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On Sequential Compactness and Semicompactness in Fuzzy Topology*

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In [1], Wong gives the definitions of sequential compactness and semicompactness in which the author himself notices the inadequacy of these definitions observing that every fuzzy topological space has the properties of sequential compactness and semicompactness. We remark that the inconvenience is due to the fact that Wong considers sequences of fuzzy sets in giving the definitions of limit and cluster. All these sequences, as the author showed, have at least a limit and a cluster. Only in a later work [3], does Wong deal again with the problem of convergence and consider the case of a sequence of fuzzy points. He does not, however, reexamine the problem of sequential compactness and semicompactness.

Moreover, it must be observed that in the general topology we would face the same inconvenience if we used sequences of sets instead of sequences of points. Therefore, we propose to give the definitions of sequential compactness and semicompactness using the concept of limit and cluster for a sequence of fuzzy points; concepts which we draw from Wong.

DEFINITION 1. Let $\{p_n\}$ be a sequence of fuzzy points and p a fuzzy point in an fts (fuzzy topological space) (X, τ). We say that $\{p_n\}$ converges to p, or p is a limit of the sequence $\{p_n\}$, and write $p_n \to p$ iff for every neighborhood (or equivalently open neighborhood) U of p there is a natural m such that $p_n \subset U$ for all $n \geqslant m$.

Notice that, given any fuzzy point p in X, every sequence $\{p_n\}$ of fuzzy point such that $p_n \subset p$ for $n \geqslant \bar{n}$, converges to p.

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DEFINITION 2. Let $\{p_n\}$ be a sequence of fuzzy points and p a fuzzy point in an fts (X, τ). Then p is said to be a limit value of the sequence $\{p_n\}$ iff there is a subsequence of $\{p_n\}$ that converges to p.

One has that every limit of a sequence is one of its limit values.

DEFINITION 3. If $\{p_n\}$ is a sequence of fuzzy points and p is a fuzzy point in an fts (X, τ), then p is said to be a cluster for the sequence $\{p_n\}$ iff for every neighborhood (or equivalently open neighborhood) U of p and for every natural m there is a natural $n \geqslant m$ such that $p_n \subset U$.

It is plain that every limit value of a sequence is a cluster for the same.

DEFINITION 4. Let p be a fuzzy point with support x and A a fuzzy set in an fts (X, τ). Then p is said to be an accumulation point of A iff for every nbhd (or, equivalently, open nbhd) U of p one has $U \cap A_p \neq \varnothing$, where A_p is the fuzzy set characterized by the following membership function:

$$\begin{aligned} \mu(z) &= 0, && \text{if} \quad z = x, \\ &= \mu_A(x), && \text{otherwise.} \end{aligned}$$

If p is an accumulation point of A, then p is an accumulation point for every other fuzzy set B whose support contains the support of A. One also has that a fuzzy point p is an accumulation point of $\{p_n(x_n, \lambda_n)\}$ iff p is an accumulation point of every fuzzy set having as support the set of the elements of the sequence $\{x_n\}$.

PROPOSITION 1. *Given a fuzzy point p and a sequence $\{p_n\}$ of fuzzy points such that the support of p_n is definitely different from the support of p, if p is a cluster for the sequence p_n, then p is an accumulation point of the same.*

From this proposition one obtains the following:

COROLLARY. *Let p be a fuzzy point and A a fuzzy set in an* fts (X, τ). *If a sequence of fuzzy points $\{p_n\}$ exists such that $p_n \subset A$ for all $n \in N$ (or at least definitely), the support of p_n differs from the support of p for all $n \in N$ (or at least definitely), and p is a cluster for the sequence $\{p_n\}$, then p is an accumulation point of A.*

One also has

PROPOSITION 2. *Let p be a fuzzy point, $\{p_n\}$ a sequence of fuzzy points, and A a fuzzy set in an* fts (X, τ). *If p is a limit (a limit value, a cluster, respectively) for the sequence $\{p_n\}$, then each fuzzy point q such that $q \supset p$ is* *a limit (a limit value, a cluster, respectively) for* $\{p_n\}$. *Analogously, if p is an accumulation point of A, then such is every other point q such that* $q \supset p$.

DEFINITION 5. An fts (X, τ) is said to be C_1 iff every fuzzy point p in X has a countable fundamental nbhd system (cfns).

LEMMA 1. *If* (X, τ) *is* C_1, *then for every fuzzy point p in X there exists a* cfns $\{V_i\}$ *such that* $V_1 \supset V_2 \supset V_3 \supset \cdots$.

Proof. By assumption, there exists a cfns $\{W_i\}$ of p. Define $V_1 = W_1$, $V_2 = W_1 \cap W_2,\ldots, V_n = \bigcap_{i=1}^{n} W_i,\ldots,$. Clearly, $V_1 \supset V_2 \supset V_3 \supset \cdots$. In order to prove that these nbhd's of p form an fns of p, let U be a nbhd of p. There exists $\bar{n} \in N$ such that $W_{\bar{n}} \subset U$. Since $p \subset W_i$ for every $i = 1, 2,\ldots, \bar{n}$, then $p \subset \bigcap_{i=1}^{\bar{n}} W_i = V_{\bar{n}} \subset U$.

We can now prove

THEOREM 1. *Let* (X, τ) *be a* C_1 fts, $\{p_n\}$ *a sequence of fuzzy points and p a fuzzy point in X. If p is a cluster for the sequence* $\{p_n\}$, *then p is one of its limit value.*

Proof. By Lemma 1 a cfns $\{V_n\}$ of p exists such that $V_1 \supset V_2 \supset \cdots$. Since p is a cluster for $\{p_n\}$, for every $n \in N$ there is $k(n) \in N$ such that $p_{k(n)} \subset V_n$. We define, in this way, a sequence of natural $\{k(n)\}$ which can be taken to be strictly increasing. To show that the subsequence $\{p_{k(n)}\}$ converge to p, let U be a nbhd of p. There exists $\bar{n} \in N$ such that $p \subset V_{\bar{n}} \subset U$; but $V_n \subset V_{\bar{n}}$ for each $n \geqslant \bar{n}$, and this implies $p_{k(n)} \subset V_n \subset V_{\bar{n}} \subset U$ for each $n \geqslant \bar{n}$.

THEOREM 2. *Let* (X, τ) *be a* C_1 fts, *A a fuzzy set and p a fuzzy point in X. If p is an accumulation point of A, a sequence of fuzzy points* $\{p_n\}$ *exists such that* $p_n \subset A$ *for every* $n \in N$ *(or at least definitely),* $p_n \to p$ *and the support of* p_n *differs definitely from the support of p.*

Proof. See [3].

We conclude with

DEFINITION 6. An fts (X, τ) is said to be semicompact iff every sequence of fuzzy points in X has a cluster. It is said to be instead sequentially compact iff every sequence of fuzzy points has a limit value.

It is easy to show Propositions 3 and 4.

PROPOSITION 3. *Every* fts *sequentially compact is semicompact.*

PROPOSITION 4. *If a* C_1 fts *is semicompact, then it is also sequentially compact.*

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