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Convergence of copulas: Critical remarks

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Abstract. We study some aspects of the relationship between the weak convergence of sequences of multivariate distribution functions and the pointwise convergence of the copulas that connect them to their respective marginals.

1. Introduction

While the convergence of univariate distribution functions (=d.f.'s) has been widely studied, comparatively little attention has been devoted to the study of weak convergence of multivariate d.f.'s especially with regard to the convergence of the copulas that connect the multivariate d.f.'s to their respective marginals. The present note is a contribution to this study in the bivariate case. The results presented here are a counterpart of the lack of uniqueness of the copula of two random variables when at least one of them has a discrete component; they also confirm the remarks made by Marshall in [1] and warn the statistician to proceed with the greatest caution whenever copulas are used in connection with probability laws or random variables that have a discrete component.

2. Preliminaries

We shall consider here univariate and bivariate d.f.'s; a univariate d.f. is a map F from the set of the extended real numbers $\overline{\mathbb{R}} := [-\infty, +\infty]$ into the unit interval $[0, 1]$ that is non–decreasing, left–continuous on the reals $\mathbb{R}$ and such that $F(-\infty) = 0$ and $F(+\infty) = 1$. A bivariate d.f. is a function H from $\overline{\mathbb{R}}^2$ into $[0, 1]$ that is left–continuous on $\mathbb{R}^2$ in each variable, satisfies $H(+\infty, +\infty) = 1$ and, for every $t \in \overline{\mathbb{R}}$,

$$H(t, -\infty) = H(-\infty, t) = 0;$$

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moreover, H satisfies, for all x, x', y, y' in $\overline{\mathbb{R}}$, with $x \le x'$ and $y \le y'$, the inequality

$$H(x', y') - H(x, y') - H(x', y) + H(x, y) \ge 0.$$

The space of univariate and bivariate d.f.'s will be denoted by Δ and by Δ_2 respectively.

Copulas were introduced by Sklar in 1959 ([6]). The reader is referred also to [7, 4, 3, 8, 2].

A copula is a function $C : [0,1]^2 \to [0,1]$ that satisfies the following properties:

- for all t in $[0,1]$, $C(t,0) = C(0,t) = 0$;
- for all t in $[0,1]$, $C(t,1) = C(1,t) = t$;
- if s, s', t, t' are in $[0,1]$ with $s \le s'$ and $t \le t'$, then

$$C(s', t') - C(s, t') - C(s', t) + C(s, t) \ge 0.$$

As a consequence of these properties it follows that (a) C satisfies the Lipschitz condition

$$|C(s,t) - C(s',t')| \le |s - s'| + |t - t'| \qquad s, s', t, t' \in [0,1]\,;$$

(b) that it is non–decreasing in each variable and (c) that it is absolutely continuous. In other words, a copula is the restriction to the unit square $[0,1]^2$ of a two–dimensional distribution function that concentrates all the probability mass on the unit square and which has uniform marginals.

The importance of the concept of copula stems from the following

Theorem 1 (Sklar). *Let X and Y be two random variables on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$ having H as their joint distribution function and let F and G be the marginals of H,*

$$F(x) = H(x, +\infty), \qquad\qquad G(y) = H(+\infty, y).$$

Then there exists (at least) a copula C such that

$$H(x, y) = C\left(F(x), G(y)\right). \tag{1}$$

If both F and G are continuous, then the copula C is uniquely determined; otherwise, C is uniquely determined on Ran $F \times$ Ran G. *Conversely, if C is a copula and F and G are univariate distribution functions, then the function H defined by* (1) *is a joint distribution function with marginals given by F and G.*

If either F or G, or both, is not continuous, then there may be more than one copula may satisfy (1); all of these coincide on the set Ran $F\times$ Ran G. However, by a method of bilinear interpolation, it is always possible to choose a single copula that satisfies (1). For this method see [2, Section 2.3.5]. The probabilistic meaning of the choice of the method of bilinear interpolation is still not clear.

3. Main results

In this section we consider the relationship between the (weak) convergence of two sequences of univariate d.f.'s $\{F_n\}$ and $\{G_n\}$, the sequence of bivariate d.f.'s $\{H_n\}$ of which they are the marginals, and the sequence $\{C_n\}$ of connecting copulas. It is, of course, obvious that pointwise convergence of copulas is equivalent to their uniform convergence.

Theorem 2. *Let $\{F_n : n \in \mathbb{Z}_+\}$ and $\{G_n : n \in \mathbb{Z}_+\}$ two sequences of univariate d.f.'s and let $\{C_n : n \in \mathbb{Z}_+\}$ be a sequence of copulas; then, for every n in $\mathbb{Z}_+$, a bivariate d.f. H_n is defined through*

$$H_n(x, y) := C_n (F_n(x), G_n(y)) .$$

If the sequences $\{F_n\}$ and $\{G_n\}$ converge to F_0 and to G_0 respectively in the weak convergence of Δ, and if the sequence of copulas $\{C_n\}$ converges to the copula C_0 pointwise in $[0,1]^2$, then the sequence $\{H_n\}$ converges to H_0 in the weak topology of Δ_2.

Proof. Let F_0 and G_0 be continuous at $x \in \overline{\mathbb{R}}$ and at $y \in \overline{\mathbb{R}}$ respectively, and let $s = F_0(x)$ and $t = G_0(y)$. Since C_0 is continuous at $(s, t) \in [0,1]^2$, H_0 is continuous at $(x, y) \in \overline{\mathbb{R}}^2$. The points $(x, y) \in \overline{\mathbb{R}}^2$ at which H_0 is continuous are dense in $\overline{\mathbb{R}}^2$, since the closure of the cartesian product of the sets $C(F_0)$ and $C(G_0)$ of continuity points of F_0 and G_0 is dense in $\overline{\mathbb{R}}^2$,

$$\overline{C(F_0) \times C(G_0)} = \overline{C(F_0)} \times \overline{C(G_0)} = \overline{\mathbb{R}} \times \overline{\mathbb{R}} = \overline{\mathbb{R}}^2.$$

Then

$$\lim_{n\to+\infty} F_n(x) = F_0(x) \qquad \text{and} \qquad \lim_{n\to+\infty} G_n(y) = G_0(y).$$

Therefore

$$\begin{aligned}
|H_0(x, y) - H_n(x, y)| &= |C_0 (F_0(x), G_0(y)) - C_n (F_n(x), G_n(y))| \\
&\le |C_0 (F_0(x), G_0(y)) - C_n (F_0(x), G_0(y))| \\
&\quad + |C_n (F_0(x), G_0(y)) - C_n (F_n(x), G_0(y))| \\
&\quad + |C_n (F_n(x), G_0(y)) - C_n (F_n(x), G_n(y))|
\end{aligned}$$

Because of the Lipschitz condition, one has

$$|C_n(F_n(x), G_n(y)) - C_n(F_n(x), G_0(y))| \le |G_n(y) - G_0(y)| \xrightarrow[n\to+\infty]{} 0$$

and

$$|C_n(F_n(x), G_0(y)) - C_n(F_0(x), G_0(y))| \le |F_n(x) - F_0(x)| \xrightarrow[n\to+\infty]{} 0,$$

while, by assumption,

$$|C_n(F_0(x), G_0(y)) - C_0(F_0(x), G_0(y))| \xrightarrow[n\to+\infty]{} 0.$$

This proves the assertion. □

Theorem 3. *Let $\{H_n : n \in \mathbb{Z}_+\}$ be a sequence of bivariate d.f.'s, and, for every $n \in \mathbb{Z}_+$, let F_n and G_n be the marginals of H_n,*

$$F_n(x) := H_n(x, +\infty) \qquad \text{and} \qquad G_n(y) := H_n(+\infty, y).$$

For every $n \in \mathbb{N}$, let C_n be the connecting copula, so that, for every $(x, y) \in \overline{\mathbb{R}}^2$,

$$H_n(x, y) = C_n(F_n(x), G_n(y)).$$

If $H_n \xrightarrow[n\to+\infty]{} H_0$ weakly, and if the marginals F_n and G_n converge weakly to F_0 and G_0 respectively, then there exists a dense subset D of $\operatorname{Ran} F_0 \times \operatorname{Ran} G_0$ such that, for every point $(s, t) \in D$, one has

$$C_n(s, t) \longrightarrow_{n\to+\infty} C_0(s, t).$$

Proof. The set A of continuity points of F_0 is, as is well known, the whole extended real line $\overline{\mathbb{R}}$ with the exception of at most a countably infinity of points, say

$$A = \overline{\mathbb{R}} \setminus (\cup_{n\in\mathbb{N}} x_n).$$

Then, one has

$$F_0(A) = [0,1] \setminus (\cup_{n\in\mathbb{N}} [F_0(x_n - 0), F_0(x_n + 0)]),$$
$$F_0(\overline{A}) = F_0(\overline{\mathbb{R}}) = \text{Ran } F_0 = [0,1] \setminus (\cup_{n\in\mathbb{N}}]F_0(x_n - 0), F_0(x_n + 0)]),$$
$$\overline{F_0(A)} = [0,1] \setminus (\cup_{n\in\mathbb{N}}]F_0(x_n - 0), F_0(x_n + 0)[),$$

so that

$$\overline{F_0(A)} \supset F_0(\overline{A}) = \text{Ran } F_0.$$

Thus, if $B := C(G_0)$ is the set of continuity points for G_0, one has

$$\begin{aligned}\overline{F_0(A) \times G_0(B)} &= \overline{F_0(A)} \times \overline{G_0(B)} \supset F_0\left(\overline{A}\right) \times G_0\left(\overline{B}\right)\\ &= F_0\left(\overline{\mathbb{R}}\right) \times G_0\left(\overline{\mathbb{R}}\right) = \operatorname{Ran} F_0 \times \operatorname{Ran} G_0.\end{aligned}$$

Now, it suffices to set $D := F_0(A) \times G_0(B)$. Let (s,t) belong to D; then $s = F_0(x)$ and $t = G_0(y)$ for some x in $C(F_0)$ and some y in $C(G_0)$. Therefore

$$\begin{aligned}|C_n(s,t) - C_0(s,t)| &= |C_n\left(F_0(x), G_0(y)\right) - C_0\left(F_0(x), G_0(y)\right)|\\ &\le |C_n\left(F_0(x), G_0(y)\right) - C_n\left(F_n(x), G_n(y)\right)|\\ &\quad + |C_n\left(F_n(x), G_n(y)\right) - C_0\left(F_0(x), G_0(y)\right)|.\end{aligned}$$

Since $C(F_0) \times C(G_0)$ is included in $C(H_0)$, the subset of $\mathbb{R}^2$ consisting of continuity points for H_0, one has

$$|C_n\left(F_n(x), G_n(y)\right) - C_0\left(F_0(x), G_0(y)\right)| = |H_n(x,y) - H_0(x,y)| \xrightarrow[n\to+\infty]{} 0.$$

On the other hand, the Lipschitz condition satisfied by all copulas and the weak convergence of the marginals imply

$$\begin{aligned}|C_n\left(F_0(x), G_0(y)\right) &- C_n\left(F_n(x), G_n(y)\right)|\\ &\le |F_n(x) - F_0(x)| + |G_n(y) - G_0(y)| \xrightarrow[n\to+\infty]{} 0.\end{aligned}$$

These latter relationships prove the assertion. □

We recall that the assumption of the convergence of the marginals in Theorem 3 is needed because the weak convergence of $\{H_n\}$ does not suffice to guarantee that the marginals will converge weakly as the following example taken from [5] shows.

Example 1. Let $\epsilon_a \in \Delta$ be the d.f. of any random variable that takes almost certainly the value a; it is then defined, for $a \in \mathbb{R}$, by

$$\epsilon_a(t) := \begin{cases} 0, & t \le a,\\ 1, & t > a;\end{cases}$$

if $a = -\infty$, then $\epsilon_{-\infty}(t) = 1$ for all $t \neq -\infty$, while if $a = +\infty$, then $\epsilon_{+\infty}(t) = 0$ for all $t \in \mathbb{R}$.

Consider now the two sequences $\{F^{(n)} : n \in \mathbb{Z}_+\}$ and $\{G^{(n)} : n \in \mathbb{Z}_+\}$ of univariate d.f.'s defined by

$$\begin{aligned}F^{(n)} &:= \epsilon_n, &&\text{and}\qquad G^{(n)} := \frac{\epsilon_n}{2} + \frac{\epsilon_{+\infty}}{2} \qquad (n \in \mathbb{N})\\ F^{(0)} &:= \frac{\epsilon_0}{2} + \frac{\epsilon_{+\infty}}{2}, &&\text{and}\qquad G^{(0)} := \epsilon_{+\infty} \qquad (n = 0).\end{aligned}$$

Define the sequence $\{H^{(n)} : n \in \mathbb{Z}_+\}$ of bivariate d.f.'s by

$$H^{(n)}(x,y) := F^{(n)}(x)\, G^{(n)}(y) \qquad (x,y) \in \overline{\mathbb{R}}^2; n \in \mathbb{Z}_+.$$

Here the connecting copula is that of independence, i.e.

$$H^{(n)} = \Pi\,(F^{(n)}, G^{(n)}) \qquad n \in \mathbb{Z}_+.$$

The d.f. $H^{(0)}$ is continuous on $\mathbb{R}^2$, since $H^{(0)}(x,y) = 0$ for all $(x,y) \in \mathbb{R}^2$, but not on $\overline{\mathbb{R}}^2$, because its marginals $F_1^{(0)}$ and $F_2^{(0)}$ are given, respectively by

$$F_1^{(0)}(x) = H^{(0)}(x,+\infty) = F^{(0)}(x) \qquad \text{and} \qquad F_2^{(0)}(y) = H^{(0)}(+\infty,y) = G^{(0)}(y).$$

At every point $(x,y) \in \mathbb{R}^2$ one has

$$H^{(n)}(x,y) = \frac{\epsilon_n(x)\,\epsilon_n(y)}{2} \qquad n \in \mathbb{N},$$

so that $H^{(n)} \to 0$ on $\mathbb{R}^2$ as n tends to $+\infty$. Thus $\{H^{(n)} : n \in \mathbb{N}\}$ converges weakly to $H^{(0)}$. However, the sequence of the marginals $F_1^{(n)}$ of $H^{(n)}$ does not converges weakly to the marginal $F^{(0)}$ of $H^{(0)}$. Indeed, one has $F^{(0)}(x) = 1/2$ for every real $x > 0$, while $F^{(n)}(x) = \epsilon_n(x)$ that goes to zero as n tends to $+\infty$. □

It is important to stress that, although

$$\lim_{n\to+\infty} C_n(s,t) = C_0(s,t)$$

at the points (s,t) of the set D, the sequence $\{C_n(s,t)\}$ need not converge to $C_0(s,t)$ at all the points of $[0,1]^2$, as the following example shows.

Example 2. Let $\{\lambda_n : n \in \mathbb{N}\}$ be a sequence of real numbers in $]0,1[$ that converges to $1/2$, let Φ be the d.f. of the standard normal law $N(0,1)$. Introduce the d.f.'s

$$F_n := \lambda_n\, \Phi + (1-\lambda_n)\, \epsilon_{1/n} \qquad G_n := \Phi \qquad (n \in \mathbb{N}),$$

and

$$F_0 := \frac{1}{2}\,(\Phi + \epsilon_0) \qquad G_0 := \Phi.$$

Then, obviously, one has $F_n \to F_0$ and $G_n \to G_0$ weakly.

Let $\widetilde{C}$ be the ordinal sum (see [2, Section 3.2.2]) of Π, M and Π, respectively, over the partition $((0,1/4),(1/4,3/4),(3/4,1))$; then

$$\widetilde{C}(s,t)=\begin{cases} 4\,st, & (s,t)\in\left[0,\dfrac{1}{4}\right]^2, \\ \dfrac{1}{4}+M\left(s-\dfrac{1}{4},t-\dfrac{1}{4}\right), & (s,t)\in\left[\dfrac{1}{4},\dfrac{3}{4}\right]^2, \\ \dfrac{3}{4}+\left(s-\dfrac{3}{4}\right)\left(t-\dfrac{3}{4}\right), & (s,t)\in\left[\dfrac{3}{4},1\right]^2, \\ M(s,t), & \text{elsewhere.} \end{cases}$$

Notice that

$$\overline{D}=\overline{\operatorname{Ran}F_0}\times\ \overline{\operatorname{Ran}G_0}=\left(\left[0,\frac{1}{4}\right]\cup\left[\frac{3}{4},1\right]\right)\times[0,1]\,.$$

Consider now the copula C_0 that coincides with $\widetilde{C}$ on $\overline{D}$, i.e.

$$\forall (s,t)\in\overline{D}\qquad C_0(s,t):=\widetilde{C}(s,t),$$

while on the complement of D, viz. on the set $[1/4,3/4]\times[0,1]$, C_0 is defined as the bilinear interpolation (see [2, Lemma 2.3.5]) of the restriction of $\widetilde{C}$ to $\overline{D}$, which is a sub–copula. Thus, if s is in $(1/4,3/4)$, then

$$C_0(s,t)=\left(\frac{3}{2}-2s\right)\widetilde{C}\left(\frac{1}{4},t\right)+\left(2s-\frac{1}{2}\right)\widetilde{C}\left(\frac{3}{4},t\right).$$

Let d.f.'s in Δ_2 be defined through

$$H_n:=C_n\left(F_n,G_n\right),\qquad H_0:=C_0\left(F_0,G_0\right),$$

where, for every $n\in\mathbb{N}$ one has $C_n:=\widetilde{C}$. Then the sequence $\{H_n:n\in\mathbb{N}\}$ by construction converges weakly to H_0, and, by Theorem 3, $\{C_n\}$ converges to C_0 at the points of $\overline{D}$. However, it is easily checked that, if s belongs the open interval $]1/4,3/4[$, then $\widetilde{C}(s,t)\neq C_0(s,t)$, i.e.$\{C_n(s,t)\}$ does not converge to $C_0(s,t)$; in fact, for s in $]1/4,3/4[$, one has

$$C_0(s,t)=\begin{cases} t, & t\in\left[0,\dfrac{1}{4}\right], \\ 2st-\dfrac{1}{2}s-\dfrac{1}{2}t+\dfrac{3}{8}, & t\in\left[\dfrac{1}{4},\dfrac{3}{4}\right], \\ s, & t\in\left[\dfrac{3}{4},1\right]. \end{cases}$$

Therefore $\{C_n(s,t)\}$ does not converge to $C_0(s,t)$. Notice also that the limit copula in Theorem 2 may not be unique; in fact, since convergence occurs on a set that does not coincide with all of $[0,1]^2$, if at least one of the d.f.'s F_0 or G_0 is not continuous on $\overline{\mathbb{R}}$, one has, in the present example, that the sequence $\{C_n(s,t)\}$ converges on the set Ran $F_0\times$ Ran G_0 to both $\widetilde{C}(s,t)$ and $C_0(s,t)$, which are equal on that set but which are not equal on the entire unit square $[0,1]^2$.

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Sadržaj

U radu se proučavaju neki aspekti odnosa izmedju slabe konvergencije nizova multivariantnih funkcija distribucije i konvergencije po tačkama kopule koje ih povezuju sa njihovim marginalima.