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Multivariate Analysis

Copulas Constructed from Horizontal Sections

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In analogy with the study of copulas whose diagonal sections have been fixed, we study the set $\mathscr{C}_h$ of copulas for which a horizontal section h has been given. We first show that this set is not empty, by explicitly writing one such copula, which we call horizontal copula. Then we find the copulas that bound both below and above the set $\mathscr{C}_h$. Finally, we determine the expressions for Kendall's tau and Spearman's rho for the horizontal and the bounding copulas.

Keywords Copula; Horizontal section; Kendall's tau; Spearman's rho.

Mathematics Subject Classification Primary 60E05; Secondary 62E10, 26E99.

1. Introduction

We briefly recall that a copula (see Sklar, 1959; Schweizer and Sklar, 1983) is a function $C : [0, 1] \times [0, 1] \to [0, 1]$ that satisfies the following properties:

(i) for all $x \in [0, 1]$, $C(x, 0) = C(0, x) = 0$;
(ii) for all $x \in [0, 1]$, $C(1, x) = C(1, x) = x$;
(iii) it is 2-increasing, i.e., for all x, x^*, y, and y^* in [0,1] with $x \leq x^*$ and $y \leq y^*$ one has

$$C(x^*, y^*) - C(x^*, y) - C(x, y^*) + C(x, y) \geq 0. \tag{1}$$

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As a consequence of these properties, a copula C satisfies the Lipschitz condition

$$|C(x^*, y^*) - C(x, y)| \leq |x^*, x| + |y^* - y|,$$

is non decreasing in each variable and is absolutely continuous in each variable.

A copula is the restriction of a bivariate distribution function that concentrates all the probability on the unit square $[0, 1]^2$ and which has uniform marginals; the importance of this concept for probability and statistics stems from Sklar's theorem (see Sklar, 1959; Nelsen, 1999).

Given a copula C, its *diagonal section* is the function $\delta : [0, 1] \to [0, 1]$ defined by $\delta(u) = C(u, u)$. If C is the copula of the random variables X and Y then for its diagonal section δ we have

$$\delta(u) = P(X \leq Q_X(u), Y \leq Q_Y(u)),$$

where $Q_X(u)$, $Q_Y(u)$ are the u-quantiles of the random variables X and Y, respectively.

The problem of finding a copula C if only the diagonal section δ of C is known has been the object of several papers. Bertino (1977) found the smallest copula with a given diagonal section δ (compare also Fredricks and Nelsen, 1997, 2002; Nelsen, 1999). In Fredricks and Nelsen (1997), through

$$C_\delta(x, y) = \min\left(x, y, \frac{\delta(x) + \delta(y)}{2}\right).$$

a copula was introduced whose diagonal section coincides with a prescribed suitable function δ that plays rôle of diagonal; this copula was called *diagonal copula*. Other related results can also be found in Nelsen et al. (2004) and Klement and Kolesárová (2005).

The aim of this article is to study copulas with a given horizontal section. The article is organized as follows. In the following section, copulas with given horizontal sections are discussed. In Sec. 3, the greatest and smallest copula with a prescribed horizontal section are given, and in Sec. 4, some dependence parameters of these copulas are introduced.

2. Horizontal Sections of Copulas

Let $b \in]0, 1[$ be a fixed number. The horizontal b-section of a copula C is the function $h_{C,b} : [0, 1] \to [0, 1]$ given by $h_{C,b}(x) = C(x, b)$. In general, we shall speak of a horizontal section h. Given two random variables, U and V, having uniform distribution on [0,1] and linked through the copula C, we have

$$h_{C,b}(u) = P(U \leq u, V \leq b) = bP(U \leq u \mid V \leq b) = bF_{U \mid V \leq b}(u),$$

i.e., the horizontal section $h_{C,b}$ is b times the conditional distribution function of U under the condition $V \leq b$. In general, when no restrictions are imposed on the random variables X and Y linked by the copula C, for each $t \in \mathbb{R}$ we have

$$h_{C,b}(F_X(t)) = bF_{X \mid Y \leq Q_Y(b)}(t),$$

where Q_Y is the quantile function of Y. Thus horizontal sections express our knowledge about the random variable X under the condition that Y does not exceed some prescribed fixed value. This corresponds to truncation (on the right) of the random variable Y, a well-known statistical practice (see, e.g., Kendall and Stuart, 1973). For instance, if Y respresents the distance from the center of a vertical circular target of fixed radius b on a shooting range, then one can only observe Y for the shots that actually hit the target, namely for those shots for which $Y \leq b$.

In this article, we wish to study possible copulas C linking X and Y if only such partial knowledge expressed by a horizontal section is available.

It follows easily from the properties of a copula that all horizontal b-sections of copulas are non decreasing, 1-Lipschitz functions. Since the copulas W and M given by $W(x, y) = \max(x + y - 1, 0)$ and $M(x, y) = \min(x, y)$ are the Fréchet–Hoeffding lower and upper bounds of copulas, we have

$$h_{W,b} \leq h_{C,b} \leq h_{M,b} \tag{2}$$

i.e., for all $x \in [0, 1]$,

$$\max(x + b - 1, 0) \leq h_{C,b}(x) \leq \min(x, b).$$

In particular, $h_{C,b}(0) = 0$ and $h_{C,b}(1) = b$ for the horizontal section of every copula C.

For fixed $b \in]0, 1[$, let $\mathscr{H}_b$ be the set of all non decreasing 1-Lipschitz functions h satisfying the same bounds as in (2). The question arises whether for each $h \in \mathscr{H}_b$ there is a copula C whose horizontal b-section coincides with h, i.e., $C(x, b) = h(x)$ for all $x \in [0, 1]$.

Proposition 2.1. *Let $b \in]0, 1[$, $h \in \mathscr{H}_b$. Then the function $\widetilde{C}_h : [0, 1]^2 \to [0, 1]$ defined by*

$$\widetilde{C}_h(x, y) = \begin{cases} \dfrac{yh(x)}{b} & \text{if } y \leq b, \\ \dfrac{(1-y)h(x) + (y-b)x}{1-b} & \text{otherwise}, \end{cases} \tag{3}$$

is a copula whose horizontal b-section coincides with h.

Proof. We first verify the boundary conditions. Let x and y belong to $[0, 1]$. Because of $h(1) = b$ and $h(0) = 0$, we have $\widetilde{C}_h(x, 1) = \frac{(1-b)x}{1-b} = x$ and $\widetilde{C}_h(1, y) = \frac{yh(1)}{b} = y$ in the case $y \leq b$, while, if $y > b$, then we obtain $\widetilde{C}_h(1, y) = \frac{(1-y)b+(y-b)}{1-b} = y$. Similarly, $\widetilde{C}_h(x, 0) = \widetilde{C}_h(0, y) = 0$.

We next verify the 2-increasing property of $\widetilde{C}_h$. Put $R = [x, x^*] \times [y, y^*]$ with $0 \leq x \leq x^* \leq 1$ and $0 \leq y \leq y^* \leq 1$, and denote by $V_{\widetilde{C}_h}(R)$ the $\widetilde{C}_h$-volume of the rectangle R, i.e.,

$$V_{\widetilde{C}_h}(R) = \widetilde{C}_h(x, y) + \widetilde{C}_h(x^*, y^*) - \widetilde{C}_h(x^*, y) - \widetilde{C}_h(x, y^*). \tag{4}$$

If $R \subseteq [0, 1] \times [0, b]$, then from (3) and since h is non decreasing, we obtain

$$V_{\widetilde{C}_h}(R) = \frac{1}{b}(h(x^*) - h(x))(y^* - y) \geq 0.$$

If $R \subseteq [0, 1] \times [b, 1]$, then from (3) and the 1-Lipschitz property of h we have

$$V_{\widetilde{C}_h}(R) = \frac{1}{1-b}(y^* - y)(h(x) - h(x^*) + x^* - x) \geq 0.$$

The 2-increasing property of $\widetilde{C}_h$ in general follows from the additivity of the $\widetilde{C}_h$-volume. □

Copulas defined by (3) will be called *horizontal copulas*. Note that all the horizontal copulas $\widetilde{C}_h$ are absolutely continuous.

Remark 2.1. Formally, we can also deal with $b \in \{0, 1\}$. However, then the sets $\mathscr{H}_b$ are trivial. For $b \in \{0, 1\}$, the horizontal copula is $\widetilde{C}_h = \Pi$, the product copula Π defined by $\Pi(x, y) = xy$, see (3) (using the convention $\frac{0}{0} = 0$).

Moreover, if $h \in \mathscr{H}_b$ is piecewise linear then the horizontal copula $\widetilde{C}_h$ is piecewise bilinear (compare the checkerboard approximation copulas in Li et al., 1998, and the bilinear extension of discrete copulas in Sungur and Ng, 2005).

Example 2.1. Fix $b \in]0, 1[$. The horizontal copula for the greatest element h_M of $\mathscr{H}_b$ given by $h_M(x) = \min(x, b)$, is the ordinal sum $\widetilde{C}_{h_M} = (\langle 0, b, \Pi\rangle, \langle b, 1, \Pi\rangle)$.

The smallest element h_W of $\mathscr{H}_b$ given by $h_W(x) = \max(x + b - 1, 0)$, leads to the horizontal copula $\widetilde{C}_{h_W}$, which is a W-ordinal sum of Π-summands (compare Mesiar and Szolgay, 2004),

$$\widetilde{C}_{h_W} = W\text{-}(\langle 0, 1-b, \Pi\rangle, \langle 1-b, 1, \Pi\rangle).$$

3. Properties of the Class $\mathscr{C}_h$

The proof of the following result is trivial and therefore omitted.

Proposition 3.1. *Let $b \in]0, 1[$ and $h \in \mathscr{H}_b$. The set $\mathscr{C}_h$ of all copulas, whose horizontal b-section is h, is a convex and compact subset of the space of all continuous real valued functions defined on $[0, 1]^2$ with respect to the topology of uniform convergence.*

Next, we show that the set $\mathscr{C}_h$ has a greatest and a smallest element.

Theorem 3.1. *Let $b \in]0, 1[$, $h \in \mathscr{H}_b$. Then the function $\overline{C}_h : [0, 1]^2 \to [0, 1]$ defined by*

$$\overline{C}_h(x, y) = \begin{cases} y & \text{if } y \leq h(x), \\ h(x) & \text{if } h(x) < y \leq b, \\ y - b + h(x) & \text{if } b < y \leq x + b - h(x), \\ x & \text{otherwise}, \end{cases} \tag{5}$$

is the greatest copula whose horizontal b-section coincides with h.

Proof. Using the properties $h(0) = 0, h(1) = b$ and the definition (5) of $\overline{C}_h$, we obtain, for all x and y in $[0, 1]$,

$$\overline{C}_h(x, 1) = x \quad \text{and} \quad \overline{C}_h(1, y) = \begin{cases} y & \text{if } y \leq b, \\ y - b + h(1) = y & \text{otherwise.} \end{cases}$$

Similarly, for all x and y in $[0, 1]$, $\overline{C}_h(x, 0) = \overline{C}_h(0, y) = 0$, which means that $\overline{C}_h$ satisfies the boundary conditions of copulas.

Next, consider a rectangle $R = [x, x^*] \times [y, y^*]$ with $0 \leq x \leq x^* \leq 1$ and $0 \leq y \leq y^* \leq 1$. The 2-increasing property of $\overline{C}_h$ is equivalent to the property $V_{\overline{C}_h}(R) \geq 0$ for each $R \subseteq [0, 1]^2$, where $V_{\overline{C}_h}(R)$ is the $\overline{C}_h$-volume of R. The unit square is divided by the curves $y = h(x)$, $y = b$, $y = x + b - h(x)$ into four subsets. Evidently, if R is entirely contained in one of these sets, then $V_{\overline{C}_h}(R) = 0$. The only two interesting cases where the 2-increasing property of $\overline{C}_h$ must be verified are the following ones.

Consider first that $y = x + b - h(x)$ and $y^* = x^* + b - h(x^*)$. Then

$$V_{\overline{C}_h}(R) = x^* - y + b - h(x^*) = y^* - y \geq 0.$$

Next, if $y = h(x)$ and $y^* = h(x^*)$, then

$$V_{\overline{C}_h}(R) = y^* - h(x) = y^* - y \geq 0.$$

Note that any other rectangle $R \subseteq [0, 1]^2$ is always a finite union of rectangles of the types discussed above and the 2-increasing property of $\overline{C}_h$ follows from the additivity of $V_{\overline{C}_h}$. Evidently, $\overline{C}_h(x, b) = h(x)$, for every $x \in [0, 1]$.

Next, notice that $\overline{C}_h$ can be written in the form $\overline{C}_h = G_h \wedge M$, where

$$G_h(x, y) = \begin{cases} h(x) & \text{if } y \leq b, \\ y - b + h(x) & \text{otherwise,} \end{cases} \tag{6}$$

is the greatest 1-Lipschitz non decreasing function from $[0, 1]^2$ into $\mathbb{R}$ such that $G_h(x, b) = h(x)$ (see Kolesárová and Klement, 2005). Since M is the upper Fréchet–Hoeffding bound for copulas, each copula $C \in \mathscr{C}_h$ is bounded by $G_h \wedge M = \overline{C}_h$. $\square$

Theorem 3.2. *Let $b \in]0, 1[$, $h \in \mathscr{H}_b$. Then the function $\underline{C}_h : [0, 1]^2 \to [0, 1]$ defined by*

$$\underline{C}_h(x, y) = \begin{cases} 0 & \text{if } y \leq b - h(x), \\ y - b + h(x) & \text{if } b - h(x) < y \leq b, \\ h(x) & \text{if } b < y \leq 1 - x + h(x), \\ x + y - 1 & \text{otherwise,} \end{cases}$$

is the smallest copula whose horizontal b-section coincides with h.

Proof. The proof is similar to that of Theorem 3.1. We omit the details of verification of the boundary conditions. The unit square is again divided into four subsets by the curves $y = b - h(x)$, $y = b$, and $y = 1 - x + h(x)$. The 2-increasing property of $\underline{C}_h$ will be examined only in the following two relevant cases.

Consider first the rectangle $R = [x, x^*] \times [y, y^*]$ with $y = b - h(x^*)$ and $y^* = b - h(x)$. Then, from the isotony of h we have

$$V_{\underline{C}_h}(R) = y^* - b + h(x^*) = h(x^*) - h(x) \geq 0.$$

Next, if $R = [x, x^*] \times [y, y^*]$ with $y = 1 - x^* + h(x^*)$ and $y^* = 1 - x + h(x)$, then

$$\begin{aligned} V_{\underline{C}_h}(R) &= h(x) + x^* + y^* - 1 - h(x) - h(x^*) = x^* + 1 - x + h(x) - 1 - h(x^*) \\ &= x^* - x + h(x) - h(x^*) \geq 0. \end{aligned}$$

The last inequality follows from the 1-Lipschitz property of h.

Since any other rectangle $R \subseteq [0, 1]^2$ is a finite union either of rectangles that are of the two types described above or of rectangles that are subsets of one of the mentioned four areas, where $V_{\underline{C}_h}(R) = 0$, the 2-increasing property is proved. It is clear that, for all $x \in [0, 1]$, one has $\underline{C}_h(x, b) = h(x)$.

Observe that $\underline{C}_h$ may be expressed in the form $\underline{C}_h = F_h \vee W$, where

$$F_h(x, y) = \begin{cases} y - b + h(x) & \text{if } y \leq b, \\ h(x) & \text{otherwise,} \end{cases}$$

i.e., F_h is the smallest non decreasing 1-Lipschitz function on $[0, 1]^2$ with horizontal b-section given by h. Since the lower Fréchet–Hoeffding bound for copulas is W, $\underline{C}_h$ is the smallest copula with the required property. □

Corollary 3.1. *For every function $h \in \mathscr{H}_b$, and for every copula C, the following statements are equivalent*:

(i) $C \in \mathscr{C}_h$;
(ii) $\underline{C}_h \leq C \leq \overline{C}_h$.

Remark 3.1.

(i) For all $b \in]0, 1[$ and for every $h \in \mathscr{H}_b$, $\overline{C}_h$, and $\underline{C}_h$ are singular copulas with supports on the curves $y = h(x)$, $y = x + b - h(x)$ and $y = b - h(x)$, $y = 1 - x + h(x)$, respectively.

(ii) The upper bound $\overline{C}_h$ and the lower bound $\underline{C}_h$ of copulas in $\mathscr{C}_h$ can also be written as

$$\overline{C}_h(x, y) = \min(h(x) + \max(y - b, 0), M(x, y)),$$

$$\underline{C}_h(x, y) = \max(h(x) + \min(y - b, 0), W(x, y)).$$

(iii) Observe that quasi-copulas (see Alsina et al., 1993; Genest et al., 1999) have the same set $\mathscr{H}_b$ of b-horizontal sections as copulas. Moreover, for each $b \in]0, 1[$ and $h \in \mathscr{H}_b$, $\overline{C}_h$ and $\underline{C}_h$ are the greatest and smallest quasi-copula with given horizontal section h, respectively. However, the quasi-copula $Q = \text{med}\big(\overline{C}_h, \underline{C}_h, b \cdot \mathbf{1}_{[0,1]^2}\big)$ has horizontal section h, but need not be a copula whenever $h \notin \{h_W, h_M\}$.

Example 3.1.

(i) For the smallest element h_W of $\mathscr{H}_b$ the greatest copula with b-horizontal section given by h_W is given by

$$\overline{C}_{h_W}(x, y) = \begin{cases} 0 & \text{if } (x, y) \in [0, 1-b] \times [0, b], \\ \min(x, y-b) & \text{if } (x, y) \in [0, 1-b] \times]b, 1], \\ \min(y, x+b-1) & \text{if } (x, y) \in]1-b, 1] \times [0, b], \\ x+y-1 & \text{otherwise.} \end{cases}$$

Note that $\overline{C}_{h_W}$ is the W-ordinal sum W-$(\langle 0, 1-b, M\rangle, \langle 1-b, 1, M\rangle)$ which is a shuffle of M (see Nelsen, 1999).

(ii) It can be easily shown that, for the greatest element h_M of $\mathscr{H}_b$, the smallest copula $\underline{C}_{h_M}$ with b-horizontal section given by h_M is the ordinal sum $(\langle 0, b, W\rangle, \langle b, 1, W\rangle)$.

For $b \in]0, 1[$ and a horizontal section $h \in \mathscr{H}_b$, define $h^-, \hat{h} : [0, 1] \to [0, 1]$ by $h^-(x) = x - h(x)$ and $\hat{h}(x) = x - b + h(1-x)$. Obviously, both h^- and $\hat{h}$ belong to $\mathscr{H}_{1-b}$. Moreover, we have $\widehat{(\widetilde{C}_h)} = \widetilde{C}_{\hat{h}}$, where $\widehat{C}$ is the survival copula of the copula C. Similarly, we obtain $\widehat{(\overline{C}_h)} = \overline{C}_{\hat{h}}$ and $\widehat{(\underline{C}_h)} = \underline{C}_{\hat{h}}$.

Given a copula C, define the copula $C^- : [0, 1] \to [0, 1]$ by

$$C^-(x, y) = x - C(x, 1-y)$$

(observe that if C is the copula of the random variables X and Y then C^- is the copula of the random variables X and $-Y$). Then we have $(\widetilde{C}_h)^- = \widetilde{C}_{h^-}$, $(\overline{C}_h)^- = \underline{C}_{h^-}$ and $(\underline{C}_h)^- = \overline{C}_{h^-}$.

4. Kendall's Tau and Spearman's Rho in $\mathscr{C}_h$

If X and Y are continuous random variables with copula C, then the population version of Kendall's tau is given (see Nelsen, 1999) by

$$\tau_C = 4 \iint_{[0,1]^2} C(x, y)\, dC(x, y) - 1. \tag{7}$$

By recourse to Theorem 3.1 in Li et al. (2002), (7) may be written in the form

$$\tau_C = 1 - 4 \iint_{[0,1]^2} D_1 C(x, y)\, D_2 C(x, y) dx\, dy, \tag{8}$$

where D_1 and D_2 are the partial derivatives with respect to the first and second coordinate, respectively.

Since the partial derivatives of $\widetilde{C}_h$ exist almost everywhere in $[0, 1]^2$, one has

$$D_1 \widetilde{C}_h(x, y) = \frac{y}{b} h'(x) \mathbf{1}_{[0,b]}(y) + \frac{(1-y)h'(x) + y - b}{1-b} \mathbf{1}_{[b,1]}(y) \quad \text{a.e.,}$$
$$D_2 \widetilde{C}_h(x, y) = \frac{h(x)}{b} \mathbf{1}_{[0,b]}(y) + \frac{x - h(x)}{1-b} \mathbf{1}_{[b,1]}(y) \quad \text{a.e.,}$$

so that

$$\begin{aligned}
&\iint_{[0,1]^2} D_1\widetilde{C}_h(x,y)\, D_2\widetilde{C}_h(x,y)dx\,dy \\
&\quad = \frac{b^2}{4} + \frac{1-b^2}{2(1-b)^2}\int_0^1 (x - xh'(x) - h(x) + h(x)h'(x))dx \\
&\qquad + \frac{1}{1-b}\int_0^1 (xh'(x) - bx - h(x)h'(x) + bh(x))dx.
\end{aligned}$$

After some calculation we obtain $\tau_{\widetilde{C}_h} = 4\int_0^1 h(x)dx - 2b$.

As for $\overline{C}_h$, the greatest copula in $\mathscr{H}_b$, one easily computes its partial derivatives

$$\begin{aligned}
D_1\overline{C}_h(x,y) &= h'(x)\mathbf{1}_{[h(x),x+b-h(x)]}(y) + \mathbf{1}_{[x+b-h(x),1]}(y) \quad \text{a.e.,} \\
D_2\overline{C}_h(x,y) &= \mathbf{1}_{[0,h(x)]}(y) + \mathbf{1}_{[b,x+b-h(x)]}(y) \quad \text{a.e.}
\end{aligned}$$

Hence, $\tau_{\overline{C}_h} = 4\int_0^1 h(x)\,dx - 1 + 2(1-b)^2$.

Finally, we compute Kendall's tau for $\underline{C}_h$, the smallest copula in $\mathscr{H}_b$. Its partial derivatives are

$$\begin{aligned}
D_1\underline{C}_h(x,y) &= h'(x)\mathbf{1}_{[b-h(x),\mathbf{1}-x+h(x)]}(y) + \mathbf{1}_{[1-x+h(x),1]}(y) \quad \text{a.e.,} \\
D_2\underline{C}_h(x,y) &= \mathbf{1}_{[b-h(x),b]}(y) + \mathbf{1}_{[1-x+h(x),1]}(y) \quad \text{a.e.}
\end{aligned}$$

Thus $\tau_{\underline{C}_h} = 4\int_0^1 h(x)\,dx - 1 - 2b^2$.

Summarizing these results, we have the following.

Proposition 4.1. *Let $b \in]0,1[$ and $h \in \mathscr{H}_b$. Then*:

(i) $\tau_{\widetilde{C}_h} = 4\int_0^1 h(x)dx - 2b$,
(ii) $\tau_{\widetilde{C}_h} = 4\int_0^1 h(x)dx - 1 + 2(1-b)^2$,
(iii) $\tau_{\underline{C}_h} = 4\int_0^1 h(x)dx - 1 - 2b^2$,
(vi) *for each copula $C \in \mathscr{C}_h$ we have $\tau_C = 4\int_0^1 h(x)dx + c$, where the constant c satisfies $c \in [-1-2b^2, -1+2(1-b)^2]$.*

Observe that for each $b \in]0,1[$ and each $h \in \mathscr{H}_b$ we have

$$\tau_{\widetilde{C}_h} = \frac{\tau_{\overline{C}_h} + \tau_{\underline{C}_h}}{2}.$$

Moreover, in particular we get

$$\begin{aligned}
\tau_{\widetilde{C}_{h_M}} &= 2b(1-b), \\
\tau_{\widetilde{C}_{h_W}} &= -2b(1-b), \\
\tau_{\overline{C}_{h_W}} &= (1-2b)^2 \quad \big(\text{and obviously } \tau_{\overline{C}_{h_M}} = \tau_M = 1\big), \\
\tau_{\underline{C}_{h_M}} &= -(1-2b)^2 \quad (\text{and } \tau_{\underline{C}_{h_W}} = \tau_W = -1).
\end{aligned}$$

Turning our attention to Spearman's rho, recall that for continuous random variables X and Y whose copula is C, the population version of Spearman's rho (denoted by ϱ_C) is given by (see Nelsen, 1999)

$$\varrho_C = 12 \iint_{[0,1]^2} C(x, y)\,dx\,dy - 3. \tag{9}$$

It is not difficult to compute Spearman's rho for distinguished copulas in the class $\mathscr{C}_h$. The results are summarized below.

Proposition 4.2. *Let* $b \in]0, 1[$ *and* $h \in \mathscr{H}_b$. *Then*

(i) $\varrho_{\widetilde{C}_h} = 6 \int_0^1 h(x)dx - 3b$,
(ii) $\varrho_{\overline{C}_h} = 1 - 6b + 12 \int_0^1 h(x)(x + b - h(x))dx$,
(iii) $\varrho_{\underline{C}_h} = -1 + 12 \int_0^1 h(x)(h(x) - x - b + 1))dx$.

Observe that the copulas $\widetilde{C}_h$ and $\frac{\overline{C}_h + \underline{C}_h}{2}$ have the same Spearman's rho, i.e.,

$$\varrho_{\widetilde{C}_h} = \varrho_{\frac{\overline{C}_h + \underline{C}_h}{2}} = \frac{\varrho_{\overline{C}_h} + \varrho_{\underline{C}_h}}{2}.$$

Moreover, for each copula $C \in \mathscr{C}_h$ we have $\varrho_C \in [\varrho_{\underline{C}_h}, \varrho_{\overline{C}_h}]$. In particular, we get

$$\begin{aligned}
\varrho_{\widetilde{C}_{h_M}} &= 3b(1 - b),\\
\varrho_{\widetilde{C}_{h_W}} &= -3b(1 - b),\\
\varrho_{\overline{C}_{h_W}} &= 1 - 6b + 6b^2 \quad (\text{and } \varrho_{\overline{C}_{h_M}} = \varrho_M = 1),\\
\varrho_{\underline{C}_{h_M}} &= -1 + 6b - 6b^2 \quad (\text{and } \varrho_{\underline{C}_{h_W}} = \varrho_W = -1).
\end{aligned}$$

5. Concluding Remarks

We have discussed possible extensions of copulas when only one horizontal section $h \in \mathscr{H}_b$ for some $b \in]0, 1[$ is known. Evidently, for all $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \leq 1$ the convex combination

$$C_{\alpha,\beta;b,h} = \alpha\overline{C}_h + \beta\underline{C}_h + (1 - \alpha - \beta)\widetilde{C}_h \tag{10}$$

also is an element of $\mathscr{C}_h$. Observe that, in the case $b \in \{0, 1\}$, the copulas given by (10) form the Fréchet family of copulas (see Nelsen, 1999).

For the possible identification of a copula $C_{\alpha,\beta;b,h}$ where not only some horizontal section $h \in \mathscr{H}_b$ with $b \in]0, 1[$, but also $\tau_{C_{\alpha,\beta;b,h}}$ or $\varrho_{C_{\alpha,\beta;b,h}}$ are known, we give the equations for these dependence parameters $\tau_{C_{\alpha,\beta;b,h}}$ or $\varrho_{C_{\alpha,\beta;b,h}}$ as functions of b, h, α, and β:

$$\begin{aligned}
\tau_{C_{\alpha,\beta;b,h}} = 4\int_0^1 h(x)dx &+ 2\alpha^2(1 - b)^2 - 2\beta^2 b^2 + (1 - \alpha - \beta)^2(1 - 2b)\\
&+ 4\alpha\beta(1 - 2b) + \alpha(1 - \alpha - \beta)\frac{8 - 16b + 4b^2}{3}\\
&+ \beta(1 - \alpha - \beta)\frac{4 - 8b - 4b^2}{3} - 1,
\end{aligned}$$

$$\varrho_{C_{\alpha,\beta;b,h}} = 12\int_0^1 h(x)\cdot((\alpha-\beta)(x+b-h(x))+\beta-2(1-\alpha-\beta))dx$$
$$+\alpha(1-6\beta)-\beta-3(1-\alpha-\beta)b,$$

making it possible to transform the first equation by means of the second one into a quadratic equation in one variable.

Moreover, each element of the class $\{\widetilde{C}_h \mid h \in \mathscr{H}_b, b \in [0,1]\}$, i.e., each horizontal copula satisfies the remarkable property $3\tau_{\widetilde{C}_h} = 2\varrho_{\widetilde{C}_h}$ (note that the members of the family of Farlie–Gumbel–Morgenstern copulas as considered in Eyraud, 1938; Morgenstern, 1956; Gumbel, 1958; Farlie, 1960, and Nelsen, 1999 have the same property).

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