# 2-Increasing binary aggregation operators 

Fabrizio Durante ${ }^{\mathrm{a}, 1}$, Radko Mesiar ${ }^{\mathrm{b}, 2}$, Pier Luigi Papini ${ }^{\mathrm{c}}$, Carlo Sempi ${ }^{\mathrm{a}, *, 1}$<br>${ }^{a}$ Università di Lecce, Dipartimento di Matematica, "E. De Giorgi", 73100 Lecce, Italy<br>${ }^{\text {b }}$ Department of Mathematics and Descriptive Geometry, Faculty of Civil Engineering, Slovak University of Technology, Bratislava, Slovakia<br>${ }^{\text {c }}$ Università di Bologna, Dipartimento di Matematica, Piazza Porta S. Donato 5, 40126 Bologna, Italy

Received 19 December 2005; received in revised form 11 April 2006; accepted 19 April 2006


#### Abstract

In this work we investigate the class of binary aggregation operators (=agops) satisfying the 2-increasing property, obtaining some characterizations for agops having other special properties (e.g., quasi-arithmetic mean, Choquet-integral based, modularity) and presenting some construction methods. In particular, the notion of $P$-increasing function is used in order to characterize the composition of 2-increasing agops. The lattice structure (with respect to the pointwise order) of some subclasses of 2 -increasing agops is presented. Finally, a method is given for constructing copulas beginning from 2increasing and 1-Lipschitz agops.


© 2006 Elsevier Inc. All rights reserved.
Keywords: Aggregation operator; Copula; 2-Increasing property; Supermodularity

## 1. Introduction

The aggregation of different pieces of information is one of the major problem for all kinds of knowledge based systems. Here the aggregations of a finite number of input values, which belong to the unit interval $[0,1]$, into an output value, belonging to the same interval, are considered, following the ideas and the notations of the books [2,4]. Specifically, binary aggregation operators (agops, for short) satisfying the 2 -increasing property are analysed in details.

The 2 -increasing property has a relevant connection with the theory of copulas, whose interest in statistics is well known (see [18]). Moreover, associative copulas form a distinguished subset of the class of continuous triangular norms, which are largely used in the treatment of vague information (see, for example, [11]). The 2 -increasing property has been considered, for instance, in the fuzzy inference processing, when one imposes

[^0]some special conditions (see [22,17]). However, this property has a relevant rôle also in the theory of fuzzy measures, where it is also known as "supermodularity" (see [6]).

Basic definitions and properties of 2-increasing binary aggregation operators are considered in Section 2. In Section 3, the 2-increasing property is characterized in some subclasses of agops, while construction methods are investigated in Sections 4 and 5. The lattice structure of some subclasses are, then, considered in Section 6. Finally, we present a method for constructing a copula starting from a 2 -increasing and 1-Lipschitz agop (Section 7).

## 2. Definitions and basic properties

A binary aggregation operator (shortly, agop) is a mapping $A$ from $[0,1]^{2}$ into $[0,1]$ that satisfies the following properties:
(A1) $A(0,0)=0$ and $A(1,1)=1$;
(A2) $A(x, y) \leqslant A\left(x^{\prime}, y^{\prime}\right)$ for $x \leqslant x^{\prime}$ and $y \leqslant y^{\prime}$.
The class of agops will be denoted by $\mathscr{A}$.
An agop $A$ has a neutral element $e \in[0,1]$ if

$$
\begin{equation*}
A(x, e)=x=A(e, x) \quad \text { for every } x \in[0,1], \tag{2.1}
\end{equation*}
$$

and it has an annihilator $a \in[0,1]$ if

$$
\begin{equation*}
A(x, a)=a=A(a, x) \quad \text { for every } x \in[0,1] . \tag{2.2}
\end{equation*}
$$

Given an agop $A$,

- the horizontal section of $A$ at $b \in[0,1]$ is the function $h_{b}:[0,1] \rightarrow[0,1]$ defined by $h_{b}(x):=A(x, b)$;
- the vertical section of $A$ at $a \in[0,1]$ is the function $v_{a}:[0,1] \rightarrow[0,1]$ defined by $v_{a}(y):=A(a, y)$;
- the diagonal section of $A$ is the function $\delta_{A}:[0,1] \rightarrow[0,1]$ defined by $\delta_{A}(t):=A(t, t)$.

The sections $h_{0}, h_{1}, v_{0}$, and $v_{1}$ are called marginals of $A$.
An agop $A$ is said to be 2-increasing if, for all $x_{1}, x_{2}, y_{1}$, and $y_{2}$ in $[0,1]$ with $x_{1} \leqslant x_{2}$ and $y_{1} \leqslant y_{2}$, one has

$$
\begin{equation*}
V_{A}\left(\left[x_{1}, x_{2}\right] \times\left[y_{1}, y_{2}\right]\right):=A\left(x_{1}, y_{1}\right)+A\left(x_{2}, y_{2}\right)-A\left(x_{1}, y_{2}\right)-A\left(x_{2}, y_{1}\right) \geqslant 0 . \tag{2.3}
\end{equation*}
$$

Inequality (2.3) is called rectangular inequality and $V_{A}\left(\left[x_{1}, x_{2}\right] \times\left[y_{1}, y_{2}\right]\right)$ is said to be the $A$-volume of $\left[x_{1}, x_{2}\right] \times\left[y_{1}, y_{2}\right]$. Notice that inequality (2.3) is equivalent to the fact that both the functions

$$
t \mapsto A\left(x_{2}, t\right)-A\left(x_{1}, t\right) \quad \text { and } \quad s \mapsto A\left(s, y_{2}\right)-A\left(s, y_{1}\right)
$$

are increasing for $x_{1} \leqslant x_{2}$ and for $y_{1} \leqslant y_{2}$, respectively. This property is also known as moderate growth (see [17]).

A 2-increasing agop is also supermodular with respect to the pointwise order on $[0,1]^{2}$. In fact, if $\mathbf{x} \wedge \mathbf{y}$ and $\mathbf{x} \vee \mathbf{y}$ denote, respectively, the componentwise minimum and maximum of two points $\mathbf{x}$ and $\mathbf{y}$ of $[0,1]^{2}$, then inequality (2.3) can be rewritten in the form

$$
A(\mathbf{x} \wedge \mathbf{y})+A(\mathbf{x} \vee \mathbf{y}) \geqslant A(\mathbf{x})+A(\mathbf{y})
$$

The class of 2-increasing agops will be denoted by $\mathscr{A}_{2}$.
Remark 2.1. Let $A$ be a 2 -increasing agop, $A \in \mathscr{A}_{2}$. Then the dual of $A$ is defined, for every point ( $x, y$ ) in $[0,1]^{2}$, by

$$
A^{d}(x, y):=1-A(1-x, 1-y) .
$$

It is immediately seen that $A^{d}$ is 2-decreasing, in the sense that, for its $A^{d}$-volume, $V_{A^{d}}\left(\left[x_{1}, x_{2}\right] \times\left[y_{1}, y_{2}\right]\right) \leqslant 0$ holds. Analogously, if $A$ is 2 -decreasing, then its dual is 2 -increasing. Therefore one may limit oneself to considering only 2 -increasing agops, since the analogous results for the 2 -decreasing ones are obtained by duality.

Some important examples of 2-increasing agops are the restrictions to $[0,1]^{2}$ of bivariate distribution functions $F$ such that $F(0,0)=0$ and $F(1,1)=1$, in particular copulas, which correspond to the case when the marginal distributions are uniform on the unit interval (and in such a case, they possess a neutral element 1) (see [18]), the smallest agop, defined by

$$
A_{S}(x, y)= \begin{cases}1, & \text { if }(x, y)=(1,1) \\ 0, & \text { otherwise }\end{cases}
$$

and the weighted arithmetic means (see [3]).
In order to prove that a function $F:[0,1]^{2} \rightarrow[0,1]$ satisfies the rectangular inequality (2.3), the following technical result will be useful. But, first, let $\Delta_{+}$and $\Delta_{-}$denote the subsets of the unit square given by

$$
\Delta_{+}:=\left\{(x, y) \in[0,1]^{2}: x \geqslant y\right\}, \quad \Delta_{-}:=\left\{(x, y) \in[0,1]^{2}: x \leqslant y\right\}
$$

then one has
Lemma 2.2. For every agop $A$, the $A$-volume $V_{A}(R)$ of any rectangle $R \subseteq[0,1]^{2}$ can be expressed as the sum

$$
\sum_{i} V_{A}\left(R_{i}\right)
$$

of at most three terms, where the rectangles $R_{i}$ may have a side in common and belong to one of the following types:
(a) $R_{i} \subseteq \Delta_{+}$;
(b) $R_{i} \subseteq \Delta_{-}$;
(c) $R_{i}=[s, t] \times[s, t]$ for some $s$ and $t$ in $[0,1], s<t$.

Proof. Let a rectangle $R \subseteq[0,1]^{2}$ be given; if it belongs to one of the three types (a), (b) or (c) there is nothing to prove. Then consider the other possible cases: $R$ may have one, two or three vertices in $\Delta_{-}$.

Let $R=\left[x_{1}, x_{2}\right] \times\left[y_{1}, y_{2}\right]$ have one vertex in $\Delta_{+}$and three in $\Delta_{-}$; then, since $y_{2}>x_{2}>y_{1}>x_{1}$, one can write

$$
R=\left(\left[x_{1}, y_{1}\right] \times\left[y_{1}, y_{2}\right]\right) \cup\left(\left[y_{1}, x_{2}\right] \times\left[y_{1}, x_{2}\right]\right) \cup\left(\left[y_{1}, x_{2}\right] \times\left[x_{2}, y_{2}\right]\right) ;
$$

of these rectangles, the first and the third one are of type (b), while the second one is of type (c). Now

$$
\begin{aligned}
& V_{A}\left(\left[x_{1}, y_{1}\right] \times\left[y_{1}, y_{2}\right]\right)=A\left(y_{1}, y_{2}\right)-A\left(y_{1}, y_{1}\right)-A\left(x_{1}, y_{2}\right)+A\left(x_{1}, y_{1}\right), \\
& V_{A}\left(\left[y_{1}, x_{2}\right] \times\left[y_{1}, x_{2}\right]\right)=A\left(x_{2}, x_{2}\right)-A\left(x_{2}, y_{1}\right)-A\left(y_{1}, x_{2}\right)+A\left(y_{1}, y_{1}\right), \\
& V_{A}\left(\left[y_{1}, x_{2}\right] \times\left[x_{2}, y_{2}\right]\right)=A\left(x_{2}, y_{2}\right)-A\left(x_{2}, x_{2}\right)-A\left(y_{1}, y_{2}\right)+A\left(y_{1}, x_{2}\right) .
\end{aligned}
$$

Therefore, summing these equalities yields

$$
\begin{aligned}
& V_{A}\left(\left[x_{1}, y_{1}\right] \times\left[y_{1}, y_{2}\right]\right)+V_{A}\left(\left[y_{1}, x_{2}\right] \times\left[y_{1}, x_{2}\right]\right)+V_{A}\left(\left[y_{1}, x_{2}\right] \times\left[x_{2}, y_{2}\right]\right) \\
& \quad=A\left(x_{2}, y_{2}\right)-A\left(x_{2}, y_{1}\right)-A\left(x_{1}, y_{2}\right)+A\left(x_{1}, y_{1}\right)=V_{A}\left(\left[x_{1}, x_{2}\right] \times\left[y_{1}, y_{2}\right]\right)
\end{aligned}
$$

which proves the assertion in this case. The other cases are proved in a similar manner.
Proposition 2.3. An agop $A:[0,1]^{2} \rightarrow[0,1]$ is 2-increasing if, and only if, the three following conditions hold:
(i) $V_{A}(R) \geqslant 0$ for every rectangle $R \subseteq \Delta_{+}$;
(ii) $V_{A}(R) \geqslant 0$ for every rectangle $R \subseteq \Delta_{-}$;
(iii) $V_{A}(R) \geqslant 0$ for every square $R \subseteq[0,1]^{2}, R=[s, t] \times[s, t]$.

Proof. If $A$ is 2 -increasing, (i), (ii) and (iii) follow easily. Conversely, let $R$ be a rectangle contained in $[0,1]^{2}$. Then, because of the previous lemma, $R$ can be decomposed into the union of at most three sub-rectangles $R_{i}$ of type (a), (b) or (c); and for each of them $V_{A}\left(R_{i}\right) \geqslant 0$ holds. Therefore $V_{A}(R)=\sum V_{A}\left(R_{i}\right) \geqslant 0$.

Remark 2.4. Every 2 -increasing agop $A$ induces a positive and $\sigma$-additive measure $\mu_{A}$ on $[0,1]^{2}$ given, for every rectangle $\left[x_{1}, x_{2}\right] \times\left[y_{1}, y_{2}\right]$, by

$$
\mu_{A}\left(\left[x_{1}, x_{2}\right] \times\left[y_{1}, y_{2}\right]\right):=V_{A}\left(\left[x_{1}, x_{2}\right] \times\left[y_{1}, y_{2}\right]\right)
$$

and extended on the whole family of Borel sets of $[0,1]^{2}$ using classical measure theoretic techniques.

## 3. Characterizations of some subclasses of 2 -increasing agops

In this section, some subclasses of agops satisfying the 2 -increasing property are characterized.
Proposition 3.1. Let A be a 2-increasing agop. Then:
(a) the neutral element $e \in[0,1]$ of $A$, if it exists, is equal to 1 ;
(b) the annihilator $a \in[0,1]$ of $A$, if it exists, is equal to 0 .
(c) if $A$ is continuous on the border of $[0,1]^{2}$, then $A$ is continuous on $[0,1]^{2}$.

Proof. Let $A$ be a 2-increasing agop.
If $A$ has neutral element $e \in[0,1[$, then

$$
A(1,1)+A(e, e)=1+A(e, e) \geqslant A(e, 1)+A(1, e)=1+1,
$$

a contradiction. Therefore $e=1 \mathrm{and}$, as a consequence, $A$ is a copula.
Let $A$ have an annihilator $a \in[0,1]$. Assume, if possible that $a>0$; then

$$
A(a, a)-A(a, 0)-A(0, a)+A(0,0)=-a \geqslant 0
$$

a contradiction; as a consequence, $a=0$.
Let $A$ be continuous on the border of $[0,1]^{2}$ and let $\left(x_{0}, y_{0}\right)$ be a point in $] 0,1\left[^{2}\right.$ such that $A$ is not continuous in ( $x_{0}, y_{0}$ ). Suppose, without loss of generality, that there exists a sequence $\left\{x_{n}\right\}_{n \in \mathbb{N}}$ in $[0,1], x_{n} \leqslant x_{0}$ for every $n \in \mathbb{N}$, such that $\left\{x_{n}\right\}_{n \in \mathbb{N}}$ tends to $x_{0}$ as $n \rightarrow+\infty$ and

$$
\lim _{n \rightarrow+\infty} A\left(x_{n}, y_{0}\right)<A\left(x_{0}, y_{0}\right) .
$$

Therefore, there exist $\epsilon>0$ and $n_{0} \in \mathbb{N}$ such that $A\left(x_{0}, y_{0}\right)-A\left(x_{n}, y_{n}\right)>\epsilon$ for every $n \geqslant n_{0}$. But, because $A$ is continuous on the border of the unit square, there exists $\bar{n}>n_{0}$ such that $A\left(x_{0}, 1\right)-A\left(x_{\bar{n}}, 1\right)<\epsilon$. But this violates the 2 -increasing property, because, in this case,

$$
V\left(\left[x_{\bar{n}}, x_{0}\right] \times\left[y_{0}, 1\right]\right)<0 .
$$

Thus the only possibility is that $A$ is continuous on $[0,1]^{2}$.
Proposition 3.2. Let $M_{f}$ be a quasi-arithmetic mean, viz. let a continuous strictly monotone function $f:[0,1] \rightarrow \mathbb{R}$ exist such that

$$
M_{f}(x, y):=f^{-1}\left(\frac{f(x)+f(y)}{2}\right) .
$$

Then $M_{f}$ is 2-increasing if, and only if, $f^{-1}$ is convex.
Proof. Let $s$ and $t$ be real numbers and set $a:=f^{-1}(s)$ and $b:=f^{-1}(t)$. If $M_{f}$ is 2-increasing, then

$$
\begin{equation*}
M_{f}(a, a)+M_{f}(b, b) \geqslant M_{f}(a, b)+M_{f}(b, a)=2 M_{f}(a, b), \tag{3.1}
\end{equation*}
$$

because $M_{f}$ is also commutative, and (3.1) is equivalent to

$$
f^{-1}(s)+f^{-1}(t) \geqslant 2 f^{-1}\left(\frac{s+t}{2}\right)
$$

This shows that $f^{-1}$ is Jensen-convex and hence convex.

Conversely, let $f^{-1}$ be convex; one has to prove that, whenever $x_{1} \leqslant x_{2}$ and $y_{1} \leqslant y_{2}$,

$$
M_{f}\left(x_{1}, y_{1}\right)+M_{f}\left(x_{2}, y_{2}\right) \geqslant M_{f}\left(x_{2}, y_{1}\right)+M_{f}\left(x_{1}, y_{2}\right),
$$

or, equivalently, that

$$
f^{-1}\left(s_{1}\right)+f^{-1}\left(s_{4}\right) \geqslant f^{-1}\left(s_{2}\right)+f^{-1}\left(s_{3}\right),
$$

where

$$
\begin{array}{ll}
s_{1}:=\frac{f\left(x_{1}\right)+f\left(y_{1}\right)}{2}, & s_{4}:=\frac{f\left(x_{2}\right)+f\left(y_{2}\right)}{2}, \\
s_{2}:=\frac{f\left(x_{2}\right)+f\left(y_{1}\right)}{2}, & s_{3}:=\frac{f\left(x_{1}\right)+f\left(y_{2}\right)}{2} .
\end{array}
$$

Assume now that $f$ is (strictly) increasing; setting

$$
\alpha:=\frac{s_{4}-s_{2}}{s_{4}-s_{1}},
$$

one has $\alpha \in[0,1]$ and

$$
s_{2}=\alpha s_{1}+(1-\alpha) s_{4}, \quad s_{3}=(1-\alpha) s_{1}+\alpha s_{4} .
$$

Because $f^{-1}$ is convex, one has

$$
f^{-1}\left(s_{2}\right)+f^{-1}\left(s_{3}\right) \leqslant f^{-1}\left(s_{1}\right)+f^{-1}\left(s_{4}\right),
$$

namely the assertion.
If, on the other hand, $f$ is (strictly) decreasing, then one sets

$$
\alpha:=\frac{s_{1}-s_{2}}{s_{1}-s_{4}}
$$

in order to reach the same conclusion.
Remark 3.3. Notice that, if $M_{f}$ is a quasi-arithmetic mean generated by a function $f$, with $f^{-1}$ convex, then

$$
M_{f}(x, y) \leqslant \frac{x+y}{2} \quad \text { for every }(x, y) \in[0,1]^{2} .
$$

In fact, if $f$ is increasing, so is $f^{-1}$, and $M_{f}(x, y) \leqslant \frac{x+y}{2}$ is equivalent to the fact that $f$ is Jensen-concave and, thus, $f^{-1}$ convex. Instead, if $f$ is decreasing, so is $f^{-1}$, and $M_{f}(x, y) \leqslant \frac{x+y}{2}$ is equivalent to the fact that $f$ is Jensen-convex and, thus, $f^{-1}$ convex.

The 2 -increasing property can be considered also in the class of Choquet integral-based agop, a class of agops based on the ideas of [5] (see, also, $[1,3]$ ).
Proposition 3.4. The Choquet integral-based agop, defined for a and $b$ in $[0,1]$ by

$$
A_{\mathrm{Ch}}(x, y)= \begin{cases}(1-b) x+b y, & \text { if } x \leqslant y, \\ a x+(1-a) y, & \text { if } x>y,\end{cases}
$$

is 2-increasing if, and only if, $a+b \leqslant 1$.
Proof. It is easily proved that $A_{\mathrm{Ch}}$ is 2-increasing on every rectangle contained either in $\Delta_{+}$or in $\Delta_{-}$. Now, let $R:=[s, t] \times[s, t], s<t$. Then

$$
V_{A_{\mathrm{Ch}}}([s, t] \times[s, t])=s+t-[(1-b) s+b t]-[a t+(1-a) s]
$$

is greater than 0 for all $s$ and $t$ such that $0 \leqslant s<t \leqslant 1$ if, and only if, $a+b \leqslant 1$. Therefore, we have the desired assertion by using Proposition 2.3.

Notice that, if $a+b=1$, we have the weighted arithmetic mean. Instead, if $a=b \leqslant 1 / 2$, we have the OWA operator $A_{\mathrm{Ch}}(x, y)=(1-a) \min \{x, y\}+a \max \{x, y\}$. OWA operators have been introduced by Yager (see [21]) and they are agops that lie between the minimum and the maximum. A recent application of OWA operators is given in [20].
Remark 3.5. The above proposition can be also proved by using some known results on fuzzy measures. In fact, following [6], it is known that a Choquet integral operator based on a fuzzy measure $m$ is supermodular if, and only if, the fuzzy measure $m$ is supermodular. But, in the case of 2 inputs, say $\mathbb{X}_{2}:=\{1,2\}$, we can define a fuzzy measure $m$ on $2^{\mathbb{X}_{2}}$ by giving the values $m(\{1\})=a$ and $m(\{2\})=b$, where $a$ and $b$ are in $[0,1]$. Moreover, it is also known that $m$ is supermodular if, and only if, $a+b \leqslant 1$.

A special subclass of 2 -increasing agops is that formed by modular agops, i.e. those $A$ 's for which $V_{A}(R)=0$ for every rectangle $R \subseteq[0,1]^{2}$. For these operators the following characterization holds.
Proposition 3.6. For an agop $A$ the following statements are equivalent:
(a) $A$ is modular;
(b) increasing functions $f$ and $g$ from $[0,1]$ into $[0,1]$ exist such that $f(0)=g(0)=0, f(1)+g(1)=1$, and

$$
\begin{equation*}
A(x, y)=f(x)+g(y) . \tag{3.2}
\end{equation*}
$$

Proof. If $A$ is modular, set $f(x):=A(x, 0)$ and $g(y):=A(0, y)$. From the modularity of $A$

$$
0=V_{A}([0, x] \times[0, y])=A(x, y)-f(x)-g(y)+A(0,0),
$$

which implies (b). Viceversa, it is clear that every function of type (3.2) is modular.

## 4. Construction of 2 -increasing agops

In the literature, there is a variety of construction methods for agops (see [3] and the references therein). In this section, some of these methods are used to obtain an agop satisfying the 2 -increasing property.
Proposition 4.1. Let $\varphi:[0,1] \rightarrow[0,1]$ be a continuous and strictly increasing function with $\varphi(0)=0$ and $\varphi(1)=1$. The following statements are equivalent:
(a) $\varphi$ is concave;
(b) for every $A \in \mathscr{A}_{2}$, the function

$$
A_{\varphi}(x, y):=\varphi^{-1}(A(\varphi(x), \varphi(y)))
$$

is a 2-increasing agop.

Proof. (a) $\Rightarrow$ (b) Let $A$ be a 2-increasing agop and let $R=\left[x_{1}, x_{2}\right] \times\left[y_{1}, y_{2}\right]$ be any rectangle contained in $[0,1]^{2}$. Then

$$
V_{A_{\varphi}}(R)=\varphi^{-1}\left(A\left(\varphi\left(x_{1}\right), \varphi\left(y_{1}\right)\right)\right)+\varphi^{-1}\left(A\left(\varphi\left(x_{2}\right), \varphi\left(y_{2}\right)\right)\right)-\varphi^{-1}\left(A\left(\varphi\left(x_{2}\right), \varphi\left(y_{1}\right)\right)\right)-\varphi^{-1}\left(A\left(\varphi\left(x_{1}\right), \varphi\left(y_{2}\right)\right)\right) .
$$

Setting

$$
\begin{array}{ll}
s_{1}:=A\left(\varphi\left(x_{1}\right), \varphi\left(y_{1}\right)\right), & s_{4}:=A\left(\varphi\left(x_{2}\right), \varphi\left(y_{2}\right)\right), \\
s_{2}:=A\left(\varphi\left(x_{2}\right), \varphi\left(y_{1}\right)\right), & s_{3}:=A\left(\varphi\left(x_{1}\right), \varphi\left(y_{2}\right)\right),
\end{array}
$$

yields

$$
s_{1} \leqslant s_{2} \leqslant s_{4}, \quad s_{1} \leqslant s_{3} \leqslant s_{4} \quad \text { and } \quad s_{1}+s_{4} \geqslant s_{2}+s_{3} .
$$

Thus $w:=s_{2}+s_{3}-s_{1}$ belongs to the interval $\left[\max \left\{s_{2}, s_{3}\right\}, s_{4}\right]$; moreover, $s_{1}+w=s_{2}+s_{3}$. Put $\alpha:=\left(w-s_{2}\right) /$ ( $w-s_{1}$ ) and notice that $\alpha$ so defined belongs to $[0,1]$. Now

$$
s_{2}=\alpha s_{1}+(1-\alpha) w \quad \text { and } \quad s_{3}=(1-\alpha) s_{1}+\alpha w .
$$

Since $\varphi^{-1}$ is convex and increasing, one has

$$
\varphi^{-1}\left(s_{2}\right)+\varphi^{-1}\left(s_{3}\right) \leqslant \varphi^{-1}\left(s_{1}\right)+\varphi^{-1}(w) \leqslant \varphi^{-1}\left(s_{1}\right)+\varphi^{-1}\left(s_{4}\right),
$$

namely $V_{A_{\varphi}}(R) \geqslant 0$.
(b) $\Rightarrow$ (a) Consider the 2-increasing agop given by

$$
A(x, y):=\frac{\varphi^{-1}(x)+\varphi^{-1}(y)}{2}
$$

Since $A_{\varphi}$ is 2-increasing, one has, for $s \leqslant t$,

$$
A_{\varphi}(s, s)+A_{\varphi}(t, t) \geqslant A_{\varphi}(s, t)+A_{\varphi}(t, s),
$$

or, equivalently,

$$
\varphi^{-1}(s)+\varphi^{-1}(t) \geqslant 2 \varphi^{-1}\left(\frac{s+t}{2}\right)
$$

Thus $\varphi^{-1}$ is Jensen-convex, and, because of its continuity, convex; therefore, $\varphi$ is concave.
Proposition 4.2. Let $f$ and $g$ be increasing functions from $[0,1]$ into $[0,1]$ such that $f(0)=g(0)=0$ and $f(1)=g(1)=1$. Let $A$ be a 2-increasing agop. Then, the function defined by

$$
A_{f, g}(x, y):=A(f(x), g(y))
$$

is a 2-increasing agop.
Proof. It is obvious that $A_{f, g}(0,0)=0, A_{f, g}(1,1)=1$ and $A_{f, g}$ is increasing in each place, since it is the composition of increasing functions. Moreover, given $R=\left[x_{1}, x_{2}\right] \times\left[y_{1}, y_{2}\right]$, one obtains

$$
V_{A_{f, g}}(R)=V_{A}\left(\left[f\left(x_{1}\right), g\left(x_{2}\right)\right] \times\left[f\left(y_{1}\right), g\left(y_{2}\right)\right]\right) \geqslant 0
$$

which is the desired assertion.
Example 4.3. Let $f$ and $g$ be increasing functions from $[0,1]$ into $[0,1]$ with $f(0)=g(0)=0$ and $f(1)=g(1)=1$.
Then

$$
A_{f, g}(x, y):=f(x) \cdot g(y)
$$

is a 2 -increasing agop as a consequence of the previous proposition, by taking $A=\Pi$, the product agop.
From Sklar's theorem (see [18]), the following is easily derived.
Corollary 4.4. The following statements are equivalent:
(a) $H$ is the restriction to the unit square $[0,1]^{2}$ of a bivariate probability distribution function on $[0,1]^{2}$, with $H(0,0)=0$ and $H(1,1)=1$;
(b) there exist a copula $C$ and increasing and left continuous functions $f$ and $g$ from $[0,1]$ into $[0,1]$ such that $f(0)=g(0)=0$ and $f(1)=g(1)=1$, and one has

$$
H(x, y):=C(f(x), g(y)) .
$$

Remark 4.5. In view of Proposition 4.2, we notice that a copula can be constructed from every 2 -increasing agop that is continuous on the boundary and which has an annihilator (necessarily equal to zero). Indeed, it suffices to put

$$
f(x):=\sup \{t \in[0,1]: A(t, 1)=x\}, \quad g(y):=\sup \{t \in[0,1]: A(1, t)=y\} .
$$

For example, let $B$ and $C$ be copulas and consider the function $A^{\prime}(x, y)=B(x, y) \cdot C(x, y)$. As one will show in the sequel (see Section 5), $A^{\prime}$ is a continuous 2 -increasing agop with annihilator 0 . Moreover, one has

$$
f(x)=g(x)=\sup \left\{t \in[0,1]: A^{\prime}(t, 1)=x\right\}=\sqrt{x}
$$

Therefore, in view of Proposition 4.2 the function

$$
A^{\prime}(f(x), g(y))=B(\sqrt{x}, \sqrt{y}) \cdot C(\sqrt{x}, \sqrt{y})
$$

is a copula. The same procedure holds if one considers $A^{\prime}(x, y)=B^{\alpha}(x, y) \cdot C^{\beta}(x, y)$ for all $\alpha, \beta \geqslant 1$, as consequence of the results in Section 5.

Sometimes, it is useful to construct an agop with specified values on its diagonal, horizontal or vertical section (see, for example, [12,13]). Specifically, given a suitable prescribed function $f$, the problem is whether there is a 2 -increasing agop with (diagonal, horizontal or vertical) section equal to $f$. The following result, whose proof can easily be produced by the reader, answers the question.

Proposition 4.6. Let $h, v$ and $\delta$ be increasing functions from $[0,1]$ into $[0,1], \delta(0)=0$ and $\delta(1)=1$. Then:

- $A_{\delta}(x, y)=\delta(x)$ is an agop whose diagonal section is $\delta$;
- an agop $A_{h}$ with horizontal section at $\left.b \in\right] 0,1[$ equal to $h$ is given by

$$
A_{h}(x, y)= \begin{cases}1, & \text { if }(x, y)=(1,1) \\ 0, & \text { if } y=0 \\ h(x), & \text { otherwise }\end{cases}
$$

- an agop $A_{v}$ with vertical section at $\left.a \in\right] 0,1[$ equal to $v$ is given by

$$
A_{v}(x, y)= \begin{cases}1, & \text { if }(x, y)=(1,1) \\ 0, & \text { if } x=0 \\ v(y), & \text { otherwise }\end{cases}
$$

In [16] (see also [3]), an ordinal sum construction for agops is given. Here, we modify the method given there in order to ensure that an ordinal sum of 2-increasing agops is again a 2 -increasing agop.

Consider a partition of the unit interval $[0,1]$ by the points $0=a_{0}<a_{1}<\cdots<a_{n}=1$ and let $A_{1}, A_{2}, \ldots, A_{n}$ be $n 2$-increasing agops. For $i=1,2, \ldots, n$, consider the function $\tilde{A}_{i}$ defined on the square $\left[a_{i}, a_{i+1}\right]^{2}$ by

$$
\widetilde{A}_{i}(x, y)=a_{i}+\left(a_{i+1}-a_{i}\right) A_{i}\left(\frac{x-a_{i}}{a_{i+1}-a_{i}}, \frac{y-a_{i}}{a_{i+1}-a_{i}}\right) .
$$

Then $\widetilde{A}_{i}$ is 2-increasing on $\left[a_{i}, a_{i+1}\right]^{2}$, because it is obtained by $A$ via the transformation $\varphi(t):=\left(t-a_{i}\right) /$ $\left(a_{i+1}-a_{i}\right)$ (see Proposition 4.1); in fact, $\widetilde{A}_{i}$ is the rescaling of the 2-increasing agop $A_{i}$ in $\left[a_{i}, a_{i+1}\right]^{2}$. Now, define, for every point $(x, y)$ such that $a_{i} \leqslant \min \{x, y\}<a_{i+1}$,

$$
\begin{equation*}
A_{1, n}(x, y):=\widetilde{A}_{i}\left(\min \left\{x, a_{i+1}\right\}, \min \left\{y, a_{i+1}\right\}\right) . \tag{4.1}
\end{equation*}
$$

Therefore, it is not hard to prove that $A_{1, n}$ is also a 2-increasing agop, called the ordinal sum of the agops $\left\{A_{i}: i=1,2, \ldots, n\right\}$; we write

$$
A_{1, n}=\left(\left\langle a_{i}, a_{i+1}, A_{i}\right\rangle: i=1,2, \ldots, n\right) .
$$

Example 4.7. Consider a partition of [0,1] by means of the points $0=a_{0}<a_{1}<\cdots<a_{n}=1$. Let $\left\{A_{i}: i=1,2, \ldots, n\right\}$ be 2 -increasing agops such that, for every index $i, A_{i}=A_{S}$, the smallest agop. Let $A_{1, n}$ be the ordinal sum $\left(\left\langle a_{i}, a_{i+1}, A_{i}\right\rangle_{i=1,2, \ldots, n}\right)$. For every point $(x, y)$ such that $a_{i} \leqslant \min \{x, y\}<a_{i+1}, A_{1, n}(x, y)=a_{i}$.

## 5. Composition of 2 -increasing agops: the $P$-increasing property

The composition of two agops poses interesting problems. In particular, in order to preserve the 2 -increasing property for the composed agop, the notion of $P$-increasing function is useful. It was introduced in [8] and it is here reproduced.

A function $\psi:[0,1]^{2} \rightarrow[0,1]$ is said to be $P$-increasing if, and only if,

$$
\begin{equation*}
\psi\left(s_{1}, t_{1}\right)+\psi\left(s_{4}, t_{4}\right) \geqslant \max \left\{\psi\left(s_{2}, t_{2}\right)+\psi\left(s_{3}, t_{3}\right), \psi\left(s_{3}, t_{2}\right)+\psi\left(s_{2}, t_{3}\right)\right\}, \tag{5.1}
\end{equation*}
$$

for all $s_{i}, t_{i} \in[0,1](i \in\{1,2,3,4\})$ such that

$$
\begin{align*}
& s_{1} \leqslant s_{2} \wedge s_{3} \leqslant s_{2} \vee s_{3} \leqslant s_{4}, \quad t_{1} \leqslant t_{2} \wedge t_{3} \leqslant t_{2} \vee t_{3} \leqslant t_{4},  \tag{5.2}\\
& s_{1}+s_{4} \geqslant s_{2}+s_{3}, \quad t_{1}+t_{4} \geqslant t_{2}+t_{3} . \tag{5.3}
\end{align*}
$$

The term " $P$-increasing" stands for " probabilistically increasing", where the first word arises from the fact that functions of this type have been used to characterize some operations on bivariate distribution functions (see [8]).

The above conditions, which, at first sight, may appear impractical, can be characterized in the following simpler manner (for the proof see [8]).
Theorem 5.1. A function $\psi:[0,1]^{2} \rightarrow[0,1]$ is P-increasing if, and only if, the following statements hold:
(a) $\psi$ is 2-increasing;
(b) $\psi$ is increasing in each place;
(c) $\psi$ is convex in each place.

Notice that, in view of the above (c), every $P$-increasing function has horizontal and vertical sections that are continuous on $] 0,1[$.

The characterization of the composition of 2-increasing agops is given through the $P$-increasing property.
Theorem 5.2. Let $H$ be an agop. The following statements are equivalent:
(a) $H$ is $P$-increasing;
(b) for every $(A, B) \in \mathscr{A}_{2} \times \mathscr{A}_{2}, F(x, y)=H(A(x, y), B(x, y))$ is a 2-increasing agop.

Proof. Part $(\mathrm{a}) \Rightarrow(\mathrm{b})$ is a particular case of the generalized composition of 2-increasing agops studied in [7]. Conversely, let $s_{i}, t_{i} \in[0,1](i \in\{1,2,3,4\})$ such that (5.2) and (5.3) hold. Define the following two agops:

$$
\begin{aligned}
& A(x, y):= \begin{cases}0, & \text { if } \min \{x, y\}=0 ; \\
s_{1} ; & \text { if }(x, y) \in] 0,1 / 2] \times] 0,1 / 2] ; \\
s_{2} ; & \text { if }(x, y) \in] 0,1 / 2] \times] 1 / 2,1] ; \\
s_{3} ; & \text { if }(x, y) \in] 1 / 2,1] \times] 0,1 / 2] ; \\
s_{4} ; & \text { if }(x, y) \in] 1 / 2,1[\times] 1 / 2,1[; \\
1 ; & \text { if }(x, y)=(1,1) ;\end{cases} \\
& B(x, y):= \begin{cases}0, & \text { if } \min \{x, y\}=0 ; \\
t_{1} ; & \text { if }(x, y) \in] 0,1 / 2] \times] 0,1 / 2] ; \\
t_{2} ; & \text { if }(x, y) \in] 0,1 / 2] \times] 1 / 2,1] ; \\
t_{3} ; & \text { if }(x, y) \in] 1 / 2,1] \times] 0,1 / 2] ; \\
t_{4} ; & \text { if }(x, y) \in] 1 / 2,1[\times] 1 / 2,1[; \\
1 ; & \text { if }(x, y)=(1,1) .\end{cases}
\end{aligned}
$$

Let $F(x, y)=H(A(x, y), B(x, y))$ be the composition of $A$ and $B$. Then $A, B \in \mathscr{A}_{2}$ and from $F \in \mathscr{A}_{2}$ we have

$$
V_{F}\left([1 / 3,2 / 3]^{2}\right)=H\left(s_{1}, t_{1}\right)+H\left(s_{4}, t_{4}\right)-H\left(s_{2}, t_{2}\right)-H\left(s_{3}, t_{3}\right) \geqslant 0,
$$

viz. $H$ is $P$-increasing.

Corollary 5.3. Let $f:[0,1] \rightarrow[0,1]$ be such that $f(0)=0$ and $f(1)=1$. The following statements are equivalent:
(a) $f$ is convex and increasing;
(b) for every $A \in \mathscr{A}_{2}, F(x, y)=f(A(x, y))$ is a 2-increasing agop.

Proof. It suffices to apply the above theorem to the function $H(x, y)=f(x)$, which is $P$-increasing because of Theorem 5.1.

Remark 5.4. Let $A$ and $B$ be 2-increasing agops and consider their product $F(x, y)=A(x, y) \cdot B(x, y)$. Then $F$ is 2 -increasing, because the function $\Pi(x, y)=x y$ is $P$-increasing and Theorem 5.2 holds. However, the minimum $M$ is not $P$-increasing: take, for example, the points $(0,1 / 10),(1 / 2,3 / 10),(1 / 2,3 / 10)$ and $(1,1 / 2)$, then

$$
M(0,1 / 10)-M(1 / 2,3 / 10)-M(1 / 2,3 / 10)+M(1,1 / 2)=-1 / 10<0
$$

Therefore $\min \{A(x, y), B(x, y)\}$ need not be 2 -increasing.
The composition of two copulas is still an open problem, even though some partial results can be found in [8] (see also [7] for generalized composition). As in the general case, the notion of $P$-increasingness is here essential and the two following results may, therefore, turn out to be useful.

Proposition 5.5. Let $A$ be an agop such that $A(x, x) \geqslant x$ for every $x \in[0,1]$. Then $A$ is $P$-increasing if, and only if, there exists $a \in[0,1]$ such that $A(x, y)=a x+(1-a) y$.

Proof. Let $A$ be a $P$-increasing agop such that $A(x, x) \geqslant x$ for every $x \in[0,1]$. In particular, on account of Theorem 5.1, $A$ is 2-increasing and its horizontal and vertical sections are convex. Set $a:=A(1,0)$ and $b:=A(0,1)$ and notice that $a+b \leqslant 1$.

In view of the 2 -increasing property, for every $y \in[0,1]$ one has

$$
\begin{equation*}
A(0, y)+A(y, 1) \geqslant A(y, y)+A(0,1) \geqslant y+b \tag{5.4}
\end{equation*}
$$

and, from the convexity of $y \mapsto A(0, y)$,

$$
A(0, y) \leqslant y A(0,1)+(1-y) A(0,0)=b y .
$$

Therefore, connecting the two inequalities above, one obtains $A(y, 1) \geqslant y+(1-y) b$. On the other hand, from the convexity of $y \mapsto A(y, 1)$,

$$
A(y, 1) \leqslant y A(1,1)+(1-y) A(0,1)=y+(1-y) b,
$$

viz. $A(y, 1)=y+(1-y) b$. Analogously $A(1, y)=(1-a) y+a$.
From (5.4), it follows also that:

$$
A(0, y) \geqslant y+b-(1-b) y-b=b y
$$

and, because $A(0, y) \leqslant y A(0,1)=b y$, one has $A(0, y)=b y$. In the same manner, $A(x, 0)=a x$.
Now, because $A$ is 2-increasing, for every $y \geqslant x$, one has

$$
A(x, y) \geqslant A(x, 1)+A(y, y)-A(y, 1) \geqslant(1-b) x+b y
$$

and

$$
A(x, y) \leqslant A(x, 1)+A(0, y)-b=(1-b) x+b y
$$

viz. $A(x, y)=(1-b) x+b y$. In the same manner, for every $x \geqslant y$, one obtains $A(x, y)=a x+(1-a) y$.
Finally, notice that

$$
A(x, 1 / 2)= \begin{cases}(1-b) x+b / 2, & \text { if } x \leqslant 1 / 2 \\ a x+(1-a) / 2, & \text { if } x>1 / 2\end{cases}
$$

and, from the convexity of $x \mapsto A(x, 1 / 2)$, one has

$$
A\left(\frac{1}{2}, \frac{1}{2}\right) \leqslant \frac{1}{2} A\left(0, \frac{1}{2}\right)+\frac{1}{2} A\left(1, \frac{1}{2}\right)
$$

which is equivalent to $a+b \geqslant 1$. Therefore $a+b=1$ and, for every $(x, y) \in[0,1]^{2}, A(x, y)=a x+$ $(1-a) y$.

Corollary 5.6. Let $A$ be a P-increasing agop. Then the following statements are equivalent:
(a) $A$ is idempotent;
(b) there exists $a \in[0,1]$ such that $A(x, y)=a x+(1-a) y$.

Proposition 5.7. Let $f$ and $g$ be increasing and convex functions from $[0,1]$ into $[0,1]$ such that $f(0)=g(0)=0$ and $f(1)=g(1)=1$. Let A be a P-increasing agop. Then, the function defined by

$$
A_{f, g}(x, y):=A(f(x), g(y))
$$

is a $P$-increasing agop.
Proof. From Proposition 4.2, it follows that $A_{f, g}$ is a 2-increasing agop. Moreover, every horizontal (resp., vertical) section of $A_{f, g}$ is convex, because it is composition of the convex and increasing horizontal (resp., vertical) section of $A$ with $f$ (resp. g). Now, the desired assertion follows from Theorem 5.1.

## 6. Bounds on arbitrary subsets of 2 -increasing agops

Given a (2-increasing) agop $A$, it is obvious that

$$
A_{S}(x, y) \leqslant A(x, y) \quad \text { for every }(x, y) \quad \text { in }[0,1],
$$

and $A_{S}$ is the best-possible lower bound in the set $\mathscr{A}_{2}$, because it is 2 -increasing.
Moreover, the best-possible upper bound in $\mathscr{A}_{2}$ is the greatest agop

$$
A_{G}(x, y)= \begin{cases}0, & \text { if }(x, y)=(0,0) \\ 1, & \text { otherwise }\end{cases}
$$

Notice that $A_{G}$ is not 2-increasing, e.g. $V_{A_{G}}\left([0,1]^{2}\right)=-1$, but it is the pointwise limit of the sequence $\left\{A_{n}\right\}$ of 2increasing agops, defined by

$$
A_{n}(x, y)= \begin{cases}1, & \text { if }(x, y) \in[1 / n, 1]^{2} \\ 0, & \text { otherwise }\end{cases}
$$

In particular, $\left(\mathscr{A}_{2}, \leqslant\right)$ is not a complete lattice.
But, the following result holds.
Proposition 6.1. Every agop is the supremum of a suitable subset of $\mathscr{A}_{2}$.
Proof. Let $A$ be an agop; we may (and, in fact do) suppose that $A \neq A_{G}$, since this case has already been considered, and that $A$ is not 2 -increasing, this case being trivial. For every $\left(x_{0}, y_{0}\right)$ in $[0,1]$, let $z_{0}=A\left(x_{0}, y_{0}\right)$ and consider the following 2 -increasing agop:

$$
\widehat{A}_{x_{0}, y_{0}}:= \begin{cases}1, & \text { if }(x, y)=(1,1) \\ z_{0}, & \text { if }(x, y) \in\left[x_{0}, 1\right] \times\left[y_{0}, 1\right] \backslash\{(1,1)\} \\ 0, & \text { otherwise }\end{cases}
$$

Then one has

$$
A(x, y)=\sup \left\{\widehat{A}_{x_{0}, y_{0}}:\left(x_{0}, y_{0}\right) \in[0,1]^{2}\right\}
$$

For the case of copulas, the lattice structure was considered in [19]. Here, other cases will be considered.

Proposition 6.2. Let $A$ be a 2-increasing agop with marginals $h_{0}, h_{1}, v_{0}$ and $v_{1}$. Let

$$
\begin{equation*}
A_{*}(x, y):=\max \left\{h_{0}(x)+v_{0}(y), h_{1}(x)+v_{1}(y)-1\right\} \tag{6.1}
\end{equation*}
$$

and

$$
\begin{equation*}
A^{*}(x, y):=\min \left\{h_{1}(x)+v_{0}(y)-A(0,1), h_{0}(x)+v_{1}(y)-A(1,0)\right\} . \tag{6.2}
\end{equation*}
$$

Then, for every $(x, y)$ in $[0,1]^{2}, A_{*}$ and $A^{*}$ have the same marginals of $A$ and

$$
\begin{equation*}
A_{*}(x, y) \leqslant A(x, y) \leqslant A^{*}(x, y) . \tag{6.3}
\end{equation*}
$$

Proof. Let $A$ be a 2 -increasing agop. Let $(x, y)$ be a point in $] 0,1\left[^{2}\right.$. In view of inequality (2.3), we have

$$
\begin{aligned}
& A(x, y) \geqslant A(x, 0)+A(0, y)=h_{0}(x)+v_{0}(y) \\
& A(x, y) \geqslant A(x, 1)+A(1, y)-1=h_{1}(x)+v_{1}(y)-1
\end{aligned}
$$

which together yield the first of the inequalities (6.3). Analogously,

$$
\begin{aligned}
& A(x, y) \leqslant A(0, y)+A(x, 1)-A(0,1)=h_{1}(x)+v_{0}(y)-A(0,1), \\
& A(x, y) \leqslant A(x, 0)+A(1, y)-A(1,0)=h_{0}(x)+v_{1}(y)-A(1,0),
\end{aligned}
$$

namely the second of the inequalities (6.3).
It should be noticed that, in the special case of copulas, the bounds of (6.3) coincide with the usual FréchetHoeffding bounds (see [18]).

The subclasses of 2-increasing agops with prescribed marginals have the smallest and the greatest element (in the pointwise ordering), as stated here.
Theorem 6.3. For every 2-increasing agop $A$ with marginals $h_{0}, h_{1}, v_{0}$ and $v_{1}$, the bounds, $A_{*}$ and $A^{*}$ defined by (6.1) and (6.2) are 2-increasing agops.

Proof. The functions $A_{*}$ and $A^{*}$ defined by (6.1) and (6.2), respectively, are obviously agops. Below we shall prove that they are also 2 -increasing. To this end let $R=\left[x, x^{\prime}\right] \times\left[y, y^{\prime}\right]$ be any rectangle contained in the unit square.

Consider, first, the case of $A^{*}$. Then

$$
\begin{aligned}
& A^{*}\left(x^{\prime}, y^{\prime}\right):=\min \left\{h_{1}\left(x^{\prime}\right)+v_{0}\left(y^{\prime}\right)-A(0,1), h_{0}\left(x^{\prime}\right)+v_{1}\left(y^{\prime}\right)-A(1,0)\right\}, \\
& A^{*}(x, y):=\min \left\{h_{1}(x)+v_{0}(y)-A(0,1), h_{0}(x)+v_{1}(y)-A(1,0)\right\}, \\
& A^{*}\left(x^{\prime}, y\right):=\min \left\{h_{1}\left(x^{\prime}\right)+v_{0}(y)-A(0,1), h_{0}\left(x^{\prime}\right)+v_{1}(y)-A(1,0)\right\}, \\
& A^{*}\left(x, y^{\prime}\right):=\min \left\{h_{1}(x)+v_{0}\left(y^{\prime}\right)-A(0,1), h_{0}(x)+v_{1}\left(y^{\prime}\right)-A(1,0)\right\} .
\end{aligned}
$$

There are four cases to be considered.
Case 1. If

$$
A^{*}\left(x^{\prime}, y^{\prime}\right)=h_{1}\left(x^{\prime}\right)+v_{0}\left(y^{\prime}\right)-A(0,1), \quad A^{*}(x, y)=h_{1}(x)+v_{0}(y)-A(0,1),
$$

then

$$
A^{*}\left(x^{\prime}, y^{\prime}\right)+A^{*}(x, y)=h_{1}\left(x^{\prime}\right)+v_{0}(y)-A(0,1)+h_{1}(x)+v_{0}\left(y^{\prime}\right)-A(0,1) \geqslant A^{*}\left(x^{\prime}, y\right)+A^{*}\left(x, y^{\prime}\right) .
$$

Case 2. If

$$
A^{*}\left(x^{\prime}, y^{\prime}\right)=h_{0}\left(x^{\prime}\right)+v_{1}\left(y^{\prime}\right)-A(1,0), \quad A^{*}(x, y)=h_{0}(x)+v_{1}(y)-A(1,0),
$$

then

$$
A^{*}\left(x^{\prime}, y^{\prime}\right)+A^{*}\left(x^{\prime}, y^{\prime}\right)=h_{0}\left(x^{\prime}\right)+v_{1}(y)-A(1,0)+h_{0}(x)+v_{1}\left(y^{\prime}\right)-A(1,0) \geqslant A^{*}\left(x^{\prime}, y\right)+A^{*}\left(x, y^{\prime}\right) .
$$

Case 3. If

$$
A^{*}\left(x^{\prime}, y^{\prime}\right)=h_{1}\left(x^{\prime}\right)+v_{0}\left(y^{\prime}\right)-A(0,1), \quad A^{*}(x, y)=h_{0}(x)+v_{1}(y)-A(1,0),
$$

then one has, since $A$ is 2-increasing, $h_{1}\left(x^{\prime}\right)+h_{0}(x) \geqslant h_{1}(x)+h_{0}\left(x^{\prime}\right)$, so that

$$
\begin{aligned}
A^{*}\left(x^{\prime}, y^{\prime}\right)+A^{*}(x, y) & =h_{1}\left(x^{\prime}\right)+h_{0}(x)-A(0,1)+v_{0}\left(y^{\prime}\right)+v_{1}(y)-A(1,0) \\
& \geqslant h_{1}(x)+v_{0}\left(y^{\prime}\right)-A(0,1)+h_{0}\left(x^{\prime}\right)+v_{1}(y)-A(0,1) \geqslant A^{*}\left(x^{\prime}, y\right)+A^{*}\left(x, y^{\prime}\right) .
\end{aligned}
$$

Case 4. If

$$
A^{*}\left(x^{\prime}, y^{\prime}\right)=h_{0}\left(x^{\prime}\right)+v_{1}\left(y^{\prime}\right)-A(1,0), \quad A^{*}(x, y)=h_{1}(x)+v_{0}(y)-A(0,1),
$$

then one has, since $A$ is 2-increasing, $v_{1}\left(y^{\prime}\right)+v_{0}(y) \geqslant v_{1}(y)+v_{0}\left(y^{\prime}\right)$, so that

$$
\begin{aligned}
A^{*}\left(x^{\prime}, y^{\prime}\right)+A^{*}(x, y) & =h_{0}\left(x^{\prime}\right)+v_{1}\left(y^{\prime}\right)-A(1,0)+h_{1}(x)+v_{0}(y)-A(0,1) \\
& \geqslant h_{0}\left(x^{\prime}\right)+v_{1}(y)-A(1,0)+h_{1}(x)+v_{0}\left(y^{\prime}\right)-A(0,1) \geqslant A^{*}\left(x^{\prime}, y\right)+A^{*}\left(x, y^{\prime}\right) .
\end{aligned}
$$

This proves that $A^{*}$ is 2-increasing. A similar proof holds for $A_{*}$. Here, again, four cases will be considered.

$$
\begin{aligned}
& A_{*}\left(x^{\prime}, y^{\prime}\right):=\max \left\{h_{0}\left(x^{\prime}\right)+v_{0}\left(y^{\prime}\right), h_{1}\left(x^{\prime}\right)+v_{1}\left(y^{\prime}\right)-1\right\}, \\
& A_{*}(x, y):=\max \left\{h_{0}(x)+v_{0}(y), h_{1}(x)+v_{1}(y)-1\right\}, \\
& A_{*}\left(x^{\prime}, y\right):=\max \left\{h_{0}\left(x^{\prime}\right)+v_{0}(y), h_{1}\left(x^{\prime}\right)+v_{1}(y)-1\right\}, \\
& A_{*}\left(x, y^{\prime}\right):=\max \left\{h_{0}(x)+v_{0}\left(y^{\prime}\right), h_{1}(x)+v_{1}\left(y^{\prime}\right)-1\right\} .
\end{aligned}
$$

Case 1. If

$$
A_{*}\left(x^{\prime}, y\right)=h_{0}\left(x^{\prime}\right)+v_{0}(y), \quad A_{*}\left(x, y^{\prime}\right)=h_{0}(x)+v_{0}\left(y^{\prime}\right),
$$

then

$$
A_{*}\left(x^{\prime}, y\right)+A_{*}\left(x, y^{\prime}\right)=h_{0}(x)+v_{0}(y)+h_{0}\left(x^{\prime}\right)+v_{0}\left(y^{\prime}\right) \leqslant A_{*}\left(x^{\prime}, y^{\prime}\right)+A_{*}(x, y) .
$$

Case 2. If

$$
A_{*}\left(x^{\prime}, y\right)=h_{0}\left(x^{\prime}\right)+v_{0}(y), \quad A_{*}\left(x, y^{\prime}\right)=h_{1}(x)+v_{1}\left(y^{\prime}\right)-1
$$

then one has, since $A$ is 2-increasing, $h_{0}\left(x^{\prime}\right)+h_{1}(x) \leqslant h_{1}\left(x^{\prime}\right)+h_{0}(x)$ so that

$$
\begin{aligned}
A_{*}\left(x^{\prime}, y\right)+A_{*}\left(x, y^{\prime}\right) & =h_{0}\left(x^{\prime}\right)+v_{0}(y)+h_{1}(x)+v_{1}\left(y^{\prime}\right)-1 \leqslant h_{1}\left(x^{\prime}\right)+v_{1}\left(y^{\prime}\right)-1+h_{0}(x)+v_{0}(y) \\
& \leqslant A_{*}\left(x^{\prime}, y^{\prime}\right)+A_{*}(x, y) .
\end{aligned}
$$

Case 3. If

$$
A_{*}\left(x^{\prime}, y\right)=h_{1}\left(x^{\prime}\right)+v_{1}(y)-1, \quad A_{*}\left(x, y^{\prime}\right)=h_{0}(x)+v_{0}\left(y^{\prime}\right)
$$

then one has, since $A$ is 2-increasing, $v_{1}(y)+v_{0}\left(y^{\prime}\right) \leqslant v_{1}\left(y^{\prime}\right)+v_{0}(y)$, so that

$$
\begin{aligned}
A_{*}\left(x^{\prime}, y\right)+A_{*}\left(x, y^{\prime}\right) & =h_{1}\left(x^{\prime}\right)+v_{1}(y)-1+h_{0}(x)+v_{0}\left(y^{\prime}\right) \leqslant h_{1}\left(x^{\prime}\right)+v_{1}\left(y^{\prime}\right)-1+h_{0}(x)+v_{0}(y) \\
& \leqslant A_{*}\left(x^{\prime}, y^{\prime}\right)+A_{*}(x, y) .
\end{aligned}
$$

Case 4. If

$$
A_{*}\left(x^{\prime}, y\right)=h_{1}\left(x^{\prime}\right)+v_{1}(y)-1, \quad A_{*}\left(x, y^{\prime}\right)=h_{1}(x)+v_{1}\left(y^{\prime}\right)-1,
$$

then

$$
A_{*}\left(x^{\prime}, y\right)+A_{*}\left(x, y^{\prime}\right)=h_{1}\left(x^{\prime}\right)+v_{1}\left(y^{\prime}\right)-1+h_{1}(x)+v_{1}(y)-1 \leqslant A_{*}\left(x^{\prime}, y^{\prime}\right)+A_{*}(x, y) .
$$

Remark 6.4. Notice that, if $A$ is a $P$-increasing agop, then $A_{*}$ is also $P$-increasing, as a consequence of Theorem 5.1 and some properties of convex functions. Instead, $A^{*}$ may not be $P$-increasing: in fact, if $A$ is a copula, then $A^{*}=M$ is not $P$-increasing.

The following result gives a necessary and sufficient condition that ensures $A_{*}=A^{*}$ in the case of a symmetric agop $A$.

Proposition 6.5. For a symmetric and 2-increasing agop A, the following are equivalent:
(a) $A_{*}=A^{*}$;
(b) there exists an interval $I:=[0, \lambda] \subseteq[0,1]$ and a constant $a \in[0,1]$ such that

$$
h_{1}(t)= \begin{cases}h_{0}(t)+a, & \text { if } t \in I,  \tag{6.4}\\ h_{0}(t)+(1-a), & \text { if } t \in[0,1] \backslash I .\end{cases}
$$

Proof. If $A$ is a symmetric agop, then $h_{0}=v_{0}$ and $h_{1}=v_{1}$. Set $a:=A(0,1)=A(1,0), a \leqslant 1 / 2$. Therefore

$$
A_{*}(x, y):=\max \left\{h_{0}(x)+h_{0}(y), h_{1}(x)+h_{1}(y)-1\right\}
$$

and

$$
A^{*}(x, y):=\min \left\{h_{1}(x)+h_{0}(y)-a, h_{0}(x)+h_{1}(y)-a\right\} .
$$

If $A=A^{*}$, then $A(x, x)=h_{1}(x)+h_{0}(x)-a$. Now, from $A=A_{*}$, one obtains also that either $A(x, x)=2 h_{0}(x)$ or $A(x, x)=2 h_{1}(x)-1$. Therefore, either

$$
\begin{equation*}
h_{1}(x)-h_{0}(x)=a, \tag{6.5}
\end{equation*}
$$

or

$$
\begin{equation*}
h_{1}(x)-h_{0}(x)=1-a . \tag{6.6}
\end{equation*}
$$

If $a=1 / 2$, then $h_{1}(x)=h_{0}(x)+a$ on [0, 1]. Otherwise, note that (6.5) holds at the point $x=0$ and (6.6) holds at the point $x=1$. Moreover, if (6.5) does not hold at a point $x_{1}$, then (6.5) does not hold also for every $x_{2}>x_{1}$. In fact, for the 2 -increasing property, one has

$$
h_{1}\left(x_{2}\right)-h_{0}\left(x_{2}\right) \geqslant h_{1}\left(x_{1}\right)-h_{0}\left(x_{1}\right)=1-a \geqslant 1 / 2 .
$$

Thus $h_{1}$ has the form of (6.4), where $I$ is an interval. The converse is just a matter of straightforward verification.

Example 6.6. Consider the arithmetic mean $A(x, y):=(x+y) / 2$, which is obviously 2 -increasing. Then, one easily evaluates $A_{*}=A^{*}=A$.

Example 6.7. Consider the 2-increasing agop given by the geometric mean $G(x, y):=\sqrt{x y}$. Then

$$
G_{*}(x, y)=\max (0, \sqrt{x}+\sqrt{y}-1) \quad \text { and } \quad G^{*}(x, y)=\min (\sqrt{x}, \sqrt{y})
$$

both of which are 2 -increasing.
Remark 6.8. In the general case of a 2 -increasing agop $A$ such that $A=A_{*}=A^{*}$, as above it can be proved that one among the following four equalities holds:

- $h_{1}(x)-h_{0}(x)=A(0,1)$;
- $h_{1}(x)-h_{0}(x)=1-A(1,0)$;
- $v_{1}(y)-v_{0}(y)=1-A(0,1)$;
- $v_{1}(y)-v_{0}(y)=A(1,0)$.

Unfortunately, one need not have explicit conditions as in the symmetric case for $h_{1}(x)-h_{0}(x)$ and $v_{1}(y)-v_{0}(y)$.

Let $h, v$ and $\delta$ be increasing functions from $[0,1]$ into $[0,1], \delta(0)=0$ and $\delta(1)=1$. Denote by $\mathscr{A}_{h}, \mathscr{A}_{v}$ and $\mathscr{A}_{\delta}$, respectively, the subclasses of 2-increasing agops with horizontal section at $b \in[0,1]$ equal to $h$, vertical section at $a \in[0,1]$ equal to $v$, diagonal section equal $\delta$. Notice that the sets $\mathscr{A}_{h}, \mathscr{A}_{v}$ and $\mathscr{A}_{\delta}$ are not empty, because of Proposition 4.6. The following results give the best-possible bounds in these subclasses.

Proposition 6.9. Let $h:[0,1] \rightarrow[0,1]$ be an increasing function. For every $A$ in $\mathscr{A}_{h}$, $A$ has the horizontal section at $b \in] 0,1[$ equal to $h$, one obtains

$$
\begin{equation*}
\left(A_{h}\right)_{*} \leqslant A(x, y) \leqslant\left(A_{h}\right)^{*} \tag{6.7}
\end{equation*}
$$

where

$$
\begin{aligned}
& \left(A_{h}\right)_{*}(x, y):= \begin{cases}1, & \text { if }(x, y)=1 ; \\
0, & \text { if } 0 \leqslant y<b ; \\
h(x), & \text { otherwise } ;\end{cases} \\
& \left(A_{h}\right)^{*}(x, y):= \begin{cases}0, & \text { if }(x, y)=(0,0) ; \\
1, & \text { if } b<y \leqslant 1 ; \\
h(x), & \text { otherwise } .\end{cases}
\end{aligned}
$$

Moreover,

$$
\left(A_{h}\right)_{*}(x, y)=\bigwedge_{A \in \mathscr{A}_{h}} A(x, y) \quad \text { and } \quad\left(A_{h}\right)^{*}(x, y)=\bigvee_{A \in \mathscr{A}_{h}} A(x, y)
$$

where $\left(A_{h}\right)_{*}$ is a 2-increasing agop and $\left(A_{h}\right)^{*}$, while it is still an agop, is not necessarily 2-increasing.
Proof. For all $(x, y) \in[0,1]^{2}$ and $A \in \mathscr{A}_{h}$, one has $A(x, y) \geqslant 0$ for every $y \in[0, b[$ and $A(x, y) \geqslant h(x)$ for every $y \in[b, 1]$, viz. $A(x, y) \geqslant\left(A_{h}\right)_{*}(x, y)$ on $[0,1]^{2}$. Analogously, $A(x, y) \leqslant h(x)$ for every $y \in[0, b]$ and $A(x, y) \leqslant 1$ for every $y \in] b, 1]$, viz. $A(x, y) \leqslant\left(A_{h}\right)^{*}(x, y)$ on $[0,1]^{2}$. Both $\left(A_{h}\right)_{*}$ and $\left(A_{h}\right)^{*}$ are agops, as is immediately seen; it is also immediate to check that $\left(A_{h}\right)_{*}$ is 2 -increasing and, therefore, that $\left(A_{h}\right)_{*}=\bigwedge_{A \in \mathscr{A}_{h}} A$. Now, suppose that $B$ is any agop greater than, or at least equal to, $\bigvee_{A \in \mathscr{A}_{h}} A$. Then $B(x, y) \geqslant A_{1}(x, y)$, where $A_{1}$ is the 2-increasing agop given by

$$
A_{1}(x, y):= \begin{cases}0, & \text { if } y=0 \\ h(x), & \text { if } 0<y \leqslant b \\ 1, & \text { if } b<y \leqslant 1\end{cases}
$$

and $B(x, y) \geqslant A_{2}(x, y)$, where $A_{2}$ is the 2 -increasing agop given by

$$
A_{2}(x, y):= \begin{cases}0, & \text { if } x=0 \\ h(x), & \text { if } x \neq 0 \quad \text { and } \quad 0<y \leqslant b \\ 1, & \text { if } x \neq 0 \quad \text { and } \quad b<y \leqslant 1\end{cases}
$$

therefore $B(x, y) \geqslant \max \left\{A_{1}(x, y), A_{2}(x, y)\right\}=\left(A_{h}\right)^{*}(x, y)$ on $[0,1]^{2}$ and one obtains $\left(A_{h}\right)^{*}=\bigvee_{A \in \mathscr{A}_{h}} A$. However $\left(A_{h}\right)^{*}$ need not be 2-increasing; in fact,

$$
V_{\left(A_{h}\right)^{*}}\left([0, b / 2]^{2}\right)=-h(b / 2),
$$

which need not be strictly less than 0 .
Analogously, one proves the following result for the class $\mathscr{A}_{v}$.
Proposition 6.10. Let $v:[0,1] \rightarrow[0,1]$ be an increasing function. For every $A$ in $\mathscr{A}_{v}, A$ has the vertical section at $a \in] 0,1[$ equal to $v$, one obtains

$$
\begin{equation*}
\left(A_{v}\right)_{*} \leqslant A(x, y) \leqslant\left(A_{v}\right)^{*} \tag{6.8}
\end{equation*}
$$

where

$$
\begin{aligned}
& \left(A_{v}\right)_{*}(x, y):= \begin{cases}1, & \text { if }(x, y)=1 ; \\
0, & \text { if } 0 \leqslant x<a ; \\
v(y), & \text { otherwise } ;\end{cases} \\
& \left(A_{v}\right)^{*}(x, y):= \begin{cases}0, & \text { if }(x, y)=(0,0) ; \\
1, & \text { if } a<x \leqslant 1 ; \\
v(y), & \text { otherwise. }\end{cases}
\end{aligned}
$$

Moreover,

$$
\left(A_{v}\right)_{*}(x, y)=\bigwedge_{A \in \mathscr{A}_{v}} A(x, y) \quad \text { and } \quad\left(A_{v}\right)^{*}(x, y)=\bigvee_{A \in \mathscr{A}_{v}} A(x, y),
$$

where $\left(A_{v}\right)_{*}$ is a 2-increasing agop and $\left(A_{v}\right)^{*}$, while it is still an agop, is not necessarily 2-increasing.
Proposition 6.11. Let $\delta$ be an increasing function with $\delta(0)=0$ and $\delta(1)=1$. For every $A$ in $\mathscr{A}_{\delta}$, one obtains

$$
\begin{equation*}
\left(A_{\delta}\right)_{*}:=\min \{\delta(x), \delta(y)\} \leqslant A(x, y) \leqslant\left(A_{\delta}\right)^{*}:=\max \{\delta(x), \delta(y)\} . \tag{6.9}
\end{equation*}
$$

Moreover, $\left(A_{\delta}\right)_{*}$ and $\left(A_{\delta}\right)^{*}$ are the best-possible bounds, in the sense that

$$
\left(A_{\delta}\right)_{*}(x, y)=\bigwedge_{A \in \mathscr{A}_{\delta}} A(x, y) \quad \text { and } \quad\left(A_{\delta}\right)^{*}(x, y)=\bigvee_{A \in \mathscr{A}_{\delta}} A(x, y),
$$

where $\left(A_{\delta}\right)_{*}$ is a 2-increasing agop and $\left(A_{\delta}\right)^{*}$, while it is still an agop, is not necessarily 2-increasing.
Proof. For all $(x, y) \in[0,1]^{2}$ and $A \in \mathscr{A}_{\delta}$, one has

$$
A(x, y) \geqslant A(x \wedge y, x \wedge y)=\min \{\delta(x), \delta(y)\}
$$

and

$$
A(x, y) \leqslant A(x \vee y, x \vee y)=\max \{\delta(x), \delta(y)\} .
$$

This proves (6.9). Both $\left(A_{\delta}\right)_{*}$ and $\left(A_{\delta}\right)^{*}$ are agops, as is immediately seen; it is also immediate to check that $\left(A_{\delta}\right)_{*}$ is 2-increasing and, therefore, that $\left(A_{\delta}\right)_{*}=\bigwedge_{A \in \mathscr{A}_{\delta}} A$. Now, suppose that $B$ is any agop greater than, or at least equal to, $\bigvee_{A \in \mathscr{A}_{\delta}} A$. Then $B(x, y) \geqslant A_{1}(x, y):=\delta(x)$ and $B(x, y) \geqslant A_{2}(x, y):=\delta(y)$, with $A_{1}$ and $A_{2}$ 2increasing agops. Thus, $B(x, y) \geqslant\left(A_{\delta}\right)^{*}$ so that $\left(A_{\delta}\right)^{*}=\bigvee_{A \in \mathscr{A}_{\delta}} A$. This proves that $\left(A_{\delta}\right)^{*}$ is the best possible upper bound for the set $\mathscr{A}_{\delta}$. However $\left(A_{\delta}\right)^{*}$ need not be 2 -increasing; in fact,

$$
V_{\left(A_{\delta}\right)^{*}}\left([0,1]^{2}\right)=\delta(0)-\delta(1)=-1<0 .
$$

Corollary 6.12. Let $\delta$ be an increasing function with $\delta(0)=0$ and $\delta(1)=1$. For every symmetric agop $A$ in $\mathscr{A}_{\delta}$, one obtains

$$
\left(A_{\delta}\right)_{*}:=\min \{\delta(x), \delta(y)\} \leqslant A(x, y) \leqslant \frac{\delta(x)+\delta(y)}{2},
$$

where the lower and upper bounds are best-possible and $(\delta(x)+\delta(y)) / 2$ is the maximal element in the class of symmetric agops.

Proof. If $A$ is symmetric and 2-increasing, one has, for every $x, y$ in $[0,1]$,

$$
\delta(x)+\delta(y)=A(x, x)+A(y, y) \geqslant 2 A(x, y) .
$$

## 7. A construction method for copulas

The construction of copulas is one of the most important problem in statistics, because, in view of Sklar's theorem, it is directly connected with the construction of classes of multivariate probability distribution functions that can be used in the simulation and in the interpretation of real data (see [18]). The main result of this section gives a simple method of constructing a copula from a 2 -increasing and 1-Lipschitz agop.
Theorem 7.1. For every 2-increasing and 1-Lipschitz agop $A$, the function

$$
\begin{equation*}
C(x, y):=\min \{x, y, A(x, y)\} \tag{7.1}
\end{equation*}
$$

is a copula.

Proof. First, in order to prove that $C$ is a copula, one notes that $C$ has neutral element 1 and annihilator 0 ; in fact, for every $x \in[0,1]$, one has

$$
\begin{aligned}
& C(x, 1)=\min \{A(x, 1), x\}=x=C(1, x), \\
& C(x, 0)=\min \{A(x, 0), 0\}=0=C(0, x) .
\end{aligned}
$$

Then, one proves that $C$ is 2 -increasing by using Proposition 2.3.
For every rectangle $R:=[s, t] \times[s, t]$ on $[0,1]^{2}$, set

$$
V_{C}(R)=\min \{A(s, s), s\}+\min \{A(t, t), t\}-\min \{A(s, t), s\}-\min \{A(t, s), s\} .
$$

One has to prove that $V_{C}(R) \geqslant 0$ and several cases are considered.
If $A(s, s) \geqslant s$, then also $A(s, t), A(t, s)$ and $A(t, t)$ are greater than $s$, because $A$ is increasing in each variable, and thus

$$
V_{C}(R)=\min \{A(t, t), t\}-s \geqslant 0
$$

If $A(s, s)<s$, then one distinguishes:

- if $A(t, t)<t$, since $A$ is 2-increasing, one has

$$
A(s, s)+A(t, t) \geqslant A(s, t)+A(t, s) \geqslant \min \{A(s, t), s\}+\min \{A(t, s), s\},
$$

viz. $V_{C}(R) \geqslant 0$;

- if $A(t, t) \geqslant t$, since $A$ is 1-Lipschitz, one has

$$
\min \{A(t, s), s\}-\min \{A(s, s), s\} \leqslant t-s \leqslant t-\min \{A(t, s), s\},
$$

and thus $V_{C}(R) \geqslant 0$.
Now, let $R=\left[x_{1}, x_{2}\right] \times\left[y_{1}, y_{2}\right]$ be a rectangle contained in $\Delta_{+}$. Then $V_{C}(R)$ is given by

$$
V_{C}(R)=\min \left\{A\left(x_{1}, y_{1}\right), y_{1}\right\}+\min \left\{A\left(x_{2}, y_{2}\right), y_{2}\right\}-\min \left\{A\left(x_{2}, y_{1}\right), y_{1}\right\}-\min \left\{A\left(x_{1}, y_{2}\right), y_{2}\right\} .
$$

If $A\left(x_{1}, y_{1}\right) \geqslant y_{1}$, then also $A\left(x_{2}, y_{1}\right), A\left(x_{1}, y_{2}\right)$ and $A\left(x_{2}, y_{2}\right)$ are greater than $y_{1}$, because $A$ is increasing in each variable, and thus

$$
V_{C}(R)=\min \left\{A\left(x_{2}, y_{2}\right), y_{2}\right\}-y_{1} \geqslant 0 .
$$

If $A\left(x_{1}, y_{1}\right)<y_{1}$, then one distinguishes:

- if $A\left(x_{2}, y_{2}\right)<y_{2}$, since $A$ is 2-increasing, one has

$$
A\left(x_{2}, y_{2}\right)+A\left(x_{1}, y_{1}\right) \geqslant A\left(x_{2}, y_{1}\right)+A\left(x_{1}, y_{2}\right) \geqslant \min \left\{A\left(x_{2}, y_{1}\right), y_{1}\right\}+\min \left\{A\left(x_{1}, y_{2}\right), y_{2}\right\},
$$

viz. $V_{C}(R) \geqslant 0$;

- if $A\left(x_{2}, y_{2}\right) \geqslant y_{2}$, one has

$$
V_{C}(R)=A\left(x_{1}, y_{1}\right)+y_{2}-A\left(x_{1}, y_{2}\right)-\min \left\{A\left(x_{2}, y_{1}\right), y_{1}\right\},
$$

because $A$ is 1-Lipschitz and, hence,

$$
A\left(x_{1}, y_{2}\right) \leqslant y_{2}-y_{1}+A\left(x_{1}, y_{1}\right) \leqslant y_{2}
$$

moreover, from the fact that

$$
A\left(x_{1}, y_{2}\right)-A\left(x_{1}, y_{1}\right) \leq y_{2}-y_{1} \leqslant y_{2}-\min \left\{A\left(x_{2}, y_{1}\right), y_{1}\right\},
$$

it follows that $V_{C}(R) \geqslant 0$.
Finally, let $R=\left[x_{1}, x_{2}\right] \times\left[y_{1}, y_{2}\right]$ be a rectangle contained in $\Delta_{-}$. Then $V_{C}(R)$ is given by

$$
V_{C}(R)=\min \left\{A\left(x_{1}, y_{1}\right), x_{1}\right\}+\min \left\{A\left(x_{2}, y_{2}\right), x_{2}\right\}-\min \left\{A\left(x_{2}, y_{1}\right), x_{2}\right\}-\min \left\{A\left(x_{1}, y_{2}\right), x_{1}\right\} .
$$

If $A\left(x_{1}, y_{1}\right) \geqslant x_{1}$, then, because $A$ is increasing in each variable,

$$
V_{C}(R)=\min \left\{A\left(x_{2}, y_{2}\right), x_{2}\right\}-\min \left\{A\left(x_{2}, y_{1}\right), x_{2}\right\} \geqslant 0 .
$$

If $A\left(x_{1}, y_{1}\right)<x_{1}$, then one distinguishes:

- if $A\left(x_{2}, y_{2}\right)<x_{2}$, since $A$ is 2-increasing, one has

$$
A\left(x_{2}, y_{2}\right)+A\left(x_{1}, y_{1}\right) \geqslant A\left(x_{2}, y_{1}\right)+A\left(x_{1}, y_{2}\right) \geqslant \min \left\{A\left(x_{2}, y_{1}\right), x_{1}\right\}+\min \left\{A\left(x_{1}, y_{2}\right), x_{2}\right\},
$$

viz. $V_{C}(R) \geqslant 0$;

- if $A\left(x_{2}, y_{2}\right) \geqslant x_{2}$, one has

$$
V_{C}(R)=A\left(x_{1}, y_{1}\right)+x_{2}-\min \left\{A\left(x_{1}, y_{2}\right), x_{1}\right\}-\min \left\{A\left(x_{2}, y_{1}\right), x_{2}\right\},
$$

moreover, from the inequality

$$
\min \left\{A\left(x_{2}, y_{1}\right), x_{2}\right\}-A\left(x_{1}, y_{1}\right) \leqslant A\left(x_{2}, y_{1}\right)-A\left(x_{1}, y_{1}\right) \leqslant x_{2}-x_{1} \leqslant x_{2}-\min \left\{A\left(x_{1}, y_{2}\right), x_{1}\right\},
$$

it follows that $V_{C}(R) \geqslant 0$.

Notice that agops satisfying the assumptions of Theorem 7.1 are stable under convex combinations. Thus, many examples can be provided: it suffices to consider, for examples, copulas, quasi-arithmetic means bounded from above by the arithmetic mean, and their convex combinations.
Example 7.2. Let $A$ be the modular agop $A(x, y)=(\delta(x)+\delta(y)) / 2$, where $\delta:[0,1] \rightarrow[0,1]$ is an increasing and 2-Lipschitz function with $\delta(0)=0$ and $\delta(1)=1$. Then $A$ satisfies the assumptions of Theorem 7.1 and it generates the following copula:

$$
C_{\delta}(x, y)=\min \left\{x, y, \frac{\delta(x)+\delta(y)}{2}\right\} .
$$

Copulas of this type were introduced in [9] and are called diagonal copulas.
Example 7.3. Let us consider the following 2-increasing and 1-Lipschitz agop

$$
A(x, y)=\lambda B(x, y)+(1-\lambda) \frac{x+y}{2}
$$

defined for every $\lambda \in[0,1]$ and for every copula $B$. This $A$ satisfies the assumptions of Theorem 7.1 and, therefore, the following class of copulas is obtained:

$$
C_{\lambda}(x, y):=\min \left\{x, y, \lambda B(x, y)+(1-\lambda) \frac{x+y}{2}\right\} .
$$

Example 7.4. Let $A$ be a 2-increasing agop of the form $A(x, y)=f(x) \cdot g(y)$ (see Example 4.3). If $A$ is 1 -Lipschitz, then $A$ satisfies the assumptions of Theorem 7.1. Consider, for instance, $f(x)=x$ and $g(y)=(y+1) / 2$, or $f(x)=(x+1) / 2$ and $g(y)=y$, which yield, respectively, the following copulas:

$$
C_{1}(x, y)=\min \left\{y, \frac{x(y+1)}{2}\right\}, \quad C_{2}(x, y)=\min \left\{x, \frac{y(x+1)}{2}\right\} .
$$

Remark 7.5. If $A$ is a copula, it is obvious that the copula $C$ defined by (7.1) is equal to $A$. But, in general, when $A$ is greater than $\min \{x, y\}$, the copula $C$ has a singular component and, therefore, does not have full support (see [18] for these notions).

Remark 7.6. If $A$ is a 1-Lipschitz agop, then $C(x, y):=\min \{x, y, A(x, y)\}$ is a 1-Lipschitz agop. In fact, the minimum of two 1 -Lipschitz agops is 1 -Lipschitz (see $[14,15]$ ) and, in this case, $C$ is the minimum between the 1-Lipschitz agop $M(x, y)=\min \{x, y\}$ and $A$. In particular, because $C$ has neutral element 1 , it follows that $C$ is a quasi-copula (see [10]).

## 8. Conclusions

We have investigated the class of binary aggregation operators satisfying the 2 -increasing property. These operators play a relevant rôle, for example, in statistics, fuzzy measures and in the treatment of vague information. We have also characterized some agops having other special properties, like quasi-arithmetic mean, Choquet-integral based, modularity, and presented some construction methods, including transformations via bijective functions and the classical composition method. To this end, we have used the new notion of $P$ increasing function and studied it in detail. The lattice structure (with respect to the pointwise order) of some subclasses of 2-increasing agops has been presented. Finally, we have given a method for constructing copulas starting from a 2 -increasing and 1-Lipschitz agop.

## References

[1] P. Benvenuti, R. Mesiar, Integrals with respect to a general fuzzy measure, in: M. Grabisch, T. Murofushi, M. Sugeno (Eds.), Fuzzy Measures and Integrals: Theory and Applications, Studies in Fuzziness and Soft Computing, vol. 40, Physica-Verlag, Heidelberg, 2000.
[2] B. Bouchon-Meunier (Ed.), Aggregation and Fusion of Imperfect Information, Studies in Fuzziness and Soft Computing, vol. 12, Physica-Verlag, Heidelberg, 1998.
[3] T. Calvo, A. Kolesárová, M. Komorníková, R. Mesiar, Aggregation operators: properties, classes and construction methods, in: [4], pp. 3-106.
[4] T. Calvo, R. Mesiar, G. Mayor (Eds.), Aggregation Operators. New Trends and Applications, Studies in Fuzziness and Soft Computing, vol. 97, Physica-Verlag, Heidelberg, 2002.
[5] G. Choquet, Theory of capacities, Ann. Inst. Fourier Grenoble 5 (1953-1954) 131-295.
[6] D. Denneberg, Non-Additive Measure and Integral, Kluwer Academic Publishers, Dordrecht, 1994.
[7] F. Durante, Generalized composition of binary aggregation operators, Int. J. Uncertain. Fuzz. Knowl.-Based Syst. 13 (2005) 567-577.
[8] F. Durante, C. Sempi, On the characterization of a class of binary operations on bivariate distribution functions, Publ. Math. Debrecen, in press.
[9] G.A. Fredricks, R.B. Nelsen, Copulas constructed from diagonal sections, in: V. Beneš, J. Štěpán (Eds.), Distributions with Given Marginals and Moment Problems, Kluwer, Dordrecht, 1997, pp. 129-136.
[10] C. Genest, J.J. Quesada-Molina, J.A. Rodríguez-Lallena, C. Sempi, A characterization of quasi-copulas, J. Multivariate Anal. 69 (1999) 193-205.
[11] S. Gottwald, Mathematical fuzzy logic as a tool for the treatment of vague information, Inform. Sci. 172 (2005) 41-71.
[12] E.P. Klement, A. Kolesárová, R. Mesiar, C. Sempi, Copulas constructed from the horizontal section, submitted for publication.
[13] A. Kolesárová, E.P. Klement, On affine sections of 1-Lipschitz aggregation operators, in: Proceedings Joint EUSFLAT-LFA Conference, Barcelona, 2005, pp. 1293-1296.
[14] A. Kolesárová, 1-Lipschitz aggregation operators and quasi-copulas, Kybernetika 39 (2003) 615-629.
[15] A. Kolesárová, J. Mordelová, E. Muel, Kernel aggregation operators and their marginals, Fuzzy Sets Syst. 142 (2004) 35-50.
[16] R. Mesiar, B. De Baets, New construction methods for aggregation operators, in: Proceedings IPMU, Madrid, 2000, pp. 701-706.
[17] R. Mesiar, A note on moderate growth of $t$-conorms, Fuzzy Sets Syst. 122 (2001) 357-359.
[18] R.B. Nelsen, An Introduction to Copulas, Lecture Notes in Statistics, vol. 139, Springer, New York, 1999.
[19] R.B. Nelsen, M. Úbeda Flores, The lattice-theoretic structure of sets of bivariate copulas and quasi-copulas, C.R. Math. Acad. Sci. Paris 341 (2005) 583-586.
[20] G. Pasi, R.R. Yager, Modeling the concept of majority opinion in group decision making, Inform. Sci. 176 (2006) 390-414.
[21] R.R. Yager, On ordered weighted averaging aggregation operators in multicriteria decision making, IEEE Trans. Syst. Man. Cybernet. 18 (1988) 183-190.
[22] R.R. Yager, On global requirements for implication operators in fuzzy modus ponens, Fuzzy Sets Syst. 106 (1999) 3-10.


[^0]:    * Corresponding author. Tel.: +39832 323136; fax: +39832297410.

    E-mail addresses: fabrizio.durante@unile.it (F. Durante), mesiar@math.sk (R. Mesiar), papini@dm.unibo.it (P.L. Papini), carlo.sempi@unile.it (C. Sempi).
    ${ }^{1}$ The support of M.I.U.R. through the project "Metodi stocastici in finanza matematica" is gratefully acknowledged.
    ${ }^{2}$ The support of grants VEGA 1/1145/05 and APVT 20-003204 is kindly acknowledged.

