of residual implicators, which, by deresiduation, allow to obtain new constructions of conjunctors. We start with the following result.

Proposition 5.1. *For every $R \in \mathcal{R}$, the mapping $R^* : [0,1]^2 \to [0,1]$ given by*

$$R^*(x,y) = \begin{cases} R(x,y), & \text{if } x + y \geq 1, \\ R(1-y, 1-x), & \text{otherwise,} \end{cases}$$

is also in $\mathcal{R}$.

Proof. Given an R–implicator R, we have to prove that R^* satisfies conditions (R1)–(R6). Conditions (R1), (R2), (R5) and (R6) are immediate. In order to show that R^* is decreasing in the first variable, let x, x', y be in $[0,1]$ with $x \leq x'$. If $x' + y < 1$, then

$$R^*(x', y) = R(1-y, 1-x') \leq R(1-y, 1-x) = R^*(x,y),$$

because R satisfies (R3). If $x' + y \geq 1$ and $x + y < 1$, then

$$R^*(x', y) = R(x', y) \leq R(1-y, 1-x) = R^*(x,y),$$

because R satisfies (R3) and (R4). If $x+y \geq 1$, then $R^*(x', y) \leq R^*(x, y)$, because R satisfies (R3). Therefore R^* satisfies (R3). Analogously, we prove that R^* satisfies (R4). □

Starting with the method given in Proposition 5.1, we can consider a new method for constructing a left–continuous semi–copula. For each C in $\mathcal{L}$, we introduce the new conjunctor C^* defined by $C^* = \Psi^{-1}((R_C)^*)$, viz. C^* is obtained by deresiduation of $(R_C)^*$. For example, $(T_{\mathbf{P}})^*(x, y) = \max\{0, \min\{xy, \frac{x+y-1}{y}\}\}$.

Given an R–implicator R, R^* satisfies the *contrapositive symmetry* with respect to the strong negation $n(t) = 1 - t$, viz.

(CS) $R^*(x, y) = R^*(1 - y, 1 - x)$ for every x, y in $[0, 1]$.

Contrapositive symmetry of residual implicators was considered, for the first time, in [13] (see also [22]), in order to investigate whether, in some fuzzy logics, the truth value of $a \Rightarrow b$ is the same as the truth value of NOT $b \Rightarrow$ NOT a. In particular, it is known from [22] that, if T is a t–norm such that R_T satisfies (CS), then T belongs to the family $(T_a)_{a\in[1/2,1]}$ of t–norms given by

$$T_a(x, y) = \begin{cases} 0, & \text{if } x + y \leq 1, \\ x + y - a, & \text{if } x + y > 1,\ (x, y) \in [1 - a, a]^2, \\ \min\{x, y\}, & \text{otherwise.} \end{cases}$$

Notice that every member of this class can be obtained by means of the construction of conjunctors derived by Proposition 5.1. In fact, if we consider a member of the Mayor–Torrens family of t–norms $(T_a^{MT})_{a\in[0,1]}$, which is an ordinal sum of the type $T_a^{MT} = (\langle 0, a, T_{\mathbf{L}}\rangle)$ [24], then we have that $(T_a^{MT})^* = T_{\max\{a,0.5\}}$.

Finally, we present a construction method for quasi-copulas, which is based on the following result.

Proposition 5.2. *Let R_1 and R_2 be in $\mathcal{R}$. Then, for every $\lambda \in [0, 1]$, $R = \lambda R_1 + (1 - \lambda R_2)$ is also in $\mathcal{R}$. Moreover, if R_1 and R_2 also satisfy* (R9)–(R10)*, then R also satisfies* (R9)–(R10)*.*

Proof. This is an immediate consequence of the fact that conditions (R1)–(R6) and (R9)–(R10) are preserved under convex combinations. □

As a consequence, given Q_1 and Q_2 be in $\mathcal{Q}$, let R_{Q_1} and R_{Q_2} be the R–implicators associated to Q_1 and Q_2, respectively. Then, for every $\lambda \in [0, 1]$, $R = \lambda R_{Q_1} + (1 - \lambda R_{Q_2})$ is also an R–implicator satisfying (R9)–(R10). Moreover, the deresiduum Q_R is a quasi–copula.

Example 5.3. Let $R_{T_{\mathbf{P}}}$ be the residual implicator of $T_{\mathbf{P}}$, and let R_C be the residual implicator of the copula C given by $C(x, y) = \sqrt{T_{\mathbf{P}}(x, y) \cdot T_{\mathbf{M}}(x, y)}$ (see [31]). Consider the residual implicator

$$R(x, y) = \frac{R_C(x, y) + R_{T_{\mathbf{P}}}(x, y)}{2}.$$

By deresiduation, R generates the following quasi–copula

$$Q_R(x,y) = \min\left\{\frac{x(\sqrt{1+8y}-1)}{2}, \frac{2xy}{1+\sqrt{x}}\right\}.$$

Notice that the conjunctors C_{R_α} and Q_R, respectively, in Examples 4.9 and 5.3, were obtained by the convex combination of the R–implicators of two copulas, and they are, in fact, copulas. However, it is unknown whether Proposition 5.2 can be applied in the case of copulas. In the affirmative case, we will have another tool to build classes of bivariate probability distribution functions [31].

6. Conclusions

In many practical applications of fuzzy set theory, it seems clear that one needs more flexibility in the choice of the connectives and that, in particular, the associativity and the commutativity of a conjunction can be removed.

Motivated by these considerations, we present several classes of conjunctors and characterize their respective residual implicators. In particular, we obtain a new characterization for the residual implicator of a left–continuous triangular norm.

We have hence established a one-to-one correspondence between some construction methods for conjunctors and some construction methods for residual implicators. Specifically, we have that:

(i) an ordinal sum of conjunctors is transformed into an R-ordinal sum of the corresponding residual implicators;
(ii) a φ–transform of a conjunctor is transformed into a φ–transform of the corresponding residual implicators;
(iii) the maximum of two conjunctors is transformed into the minimum of the corresponding residual implicators.

Finally, we give directly two construction methods in the class of residual implicators, and, by using a deresiduation procedure, we obtain new conjunctors. In particular, we use this tool to construct new quasi–copulas.

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