NUMERICAL ALGEBRA,
CONTROL AND OPTIMIZATION
Volume **3**, Number **1**, March **2013**

doi:10.3934/naco.2013.3.145

pp. **145–160**

NECESSARY OPTIMALITY CONDITIONS FOR INFINITE HORIZON VARIATIONAL PROBLEMS ON TIME SCALES

Monika Dryl and Delfim F. M. Torres

Center for Research and Development in Mathematics and Applications
Department of Mathematics, University of Aveiro, 3810-193 Aveiro, Portugal

Abstract. We prove Euler–Lagrange type equations and transversality conditions for generalized infinite horizon problems of the calculus of variations on time scales. Here the Lagrangian depends on the independent variable, an unknown function and its nabla derivative, as well as a nabla indefinite integral that depends on the unknown function.

1. **Introduction.** In recent years, it has been shown that the behavior of many systems is described more accurately using dynamic equations on a time scale or a measure chain [1, 2, 8]. If the variational principle holds as a unified law [5], then the above time scale differential equations must also come from minimization of some delta or nabla functional with a Lagrangian containing delta or nabla derivative terms [7, 10, 12]. Here we consider the following infinite horizon variational problem:

$$\mathcal{J}(x) = \int_a^{\infty} L\left(t, x^{\rho}(t), x^{\nabla}(t), z(t)\right) \nabla t \longrightarrow \text{extr}, \tag{1}$$

where "extr" means "minimize" or "maximize". The variable $z(t)$ is defined by

$$z(t) = \int_a^t g\left(\tau, x^{\rho}(\tau), x^{\nabla}(\tau)\right) \nabla \tau.$$

Integral (1) does not necessarily converge, being possible to diverge to plus or minus infinity or oscillate. Problem (1) generalizes the ones recently studied in [6, 11].

The paper is organized as follows. In Section 2 we collect the necessary definitions and results of the nabla calculus on time scales, which are necessary in the sequel. In Section 3 we state and prove the new results: we prove necessary optimality conditions to problem (1), obtaining Euler–Lagrange type equations in the class of functions $x \in C^1_{ld}(\mathbb{T}, \mathbb{R}^n)$ and new transversality conditions (Theorems 3.4 and 3.5).

2010 *Mathematics Subject Classification.* Primary: 49K05; Secondary: 34N05.
Key words and phrases. Time scales, calculus of variations, Euler–Lagrange equations, transversality conditions, infinite horizon.
Part of first author's Ph.D., which is carried out at the University of Aveiro under the *Doctoral Programme in Mathematics and Applications* of Universities of Aveiro and Minho.

2. **Preliminaries.** In this section we introduce basic definitions and theorems that are needed in Section 3. For more on the time scale theory we refer the reader to [1, 2, 13]. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of $\mathbb{R}$. All the intervals in this paper are time scale intervals with respect to a given time scale $\mathbb{T}$ (for example, by $[a, b]$ we mean $[a, b] \cap \mathbb{T}$).

Definition 2.1 (e.g., Section 2.1 of [13]). The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$ for $t \neq \sup \mathbb{T}$ and $\sigma(\sup \mathbb{T}) = \sup \mathbb{T}$ if $\sup \mathbb{T} < +\infty$. Similarly, the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$ for $t \neq \inf \mathbb{T}$ and $\rho(\inf \mathbb{T}) = \inf \mathbb{T}$ if $\inf \mathbb{T} > -\infty$.

Definition 2.2 (e.g., Section 2.1 of [13]). The backward graininess function $\nu : \mathbb{T} \to [0, \infty)$ is defined by $\nu(t) := t - \rho(t)$.

A point $t \in \mathbb{T}$ is called right-dense, right-scattered, left-dense or left-scattered if $\sigma(t) = t$, $\sigma(t) > t$, $\rho(t) = t$, $\rho(t) < t$, respectively. We say that t is isolated if $\rho(t) < t < \sigma(t)$, that t is dense if $\rho(t) = t = \sigma(t)$. If $\mathbb{T}$ has a right-scattered minimum m, then define $\mathbb{T}_\kappa := \mathbb{T} - \{m\}$; otherwise, set $\mathbb{T}_\kappa := \mathbb{T}$. To simplify the notation, let $f^\rho(t) := f(\rho(t))$.

Definition 2.3 (e.g., Section 2.2 of [13]). We say that function $f : \mathbb{T} \to \mathbb{R}$ is nabla differentiable at $t \in \mathbb{T}_\kappa$ if there is a number $f^\nabla(t)$ such that for all $\epsilon > 0$ there exists a neighborhood U of t such that

$$|f^\rho(t) - f(s) - f^\nabla(t)(\rho(t) - s)| \leq \epsilon|\rho(t) - s| \text{ for all } s \in U.$$

We call $f^\nabla(t)$ the nabla derivative of f at t. Moreover, f is nabla differentiable on $\mathbb{T}$ provided $f^\nabla(t)$ exists for all $t \in \mathbb{T}_\kappa$.

Theorem 2.4 (e.g., Theorem 8.41 of [1]). *Assume $f, g : \mathbb{T} \to \mathbb{R}$ are nabla differentiable at $t \in \mathbb{T}_\kappa$. Then:*

1. *The sum $f + g : \mathbb{T} \to \mathbb{R}$ is nabla differentiable at t with*
$$(f + g)^\nabla(t) = f^\nabla(t) + g^\nabla(t).$$
2. *For any constant α, $\alpha f : \mathbb{T} \to \mathbb{R}$ is nabla differentiable at t and*
$$(\alpha f)^\nabla(t) = \alpha f^\nabla(t).$$
3. *The product $fg : \mathbb{T} \to \mathbb{R}$ is nabla differentiable at t and the following product rules hold:*
$$(fg)^\nabla(t) = f^\nabla(t)g(t) + f^\rho g^\nabla(t) = f^\nabla(t)g^\rho(t) + f(t)g^\nabla(t).$$
4. *If $g(t)g^\rho(t) \neq 0$, then f/g is nabla differentiable at t and the following quotient rule hold:*
$$\left(\frac{f}{g}\right)^\nabla(t) = \frac{f^\nabla(t)g(t) - f(t)g^\nabla(t)}{g(t)g^\rho(t)}.$$

Definition 2.5 (e.g., Section 3.1 of [2]). Let $\mathbb{T}$ be a time scale, $f : \mathbb{T} \to \mathbb{R}$. We say that function f is ld-continuous if it is continuous at left-dense points and its right-sided limits exist (finite) at all right-dense points.

Definition 2.6 (e.g., Definition 9 of [13]). A function $F : \mathbb{T} \to \mathbb{R}$ is called a nabla antiderivative of $f : \mathbb{T} \to \mathbb{R}$ provided $F^\nabla(t) = f(t)$ for all $t \in \mathbb{T}_\kappa$. In this case we define the nabla integral of f from a to b ($a, b \in \mathbb{T}$) by

$$\int_a^b f(t)\nabla t := F(b) - F(a).$$

The set of all ld-continuous functions $f:\mathbb{T}\to\mathbb{R}$ is denoted by $C_{ld}=C_{ld}(\mathbb{T},\mathbb{R})$, and the set of all nabla differentiable functions with ld-continuous derivative by $C^1_{ld}=C^1_{ld}(\mathbb{T},\mathbb{R})$.

Theorem 2.7 (e.g., Theorem 8.45 of [1] or Theorem 11 of [13])**.** *Every ld-continuous function f has a nabla antiderivative F. In particular, if $a\in\mathbb{T}$, then F defined by*

$$F(t)=\int_a^t f(\tau)\nabla\tau, \quad t\in\mathbb{T},$$

is a nabla antiderivative of f.

Theorem 2.8 (e.g., Theorem 8.47 of [1] or Theorem 12 of [13])**.** *If $a,b\in\mathbb{T}$, $a\le b$, and $f,g\in C_{ld}(\mathbb{T},\mathbb{R})$, then*

1. $\int_a^b (f(t)+g(t))\nabla t=\int_a^b f(t)\nabla t+\int_a^b g(t)\nabla t$;
2. $\int_a^a f(t)\nabla t=0$;
3. $\int_a^b f(t)g^{\nabla}(t)\nabla t= f(t)g(t)|_{t=a}^{t=b}-\int_a^b f^{\nabla}(t)g^{\rho}(t)\nabla t$;
4. *If $f(t)>0$ for all $a<t\le b$, then* $\int_a^b f(t)\nabla t>0$;
5. *If $t\in\mathbb{T}_\kappa$, then* $\int_{\rho(t)}^t f(\tau)\nabla\tau=\nu(t)f(t)$.

Definition 2.9. If $a\in\mathbb{T}$, $\sup\mathbb{T}=+\infty$ and $f\in C_{ld}([a,+\infty[,\mathbb{R})$, then we define the improper nabla integral by

$$\int_a^{+\infty} f(t)\nabla t:=\lim_{b\to+\infty}\int_a^b f(t)\nabla t,$$

provided this limit exists (in $\overline{\mathbb{R}}:=\mathbb{R}\cup\{-\infty,+\infty\}$).

Theorem 2.10 (e.g., [4])**.** *Let S and T be subsets of a normed vector space. Let f be a map defined on $T\times S$, having values in some complete normed vector space. Let v be adherent to S and w adherent to T. Assume that*

1. $\lim_{x\to v} f(t,x)$ *exists for each $t\in T$;*
2. $\lim_{t\to w} f(t,x)$ *exists uniformly for $x\in S$.*

Then, $\lim_{t\to w}\lim_{x\to v} f(t,x)$, $\lim_{x\to v}\lim_{t\to w} f(t,x)$ *and* $\lim_{(t,x)\to(w,v)} f(t,x)$ *all exist and are equal.*

The next result can be easily obtained from Theorem 4 of [11] by using the delta-nabla duality theory of time scales [3, 9, 10].

Theorem 2.11. *Suppose that $x_\star$ is a local minimizer or local maximizer to problem*

$$\mathcal{L}(x)=\int_a^b L\left(t,x^{\rho}(t),x^{\nabla}(t),z(t)\right)\nabla t\longrightarrow extr,$$

where the variable z is the integral defined by

$$z(t) = \int_a^t g\left(\tau, x^\rho(\tau), x^\nabla(\tau)\right) \nabla\tau,$$

in the class of functions $x \in C^1_{ld}(\mathbb{T}, \mathbb{R}^n)$ satisfying the boundary conditions $x(a) = \alpha$ and $x(b) = \beta$. Then, $x_\star$ satisfies the Euler–Lagrange system of equations

$$g_x\langle x\rangle(t) \int_{\rho(t)}^b L_z[x,z](\tau)\nabla\tau - \left(g_v\langle x\rangle(t) \int_{\rho(t)}^b L_z[x,z](\tau)\nabla\tau\right)^\nabla + L_x[x,z](t) - L_v^\nabla[x,z](t) = 0$$

for all $t \in [a,b]_\kappa$, where L_x, L_v and L_z are, respectively, the partial derivatives of $L(\cdot,\cdot,\cdot,\cdot)$ with respect to its second, third and fourth argument, g_x and g_v are, respectively, the partial derivatives of $g(\cdot,\cdot,\cdot)$ with respect to its second and third argument, and the operators $[\cdot,\cdot]$ and $\langle\cdot\rangle$ are defined by $[x,z](t) := \left(t, x^\rho(t), x^\nabla(t), z(t)\right)$ and $\langle x\rangle(t) := \left(t, x^\rho(t), x^\nabla(t)\right)$.

3. **Main Results.** Let $\mathbb{T}$ be a time scale such that $\sup \mathbb{T} = +\infty$. Suppose that a, T, $T' \in \mathbb{T}$ are such that $T > a$ and $T' > a$. The meaning of $L[x,z](t)$, $g\langle x\rangle(t)$ and that of partial derivatives $L_x[x,z](t)$, $L_v[x,z](t)$, $L_z[x,z](t)$, $g_x\langle x\rangle(t)$ and $g_v\langle x\rangle(t)$ is given in Theorem 2.11. Let us consider the following variational problem on $\mathbb{T}$:

$$\mathcal{J}(x) := \int_a^\infty L[x,z](t)\nabla t = \int_a^\infty L\left(t, x^\rho(t), x^\nabla(t), z(t)\right)\nabla t \longrightarrow \max \tag{2}$$

subject to $x(a) = x_a$. The variable z is the integral defined by

$$z(t) := \int_a^t g\langle x\rangle(\tau)\nabla\tau = \int_a^t g\left(\tau, x^\rho(\tau), x^\nabla(\tau)\right)\nabla\tau.$$

We assume that $x_a \in \mathbb{R}^n$, $n \in \mathbb{N}$, $(u,v,w) \to L(t,u,v,w)$ is a $C^1_{ld}(\mathbb{R}^{2n+1}, \mathbb{R})$ and $(u,v) \to g(t,u,v)$ a $C^1_{ld}(\mathbb{R}^{2n}, \mathbb{R})$ function for any $t \in \mathbb{T}$, and functions $t \to L_v[x,z](t)$ and $t \to g_v\langle x\rangle(t)$ are nabla differentiable for all $x \in C^1_{ld}(\mathbb{T}, \mathbb{R}^n)$.

Definition 3.1. We say that x is an admissible path for problem (2) if $x \in C^1_{ld}(\mathbb{T}, \mathbb{R}^n)$ and $x(a) = x_a$.

Definition 3.2. We say that $x_\star$ is a weak maximizer to problem (2) if $x_\star$ is an admissible path and, moreover,

$$\lim_{T\to+\infty} \inf_{T'\geq T} \int_a^{T'} \left(L[x,z](t) - L[x_\star, z_\star](t)\right)\nabla t \leq 0$$

for all admissible path x.

Lemma 3.3. *Let $g \in C_{ld}(\mathbb{T}, \mathbb{R})$. Then,*

$$\lim_{T\to\infty} \inf_{T'\geq T} \int_a^{T'} g(t)\eta^\rho(t)\nabla t = 0$$

for all $\eta \in C_{ld}(\mathbb{T}, \mathbb{R})$ such that $\eta(a) = 0$ if, and only if, $g(t) = 0$ on $[a, +\infty[$.

Proof. The implication $\Leftarrow$ is obvious. Let us prove the implication $\Rightarrow$ by contradiction. Suppose that $g(t) \not\equiv 0$. Let t_0 be a point on $[a, +\infty[$ such that $g(t_0) \neq 0$. Suppose, without loss of generality, that $g(t_0) > 0$. Two situations may occur: t_0 is left-dense (case I) or t_0 is left-scattered (case II). Case I: if t_0 is left-dense, then function g is positive on $[t_1, t_0]$ for $t_1 < t_0$. Define:

$$\eta(t) = \begin{cases} (t_0 - t)(t - t_1) \text{ for } t \in [t_1, t_0], \\ 0 \text{ otherwise.} \end{cases}$$

Then,

$$\eta(\rho(t)) = \begin{cases} (t_0 - \rho(t))\,(\rho(t) - t_1) \text{ for } \rho(t) \in [t_1, t_0], \\ 0 \text{ otherwise.} \end{cases}$$

If $\rho(t) \in [t_1, t_0]$, then $\eta(\rho(t)) = (t_0 - \rho(t))(\rho(t) - t_1) > 0$. Thus,

$$\lim_{T \to +\infty} \inf_{T' \geq T} \int_a^{T'} g(t)\eta^\rho(t)\nabla t = \int_{t_1}^{t_0} g(t)\eta(\rho(t))\nabla t > 0$$

and we obtain a contradiction. Case II: t_0 is left-scattered. Two situations are then possible: $\rho(t_0)$ is left-scattered or $\rho(t_0)$ is left-dense. If $\rho(t_0)$ is left-scattered, then $\rho(\rho(t_0)) < \rho(t_0) < t_0$. Let $t \in [\rho(t_0), t_0]$. Define

$$\eta(t) = \begin{cases} g(t_0) \text{ for } t = \rho(t_0), \\ 0 \text{ otherwise.} \end{cases}$$

Then,

$$\eta(\rho(t_0)) = \begin{cases} g(t_0) \text{ for } \rho(t_0) = \rho(\rho(t_0)), \\ 0 \text{ otherwise.} \end{cases}$$

It means that $\eta(\rho(t_0)) = g(t_0) > 0$. From point 5 of Theorem 2.8, we obtain

$$\begin{aligned} \lim_{T \to +\infty} \inf_{T' \geq T} \int_a^{T'} g(t)\eta^\rho(t)\nabla t &= \int_{\rho(t_0)}^{t_0} g(t)\eta^\rho(t)\nabla t \\ &= g(t_0)\eta(\rho(t_0))\nu(t_0) = g(t_0)g(t_0)(t_0 - \rho(t_0)) > 0, \end{aligned}$$

which is a contradiction. It remains to consider the situation when $\rho(t_0)$ is left-dense. Two cases are then possible: $g(\rho(t_0)) \neq 0$ or $g(\rho(t_0)) = 0$. If $g(\rho(t_0)) \neq 0$, then we can assume that $g(\rho(t_0)) > 0$ and g is also positive in $[t_2, \rho(t_0)]$ for $t_2 < \rho(t_0)$. Define

$$\eta(t) = \begin{cases} (\rho(t_0) - t)\,(t - t_2) \text{ for } t \in [t_2, \rho(t_0)], \\ 0 \text{ otherwise.} \end{cases}$$

Then,

$$\eta(\rho(t)) = \begin{cases} (\rho(t_0) - \rho(t))\,(\rho(t) - t_2) \text{ for } \rho(t) \in [t_2, \rho(t_0)], \\ 0 \text{ otherwise.} \end{cases}$$

On the interval $[t_2, \rho(t_0)]$ the function $\eta(\rho(t))$ is greater than 0. Then,

$$\lim_{T \to +\infty} \inf_{T' \geq T} \int_a^{T'} g(t)\eta^\rho(t)\nabla t = \int_{t_2}^{\rho(t_0)} g(t)\eta^\rho(t)\nabla t > 0,$$

which is a contradiction. Suppose that $g(\rho(t_0)) = 0$. Here two situations may occur: (i) $g(t) = 0$ on $[t_3, \rho(t_0)]$ for some $t_3 < \rho(t_0)$ or (ii) for all $t_3 < \rho(t_0)$ there exists

$t \in [t_3, \rho(t_0)]$ such that $g(t) \neq 0$. In case (i) $t_3 < \rho(t_0) < t_0$. Let us define

$$\eta(t) = \begin{cases} g(t_0) & \text{for } t = \rho(t_0), \\ \varphi(t) & \text{for } t \in [t_3, \rho(t_0)[, \\ 0 & \text{otherwise}, \end{cases}$$

for function φ such that $\varphi \in C_{ld}$, $\varphi(t_3) = 0$ and $\varphi(\rho(t_0)) = g(t_0)$. Then,

$$\eta(\rho(t)) = \begin{cases} g(t_0) & \text{for } \rho(t) = \rho(t_0), \\ \varphi(\rho(t)) & \text{for } \rho(t) \in [t_3, \rho(t_0)[, \\ 0 & \text{otherwise}. \end{cases}$$

It follows from point 5 of Theorem 2.8 that

$$\begin{aligned}\lim_{T\to+\infty} \inf_{T'\geq T} \int_a^{T'} g(t)\eta^\rho(t)\nabla t = \int_{t_3}^{t_0} g(t)\eta^\rho(t)\nabla t = \int_{\rho(t_0)}^{t_0} g(t)\eta^\rho(t)\nabla t \\ = \nu(t_0)g(t_0)\eta(\rho(t_0)) = (t_0 - \rho(t_0))g(t_0)\eta(\rho(t_0)) > 0,\end{aligned}$$

which is a contradiction. In case (ii), $t_3 < \rho(t_0) < t_0$. When $\rho(t_0)$ is left-dense, then there exists a strictly increasing sequence $S = \{s_k : k \in \mathbb{N}\} \subseteq \mathbb{T}$ such that $\lim_{k\to\infty} s_k = \rho(t_0)$ and $g(s_k) \neq 0$ for all $k \in \mathbb{N}$. If there exists a left-dense s_k, then we have Case I with $t_0 := s_k$. If all points of the sequence S are left-scattered, then we have Case II with $t_0 := s_i$, $i \in \mathbb{N}$. Since $\rho(t_0)$ is a left-scattered point, we are in the first situation of case II and we obtain a contradiction. Therefore, we conclude that $g \equiv 0$ on $[a, +\infty[$. □

Corollary 1. *Let $h \in C_{ld}(\mathbb{T}, \mathbb{R})$. Then,*

$$\lim_{T\to\infty} \inf_{T'\geq T} \int_a^{T'} h(t)\eta^\nabla(t)\nabla t = 0 \tag{3}$$

for all $\eta \in C_{ld}(\mathbb{T}, \mathbb{R})$ such that $\eta(a) = 0$ if, and only if, $h(t) = c$, $c \in \mathbb{R}$, on $[a, +\infty[$.

Proof. Using integration by parts (third item of Theorem 2.8),

$$\int_a^{T'} h(t)\eta^\nabla(t)\nabla t = h(t)\eta(t)|_{t=a}^{t=T'} - \int_a^{T'} h^\nabla(t)\eta^\rho(t)\nabla t = h(T')\eta(T') - \int_a^{T'} h^\nabla(t)\eta^\rho(t)\nabla t$$

holds for all $\eta \in C_{ld}(\mathbb{T}, \mathbb{R})$. In particular, it holds for the subclass of η with $\eta(T') = 0$ and (3) is equivalent to

$$\lim_{T\to\infty} \inf_{T'\geq T} \int_a^{T'} h^\nabla(t)\eta^\rho(t)\nabla t = 0.$$

Using Lemma 3.3, we obtain $h^\nabla(t) = 0$, i.e., $h(t) = c$, $c \in \mathbb{R}$, on $[a, +\infty[$. □

Theorem 3.4. *Suppose that a weak maximizer to problem (2) exists and is given by $x_\star$. Let $p \in C^1_{ld}(\mathbb{T}, \mathbb{R}^n)$ be such that $p(a) = 0$. Define*

$$A(\varepsilon, T') := \int_a^{T'} \frac{L\left(t, x_\star^\rho(t) + \varepsilon p^\rho(t), x_\star^\nabla(t) + \varepsilon p^\nabla(t), z_\star(t,p)\right) - L\left[x_\star, z_\star\right](t)}{\varepsilon} \nabla t,$$

where

$$z_\star(t,p) = \int_a^t g\langle x_\star + \varepsilon p\rangle(\tau)\nabla\tau,$$

$$z_\star(t) = \int_a^t g\langle x_\star\rangle(\tau)\nabla\tau,$$

and

$$V(\varepsilon,T) := \inf_{T'\geq T} \varepsilon A(\varepsilon,T'),$$
$$V(\varepsilon) := \lim_{T\to\infty} V(\varepsilon,T).$$

Suppose that

1. $\lim\limits_{\varepsilon\to 0}\frac{V(\varepsilon,T)}{\varepsilon}$ *exists for all* T*;*
2. $\lim\limits_{T\to\infty}\frac{V(\varepsilon,T)}{\varepsilon}$ *exists uniformly for* ε*;*
3. *for every* $T' > a$*,* $T > a$*,* $\varepsilon \in \mathbb{R}\setminus\{0\}$*, there exists a sequence* $(A(\varepsilon,T'_n))_{n\in\mathbb{N}}$ *such that* $\lim\limits_{n\to\infty} A(\varepsilon,T'_n) = \inf\limits_{T'\geq T} A(\varepsilon,T')$ *uniformly for* ε*.*

Then, $x_\star$ *satisfies the Euler–Lagrange system of* n *equations*

$$\lim_{T\to\infty}\inf_{T'\geq T}\left\{ g_x\langle x\rangle(t)\int_{\rho(t)}^{T'} L_z[x,z](\tau)\nabla\tau - \left(g_v\langle x\rangle(t)\int_{\rho(t)}^{T'} L_z[x,z](\tau)\nabla\tau\right)^{\nabla}\right\} + L_x[x,z](t) - L_v^{\nabla}[x,z](t) = 0 \quad (4)$$

for all $t\in[a,+\infty[$ *and the transversality condition*

$$\lim_{T\to\infty}\inf_{T'\geq T}\left\{x(T')\cdot\left[L_v[x,z](T') + g_v\langle x\rangle(T')\nu(T')L_z[x,z](T')\right]\right\} = 0. \quad (5)$$

Proof. If $x_\star$ is optimal, in the sense of Definition 3.2, then $V(\varepsilon)\leq 0$ for any $\varepsilon\in\mathbb{R}$. Because $V(0)=0$, then 0 is a maximizer of V. We prove that V is differentiable at 0, thus $V'(0)=0$. Note that

$$\begin{aligned}
0 = V'(0) &= \lim_{\varepsilon\to 0}\frac{V(\varepsilon)}{\varepsilon} = \lim_{\varepsilon\to 0}\lim_{T\to\infty}\frac{V(\varepsilon,T)}{\varepsilon} = \lim_{T\to\infty}\lim_{\varepsilon\to 0}\frac{V(\varepsilon,T)}{\varepsilon}\\
&= \lim_{T\to\infty}\lim_{\varepsilon\to 0}\inf_{T'\geq T} A(\varepsilon,T') = \lim_{T\to\infty}\lim_{\varepsilon\to 0}\lim_{n\to\infty} A(\varepsilon,T_n^{'})\\
&= \lim_{T\to\infty}\lim_{n\to\infty}\lim_{\varepsilon\to 0} A(\varepsilon,T_n^{'}) = \lim_{T\to\infty}\inf_{T'\geq T}\lim_{\varepsilon\to 0} A(\varepsilon,T')\\
&= \lim_{T\to\infty}\inf_{T'\geq T}\lim_{\varepsilon\to 0}\int_a^{T'}\frac{L\left(t,x_\star^\rho(t)+\varepsilon p^\rho(t),x_\star^\nabla(t)+\varepsilon p^\nabla(t),z_\star(t,p)\right) - L\left[x_\star,z_\star\right](t)}{\varepsilon}\nabla t\\
&= \lim_{T\to\infty}\inf_{T'\geq T}\int_a^{T'}\lim_{\varepsilon\to 0}\frac{L\left(t,x_\star^\rho(t)+\varepsilon p^\rho(t),x_\star^\nabla(t)+\varepsilon p^\nabla(t),z_\star(t,p)\right) - L\left[x_\star,z_\star\right](t)}{\varepsilon}\nabla t,
\end{aligned}$$

that is,

$$\lim_{T\to\infty}\inf_{T'\geq T}\int\limits_a^{T'}\Bigg[L_x[x_\star,z_\star](t)\cdot p^\rho(t)+L_v[x_\star,z_\star](t)\cdot p^\nabla(t)$$
$$+L_z[x_\star,z_\star](t)\int\limits_a^t\big(g_x\langle x_\star\rangle(\tau)\cdot p^\rho(\tau)+g_v\langle x_\star\rangle(\tau)\cdot p^\nabla(\tau)\big)\nabla\tau\Bigg]\nabla t=0. \quad (6)$$

Using the integration by parts formula given by point 3 of Theorem 2.8, we obtain:

$$\int\limits_a^{T'}L_v[x_\star,z_\star](t)\cdot p^\nabla(t)\nabla t= L_v[x_\star,z_\star](t)\cdot p(t)|_{t=a}^{t=T'}-\int\limits_a^{T'}L_v^\nabla[x_\star,z_\star](t)\cdot p^\rho(t)\nabla t$$
$$=L_v[x_\star,z_\star](T')\cdot p(T')-\int\limits_a^{T'}L_v^\nabla[x_\star,z_\star](t)\cdot p^\rho(t)\nabla t.$$

Next, we consider the last part of equation (6). First we use the third nabla differentiation formula of Theorem 2.4:

$$\left[\int\limits_t^{T'}L_z[x_\star,z_\star](\tau)\nabla\tau\int\limits_a^t\big(g_x\langle x_\star\rangle(\tau)\cdot p^\rho(\tau)+g_v\langle x_\star\rangle(\tau)\cdot p^\nabla(\tau)\big)\nabla\tau\right]^\nabla$$
$$=\left(\int\limits_t^{T'}L_z[x_\star,z_\star](\tau)\nabla\tau\right)^\nabla\int\limits_a^t\big(g_x\langle x_\star\rangle(\tau)\cdot p^\rho(\tau)+g_v\langle x_\star\rangle(\tau)\cdot p^\nabla(\tau)\big)\nabla\tau$$
$$+\left(\int\limits_{\rho(t)}^{T'}L_z[x_\star,z_\star](\tau)\nabla\tau\right)\left(\int\limits_a^t\big(g_x\langle x_\star\rangle(\tau)\cdot p^\rho(\tau)+g_v\langle x_\star\rangle(\tau)\cdot p^\nabla(\tau)\big)\nabla\tau\right)^\nabla$$
$$=-L_z[x_\star,z_\star](t)\int\limits_a^t\big(g_x\langle x_\star\rangle(\tau)\cdot p^\rho(\tau)+g_v\langle x_\star\rangle(\tau)\cdot p^\nabla(\tau)\big)\nabla\tau$$
$$+\left(\int\limits_{\rho(t)}^{T'}L_z[x_\star,z_\star](\tau)\nabla\tau\right)\big(g_x\langle x_\star\rangle(t)\cdot p^\rho(t)+g_v\langle x_\star\rangle(t)\cdot p^\nabla(t)\big).$$

Integrating both sides from $t=a$ to $t=T'$,

$$\int\limits_a^{T'}\left[\int\limits_t^{T'}L_z[x_\star,z_\star](\tau)\nabla\tau\int\limits_a^t\big(g_x\langle x_\star\rangle(\tau)\cdot p^\rho(\tau)+g_v\langle x_\star\rangle(\tau)\cdot p^\nabla(\tau)\big)\nabla\tau\right]^\nabla\nabla t$$
$$=-\int\limits_a^{T'}\left[L_z[x_\star,z_\star](t)\int\limits_a^t\big(g_x\langle x_\star\rangle(\tau)\cdot p^\rho(\tau)+g_v\langle x_\star\rangle(\tau)\cdot p^\nabla(\tau)\big)\nabla\tau\right]\nabla t$$
$$+\int\limits_a^{T'}\left[\int\limits_{\rho(t)}^{T'}L_z[x_\star,z_\star](\tau)\nabla\tau\big(g_x\langle x_\star\rangle(t)\cdot p^\rho(t)+g_v\langle x_\star\rangle(t)\cdot p^\nabla(t)\big)\right]\nabla t.$$

The left-hand side of above equation is zero,

$$\int_a^{T'} \left[\int_t^{T'} L_z[x_\star, z_\star](\tau)\nabla\tau \int_a^t \left(g_x\langle x_\star\rangle(\tau)\cdot p^\rho(\tau) + g_v\langle x_\star\rangle(\tau)\cdot p^\nabla(\tau)\right)\nabla\tau \right]^\nabla \nabla t$$
$$= \int_t^{T'} L_z[x_\star, z_\star](\tau)\nabla\tau \int_a^t \left(g_x\langle x_\star\rangle(\tau)\cdot p^\rho(\tau) + g_v\langle x_\star\rangle(\tau)\cdot p^\nabla(\tau)\right)\nabla\tau \Bigg|_{t=a}^{t=T'} = 0,$$

and, therefore,

$$\begin{aligned}
&\int_a^{T'} \left[L_z[x_\star, z_\star](t) \int_a^t \left(g_x\langle x_\star\rangle(\tau)\cdot p^\rho(\tau) + g_v\langle x_\star\rangle(\tau)\cdot p^\nabla(\tau)\right)\nabla\tau \right]\nabla t \\
&\quad = \int_a^{T'} \left[\int_{\rho(t)}^{T'} L_z[x_\star, z_\star](\tau)\nabla\tau \left(g_x\langle x_\star\rangle(t)\cdot p^\rho(t) + g_v\langle x_\star\rangle(t)\cdot p^\nabla(t)\right) \right]\nabla t \\
&\quad = \int_a^{T'} \left[g_x\langle x_\star\rangle(t)\cdot p^\rho(t) \int_{\rho(t)}^{T'} L_z[x_\star, z_\star](\tau)\nabla\tau \right]\nabla t \\
&\qquad + \int_a^{T'} \left[p^\nabla(t)\cdot g_v\langle x_\star\rangle(t) \int_{\rho(t)}^{T'} L_z[x_\star, z_\star](\tau)\nabla\tau \right]\nabla t.
\end{aligned} \tag{7}$$

Using point 3 of Theorem 2.8 and the fact that $p(a) = 0$,

$$\begin{aligned}
&\int_a^{T'} \left[p^\nabla(t)\cdot g_v\langle x_\star\rangle(t) \int_{\rho(t)}^{T'} L_z[x_\star, z_\star](\tau)\nabla\tau \right]\nabla t \\
&\qquad = p(T')\cdot g_v\langle x_\star\rangle(T') \int_{\rho(T')}^{T'} L_z[x_\star, z_\star](\tau)\nabla\tau \\
&\qquad\quad - \int_a^{T'} \left(g_v\langle x_\star\rangle(t) \int_{\rho(t)}^{T'} L_z[x_\star, z_\star](\tau)\nabla\tau \right)^\nabla \cdot p^\rho(t)\nabla t.
\end{aligned}$$

Then, from (6),

$$\begin{aligned}
&\lim_{T\to\infty} \inf_{T'\geq T} \int_a^{T'} L_x[x_\star, z_\star](t)\cdot p^\rho(t)\nabla t + L_v[x_\star, z_\star](T')\cdot p(T') - \int_a^{T'} L_v^\nabla[x_\star, z_\star](t)\cdot p^\rho(t)\nabla t \\
&\quad + \int_a^{T'} \left(\int_{\rho(t)}^{T'} L_z[x_\star, z_\star](\tau)\nabla\tau \left(g_x\langle x_\star\rangle(t)\cdot p^\rho(t) + g_v\langle x_\star\rangle(t)\cdot p^\nabla(t)\right) \right)\nabla t
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{T\to\infty} \inf_{T'\geq T} \int\limits_a^{T'} L_x[x_\star, z_\star](t)\cdot p^\rho(t)\nabla t + L_v[x_\star, z_\star](T')\cdot p(T')\\
&\quad - \int\limits_a^{T'} \left\{ L_v^\nabla[x_\star, z_\star](t)\cdot p^\rho(t) + \int\limits_{\rho(t)}^{T'} L_z[x_\star, z_\star](\tau)\nabla\tau g_x\langle x_\star\rangle(t)\cdot p^\rho(t)\right\}\nabla t\\
&\quad + g_v\langle x_\star\rangle(T') \int\limits_{\rho(T')}^{T'} L_z[x_\star, z_\star](\tau)\nabla\tau\cdot p(T')\\
&\quad - \int\limits_a^{T'} \left(g_v\langle x_\star\rangle(t) \int\limits_{\rho(t)}^{T'} L_z[x_\star, z_\star](\tau)\nabla\tau\right)^\nabla \cdot p^\rho(t)\nabla t\\
&= \lim_{T\to\infty} \inf_{T'\geq T} \left\{ \int\limits_a^{T'} p^\rho(t)\cdot \left[L_x[x_\star, z_\star](t) - L_v^\nabla[x_\star, z_\star](t) \right.\right.\\
&\quad \left. + g_x\langle x_\star\rangle(t) \int\limits_{\rho(t)}^{T'} L_z[x_\star, z_\star](\tau)\nabla\tau - \left(g_v\langle x_\star\rangle(t) \int\limits_{\rho(t)}^{T'} L_z[x_\star, z_\star](\tau)\nabla\tau\right)^\nabla \right]\nabla t\\
&\quad \left. + L_v[x_\star, z_\star](T')\cdot p(T') + \left(g_v\langle x_\star\rangle(T') \int\limits_{\rho(T')}^{T'} L_z[x_\star, z_\star](\tau)\nabla\tau\right)\cdot p(T')\right\} = 0.
\end{aligned}
\tag{8}
$$

We know that equation (8) holds for all $p \in C_{ld}^1$ such that $p(a) = 0$, then, in particular, it also holds for the subclass of p with $p(T') = 0$. Therefore,

$$
\begin{aligned}
\lim_{T\to\infty} \inf_{T'\geq T} \int\limits_a^{T'} p^\rho(t)\cdot &\left[L_x[x_\star, z_\star](t) - L_v^\nabla[x_\star, z_\star](t) + g_x\langle x_\star\rangle(t) \int\limits_{\rho(t)}^{T'} L_z[x_\star, z_\star](\tau)\nabla\tau \right.\\
&\left. - \left(g_v\langle x_\star\rangle(t) \int\limits_{\rho(t)}^{T'} L_z[x_\star, z_\star](\tau)\nabla\tau\right)^\nabla \right]\nabla t = 0.
\end{aligned}
$$

Choosing $p = (p_1, \ldots, p_n)$ such that $p_2 \equiv \cdots \equiv p_n \equiv 0$,

$$
\begin{aligned}
\lim_{T\to\infty} \inf_{T'\geq T} \int\limits_a^{T'} p_1^\rho(t) &\left[L_{x_1}[x_\star, z_\star](t) + g_{x_1}\langle x_\star\rangle(t) \int\limits_{\rho(t)}^{T'} L_z[x_\star, z_\star](\tau)\nabla\tau \right.\\
&\left. - L_{v_1}^\nabla[x_\star, z_\star](t) - \left(g_{v_1}\langle x_\star\rangle(t) \int\limits_{\rho(t)}^{T'} L_z[x_\star, z_\star](\tau)\nabla\tau\right)^\nabla \right]\nabla t = 0.
\end{aligned}
$$

Using Lemma 3.3,

$$g_{x_1}\langle x_\star\rangle(t)\int_{\rho(t)}^{T'} L_z[x_\star,z_\star](\tau)\nabla\tau-\left(g_{v_1}\langle x_\star\rangle(t)\int_{\rho(t)}^{T'} L_z[x_\star,z_\star](\tau)\nabla\tau\right)^{\nabla} + L_{x_1}[x_\star,z_\star](t)-L_{v_1}^{\nabla}[x_\star,z_\star](t)=0$$

for all $t\in[a,+\infty[$ and all $T'\geq t$. We can do the same for other coordinates. For all $i=1,\ldots,n$ we obtain the equation

$$g_{x_i}\langle x_\star\rangle(t)\int_{\rho(t)}^{T'} L_z[x_\star,z_\star](\tau)\nabla\tau-\left(g_{v_i}\langle x_\star\rangle(t)\int_{\rho(t)}^{T'} L_z[x_\star,z_\star](\tau)\nabla\tau\right)^{\nabla} + L_{x_i}[x_\star,z_\star](t)-L_{v_i}^{\nabla}[x_\star,z_\star](t)=0$$

for all $t\in[a,+\infty[$ and all $T'\geq t$. These n conditions can be written in vector form as

$$g_{x}\langle x_\star\rangle(t)\int_{\rho(t)}^{T'} L_z[x_\star,z_\star](\tau)\nabla\tau-\left(g_{v}\langle x_\star\rangle(t)\int_{\rho(t)}^{T'} L_z[x_\star,z_\star](\tau)\nabla\tau\right)^{\nabla} + L_{x}[x_\star,z_\star](t)-L_{v}^{\nabla}[x_\star,z_\star](t)=0 \quad (9)$$

for all $t\in[a,+\infty[$ and all $T'\geq t$, which implies the Euler–Lagrange system of n equations (4). From the system of equations (9) and equation (8), we conclude that

$$\lim_{T\to\infty}\inf_{T'\geq T}\left\{\left(L_v[x_\star,z_\star](T')+g_v\langle x_\star\rangle(T')\int_{\rho(T')}^{T'} L_z[x_\star,z_\star](\tau)\nabla\tau\right)\cdot p(T')\right\}=0. \quad (10)$$

Next, we define a special curve p: for all $t\in[a,\infty[$

$$p(t)=\alpha(t)x_\star(t)\,, \quad (11)$$

where $\alpha:[a,\infty[\to\mathbb{R}$ is a C_{ld}^1 function satisfying $\alpha(a)=0$ and for which there exists $T_0\in\mathbb{T}$ such that $\alpha(t)=\beta\in\mathbb{R}\backslash\{0\}$ for all $t>T_0$. Substituting $p(T')=\alpha(T')x_\star(T')$ into (10), we conclude that

$$\lim_{T\to\infty}\inf_{T'\geq T}\left\{L_v[x_\star,z_\star](T')\cdot\beta x_\star(T')+g_v\langle x_\star\rangle(T')\int_{\rho(T')}^{T'} L_z[x_\star,z_\star](\tau)\nabla\tau\cdot\beta x_\star(T')\right\}$$

vanishes and, therefore,

$$\lim_{T\to\infty}\inf_{T'\geq T}\left\{x_\star(T')\cdot\left[L_v[x_\star,z_\star](T')+g_v\langle x_\star\rangle(T')\int_{\rho(T')}^{T'} L_z[x_\star,z_\star](\tau)\nabla\tau\right]\right\}=0.$$

From item 5 of Theorem 2.8, $x_\star$ satisfies the transversality condition (5). □

In contrast with Theorem 3.4, the following theorem is proved by manipulating equation (6) differently: using integration by parts and nabla differentiation formulas, we transform the items which consist of p^ρ into p^∇. Thanks to that, we apply Corollary 1 instead of Lemma 3.3 to obtain the intended conclusions.

Theorem 3.5. *Under assumptions of Theorem 3.4, the Euler–Lagrange system of n equations*

$$\lim_{T\to\infty}\inf_{T'\geq T}\left\{\int_t^{T'} g_x\langle x_\star\rangle(\tau)\int_{\rho(\tau)}^{T'} L_z[x_\star,z_\star](s)\nabla s\nabla\tau + g_v\langle x_\star\rangle(t)\int_{\rho(t)}^{T'} L_z[x_\star,z_\star](\tau)\nabla\tau\right\} + L_v[x_\star,z_\star](t) - \int_a^t L_x[x_\star,z_\star](\tau)\nabla\tau = c \qquad (12)$$

holds for all $t\in[a,\infty[$, $c\in\mathbb{R}^n$, together with the transversality condition

$$\lim_{T\to\infty}\inf_{T'\geq T}\left\{x_\star(T')\cdot\int_a^{T'} L_x[x_\star,z_\star](\tau)\nabla\tau\right\} = 0. \qquad (13)$$

Proof. We use the necessary optimality condition (6) found in the proof of Theorem 3.4. Using point 3 of Theorem 2.4,

$$\begin{aligned}&\left[p(t)\cdot\int_a^t L_x[x_\star,z_\star](\tau)\nabla\tau\right]^\nabla\\ &\quad= p^\nabla(t)\cdot\int_a^t L_x[x_\star,z_\star](\tau)\nabla\tau + p^\rho(t)\cdot\left[\int_a^t L_x[x_\star,z_\star](\tau)\nabla\tau\right]^\nabla\\ &\quad= p^\nabla(t)\cdot\int_a^t L_x[x_\star,z_\star](\tau)\nabla\tau + p^\rho(t)\cdot L_x[x_\star,z_\star](t).\end{aligned}$$

Then, integrating both sides from $t=a$ to $t=T'$,

$$\begin{aligned}&\int_a^{T'}\left[p(t)\cdot\int_a^t L_x[x_\star,z_\star](\tau)\nabla\tau\right]^\nabla\nabla t\\ &\quad=\int_a^{T'}\left(p^\nabla(t)\cdot\int_a^t L_x[x_\star,z_\star](\tau)\nabla\tau\right)\nabla t + \int_a^{T'} p^\rho(t)\cdot L_x[x_\star,z_\star](t)\nabla t.\end{aligned}$$

Therefore,

$$\begin{aligned}&p(t)\cdot\int_a^t L_x[x_\star,z_\star](\tau)\nabla\tau\Bigg|_{t=a}^{t=T'}\\ &\quad=\int_a^{T'}\left(p^\nabla(t)\cdot\int_a^t L_x[x_\star,z_\star](\tau)\nabla\tau\right)\nabla t + \int_a^{T'} p^\rho(t)\cdot L_x[x_\star,z_\star](t)\nabla t.\end{aligned}$$

Since $p(a)=0$, we obtain that $p(T')\cdot\int_a^{T'} L_x[x_\star,z_\star](\tau)\nabla\tau$ is equal to

$$\int_a^{T'} p^\nabla(t)\cdot\left(\int_a^t L_x[x_\star,z_\star](\tau)\nabla\tau\right)\nabla t+\int_a^{T'} p^\rho(t)\cdot L_x[x_\star,z_\star](t)\nabla t,$$

that is,

$$\begin{aligned}\int_a^{T'} & p^\rho(t)\cdot L_x[x_\star,z_\star](t)\nabla t\\ &=-\int_a^{T'} p^\nabla(t)\cdot\left(\int_a^t L_x[x_\star,z_\star](\tau)\nabla\tau\right)\nabla t+p(T')\cdot\int_a^{T'} L_x[x_\star,z_\star](\tau)\nabla\tau.\end{aligned}$$

Making the same calculations as in the proof of Theorem 3.4, we obtain (7). Using again point 3 of Theorem 2.4,

$$\begin{aligned}&\left[p(t)\cdot\int_t^{T'}\left(g_x\langle x_\star\rangle(\tau)\int_{\rho(\tau)}^{T'} L_z[x_\star,z_\star](s)\nabla s\right)\nabla\tau\right]^\nabla\\ &\quad=p^\nabla(t)\cdot\int_t^{T'}\left(g_x\langle x_\star\rangle(\tau)\int_{\rho(\tau)}^{T'} L_z[x_\star,z_\star](s)\nabla s\right)\nabla\tau\\ &\qquad+p^\rho(t)\cdot\left[\int_t^{T'}\left(g_x\langle x_\star\rangle(\tau)\int_{\rho(\tau)}^{T'} L_z[x_\star,z_\star](s)\nabla s\right)\nabla\tau\right]^\nabla\\ &\quad=p^\nabla(t)\cdot\int_t^{T'}\left(g_x\langle x_\star\rangle(\tau)\int_{\rho(\tau)}^{T'} L_z[x_\star,z_\star](s)\nabla s\right)\nabla\tau\\ &\qquad-p^\rho(t)\cdot g_x\langle x_\star\rangle(t)\int_{\rho(t)}^{T'} L_z[x_\star,z_\star](\tau)\nabla\tau.\end{aligned}$$

Integrating both sides from $t=a$ to $t=T'$, and because of point 2 of Theorem 2.8 and $p(a)=0$,

$$\begin{aligned}\int_a^{T'}&\left[p(t)\cdot\int_t^{T'}\left(g_x\langle x_\star\rangle(\tau)\int_{\rho(\tau)}^{T'} L_z[x_\star,z_\star](s)\nabla s\right)\nabla\tau\right]^\nabla\nabla t\\ &=p(t)\cdot\int_t^{T'}\left(g_x\langle x_\star\rangle(\tau)\int_{\rho(\tau)}^{T'} L_z[x_\star,z_\star](s)\nabla s\right)\nabla\tau\Bigg|_{t=a}^{t=T'}=0.\end{aligned}$$

Then,

$$\int_a^{T'} p^\rho(t)\cdot\left(g_x\langle x_\star\rangle(t)\int_{\rho(t)}^{T'} L_z[x_\star,z_\star](s)\nabla s\right)\nabla t$$
$$=\int_a^{T'} p^\nabla(t)\cdot\left[\int_t^{T'}\left(g_x\langle x_\star\rangle(\tau)\int_{\rho(\tau)}^{T'} L_z[x_\star,z_\star](s)\nabla s\right)\nabla\tau\right]\nabla t.$$

From (7) and previous relations, we write (6) in the following way:

$$\begin{aligned}
&\lim_{T\to\infty}\inf_{T'\geq T}\Bigg\{-\int_a^{T'} p^\nabla(t)\cdot\int_a^{t} L_x[x_\star,z_\star](\tau)\nabla\tau\nabla t+p(T')\cdot\int_a^{T'} L_x[x_\star,z_\star](\tau)\nabla\tau\\
&+\int_a^{T'} L_v[x_\star,z_\star](t)\cdot p^\nabla(t)\nabla t+\int_a^{T'} p^\nabla(t)\cdot\int_t^{T'}\left(g_x\langle x_\star\rangle(\tau)\int_{\rho(\tau)}^{T'} L_z[x_\star,z_\star](s)\nabla s\right)\nabla\tau\nabla t\\
&+\int_a^{T'} g_v\langle x_\star\rangle(t)\cdot\left(p^\nabla(t)\int_{\rho(t)}^{T'} L_z[x_\star,z_\star](\tau)\nabla\tau\right)\nabla t\Bigg\}\\
&=\lim_{T\to\infty}\inf_{T'\geq T}\Bigg\{\int_a^{T'} p^\nabla(t)\cdot\Bigg[\int_a^{t}-L_x[x_\star,z_\star](\tau)\nabla\tau+L_v[x_\star,z_\star](t)\\
&\quad+\int_t^{T'}\left(g_x\langle x_\star\rangle(\tau)\int_{\rho(\tau)}^{T'} L_z[x_\star,z_\star](s)\nabla s\right)\nabla\tau\\
&\quad+g_v\langle x_\star\rangle(t)\int_{\rho(t)}^{T'} L_z[x_\star,z_\star](\tau)\nabla\tau\Bigg]\nabla t+p(T')\cdot\int_a^{T'} L_x[x_\star,z_\star](\tau)\nabla\tau\Bigg\}=0.
\end{aligned}\tag{14}$$

Because (14) holds for all $p\in C_{ld}$ with $p(a)=0$, in particular it also holds in the subclass of functions $p\in C_{ld}$ with $p(a)=p(T')=0$. Let $i\in\{1,\ldots,n\}$. Choosing $p=(p_1,\ldots,p_n)$ such that all $p_j\equiv 0$, $j\neq i$, and $p_i\in C_{ld}$ with $p_i(a)=p_i(T')=0$, we conclude that

$$\begin{aligned}
&\lim_{T\to\infty}\inf_{T'\geq T}\int_a^{T'} p_i^\nabla(t)\Bigg\{\int_a^{t}-L_{x_i}[x_\star,z_\star](\tau)\nabla\tau+L_{v_i}[x_\star,z_\star](t)\\
&+\int_t^{T'} g_{x_i}\langle x_\star\rangle(\tau)\int_{\rho(\tau)}^{T'} L_z[x_\star,z_\star](s)\nabla s\nabla\tau+g_{v_i}\langle x_\star\rangle(t)\int_{\rho(t)}^{T'} L_z[x_\star,z_\star](\tau)\nabla\tau\Bigg\}\nabla t=0.
\end{aligned}$$

From Corollary 1 we obtain the equations

$$L_{v_i}[x_\star, z_\star](t) - \int_a^t L_{x_i}[x_\star, z_\star](\tau)\nabla\tau + \int_t^{T'} \left(g_{x_i}\langle x_\star\rangle(\tau) \int_{\rho(\tau)}^{T'} L_z[x_\star, z_\star](s)\nabla s \right) \nabla\tau + g_{v_i}\langle x_\star\rangle(t) \int_{\rho(t)}^{T'} L_z[x_\star, z_\star](\tau)\nabla\tau = c_i, \quad (15)$$

$c_i \in \mathbb{R}$, $i = 1, \ldots, n$, for all $t \in [a, +\infty[$ and all $T' \geq t$. These n conditions imply the Euler–Lagrange system of equations (12). From (14) and (15), we conclude that

$$\lim_{T\to\infty} \inf_{T'\geq T} \left\{ p(T') \cdot \int_a^{T'} L_x[x_\star, z_\star](\tau)\nabla\tau \right\} = 0. \quad (16)$$

Using the special curve p defined by (11), we obtain from equation (16) that

$$\lim_{T\to\infty} \inf_{T'\geq T} \left\{ \beta x_\star(T') \cdot \int_a^{T'} L_x[x_\star, z_\star](\tau)\nabla\tau \right\} = 0.$$

Therefore, $x_\star$ satisfies the transversality condition (13). □

Acknowledgments. Work supported by *FEDER* funds through *COMPETE* — Operational Programme Factors of Competitiveness ("Programa Operacional Factores de Competitividade") and by Portuguese funds through the *Center for Research and Development in Mathematics and Applications* (University of Aveiro) and the Portuguese Foundation for Science and Technology ("FCT — Fundação para a Ciência e a Tecnologia"), within project PEst-C/MAT/UI4106/2011 with COMPETE number FCOMP-01-0124-FEDER-022690. Dryl was also supported by FCT through the Ph.D. fellowship SFRH/BD/51163/2010; Torres by FCT through the project PTDC/MAT/113470/2009. The authors are grateful to N. Martins for helpful suggestions and for a detailed reading of a preliminary version of the paper.

REFERENCES

[1] M. Bohner and A. Peterson, "Dynamic Equations on Time Scales," Birkhäuser Boston, Boston, MA, 2001.

[2] M. Bohner and A. Peterson, "Advances in Dynamic Equations on Time Scales," Birkhäuser Boston, Boston, MA, 2003.

[3] M. C. Caputo, *Time scales: from nabla calculus to delta calculus and vice versa via duality*, Int. J. Difference Equ., **5** (2010), 25–40.

[4] S. Lang, "Undergraduate Analysis," 2^{nd} edition, Undergraduate Texts in Mathematics, Springer, New York, 1997.

[5] G. Leitmann, "The Calculus of Variations and Optimal Control," Mathematical Concepts and Methods in Science and Engineering, 24, Plenum, New York, 1981.

[6] A. B. Malinowska, N. Martins and D. F. M. Torres, *Transversality conditions for infinite horizon variational problems on time scales*, Optim. Lett., **5** (2011), 41–53.

[7] A. B. Malinowska and D. F. M. Torres, *Strong minimizers of the calculus of variations on time scales and the Weierstrass condition*, Proc. Est. Acad. Sci., **58** (2009), 205–212.

[8] A. B. Malinowska and D. F. M. Torres, *Leitmann's direct method of optimization for absolute extrema of certain problems of the calculus of variations on time scales*, Appl. Math. Comput., **217** (2010), 1158–1162.

[9] A. B. Malinowska and D. F. M. Torres, *A general backwards calculus of variations via duality*, Optim. Lett., **5** (2011), 587–599.

[10] N. Martins and D. F. M. Torres, *Noether's symmetry theorem for nabla problems of the calculus of variations*, Appl. Math. Lett., **23** (2010), 1432–1438.
[11] N. Martins and D. F. M. Torres, *Generalizing the variational theory on time scales to include the delta indefinite integral*, Comput. Math. Appl., **61** (2011), 2424–2435.
[12] N. Martins and D. F. M. Torres, *Higher-order infinite horizon variational problems in discrete quantum calculus*, Comput. Math. Appl., **64** (2012), 2166–2175.
[13] D. F. M. Torres, *The variational calculus on time scales*, Int. J. Simul. Multidisci. Des. Optim., **4** (2010), 11–25.

Received December 2011; 1^{st} revision October 2012; 2^{nd} revision November 2012.

E-mail address: monikadryl@ua.pt
E-mail address: delfim@ua.pt