Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2013, Article ID 145050, 12 pages
http://dx.doi.org/10.1155/2013/145050

Research Article

Existence of Three Positive Solutions to Some *p*-Laplacian Boundary Value Problems

Moulay Rchid Sidi Ammi,[1] and Delfim F. M. Torres[2]

[1] *Department of Mathematics, AMNEA Group, Faculty of Sciences and Technics, Moulay Ismail University, P.O. Box 509, Errachidia, Morocco*
[2] *Department of Mathematics, Center for Research and Development in Mathematics and Applications, University of Aveiro, 3810-193 Aveiro, Portugal*

Correspondence should be addressed to Delfim F. M. Torres; delfim@ua.pt

Received 24 March 2012; Revised 5 August 2012; Accepted 19 October 2012

Academic Editor: Zengji Du

We obtain, by using the Leggett-Williams fixed point theorem, sufficient conditions that ensure the existence of at least three positive solutions to some p-Laplacian boundary value problems on time scales.

1. Introduction

The study of dynamic equations on time scales goes back to the 1989 Ph.D. thesis of Hilger [1, 2] and is currently an area of mathematics receiving considerable attention [3–7]. Although the basic aim of the theory of time scales is to unify the study of differential and difference equations in one and the same subject, it also extends these classical domains to hybrid and in-between cases. A great deal of work has been done since the eighties of the XX century in unifying the theories of differential and difference equations by establishing more general results in the time scale setting [8–12].

Boundary value p-Laplacian problems for differential equations and finite difference equations have been studied extensively (see, e.g., [13] and references therein). Although many existence results for dynamic equations on time scales are available [14, 15], there are not many results concerning p-Laplacian problems on time scales [16–19]. In this paper we prove new existence results for three classes of p-Laplacian boundary value problems on time scales. In contrast with our previous works [17, 18], which make use of the Krasnoselskii fixed point theorem and the fixed point index theory, respectively, here we use the Leggett-Williams fixed point theorem [20, 21] obtaining multiplicity of positive solutions. The application of the Leggett-Williams fixed point theorem for proving multiplicity of solutions for boundary value problems on time scales was first introduced by Agarwal and O'Regan [22] and is now recognized as an important tool to prove existence of positive solutions for boundary value problems on time scales [23–28].

The paper is organized as follows. In Section 2 we present some necessary results from the theory of time scales (Section 2.1) and the theory of cones in Banach spaces (Section 2.2). We end Section 2.2 with the Leggett-Williams fixed point theorem for a cone-preserving operator, which is our main tool in proving existence of positive solutions to the boundary value problems on time scales we consider in Section 3. The contribution of the paper is Section 3, which is divided into three parts. The purpose of the first part (Section 3.1) is to prove existence of positive solutions to the nonlocal p-Laplacian dynamic equation on time scales

$$-\left(\phi_p\left(u^{\Delta}(t)\right)\right)^{\nabla}=\frac{\lambda f(u(t))}{\left(\int_0^T f(u(\tau))\nabla\tau\right)^2},\quad t\in(0,T)_{\mathbb{T}},\tag{1}$$

satisfying the boundary conditions

$$\begin{gathered}\phi_p\left(u^{\Delta}(0)\right)-\beta\left(\phi_p\left(u^{\Delta}(\eta)\right)\right)=0,\\ u(T)-\beta u(\eta)=0,\end{gathered}\tag{2}$$

where $\eta \in (0,T)_{\mathbb{T}}$, $\phi_p(\cdot)$ is the p-Laplacian operator defined by $\phi_p(s) = |s|^{p-2}s$, $p > 1$, and $(\phi_p)^{-1} = \phi_q$ with q the Holder conjugate of p, that is, $1/p + 1/q = 1$. The concrete value of p is connected with the application at hands. For $p = 2$, problem (1)-(2) describes the operation of a device flowed by an electric current, for example, thermistors [29], which are devices made from materials whose electrical conductivity is highly dependent on the temperature. Thermistors have the advantage of being temperature measurement devices of low cost, high resolution, and flexibility in size and shape. Constant λ in (1) is a dimensionless parameter that can be identified with the square of the applied potential difference at the ends of a conductor, $f(u)$ is the temperature-dependent resistivity of the conductor, and β in (2) is a transfer coefficient supposed to verify $0 < \beta < 1$. For a more detailed discussion about the physical justification of (1)-(2) the reader is referred to [17]. Theoretical analysis (existence, uniqueness, regularity, and asymptotic results) for thermistor problems with various types of boundary and initial conditions has received significant attention in the last few years for the particular case $\mathbb{T} = \mathbb{R}$ [30–34]. The second part of our results (Section 3.2) is concerned with the following quasilinear elliptic problem:

$$\begin{gathered} -\left(\phi_p\left(u^{\Delta}(t)\right)\right)^{\nabla} = f(u(t)) + h(t), \quad t \in (0,T)_{\mathbb{T}}, \\ u^{\Delta}(0) = 0, \qquad u(T) - u(\eta) = 0, \end{gathered} \tag{3}$$

where $\eta \in (0,T)_{\mathbb{T}}$. Results on existence of infinitely many radial solutions to (3) are proved in the literature using (i) variational methods, where solutions are obtained as critical points of some energy functional on a Sobolev space, with f satisfying appropriate conditions [35, 36]; (ii) methods based on phase-plane analysis and the shooting method [37]; (iii) the technique of time maps [38]. For $p = 2$, $h \equiv 0$, and $\mathbb{T} = \mathbb{R}$, problem (3) becomes a well-known boundary value problem of differential equations. Our results generalize earlier works to the case of a generic time scale $\mathbb{T}$, $p \neq 2$, and h not identically zero. Finally, the third part of our contribution (Section 3.3) is devoted to the existence of positive solutions to the p-Laplacian dynamic equation

$$\begin{aligned} &\left(\phi_p\left(u^{\Delta}(t)\right)\right)^{\nabla} + \lambda a(t) f(u(t), u(\omega(t))) = 0, \\ &\quad t \in (0,T)_{\mathbb{T}}, \\ &u(t) = \psi(t), \quad t \in [-r, 0]_{\mathbb{T}}, \\ &u(0) - B_0\left(u^{\Delta}(0)\right) = 0, \qquad u^{\Delta}(T) = 0 \end{aligned} \tag{4}$$

on a time scale $\mathbb{T}$ such that $0, T \in \mathbb{T}^{\kappa}_{\kappa}$, $-r \in \mathbb{T}$ with $-r \le 0 < T$, where $\lambda > 0$. This problem is considered in [39] where the author applies the Krasnoselskii fixed point theorem to obtain one positive solution to (4). Here we use the same conditions as in [39], but applying Leggett-Williams' theorem we are able to obtain more: we prove existence of at least three positive solutions and we are able to localize them.

2. Preliminaries

Here we just recall the basic concepts and results needed in the sequel. For an introduction to time scales the reader is referred to [3, 8–10, 40, 41] and references therein; for a good introduction to the theory of cones in Banach spaces we refer the reader to the book [42].

2.1. Time Scales. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. The operators σ and ρ from $\mathbb{T}$ to $\mathbb{T}$ are defined in [1, 2] as

$$\begin{gathered} \sigma(t) = \inf\{\tau \in \mathbb{T} \mid \tau > t\} \in \mathbb{T}, \\ \rho(t) = \sup\{\tau \in \mathbb{T} \mid \tau < t\} \in \mathbb{T} \end{gathered} \tag{5}$$

and are called the forward jump operator and the backward jump operator, respectively. A point $t \in \mathbb{T}$ is left-dense, left-scattered, right-dense, and right-scattered if $\rho(t) = t$, $\rho(t) < t$, and $\sigma(t) = t$, $\sigma(t) > t$, respectively. If $\mathbb{T}$ has a right-scattered minimum m, define $\mathbb{T}_{\kappa} = \mathbb{T} - \{m\}$; otherwise set $\mathbb{T}_{\kappa} = \mathbb{T}$. If $\mathbb{T}$ has a left-scattered maximum M, define $\mathbb{T}^{\kappa} = \mathbb{T} - \{M\}$; otherwise set $\mathbb{T}^{\kappa} = \mathbb{T}$. Following [43], we also introduce the set $\mathbb{T}^{\kappa}_{\kappa} = \mathbb{T}^{\kappa} \cap \mathbb{T}_{\kappa}$.

Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^{\kappa}$ (assume t is not left-scattered if $t = \sup \mathbb{T}$), then the delta derivative of f at the point t is defined to be the number $f^{\Delta}(t)$ (provided it exists) with the property that for each $\epsilon > 0$ there is a neighborhood U of t such that

$$\left| f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s) \right| \le |\sigma(t) - s| \quad \forall s \in U. \tag{6}$$

Similarly, for $t \in \mathbb{T}_{\kappa}$ (assume t is not right-scattered if $t = \inf \mathbb{T}$), the nabla derivative of f at the point t is defined in [44] to be the number $f^{\nabla}(t)$ (provided it exists) with the property that for each $\epsilon > 0$ there is a neighborhood U of t such that

$$\left| f(\rho(t)) - f(s) - f^{\nabla}(t)(\rho(t) - s) \right| \le |\rho(t) - s| \quad \forall s \in U. \tag{7}$$

If $\mathbb{T} = \mathbb{R}$, then $f^{\Delta}(t) = f^{\nabla}(t) = f'(t)$. If $\mathbb{T} = \mathbb{Z}$, then $f^{\Delta}(t) = f(t+1) - f(t)$ is the forward difference operator while $f^{\nabla}(t) = f(t) - f(t-1)$ is the backward difference operator.

A function f is left-dense continuous (i.e., ld-continuous), if f is continuous at each left-dense point in $\mathbb{T}$ and its right-sided limit exists at each right-dense point in $\mathbb{T}$. If f is ld-continuous, then there exists F such that $F^{\nabla}(t) = f(t)$ for any $t \in \mathbb{T}_{\kappa}$. We then introduce the nabla integral by

$$\int_a^b f(t) \nabla t = F(b) - F(a). \tag{8}$$

We define right-dense continuous (rd-continuous) functions in a similar way. If f is rd-continuous, then there exists F such that $F^{\Delta}(t) = f(t)$ for any $t \in \mathbb{T}^{\kappa}$, and we define the delta integral by

$$\int_a^b f(t) \Delta t = F(b) - F(a). \tag{9}$$

2.2. Cones in Banach Spaces. In this paper $\mathbb{T}$ is a time scale with $0 \in \mathbb{T}_\kappa$ and $T \in \mathbb{T}^\kappa$. We use $\mathbb{R}^+$ and $\mathbb{R}_0^+$ to denote, respectively, the set of positive and nonnegative real numbers. By $[0,T]_\mathbb{T}$ we denote the set $[0,T] \cap \mathbb{T}$. Similarly, $(0,T)_\mathbb{T} = (0,T) \cap \mathbb{T}$. Let $E = \mathbb{C}_{ld}([0,T]_\mathbb{T}, \mathbb{R})$. It follows that E is a Banach space with the norm $\|u\| = \max_{[0,T]_\mathbb{T}} |u(t)|$.

Definition 1. Let E be a real Banach space. A nonempty, closed, and convex set $P \subset E$ is called a cone if it satisfies the following two conditions:

(i) $u \in P$, $\lambda \geq 0$, implies $\lambda u \in P$;

(ii) $u \in P$, $-u \in P$, implies $u = 0$.

Every cone $P \subset E$ induces an ordering in E given by

$$u \leq v \quad \text{iff } v - u \in P. \tag{10}$$

Definition 2. Let E be a real Banach space and $P \subset E$ a cone. A function $\alpha : P \to \mathbb{R}_0^+$ is called a nonnegative continuous concave functional if α is continuous and

$$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y) \tag{11}$$

for all $x, y \in P$ and $0 \leq t \leq 1$.

Let a, b, and r be positive constants, $P_r = \{u \in P \mid \|u\| < r\}$, $P(\alpha, a, b) = \{u \in P \mid a \leq \alpha(u), \|u\| < b\}$. The following fixed point theorem provides the existence of at least three positive solutions. The origin in E is denoted by $\varnothing$. The proof of the Leggett-Williams fixed point theorem can be found in Guo and Lakshmikantham [42] or Leggett and Williams [45].

Theorem 3 (Leggett-Williams' Theorem). *Let P be a cone in a real Banach space E. Let $G : \overline{P_c} \to \overline{P_c}$ be a completely continuous map and α a nonnegative continuous concave functional on P such that $\alpha(u) \leq \|u\|$ for all $u \in \overline{P_c}$. Suppose there exist positive constants a, b, and d with $0 < a < b < d \leq c$ such that*

(i) *$\{u \in P(\alpha, b, d) \mid \alpha(u) > b\} \neq \varnothing$ and $\alpha(Gu) > b$ for all $u \in P(\alpha, b, d)$;*

(ii) *$\|Gu\| < a$ for all $u \in \overline{P_a}$;*

(iii) *$\alpha(Gu) > b$ for all $u \in P(\alpha, b, c)$ with $\|Gu\| > d$.*

Then G has at least three fixed points u_1, u_2, and u_3 satisfying

$$\|u_1\| < a, \qquad b < \alpha(u_2), \qquad \|u_3\| > a, \qquad \alpha(u_3) < b. \tag{12}$$

3. Main Results

We prove existence of three positive solutions to different p-Laplacian problems on time scales: in Section 3.1 we study problem (1)-(2), in Section 3.2 problem (3), and finally (4) in Section 3.3.

3.1. Nonlocal Thermistor Problem. By a solution $u : \mathbb{T} \to \mathbb{R}$ of (1)-(2) we mean a delta differentiable function such that u^Δ and $(|u^\Delta|^{p-2}u^\Delta)^\nabla$ are both continuous on $\mathbb{T}_\kappa^\kappa$ and u satisfies (1)-(2). We consider the following hypothesis:

(H1) $f : \mathbb{R} \to \mathbb{R}^+$ is a continuous function.

Lemma 4 (see Lemma 3.1 of [17]). *Assume that hypothesis (H1) on function f is satisfied. Then u is a solution to (1)-(2) if and only if $u \in E$ is a solution to the integral equation*

$$u(t) = -\int_0^t \phi_q(g(s))\,\Delta s + B, \tag{13}$$

where

$$\begin{aligned} g(s) &= \int_0^s \lambda h(u(r))\,\nabla r - A, \\ A = \phi_p(u^\Delta(0)) &= -\frac{\lambda\beta}{1-\beta}\int_0^\eta h(u(r))\,\nabla r, \\ h(u(t)) &= \frac{\lambda f(u(t))}{\left(\int_0^T f(u(\tau))\,\nabla\tau\right)^2}, \\ B = u(0) &= \frac{1}{1-\beta}\left\{\int_0^T \phi_q(g(s))\,\Delta s - \beta\int_0^\eta \phi_q(g(s))\,\Delta s\right\}. \end{aligned} \tag{14}$$

Lemma 5. *Suppose (H1) holds. Then a solution u to (1)-(2) satisfies $u(t) \geq 0$ for $t \in (0,T)_\mathbb{T}$.*

Proof. We have $A = -\lambda\beta/(1-\beta)\int_0^\eta h(u(r))\nabla r \leq 0$. Then, $g(s) = \lambda\int_0^s h(u(r))\nabla r - A \geq 0$. It follows that $\phi_p(g(s)) \geq 0$. Since $0 < \beta < 1$, we also have

$$\begin{aligned} u(0) &= B \\ &= \frac{1}{1-\beta}\left\{\int_0^T \phi_q(g(s))\,\Delta s - \beta\int_0^\eta \phi_q(g(s))\,\Delta s\right\} \\ &\geq \frac{1}{1-\beta}\left\{\beta\int_0^T \phi_q(g(s))\,\Delta s - \beta\int_0^\eta \phi_q(g(s))\,\Delta s\right\} \\ &\geq 0, \\ u(T) &= u(0) - \int_0^T \phi_q(g(s))\,\Delta s \\ &= \frac{-\beta}{1-\beta}\int_0^\eta \phi_q(g(s))\,\Delta s \\ &\quad + \frac{1}{1-\beta}\int_0^T \phi_q(g(s))\,\Delta s - \int_0^T \phi_q(g(s))\,\Delta s \\ &= \frac{-\beta}{1-\beta}\int_0^\eta \phi_q(g(s))\,\Delta s + \frac{\beta}{1-\beta}\int_0^T \phi_q(g(s))\,\Delta s \\ &= \frac{\beta}{1-\beta}\left\{\int_0^T \phi_q(g(s))\,\Delta s - \int_0^\eta \phi_q(g(s))\,\Delta s\right\} \\ &\geq 0. \end{aligned} \tag{15}$$

If $t \in (0,T)_{\mathbb{T}}$, then

$$\begin{aligned} u(t) &= u(0) - \int_0^t \phi_q(g(s))\,\Delta s \\ &\geq -\int_0^T \phi_q(g(s))\,\Delta s + u(0) = u(T) \\ &\geq 0. \end{aligned} \tag{16}$$

Consequently, $u(t) \geq 0$ for $t \in (0,T)_{\mathbb{T}}$. □

On the other hand, we have $\phi_p(u^\Delta(s)) = \phi_p(u^\Delta(0)) - \int_0^s \lambda h(u(r))\nabla r \leq 0$. Since $A = \phi_p(u^\Delta(0)) \leq 0$, then $u^\Delta \leq 0$. This means that $\|u\| = u(0)$, $\inf_{t\in(0,T)_{\mathbb{T}}} u(t) = u(T)$. Moreover, $\phi_p(u^\Delta(s))$ is nonincreasing, which implies with the monotonicity of ϕ_p that u^Δ is a nonincreasing function on $(0,T)_{\mathbb{T}}$. Hence, u is concave. In order to apply Theorem 3, let us define the cone $P \subset E$ by

$$\begin{aligned} P = \{&u \in E \mid u \text{ is nonnegative}, \\ &\text{decreasing on } [0,T]_{\mathbb{T}} \text{ and concave on } E\}. \end{aligned} \tag{17}$$

We also define the nonnegative continuous concave functional $\alpha : P \to \mathbb{R}_0^+$ by

$$\alpha(u) = \min_{t\in[\xi,T-\xi]_{\mathbb{T}}} u(t), \quad \xi \in \left(0, \frac{T}{2}\right), \quad \forall u \in P. \tag{18}$$

It is easy to see that problem (1)-(2) has a solution $u = u(t)$ if and only if u is a fixed point of the operator $G : P \to E$ defined by

$$Gu(t) = -\int_0^t \phi_q(g(s))\,\Delta s + B, \tag{19}$$

where g and B are as in Lemma 4.

Lemma 6. *Let G be defined by* (19). *Then,*

(i) $G(P) \subseteq P$;

(ii) $G : P \to P$ *is completely continuous.*

Proof. (i) holds clearly from above. (ii) Suppose that $D \subseteq P$ is a bounded set and let $u \in D$. Then,

$$\begin{aligned} |Gu(t)| &= \left| -\int_0^t \phi_q(g(s))\,\Delta s + B \right| \\ &\leq \left| -\int_0^t \phi_q\left(\int_0^s \frac{\lambda f(u(r))}{\left(\int_0^T f(u(\tau))\nabla\tau\right)^2}\nabla r - A \right)\Delta s \right| + |B| \\ &\leq \int_0^T \phi_q\left(\int_0^s \frac{\lambda \sup_{u\in D} f(u)}{(T \inf_{u\in D} f(u))^2}\,\nabla r - A \right)\Delta s + |B|, \\ |A| &= \left| \frac{\lambda\beta}{1-\beta} \int_0^\eta h(u(r))\nabla r \right| \\ &= \left| \frac{\lambda\beta}{1-\beta} \int_0^\eta \frac{f(u(r))}{\left(\int_0^T f(u(\tau))\nabla\tau\right)^2}\nabla r \right| \\ &\leq \frac{\lambda\beta}{1-\beta} \frac{\sup_{u\in D} f(u)}{(T\inf_{u\in D} f(u))^2}\,\eta. \end{aligned} \tag{20}$$

In the same way, we have

$$\begin{aligned} |B| &\leq \frac{1}{1-\beta} \int_0^T \phi_q(g(s))\,\Delta s \\ &\leq \frac{1}{1-\beta} \int_0^T \phi_q\left(\frac{\lambda \sup_{u\in D} f(u)}{(T\inf_{u\in D} f(u))^2} \left(s + \frac{\beta}{1-\beta}\eta \right) \right)\Delta s. \end{aligned} \tag{21}$$

It follows that

$$|Gu(t)| \leq \int_0^T \phi_q\left(\frac{\lambda \sup_{u\in D} f(u)}{(T\inf_{u\in D} f(u))^2} \left(s + \frac{\beta\eta}{1-\beta} \right) \right)\Delta s + |B|. \tag{22}$$

As a consequence, we get

$$\begin{aligned} \|Gu\| &\leq \frac{2-\beta}{1-\beta} \int_0^T \phi_q\left(\frac{\lambda \sup_{u\in D} f(u)}{(T\inf_{u\in D} f(u))^2} \left(s + \frac{\beta\eta}{1-\beta} \right) \right)\Delta s \\ &\leq \frac{2}{1-\beta}\phi_q\left(\frac{\lambda \sup_{u\in D} f(u)}{(T\inf_{u\in D} f(u))^2} \right) \int_0^T \phi_q\left(s + \frac{\beta\eta}{1-\beta} \right)\Delta s. \end{aligned} \tag{23}$$

Then $G(D)$ is bounded on the whole bounded set D. Moreover, if $t_1, t_2 \in [0,T]_{\mathbb{T}}$ and $u \in D$, then we have for a positive constant c

$$|Gu(t_2) - Gu(t_1)| \leq \left| \int_{t_1}^{t_2} \phi_q(g(s))\,\Delta s \right| \leq c\,|t_2 - t_1|. \tag{24}$$

We see that the right-hand side of the above inequality goes uniformly to zero when $|t_2 - t_1| \to 0$. Then by a standard application of the Arzela-Ascoli theorem we have that $G : P \to P$ is completely continuous. □

We can also easily obtain the following properties.

Lemma 7. (i) $\alpha(u) = u(T-\xi) \le \|u\|$ for all $u \in P$;
(ii) $\alpha(Gu) = (Gu)(T-\xi)$;
(iii) $\|Gu\| = Gu(0)$.

We now state the main result of Section 3.1.

Theorem 8. *Suppose that $(H1)$ is verified and there exist positive constants a, b, c, and d such that $0 < \zeta a = a_1 < b < d - T^2 c\phi_q(1/T) < d < \zeta c = c_1$, with $\zeta = (T^2/(1-\beta))\,\phi_q(1/T)$. One further imposes f to satisfy the following hypotheses:*

(H2) $\min_{0\le u\le a_1} f(u) \ge \lambda^2/(T(1-\beta)\phi_p(a))$ *uniformly for all* $t \in [0,T]_{\mathbb{T}}$;

(H3) $\min_{0\le u\le c_1} f(u) \ge \lambda^2/(T(1-\beta)\phi_p(c))$ *uniformly for all* $t \in [0,T]_{\mathbb{T}}$;

(H4) $\min_{b\le u\le d} f(u) \ge \phi_p(bB_1)$ *uniformly for all* $t \in [0,T]_{\mathbb{T}}$, *where*

$$B_1 = \frac{(1-\beta)}{\beta\xi}\left|\phi_p(T-\xi)\right|\phi_p\left(\frac{\lambda}{(T\sup_{b\le u\le d} f(u))^2}\right). \quad (25)$$

Then the boundary value problem (1)-(2) has at least three positive solutions u_1, u_2, and u_3, verifying

$$\|u_1\| < a, \qquad b < \alpha(u_2), \qquad \|u_3\| > a, \qquad \alpha(u_3) < b. \quad (26)$$

Proof. The proof passes by several lemmas. We have already seen in Lemma 6 that the operator G is completely continuous. We now show the following Lemma.

Lemma 9. *The following relations hold:*

$$G\overline{P_{c_1}} \subset \overline{P_{c_1}}, \qquad G\overline{P_{a_1}} \subset \overline{P_{a_1}}. \quad (27)$$

Proof. Obviously, $G\overline{P_{a_1}} \subset P$. Moreover, for all $u \in \overline{P_{a_1}}$, we have $0 \le u(t) \le a_1$. On the other hand we have

$$\begin{aligned} Gu(t) = &-\int_0^t \phi_q(g(s))\,\Delta s + \frac{1}{1-\beta}\int_0^T \phi_q(g(s))\,\Delta s \\ &- \frac{\beta}{1-\beta}\int_0^\eta \phi_q(g(s))\,\Delta s, \end{aligned} \quad (28)$$

and for all $u \in G\overline{P_{a_1}}$ we have $0 \le u(t) \le a_1$. Then,

$$|Gu| \le \frac{1}{1-\beta}\int_0^T \phi_q(g(s))\,\Delta s. \quad (29)$$

We have

$$\begin{aligned} h(u(t)) &= \frac{\lambda f(u(t))}{\left(\int_0^T f(u(\tau))\,\nabla\tau\right)^2}, \\ g(s) &= \int_0^s \lambda h(u(r))\,\nabla r + \frac{\lambda\beta}{1-\beta}\int_0^\eta h(u(r))\,\nabla r \\ &\le \left(\lambda + \frac{\lambda\beta}{1-\beta}\int_0^T h(u(r))\,\nabla r\right) \\ &\le \frac{\lambda}{1-\beta}\int_0^T h(u(r))\,\nabla r. \end{aligned} \quad (30)$$

Using (H2) it follows that

$$\phi_q(g(s)) \le aT\phi_q\left(\frac{1}{T}\right). \quad (31)$$

Then we get

$$|Gu| \le a_1, \qquad G\overline{P_{a_1}} \subset \overline{P_{a_1}}. \quad (32)$$

Similarly, using (H3) we get $G\overline{P_{c_1}} \subset \overline{P_{c_1}}$. □

Lemma 10. *The set*

$$\{u \in P(\alpha,b,d) \mid \alpha(u) > b\} \quad (33)$$

is nonempty, and

$$\alpha(Gu) > b, \quad \textit{if } u \in P(\alpha,b,d). \quad (34)$$

Proof. Let $u = (b+d)/2$. Then, $u \in P$, $\|u\| = (b+d)/2 \le d$, and $\alpha(u) \ge (b+d)/2 > b$. The first part of the lemma is proved. For $u \in P(\alpha,b,d)$ we have $b \le u \le d$. If $t \in [\xi,T]_{\mathbb{T}}$, then

$$\begin{aligned} \alpha(Gu) &= (Gu)(T-\xi) \\ &= -\int_0^{T-\xi} \phi_q(g(s))\,\Delta s + B \\ &\ge \frac{\beta}{1-\beta}\int_{T-\xi}^T \phi_q(g(s))\,\Delta s. \end{aligned} \quad (35)$$

Since $A \le 0$, we have by using (H4)

$$\begin{aligned} g(s) &= \lambda\int_0^s h(u(r))\,\nabla r - A \\ &\ge \lambda\int_0^s h(u(r))\,\nabla r \\ &\ge \lambda\int_0^s \frac{f(u)}{(T\sup_{b\le u\le d} f(u))^2}\nabla u \\ &\ge \lambda\frac{(bB_1)^{p-1}}{(T\sup_{b\le u\le d} f(u))^2}s. \end{aligned} \quad (36)$$

Using the fact that ϕ_q is nondecreasing we get

$$\begin{aligned}\phi_q(g(s)) &\geq \phi_q\left(\lambda\frac{(bB_1)^{p-1}}{(T\sup_{b\leq u\leq d}f(u))^2}s\right)\\&\geq bB_1\phi_q\left(\frac{\lambda}{(T\sup_{b\leq u\leq d}f(u))^2}\right)\phi_q(s).\end{aligned} \tag{37}$$

Using the expression of B_1

$$\begin{aligned}\alpha(Gu) &\geq \frac{\beta}{1-\beta}bB_1\phi_q\left(\frac{\lambda}{(T\sup_{b\leq u\leq d}f(u))^2}\right)\int_{T-\xi}^{T}\phi_q(s)\Delta s\\&\geq bB_1\frac{\beta}{1-\beta}\phi_q\left(\frac{\lambda}{(T\sup_{b\leq u\leq d}f(u))^2}\right)\phi_q(T-\xi)\xi\\&\geq b.\end{aligned} \tag{38}$$

□

Lemma 11. *For all $u\in P(\alpha,b,c_1)$ with $\|Gu\|>d$ one has*

$$\alpha(Gu)>b. \tag{39}$$

Proof. If $u\in P(\alpha,b,c_1)$ and $\|Gu\|>d$, then $0\leq u(t)\leq c_1$. Using hypothesis (H3) and the fact that $0<\beta<1$, it follows that

$$\begin{aligned}\alpha(Gu) &= Gu(T-\xi)\\&= -\int_0^{T-\xi}\phi_q(g(s))\Delta s+B\\&\geq -\int_0^{T}\phi_q(g(s))\Delta s+Gu(0)\\&\geq \|Gu\|-T^2c\phi_q\left(\frac{1}{T}\right)\\&\geq d-T^2c\phi_q\left(\frac{1}{T}\right)\\&> b.\end{aligned} \tag{40}$$

□

Gathering Lemmas 4 to 11 and applying Theorem 3, there exist at least three positive solutions u_1, u_2, and u_3 to (1)-(2) verifying

$$\|u_1\|<a,\qquad b<\alpha(u_2),\qquad \|u_3\|>a,\qquad \alpha(u_3)<b. \tag{41}$$

□

Example 12. Let $\mathbb{T}=\{1-(1/2)^{\mathbb{N}_0}\}\cup\{1\}$, where $\mathbb{N}_0$ denotes the set of all nonnegative integers. Consider the p-Laplacian dynamic equation

$$-\left(\phi_p\left(u^{\Delta}(t)\right)\right)^{\nabla}=\frac{\lambda f(u(t))}{\left(\int_0^T f(u(\tau))\nabla\tau\right)^2},\quad t\in(0,T)_{\mathbb{T}}, \tag{42}$$

satisfying the boundary conditions

$$\begin{aligned}\phi_p\left(u^{\Delta}(0)\right)-\beta\left(\phi_p\left(u^{\Delta}\left(\frac{1}{4}\right)\right)\right)&=0,\\u(1)-\beta u\left(\frac{1}{4}\right)&=0,\end{aligned} \tag{43}$$

where $p=3/2$, $q=3$, $\eta=1/4$, $\beta=1/2$, $\lambda=1$, $T=1$, and

$$f(u)=\begin{cases}2\sqrt{2}, & 0\leq u\leq 1,\\ 4(u-1)+2\sqrt{2}, & 1\leq u\leq \dfrac{3}{2},\\ 2+2\sqrt{2}, & \dfrac{3}{2}\leq u\leq 10,\\ 2u+2\sqrt{2}-18, & 10\leq u\leq 16.\end{cases} \tag{44}$$

Choose $a_1=1=2a$, $b=3/2$, $c_1=16=2c$, and $d=10$. It is easy to see that $\zeta=2$, $B_1=1/(2(2+\sqrt{2}))$ and

$$\begin{aligned}&\min\{f(u):u\in[0,a_1]\}=2\sqrt{2}\\&\quad\geq\frac{\lambda^2}{T(1-\beta)\phi_p(a)}=\frac{2}{\sqrt{a}}=2\sqrt{2},\\&\min\{f(u):u\in[0,c_1]\}=2\sqrt{2}\\&\quad\geq\frac{\lambda^2}{T(1-\beta)\phi_p(c)}=\frac{2}{\sqrt{c}}=\frac{1}{\sqrt{2}},\\&\min\{f(u):u\in[b,d]\}=2+2\sqrt{2}\\&\quad\geq\phi_p(bB_1)=\sqrt{bB_1}=\sqrt{\frac{3}{4(2+2\sqrt{2})}}.\end{aligned} \tag{45}$$

Then, hypotheses (H1)–(H4) are satisfied. Therefore, by Theorem 8, problem (42)-(43) has at least three positive solutions.

3.2. Quasilinear Elliptic Problem. We are interested in this section in the study of the following quasilinear elliptic problem:

$$\begin{aligned}-\left(\phi_p\left(u^{\Delta}(t)\right)\right)^{\nabla}&=f(u(t))+h(t),\quad t\in(0,T)_{\mathbb{T}},\\u^{\Delta}(0)=0,&\qquad u(T)-u(\eta)=0,\end{aligned} \tag{46}$$

where $\eta\in(0,T)_{\mathbb{T}}$. We assume the following hypotheses:

(A1) function $f:\mathbb{R}\to\mathbb{R}_0^+$ is continuous;

(A2) function $h:(0,T)_{\mathbb{T}}\to\mathbb{R}_0^+$ is left dense continuous, that is,

$$h\in\mathbb{C}_{ld}\left((0,T)_{\mathbb{T}},\mathbb{R}_0^+\right),\quad h\in L^{\infty}. \tag{47}$$

Similarly as in Section 3.1, we prove existence of solutions by constructing an operator whose fixed points are solutions to (46). The main ingredient is, again, the Leggett-Williams fixed point theorem (Theorem 3). We can easily see that (46) is equivalent to the integral equation

$$u(t)=\phi_q\left(\int_\eta^T (f(u(r))+h(r))\nabla r\right)+\int_0^t \phi_q\left(\int_s^T (f(u(r))+h(r))\nabla r\right)\Delta s. \tag{48}$$

On the other hand, we have $-(\phi_p(u^\Delta))^\nabla = f(u(t))+h(t)$. Since $f, h \geq 0$, we have $(\phi_p(u^\Delta))^\nabla \leq 0$ and $\phi_p(u^\Delta(t_2)) \leq \phi_p(u^\Delta(t_1))$ for any $t_1, t_2 \in [0,T]_{\mathbb{T}}$ with $t_1 \leq t_2$. It follows that $u^\Delta(t_2) \leq u^\Delta(t_1)$ for $t_1 \leq t_2$. Hence, $u^\Delta(t)$ is a decreasing function on $[0,T]_{\mathbb{T}}$. Then, u is concave. In order to apply Theorem 3 we define the cone

$$P=\{u \in E \mid u \text{ is nonnegative, increasing on } [0,T]_{\mathbb{T}}, \text{and concave on } E\}. \tag{49}$$

For $\xi \in (0,T/2)$ we also define the nonnegative continuous concave functional $\alpha : P \to \mathbb{R}_0^+$ by

$$\alpha(u)=\min_{t\in[\xi,T-\xi]_{\mathbb{T}}} u(t), \quad u \in P \tag{50}$$

and the operator $F : P \to E$ by

$$Fu(t)=\phi_q\left(\int_\eta^T (f(u(r))+h(r))\nabla r\right)+\int_0^t \phi_q\left(\int_s^T (f(u(r))+h(r))\nabla r\right)\Delta s. \tag{51}$$

It is easy to see that (46) has a solution $u=u(t)$ if and only if u is a fixed point of the operator F. For convenience, we introduce the following notation:

$$\begin{gathered}\gamma=(1+T)\phi_q(T),\\ A=\frac{a-\alpha\|h\|_\infty^{1/(p-1)}}{\alpha a}, \quad \text{where } \alpha=\phi_q\left(2^{p-2}\right)\phi_q(T)(T+1),\\ B=\phi_p(T-\eta).\end{gathered} \tag{52}$$

Theorem 13. *Suppose that hypotheses (A1) and (A2) are satisfied; there exist positive constants a, b, c, and d with*

$$\begin{aligned}0&<\gamma a\\ &=a_1<b<d-2^{p-2}(T-\xi)\phi_q(T-\xi)\left(\|h\|^{1/(p-1)}+bB\right)\\ &<d<\gamma c=c_1\end{aligned} \tag{53}$$

and, in addition to (A1) and (A2), that f satisfies

(A3) $\max_{0\leq u\leq a} f(u) \leq \phi_p(aA)$;

(A4) $\max_{0\leq u\leq c} f(u) \leq \phi_p(cA)$;

(A5) $\min_{b\leq u\leq d} f(u) \geq \phi_p(bB)$.

Then problem (46) has at least three positive solutions u_1, u_2, and u_3, verifying

$$\|u_1\|<a, \qquad b<\alpha(u_2), \qquad \|u_3\|>a, \qquad \alpha(u_3)<b. \tag{54}$$

Proof. As done for Theorem 8, the proof is divided in several steps. We first show that $F : P \to P$ is completely continuous. Indeed, F is obviously continuous. Let

$$U_\delta=\{u\in P \mid \|u\|\leq\delta\}. \tag{55}$$

It is easy to see that for $u \in U_\delta$ there exists a constant $c>0$ such that $|Fu(t)|\leq c$. On the other hand, let $t_1, t_2 \in (0,T)_{\mathbb{T}}$, $u \in U_\delta$. Then there exists a positive constant c such that

$$|Fu(t_2)-Fu(t_1)|\leq c|t_2-t_1|, \tag{56}$$

which converges uniformly to zero when $|t_2-t_1|$ tends to zero. Using the Arzela-Ascoli theorem we conclude that $F : P \to P$ is completely continuous.

We now show that

$$F\overline{P_{c_1}}\subset\overline{P_{c_1}}, \qquad F\overline{P_{a_1}}\subset\overline{P_{a_1}}. \tag{57}$$

For all $u\in\overline{P_{a_1}}$ we have $0\leq u\leq a_1$ and

$$\begin{aligned}\|F(u)\|&\leq\phi_q\left(\int_\eta^T\left((aA)^{p-1}+\|h\|_\infty\right)\nabla r\right)\\&\quad+\int_0^T\phi_q\left(\int_s^T\left((aA)^{p-1}+\|h\|_\infty\right)\nabla r\right)\Delta s\\&\leq\phi_q\left(\left((aA)^{p-1}+\|h\|_\infty\right)(T-\eta)\right)\\&\quad+\int_0^T\phi_q\left(\left((aA)^{p-1}+\|h\|_\infty\right)(T-s)\right)\Delta s\\&\leq\phi_q\left((aA)^{p-1}+\|h\|_\infty\right)\phi_q(T)\\&\quad+\phi_q\left((aA)^{p-1}+\|h\|_\infty\right)\int_0^T\phi_q(T-s)\Delta s\\&\leq\phi_q\left((aA)^{p-1}+\left(\|h\|_\infty^{1/(p-1)}\right)^{p-1}\right)\phi_q(T)\\&\quad+\phi_q\left((aA)^{p-1}+\left(\|h\|_\infty^{1/(p-1)}\right)^{p-1}\right)\phi_q(T)T\\&\leq\phi_q\left((aA)^{p-1}+\left(\|h\|_\infty^{1/(p-1)}\right)^{p-1}\right)\phi_q(T)(T+1).\end{aligned} \tag{58}$$

Using the elementary inequality

$$x^p+y^p\leq 2^{p-1}(x+y)^p \tag{59}$$

and the form of A, it follows that

$$\begin{aligned}\|F(u)\| &\leq \phi_q(T+1)\left(2^{p-2}\right)\left(aA+\|h\|_\infty^{1/(p-1)}\right)\\ &\leq \phi_q(T+1)\left(2^{p-2}\right)a=\gamma a=a_1.\end{aligned} \tag{60}$$

Then $F\overline{P_{a_1}}\subset\overline{P_{a_1}}$. In a similar way we prove that $F\overline{P_{c_1}}\subset\overline{P_{c_1}}$.

Our following step is to show that

$$\begin{gathered}\{u\in P(\alpha,b,d)\mid \alpha(u)>b\}\neq\emptyset,\\ \alpha(Fu)>b,\quad \text{if } u\in P(\alpha,b,d).\end{gathered} \tag{61}$$

The first point is obvious. Let us prove the second part of (61). For $u\in P(\alpha,b,d)$ we have $b\leq u\leq d$, if $t\in[\xi,T]_{\mathbb{T}}$. Then, using (A2) we have

$$\begin{aligned}\alpha(Fu)&=Fu(\xi)\\ &\geq\phi_q\left(\int_\eta^T f(u(r))\,\nabla r\right)\\ &\quad+\int_0^\xi\phi_q\left(\int_s^T f(u(r))\nabla r\right)\Delta s\\ &\geq\phi_q\left(\int_\eta^T f(u(r))\,\nabla r\right)\\ &\geq bB\phi_q(T-\xi)\\ &\geq b.\end{aligned} \tag{62}$$

Finally we prove that $\alpha(Fu)>b$ for all $u\in P(\alpha,b,c_1)$ and $\|Fu\|>d$:

$$\begin{aligned}\alpha(Fu)&=Fu(\xi)\\ &=\phi_q\left(\int_\eta^T(f(u(r))+h(r))\nabla r\right)\\ &\quad+\int_0^\xi\phi_q\left(\int_s^T(f(u(r))+h(r))\nabla r\right)\Delta s\\ &=\phi_q\left(\int_\eta^T(f(u(r))+h(r))\nabla r\right)\\ &\quad+\int_0^T\phi_q\left(\int_s^T(f(u(r))+h(r))\nabla r\right)\Delta s\\ &\quad-\int_\xi^T\phi_q\left(\int_s^T(f(u(r))+h(r))\nabla r\right)\Delta s\\ &\geq\|Fu\|-\int_\xi^T\phi_q\left(\int_s^T(f(u(r))+h(r))\nabla r\right)\Delta s\\ &\geq\|Fu\|-\int_\xi^T\phi_q\left(\int_\xi^T(f(u(r))+h(r))\nabla r\right)\Delta s\\ &\geq\|Fu\|-\int_\xi^T\phi_q\left((T-\xi)\left(\|h\|+\phi_p(bB)\right)\right)\Delta s\\ &\geq\|Fu\|-(T-\xi)\phi_q(T-\xi)\phi_q\left(\|h\|+\phi_p(bB)\right).\end{aligned} \tag{63}$$

Using again the elementary inequality $x^p+y^p\leq 2^{p-1}(x+y)^p$ we get that

$$\begin{aligned}\alpha(Fu)&\geq\|Fu\|-(T-\xi)\phi_q(T-\xi)\left(\|h\|^{1/(p-1)}+bB\right)\\ &\geq d-2^{p-2}(T-\xi)\phi_q(T-\xi)\left(\|h\|^{1/(p-1)}+bB\right)\\ &\geq b.\end{aligned} \tag{64}$$

By Theorem 3 there exist at least three positive solutions u_1, u_2, and u_3 to (46) satisfying $\|u_1\|<a$, $b<\alpha(u_2)$, $\|u_3\|>a$, and $\alpha(u_3)<b$. □

Example 14. Let $\mathbb{T}=\{1-(1/2)^{\mathbb{N}_0}\}\cup\{1\}$, where $\mathbb{N}_0$ denotes the set of all nonnegative integers. Consider the p-Laplacian dynamic equation

$$\left[\phi_p\left(u^\Delta(t)\right)\right]^\nabla+f(u(t))=0,\quad t\in[0,1]_{\mathbb{T}}, \tag{65}$$

satisfying the boundary conditions

$$u(1)-u\left(\frac{1}{2}\right)=0,\qquad u^\Delta(0)=0, \tag{66}$$

where $p=3/2$, $q=3$, $a(t)\equiv 1$, $h\equiv 0$, $T=1$, and

$$f(u)=\begin{cases}\dfrac{\sqrt{2}}{2}, & 0\leq u\leq\dfrac{1}{2},\\ 4\left(u-\dfrac{1}{2}\right)+\dfrac{\sqrt{2}}{2}, & \dfrac{1}{2}\leq u\leq\dfrac{3}{2},\\ 4+\dfrac{\sqrt{2}}{2}, & \dfrac{3}{2}\leq u\leq 25.\end{cases} \tag{67}$$

Choose $a = 1/2$, $b = 3/2$, $c = 25$, and $d = 3$. It is easy to see that $\gamma = 2$, $A = 1$, $B = \sqrt{2}/2$, $\alpha = 1$, and

$$\begin{aligned}
&\max\left\{f(u) : u \in \left[0, \frac{1}{2}\right]\right\} = \frac{\sqrt{2}}{2} \le (aA)^{p-1} = \sqrt{a} = \frac{\sqrt{2}}{2}, \\
&\max\{f(u) : u \in [0, 25]\} = 4 + \frac{\sqrt{2}}{2} \simeq 4,707 \\
&\le (cA)^{p-1} = c^{p-1} = \sqrt{c} = 5, \\
&\min\left\{f(u) : u \in \left[\frac{3}{2}, 3\right]\right\} = 4 + \frac{\sqrt{2}}{2} \simeq 4,707 \\
&\ge (bB)^{p-1} = \sqrt{bB} \simeq 1,02.
\end{aligned} \tag{68}$$

Therefore, by Theorem 13, problem (65)-(66) has at least three positive solutions.

3.3. A p-Laplacian Functional Dynamic Equation on Time Scales with Delay. Let $\mathbb{T}$ be a time scale with $0, T \in \mathbb{T}_\kappa^\kappa$, $-r \in \mathbb{T}$ with $-r \le 0 < T$. We are concerned in this section with the existence of positive solutions to the p-Laplacian dynamic equation

$$\begin{aligned}
&\left(\phi_p\left(u^\Delta(t)\right)\right)^\nabla + \lambda a(t) f(u(t), u(\omega(t))) = 0, \quad t \in (0, T)_\mathbb{T}, \\
&u(t) = \psi(t), \quad t \in [-r, 0]_\mathbb{T}, \\
&u(0) - B_0\left(u^\Delta(0)\right) = 0, \qquad u^\Delta(T) = 0,
\end{aligned} \tag{69}$$

where $\lambda > 0$. We define $X = \mathbb{C}_{ld}([0, T]_\mathbb{T}, \mathbb{R})$, which is a Banach space with the maximum norm $\|u\| = \max_{[0,T]_\mathbb{T}} |u(t)|$. We note that u is a solution to (69) if and only if

$$u(t) = \begin{cases} B_0\left(\phi_q\left(\displaystyle\int_0^T \lambda a(r) \times f(u(r), u(\omega(r))) \nabla r\right)\right) \\ \quad + \displaystyle\int_0^t \phi_q\left(\int_s^T \lambda a(r) \times f(u(r), u(\omega(r))) \nabla r\right) \Delta s & \text{if } t \in [0, T]_\mathbb{T}, \\ \psi(t) & \text{if } t \in [-r, 0]_\mathbb{T}. \end{cases} \tag{70}$$

Let

$$K = \{u \in X \mid u \text{ is nonnegative and concave on } E\}. \tag{71}$$

Clearly K is a cone in the Banach space X. For each $u \in X$, we extend u to $[-r, 0]_\mathbb{T}$ with $u(t) = \psi(t)$ for $t \in [-r, 0]_\mathbb{T}$. We also define the nonnegative continuous concave functional $\alpha : P \to \mathbb{R}_0^+$ by

$$\alpha(u) = \min_{t \in [\xi, T-\xi]_\mathbb{T}} u(t), \quad \xi \in \left(0, \frac{T}{2}\right), \quad \forall u \in K. \tag{72}$$

For $t \in [0, T]_\mathbb{T}$, define $Q : K \to X$ as

$$\begin{aligned}
Qu(t) = {} & B_0\left(\phi_q\left(\int_0^T \lambda a(r) f(u(r), u(\omega(r))) \nabla r\right)\right) \\
& + \int_0^t \phi_q\left(\int_s^T \lambda a(r) f(u(r), u(\omega(r))) \nabla r\right) \Delta s.
\end{aligned} \tag{73}$$

Lemma 15. *Let u_1 be a fixed point of Q in the cone K. Define*

$$u(t) = \begin{cases} u_1, & t \in [0, T]_\mathbb{T}, \\ \psi(t), & t \in [-r, 0]_\mathbb{T}. \end{cases} \tag{74}$$

It follows that (74) *is a positive solution to* (69) *satisfying*

$$\begin{aligned}
\|Qu\| \le {} & (T + \gamma) \lambda^{q-1} \\
& \times \phi_q\left(\int_0^T a(r) f(u(r), u(\omega(r))) \nabla r\right) \quad \textit{for } t \in [0, T]_\mathbb{T}.
\end{aligned} \tag{75}$$

Proof.

$$\begin{aligned}
\|Qu\| = {} & (Qu)(T) \\
= {} & B_0\left(\phi_q\left(\int_0^T \lambda a(r) f(u(r), u(\omega(r))) \nabla r\right)\right) \\
& + \int_0^T \phi_q\left(\int_s^T \lambda a(r) f(u(r), u(\omega(r))) \nabla r\right) \Delta s \\
\le {} & (T + \gamma) \lambda^{q-1} \phi_q\left(\int_0^T a(r) f(u(r), u(\omega(r))) \nabla r\right).
\end{aligned} \tag{76}$$

□

From (73) and (75) it follows that

(i) $Q(K) \subset K$;

(ii) $Q : K \to K$ is completely continuous;

(iii) $u(t) \ge \delta/(T + \gamma)\|u\|$, $t \in [0, T]_\mathbb{T}$.

Depending on the signature of the delay ω, we set the following two subsets of $[0, T]_\mathbb{T}$:

$$\begin{aligned}
Y_1 &:= \{t \in [0, T]_\mathbb{T} \mid \omega(t) < 0\}; \\
Y_2 &:= \{t \in [0, T]_\mathbb{T} \mid \omega(t) \ge 0\}.
\end{aligned} \tag{77}$$

In the remainder of this section, we suppose that Y_1 is nonempty and $\int_{Y_1} a(r) \nabla r > 0$. For convenience we also denote

$$l := \frac{\phi_p\left(\int_0^T a(r) \nabla r\right)}{\lambda^{q-1}(T + \gamma)}, \qquad m := \frac{\phi_p\left(\int_0^T a(r) \nabla r\right)}{\delta \lambda^{q-1}}. \tag{78}$$

Theorem 16. *Suppose that there exist positive constants a, b, c, and d such that* $0 < a < b < \delta d/(T + \gamma) < d < c$. *Assume that the following hypotheses (C1)–(C8) hold:*

(C1) $f : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *is continuous;*

(C2) *function* $a : (0,T)_{\mathbb{T}} \to \mathbb{R}_0^+$ *is left-dense continuous;*

(C3) $\psi : [-r,0]_{\mathbb{T}} \to \mathbb{R}_0^+$ *is continuous;*

(C4) $\omega : [0,T]_{\mathbb{T}} \to [-r,T]_{\mathbb{T}}$ *is continuous,* $\omega(t) \le t$ *for all* t;

(C5) $B_0 : \mathbb{R} \to \mathbb{R}$ *is continuous and there are* $0 < \delta \le \gamma$ *such that*

$$\delta s \le B_0(s) \le \gamma s \quad \text{for } s \in \mathbb{R}_0^+; \tag{79}$$

(C6) $\lim_{x\to 0^+} f(x,\psi(s))/x^{p-1} < l^{p-1}$, *uniformly in* $s \in [-r,0]_{\mathbb{T}}$;

(C7) $\lim_{x_1\to 0^+; x_2\to 0^+} f(x_1,x_2)/\max\{x_1^{p-1},x_2^{p-1}\} < l^{p-1}$;

(C8) $\lim_{x\to\infty} f(x,\psi(s))/x^{p-1} > m^{p-1}$, *uniformly in* $s \in [-r,0]_{\mathbb{T}}$.

Then, for each $0 < \lambda < \infty$ *the boundary value problem* (69) *has at least three positive solutions* u_1, u_2, *and* u_3 *verifying*

$$\|u_1\| < a, \qquad b < \alpha(u_2), \qquad \|u_3\| > a, \qquad \alpha(u_3) < b. \tag{80}$$

Proof. The proof passes by three lemmas.

Lemma 17. *The following relations hold:*

$$Q\overline{P_a} \subset \overline{P_a}, \qquad Q\overline{P_c} \subset \overline{P_c}. \tag{81}$$

Proof. Using condition (C6) for $\varepsilon_1 > 0$ such that $0 < x \le \varepsilon_1$, we have

$$f(x,\psi(s)) < (lx)^{p-1} \quad \text{for each } s \in [-r,0]_{\mathbb{T}}. \tag{82}$$

Applying condition (C7) we get

$$f(x_1,x_2) < \max\left\{x_1^{p-1},x_2^{p-1}\right\} l^{p-1} \tag{83}$$

for $\varepsilon_2 > 0$ such that $0 < x_1 \le \varepsilon_2$, $0 < x_2 \le \varepsilon_2$. Put $\varepsilon = \min\{\varepsilon_1,\varepsilon_2\}$. Then for $\|u\| \le a$ and from (75) we have

$$\begin{aligned}
|Qu| &\le \|Qu\| \\
&\le (T+\gamma)\lambda^{q-1}\phi_q\left(\int_0^T a(r) f(u(r),u(\omega(r)))\nabla r\right) \\
&= (T+\gamma)\lambda^{q-1}\left[\phi_q\left(\int_{Y_1} a(r) f(u(r),\psi(\omega(r)))\nabla r\right.\right. \\
&\qquad \left.\left. + \int_{Y_2} a(r) f(u(r),u(\omega(r)))\nabla r\right)\right] \\
&\le l(T+\gamma)\lambda^{q-1}\|u\|\phi_q\left(\int_0^T a(r)\nabla r\right) \\
&\le l(T+\gamma)\lambda^{q-1}a\phi_q\left(\int_0^T a(r)\nabla r\right) \\
&= a.
\end{aligned} \tag{84}$$

Then $Q\overline{P_a} \subset \overline{P_a}$. Similarly one can show that $Q\overline{P_c} \subset \overline{P_c}$. □

Lemma 18. *The set*

$$\{u \in P(\alpha,b,d) \mid \alpha(u) > b\} \tag{85}$$

is nonempty, and

$$\alpha(Qu) > b, \quad \text{if } u \in P(\alpha,b,d). \tag{86}$$

Proof. Applying hypothesis (C8) we have

$$f(u,\psi(s)) > \phi_p(mu) \quad \text{for each } s \in [-r,0]_{\mathbb{T}}. \tag{87}$$

We also have $u(t) \ge \delta/(T+\gamma)\|u\|$. Let $u \in P(\alpha,b,d)$. Then, $b \le u(t) \le d$. Hence,

$$\begin{aligned}
\alpha(Qu) &= (Qu)(T-\xi) \\
&\ge \frac{\delta}{T+\gamma}\|Qu\| \\
&\ge \delta\phi_q\left(\int_0^T \lambda a(r) f(u(r),u(\omega(r)))\nabla r\right) \\
&\ge \delta\lambda^{q-1}\phi_q\left(\int_{Y_1} a(r) f(u(r),\psi(\omega(r)))\nabla r\right. \\
&\qquad \left. + \int_{Y_2} a(r) f(u(r),\psi(\omega(r)))\nabla r\right) \\
&\ge \delta\lambda^{q-1}\phi_q\left(\int_{Y_1} a(r) f(u(r),\psi(\omega(r)))\nabla r\right) \\
&\ge m\delta\lambda^{q-1}\min_{t\in Y_1}\{u(t)\}\phi_q\left(\int_{Y_1} a(r)\nabla r\right) \\
&\ge bm\delta\lambda^{q-1}\phi_q\left(\int_{Y_1} a(r)\nabla r\right) \\
&\ge b.
\end{aligned} \tag{88}$$

□

Lemma 19. *For all* $u \in P(\alpha,b,c)$ *and* $\|Qu\| > d$ *one has* $\alpha(Qu) > b$.

Proof. Using the fact that $\delta \le T+\gamma$, we have

$$\begin{aligned}
\alpha(Qu) &= (Qu)(T-\xi) \\
&\ge \frac{\delta}{T+\gamma}\|Qu\| \\
&> \frac{\delta d}{T+\gamma} \\
&\ge b.
\end{aligned} \tag{89}$$

□

Applying the Leggett-Williams theorem (Theorem 3), the proof of Theorem 16 is complete. □

Example 20. Let $\mathbb{T} = [-3/4, -1/4] \cup \{0, 3/4\} \cup \{(1/2)^{\mathbb{N}_0}\}$, where $\mathbb{N}_0$ denotes the set of all nonnegative integers. Consider the following p-Laplacian functional dynamic equation on the time scale $\mathbb{T}$:

$$\left[\Phi_p\left(u^{\Delta}(t)\right)\right]^{\nabla} + (u_1 + u_2)^2 = 0, \quad t \in (0,1)_{\mathbb{T}},$$
$$\psi(t) \equiv 0, \quad t \in \left[-\frac{3}{4}, 0\right]_{\mathbb{T}}, \tag{90}$$
$$u(0) - B_0\left(u^{\Delta}\left(\frac{1}{4}\right)\right) = 0, \qquad u^{\Delta}(1) = 0,$$

where $T = 1$, $p = 3/2$, $q = 3$, $a(t) \equiv 1$, $B_0(s) = s$, $w(t) : [0,1]_{\mathbb{T}} \to [-3/4, 1]_{\mathbb{T}}$ with $w(t) = t - 3/4$, $r = 3/4$, $\eta = 1/4$, $l = 1/2$, $m = 1$, and $f(u, \psi(t)) = u^2$, $f(u_1, u_2) = (u_1 + u_2)^2$. We deduce that $Y_1 = [0, 3/4)_{\mathbb{T}}$, $Y_2 = [3/4, 1]_{\mathbb{T}}$. It is easy to see that hypotheses (C1)–(C5) are verified. On the other hand, notice that $\lim_{x \to 0^+} f(x, \psi(s))/x^{p-1} = 0 < l^{p-1}$ and $\lim_{x \to \infty} f(x, \psi(s))/(x^{p-1}) = +\infty > m^{p-1}$. Thus, hypotheses (C6)–(C8) are obviously satisfied. Then, by Theorem 16, problem (90) has at least three positive solutions of the form

$$u(t) = \begin{cases} u_i(t), & t \in [0,1]_{\mathbb{T}},\ i = 1,2,3, \\ \psi(t), & t \in \left[-\frac{3}{4}, 0\right]_{\mathbb{T}}. \end{cases} \tag{91}$$

Acknowledgments

This work was supported by FEDER funds through COMPETE—Operational Programme Factors of Competitiveness (Programa Operacional Factores de Competitividade)—by Portuguese funds through the Center for Research and Development in Mathematics and Applications (University of Aveiro), and the Portuguese Foundation for Science and Technology (FCT—Fundação para a Ciência e a Tecnologia), within project PEst-C/MAT/UI4106/2011 with COMPETE no. FCOMP-01-0124-FEDER-022690. The authors were also supported by the project New Explorations in Control Theory Through Advanced Research (NECTAR) cofinanced by FCT, Portugal and the Centre National de la Recherche Scientifique et Technique (CNRST), Morocco.

References

[1] S. Hilger, "Analysis on measure chains—a unified approach to continuous and discrete calculus," *Results in Mathematics*, vol. 18, no. 1-2, pp. 18–56, 1990.

[2] S. Hilger, "Differential and difference calculus—unified!," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 30, no. 5, pp. 2683–2694, 1997.

[3] R. Agarwal, M. Bohner, D. O'Regan, and A. Peterson, "Dynamic equations on time scales: a survey," *Journal of Computational and Applied Mathematics*, vol. 141, no. 1-2, pp. 1–26, 2002.

[4] E. Girejko, A. B. Malinowska, and D. F. M. Torres, "The contingent epiderivative and the calculus of variations on time scales," *Optimization*, vol. 61, no. 3, pp. 251–264, 2012.

[5] E. Girejko and D. F. M. Torres, "The existence of solutions for dynamic inclusions on time scales via duality," *Applied Mathematics Letters*, vol. 25, no. 11, pp. 1632–1637, 2012.

[6] N. Martins and D. F. M. Torres, "Generalizing the variational theory on time scales to include the delta indefinite integral," *Computers & Mathematics with Applications*, vol. 61, no. 9, pp. 2424–2435, 2011.

[7] S. Yu, H. Wu, and J. Chen, "Multiple periodic solutions of delayed predator-prey systems with type IV functional responses on time scales," *Discrete Dynamics in Nature and Society*, vol. 2012, Article ID 271672, 12 pages, 2012.

[8] R. P. Agarwal and D. O'Regan, "Nonlinear boundary value problems on time scales," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 44, no. 4, pp. 527–535, 2001.

[9] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales*, Birkhäuser, Boston, Mass, USA, 2001.

[10] M. Bohner and A. Peterson, *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston, Mass, USA, 2003.

[11] M. Hu and L. Wang, "Unique existence theorem of solution of almost periodic differential equations on time scales," *Discrete Dynamics in Nature and Society*, vol. 2012, Article ID 240735, 8 pages, 2012.

[12] K. R. Prasad, P. Murali, and S. N. Rao, "Existence of multiple positive solutions to three-point boundary value problems on time scales," *International Journal of Difference Equations*, vol. 4, no. 2, pp. 219–232, 2009.

[13] R. P. Agarwal, H. Lü, and D. O'Regan, "Eigenvalues and the one-dimensional p-Laplacian," *Journal of Mathematical Analysis and Applications*, vol. 266, no. 2, pp. 383–400, 2002.

[14] D. R. Anderson, "Solutions to second-order three-point problems on time scales," *Journal of Difference Equations and Applications*, vol. 8, no. 8, pp. 673–688, 2002.

[15] E. R. Kaufmann, "Positive solutions of a three-point boundary-value problem on a time scale," *Electronic Journal of Differential Equations*, vol. 2003, no. 82, pp. 1–11, 2003.

[16] D. Anderson, R. Avery, and J. Henderson, "Existence of solutions for a one dimensional p-Laplacian on time-scales," *Journal of Difference Equations and Applications*, vol. 10, no. 10, pp. 889–896, 2004.

[17] M. R. Sidi Ammi and D. F. M. Torres, "Existence of positive solutions for non local p-Laplacian thermistor problems on time scales," *JIPAM*, vol. 8, no. 3, article 69, p. 10, 2007.

[18] M. R. S. Ammi and D. F. M. Torres, "Existence of infinitely many solutions for a quasilinear elliptic problem on time scales," *International Journal of Pure and Applied Mathematics*, vol. 39, no. 2, pp. 239–248, 2007.

[19] D.-B. Wang, "Existence, multiplicity and infinite solvability of positive solutions for p-Laplacian dynamic equations on time scales," *Electronic Journal of Differential Equations*, vol. 2006, no. 96, pp. 1–10, 2006.

[20] M. A. Krasnosel'skiĭ, *Positive Solutions of Operator Equations*, Translated from the Russian by Richard E. Flaherty; edited by Leo F. Boron, P. Noordhoff Ltd. Groningen, Groningen, The Netherlands, 1964.

[21] D. Zhao, H. Wang, and W. Ge, "Existence of triple positive solutions to a class of p-Laplacian boundary value problems," *Journal of Mathematical Analysis and Applications*, vol. 328, no. 2, pp. 972–983, 2007.

[22] R. P. Agarwal and D. O'Regan, "Triple solutions to boundary value problems on time scales," *Applied Mathematics Letters*, vol. 13, no. 4, pp. 7–11, 2000.

[23] K. L. Boey and P. J. Y. Wong, "Existence of triple positive solutions of two-point right focal boundary value problems on time scales," *Computers & Mathematics with Applications*, vol. 50, no. 10–12, pp. 1603–1620, 2005.

[24] Y. Li, T. Zhang, and J. Suo, "Existence of three positive periodic solutions for differential systems with feedback controls on time scales," *Electronic Journal of Differential Equations*, vol. 2010, no. 118, pp. 1–16, 2010.

[25] A. K. Rao and S. N. Rao, "Existence of multiple positive solutions for even order Sturm-Liouville dynamic equations," *Applied Mathematical Sciences*, vol. 4, no. 1–4, pp. 31–40, 2010.

[26] Y. Zhang and S. Qiao, "Existence of positive solutions for m-point boundary value problems on time scales," *Discrete Dynamics in Nature and Society*, vol. 2009, Article ID 189768, 12 pages, 2009.

[27] X. Zhao and W. Ge, "Multiple positive solutions for time scale boundary value problems on infinite intervals," *Acta Applicandae Mathematicae*, vol. 106, no. 2, pp. 265–273, 2009.

[28] X. Zhao and W. Ge, "Erratum to 'Multiple positive solutions for time scale boundary value problems on infinite intervals,'" *Acta Applicandae Mathematicae*, vol. 109, no. 3, article 1179, 2010.

[29] M. R. Sidi Ammi and D. F. M. Torres, "Numerical analysis of a nonlocal parabolic problem resulting from thermistor problem," *Mathematics and Computers in Simulation*, vol. 77, no. 2-3, pp. 291–300, 2008.

[30] S. N. Antontsev and M. Chipot, "The thermistor problem: existence, smoothness uniqueness, blowup," *SIAM Journal on Mathematical Analysis*, vol. 25, no. 4, pp. 1128–1156, 1994.

[31] G. Cimatti, "Existence of weak solutions for the nonstationary problem of the joule heating of a conductor," *Annali di Matematica Pura ed Applicata. Serie Quarta*, vol. 162, pp. 33–42, 1992.

[32] A. C. Fowler, I. Frigaard, and S. D. Howison, "Temperature surges in current-limiting circuit devices," *SIAM Journal on Applied Mathematics*, vol. 52, no. 4, pp. 998–1011, 1992.

[33] P. Shi, M. Shillor, and X. Xu, "Existence of a solution to the Stefan problem with Joule's heating," *Journal of Differential Equations*, vol. 105, no. 2, pp. 239–263, 1993.

[34] X. Xu, "Local regularity theorems for the stationary thermistor problem," *Proceedings of the Royal Society of Edinburgh. Section A. Mathematics*, vol. 134, no. 4, pp. 773–782, 2004.

[35] A. Ambrosetti and P. H. Rabinowitz, "Dual variational methods in critical point theory and applications," *Journal of Functional Analysis*, vol. 14, pp. 349–381, 1973.

[36] A. Castro and A. Kurepa, "Energy analysis of a nonlinear singular differential equation and applications," *Revista Colombiana de Matemáticas*, vol. 21, no. 1, pp. 155–166, 1987.

[37] A. El Hachimi and F. de Thelin, "Infinitely many radially symmetric solutions for a quasilinear elliptic problem in a ball," *Journal of Differential Equations*, vol. 128, no. 1, pp. 78–102, 1996.

[38] A. El Hachimi and J.-P. Gossez, "A note on a nonresonance condition for a quasilinear elliptic problem," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 22, no. 2, pp. 229–236, 1994.

[39] C. Song, "Existence of solutions for p-Laplacian functional dynamic equations on time scales," *Electronic Journal of Differential Equations*, vol. 2006, no. 113, pp. 1–8, 2006.

[40] R. P. Agarwal and M. Bohner, "Basic calculus on time scales and some of its applications," *Results in Mathematics*, vol. 35, no. 1-2, pp. 3–22, 1999.

[41] R. P. Agarwal, M. Bohner, and P. J. Y. Wong, "Sturm-Liouville eigenvalue problems on time scales," *Applied Mathematics and Computation*, vol. 99, no. 2-3, pp. 153–166, 1999.

[42] D. J. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, vol. 5 of *Notes and Reports in Mathematics in Science and Engineering*, Academic Press, Boston, Mass, USA, 1988.

[43] A. B. Malinowska and D. F. M. Torres, "On the diamond-alpha Riemann integral and mean value theorems on time scales," *Dynamic Systems and Applications*, vol. 18, no. 3-4, pp. 469–481, 2009.

[44] F. M. Atici and G. Sh. Guseinov, "On Green's functions and positive solutions for boundary value problems on time scales," *Journal of Computational and Applied Mathematics*, vol. 141, no. 1-2, pp. 75–99, 2002.

[45] R. W. Leggett and L. R. Williams, "Multiple positive fixed points of nonlinear operators on ordered Banach spaces," *Indiana University Mathematics Journal*, vol. 28, no. 4, pp. 673–688, 1979.