On the Probabilistic Behaviour of Multivariate Lacunary Systems

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Discrepancy and Uniform Distribution

A sequence of vectors $(x_n)_{n\geq 1} = (x_{n,1}, \ldots, x_{n,d})_{n\geq 1}$ of real numbers in $[0,1)^d$ is called *uniformly distributed modulo one* if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathbf{1}_{\mathcal{A}}(x_n) = \lambda(\mathcal{A}) \tag{1.0.1}$$

for any axis-parallel box $\mathcal{A} \subset [0,1)^d$ where $\mathbf{1}_{\mathcal{A}}$ denotes the indicator function on the set $\mathcal{A}$ and λ denotes the Lebesgue-measure on $[0,1)^d$. The discrepancy resp. the star discrepancy of the first N elements of $(x_n)_{n\geq 1}$ is defined by

$$\begin{aligned} D_N(x_1, \ldots, x_N) &= \sup_{\mathcal{A}\in\mathcal{B}} \left| \frac{1}{N} \sum_{n=1}^{N} \mathbf{1}_{\mathcal{A}}(x_n) - \lambda(\mathcal{A}) \right|, \\ D_N^*(x_1, \ldots, x_N) &= \sup_{\mathcal{A}\in\mathcal{B}^*} \left| \frac{1}{N} \sum_{n=1}^{N} \mathbf{1}_{\mathcal{A}}(x_n) - \lambda(\mathcal{A}) \right|, \end{aligned} \tag{1.0.2}$$

where $\mathcal{B}$ denotes the set of all axis-parallel boxes $\mathcal{A} = \prod_{i=1}^{d} [\alpha_i, \beta_i) \subset [0,1)^d$ and furthermore $\mathcal{B}^*$ denotes the set of all axis-parallel boxes $\mathcal{A} = \prod_{i=1}^{d} [0, \beta_i) \subset [0,1)^d$ with one corner in 0. It is well-known that (1.0.1) is equivalent to $D_N(x_1, \ldots, x_N) \to 0$ resp. $D_N^*(x_1, \ldots, x_N) \to 0$ for $N \to \infty$. By a classical result of Weyl [57] it is known that for any increasing sequence $(M_n)_{n\geq 1}$ of positive integers the sequence $(\langle M_n x\rangle)_{n\geq 1}$, where $\langle\cdot\rangle$ denotes the fractional part, is uniformly distributed modulo one for almost all $x \in [0,1)$. This result naturally extends to the multidimensional case. Sequences with vanishing star-discrepancy have applications in the theory of numerical integration. The connection is established by the Koksma-Hlawka inequality (see [23]) which states that for any sequence of vectors $(x_n)_{n\geq 1} \subset [0,1)^d$ we have

$$\left| \frac{1}{N} \sum_{n=1}^{N} f(x_n) - \int_{[0,1)^d} f(x)\, dx \right| \leq D_N^*(x_1, \ldots, x_N) \cdot V_{HK}(f) \tag{1.0.3}$$

for any function f on $[0,1)^d$ where V_{HK} denotes the total variation in the sense of Hardy and Krause. Thus the integral can be approximated by the mean of the values which some points have under f where the approximation error is given by the total variation of f and the star-discrepancy of the points. Although (1.0.3) is interesting form a theoretical point of view, it is of little use in practice. In general the total variation is more difficult to compute than the integral. But nevertheless, it becomes evident that sequences of points with low discrepancy give small approximation errors. Therefore we are not only interested in sequences such that the star-discrepancy tends to 0, but also in the speed of convergence. For $d = 1$ Erdős and Koksma [25] and independently Cassels [16] showed

$$D_N(M_1x, \ldots, M_nx) = \mathcal{O}\left(\frac{(\log(N))^{5/2+\varepsilon}}{N^{1/2}}\right) \quad \text{a.e.}$$

for any sequence $(M_n)_{n\geq 1}$ of distinct integers and any $\varepsilon > 0$ where for simplicity we write $D_N(M_1x, \ldots, M_nx) = D_N(\langle M_1x\rangle, \ldots, \langle M_nx\rangle)$. Later Baker [9] improved the result by replacing $5/2$ by $3/2$. By constructing a particular sequence Berkes and Philipp [13] gave the lower bound

$$D_N(M_1x, \ldots, M_nx) = \Omega\left(\frac{(\log(N))^{1/2}}{N^{1/2}}\right) \quad \text{a.e.}$$

Lacunary sequences

If the sequence $(M_n)_{n\geq 1}$ is growing fast enough, better results can be achieved. Thus we now introduce lacunary sequences, i.e. sequences which grow very rapidly. We already give the definition for general dimension $d \geq 1$. Let $(M_n)_{n\geq 1}$ be a sequence of non-singular integer-valued $d \times d$-matrices satisfying a Hadamard gap type condition of the form

$$||M_{n+k}^T j||_\infty \geq q^k ||M_n^T||_\infty \tag{1.0.4}$$

for all $j \in \mathbb{Z}^d \backslash \{0\}$, $n \in \mathbb{N}$, $k \geq \log_q(||j||_\infty)$ and some absolute constant $q > 1$. Here A^T denotes the transpose of a matrix A. Since this extends the definition of lacunary sequences for $d = 1$ to the multivariate case we call this system a multivariate lacunary sequence satisfying a Hadamard gap condition. Now let $(M_n)_{n\geq 1}$ be a sequence of $d \times d$-matrices such that $M_{n,i,i'} = 0$ for all $n \in \mathbb{N}$ and $i, i' \in \{1, \ldots, d\}$ with $i \neq i'$. If any sequence $(M_{n,i,i})_{n\geq 1}$ for $i \in \{1, \ldots, d\}$ is an increasing sequence of integers and if

$$||M_{n+1}^T||_\infty \geq q||M_n^T||_\infty \tag{1.0.5}$$

for all $n \in \mathbb{N}$ and some absolute constant $q > 1$ we call the sequence $(M_n)_{n\geq 1}$ a lacunary sequence satisfying a weak Hadamard gap condition. The first function system related to lacunary sequences which was investigated is the system of Rademacher functions which are defined by

$$r_n(x) = \operatorname{sgn}(\sin(2^n \pi x))$$

for all $n \in \mathbb{N}$. Khintchine [40] showed the Law of the Iterated Logarithm for Rademacher functions, i.e.

$$\limsup_{N\to\infty} \frac{\sum_{n=1}^N r_n(x)}{\sqrt{2N\log(\log(N))}} = 1 \quad \text{a.e.}$$

With $f = r_1$ and $M_n = 2^{n-1}$ for all $n \in \mathbb{N}$ we have $f(M_n x) = r_n(x)$ and thus Rademacher functions form a lacunary function system. Since their behaviour is similar to that of independent, identically distributed random variables, it is reasonable to study systems $(f(M_n x))_{n\geq 1}$ for general one-periodic functions f of mean zero. Of particular interest are functions of the form $\mathbf{1}_{\mathcal{A}}(\langle\cdot\rangle) - \lambda(\mathcal{A})$ for some box $\mathcal{A} \subset [0,1)^d$ because they are used to define (1.0.2). For $d = 1$ and some sequence $(M_n)_{n\geq 1}$ satisfying (1.0.4) Salem and Zygmund [49] proved that for any sequence of integers $(a_n)_{n\geq 1}$ with

$$a_N = o(A_N) \quad \text{for} \quad A_N = \frac{1}{2}\left(\sum_{n=1}^{N} a_n^2\right)^{1/2}$$

we have

$$\lim_{N\to\infty} \mathbb{P}\left(\frac{1}{A_N}\sum_{n=1}^{N} a_n \cos(2\pi M_n x) \leq t\right) = \Phi(t) \tag{1.0.6}$$

where $\mathbb{P}$ denotes the probability measure induced by the Lebesgue measure on $[0,1)^d$ and Φ denotes the standard normal distribution, i.e. for all $t \in \mathbb{R}$ we have

$$\Phi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-\frac{1}{2}y^2}\, dy.$$

Furthermore Weiss [56] (see also Salem and Zygmund [50], Erdős and Gál [24]) showed that

$$\limsup_{N\to\infty} \frac{\sum_{n=1}^{N} a_n \cos(2\pi M_n x)}{\sqrt{2A_N^2 \log\log(N)}} = 1 \quad a.e. \tag{1.0.7}$$

under the condition

$$a_N = o\left(\frac{A_N}{\sqrt{\log(\log(A_N))}}\right).$$

With coefficients satisfying $a_N = o(A_N^{1-\delta})$ for some $\delta > 0$ Philipp and Stout [47] showed that there exists a Brownian Motion $\{W(t) : t \geq 0\}$ such that

$$\sum_{n=1}^{N} a_n \cos(2\pi M_n x) = W(A_N) + \mathcal{O}(A_N^{1/2-\varrho}) \quad \text{a.e.}$$

for some $\varrho > 0$. Therefore for lacunary $(M_n)_{n\geq 1}$ the sequence $(a_n \cos(2\pi M_n x))_{n\geq 1}$ shows a behaviour typical for independent, identically distributed random variables. One could ask whether this holds for other periodic functions as well. The answer is negative in general. By a result of Erdős and Fortet (see [39]) for $f(x) = \cos(2\pi x) + \cos(4\pi x)$ and $M_n = 2^n - 1$ we have

$$\lim_{N\to\infty} \mathbb{P}\left(\frac{1}{\sqrt{N}}\sum_{n=1}^{N} f(M_n x) \leq t\right) = \frac{1}{\sqrt{\pi}} \int_0^1 \int_{-\infty}^{t|\cos(\pi s)|/2} e^{-u^2}\, du ds \tag{1.0.8}$$

and

$$\limsup_{N\to\infty} \frac{\sum_{n=1}^{N} f(M_n x)}{\sqrt{N\log(\log(N))}} = 2\cos(\pi x) \quad a.e. \tag{1.0.9}$$

Thus neither the Central Limit Theorem nor the Law of the Iterated Logarithm is satisfied. This result was later generalized by Conze and Le Borgne [18] (see also [4] for further information). On the other hand Kac [38] showed that any one-periodic function $f : \mathbb{R} \to \mathbb{R}$ of mean zero which is of bounded variation on $[0, 1)$ or Lipschitz-continuous satisfies

$$\lim_{N\to\infty} \mathbb{P}\left(\frac{1}{\sqrt{N}}\sum_{n=1}^{N} f(2^n x) \le t\sigma\right) = \Phi(t) \tag{1.0.10}$$

if

$$\sigma^2 = \mathbb{E}[f] + 2\sum_{n=1}^{\infty} \mathbb{E}[f(x)f(2^n x)] \neq 0. \tag{1.0.11}$$

Furthermore Maruyama [43] and Izumi [37] proved

$$\limsup_{N\to\infty} \frac{\sum_{n=1}^{N} f(2^n x)}{\sqrt{2N\log(\log(N))}} = \sigma \quad \text{a.e.} \tag{1.0.12}$$

This illustrates that the behaviour of $(f(M_n x))_{n\geq 1}$ does not only depend on the speed of growth of $(M_n)_{n\geq 1}$ but also on number theoretic properties of the sequence $(M_n)_{n\geq 1}$. Later on the Central Limit Theorem was shown for more general lacunary sequences. By a result of Gaposhkin [30]

$$\lim_{N\to\infty} \mathbb{P}\left(\sum_{n=1}^{N} f(M_n x) \le t\sigma_N\right) = \Phi(t) \tag{1.0.13}$$

holds for sequences $(M_n)_{n\geq 1}$ satisfying

$$\sigma_N^2 = \int_0^1 \left(\sum_{i=1}^{N} f(M_n x)\right)^2 dx \geq CN, \tag{1.0.14}$$

for an absolute constant $C > 0$ and one of the following conditions

- $\frac{M_{n+1}}{M_n} \in \mathbb{N}$, for all $n \in \mathbb{N}$,
- $\lim_{n\to\infty} \frac{M_{n+1}}{M_n} = \theta$, such that θ^r irrational for all $r \in \mathbb{N}$.

Takahashi [53] showed (1.0.13) for $M_{n+1}/M_n \to \infty$ and α-Lipschitz-continuous functions. The connection between the Central Limit Theorem and the number of solutions of certain Diophantine equations is due to Gaposhkin [31]. The Central Limit Theorem holds for lacunary sequences $(M_n)_{n\geq 1}$ satisfying (1.0.14), if for any fixed j, j', ν the number of solutions of the Diophantine equation

$$M_n j \pm M_{n'} j' = \nu \tag{1.0.15}$$

is bounded by an absolute constant $C_{j,j'} > 0$ which is independent of ν. Observe that "nice" periodic functions can be approximated by trigonometric polynomials very well. Thus because of the product-to-sum identities of trigonometric functions the behaviour of the moments of $\sum f(M_n x)$ depends on the number of solutions of Diophantine equations of certain length. Recently Aistleitner and Berkes [2] improved this result: For a lacunary sequence $(M_n)_{n\geq 1}$ satisfying the Hadamard gap condition set

$$\begin{aligned} L(N,G,\nu) &= |\{1 \leq |j|, |j'| \leq G, 1 \leq n, n' \leq N : M_n j \pm M_{n'} j' = \nu\}|, \\ L^*(N,G,\nu) &= |\{1 \leq |j|, |j'| \leq G, 1 \leq n, n' \leq N, n \neq n' : M_n \pm j M_{n'} j' = \nu\}|, \\ L(N,G) &= \sup_{\nu \neq 0} L(N,G,\nu) \end{aligned} \tag{1.0.16}$$

for any $G \geq 1$, $\nu \in \mathbb{Z}$. Let $f : \mathbb{R} \to \mathbb{R}$ be some function of finite total variation which is one-periodic and satisfies $\mathbb{E}[f] = 0$ as well as (1.0.14) for some lacunary sequence satisfying the Hadamard gap condition (1.0.4). Aistleitner and Berkes showed that if $L(N,G) = o(N)$ for $N \to \infty$ and any fixed $G \geq 1$ then (1.0.13) holds.

Law of the Iterated Logarithm for the discrepancy of lacunary point sets

The Law of the Iterated Logarithm for the discrepancy of an one-dimensional lacunary point set was shown by Philipp [45]. He proved

$$\frac{1}{4\sqrt{2}} \leq \limsup_{N\to\infty} \frac{N D_N(M_1 x, \ldots, M_N x)}{\sqrt{2N \log(\log(N))}} \leq C \quad \text{a.e.}$$

where the constant $C > 0$ depends on q only. This corresponds to the Chung-Smirnov Law of the Iterated Logarithm, that is

$$\limsup_{N\to\infty} \frac{N D_N(\xi_1, \ldots, \xi_N)}{\sqrt{2N \log(\log(N))}} = \frac{1}{2} \quad \text{a.s.} \tag{1.0.17}$$

for any sequence of independent, identically distributed non-degenerate random variables $(\xi_n)_{n\geq 1}$ in $[0,1)$ with $\mathbb{E}[\xi_1] = 0$ and $\mathbb{E}[\xi_1^2] = 1$. For sequences of type $(M_n)_{n\geq 1} = (\theta^n)_{n\geq 1}$ for $\theta > 0$ the precise value of the Law of the Iterated Logarithm was determined by Fukuyama [28], i.e. for a.e. x we have

$$\limsup_{N\to\infty} \frac{N D_N(\theta^1 x, \ldots, \theta^N x)}{\sqrt{2N \log(\log(N))}} = \begin{cases} \sqrt{42}/9, & \text{if } \theta = 2, \\ \frac{\sqrt{(\theta+1)\theta(\theta-2)}}{2\sqrt{(\theta-1)^3}}, & \text{if } \theta \geq 4 \text{ is an even integer}, \\ \frac{\sqrt{(\theta+1)}}{2\sqrt{\theta-1}}, & \text{if } \theta \geq 3 \text{ is an odd integer}, \\ 1/2, & \text{if } \theta^r \notin \mathbb{Q} \text{ for all } r \in \mathbb{N}. \end{cases}$$

Therefore the probabilistic analogy is not complete. The precise value depends sensitively on number theoretic properties of the sequence $(M_n)_{n\geq 1}$, mainly on the number of non-trivial solutions of the Diophantine equations $M_n^T j \pm M_{n'}^T j' = 0$.

Aistleitner [1] used the method applied in [2] to prove the Law of the Iterated Logarithm for function systems $(f(M_n x))_{n\geq 1}$ as well as for the discrepancy $D_N(M_1 x, \ldots, M_N x)$ for point sets defined by a lacunary sequence $(M_n)_{n\geq 1}$ satisfying the Hadamard gap condition (1.0.4) under the condition $\max(L(N,G), L^*(N,G,0)) = \mathcal{O}(N/(\log(N))^{1+\varepsilon})$. Later Aistleitner, Fukuyama and Furuya [5] improved this result by proving sufficiency of $L(N,G) = \mathcal{O}(N/(\log(N))^{1+\varepsilon})$ for the Law of the Iterated Logarithm for lacunary function systems and $L^*(N,G,0) = o(N)$ in addition to the former condition for the Law of the Iterated Logarithm for discrepancy of lacunary point sets.

Functions in several variables

The Central Limit Theorem for lacunary sequences $(M_n)_{n\geq 1}$ of $d\times d$-matrices satisfying (1.0.4) was proved by Conze, Le Borgne and Roger [19]. There it was shown that the Central Limit Theorem holds if the sequence is satisfying a strong number theoretic condition, i.e. there is an absolute constant C such that for any integers G and N the following condition holds: For $2s$ integers $1 \leq n_1 \leq n_1' < n_2 \leq n_2' < \cdots < n_s \leq n_s' \leq N$ with $n_{k+1} \geq n_k' + C\log_q(G)$ for $k \in \{1, \ldots, s-1\}$ and vectors $j_1, j_1', \ldots, j_s, j_s'$ with $||j_k||_\infty, ||j_k'||_\infty \leq G$ for $k \in \{1, \ldots, s\}$ we have

$$M_{n_s}^T j_s + M_{n_s'}^T j_s' \neq 0 \quad \Longrightarrow \quad \sum_{k=1}^{s} M_{n_k}^T j_k + M_{n_k'}^T j_k' \neq 0.$$

Such condition for example is satisfied in the product case, i.e. there exists a sequence of matrices $(A_n)_{n\geq 1}$ with $M_n^T = A_1^T \cdots A_n^T$ for all $n \in \mathbb{N}$. Although it was stated that the Central Limit Theorem is satisfied with a rate of convergence of order $\mathcal{O}((\log(N))^{2/5} N^{-1/20})$ with an implied constant independent of d, it was actually only proved for some constant depending on d since it was only shown that

$$\int_{[0,1)} \left(\sum_{n=1}^{N} f(M_n x) \right)^2 dx \leq CN$$

for some constant $C > 0$ depending on d.

Sublacunary sequences

The probabilistic behaviour becomes more complicated in the case of sequences $(M_n)_{n\geq 1}$ with subexponential growth, i.e. $||M_{n+1}^T||_\infty / ||M_n^T||_\infty \to 1$. Although there are such sequences which show a similar probabilistic behaviour, there are others for which the Central Limit Theorem or the Law of the Iterated Logarithm is not satisfied even for $f(x) = \cos(2\pi x)$. Bobkov and Götze [14] proved that any orthonormal system $(X_n)_{n\geq 1}$ where $1/\sqrt{N} \cdot \sum_{n=1}^{N} X_{M_n}$ for some strictly increasing sequence of integers $(M_n)_{n\geq 1}$ converges weakly to some random variable ξ satisfies $\mathbb{E}[\xi^2] \leq 1 - \lim_{n\to\infty} n/M_n$. Thus for systems like $X_n = \sqrt{2} \cdot \cos(2\pi M_n x)$ with some sublacunary integer-valued sequence

$(M_n)_{n\geq 1}$ the Central Limit Theorem does not hold in general. For further information on the behaviour of sublacunary systems see e.g. [11].
One of the most interesting examples of sub-lacunary sequences which show the behaviour of independent random variables are so-called Hardy-Littlewood-Pólya-sequences, i.e. sequences which consist of elements of a semigroup generated by a finite number of coprime integers sorted in increasing order. Philipp [46] showed the almost sure invariance principle for $(\cos(2\pi M_n x))_{n\geq 1}$ for some Hardy-Littlewood-Pólya sequence $(M_n)_{n\geq 1}$ and furthermore the Law of the Iterated Logarithm for the discrepancy of $(M_n)_{n\geq 1}$. Later on Fukuyama and Petit [29] showed the Central Limit Theorem for this kind of sequences. This result was later extended to the multidimensional case. Let $\mathcal{S}$ be an abelian semigroup of endomorphisms of the d-dimensional torus with finitely many generators $\mathcal{A}_1, \dots, \mathcal{A}_s$. For suitable periodic functions f of mean zero Cohen and Conze [17] and Levin [42] proved

$$\lim_{\min(N_1,\dots,N_d)\to\infty} \mathbb{P}\left(\sum_{n_1=1}^{N_1}\cdots\sum_{n_d=1}^{N_d} f\left(\mathcal{A}_1^{N_1}\cdots\mathcal{A}_d^{N_d}x\right) \leq t\sigma_{N_1,\dots,N_d}\right) = \Phi(t) \qquad (1.0.18)$$

where

$$\sigma^2_{N_1,\dots,N_d} = \int_{[0,1)^2}\left(\sum_{n_1=1}^{N_1}\cdots\sum_{n_d=1}^{N_d} f\left(\mathcal{A}_1^{N_1}\cdots\mathcal{A}_d^{N_d}x\right)\right)^2 dx.$$

Results of this thesis

The methods used in early results in this area are based on substantial use of Fourier analysis such as bounding the size of Fourier coefficients, the tails of Fourier series etc. In [2] a new method effectively reducing the use of Fourier analysis was introduced which was used in [1] resp. [5] to show the Law of the Iterated Logarithm.
In chapter 3 we adopt this method to prove the Central Limit Theorem and the Law of the Iterated Logarithm for lacunary sequences $(M_n)_{n\geq 1}$ satisfying a Hadamard gap condition (1.0.4) under some weak conditions on the number of solutions of the Diophantine equation $M_n^T j \pm M_{n'}^T j' = \nu$. We extend (1.0.16) to the multivariate case by setting

$$\begin{aligned} L(N,G,\nu) &= |\{1 \leq n,n' \leq N : \\ &\quad \exists j,j' \in \mathbb{Z}^d, 1 \leq ||j||_\infty|, ||j'||_\infty \leq G, M_n^T j \pm M_{n'}^T j' = \nu\}|, \\ L^*(N,G,\nu) &= |\{1 \leq n,n' \leq N, n \neq n' : \\ &\quad \exists j,j' \in \mathbb{Z}^d, 1 \leq ||j||_\infty|, ||j'||_\infty \leq G, M_n^T j \pm M_{n'}^T j' = \nu\}|, \\ L(N,G) &= \sup_{\nu\neq 0} L(N,G,\nu). \end{aligned} \qquad (1.0.19)$$

and for all $N \geq 1$ we assume

$$\sigma_N^2 := \int_{[0,1)^d}\left(\sum_{n=1}^{N} f(M_n x)\right)^2 dx \geq C \cdot N$$

for some absolute constant $C > 0$. We prove that for $L(N, G) = o(N)$ for all $G \geq 1$ we have

$$\lim_{N\to\infty} \left| \mathbb{P}\left(\sum_{n=1}^{N} f(M_n x) \leq t\sigma_N \right) - \Phi(t) \right| = 0.$$

If furthermore there exists some $0 < \beta < 1$ such that $L(N, G_N) = o(N^{\beta})$ for any sequences $(G_N)_{N\geq 1}$ we show an upper bound on the rate of convergence. Moreover if $\Sigma_{f,M_n} = \lim_{N\to\infty} \sigma_N^2 > 0$ and $L^*(N, G) = L(N, G) = \mathcal{O}(N/(\log N)^{1+\varepsilon})$ for some $\varepsilon > 0$ we prove the Law of the Iterated Logarithm, i.e.

$$\limsup_{N\to\infty} \frac{|\sum_{n=1}^{N} f(M_n x)|}{\sqrt{2N \log(\log(N))}} = \sqrt{\Sigma_{f,M_n}} \quad \text{a.e.}$$

If the sequence $(M_n)_{n\geq 1}$ furthermore satisfies $L(N, G, 0) = o(N)$ we obtain the Law of the Iterated Logarithm for the star-discrepancy, i.e.

$$\limsup_{N\to\infty} \frac{N D_N^*(M_1 x, \ldots, M_n x)}{\sqrt{2N \log(\log(N))}} = \frac{1}{2} \quad \text{a.e.}$$

We also prove similar results for lacunary sequences satisfying only a weak Hadamard gap condition (1.0.5).
The main idea in the proof of the Central Limit Theorem is to apply a Theorem due to Heyde and Brown [36] which ensures the Central Limit Theorem for martingale differences sequences satisfying certain moment conditions resp. a consequence of Strassen's almost sure invariance principle [52] for which we get the Law of the Iterated Logarithm under similar moments conditions. The elements of the martingale differences are defined by sums of the form $\sum_{n\in\Delta_k} \varphi_n(x)$ where $\varphi_n(x)$ is a piecewise constant function approximating $f(M_n x)$. The blocks Δ_k form a sequence of growing blocks which decomposes the set of natural numbers except for small gaps between consecutive blocks. The filtration is defined by a sequence of σ-fields which are generated by a decomposition of $[0,1)^d$ into "dyadic" blocks, i.e. their side lengths are negative powers of 2 where the exponents depend on the magnitude of the "frequencies" M_n in the corresponding block in a certain manner. The Central Limit Theorem is ensured by a Berry-Esseen type inequality which gives an upper bound on the mutual distance between the distribution function of the normalized sums of the martingales and the distribution function of a standard normal distributed random variable which only depends on second and fourth moments conditions. In fact only for an upper bound of the conditional variances a condition on the number of solutions of the Diophantine equations are necessary since all other moments for which we need upper bounds can be estimated in a different way.

Double infinite matrices and the inverse of the discrepancy

Low-discrepancy sequences

Levin proved the Central Limit Theorem for the discrepancy of particular sequences with small discrepancy which are constructed by ergodic transformations of $[0,1)^d$ or sequences obtained from lattices (see [42] and the references therein for further information). One class of this sequences which are obtained via ergodic transformations is the class of Halton sequences which we introduce now. Halton sequences extend the definition of Van der Corput sequences to the multidimensional case. For an integer $d \geq 1$ let $(p_i)_{1 \leq i \leq d}$ be a system of d pairwise coprime integers. Then for any integer n and $i \in \{1, \ldots, d\}$ let the p_i-adic decomposition of n be given by

$$n = \sum_{j=0}^{\infty} \alpha(j,i) p_i^j$$

with $\alpha(j,i) \in \{0, \ldots, p_i - 1\}$ for all $j \geq 0$ where $|\{j \in \mathbb{N} \cup \{0\} : \alpha(j,i) \neq 0\}| < \infty$ for all $n \in \mathbb{N}$ and $i \in \{1, \ldots, d\}$. For $n \in \mathbb{N}$ and $i \in \{1, \ldots, d\}$ define

$$x_{n,i} = \sum_{j=0}^{\infty} \alpha(j,i) p_i^{-j-1}.$$

The sequence $(x_n)_{n \geq 1} = (x_{n,1}, \ldots, x_{n,d})_{n \geq 1}$ is called a Halton sequence in base $(p_1, \ldots, p_d)$. The discrepancy of a Halton sequence satisfies

$$D_N(x_1, \ldots, x_N) \leq C_d \frac{\log(N)^d}{N} \tag{1.0.20}$$

with some constant $C_d > 0$ which depends on d (see [33]). Observe that the one-dimensional projection can be represented as the orbit of a von Neumann-Kakutani transformation. By using a randomly chosen starting point for this transformation Wang and Hickernell [55] introduced the so-called randomized Halton sequences. Sequences with a discrepancy satisfying (1.0.20) are called low-discrepancy sequences. Numerical integration using deterministic low-discrepancy sequences is called Quasi-Monte Carlo (QMC) integration in contrast to classical Monte Carlo integration which uses independent randomly chosen points. Many examples of deterministic low-discrepancy sequences can be found in the books of Dick and Pillichshammer [21] and also Niederreiter [44]. A lower bound on the discrepancy was given by Roth [48] who proved

$$D_N(x_1, \ldots, x_N) \geq C_d \frac{\log(N)^{d/2}}{N}$$

for infinitely many N, some constant $C_d > 0$ depending only on d and any sequence of points $(x_n)_{n \geq 1}$.

Although low-discrepancy sequences have best known asymptotic bounds there are difficulties in applying them in practice. There are many applications which demand evaluation of high-dimensional integrals. For example a typical task in finance is to calculate the expected payoff of a derivative which usually depends on the values of underlying assets for a long period of time which basically means to evaluate an integral over some d-dimensional domain where d mainly is the product of the number of time steps and the number of underlying assets. Therefore dimensions $d \geq 365$ are not unlikely in practical applications. Another example is sampling from the stationary distribution of an

aperiodic irreducible Markov Chain with a large state space. In the Glauber Dynamics and also in other Markov Chains the probability of a single state is known only up to some normalizing factor but the transition probabilities are precisely known. Therefore it is reasonable to start with some fixed initial state and run the Markov Chain by using a proposal function which depending on the current state and some random bits determines the next state of the Chain. Observe that for each step a new random variable is needed. If the number of time steps is large enough and therefore the number of random variables is large, the resulting empirical distribution gives a good approximation for the stationary distribution.
The upper bound on the right-hand side of (1.0.20) only is vanishing if $N \geq e^d$ and thus such an upper bound is not feasible for high-dimensional integration in practice. There are some particular low-discrepancy sequences which provide good results in some special applications. For example, Atanassov [8] modified the definition of a Halton sequence obtaining a constant C_d on the right-hand side of (1.0.20) vanishing exponentially in d. But in general the situation is dissatisfying. Therefore randomized Quasi-Monte Carlo methods were introduced which try to combine the advantages of Quasi-Monte Carlo methods and classical Monte Carlo methods. Observe that the latter ones provide error bounds which are independent of the dimension while the former ones provide good asymptotic error bounds. Randomized Halton sequences are one example. For further example see the book of Lemieux [41] and the references therein.

Inverse of the discrepancy

Since low-discrepancy sequence only give good error bounds if the number of points is large in comparison with the dimension, one could ask about sequences which have small discrepancy in the special case of a "small" number of sample points in comparison with the dimension. This led to the introduction of the "inverse of the star-discrepancy"

$$n(d,\varepsilon) = \min\{N \in \mathbb{N} : \exists x_1, \dots, x_N \in [0,1)^d, D_N^*(x_1, \dots, x_N) \leq \varepsilon\}$$

which states the smallest number of points in $[0,1)^d$ having the upper bound ε on the star-discrepancy. Heinrich, Novak, Wasilkowski and Woźniakowski [34] showed

$$n(d,\varepsilon) = \mathcal{O}(d\varepsilon^{-2}) \tag{1.0.21}$$

with some implied constant which is independent of d and ε. Thus there is a sequence of points in $[0,1)^d$ with

$$D_N^*(x_1, \dots, x_N) \leq C \frac{\sqrt{d}}{\sqrt{N}} \tag{1.0.22}$$

which for small N compared with d gives a better bound than (1.0.20). Furthermore Hinrichs [35] proved

$$n(d,\varepsilon) = \Omega(d\varepsilon^{-1}). \tag{1.0.23}$$

Thus the dependence of d in (1.0.22) is optimal, only the precise order of ε is unknown. In applications it often is desirable to have a sequence which is extendable not only in the number of points but also in dimension. Therefore Dick [20] proved that there exists a double infinite matrix $(x_{n,i})_{n\geq 1, i\geq 1}$ with numbers $x_{n,i} \in [0,1)$ such that for any pair of natural numbers $N, d \geq 1$ the projection $(x_{n,i})_{1\leq n\leq N, 1\leq i\leq d}$ defines an N-element sequence of points

$$\{(x_{1,1},\ldots,x_{1,d}),\ldots,(x_{N,1},\ldots,x_{N,d})\} \subset [0,1)^d$$

with star-discrepancy

$$D_N^*(x_1,\ldots,x_N) \leq C\frac{\sqrt{d\log(N)}}{\sqrt{N}} \tag{1.0.24}$$

for some absolute constant $C > 0$ independent of d. Observe that the logarithmic term is due to the fact that Dick actually proved that any matrix generated by independent uniformly distributed random variables satisfies the upper bound with positive probability. The result was later improved by Doerr, Gnewuch, Kritzer and Pillichshammer [22] who showed

$$D_N^*(x_1,\ldots,x_N) \leq C\frac{\sqrt{d\log(1+N/d)}}{\sqrt{N}}. \tag{1.0.25}$$

Aistleitner and Weimar [6] later obtained

$$D_N^*(x_1,\ldots,x_N) \leq \frac{\sqrt{C_1 d + C_2\log(\log(N))}}{\sqrt{N}} \tag{1.0.26}$$

which is the best possible result because of the Chung-Smirnov Law of the Iterated Logarithm (1.0.17).
To avoid the iterated logarithm term in the upper bound hybrid sequences which are partly constructed by random numbers and partly by elements of a low-discrepancy sequence were introduced. Aistleitner [3] constructed a matrix where for large n compared to i the entries $x_{n,i}$ are taken from a Halton sequence while for small n compared to i they are randomly chosen. He proved that there exists a matrix which satisfies (1.0.22) uniformly in N and d. In order to prove the upper bound he defined for any dimension d a system of subsets of $[0,1)^d$ such that each subset is a finite union of axis-parallel boxes and any block $\mathcal{A} \subset \mathcal{B}^*$ can be approximated well enough by a disjoint union of a subsystem of this subsets. Furthermore he used the maximal Bernstein inequality to show that for any of this subsets $\mathcal{A}$ and any number N of points the relative number of points inside $\mathcal{A}$ deviates from its Lebesgue measure by $C\sqrt{d}/\sqrt{N}\cdot g(\mathcal{A})$ only with very small probability. Here $C > 0$ is some absolute constant and $g(\mathcal{A})$ is some function which decays fast enough for $\lambda(\mathcal{A}) \to 0$.

Results of this thesis

In chapter 4 we consider a similar constructed double infinite matrix. We define a double infinite matrix $(x_{n,i})_{n,i\geq 1}$ where $(x_{1,i})_{i\geq 1}$ forms a family of independent uniformly in $[0,1)$ distributed random variables. While for large n compared to i, i.e. $2^{12\cdot 2^i} < n-1$ or $i = 1$, we define $x_{n+1,i}$ by taking elements of randomized Halton sequences, for small

n compared to i we take fractional parts of some lacunary sequences, i.e. we have $x_{n+1,i} = \langle 2^{(\lceil \log_2(i)+1 \rceil)n} x_{1,i} \rangle$, instead of independent random numbers $x_{n,i}$. We prove that for any $\varepsilon > 0$ we have

$$D_N^*(x_1, \ldots, x_N) \leq (2576 + 357 \log(\varepsilon^{-1})) \frac{\sqrt{d}}{\sqrt{N}}$$

with probability at least $1 - \varepsilon$. Observe that in this setup we can not directly use the maximal Bernstein inequality any more since this inequality only holds for martingales like the sum of independent random variables of mean zero. Nevertheless, if the system of subsets is chosen well enough, i.e. any subset has corners of the form $(2^{-(\lceil \log_2(1)+1 \rceil)h} a_1, \ldots, 2^{-(\lceil \log_2(d)+1 \rceil)h} a_d)$ for integers $a_1, \ldots, a_d$ and some small enough integer h, the points defined by any subsequence $n_k = k \cdot 2^\kappa + \gamma$ for suitable integers $\kappa, \gamma \geq 1$ depending on h are stochastically independent. Thus by decomposing the sequence of points into such subsequences we may still apply the maximal Bernstein inequality. The practical purpose of having points defined by such a lacunary sequence instead of independent random points is reducing the number of digits which are necessary to simulate those points. To simulate N random points in $[0,1)^d$ with a precision of H digits requires a simulation of dHN digits while by using points from such a lacunary sequence this number may be reduced to $\mathcal{O}(dH + d\log(d)N)$.

Random Matrix ensembles with correlated entries

Previous results

Lacunary function systems also have applications in random matrix theory. Random matrices are of particular interest in theoretic physics. Initially they were introduced by Wigner [58] to describe properties of atoms with heavy nuclei. Consider an ensemble $X_N = (\frac{1}{\sqrt{N}} X_{n,n'})_{1 \leq n,n' \leq N}$ of symmetric random $N \times N$ matrices such that except for the symmetry condition the entries $X_{n,n'}$ are independent random variables with mean zero, unit variance and universally bounded moments. Wigner showed that the mean empirical eigenvalue distribution converges weakly to the semicircle law as the size of the matrix tends to infinity. Being more precise he proved

$$\lim_{N \to \infty} \mathbb{E}\left[\frac{1}{N} \operatorname{Tr}\left(X_N^K\right)\right] = \begin{cases} C_{K/2} = \frac{K!}{(K/2)!(K/2+1)!}, & K \text{ even}, \\ 0, & K \text{ odd} \end{cases} \tag{1.0.27}$$

where the coefficients C_K denote the Catalan numbers. Note that many problems in combinatorics have solutions which are related to Catalan numbers. For example, $C_{K/2}$ is the number of non-crossing pair partitions of $\{1, \ldots, K\}$ for some even K, i.e. there exist precisely $C_{K/2}$ pair partitions Δ such that there are no $k_1 < k_2 < k_3 < k_4$ with $k_1, k_3 \in \mathcal{S}$ and $k_2, k_4 \in \mathcal{S}'$ for some $\mathcal{S}, \mathcal{S}' \in \Delta$ with $\mathcal{S} \neq \mathcal{S}'$. This fact plays an important part in the proof that in the limit the mean expectation the values of the traces of X_N^K coincide with the moments of the semicircle law which has density $1/2\pi \cdot \sqrt{4 - x^2} \cdot \mathbf{1}_{x^2 \leq 4}$.

Having many other applications in physics, e.g. in quantum chaos or in telecommunications, in pure mathematics, e.g. in number theory, and further areas, random matrix models have been studied intensively in the last decades. In recent years the question arose whether the asymptotic behaviour still holds if the independence condition is weakened. Besides investigations on some specific models so far there are only some few attempts in this area. Schenker and Schulz-Baldes [51] defined an ensemble of random matrices X_N where for any N there exists an equivalence relation $\sim_N$ on the set of entries of X_N such that entries from different equivalence classes are independent while entries from the same class may be correlated. Observe that this model generalizes the classical case where the equivalence classes are of the form $\{X_{n,n'}, X_{n',n}\}$. They showed that if the classes are not too large, i.e.

$$\max_{n\in\{1,\dots,N\}} |\{n', m, m' \in \{1, \dots, N\} : (n, n') \sim_N (m, m')\}| = o(N^2)$$

and

$$\max_{n,n',m\in\{1,\dots,N\}} |\{m' \in \{1, \dots, N\} : (n, n') \sim_N (m, m')\}| = C$$

for some absolute constant $C > 0$, and if not too many different entries of the same class lie on the same row resp. the same column, i.e.

$$|\{n, n', m' \in \{1, \dots, N\} : (n, n') \sim_N (n', m'), n \neq m'\}| = o(N^2),$$

then the mean empirical eigenvalue distribution converges weakly to the semicircle law. They proved the result without any further condition on the correlation of two entries in the same equivalence class. It is natural to consider random matrices such the correlation decays with the distance of two entries. Therefore it is reasonable to study matrix models where each entry has except for some small errors only a finite range of dependence. As Anderson and Zeitouni [7] showed the converge to the semicircle law does not hold in general under this assumption, but further conditions are necessary. Thus although the conditions given by Schenker and Schulz-Baldes appear to be too strict they can not be weakened without further constraints on the correlation of different entries. Friesen and Löwe [27] studied random matrix ensembles with stochastically independent diagonals but correlated entries on the diagonal. They showed convergence to the semicircle law for

$$\max_{n,n'\in\{1,\dots,N\}} \mathbb{E}[X_{n,n'} X_{n+t,n'+t}] \leq Ct^{-\varepsilon}$$

for some absolute constants $C > 0$ and $\varepsilon > 0$. Although it seems to be more reasonable to study ensembles with independent rows or columns rather than ensembles with independent diagonals these ensembles provide some difficulties. Since the symmetry condition is necessary to have real eigenvalues it implies that the columnwise independence and rowwise dependence turns into rowwise independence and columnwise dependence by crossing the main diagonal. This not only seems to be not natural from a stochastic point of view, but also the mean empirical eigenvalue does not appear to converge to the semicircle law in general as simulations show.

Results of this thesis

In chapter 5 we introduce random matrix ensembles with

$$X_{n,n'} = f(M_{n+n',1}x_1, M_{|n-n'|,2}x_2)$$

for two integer-valued lacunary sequences $(M_{n,1})_{n\geq 2}, (M_{n,2})_{n\geq 0}$ and suitable functions f. We prove weak convergence of the mean empirical eigenvalue distribution towards the semicircle law under some further number theoretic properties of the sequence $(M_{n,1})_{n\geq 1}$. To prove this result we show (1.0.27). Furthermore we give examples to show that even in this particular class of random matrix ensembles the asymptotic behaviour of the spectrum becomes delicate. We prove that the empirical spectral distribution does not converge to the semicircle law in general even if the correlation of two entries decays exponentially in the distance. For $f(x_1, x_2) = 1/\sqrt{2}\cdot(\cos(2\pi(x_1+x_2))+\cos(4\pi(x_1+x_2)))$ and $M_{n,1} = 2^n$ we show that the mean empirical spectral distribution does not converge to semicircle law while for any sequence $(M_{n,1})_{n\geq 1}$ with $M_{n+1,1}/M_{n,1} \to \infty$ for $n \to \infty$ and any periodic function f of finite total variation in the sense of Hardy and Krause with mean zero and unit variance the mean spectral distribution converges to the semicircle law.

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2 Preliminaries

In this chapter we repeat some basic results on periodic functions of finite total variation resp. lacunary sequences which are going to be used in the subsequent chapters.
For some integer $d \geq 1$ set $I = \{1, \ldots, d\}$. We now introduce the total variation in the sense of Hardy and Krause for periodic functions on $\mathbb{R}^d$. Let $f : \mathbb{R}^d \to \mathbb{R}$ be some periodic function, i.e. f satisfies $f(x+z) = f(x)$ for all $x \in \mathbb{R}^d$ and $z \in \mathbb{Z}^d$. For some subset $J \subseteq I$ and points $a, b \in [0,1)^{|J|}$ with $a_i \leq b_i$ for all $i \in J$ and some $z \in [0,1)^{|I \setminus J|}$ define

$$\Delta_J(f, a, b, z) = \sum_{\delta \in \{0,1\}^{|J|}} (-1)^{\sum_{i \in J} \delta_i} f(c_\delta)$$

where $c_\delta = (c_{\delta,1}, \ldots, c_{\delta,d})$ is defined by $c_{\delta,i} = \delta_i a_i + (1-\delta_i) b_i$ for $i \in J$ and $c_i = z_i$ for $i \notin J$. A finite set $\mathcal{Y}_i = \{y_1 \ldots, y_{m(i)}\} \subset [0,1)$ with $0 = y_1 < \ldots < y_{m(i)} < 1$ for some positive integer $m(i)$ is called a ladder. A multidimensional ladder on $[0,1)^d$ has the form $\mathcal{Y} = \prod_{i \in I} \mathcal{Y}_i$. For a multidimensional ladder $\mathcal{Y}$, a subset $J \subseteq I$ and $z \in [0,1)^d$ set $\mathcal{Y}_{J,z} = \prod_{i \in J} \mathcal{Y}_i \times \prod_{i \notin J} \{z_i\}$. For $y \in \mathcal{Y}_{J,z}$ define $y_+ \in [0,1)^{|J|} \times \prod_{i \notin J} \{z_i\}$ such that $y_{+,i}$ is the successor of y_i in $\mathcal{Y}_i$ resp. 1 if y_i is the largest element in $\mathcal{Y}_i$. Then we define the variation of f over $\mathcal{Y}_J$ by

$$V_{\mathcal{Y}_{J,z}}(f) = \sum_{y \in \mathcal{Y}_{J,z}} |\Delta_J(f, y, y_+, z)|.$$

Denote the set of all ladders $\mathcal{Y}_{J,z}$ by $\mathbb{Y}_{J,z}$. Then the total variation of f over $[0,1)^{|J|}$ is defined by

$$V_J(f) = \sup_{z \in [0,1)^{|I|}} \sup_{\mathcal{Y}_{J,z} \in \mathbb{Y}_{J,z}} V_{\mathcal{Y}_{J,z}}(f).$$

The total variation of f on $[0,1)^d$ in the sense of Hardy and Krause is

$$V_{HK}(f) = \sum_{J \subseteq I, J \neq \emptyset} V_J(f).$$

A function f is called to be of finite total variation if $V_{HK}(f) < \infty$.
The following Lemma was proved in [59]:

Lemma 2.0.1 *Let $f(x) = \sum_{j \in \mathbb{Z}^d \setminus \{0\}} a_j \cos(2\pi \langle j, x \rangle) + b_j \sin(2\pi \langle j, x \rangle)$ be a periodic function of finite total variation in the sense of Hardy and Krause. Then we have*

$$|a_j|, |b_j| \leq C \left(\prod_{i \in I, j_i \neq 0} \frac{1}{2\pi |j_i|} \right) V_{\{i \in I : j_i \neq 0\}}(f).$$

for some absolute constant $C > 0$ and all $j \in \mathbb{Z}^d \setminus \{0\}$.

Observe that hereafter we always write C for some absolute constant which may vary from line to line. Furthermore we always assume that $f : \mathbb{R}^d \to \mathbb{R}$ is a periodic function of mean zero such that the Fourier series of f exists and converges to f.
Thus we have $f(x) = \sum_{j\in\mathbb{Z}^d\setminus\{0\}} a_j \cos(2\pi\langle j, x\rangle) + b_j \sin(2\pi\langle j, x\rangle)$ with suitable numbers $a_j, b_j \in \mathbb{R}$ for all $j \in \mathbb{Z}^d\setminus\{0\}$. For $\Gamma \in \mathbb{N}_0^d$ we denote the Γth partial sum of f by

$$\psi_\Gamma(x) = \sum_{j\in\mathbb{Z}^d, |j_i|\le\Gamma_i} a_j \cos(2\pi\langle j, x\rangle) + b_j \sin(2\pi\langle j, x\rangle) \tag{2.0.1}$$

and set $\rho_\Gamma(x) = f(x) - p(x)$. If $\Gamma_i = G$ for all $i \in I$ we simply write $\psi_G(x)$ resp. $\rho_G(x)$. The Gth Fejér mean of f is defined by

$$\begin{aligned} p_G(x) &= \frac{1}{(G+1)^d} \sum_{\Gamma\in\mathbb{N}_0^d, ||\Gamma||_\infty\le G} \psi_\Gamma(x) \\ &= \sum_{j\in\mathbb{Z}^d, |j_i|\le G} a'_j \cos(2\pi\langle j, x\rangle) + b'_j \sin(2\pi\langle j, x\rangle) \end{aligned} \tag{2.0.2}$$

where

$$a'_j = a_j \prod_{i\in I} \frac{G+1-|j_i|}{G+1}, \quad b'_j = b_j \prod_{i\in I} \frac{G+1-|j_i|}{G+1}$$

for all $j \in \mathbb{Z}^d$ with $||j||_\infty \le G$. Observe that

$$p_G(x) = \int_{[0,1)^d} f(x+t) K_G(t)\, dt$$

where $K_G(t) = K_\Gamma(t)$ with $\Gamma_i = G$ for all $i \in I$ and $K_\Gamma(t) = \prod_{i\in I} K_{\Gamma_i}(t_i) = \prod_{i\in I} K_G(t_i)$ is the d-dimensional Gth Fejér kernel and $K_G(t_i)$ is the one-dimensional Gth Fejér kernel defined by

$$\begin{aligned} K_G(t_i) &= \frac{1}{G+1} \sum_{l=0}^{G} \sum_{j_i=-l}^{l} \cos(2\pi\langle j_i, x\rangle) \\ &= \frac{1}{G+1} \sum_{l=0}^{G} \frac{\sin(2\pi\langle l+1/2, t_i\rangle)}{\sin(2\pi\langle 1/2, t_i\rangle)} \\ &= \frac{1}{G+1} \frac{(\sin(2\pi\langle (G+1)/2, t_i\rangle))^2}{2(\sin(2\pi\langle 1/2, t_i\rangle))^2} \\ &\ge 0. \end{aligned}$$

Therefore we have

$$\begin{aligned} |p_G(x)| &= \left| \int_{[0,1)^d} f(x+t) K_G(t)\, dt \right| \\ &\le ||f||_\infty \left| \int_{[0,1)^d} K_G(t)\, dt \right| \\ &\le ||f||_\infty. \end{aligned}$$

Now we define $r_G(x) = f(x) - p_G(x)$. Thus we have

$$r_G(x) = \sum_{j \in \mathbb{Z}^d \setminus \{0\}} \tilde{a}_j \cos(2\pi \langle j, x \rangle) + \tilde{b}_j \sin(2\pi \langle j, x \rangle) \tag{2.0.3}$$

where

$$\tilde{a}_j = \begin{cases} a_j, & ||j||_\infty > G, \\ a_j \left(1 - \prod_{i \in I} \left(1 - \frac{|j_i|}{G+1}\right)\right), & ||j||_\infty \leq G \end{cases}$$

and $\tilde{b}_j$ is defined analogously.

Lemma 2.0.2 *Let $f(x) = \sum_{j \in \mathbb{Z}^d \setminus \{0\}} a_j \cos(2\pi \langle j, x \rangle) + b_j \sin(2\pi \langle j, x \rangle)$ be some periodic function satisfying $V_{HK}(f) \leq 1$. Then there exists some absolute constant $C > 0$ such that for any $G \geq d$ the function r_G as defined in (2.0.3) satisfies*

$$||r_G||_2^2 \leq CdG^{-1}.$$

Proof. We have

$$||r_G||_2^2 = \sum_{j \in \mathbb{Z}^d \setminus \{0\}} \tilde{a}_j^2 + \tilde{b}_j^2 = \underbrace{\sum_{0 < ||j||_\infty \leq G} \tilde{a}_j^2 + \tilde{b}_j^2}_{(*)} + \underbrace{\sum_{||j||_\infty > G} \tilde{a}_j^2 + \tilde{b}_j^2}_{(**)}. \tag{2.0.4}$$

For some given nonempty $J \subseteq I$ set

$$\begin{aligned} D(G, J) &= \{j \in \mathbb{Z}^d : 1 \leq |j_i| \leq G \text{ for } i \in J, j_i = 0 \text{ for } i \notin J\}, \\ D'(G, J) &= \{j \in \mathbb{Z}^d : j_i \neq 0 \text{ for } i \in J, j_i = 0 \text{ for } i \notin J, j \notin D(G, J)\}. \end{aligned}$$

To estimate $(*)$ we first by Lemma 2.0.1 observe

$$\sum_{0 < ||j||_\infty \leq G} \tilde{a}_j^2 + \tilde{b}_j^2 \leq 2 \sum_{J \subseteq I, J \neq \emptyset} \sum_{j \in D(G,J)} \left(\prod_{i \in J} \frac{1}{2\pi |j_i|}\right)^2 \left(1 - \prod_{i \in J} \left(1 - \frac{|j_i|}{G+1}\right)\right)^2 V_J(f)^2.$$

By definition of $V_{HK}(f)$ it is enough to show

$$V(G, J) = 2 \sum_{j \in D(G,J)} \left(\prod_{i \in J} \frac{1}{2\pi |j_i|}\right)^2 \left(1 - \prod_{i \in J} \left(1 - \frac{|j_i|}{G+1}\right)\right)^2 \leq CG^{-1} \tag{2.0.5}$$

for some absolute constant $C > 0$. By decomposing we have

$$\begin{aligned} V(G, J) = 2 \sum_{K, K' \subseteq J, K, K' \neq 0} \sum_{j \in D(G,J)} \prod_{i \in K} \frac{1}{2\pi |j_i|} \frac{|j_i|}{G+1} \prod_{i \in J \setminus K} \frac{1}{2\pi |j_i|} \left(1 - \frac{|j_i|}{G+1}\right) \\ \cdot \prod_{i \in K'} \frac{1}{2\pi |j_i|} \frac{|j_i|}{G+1} \prod_{i \in J \setminus K'} \frac{1}{2\pi |j_i|} \left(1 - \frac{|j_i|}{G+1}\right). \end{aligned}$$

Thus we get

$$V(G,J) = 2 \sum_{K,K' \subseteq J, K,K' \neq 0} W_1(K,K') \cdot W_2(K,K') \cdot W_3(K,K') \cdot W_4(K,K')$$

where

$$\begin{aligned}
W_1(K,K') &= \prod_{i \in K \cap K'} \frac{1}{(2\pi)^2} \sum_{j_i=-G}^{G} \frac{1}{(G+1)^2} \leq \prod_{i \in K \cap K'} \frac{1}{4G}, \\
W_2(K,K') &= \prod_{i \in K \cap J \setminus K'} \frac{1}{(2\pi)^2} \sum_{j_i=-G}^{G} \frac{1}{G+1} \left(\frac{1}{|j_i|} - \frac{1}{G+1} \right) \leq \prod_{i \in K \cap J \setminus K'} \frac{\log(G)}{4G}, \\
W_3(K,K') &= \prod_{i \in J \setminus K \cap K'} \frac{1}{(2\pi)^2} \sum_{j_i=-G}^{G} \left(\frac{1}{|j_i|} - \frac{1}{G+1} \right) \frac{1}{G+1} \leq \prod_{i \in J \setminus K \cap K'} \frac{\log(G)}{4G}, \\
W_4(K,K') &= \prod_{i \in J \setminus K \cap J \setminus K'} \frac{1}{(2\pi)^2} \sum_{j_i=-G}^{G} \left(\frac{1}{|j_i|} - \frac{1}{G+1} \right)^2 \leq \prod_{i \in J \setminus K \cap J \setminus K'} \frac{1}{4}.
\end{aligned}$$

Since $K \cap K' \neq \emptyset$ or $K \cap J \setminus K' \neq \emptyset$ and $J \setminus K \cap K' \neq \emptyset$ we conclude

$$V(G,J) \leq \sum_{K,K' \subseteq J, K,K' \neq 0} \frac{C}{4^{|J|} G} \leq C G^{-1} \tag{2.0.6}$$

for some absolute constant $C > 0$ and therefore (2.0.5) is verified. We now estimate $(**)$. By Lemma 2.0.1 we have

$$\begin{aligned}
\sum_{||j||_\infty > G} \tilde{a}_j^2 + \tilde{b}_j^2 &\leq \sum_{J \subseteq I, J \neq \emptyset} \sum_{j \in D'(G,J)} a_j^2 + b_j^2 \\
&\leq 2 \sum_{J \subseteq I, J \neq \emptyset} \sum_{j \in D'(G,J)} \left(\prod_{i \in J} \frac{1}{(2\pi |j_i|)^2} \right) V_j(f)^2.
\end{aligned}$$

We furthermore for some nonempty $J \subseteq I$ get

$$\begin{aligned}
\sum_{j \in D'(G,J)} \left(\prod_{i \in J} \frac{1}{(2\pi |j_i|)^2} \right) &\leq \sum_{l=1}^{|J|} \binom{|J|}{l} \left(2 \sum_{j=1}^{G} \frac{1}{(2\pi j)^2} \right)^{|J|-l} \cdot \left(2 \sum_{j=G+1}^{\infty} \frac{1}{(2\pi j)^2} \right)^l \\
&\leq \sum_{l=1}^{|J|} \binom{|J|}{l} \left(\frac{1}{2\pi^2 G} \right)^l \\
&\leq \sum_{l=1}^{|J|} \left(\frac{d}{2\pi^2 G} \right)^l \\
&\leq C d G^{-1}
\end{aligned}$$

for some absolute constant $C > 0$. Therefore we have

$$\sum_{||j||_\infty > G} \tilde{a}_j^2 + \tilde{b}_j^2 \leq CdG^{-1}$$

and the Lemma is proved.

With $L(N, G, \nu)$ as defined in (1.0.19) we have

Lemma 2.0.3 *Let $f : \mathbb{R}^d \to \mathbb{R}$ be a periodic function satisfying $V_{HK}(f) \leq 1$ and let $(M_n)_{n\geq 0}$ be a lacunary sequence of matrices satisfying the Hadamard gap condition (1.0.4). Then we have*

$$\int_{[0,1)^d} \left(\sum_{n=1}^{N} f(M_n x) \right)^2 dx \leq C(\log(d)||f||_2^2 + ||f||_2)N \tag{2.0.7}$$

where $C > 0$ is an absolute constant depending only on q. If the sequence $(M_n)_{n\geq 1}$ furthermore satisfies $L(N, G, 0) = o(N)$ for any fixed $G \geq 1$ then we have

$$\lim_{N\to\infty} \frac{1}{N} \int_{[0,1)^d} \left(\sum_{n=1}^{N} f(M_n x) \right)^2 dx = ||f||_2^2. \tag{2.0.8}$$

Note that hereafter we write $\log(x)$ for $\max(1, \log(x))$.

Proof. For $||j||_\infty, ||j'||_\infty \leq G$ and $k > \log_q(G)$ we have

$$||M_n^T j'||_\infty \leq ||M_n^T||_\infty ||j'||_\infty \leq q^{-k} ||M_{n+k}^T j||_\infty ||j'||_\infty < ||M_{n+k}^T j||_\infty. \tag{2.0.9}$$

Therefore we obtain $M_n^T j' \neq M_{n+k}^T j$. Now let p_G be the Gth Fejér mean of f. Then for some $k > \log_q(G)$ we have

$$p_G(M_n x) p_G(M_{n+k} x) = \sum_u \alpha_u \cos(2\pi\langle u, x\rangle) + \beta_u \sin(2\pi\langle u, x\rangle)$$

where any u is of the form $M_n^T j \pm M_{n+k}^T j'$ for some $1 \leq ||j||_\infty, ||j'||_\infty \leq G$. Therefore by (2.0.9) we get

$$\int_{[0,1)^d} p_G(M_n x) p_G(M_{n+k} x)\, dx = 0 \tag{2.0.10}$$

for $k > \log_q(G)$. By Lemma 2.0.2 for any $k \geq 1$ there is a trigonometric polynomial g_k with $dq^{2k} - 1 < \deg(g_k) \leq dq^{2k}$ such that

$$||f - g_k||_2 \leq Cq^{-k}.$$

Therefore for $k' > \log_q(dq^{2k})$ by (2.0.10) and Cauchy-Schwarz inequality we have

$$\begin{aligned}\left|\int_{[0,1)^d} f(M_n x) f'(M_{n+k'}x)\,dx\right| &\leq \left|\int_{[0,1)^d} (f-g_k)(M_n x) f(M_{n+k'}x)\,dx\right| \\ &\quad + \left|\int_{[0,1)^d} g_k(M_n x) g_k(M_{n+k'}x)\,dx\right| \\ &\quad + \left|\int_{[0,1)^d} g_k(M_n x)(f-g_k)(M_{n+k'}x)\,dx\right| \\ &\leq 2C\|f\|_2 q^{-k}\end{aligned} \tag{2.0.11}$$

since $||g_k||_2 \leq ||f||_2$. We obtain

$$\begin{aligned}&\int_{[0,1)^d}\left(\sum_{n=1}^N f(M_n x)\right)^2 dx \\ &\leq N||f||_2^2 + 2\sum_{n=1}^N\sum_{k'=1}^{N-n}\left|\int_{[0,1)^d} f(M_n x) f(M_{n+k'}x)\,dx\right| \\ &\leq N||f||_2^2 + 2N(\log_q(d)+2)||f||_2^2 + 2\sum_{n=1}^N\sum_{k'=\log_q(d)+3}^{N-n}\left|\int_{[0,1)^d} f(M_n x) f(M_{n+k'}x)\,dx\right| \\ &\leq N||f||_2^2 + CN\log_q(d)||f||_2^2 + \sum_{n=1}^N\sum_{k=1}^{\infty} C||f||_2 q^{-k} \\ &\leq C(\log(d)||f||_2^2 + ||f||_2)N.\end{aligned}$$

Thus (2.0.7) is shown.
The proof of (2.0.8) is similar. For $k' > \log_q(q^{2k})$ instead of (2.0.11) we get

$$\left|\int_{[0,1)^d} f(M_n x) f(M_{n+k}x)\,dx\right| \leq C||f||_2 d^{1/2} q^{-k}$$

and similarly

$$\left|\int_{[0,1)^d} s_1(M_n x) s_2(M_{n+k}x)\,dx\right| \leq C||f||_2 d^{1/2} q^{-k}$$

where for $i \in \{1,2\}$ the function s_i is of the form p_{G_i} or r_{G_i} for some suitable number $G_i > 0$.

For any $G \geq 1$ we obtain

$$\left| \frac{1}{N} \int_{[0,1)^d} \left(\sum_{n=1}^{N} f(M_n x) \right)^2 dx - ||f||_2^2 \right|$$

$$\begin{aligned}
&\leq \frac{2}{N} \sum_{n=1}^{N} \sum_{k=1}^{N-n} \left| \int_{[0,1)^d} f(M_n x) f(M_{n+k} x)\, dx \right| \\
&\leq \frac{C}{N} \sum_{n=1}^{N} \sum_{k=1}^{N-n} ||f||_2 d^{1/2} \min(G^{-1/2}, q^{-k}) \\
&\quad + \frac{2}{N} \sum_{n=1}^{N} \sum_{k=1}^{N-n} \left| \int_{[0,1)^d} p_G(M_n x) p_G(M_{n+k} x)\, dx \right| \\
&\leq C ||f||_2 d^{1/2} G^{-1/2} \\
&\quad + (2G+1)^{2d} \frac{h(N)}{N}
\end{aligned}$$

for some function $h(N)$ with $h(N)/N \to 0$. Observe that such a function exists by assumption on $L(N, G, 0)$. Since the constant $G \geq 1$ can be chosen arbitrary, (2.0.8) is shown.

Lemma 2.0.4 *Let $(M_n)_{n \geq 1}$ be some lacunary sequence satisfying the Hadamard gap condition (1.0.4). For any $n, n' \in \mathbb{N}$ and $j \in \mathbb{Z}^d$ with $||j||_\infty \leq G$ for some $G \geq 2$ there exists at most one $j' \in \mathbb{Z}^d$ with $||j'||_\infty \leq G$ such that*

$$||M_n^T j \pm M_{n'}^T j'||_\infty < ||M_{n''}^T||_\infty \tag{2.0.12}$$

where $n'' \leq \min(n, n') - \log_q(G)$.

Here (2.0.12) has to be understood in the following sense: For n, n', j there exists at most one j' with $||M_n^T j + M_{n'}^T j'||_\infty < ||M_{n''}^T||_\infty$ and at most one possibly different j' with $||M_n^T j - M_{n'}^T j'||_\infty < ||M_{n''}^T||_\infty$.

Proof. Suppose that (2.0.12) is satisfied for some n, n', j, j'. Now take some $j'' \neq j'$. We observe

$$\begin{aligned}
||M_n^T j \pm M_{n'}^T j''||_\infty &\geq ||M_{n'}^T (j'' - j')||_\infty - ||M_n^T j \pm M_{n'}^T j'||_\infty \\
&> G||M_{n''}^T||_\infty - ||M_{n''}^T||_\infty \\
&\geq ||M_{n''}^T||_\infty.
\end{aligned}$$

Therefore the Lemma is proved.

Lemma 2.0.5 *Let* $p(x) = \sum_{j \in \mathbb{Z}^d, 0 < ||j||_\infty \leq G} a_j \cos(2\pi \langle j, x \rangle) + b_j \sin(2\pi \langle j, x \rangle)$ *with* $G \geq 2$ *be some trigonometric polynomial satisfying* $||p||_\infty \leq 1$ *and* $V_{HK}(p) \leq 1$*. Let* $(M_n)_{n \geq 1}$ *be a lacunary sequence satisfying the Hadamard gap condition (1.0.4). Then we have*

$$\int_{[0,1)^d} \left(\sum_{n=1}^{N} p(M_n x) \right)^4 dx \leq C N^2 \tag{2.0.13}$$

for some constant $C > 0$ *which depends on* q *and* d*. If furthermore* $N \geq \max(G d^{-1}, d)$ *then we have*

$$\int_{[0,1)^d} \left(\sum_{n=1}^{N} p(M_n x) \right)^4 dx \leq C (\log(d))^{2/3} N^2 \tag{2.0.14}$$

for some absolute constant $C > 0$ *depending only on* q*.*

Proof. We only prove (2.0.14). The proof of (2.0.13) is essentially the same.
First we are going to show

$$\mathbb{P}\left(\left| \sum_{n=1}^{N} p(M_n x) \right| > t\sqrt{N} \right) \leq C \exp\left(-\frac{t^{3/2}}{8C \log(d)} \right) \tag{2.0.15}$$

for some constant $C > 0$ depending only on q. For some $0 < \beta < 1$ set $P = \lfloor N^\beta \rfloor$ and $l = \lceil N/2P \rceil$. Without loss of generality we may assume $3N^4 \leq q^{N^\beta - 1}$. There exists some $N_0 \in \mathbb{N}$ depending only on q with $3N^4 \leq q^{N^\beta - 1}$ for $N \geq N_0$. Therefore there is some constant $C_q > 0$ which depends only on q and β such that (2.0.14) is satisfied with C_q and any $N < N_0$. Since $N \geq G^2$ we have $\log_q(G) + \log_q(3G) \leq \lfloor N^\beta \rfloor$ for $N \geq N_0$. Now define

$$U_m(x) = \sum_{n=Pm+1}^{\min(P(m+1),N)} p(M_n x).$$

By Markov's inequality and Cauchy-Schwarz inequality we obtain

$$\begin{aligned} \mathbb{P}\left(\left| \sum_{n=1}^{N} p(M_n x) \right| > t\sqrt{N} \right) &\leq 2\exp(-\kappa_N t\sqrt{N}) \int_{[0,1)^d} \exp\left(\kappa_N \sum_{n=1}^{N} p(M_n x) \right) dx \\ &\leq 2\exp(-\kappa_N t\sqrt{N}) I(\kappa_N, l)^{1/2} I'(\kappa_N, l)^{1/2} \end{aligned} \tag{2.0.16}$$

where $\kappa_N > 0$ and

$$\begin{aligned} I(\kappa_N, l) &= \int_{[0,1)^d} \prod_{m=0}^{l-1} \exp\left(2\kappa_N U_{2m}(x)\right) dx, \\ I'(\kappa_N, l) &= \int_{[0,1)^d} \prod_{m=1}^{l} \exp\left(2\kappa_N U_{2m-1}(x)\right) dx. \end{aligned}$$

Since $I(\kappa_N, l)$ and $I(\kappa_N, l)$ can be estimated similarly, we only estimate the first one.

Using $e^{|z|} \leq (1+z+z^2)e^{|z|^3}$ we observe

$$I(\kappa_N, l) \leq \exp\left(\sum_{m=0}^{l-1} |\kappa_N U_{2m}(x)|^3\right) \int_{[0,1)^d} \prod_{m=0}^{l-1} (1+\kappa_N U_{2m}(x) + (\kappa_N U_{2m}(x))^2)\, dx \tag{2.0.17}$$

For some $m \geq 0$ we have

$$U_{2m}^2(x) = \sum_{j\in\mathbb{Z}^d} \alpha_j \cos(2\pi\langle j, x\rangle) + \beta_j \sin(2\pi\langle j, x\rangle)$$

for suitable numbers α_j, β_j for all $j \in \mathbb{Z}^d$. Now set

$$V_{2m}(x) = \sum_{j\in\mathbb{Z}^d, ||j||_\infty < ||M_{m'}^T||_\infty} \alpha_j \cos(2\pi\langle j, x\rangle) + \beta_j \sin(2\pi\langle j, x\rangle)$$

for $m' = \lfloor 2Pm - \log_q(G)\rfloor$ and furthermore $W_{2m}(x) = U_{2m}^2(x) - V_{2m}(x)$. To estimate $|V_{2m}(x)|$ observe that by Lemma 2.0.4 for any $1 \leq n, n' \leq N$ and $j \in \mathbb{Z}^d$ with $||j||_\infty \leq G$ there is at most one $j' \in \mathbb{Z}^d$ with $||j'||_\infty \leq G$ such that $||M_n^T j \pm M_{n'}^T j'||_\infty < ||M_{n''}^T||_\infty$ where $n'' \leq \min(n, n') - \log_q(G)$. For $||j||_\infty, ||j'||_\infty \leq 1/2 \cdot q^{|n-n'|}$ we have

$$||M_n^T j \pm M_{n'}^T j'||_\infty \geq \frac{1}{2} q^{|n-n'|} ||M_{n'}^T||_\infty.$$

Thus for $|n-n'| \geq \log_q(2)$ we obtain $||M_n^T j \pm M_{n'}^T j'||_\infty \geq ||M_{n''}^T||_\infty$. Then by Lemma 2.0.1 and Cauchy-Schwarz inequality we have

$$|V_{2m}(x)| \leq \sum_{n,n'=1}^{N} \sum_{\substack{1\leq ||j||_\infty, ||j'||_\infty \leq G, \\ ||M_n^T j \pm M_{n'}^T j'||_\infty < ||M_{m'}^T||_\infty}} (|a_j| + |b_j|)(|a_{j'}| + |b_{j'}|) \leq C||p||_2 N \tag{2.0.18}$$

where the constant $C > 0$ depends only on q. Since

$$0 \leq 1 + \kappa_n U_{2m}(x) + (\kappa_n U_{2m}(x))^2$$

by Lemma 2.0.3 we obtain

$$\begin{aligned} I(\kappa_N, l) &\leq \exp(C\kappa_N^3 N^{1+2\beta}) \\ &\cdot \int_{[0,1)^d} \prod_{m=0}^{l-1} (1 + C(\log(d)||p||_2^2 + ||p||_2)P + \kappa_N U_{2m}(x) + \kappa_N^2 W_{2m}(x))\, dx. \end{aligned} \tag{2.0.19}$$

Furthermore we get

$$\kappa_N U_{2m}(x) + \kappa_N^2 W_{2m}(x) = \sum_j \alpha_j \cos(2\pi\langle j, x\rangle) + \beta_j \sin(2\pi\langle j, x\rangle)$$

for suitable α_j, β_j and $j \in \mathbb{Z}^d$ satisfying $||M_{m'}^T||_\infty \leq ||j||_\infty \leq 2G||M_{P(2m+1)}^T||_\infty$.

Now for $0 \le k \le m$ let j_{2k} be any frequency vector of the trigonometric polynomial $1 + C(\log(d)||p||_2^2 + ||p||_2)N + \kappa_N U_{2m}(x) + \kappa_N^2 W_{2m}(x)$. If $j_{2m} \neq 0$ then we get

$$\begin{aligned}
||j_{2m}||_\infty - \sum_{k=0}^{m-1} ||j_{2k}||_\infty & \\
&\ge ||M_{m'}^T||_\infty - 2G \sum_{k=0}^{m-1} ||M_{P(2k+1)^T}||_\infty \\
&\ge 3G||M_{P(2m-1)^T}||_\infty - 2G||M_{P(2m-1)^T}||_\infty \sum_{k=0}^{m-1} q^{P(2k+1)-P(2m-1)} \\
&\ge \left(3 - 2\frac{1}{1-q^{-2P}}\right) G||M_{P(2m-1)}^T||_\infty \\
&> 0
\end{aligned} \tag{2.0.20}$$

where the second inequality follows by assumption on $P > \log_q(G) + \log_q(3G)$ and the fourth inequality follows by $P > \log_q(\sqrt{3})$ which without loss of generality we may assume. Therefore by (2.0.19) we have

$$I(\kappa_N, l) \le \exp(C\kappa_N^3 N^{1+2\beta}) \int_{[0,1)^d} \prod_{m=0}^{l-1} (1 + C(\log(d)||p||_2^2 + ||p||_2)P)\, dx.$$

Plugging this into (2.0.16) yields

$$\mathbb{P}\left(\left|\sum_{n=1}^N p(M_n x)\right| > t\sqrt{N}\right) \le 2\exp\left(-\kappa_N t\sqrt{N} + C_0\kappa_N^3 N^{1+2\beta} + C_0 \log(d)\kappa_N^2 N\right) \tag{2.0.21}$$

for some absolute constant C_0. Choose $\beta = 1/4$. Then for

$$\kappa_N = \frac{t}{2C_0 \log(d)\sqrt{N}}$$

and $0 \le t \le C_0 \log(d)^2$ we observe

$$\begin{aligned}
\exp&\left(-\kappa_N t\sqrt{N} + C_0\kappa_N^3 N^{1+2\beta} + C_0 \log(d)\kappa_N^2 N\right) \\
&= \exp\left(C_0 \log(d) \frac{t^2}{4C_0^2 \log(d)^2} + C_0 \frac{t^3}{8C_0^3 \log(d)^3} - \frac{t^2}{2C_0 \log(d)}\right) \\
&= \exp\left(-\frac{t^2}{4C_0 \log(d)}\left(1 - \frac{t}{2C_0 \log(d)^2}\right)\right) \\
&\le \exp\left(-\frac{t^2}{8C_0 \log(d)}\right).
\end{aligned} \tag{2.0.22}$$

A similar calculation for

$$\kappa_N = \frac{\sqrt{t}}{2\sqrt{C_0 N}}$$

and $t > C_0 \log(d)^2$ yields

$$\begin{aligned} &\exp\left(-\kappa_N t\sqrt{N} + C_0\kappa_N^3 N^{1+2\beta} + C_0\log(d)\kappa_N^2 N\right) \\ &\qquad = \exp\left(C_0\log(d)\frac{t}{4C_0} + C_0\frac{t^{3/2}}{8C_0^{3/2}} - \frac{t^{3/2}}{2C_0^{1/2}}\right) \\ &\qquad = \exp\left(-\frac{t^{3/2}}{8C_0^{1/2}}\left(3 - \frac{2C_0^{1/2}}{t^{1/2}}\log(d)\right)\right) \\ &\qquad \le \exp\left(-\frac{t^{3/2}}{8C_0^{1/2}}\right). \end{aligned} \tag{2.0.23}$$

Combining (2.0.21), (2.0.22) and (2.0.23) we get (2.0.15). Therefore we have

$$\begin{aligned} \int_{[0,1)^d}\left(\sum_{n=1}^N p(M_n x)\right)^4 dx &\le \int_0^\infty tN^2 d\mathbb{P}\left(\left|\sum_{n=1}^N p(M_n x)\right| > t^{1/4}N^{1/2}\right) \\ &\le N^2\int_0^\infty Ct\frac{t^{1/2}}{\log(d)}\exp\left(-\frac{t^{3/2}}{8C\log(d)}\right) dt \\ &\le N^2\int_0^\infty C\frac{(t')^{3/2}}{\log(d)}\exp\left(-\frac{(t')^{3/2}}{8C}\right)(\log(d))^{2/3}\, dt' \end{aligned}$$

where in the last line we substituted $t' = t/(\log(d))^{2/3}$. Thus we observe

$$\int_{[0,1)^d}\left(\sum_{n=1}^N p(M_n x)\right)^4 dx \le C(\log(d))^{2/3}N^2$$

for some constant $C > 0$ which only depends on q. Therefore the Lemma is proved.

3 Central Limit Theorem and Law of the Iterated Logarithm

In this chapter we discuss the fundamental results on the probabilistic behaviour of multivariate lacunary sequences satisfying the Hadamard gap condition (1.0.4). After presenting the main theorems in section 3.1 we prove the Central Limit Theorem in section 3.2. The Law of the Iterated Logarithm is shown in section 3.3. On this basis we establish the Law of the Iterated Logarithm for lacunary point sets in section 3.4. Furthermore in section 3.5 we demonstrate modifications of the theorems discussed so far resp. their proofs for multivariate lacunary sequences satisfying the weak Hadamard gap condition (1.0.5)

3.1 Main results

The Central Limit Theorem for multivariate lacunary systems reads as follows

Theorem 3.1.1 (Central Limit Theorem) *Let $(M_n)_{n\geq 1}$ be a lacunary sequence of non-singular $d\times d$-matrices satisfying the Hadamard gap condition (1.0.4). Furthermore assume that $L(G,N) = o(N)$ for any fixed $G \geq 2$. Let $f : \mathbb{R}^d \to \mathbb{R}$ be a bounded, periodic function of finite total variation in the sense of Hardy and Krause. Assume that the Fourier series of f exists and converges to f and assume that there exists some absolute constant $C > 0$ such that*

$$\sigma_N^2 := \int_{[0,1)^d} \left(\sum_{n=1}^{N} f(M_n x) \right)^2 dx \geq C \cdot N. \tag{3.1.1}$$

for any $N \geq 1$. Then for all $t \in \mathbb{R}$ we have

$$\lim_{N\to\infty} \left| \mathbb{P}\left(\sum_{n=1}^{N} f(M_n x) \leq t\sigma_N \right) - \Phi(t) \right| = 0. \tag{3.1.2}$$

If furthermore for some $0 < \beta < 1$ we have $L(N, G_N) = \mathcal{O}(N^\beta)$ for any sequence $(G_N)_{N\geq 1}$ with $G_N \leq dN$ for all $N \geq 1$ then for all $t \in \mathbb{R}$ and sufficiently large N we get

$$\left| \mathbb{P}\left(\sum_{n=1}^{N} f(M_n x) \leq t\sigma_N \right) - \Phi(t) \right| \leq C \frac{d^{1/5} \log(N)^{3/5} + \log(d)\log(N)}{N^{\min(1/8,(1-\beta)/5)}} \tag{3.1.3}$$

with some absolute constant $C > 0$ which only depends only on q.

Under a slightly stronger condition on the number of solutions of the Diophantine equation we also obtain

Theorem 3.1.2 (Law of the Iterated Logarithm) *Let $(M_n)_{n\geq 1}$ be a sequence of non-singular $d\times d$-matrices satisfying the Hadamard gap condition (1.0.4). Furthermore assume that for any fixed $G \geq 1$ and some $\varepsilon > 0$ we have $L(N,G) = \mathcal{O}(N/(\log N)^{1+\varepsilon})$. Let $f : \mathbb{R}^d \to \mathbb{R}$ be a bounded, periodic function of finite total variation in the sense of Hardy and Krause. Assume that the Fourier series of f exists and converges to f. Additionally, let f and $(M_n)_{n\geq 1}$ be given such that for*

$$\sigma_N^2 := \int_{[0,1)^d} \left(\sum_{n=1}^{N} f(M_n x) \right)^2 dx$$

there exists $\Sigma_{f,M_n} > 0$ with

$$\lim_{N\to\infty} \frac{\sigma_N^2}{N} = \Sigma_{f,M_n}. \tag{3.1.4}$$

Then we have

$$\limsup_{N\to\infty} \frac{|\sum_{k=1}^{N} f(M_n x)|}{\sqrt{2N \log(\log(N))}} = \sqrt{\Sigma_{f,M_n}} \quad \text{a.e.} \tag{3.1.5}$$

We now state a version of the Law of the Iterated Logarithm for the discrepancy of point sets defined by multivariate lacunary sequences with not too many non-trivial solutions of the Diophantine equation:

Theorem 3.1.3 *Let $(M_n)_{n\geq 1}$ be a lacunary sequence of non-singular $d\times d$-matrices satisfying the Hadamard gap condition (1.0.4). Assume that $L(N,G) = \mathcal{O}(N/(\log N)^{1+\varepsilon})$ and furthermore $L^*(N,G,0) = o(N)$. Then the discrepancy of $(M_n x)_{n\geq 1}$ resp. the star discrepancy satisfies the Law of the Iterated Logarithm, i.e.*

$$\limsup_{N\to\infty} \frac{ND_N(M_1 x, \dots, M_N x)}{\sqrt{2N \log(\log(N))}} = \limsup_{N\to\infty} \frac{ND_N^*(M_1 x, \dots, M_N x)}{\sqrt{2N \log(\log(N))}} = \frac{1}{2} \quad \text{a.e.} \tag{3.1.6}$$

3.2 Central Limit Theorem

The proof of Theorem 3.1.1 is essentially based on the following Theorem due to Heyde and Brown [36] which is a consequence of Strassen's almost sure invariance principle for martingale differences sequences. We will use a generalized version stated in [2].

Theorem 3.2.1 ([2, Theorem B],[36]) *Let $(X_k, \mathcal{F}_k, k \geq 1)$ be a martingale differences sequence with finite fourth moments. Set $V_K = \sum_{k=1}^{K} \mathbb{E}[X_k^2|\mathcal{F}_{k-1}]$ and let $(b_K)_{K\geq 1}$ be a sequence of positive numbers. Then we have*

$$\sup_t \left| \mathbb{P}\left(\frac{1}{\sqrt{b_K}} \sum_{k=1}^{K} X_k < t \right) - \Phi(t) \right| \leq A \left(\frac{\sum_{k=1}^{K} \mathbb{E}[X_k^4] + \mathbb{E}\left[(V_K - b_K)^2\right]}{b_K^2} \right)^{1/5}, \tag{3.2.1}$$

where A is an absolute constant.

Proof of Theorem 3.1.1. First we are going to show (3.1.3). Therefore for some fixed but large enough integer $N \geq 1$ set $G = \lfloor d \max(2, N^\alpha) \rfloor$ for some $0 < \alpha \leq 1$ and define $p = p_G$ and $r = r_G$ as in (2.0.2). Without loss of generality we may assume $||f||_\infty \leq 1$ and $V_{HK}(f) \leq 1$. Therefore it is easy to see that $||p||_2 \leq ||f||_2 \leq 1$ and $||p||_\infty \leq ||f||_\infty \leq 1$ as well as $||r||_\infty \leq ||f||_\infty + ||p||_\infty \leq 2$. We now decompose the set $\{1, \dots, N\}$ into consecutive blocks $\Delta_1, \Delta_2', \Delta_2, \dots, \Delta_k', \Delta_k, \dots$ such that the blocks have length

$$|\Delta_k'| = \lceil 2(1+2\eta) \log_q(k) + C_{d,G} \rceil, \quad |\Delta_k| = \lfloor 12\eta^{-1} C_{d,G}^{1+\eta} k^\eta \rfloor.$$

for some $0 < \eta < 1$. The constant $C_{d,G}$ is defined by

$$C_{d,G} = C(\log(G) + \log(d) + d \log(5 \log(G))) \tag{3.2.2}$$

for some large enough C depending only on q. It can easily be shown that $|\Delta_{k+1}'| \leq |\Delta_k|$ for all $k \geq 1$. In order to define a suitable martingale differences sequence we replace f by its low-frequency part p which is a finite trigonometric polynomial. Furthermore we will neglect the indices in Δ_k'. The purpose of this is having a fast enough decreasing ratio

$$\frac{||M_{(k-1)^+}||_\infty}{||M_{k^-}||_\infty} \leq k^{-2(1+2\eta)} q^{-C_{d,G}},$$

where k^+, k^- is the largest resp. the smallest integer in Δ_k. Later on it will be shown that the asymptotic size of $\sum_{n=1}^N r(M_n x)$ and $\sum_k \sum_{n \in \Delta_k'} p(M_n x)$ can be neglected. We now approximate $p(M_n x)$ by a piecewise constant function $\varphi_n(x)$ which is necessary to define the martingale differences sequence. Let

$$m(n) = \lceil \log_2(||M_{k^+}^T||_\infty) + (1+2\eta) \log_2(k) + C_{d,G}' \rceil$$

where $k = k_n$ is defined by $n \in \Delta_{k_n}$ and $C_{d,G}'$ is a constant depending on q, d and G such that

$$\begin{aligned} 2(1+\eta) \log_2(C_{d,G}) + \log_2(G) + \log_2(d) + (2d+1) \log_2(5 \log(G)) \\ \leq C_{d,G}' \\ \leq \log_2(q) C_{d,G} - 2(1+\eta) \log_2(C_{d,G}) - d \log_2(5 \log(G)). \end{aligned} \tag{3.2.3}$$

Observe that such a constant $C_{d,G}'$ exists if $C_{d,G}$ in (3.2.2) is chosen large enough. Let $\mathcal{F}_k$ be the σ-field generated all sets of the form

$$\left[\frac{v_1}{2^{m(k^+)}}, \frac{v_1+1}{2^{m(k^+)}}\right) \times \cdots \times \left[\frac{v_d}{2^{m(k^+)}}, \frac{v_d+1}{2^{m(k^+)}}\right)$$

with some $v_i \in \{0,1,\dots,2^{m(k^+)}-1\}$ and $i \in \{1,\dots,d\}$. With $x, x' \in \mathcal{A}_k$ for some atom $\mathcal{A}_k = \prod_{i=1}^d \mathcal{A}_{k,i} \in \mathcal{F}_k$ we have

$$\begin{aligned}
|p(M_n x) - p(M_n x')| &\leq \sum_{i=1}^d \sup_{y\in\mathcal{A}_k} \left|\frac{\partial}{\partial y_i} p(M_n y)\right| 2^{-m(n)} \\
&\leq Cd||M_n^T||_\infty G(5\log(G))^d 2^{-m(n)} \\
&\leq C \cdot C_{d,G}^{-2(1+\eta)} k^{-(1+2\eta)}.
\end{aligned}$$

Here the second inequality follows by Lemma 2.0.1. The third inequality follows by (3.2.3). Therefore on any atom of $\mathcal{F}_k$ we easily can find some constant function, say $\hat{\varphi}_n$, such that

$$|p(M_n x) - \hat{\varphi}_n(x)| \leq C \cdot C_{d,G}^{-2(1+\eta)} k^{-(1+2\eta)}. \tag{3.2.4}$$

We have

$$p(M_n x) = \sum_{1\leq ||j||_\infty \leq G} a'_j \cos(2\pi\langle M_n^T j, x\rangle) + b'_j \sin(2\pi\langle M_n^T j, x\rangle).$$

Thus for any atom $\mathcal{A}_{k-1} \subset \mathcal{F}_{k-1}$ we obtain

$$\begin{aligned}
&\frac{1}{\lambda(\mathcal{A}_{k-1})}\left|\int_{\mathcal{A}_{k-1}} p(M_n x)\, dx\right| \\
&\leq \frac{1}{\lambda(\mathcal{A}_{k-1})} \sum_{1\leq ||j||_\infty\leq G} |a'_j| \left|\int_{\mathcal{A}_{k-1}} \cos(2\pi\langle M_n^T j, x\rangle)\, dx\right| + |b'_j| \left|\int_{\mathcal{A}_{k-1}} \sin(2\pi\langle M_n^T j, x\rangle)\, dx\right|
\end{aligned}$$

where λ denotes the Lebesgue-measure on $[0,1)^d$. For fixed n and j choose $i \in \{1,\dots,d\}$ such that $|(M_n^T j)_i| = ||M_n^T j||_\infty$. Then we observe by angle sum and difference identities of trigonometric functions that

$$\int_{B_k} \cos(2\pi\langle M_n^T j, x\rangle)\, dx = \int_{B_k} \sin(2\pi\langle M_n^T j, x\rangle)\, dx = 0$$

for any box B_k such that $B_{k,i'} = \mathcal{A}_{k,i'}$ for $i \neq i'$ and $\lambda(B_{k,i}) = v/||M_n^T j||_\infty$ for some integer v. Thus we have

$$\begin{aligned}
&\frac{1}{\lambda(\mathcal{A}_{k-1})}\left|\int_{\mathcal{A}_{k-1}} p(M_n x)\, dx\right| \\
&\qquad \leq \sum_{1\leq ||j||_\infty\leq G} \frac{|a'_j| + |b'_j|}{||M_n^T j||_\infty} 2^{m((k-1)^+)} \\
&\qquad \leq \sum_{1\leq ||j||_\infty\leq G} (|a'_j| + |b'_j|) \cdot \frac{||M_{(k-1)^+}^T||_\infty}{q^{\lceil \log_q(G)\rceil} ||M_{k^- - \lceil \log_q(G)\rceil}^T||_\infty} k^{1+2\eta} 2^{C'_{d,G}} \qquad (3.2.5) \\
&\qquad \leq C \cdot (5\log(G))^d q^{-C_{d,G}} k^{-(1+2\eta)} 2^{C'_{d,G}} \\
&\qquad \leq C \cdot C_{d,G}^{-2(1+\eta)} k^{-(1+2\eta)}.
\end{aligned}$$

Now set

$$\varphi_n(x) = \hat{\varphi}_n(x) - \mathbb{E}[\hat{\varphi}_n|\mathcal{F}_{k-1}].$$

Then by (3.2.4) we have

$$\begin{aligned} |p(M_n x) - \varphi_n(x)| &\leq |p(M_n x) - \hat{\varphi}_n(x)| + |\hat{\varphi}_n(x) - \varphi_n(x)| \\ &\leq C \cdot C_{d,G}^{-2(1+\eta)} k^{-(1+2\eta)} \end{aligned} \tag{3.2.6}$$

for some absolute constant $C > 0$ depending only on q. Thus $p(M_n x)$ can be approximated by a function $\varphi_n(x)$, which is constant on any atom of $\mathcal{F}_k$ and satisfies:

1. $|p(M_n x) - \varphi_n(x)| \leq C \cdot C_{d,G}^{-2(1+\eta)} k^{-(1+2\eta)}$ for all $x \in [0,1)^d$,
2. $\mathbb{E}[\varphi_n|\mathcal{F}_{k-1}] = 0$ for all $n \in \Delta_k$.

We now define

$$X_k = \sum_{n \in \Delta_k} \varphi_n(x), \quad V_{K+1} = \sum_{k=1}^{K+1} \mathbb{E}[X_k^2|\mathcal{F}_{k-1}],$$

where K is given such that $N \in \Delta'_{K+1} \cup \Delta_{K+1}$. It is easy to see that $(X_k, \mathcal{F}_k)$ is a martingale differences sequence. We are going to approximate $\sum_{n=1}^N f(M_n x)$ by $\sum_{k=1}^K X_k$ in order to apply Theorem 3.2.1. Therefore we need some more notations. For $k \in \{1, \ldots, K+1\}$ define

$$Y_k = \sum_{n \in \Delta_k} p(M_n x), \quad Y'_k = \sum_{n \in \Delta'_k} p(M_n x).$$

We observe

$$N \leq \sum_{k=1}^{K+1} |\Delta'_k| + |\Delta_k| \leq 2 \sum_{k=1}^{K+1} |\Delta_k| \leq C\eta^{-1} \cdot C_{d,G}^{1+\eta} K^{1+\eta}$$

and

$$N \geq \sum_{k=1}^{K} |\Delta'_k| + |\Delta_k| \geq \sum_{k=1}^{K} |\Delta_k| \geq C\eta^{-1} \cdot C_{d,G}^{1+\eta} K^{1+\eta}.$$

There are constants $C_1, C_2 > 0$ such that

$$C_1 \eta^{-1/(1+\eta)} C_{d,G}^{-1} N^{1/(1+\eta)} \leq K \leq C_2 \eta^{-1/(1+\eta)} C_{d,G}^{-1} N^{1/(1+\eta)}. \tag{3.2.7}$$

By definition we have the following decomposition

$$\sum_{n=1}^{N} f(M_n x) = \sum_{k=1}^{K+1} X_k + \sum_{k=1}^{K+1} (Y_k - X_k) + \sum_{k=1}^{K+1} Y'_k + \sum_{n=1}^{N} r(M_n x).$$

Then standard estimates give

$$\begin{aligned}\mathbb{P}\left(\sum_{n=1}^{N} f(M_n x) \leq t\sigma_N\right) &\leq \mathbb{P}\left(\sum_{k=1}^{K+1} X_k \leq (t+\varepsilon)\sigma_N\right) + \mathbb{P}\left(\left|\sum_{k=1}^{K+1}(X_k - Y_k)\right| > \frac{\varepsilon\sigma_N}{3}\right) \\ &\quad + \mathbb{P}\left(\left|\sum_{k=1}^{K+1} Y_k'\right| > \frac{\varepsilon\sigma_N}{3}\right) + \mathbb{P}\left(\left|\sum_{n=1}^{N} r(M_n x)\right| > \frac{\varepsilon\sigma_N}{3}\right),\end{aligned} \tag{3.2.8}$$

for some $\varepsilon > 0$ as well as

$$\begin{aligned}\mathbb{P}\left(\sum_{n=1}^{N} f(M_n x) \geq t\sigma_N\right) &\geq \mathbb{P}\left(\sum_{k=1}^{K+1} X_k \geq (t-\varepsilon)\sigma_N\right) - \mathbb{P}\left(\left|\sum_{k=1}^{K+1}(X_k - Y_k)\right| > \frac{\varepsilon\sigma_N}{3}\right) \\ &\quad - \mathbb{P}\left(\left|\sum_{k=1}^{K+1} Y_k'\right| > \frac{\varepsilon\sigma_N}{3}\right) - \mathbb{P}\left(\left|\sum_{n=1}^{N} r(M_n x)\right| > \frac{\varepsilon\sigma_N}{3}\right).\end{aligned}$$

Since both inequalities can be estimated analogously we will only focus on the first one. First we estimate the three latter terms before we estimate the first one by applying Theorem 3.2.1. In order to estimate the second term by (3.2.6) we have

$$\begin{aligned}\left|\sum_{k=1}^{K+1}(X_k - Y_k)\right| &= \left|\sum_{k=1}^{K+1}\left(\sum_{n\in\Delta_k}\varphi_n(x) - \sum_{n\in\Delta_k} p(M_n x)\right)\right| \\ &\leq \sum_{k=1}^{K+1} C|\Delta_k| \cdot C_{d,G}^{-2(1+\eta)} k^{-(1+2\eta)} \\ &\leq C\eta^{-1} C_{d,G}^{-(1+\eta)}\end{aligned}$$

for some constant $C > 0$ depending only on q. Therefore by Chebyshev's inequality we see that

$$\mathbb{P}\left(\left|\sum_{k=1}^{K+1}(X_k - Y_k)\right| > \frac{\varepsilon\sigma_N}{3}\right) \leq C\eta^{-2} C_{d,G}^{-2(1+\eta)} \varepsilon^{-2} N^{-1}. \tag{3.2.9}$$

To estimate the third term we use the definition of Y_k', Lemma 2.0.3 and (3.2.7) to get

$$\begin{aligned}\left\|\sum_{k=1}^{K+1} Y_k'\right\|_2^2 &\leq C\sum_{k=1}^{K+1} |\Delta_k'| \log(d) \\ &\leq C\log(d) N^{1/(1+\eta)} \log(N).\end{aligned}$$

Thus by (3.1.1) and Chebyshev's inequality we obtain

$$\mathbb{P}\left(\left|\sum_{k=1}^{K+1} Y_k'\right| > \frac{\varepsilon\sigma_N}{3}\right) \leq C\log(d)\varepsilon^{-2} N^{-\eta/(1+\eta)} \log(N). \tag{3.2.10}$$

With another application of Chebyshev's inequality, Lemma 2.0.2 and Lemma 2.0.3 we get the following estimate for the fourth term

$$\mathbb{P}\left(\left|\sum_{n=1}^{N} r(M_n x)\right| > \frac{\varepsilon\sigma_N}{3}\right) \leq C\log(d)\varepsilon^{-2}d^{1/2}G^{-1/2}. \tag{3.2.11}$$

Thus by (3.2.8),(3.2.9),(3.2.10) and (3.2.11) we have

$$\begin{aligned}\mathbb{P}&\left(\sum_{n=1}^{N} f(M_n x) \leq t\sigma_N\right)\\ &\leq \mathbb{P}\left(\sum_{k=1}^{K+1} X_k \leq (t+\varepsilon)\sigma_N\right) + C\eta^{-2}\log(d)\varepsilon^{-2}(N^{-\eta/(1+\eta)}\log(N) + d^{1/2}G^{-1/2}).\end{aligned} \tag{3.2.12}$$

Now for any integer $k \in \{1, \ldots, K+1\}$ define

$$s_k^2 = \sum_{l=1}^{k} \int_{[0,1]^d} \left(\sum_{n\in\Delta_l} p(M_n x)\right)^2 dx.$$

Thus by we obtain

$$\begin{aligned}&\left|\mathbb{P}\left(\sum_{n=1}^{N} f(M_n x) \leq t\sigma_N\right) - \Phi(t)\right|\\ &\qquad \leq \left|\mathbb{P}\left(\sum_{k=1}^{K+1} X_k \leq s_{K+1}(t+\varepsilon)\frac{\sigma_N}{s_{K+1}}\right) - \Phi\left((t+\varepsilon)\frac{\sigma_N}{s_{K+1}}\right)\right|\\ &\qquad + \left|\Phi\left((t+\varepsilon)\frac{\sigma_N}{s_{K+1}}\right) - \Phi(t)\right|\\ &\qquad + C\eta^{-2}\log(d)\varepsilon^{-2}(N^{-\eta/(1+\eta)}\log(N) + d^{1/2}G^{-1/2}).\end{aligned} \tag{3.2.13}$$

To apply Theorem 3.2.1 we need a sequence of positive numbers $(b_K)_{K\geq 1}$. In [36] the sequence was given by $\sum_{k=1}^{K}\mathbb{E}\left[\mathbb{E}[X_k^2|\mathcal{F}_{k-1}]\right]$. Here we take s_K^2 instead since later we are going to estimate the conditional second moments of X_k by those of Y_k. In order to estimate the second term on the right-hand side of (3.2.13) we now show that for some $C > 0$ we have

$$|\sigma_N - s_{K+1}| \leq (C\eta^{-1/(1+\eta)}\log(d)(N^{1/(1+\eta)}\log(N) + Nd^{1/2}G^{-1/2}))^{1/2}. \tag{3.2.14}$$

Therefore we use standard estimates and observe

$$\begin{aligned}
&\left| s_{K+1}^2 - \int_{[0,1)^d} \left(\sum_{n\in\cup_{k=1}^{K+1}} p(M_n x) \right)^2 dx \right| \\
&\qquad \le \left| \int_{[0,1)^d} \left(\sum_{k=1}^{K+1} \left(\sum_{n\in\Delta_k} p(M_n x) \right)^2 - \left(\sum_{n\in\cup_k\Delta_k} p(M_n x) \right)^2 \right) dx \right| \\
&\qquad \le \sum_{\substack{n,n'\in\cup_k\Delta_k,\\ (n,n')\notin\Delta_k\times\Delta_k}} \left| \int_{[0,1)^d} p(M_n x) p(M_{n'} x)\, dx \right| \\
&\qquad = 0
\end{aligned} \tag{3.2.15}$$

since $||M_n^T j \pm M_{n'}^T j'||_\infty > 0$ if $1 \le ||j||_\infty, ||j'||_\infty \le G$ and $|n-n'| \ge \log_q(G)$. We now decompose

$$\sum_{n=1}^N f(M_n x) = \sum_{n\in\cup_{k=1}^{K+1}\Delta_k'} p(M_n x) + \sum_{n\in\cup_{k=1}^{K+1}\Delta_k} p(M_n x) + \sum_{n=1}^N r(M_n x)$$

By (3.2.15) we have

$$\left\| \sum_{n\in\cup_{k=1}^{K+1}\Delta_k} p(M_n x) \right\|_2 \le s_{K+1}. \tag{3.2.16}$$

Now with Lemma 2.0.3 and (3.2.7) we obtain

$$\left\| \sum_{n\in\cup_{k=1}^{K+1}\Delta_k'} p(M_n x) \right\|_2 \le \left(C\eta^{-1/(1+\eta)} \log(d) N^{1/(1+\eta)} \log(N) \right)^{1/2}. \tag{3.2.17}$$

By Lemma 2.0.2 and 2.0.3 we observe

$$\left\| \sum_{n\in\cup_{k=1}^{K+1}\Delta_k} r(M_n x) \right\|_2 \le \left(C \log(d) d^{1/2} G^{-1/2} N \right)^{1/2}. \tag{3.2.18}$$

Therefore we obtain (3.2.14). If we choose $N \in \mathbb{N}$ large enough such that

$$\frac{N}{N^{1/(1+\eta)} \log(N) + d^{1/2} G^{-1/2} N} \ge \frac{9C''\eta^{-1/(1+\eta)} \log(d)}{C'} \tag{3.2.19}$$

where the constant $C', C'' > 0$ are defined such that (3.1.1) is satisfied with $C = C'$ resp. (3.2.14) is satisfied $C = C''$ then we obtain

$$|\sigma_N - s_{K+1}| \le \frac{\sqrt{C'N}}{3}.$$

Immediately we observe

$$s_{K+1} \geq \sigma_N - |\sigma_N - s_{K+1}| \geq \frac{2\sqrt{C'N}}{3}. \tag{3.2.20}$$

We furthermore get

$$\left|\frac{\sigma_N}{s_{K+1}} - 1\right| \leq (C\eta^{-1/(1+\eta)} \log(d)(N^{-\eta/(1+\eta)} \log(N) + d^{1/2}G^{-1/2}))^{1/2} \tag{3.2.21}$$

Thus it can easily be shown that for a suitable constant $C > 0$ depending only on q and some large enough N fulfilling (3.2.19) by Mean Value Theorem we have

$$\begin{aligned}
&\left|\Phi\left((t+\varepsilon)\frac{\sigma_N}{s_{K+1}}\right) - \Phi(t)\right| \\
&\leq \left(\frac{\sigma_N}{s_{K+1}}\varepsilon + \left|\left(\frac{\sigma_N}{s_{K+1}} - 1\right)t\right|\right) \sup\left\{\frac{1}{\sqrt{2\pi}}e^{-1/2\cdot u^2} : t \leq u \leq (t+\varepsilon)\frac{\sigma_N}{s_{K+1}}\right\} \\
&\leq (C\eta^{-1/(1+\eta)} \log(d))^{1/2}(\varepsilon + (N^{-\eta/(1+\eta)} \log(N) + d^{1/2}G^{-1/2})^{1/2}).
\end{aligned} \tag{3.2.22}$$

We plug this into (3.2.13) and get

$$\begin{aligned}
&\left|\mathbb{P}\left(\sum_{n=1}^{N} f(M_n x) \leq t\sigma_N\right) - \Phi(t)\right| \\
&\qquad \leq \left|\mathbb{P}\left(\sum_{k=1}^{K+1} X_k \leq s_{K+1}t\right) - \Phi(t)\right| \\
&\qquad\quad + (C\eta^{-1/(1+\eta)} \log(d))^{1/2}(\varepsilon + (N^{-\eta/(1+\eta)} \log(N) + d^{1/2}G^{-1/2})^{1/2}) \\
&\qquad\quad + C\eta^{-2} \log(d)\varepsilon^{-2}(N^{-\eta/(1+\eta)} \log(N) + d^{1/2}G^{-1/2}).
\end{aligned} \tag{3.2.23}$$

Therefore it remains to estimate

$$\left|\mathbb{P}\left(\sum_{k=1}^{K+1} X_k \leq s_{K+1}t\right) - \Phi(t)\right|$$

for which we use Theorem 3.2.1. By Lemma 2.0.5 we easily see

$$\sum_{k=1}^{K+1} \mathbb{E}[X_k^4] \leq \sum_{k=1}^{K+1} C \log(d)^{2/3}|\Delta_k|^2 \leq C\eta^{-2} \log(d)^{2/3} C_{d,G}^{2(1+\eta)} K^{1+2\eta}. \tag{3.2.24}$$

We define

$$\varsigma_k = \int_{[0,1)^d} \left(\sum_{n\in\Delta_k} p(M_n x)\right)^2 dx.$$

By (3.2.4) we observe $|X_k^2 - Y_k^2| \le C\eta^{-2}k^{-1}$ and therefore we obtain

$$||V_{K+1} - s_{K+1}^2||_2 \le \left|\left|\sum_{k=1}^{K+1} \mathbb{E}\left[Y_k^2 - \varsigma_k|\mathcal{F}_{k-1}\right]\right|\right|_2 + C\eta^{-2}\log(K). \tag{3.2.25}$$

We now are going to decompose the terms $Y_k^2 - \varsigma_k$. Therefore we set

$$\begin{aligned} \mathcal{R}_k(x) &= \sum_{n\in\Delta_k} \sum_{\substack{n'\in\Delta_k,\\ |n-n'|\le 1+\log_q(G+1)}} p(M_n x)p(M_{n'}x), \\ Q_k(x) &= Y_k^2 - \mathcal{R}_k(x). \end{aligned}$$

Furthermore we define

$$\begin{aligned} T^+(j,j',n,n',x) =& \frac{a'_j a'_{j'}}{2}\cos(2\pi\langle M_n^T j + M_{n'}^T j', x\rangle) + \frac{a'_j b'_{j'}}{2}\sin(2\pi\langle M_n^T j + M_{n'}^T j', x\rangle) \\ &+ \frac{b'_j a'_{j'}}{2}\sin(2\pi\langle M_n^T j + M_{n'}^T j', x\rangle) - \frac{b'_j b'_{j'}}{2}\cos(2\pi\langle M_n^T j + M_{n'}^T j', x\rangle), \\ T^-(j,j',n,n',x) =& \frac{a'_j a'_{j'}}{2}\cos(2\pi\langle M_n^T j - M_{n'}^T j', x\rangle) + \frac{a'_j b'_{j'}}{2}\sin(2\pi\langle M_n^T j - M_{n'}^T j', x\rangle) \\ &+ \frac{b'_j a'_{j'}}{2}\sin(2\pi\langle M_n^T j - M_{n'}^T j', x\rangle) - \frac{b'_j b'_{j'}}{2}\cos(2\pi\langle M_n^T j - M_{n'}^T j', x\rangle), \end{aligned} \tag{3.2.26}$$

for $1 \le ||j||_\infty, ||j'||_\infty \le G$, $n, n' \in \mathbb{N}$ and $x \in [0,1)^d$. We set

$$\begin{aligned} R_k(x) &= \sum_{\substack{n,n'\in\Delta_k,\\ |n-n'|\le 1+\log_q(1+G)}} \sum_{\substack{1\le||j||_\infty,||j'||_\infty\le G,\\ ||M_n^T j+M_{n'}^T j'||_\infty \ge ||M_{(k-1)^+}^T||_\infty}} T^+(j,j',n,n',x) \\ &+ \sum_{\substack{n,n'\in\Delta_k,\\ |n-n'|\le 1+\log_q(1+G)}} \sum_{\substack{1\le||j||_\infty,||j'||_\infty\le G,\\ ||M_n^T j-M_{n'}^T j'||_\infty \ge ||M_{(k-1)^+}^T||_\infty}} T^-(j,j',n,n',x) \end{aligned} \tag{3.2.27}$$

and

$$\begin{aligned} S_k(x) &= \sum_{\substack{n,n'\in\Delta_k,\\ |n-n'|\le 1+\log_q(1+G)}} \sum_{\substack{1\le||j||_\infty,||j'||_\infty\le G,\\ 0<||M_n^T j+M_{n'}^T j'||_\infty < ||M_{(k-1)^+}^T||_\infty}} T^+(j,j',n,n',x) \\ &+ \sum_{\substack{n,n'\in\Delta_k,\\ |n-n'|\le 1+\log_q(1+G)}} \sum_{\substack{1\le||j||_\infty,||j'||_\infty\le G,\\ 0<||M_n^T j-M_{n'}^T j'||_\infty < ||M_{(k-1)^+}^T||_\infty}} T^-(j,j',n,n',x). \end{aligned} \tag{3.2.28}$$

Thus we have $Y_k^2 - \varsigma_k = Q_k + R_k + S_k$. In order to estimate $|\mathbb{E}[Q_k|\mathcal{F}_{k-1}]|$ we first observe that for $|n'-n| > 1 + \log_q(G+1)$ and $1 \le ||j||_\infty, ||j'||_\infty \le G$ we get

$$||M_n^T j \pm M_{n'}^T j'||_\infty > ||M_{\min(n,n')}^T||_\infty \ge ||M_{k^-}^T||_\infty.$$

Therefore if $C_{d,G}$ is large enough with a similar argumentation as in the proof of (3.2.6) we get

$$\begin{aligned}
|\mathbb{E}[Q_k|\mathcal{F}_{k-1}] &\leq \sum_{\substack{n,n'\in\{1,\ldots,N\},\\ |n-n'|>1+\log_q(G+1)}} \sum_{1\leq ||j||_\infty, ||j'||_\infty \leq G} \frac{1}{\lambda(\mathcal{A}_{k-1})} \\
&\quad \cdot \left| \int_{\mathcal{A}_{k-1}} T^+(j,j',n,n',x) + T^-(j,j',n,n',x)\, dx \right| \\
&\leq C(5\log(G))^{2d}|\Delta_k|^2 \frac{||M^T_{(k-1)^+}||_\infty k^{1+2\eta} 2^{C'_{d,G}}}{||M^T_{k^-}||_\infty} \\
&\leq C\eta^{-2}(5\log(G))^{2d} C_{d,G}^{2(1+\eta)} k^{2\eta} k^{1+2\eta} 2^{C'_{d,G}} k^{-2(1+2\eta)} q^{-C_{d,G}} \\
&\leq Ck^{-1}
\end{aligned}$$

for some constant $C > 0$ depending only on q. Thus we have

$$\left|\left| \sum_{k=1}^{K+1} \mathbb{E}[Q_k|\mathcal{F}_{k-1}] \right|\right|_2 \leq C\log(K). \tag{3.2.29}$$

By Lemma 2.0.4 for $1 \leq n, n' \leq N$ and $||j||_\infty \leq G$ there exists at most one $||j'||_\infty \leq G$ such that $||M_n^T j \pm M_{n'}^T j'||_\infty \leq ||M^T_{(k-1)^+}||_\infty$. Hence with Cauchy-Schwarz inequality we observe $||S_k||_\infty, ||\varsigma_k||_\infty \leq (1+\log_q(G+1))|\Delta_k|||p||_2^2 \leq (1+\log_q(G+1))|\Delta_k|$. By definition we get $||\mathcal{R}_k||_\infty \leq (1 + \log_q(G + 1))|\Delta_k|$ and therefore we also have

$$||R_k||_\infty \leq ||\mathcal{R}_k||_\infty + ||S_k||_\infty + ||\varsigma_k||_\infty \leq C(1 + \log_q(G + 1))|\Delta_k|.$$

Now we estimate $\sum_{k=1}^{K+1} \mathbb{E}[R_k|\mathcal{F}_{k-1}]$ obtaining

$$\mathbb{E}\left[\left(\sum_{k=1}^{K+1} \mathbb{E}[R_k|\mathcal{F}_{k-1}] \right)^2 \right] \leq 2\mathbb{E}\left[\sum_{k,k'=1}^{K+1} \mathbb{E}[R_k|\mathcal{F}_{k-1}]\mathbb{E}[R_{k'}|\mathcal{F}_{k'-1}] \right].$$

For $k = k'$ we have

$$\sum_{k=1}^{K+1} \mathbb{E}^2[R_k|\mathcal{F}_{k-1}] \leq \sum_{k=1}^{K+1} ||R_k||_\infty^2 \leq C\eta^{-2} C_{d,G}^{2(1+\eta)} (\log(G))^2 K^{1+2\eta}. \tag{3.2.30}$$

We may assume $k' > k$ now. Since $\mathbb{E}[R_k|\mathcal{F}_{k-1}]$ is $\mathcal{F}_{k-1}$-measurable we get

$$\begin{aligned}
&\left| \mathbb{E}\left[\sum_{1\leq k<k'\leq K+1} \mathbb{E}[R_k|\mathcal{F}_{k-1}]\mathbb{E}[R_{k'}|\mathcal{F}_{k'-1}] \middle| \mathcal{F}_{k-1} \right] \right| \\
&\qquad \leq \sum_{1\leq k<k'\leq K+1} ||R_k||_\infty |\mathbb{E}[R_{k'}|\mathcal{F}_{k-1}]| \\
&\qquad \leq \sum_{1\leq k<k'\leq K+1} C\eta^{-1} C_{d,G}^{1+\eta} k^\eta (1 + \log_q(G + 1))|\mathbb{E}[R_{k'}|\mathcal{F}_{k-1}]|.
\end{aligned}$$

Furthermore $R_{k'}$ can be represented by

$$R_{k'}(x) = \sum_{\substack{j\in\mathbb{Z}^d,\\ ||M^T_{(k'-1)^+}||_\infty \le ||j||_\infty \le 2G||M^T_{k'^+}||_\infty}} \gamma_j \cos(2\pi\langle j,x\rangle) + \delta_u \sin(2\pi\langle j,x\rangle)$$

where because of constraints on n, n', j, j' we have

$$\sum_j |\gamma_j| + |\delta_j| \le C(5\log(G))^{2d}\eta^{-1}C_{d,G}^{1+\eta}k^\eta \log(G)$$

for some constant $C > 0$ depending only on q. Thus we have by using a similar argumentation as above

$$\begin{aligned}
&|\mathbb{E}[R_{k'}|\mathcal{F}_{k-1}]| \\
&\quad\le \frac{1}{\lambda(\mathcal{A}_{k-1})}\left|\int_{\mathcal{A}_{k-1}} \sum_j \gamma_j \cos(2\pi\langle j,x\rangle) + \delta_u \sin(2\pi\langle j,x\rangle)\,dx\right| \\
&\quad\le C(5\log(G))^{2d}\eta^{-1}C_{d,G}^{1+\eta}k^\eta \log(G)\frac{||M^T_{(k-1)^+}||_\infty k^{1+2\eta}2^{C'_{d,G}}}{||M^T_{(k'-1)^+}||_\infty} \\
&\quad\le C(5\log(G))^{2d}\eta^{-1}C_{d,G}^{1+\eta}k^\eta \log(G)k^{1+2\eta}2^{C'_{d,G}}q^{-C\eta^{-1}C_{d,G}^{1+\eta}(k'-1)^\eta} \\
&\quad\le Ck^{1+3\eta}q^{-(k'-1)^\eta}
\end{aligned} \tag{3.2.31}$$

for sufficiently large $C_{d,G}$. With (3.2.30) and (3.2.31) we conclude

$$\begin{aligned}
&\left|\left|\sum_{k=1}^{K+1} \mathbb{E}[R_k|\mathcal{F}_{k-1}]\right|\right|_2 \\
&\quad\le \left(C\eta^{-2}C_{d,G}^{2(1+\eta)}\log(G)^2K^{1+2\eta} + \sum_{1\le k<k'\le K+1} k^{1+3\eta}q^{-(k'-1)^\eta}\right)^{1/2} \\
&\quad\le C\eta^{-1}C_{d,G}^{1+\eta}\log(G)K^{1/2+\eta}.
\end{aligned} \tag{3.2.32}$$

Finally we estimate S_k which can be written as

$$S_k = \sum_{0<||j||_\infty<||M^T_{(k-1)^+}||_\infty} \gamma_j \cos(2\pi\langle j,x\rangle) + \delta_j \sin(2\pi\langle j,x\rangle),$$

where $\sum_j |\gamma_j| + |\delta_j| \le C\eta^{-1}(5\log(G))^{2d}C_{d,G}^{1+\eta}k^\eta\log(G)$. The fluctuation of S_k on any atom of $\mathcal{F}_{k-1}$ is at most

$$\sum_{0<||j||_\infty\le||M^T_{(k-1)^+}||_\infty} (|\gamma_j| + |\delta_j|)2\pi G||M^T_{(k-1)^+}||_\infty \cdot 2^{-m((k-1)^+)}d \le C\eta^{-1}k^{-(1+\eta)}.$$

where the inequality follows by definition of $m((k-1)^+)$ and (3.2.3). Therefore we get

$$|\mathbb{E}[S_k|\mathcal{F}_{k-1}] - S_k| \le C\eta^{-1}k^{-(1+\eta)}.$$

Thus we have

$$\left|\left|\sum_{k=1}^{K+1} \mathbb{E}[S_k|\mathcal{F}_{k-1}]\right|\right|_2 \le \left|\left|\sum_{k=1}^{K+1} S_k\right|\right|_2 + C\eta^{-1} \tag{3.2.33}$$

for some constant depending $C > 0$ only on q. We write

$$\sum_{k=1}^{K+1} S_k(x) = \sum_{0<||j||_\infty<||M_{K+}^T||_\infty} \gamma'_j \cos(2\pi\langle j, x\rangle) + \delta'_j \sin(2\pi\langle j, x\rangle),$$

where by $L(N,G) \le CN^\beta$ we have

$$|\gamma'_j|, |\delta'_j| \le C\left(\eta^{-1}C_{d,G}^{1+\eta}K^{1+\eta}\right)^\beta. \tag{3.2.34}$$

We obtain

$$\begin{aligned}
\left|\left|\sum_{k=1}^{K+1} S_k\right|\right|_2^2 &= \sum_j \gamma_j'^2 + \delta_j'^2 \\
&\le C\left(\eta^{-1}C_{d,G}^{1+\eta}K^{1+\eta}\right)^\beta \sum_j |\gamma'_j| + |\delta'_j| \\
&\le C\left(\eta^{-1}C_{d,G}^{1+\eta}K^{1+\eta}\right)^{1+\beta}.
\end{aligned}$$

By (3.2.33) we have

$$\left|\left|\sum_{k=1}^{K+1} \mathbb{E}[S_k|\mathcal{F}_{k-1}]\right|\right|_2 \le C\left(\eta^{-1}C_{d,G}^{1+\eta}K^{1+\eta}\right)^{(1+\beta)/2} + C\eta^{-1}. \tag{3.2.35}$$

Using (3.2.25), (3.2.29), (3.2.32) and (3.2.35) we finally observe

$$\mathbb{E}\left[(V_{K+1} - s_{K+1}^2)^2\right] \le C\eta^{-2}C_{d,G}^{2(1+\eta)}\log(G)^2K^{1+\eta+\max(\eta,\beta(1+\eta))}. \tag{3.2.36}$$

With Theorem 3.2.1 and (3.2.20), (3.2.24) and (3.2.36) we obtain

$$\begin{aligned}
&\sup_t \left|\mathbb{P}\left(\sum_{k=1}^{K+1} X_k \le s_k t\right) - \Phi(t)\right| \\
&\le A\left(\frac{C\eta^{-2}C_{d,G}^{2(1+\eta)}\log(G)^2K^{1+\eta+\max(\eta,\beta(1+\eta))}}{N^2}\right)^{1/5} \\
&\le A\left(\frac{C\eta^{-2}C_{d,G}^{2(1+\eta)}\log(G)^2\left(C\eta^{-1/(1+\eta)}C_{d,G}^{-1}N^{1/(1+\eta)}\right)^{1+\eta+\max(\eta,\beta(1+\eta))}}{N^2}\right)^{1/5} \\
&\le C\eta^{-4/5}\log(G)^{2/5}C_{d,G}^{\min(\eta,(1-\beta)(1+\eta))/5}N^{\max(-1/(1+\eta),\beta-1)/5}.
\end{aligned}$$

Now we set $\alpha = 3/4$, $\eta = 3/5$ and $\varepsilon = N^{-1/8}$. Thus together with (3.2.23) we have

$$\left|\mathbb{P}\left(\sum_{k=1}^{N} f(M_n x) \le t\sigma_n\right) - \Phi(t)\right| \le C\left(\log(d)\log(N) + \log(G)^{2/5} C_{d,G}^{1/5}\right) N^{-\min(1/8,(1-\beta)/5)}$$

for some $C > 0$ which depends only on q. With (3.2.2) and $N \ge Cd$ for some constant which only depends on q and the constant used in (3.1.1) we obtain

$$\left|\mathbb{P}\left(\sum_{k=1}^{N} f(M_n x) \le t\sigma_n\right) - \Phi(t)\right| \le C\left(d^{1/5}\log(N)^{3/5} + \log(d)\log(N)\right) N^{-\min(1/8,(1-\beta)/5)}.$$

Therefore (3.1.3) is proved.
We now show (3.1.2). Therefore we take some arbitrary large $G \in \mathbb{N}$ and repeat the proof of (3.1.3). Observe that because of $L(N,G) = o(N)$ instead of (3.2.34) we get

$$|\gamma_j'|, |\delta_j'| = o\left(\eta^{-1} C_{d,G}^{1+\eta} K^{2(1+\eta)}\right)$$

and instead of (3.2.35) we also have

$$\left\|\sum_{k=1}^{K+1} \mathbb{E}[S_k|\mathcal{F}_{k-1}]\right\|_2 = o\left(\eta^{-1} C_{d,G}^{1+\eta} K^{1+\eta}\right) + C\eta^{-1}.$$

With Lemma 3.2.1 and (3.2.20), (3.2.24), (3.2.25), (3.2.29), (3.2.32), (3.2.35) and (3.2.36) we observe

$$\sup_t \left|\mathbb{P}\left(\sum_{k=1}^{K+1} X_k \le s_k t\right) - \Phi(t)\right| \le h(K)$$

for some positive function h with $\lim_{K\to\infty} h(K) = 0$. Take $\eta = 3/5$ and $\varepsilon = G^{-1/6}$ Then together with (3.2.23) for sufficiently large N we observe

$$\left|\mathbb{P}\left(\sum_{k=1}^{N} f(M_n x) \le t\sigma_n\right) - \Phi(t)\right| \le h(N) + CG^{-1/6} \tag{3.2.37}$$

for some constant $C > 0$ which depends only on q and d. Since G can be chosen arbitrary, we have shown (3.1.2) which concludes the proof of Theorem 3.1.1.

3.3 Law of the Iterated Logarithm

The following Theorem due to Strassen plays an important part in the proof of Theorem 3.1.2.

Theorem 3.3.1 ([1, Lemma 2.1],[52, Corollary 4.5]) *Let $(X_k, \mathcal{F}_k, k \geq 1)$ be a martingale differences sequence with finite fourth moments, set $V_K = \sum_{k=1}^{K} \mathbb{E}[X_k^2|\mathcal{F}_k]$ and assume $V_1 > 0$ and $V_K \to \infty$ for $K \to \infty$. Furthermore assume*

$$\lim_{K\to\infty} \frac{V_K}{b_K} = 1 \quad \text{a.s.}$$

for some sequence $(b_K)_{K\geq 1}$ of positive numbers, and

$$\sum_{K=1}^{\infty} \frac{\log(b_K)^{10}}{b_K^2} \mathbb{E}[X_K^4] < \infty.$$

Then we have

$$\limsup_{K\to\infty} \frac{\sum_{k=1}^{K} X_k}{\sqrt{2b_K \log(\log(b_K))}} = 1 \quad \text{a.s.}$$

We shall prove Theorem 3.1.2 by using this result which ensures the Law of the Iterated Logarithm for a martingale differences sequence under certain conditions. Therefore we define the martingale differences sequence in the same way as in the proof of Theorem 3.1.1, i.e. $X_k = \sum_{n\in\Delta_k} \varphi_n(x)$ where the sums are taken over a certain long blocks Δ_k with small gaps between two consecutive blocks. Furthermore the functions φ_n are piecewise constant functions which are used to approximate the trigonometric polynomials induced by the low-frequency part of f. Thus we need to give bounds for the remaining parts, i.e. the small blocks between two consecutive long blocks as well as the high-frequency part of f. Upper bounds shall be given by the following Lemma which proof is mainly based on [54] and [45] where a similar result was obtained for the one-dimensional case.

Lemma 3.3.2 *Let $(M_n)_{n\geq 1}$ be a lacunary sequence of non-singular matrices satisfying the Hadamard gap condition (1.0.4) . Let f be a bounded periodic function of finite total variation in the sense of Hardy and Krause satisfying $\mathbb{E}[f] = 0$ and $0 < ||f||_2 \leq 1$. Then we have*

$$\limsup_{N\to\infty} \frac{\left|\sum_{n=1}^{N} f(M_n x)\right|}{\sqrt{2N \log(\log(N))}} \leq C||f||_2^{1/4} \quad \text{a.e.}$$

for an absolute constant $C > 0$ depending only on q.

Proof. For some integers $R, S \in \mathbb{N}$ set $F(R,S,x) = \left|\sum_{n=1+S}^{R+S} f(M_n x)\right|$. Furthermore for $m = \max\{l \in \mathbb{N} : 2^l \leq N\}$ we have

$$\begin{aligned} &\frac{F(0,N,x)}{\sqrt{2N||f||_2^{1/2}\log\log(N))}} \\ &\qquad \leq \frac{F(0,2^m,x)}{\sqrt{2\cdot 2^m||f||_2^{1/2}\log(\log(2^m))}} + \sum_{l=\lceil m/3\rceil}^{m-1} \frac{F\left(2^m+\mu_{l+1}2^{l+1}, \delta_l 2^l, x\right)}{\sqrt{2\cdot 2^m||f||_2^{1/2}\log(\log(2^m))}} \\ &\qquad + \frac{F\left(2^m+\mu_{\lceil m/3\rceil}2^{\lceil m/3\rceil}, N^*, x\right)}{\sqrt{2\cdot 2^m||f||_2^{1/2}\log(\log(2^m))}} \end{aligned} \tag{3.3.1}$$

where $\mu_l \in \{0,\ldots,2^{m-l}-1\}$ and $\delta_l \in \{0,1\}$ for all l and the integer N^* is given by $N^* = N - 2^m - \mu_{\lceil m/3\rceil}2^{\lceil m/3\rceil}$. Let $\phi(K) = \sqrt{2K\log(\log(K))}$. Now define the sets

$$\begin{aligned} D(m) &= \left\{|F(0,2^m,x)| > 16C_1||f||_2^{1/4}\phi(2^m)\right\}, \\ E(m,l,\mu_{l+1}) &= \left\{\left|F(2^m+\mu_{l+1}2^{l+1}, 2^l, x)\right| > 16C_1\cdot 2^{(l-m)/6}||f||_2^{1/4}\phi(2^m)\right\}, \end{aligned}$$

for some absolute constant $C_1 > 0$ to be specified later. We are now going to show that for any $\varepsilon > 0$ there exists some $m_0 \in \mathbb{N}$ such that

$$\mathbb{P}\left(\bigcup_{m\geq m_0}\left(D(m) \cup \bigcup_{l=\lceil m/3\rceil}^{m-1}\bigcup_{\mu_{l+1}=0}^{2^{m-l-1}-1} E(m,l,\mu_{l+1})\right)\right) < \varepsilon. \tag{3.3.2}$$

In order to show this inequality we apply the following inequality for suitable choices of R, R', S, Z and α:

$$\begin{aligned} &\mathbb{P}\left(\left|\sum_{n=1+S}^{R+S} f(M_n x)\right| > Z||f||_2^{1/4}\sqrt{2R\log(\log(R'))}\right) \\ &\qquad \leq \frac{4C_2}{Z^2||f||_2^{1/2}R^{\alpha/2}\log(\log(R'))} + e^{-Z/4C_1\cdot||f||_2^{-1/2}\log(\log(R'))}, \end{aligned} \tag{3.3.3}$$

where C_2 is defined such that (2.0.7) is satisfied with $C = C_2\log(d)$. Let ψ_{R^α} be the R^αth partial sum of f as defined in (2.0.1) and let $\rho_{R^\alpha} = f - \psi_{R^\alpha}$. We obtain

$$\begin{aligned} &\mathbb{P}\left(\left|\sum_{n=1+S}^{R+S} f(M_N x)\right| > Z||f||_2^{1/4}\sqrt{2R\log(\log(R'))}\right) \\ &\qquad \leq \mathbb{P}\left(\left|\sum_{n=1+S}^{R+S} \psi_{R^\alpha}(M_n x)\right| > \frac{Z}{2}||f||_2^{1/4}\sqrt{2R\log(\log(R'))}\right) \\ &\qquad + \mathbb{P}\left(\left|\sum_{n=1+S}^{R+S} \rho_{R^\alpha}(M_n x)\right| > \frac{Z}{2}||f||_2^{1/4}\sqrt{2R\log(\log(R'))}\right). \end{aligned} \tag{3.3.4}$$

Using Lemma 2.0.2 and 2.0.3 and also Chebyshev's inequality the second part can be estimated by

$$\mathbb{P}\left(\left|\sum_{n=1+S}^{R+S}\rho_{R^\alpha}(M_n x)\right| > \frac{Z}{2}||f||_2^{1/4}\sqrt{2R\log(\log(R'))}\right) \le \frac{2\mathbb{E}\left[\left(\sum_{n=1+S}^{R+S}\rho_{R^\alpha}(M_n x)\right)^2\right]}{Z^2/4\cdot 2||f||_2^{1/2}R\log(\log(R'))} \le \frac{4C_2\log(d)}{Z^2||f||_2^{1/2}\cdot R^{\alpha/2}\log(\log(R'))}. \tag{3.3.5}$$

In order to estimate the first term on the right-hand side of (3.3.4) we apply the techniques used in the proof of Lemma 2.0.5. We shall show

$$\int_{[0,1)^d}\exp\left(\kappa_R\sum_{n=1+S}^{R+S}\psi_{R^\alpha}(M_n x)\right)dx \le e^{C_1\kappa_R^2||f||_2 R} \tag{3.3.6}$$

for large enough R, suitable κ_R and some absolute constant $C_1 > 0$. Therefore we set

$$\kappa_R = \frac{1}{4C_1}\sqrt{\frac{2\log(\log(R'))}{R}}$$

and $P = \lfloor R^{1/6\alpha}\rfloor$ for some $\alpha \ge 1$ as well as $l = \lfloor R/2P\rfloor$. Without loss of generality we may assume $\kappa_R P \le 1$. Again we define

$$U_m(x) = \sum_{n=Pm+1+S}^{P(m+1)+S}\psi_{R^\alpha}(x).$$

By Cauchy-Schwarz inequality we obtain

$$\int_{[0,1)^d}\exp\left(\kappa_R\sum_{n=1+S}^{R+S}\psi_{R^\alpha}(M_n x)\right)dx \le I(\kappa_R,l)^{1/2}I'(\kappa_R,l)^{1/2}$$

where $I(\kappa_R,l)$ and $I'(\kappa_R,l)$ are defined similarly as in the proof of Lemma 2.0.5. Since $|\kappa_R U_{2m}(x)| \le \kappa_R P \le 1$ we estimate $I(\kappa_R,l)$ by using $e^z \le 1+z+z^2$ which holds for $|z|\le 1$. Thus we get

$$I(\kappa_R,l) \le \int_{[0,1)^d}\prod_{m=0}^{l-1}(1+\kappa_R U_{2m}(x)+\kappa_R^2 U_{2m}(x)^2)\,dx.$$

We define

$$\begin{aligned} V_{2m}(x) &= \sum_{n,n'=2Pm+1+S}^{(2m+1)P+S}\sum_{\substack{1\le||j||_\infty,||j'||_\infty\le R^\alpha,\\ ||M_n^T j+M_{n'}^T j'||_\infty<||M_{m'}^T||_\infty}} T^+(j,j',n,n',x) \\ &\quad + \sum_{\substack{1\le||j||_\infty,||j'||_\infty\le R^\alpha,\\ ||M_n^T j-M_{n'}^T j'||_\infty<||M_{m'}^T||_\infty}} T^-(j,j',n,n',x) \end{aligned} \tag{3.3.7}$$

where T^+ and T^- are defined similarly as in (3.2.26) and $m' = \lfloor 2Pm - \alpha \log_q(R) \rfloor$. It is easy to see that because of Lemma 2.0.4 for any n, n' and j there exists at most one j' such that $||M_n^T j \pm M_{n'}^T j'||_\infty < ||M_{m'}^T||_\infty$. Furthermore for any n and n' with $||j||_\infty, ||j'||_\infty \leq 1/2 \cdot q^{|n-n'|}$ we have

$$||M_n^T j + M_{n'}^T j'||_\infty \geq \frac{1}{2} q^{|n-n'|} ||M_{\min(n,n')}^T||_\infty \geq ||M_{m'}^T||_\infty$$

where the second inequality follows for some R large enough. We conclude that for $||M_n^T j + M_{n'}^T j'||_\infty < ||M_{m'}^T||_\infty$ we have $\max(||j||_\infty, ||j'||_\infty) > 1/2 \cdot q^{|n-n'|}$. Therefore by Cauchy-Schwarz inequality and Lemma 2.0.2 we observe

$$|V_{2m}(x)| \leq C \sum_{n,n'=2Pm+1+S}^{(2m+1)P+S} ||\psi_{R^\alpha}||_2 d^{1/2} q^{-|n-n'|/2} \leq C d^{1/2} ||f||_2 P$$

for some constant $C > 0$ which only depends on q. Define $W_{2m}(x) = U_{2m}^2(x) - V_{2m}(x)$. Then we have

$$\begin{aligned}&\kappa_R U_{2m}(x) + \kappa_R^2 W_{2m}(x) \\ &\qquad = \sum_{||M_{m'}^T||_\infty \leq ||j||_\infty \leq 2R^\alpha ||M_{(2m+1)P}^T||_\infty} \alpha_j \cos(2\pi \langle j, x \rangle) + \beta_j \sin(2\pi \langle j, x \rangle)\end{aligned}$$

with suitable α_j,β_j for all $j \in \mathbb{Z}^d \backslash \{0\}$. Now for $0 \leq k \leq m$ let j_{2k} be any frequency vector of the trigonometric polynomial $Cd^{1/2}P + \kappa_R U_{2k}(x) + \kappa_R^2 W_{2k}(x)$. If R is large enough such that $P > \log_q(R^\alpha) + \log_q(3R^\alpha)$ then with a similar argumentation as in (2.0.20) we get $\sum_{k=0}^m j_{2k} \neq 0$ for $j_{2m} \neq 0$. We observe

$$\begin{aligned} I(\kappa_R, l) &\leq \int_{[0,1)^d} \prod_{m=0}^{l-1} \left(1 + Cd^{1/2}\kappa_R^2 ||f||_2 P + \kappa_R U_{2m}(x) + \kappa_R^2 W_2 m(x)\right) dx \\ &\leq \int_{[0,1)^d} (1 + Cd^{1/2}\kappa_R^2 ||f||_2 P)\, dx. \end{aligned}$$

Therefore (3.3.6) follows immediately with some constant $C_1 > 1$ which depends on q and d. Then by Markov's inequality we observe

$$\begin{aligned} &\mathbb{P}\left(\left| \sum_{n=1+S}^{R+S} \psi_{R^\alpha}(M_n x) \right| > \frac{Z}{2} ||f||_2^{1/4} \sqrt{2R \log(\log(R'))} \right) \\ &\qquad \leq 2 \frac{\mathbb{E}\left[\exp\left(\kappa_R \sum_{n=1+R}^{R+S} \psi_{R^\alpha}(M_n x)\right)\right]}{\exp\left(\kappa_R Z/2 \cdot ||f||_2^{1/4} \sqrt{2R \log(\log(R'))}\right)} \\ &\qquad \leq 2 \exp\left(C_1 \kappa_R^2 ||f||_2 R - \kappa_R \frac{Z}{2} ||f||_2^{1/4} \sqrt{2R \log(\log(R'))} \right) \qquad (3.3.8) \\ &\qquad \leq 2 \exp\left(- \frac{Z \log(\log(R'))}{4C_1 ||f||_2^{1/2}} \right) \end{aligned}$$

where the last line follows by

$$C_1\kappa_R^2||f||_2R \leq \kappa_R\frac{Z}{4}||f||_2^{1/4}\sqrt{2R\log(\log(R'))}$$

for $||f||_2 \leq 1$ and $Z \geq 1$. Now take $R = R' = 2^m$, $S = 0$, $Z = 16C_1$ and $\alpha = 2$. Then we have

$$\mathbb{P}(D(m)) \leq \frac{C_2}{256C_1^2||f||_2^{1/2}2^m\log(\log(2^m))} + e^{-4\cdot\log(\log(2^m))}.$$

It is easy to see that for any $\varepsilon > 0$ there is an $m_0 \in \mathbb{N}$ such that

$$\sum_{m\geq m_0}\mathbb{P}(D(m)) \leq \sum_{m\geq m_0} C\cdot 2^{-m} + \sum_{m\geq m_0}(\log(2)\cdot m)^{-4} \leq \frac{\varepsilon}{2}. \tag{3.3.9}$$

To estimate $\mathbb{P}(E(m,l,\mu_{l+1}))$ we take $R = 2^l$, $R' = 2^m$, $S = 2^m + \mu_{l+1}2^{l+1}$, $\alpha = 2$ and $Z = 16C_1 2^{(m-l)/3}$. First observe

$$\begin{aligned}\mathbb{P}\left(E(m,l,\mu_{l+1})\right) &= \mathbb{P}\left(F(2^m+\mu_{l+1}2^{l+1},2^l,x) > Z||f||_2^{1/4}\sqrt{2\cdot 2^l\log(\log(2^m))}\right)\\ &\leq \frac{C_2||f||_2^{-1/2}}{256C_1^2 2^{2(m-l)/3}2^l\log(\log(2^m))} + e^{-4\cdot 2^{(m-l)/3}\log(\log(2^m))}.\end{aligned}$$

We have

$$\begin{aligned}&\sum_{m\geq m_0}\sum_{l=\lceil m/3\rceil}^{m-1}\sum_{\mu_{l+1}=0}^{2^{m-l-1}-1}\frac{C_2||f||_2^{-1/2}}{256C_1^2 2^{2(m-l)/3}2^l\log(\log(2^m))}\\ &\qquad\leq C\sum_{m\geq m_0}2^{m/3}\sum_{l=\lceil m/3\rceil}^{m-1}2^{-4l/3} \leq C\sum_{m\geq m_0}2^{(1/3-4/9)m} \leq \frac{\varepsilon}{4}\end{aligned} \tag{3.3.10}$$

for m_0 sufficiently large. Furthermore we obtain

$$\begin{aligned}&\sum_{m\geq m_0}\sum_{l=\lceil m/3\rceil}^{m-1}\sum_{\mu_{l+1}=0}^{2^{m-l-1}-1}\exp\left(-4\cdot 2^{(m-l)/3}\log(\log(2^m))\right)\\ &\quad= \sum_{m\geq m_0}\sum_{l=\lceil m/3\rceil}^{m-1}\exp\left(\left(\frac{(m-l-1)\log(2)}{\log(\log(2^m))} - 4\cdot 2^{(m-l)/3}\right)\log(\log(2^m))\right)\\ &\quad\leq \sum_{m\geq m_0}\sum_{l=\lceil m/3\rceil}^{m-1}\exp\left(-2\cdot 2^{(m-l)/3}\log(\log(2^m))\right)\\ &\quad\leq \sum_{m\geq m_0}\sum_{\nu=1}^{\lfloor 2m/3\rfloor}\exp\left(-2\cdot 2^{\nu/3}\log(\log(2^m))\right)\\ &\quad\leq \sum_{m\geq m_0}\sum_{\nu=2}^{\infty}\exp(-\nu\log(\log(2^m))) \leq C\sum_{m\geq m_0}(m\log(2))^{-2} \leq \frac{C}{m_0}\\ &\quad\leq \frac{\varepsilon}{4}.\end{aligned} \tag{3.3.11}$$

Inequalities (3.3.9), (3.3.10) and (3.3.11) yield (3.3.2). With help of this inequality we know that for any $\varepsilon > 0$ there is an $N_0 \in \mathbb{N}$ such that by (3.3.1) for each $N \geq N_0$ we have

$$\frac{\left|\sum_{n=1}^{N} f(M_n x)\right|}{||f||_2^{1/4}\sqrt{2N\log(\log(N))}} \leq 8C_1\left(1+\sum_{l=1}^{\infty} 2^{-l/6}\right) + \frac{||f||_\infty}{N^{1/6}}$$

on a set of measure which is bounded from below by $1-\varepsilon$. Thus we obtain

$$\limsup_{N\to\infty} \frac{\left|\sum_{n=1}^{N} f(M_n x)\right|}{\sqrt{2N\log(\log(N))}} \leq C||f||_2^{1/4} \quad \text{a.e.}$$

for some absolute constant C which concludes the proof.

Proof of Theorem 3.1.2. In order to proof the Theorem we repeat the prove of (3.2.36). But here we take some arbitrary fixed integer G which is sufficiently large and without loss of generality we may assume $N \geq d^{-1}G$. Observe that for fixed G the definition of the blocks Δ'_k and Δ_k and therefore also the definition of the random variables $(X_k)_{k\geq 1}$ does not depend on N. We use $L(N,G) = \mathcal{O}(N/(\log(N))^{1+\varepsilon})$ where the implied constant may depend on G. Therefore instead of (3.2.36) we get

$$||V_K - s_K^2||_2 \leq c\frac{K^{1+\eta}}{(\log(K))^{(1+\varepsilon)/2}},$$

where $c > 0$ as in the remainder of this proof denotes a constant depending on q, d and G which may vary from line to line. In the remainder of this proof we follow the ideas used in [1] and [5]. Now we define a new probability space by taking the product of $[0,1)^d$ on which X_k is defined and another probability space on which independent random variables $\xi_1, \xi_2, \ldots$ with $\mathbb{P}(\xi_n = -1) = \mathbb{P}(\xi_n = 1) = 1/2$ for all $n \in \mathbb{N}$ are defined. For any $k \in \mathbb{N}$ we put $\Xi_k = \sum_{n\in\Delta_k} \xi_k$. For $m \in \mathbb{N}$ we define a martingale differences sequence $(\tilde{X}_{m,k}, \tilde{\mathcal{F}}_k)$ by taking the σ-field $\tilde{\mathcal{F}}_k = \mathcal{F}_k \times \sigma(\Xi_1, \ldots, \Xi_k)$ and setting $\tilde{X}_{m,k} = X_k + \Xi_k/m$. We further put $\tilde{s}_K^2 = s_K^2 + \sum_{k=1}^{K} |\Delta_k|/m^2$. With (3.2.4) we get $||X_k^4 - Y_k^4||_\infty \leq ||X_k^2 - Y_k^2||_\infty \cdot ||X_k^2 + Y_k^2||_\infty \leq c|\Delta_k|^4 k^{-(1+2\eta)} \leq c|\Delta_k|^2$ for some constant $c > 0$. By Lemma 2.0.5 we have $\mathbb{E}[\tilde{X}_{m,k}^4] \leq \mathbb{E}[Y_k^4] + ||Y_k^4 - X_k^4||_\infty + ck^\eta \leq c|\Delta_k|^2$. Furthermore we obtain $\mathbb{E}[\tilde{X}_{m,k}^2|\tilde{\mathcal{F}}_k] = \mathbb{E}[X_k^2|\mathcal{F}_k] + |\Delta_k|/m^2$. Thus we have

$$\tilde{V}_K = \sum_{k=1}^{K} \mathbb{E}[\tilde{X}_{m,k}^2|\tilde{\mathcal{F}}_k] = V_K + \frac{1}{m^2}\sum_{k=1}^{K} |\Delta_k| \geq \frac{1}{m^2}\sum_{k=1}^{K} |\Delta_k|$$

and

$$||\tilde{V}_K - \tilde{s}_K^2||_2 \leq c\frac{K^{1+\eta}}{(\log(K))^{(1+\varepsilon)/2}}. \tag{3.3.12}$$

We now are going to show

$$\tilde{V}_K = \tilde{s}_K^2 + o\left(\tilde{s}_K^2\left(\log\left(\tilde{s}_K^2\right)\right)^{-\varepsilon/4}\right) \quad \text{a.s.} \tag{3.3.13}$$

Since $\varsigma_K = s_K^2 - s_{K-1}^2 \leq cK^\eta$ we have $c_1 K^{1+\eta} \leq \tilde{s}_K^2 \leq c_2 K^{1+\eta}$ for constants $c_1, c_2 > 0$ which depend on q, d and G. We also get $s_{K'}^2 - s_K^2 \leq c_2(K')^\eta(K'-K)$ and furthermore $\tilde{s}_{K'}^2 - \tilde{s}_K^2 \leq c_2(K')^\eta(K'-K)$ for $K' \geq K$. Set $\alpha = 1 - \varepsilon/2 + \varepsilon^2/4$ and define $K_l = \lfloor 2^{l^\alpha} \rfloor$. Then we have

$$\frac{K_{l+1}}{K_l} = 1 + \mathcal{O}(l^{\alpha-1}) = 1 + \mathcal{O}((\log(K_l))^{(\alpha-1)/\alpha}) = 1 + o((\log(K_l))^{-\varepsilon/4})$$

resp.

$$|K_{l+1} - K_l| = o(K_l(\log(K_l))^{-\varepsilon/4}).$$

We obtain

$$\begin{aligned} 0 &\leq \tilde{s}_{K_{l+1}}^2 - \tilde{s}_{K_l}^2 \leq c_2 K_{l+1}^\eta (K_{l+1} - K_l) = K_{l+1}^\eta \cdot o\left(K_l(\log(K_l))^{-\varepsilon/4}\right) \\ &= o\left(K_l^{1+\eta}(\log(K_l))^{-\varepsilon/4}\right) = o\left(\tilde{s}_{K_l}^2(\log(K_l))^{-\varepsilon/4}\right) \end{aligned}$$

or

$$\frac{\tilde{s}_{K_{l+1}}^2}{\tilde{s}_{K_l}^2} = 1 + o\left((\log(K_l))^{-\varepsilon/4}\right).$$

Since

$$\begin{aligned} &\sum_{l=1}^{\infty} \mathbb{P}\left(\left|\tilde{V}_{K_l} - \tilde{s}_{K_l}^2\right| > \tilde{s}_{K_l}^2(\log(K_l))^{-\varepsilon/4}\right) \\ &\qquad \leq 2\sum_{l=1}^{\infty} \mathbb{E}\left[\left(\frac{\tilde{V}_{K_l} - \tilde{s}_{K_l}^2}{\tilde{s}_{K_l}^2(\log(K_l))^{-\varepsilon/4}}\right)^2\right] \leq c\sum_{l=1}^{\infty} (\log(K_l))^{-1-\varepsilon/2} < \infty \end{aligned}$$

by Borel-Cantelli-Lemma we have

$$\left|\tilde{V}_{K_l} - \tilde{s}_{K_l}^2\right| = o\left(\tilde{s}_{K_l}^2(\log(K_l))^{-\varepsilon/4}\right) \quad \text{a.s.}$$

For $K_l \leq K < K_{l+1}$ we have

$$\left(\tilde{V}_{K_l} - \tilde{s}_{K_l}^2\right) + \left(\tilde{s}_{K_l}^2 - \tilde{s}_{K_{l+1}}^2\right) \leq \tilde{V}_K - \tilde{s}_K^2 \leq \left(\tilde{V}_{K_{l+1}} - \tilde{s}_{K_{l+1}}^2\right) + \left(\tilde{s}_{K_{l+1}}^2 - \tilde{s}_{K_l}^2\right).$$

Therefore we have proved (3.3.13). By Lemma 2.0.5 we get $\mathbb{E}\left[\tilde{X}_{m,K}^4\right] \leq c|\Delta_K|^2 \leq cK^{2\eta}$. Thus we have

$$\sum_{K=1}^{\infty} \frac{(\log(\tilde{s}_K^2))^{10}}{(\tilde{s}_K^2)^2} \mathbb{E}\left[\tilde{X}_{m,K}^4\right] \leq c\sum_{K=1}^{\infty} \frac{(\log(K))^{10}}{K^2} < \infty. \tag{3.3.14}$$

Now we apply Theorem 3.3.1. Therefore we get

$$\limsup_{K\to\infty} \frac{\left|\sum_{k=1}^K \tilde{X}_{m,k}\right|}{\sqrt{2\tilde{s}_K^2 \log(\log(\tilde{s}_K^2))}} = 1 \quad \text{a.s.} \tag{3.3.15}$$

By Lemma 2.0.3 for any $N \in \mathbb{N}$ we see that

$$\begin{aligned}\left| \frac{1}{\sqrt{N}} \left\| \sum_{n=1}^{N} f(M_n x) \right\|_2 - \frac{1}{\sqrt{N}} \left\| \sum_{n=1}^{N} p(M_n x) \right\|_2 \right| \\ \leq \frac{1}{\sqrt{N}} \left\| \sum_{n=1}^{N} r(M_n x) \right\|_2 \leq C(\log(d)||r||_2^2 + ||r||_2)^{1/2}\end{aligned}$$

for some absolute constant $C > 0$ which depends only on q. Therefore we get

$$\begin{aligned}&\left| \Sigma_{f,M_n} - \limsup_{N\to\infty} \frac{1}{N} \int_{[0,1)^d} \left(\sum_{n=1}^{N} p(M_n x) \right)^2 dx \right| \\ &\leq C \left(\Sigma_{f,M_n}^{1/2} + (\log(d)||r||_2^2 + ||r||_2)^{1/2} \right) \left(\log(d)||r||_2^2 + ||r||_2 \right)^{1/2} \\ &\leq C d^{1/2} \log(d) G^{-1/2}\end{aligned}$$

for some absolute constant $C > 0$ which only depends on q. A similar estimate holds for $\liminf$. We obtain

$$\begin{aligned}\limsup_{K(N)\to\infty} \frac{\tilde{s}_K^2 \log(\log(\tilde{s}_K^2))}{s_K^2 \log(\log(s_K^2))} &\leq \frac{\limsup_{K(N)\to\infty} s_{K(N)}^2/N + 1/m^2}{\liminf_{K(N)\to\infty} s_{K(N)}^2/N} \\ &\leq \frac{\Sigma_{f,M_n} + C d^{1/2}\log(d) G^{-1/2} + 1/m^2}{\Sigma_{f,M_n} - C d^{1/2} \log(d) G^{-1/2}}.\end{aligned}$$

Hence by (3.3.15) we have

$$\limsup_{K\to\infty} \frac{\left| \sum_{k=1}^{K} \tilde{X}_{m,k} \right|}{\sqrt{2 s_K^2 \log(\log(s_K^2))}} \leq \sqrt{\frac{\Sigma_{f,M_n} + C d^{1/2}\log(d) G^{-1/2} + 1/m^2}{\Sigma_{f,M_n} - C d^{1/2} \log(d) G^{-1/2}}} \quad \text{a.s.} \tag{3.3.16}$$

Observe that there is a similar lower bound for the term on the left-hand side. By definition of $\tilde{X}_{m,k}$ we get

$$\left| \frac{\left| \sum_{k=1}^{K} X_k \right|}{\sqrt{2 s_K^2 \log(\log(s_K^2))}} - \frac{\left| \sum_{k=1}^{K} \tilde{X}_{m,k} \right|}{\sqrt{2 s_K^2 \log(\log(s_K^2))}} \right| \leq \frac{\left| \sum_{k=1}^{K} \Xi_k \right|}{m\sqrt{2 s_K^2 \log(\log(s_K^2))}}.$$

Therefore simple calculation shows that

$$\begin{aligned}\left| \limsup_{K(N)\to\infty} \frac{\left| \sum_{k=1}^{K} X_k \right|}{\sqrt{2 s_K^2 \log(\log(s_K^2))}} - \limsup_{K(N)\to\infty} \frac{\left| \sum_{k=1}^{K} \tilde{X}_{m,k} \right|}{\sqrt{2 s_K^2 \log(\log(s_K^2))}} \right| \\ \leq \limsup_{K(N)\to\infty} \frac{\left| \sum_{k=1}^{K} \Xi_k \right|}{m\sqrt{2 s_K^2 \log(\log(s_K^2))}}.\end{aligned} \tag{3.3.17}$$

Since $\sum_{k=1}^{K} \Xi_k$ is the sum of independent random variables we have

$$\limsup_{K(N)\to\infty} \frac{\left|\sum_{k=1}^{K} \Xi_k\right|}{m\sqrt{2s_K^2 \log(\log(s_K^2))}} \leq \frac{\sqrt{\Sigma_{f,M_n} + Cd^{1/2}\log(d)G^{-1/2}}}{m\sqrt{\Sigma_{f,M_n} - Cd^{1/2}\log(d)G^{-1/2}}}. \tag{3.3.18}$$

Because this inequality holds for any integer m by (3.3.16), (3.3.17) and (3.3.18) we get

$$\begin{aligned}
\sqrt{\frac{\Sigma_{f,M_n} - Cd^{1/2}\log(d)G^{-1/2}}{\Sigma_{f,M_n} + Cd^{1/2}\log(d)G^{-1/2}}} &\leq \limsup_{K(N)\to\infty} \frac{\left|\sum_{k=1}^{K} X_k\right|}{\sqrt{2s_K^2 \log(\log(s_K^2))}} \\
&\leq \sqrt{\frac{\Sigma_{f,M_n} + Cd^{1/2}\log(d)G^{-1/2}}{\Sigma_{f,M_n} - Cd^{1/2}\log(d)G^{-1/2}}} \quad \text{a.e.}
\end{aligned}$$

Observe that

$$\left|\sum_{n=1}^{N} p(M_n x)\right| \leq \left|\sum_{k=1}^{K} X_k\right| + \left|\sum_{k=1}^{K}(X_k - Y_k)\right| + \left|\sum_{k=1}^{K} Y_k'\right| + \left|\sum_{n=\tilde{N}+1}^{N} p(M_n x)\right| \tag{3.3.19}$$

where K is defined such that $N \in \Delta_{K+1}' \cup \Delta_{K+1}$ and $\hat{N} = \sum_{k=1}^{K} |\Delta_k'| + |\Delta_k|$. By using $N - \hat{N} \leq cN^{\eta/(1+\eta)}$ we have $|\sum_{k=\hat{N}+1}^{N} p(M_n x)| \leq cN^{\eta/(1+\eta)}$. By (3.2.6) we have $|\sum_{k=1}^{K}(X_k - Y_k)| \leq c$. To estimate $|\sum_{k=1}^{M}(Y_k')|$ we apply Lemma 3.3.2 and obtain

$$\left|\sum_{k=1}^{K} Y_k'\right| \leq C||p||_2^{1/4}\sqrt{N^{1/(1+\eta)}\log(N)\log(\log(N^{1/(1+\eta)}\log(N)))} \leq cN^{1/(2+2\eta)}\log(N)^2. \tag{3.3.20}$$

Plugging these estimates into (3.3.19) gives

$$\left|\sum_{n=1}^{N} p(M_n x)\right| \leq \left|\sum_{k=1}^{K} X_k\right| + cN^{1/(2+2\eta)}\log(N)^2.$$

Since $s_K^2 \geq cN$ we have

$$\frac{|\sum_{n=1}^{N} p(M_n x)|}{\sqrt{2s_K^2 \log(\log(s_K^2))}} \leq \frac{\left|\sum_{k=1}^{K} X_k\right|}{\sqrt{2s_K^2 \log(\log(s_K^2))}} + c\frac{\log(N)^2}{N^{(1-\eta)/(2+2\eta)}}.$$

It follows that

$$\begin{aligned}
\limsup_{N\to\infty} \frac{|\sum_{n=1}^{N} p(M_n x)|}{\sqrt{2s_K^2 \log(\log(s_K^2))}} &\leq \limsup_{K(N)\to\infty} \frac{\left|\sum_{k=1}^{K} X_k\right|}{\sqrt{2s_K^2 \log(\log(s_K^2))}} + \limsup_{N\to\infty} c\frac{\log(N)^2}{N^{(1-\eta)/(2+2\eta)}} \\
&\leq \sqrt{\frac{\Sigma_{f,M_n} + Cd^{1/2}\log(d)G^{-1/2}}{\Sigma_{f,M_n} - Cd^{1/2}\log(d)G^{-1/2}}} \quad \text{a.e.}
\end{aligned}$$

Similar arguments yield

$$\limsup_{N\to\infty} \frac{|\sum_{n=1}^N p(M_n x)|}{\sqrt{2s_K^2 \log(\log(s_K^2))}} \geq \sqrt{\frac{\Sigma_{f,M_n} - Cd^{1/2}\log(d)G^{-1/2}}{\Sigma_{f,M_n} + Cd^{1/2}\log(d)G^{-1/2}}} \quad \text{a.e.}$$

By a similar argumentation as in (3.2.21) we have

$$\begin{aligned}\Sigma_{f,M_n} - Cd^{1/2}\log(d)G^{-1/2} &\leq \lim_{N\to\infty} \frac{s_K^2}{N} = \lim_{N\to\infty} \frac{\sigma_N^2}{N} \\ &\leq \Sigma_{f,M_n} + Cd^{1/2}\log(d)G^{-1/2}\end{aligned}$$

and we observe

$$\begin{aligned}\frac{\Sigma_{f,M_n} - Cd^{1/2}\log(d)G^{-1/2}}{\sqrt{\Sigma_{f,M_n} + Cd^{1/2}\log(d)G^{-1/2}}} &\leq \limsup_{N\to\infty} \frac{|\sum_{n=1}^N p(M_n x)|}{\sqrt{2N\log(\log(N))}} \\ &\leq \frac{\Sigma_{f,M_n} + Cd^{1/2}\log(d)G^{-1/2}}{\sqrt{\Sigma_{f,M_n} - Cd^{1/2}\log(d)G^{-1/2}}} \quad \text{a.e.}\end{aligned} \tag{3.3.21}$$

By Lemma 3.3.2 for almost any x we get

$$\begin{aligned}\limsup_{N\to\infty} \frac{\left|\sum_{n=1}^N f(M_n x)\right|}{\sqrt{2N\log(\log(N))}} &\leq \limsup_{N\to\infty} \frac{\left|\sum_{k=1}^N p(M_n x)\right|}{\sqrt{2N\log(\log(N))}} + \limsup_{N\to\infty} \frac{\left|\sum_{n=1}^N r(M_n x)\right|}{\sqrt{2N\log(\log(N))}} \\ &\leq \frac{\Sigma_{f,M_n} + Cd^{1/2}\log(d)G^{-1/2}}{\sqrt{\Sigma_{f,M_n} - Cd^{1/2}\log(d)G^{-1/2}}} + C||r||_2^{1/4} \\ &\leq \frac{\Sigma_{f,M_n} + Cd^{1/2}\log(d)G^{-1/2}}{\sqrt{\Sigma_{f,M_n} - Cd^{1/2}\log(d)G^{-1/2}}} + C(dG^{-1})^{1/8}\end{aligned}$$

and also

$$\limsup_{N\to\infty} \frac{\left|\sum_{n=1}^N f(M_n x)\right|}{\sqrt{2N\log(\log(N))}} \geq \frac{\Sigma_{f,M_n} - Cd^{1/2}\log(d)G^{-1/2}}{\sqrt{\Sigma_{f,M_n} + Cd^{1/2}\log(d)G^{-1/2}}} - C(dG^{-1})^{1/8}.$$

Since G can be chosen arbitrary large, we finally observe that

$$\limsup_{N\to\infty} \frac{\left|\sum_{n=1}^N f(M_n x)\right|}{\sqrt{2N\log(\log(N))}} = \sqrt{\Sigma_{f,M_n}} \quad \text{a.e.} \tag{3.3.22}$$

which concludes the proof.

3.4 Law of the iterated Logarithm for the Discrepancy of Multivariate Lacunary Point Sets

The proof of this Theorem is mainly based on [45], [28] and [1]. We only show the Law of the Iterated Logarithm for D_N^*, the proof of the Law of the Iterated Logarithm for D_N is essentially the same. For some integer $h > 0$ and $\beta \in [0,1)^d$ set β_h such that $\beta_{h,i} \le \beta_i < \beta_{h,i} + 1/h$ and $\beta_h \in B_h = \{\beta \in [0,1)^d : 2^h \beta_i \in \{0, \dots, 2^h - 1\}, i = 1, \dots, d\}$. Furthermore set $\overline{[\alpha,\beta)} = [0,\beta) \backslash [0,\alpha)$ for $\alpha, \beta \in [0,1)^d$ with $\alpha_i \le \beta_i$ for all $i \in \{1, \dots, d\}$. Let $f_\beta(x) = \mathbf{1}_{[0,\beta)}(x) - \lambda([0,\beta))$ denote the centered indicator function on $[0,\beta)$ and $f_{\alpha,\beta}(x) = \mathbf{1}_{\overline{[\alpha,\beta)}}(x) - \lambda(\overline{[0,\beta)})$ the centered indicator function on $\overline{[\alpha,\beta)}$. Now choose some arbitrary fixed integer $L > 0$. We have

$$\begin{aligned} D_N^*(M_1x, \dots, M_Nx) &= \sup_{\beta \in [0,1)^d} \left| \frac{\sum_{n=1}^N f_\beta(M_nx)}{N} \right| \\ &\le \max_{\beta_L \in B_L} \left| \frac{\sum_{n=1}^N f_{\beta_L}(M_nx)}{N} \right| + \sup_{\beta \in [0,1)^d} \left| \frac{\sum_{n=1}^N f_{\beta_L,\beta}(M_nx)}{N} \right|. \end{aligned} \tag{3.4.1}$$

Then, since L can be chosen arbitrary large, (3.1.6) is shown if we prove

$$\limsup_{N\to\infty} \max_{\beta_L \in B_L} \frac{\left|\sum_{n=1}^N f_{\beta_L}(M_nx)\right|}{\sqrt{2N\log(\log(N))}} = \frac{1}{2} \quad \text{a.e.} \tag{3.4.2}$$

and

$$\limsup_{N\to\infty} \sup_{\beta \in [0,1)^d} \frac{\left|\sum_{n=1}^N f_{\beta_L,\beta}(M_nx)\right|}{\sqrt{2N\log(\log(N))}} \le C2^{-L/8} \quad \text{a.e.} \tag{3.4.3}$$

for some constant C depending only on d. For f_{β_L} by the second part of Lemma 2.0.3 we obtain $\lim_{N\to\infty} \sigma_N^2/N = ||f_{\beta_L}||_2^2 = \lambda([0,\beta_L)) - \lambda([0,\beta_L))^2 \le 1/4$ where equality holds for $\lambda([0,\beta_L)) = 1/2$. By Theorem 3.1.2 we have

$$\begin{aligned} \limsup_{N\to\infty} \max_{\beta_L \in B_L} \frac{\left|\sum_{n=1}^N f_{\beta_L}(M_nx)\right|}{\sqrt{2N\log(\log(N))}} &= \max_{\beta_L \in B_L} \limsup_{N\to\infty} \frac{\left|\sum_{n=1}^N f_{\beta_L}(M_nx)\right|}{\sqrt{2N\log(\log(N))}} \\ &= \max_{\beta_L \in B_L} ||f_{\beta_L}||_2 = \frac{1}{2} \quad \text{a.e.} \end{aligned} \tag{3.4.4}$$

Now we are going to prove (3.4.3). For some given N we set $H = \lceil m/2 + \log_2(d) \rceil$ where $m = \max\{l \in \mathbb{Z} : 2^l \le N\}$. Without loss of generality we may assume $H > L$. It is easy to see that for any $x \in [0,1)^d$ we have

$$\mathbf{1}_{[0,\beta_H)}(x) \le \mathbf{1}_{[0,\beta)}(x) \le \mathbf{1}_{[0,\beta_{H+1})}(x)$$

where for convenience we set β_{H+1} such that $\beta_{H+1,i} = \beta_{H,i} + 1/H$ for all $i \in \{1, \dots, d\}^d$. Therefore we get

$$\begin{aligned} \mathbf{1}_{[0,\beta_H)}(x) - \lambda([0,\beta_H)) - d \cdot 2^{-H} &\le \mathbf{1}_{[0,\beta)}(x) - \lambda([0,\beta)) \\ &\le \mathbf{1}_{[0,\beta_H+1/H)}(x) - \lambda([0,\beta_H + 1/H)) + d \cdot 2^{-H}. \end{aligned}$$

Thus we obtain

$$\left|\sum_{n=1}^{N}\mathbf{1}_{[0,\beta)}-\lambda([0,\beta))\right|\le\sum_{\substack{J\subset I,\\ J\neq\emptyset}}\;\sum_{h\in\{L+1,\dots,H+1\}^{|J|}}\left|\sum_{n=1}^{N}\varphi_{J,h}(x)\right|+d2^{-H}N. \tag{3.4.5}$$

Here the sum is taken over all non-empty subsets J of $I=\{1,\dots,d\}$ and $\varphi_{J,h}$ denotes the centered indicator function on the set

$$\prod_{i\in I\setminus J}[0,\beta_{L,i})\times\prod_{i\in J}[\beta_{h_i-1,i},\beta_{h_i,i})$$

for any $h\in\{L+1,\dots,H+1\}^{|J|}$. Now with $F_{\varphi_{J,h}}(R,S,x)=\left|\sum_{n=1+S}^{R+S}\varphi_{J,h}(M_nx)\right|$ we have

$$\left|\sum_{n=1}^{N}\varphi_{J,h}(x)\right|\le F_{\varphi_{J,h}}(0,2^m,x)+\sum_{l=\lceil m/3\rceil}^{m-1}F_{\varphi_{J,h}}(2^m+\mu_{l+1}2^{l+1},2^l,x)+CN^{1/3}. \tag{3.4.6}$$

As we shall later show the system of inequalities

$$\begin{array}{rcl} F_{\varphi_{J,h}}(0,2^m,x) & \le & 16C_1||\varphi_{J,h}||_2^{1/4}\sqrt{2\cdot 2^m\log(\log(2^m))},\\ F_{\varphi_{J,h}}(2^m+\mu_{l+1}2^{l+1},2^l,x) & \le & 16C_1 2^{(l-m)/6}||\varphi_{J,h}||_2^{1/4}\sqrt{2\cdot 2^m\log(\log(2^m))} \end{array} \tag{3.4.7}$$

holds for all $m\ge m_0$, $l\in\{\lceil m/3\rceil,\dots,m-1\}$, $J\subset I$, $h\in\{L+1,\dots,H+1\}^{|J|}$ with $H=\lceil m/2+\log_2(d)\rceil$ and $\beta\in[0,1)^d$ on a set of measure which is bounded from below by $1-\varepsilon$ where $\varepsilon>0$ can be chosen arbitrary and m_0 depends on the choice of ε. Simple calculation shows

$$\frac{1}{2}\prod_{i\in I\setminus J}c_i2^{-L}\prod_{i\in J}2^{-h_i}\le||\varphi_{J,h}||_2^2\le\prod_{i\in I\setminus J}c_i2^{-L}\prod_{i\in J}2^{-(h_i-1)}. \tag{3.4.8}$$

where $c_i=2^L\beta_{L,i}\in\{0,\dots,2^L-1\}$ for all $i\in I\setminus J$. Therefore we have

$$\begin{aligned}\sum_{\substack{J\subset I,\\ J\neq\emptyset}}\;\sum_{h\in\{L+1,\dots,H+1\}^{|J|}}||\varphi_{J,h}||_2^{1/4} &\le C\sum_{\substack{J\subset I,\\ J\neq\emptyset}}\;\sum_{h\in\{L+1,\dots,H+1\}^{|J|}}\prod_{i\in I\setminus J}(c_i2^{-L})^{1/8}\prod_{i\in J}2^{-h_i/8}\\ &\le C\sum_{\substack{J\subset I,\\ J\neq\emptyset}}\;\sum_{h\in\{L+1,\dots,H+1\}^{|J|}}\prod_{i\in J}2^{-h_i/8}\\ &\le C\sum_{\substack{J\subset I,\\ J\neq\emptyset}}2^{-L/8}\le C\cdot 2^{-L/8}\end{aligned} \tag{3.4.9}$$

where the constant $C>0$ depends only on d. Furthermore we have $1+\sum_{l=1}^{\infty}2^{-l/6}\le C$ for some absolute constant $C>0$. Combining (3.4.5), (3.4.6), (3.4.7) and (3.4.9) we finally obtain (3.4.3). Thus it remains to show (3.4.7). To prove (3.4.7) we apply the techniques used in the proof of Lemma 3.3.2. Since (3.4.7) shall hold for any function $\varphi_{J,h}$ we first encounter all possible choices for given J and h. We set $h_i' = h_i$ for $h_i \leq H$ and $h_i' = H$ for $h_i = H+1$. Therefore by definition $\varphi_{J,h}$ is a centered indicator function on a set of the form

$$\prod_{i \in I \setminus J} [0, 2^{-L} c_i) \times \prod_{i \in J} [2^{-(h_i-1)} a_i, 2^{-(h_i-1)} a_i + 2^{-h_i'})$$

with $c \in \{1, \ldots, 2^L\}^{|I \setminus J|}$ and $a \in \prod_{i \in J} \{0, \ldots, 2^{h_i-1} - 1\}$. Thus each choice of c and a defines a function which we denote by $\varphi_{J,h}^{(c,a)}$. We define the sets

$$\begin{aligned} &D(m, J, h, c, a) \\ &= \left\{ F_{\varphi_{J,h}^{(c,a)}}(0, 2^m, x) > 16 C_1 ||\varphi_{J,h}^{(c,a)}||_2^{1/4} \phi(2^m) \right\}, \\ &E(m, l, \mu_{l+1}, J, h, c, a) \\ &= \left\{ F_{\varphi_{J,h}^{(c,a)}}(2^m + \mu_{l+1} 2^{l+1}, 2^l, x) > 16 C_1 \cdot 2^{(l-m)/6} ||\varphi_{J,h}^{(c,a)}||_2^{1/4} \phi(2^m) \right\}, \end{aligned}$$

where $\phi(K) = \sqrt{2K \log(\log(K))}$. Now we are going to prove that for any $\varepsilon > 0$ there is some integer m_0 such that the union of all these sets with $m \geq m_0$ has total measure which is bounded from above by ε. This shall be done by estimating the measure of each set with the help of (3.3.3). We take the same choices for S, R, R' and Z as in (3.3.9), (3.3.10) and (3.3.11) but we take $\alpha = 4d + 6$. It is enough to prove

$$\sum_{\substack{J \subset I, \\ J \neq \emptyset}} \sum_{h \in \{L+1, \ldots, H+1\}^{|J|}} \sum_{c \in \{1, \ldots, 2^L\}^{|I|-|J|}} \sum_{a \in \prod_{i \in J} \{0, \ldots, 2^{h_i-1}-1\}} e^{-4 \cdot 2^{(m-l)/3} ||\varphi_{J,h}^{(c,a)}||_2^{-1/2} \log(\log(2^m))} \leq C e^{-4 \cdot 2^{(m-l)/3} \log(\log(2^m))} \quad (3.4.10)$$

for some absolute constant $C > 0$ depending only on d where the factor $2^{(m-l)/3}$ on the right-hand side of the first line in the case of $D(m, J, h, c, a)$ becomes 1 and

$$\sum_{\substack{J \subset I, \\ J \neq \emptyset}} \sum_{h \in \{L+1, \ldots, H+1\}^{|J|}} \sum_{c \in \{1, \ldots, 2^L\}^{|I|-|J|}} \sum_{a \in \prod_{i \in J} \{0, \ldots, 2^{h_i-1}-1\}} ||\varphi_{J,h}^{c,a}||_2^{-1/2} \leq C \cdot 2^{5/8 \cdot dm} \quad (3.4.11)$$

for some absolute constant $C > 0$ depending only on d. Then the conclusion follows by similar arguments as in (3.3.9), (3.3.10) and (3.3.11). Observe that with $\alpha = 4d + 6$ we get

$$\sum_{m \geq m_0} \sum_{l = \lceil m/3 \rceil}^{m-1} \sum_{\mu_{l+1}=0}^{2^{m-l-1}-1} \frac{4 C_2}{256 C_1^2 (2^l)^{\alpha/2} \log(\log(2^m))} \cdot C \cdot 2^{5/8 \cdot dm} \leq C \sum_{m \geq m_0} 2^{-dm/24} \leq \frac{\varepsilon}{4}$$

for m_0 sufficiently large and we get a similar replacement for the upper bounds on the measure of the sets $D(m,J,h,c,a)$. Without loss of generality we may assume L large enough. Therefore for $h_i > L$ for all $i \in I$ by (3.4.8) we have

$$\begin{aligned}
&\sum_{\substack{J\subset I,\\ J\neq\emptyset}} \sum_{h\in\{L+1,\dots,H+1\}^{|J|}} \sum_{c\in\{1,\dots,2^L\}^{|I|-|J|}} \sum_{a\in\prod_{i\in J}\{0,\dots,2^{h_i-1}-1\}} e^{-4\cdot 2^{(m-l)/3}||\varphi_{J,h}^{(c,a)}||_2^{-1/2}\log(\log(2^m))} \\
&\le \sum_{\substack{J\subset I,\\ J\neq\emptyset}} 2^{dL} \sum_{h\in\{L+1,\dots,H+1\}^{|J|}} e^{\log(2)\cdot\sum_{i\in J}(h_i-1)} e^{-4\cdot 2^{(m-l)/3}\prod_{i\in J} 2^{h_i/4}\log(\log(2^m))} \\
&\le \sum_{\substack{J\subset I,\\ J\neq\emptyset}} 2^{dL} \sum_{h\in\{L+1,\dots,H+1\}^{|J|}} e^{-4\cdot 2^{(m-l)/3}\prod_{i\in J} 2^{h_i/8}\log(\log(2^m))}. \\
&\le C\sum_{\substack{J\subset I,\\ J\neq\emptyset}} 2^{dL} e^{-4\cdot 2^{(m-l)/3}\prod_{i\in J} 2^{L/8}\log(\log(2^m))} \\
&\le Ce^{-4\cdot 2^{(m-l)/3}\log(\log(2^m))}
\end{aligned} \tag{3.4.12}$$

for some constant C depending only on d and thus (3.4.10) is proved. Moreover we have

$$\begin{aligned}
&\sum_{\substack{J\subset I,\\ J\neq\emptyset}} \sum_{h\in\{L+1,\dots,H+1\}^{|J|}} \sum_{c\in\{1,\dots,2^L\}^{|I|-|J|}} \sum_{a\in\prod_{i\in J}\{0,\dots,2^{h_i-1}-1\}} ||\varphi_{J,h}^{c,a}||_2^{-1/2} \\
&\qquad \le \sum_{\substack{J\subset I,\\ J\neq\emptyset}} \sum_{h\in\{L+1,\dots,H+1\}^{|J|}} 2^{5/4\cdot(|I|-|J|)L} \prod_{i\in J} 2^{5/4\cdot h_i} \\
&\qquad \le C\sum_{\substack{J\subset I,\\ J\neq\emptyset}} 2^{5/4\cdot(|I|-|J|)L} 2^{5/4\cdot|J|H} \\
&\qquad \le C\cdot 2^{5/8\cdot dm}
\end{aligned} \tag{3.4.13}$$

for some constant $C > 0$ depending only on d. Observe that the last line follows by $H \le m/2 + \log_2(d) + 1$. Thus (3.4.11) is proved which finally concludes the proof of Theorem 3.1.3.

3.5 Weakly lacunary sequences

Let $(M_n)_{n\ge 1}$ be a sequence of non-singular $d\times d$-matrices satisfying the weak Hadamard gap condition (1.0.5). Under some stronger conditions the Central Limit Theorem and the Law of the Iterated Logarithm hold for lacunary sequences which only satisfy the weak Hadamard gap condition.

Theorem 3.5.1 *Let $(M_n)_{n\geq 1}$ be a lacunary sequence of non-singular $d\times d$-matrices satisfying the weak Hadamard gap condition (1.0.5). Assume that $L(G,N)=o(N)$ for any fixed $G\geq 1$. Let $f:\mathbb{R}^d\to\mathbb{R}$ be a bounded, periodic function of finite total variation in the sense of Hardy and Krause such that the Fourier series of f exists and converges to f and is of the form*

$$f(x)=\sum_{j\in(\mathbb{Z}\backslash\{0\})^d} a_j\cos(2\pi\langle j,x\rangle)+b_j\sin(2\pi\langle j,x\rangle).$$

Furthermore assume that there exists some absolute constant $C>0$ such that

$$\sigma_N^2:=\int_{[0,1)^d}\left(\sum_{n=1}^N f(M_nx)\right)^2 dx\geq C\cdot N. \tag{3.5.1}$$

for any $N\geq 1$. Then for all $t\in\mathbb{R}$ we have

$$\lim_{N\to\infty}\left|\mathbb{P}\left(\sum_{n=1}^N f(M_nx)\leq t\sigma_N\right)-\Phi(t)\right|=0. \tag{3.5.2}$$

Theorem 3.5.2 *Let $(M_n)_{n\geq 1}$ be a sequence of non-singular $d\times d$-matrices satisfying the weak Hadamard gap condition (1.0.5). Assume that $L(N,G)=\mathcal{O}(N/(\log N)^{1+\varepsilon})$ for any fixed $G\geq 1$ and some $\varepsilon>0$. Let $p:\mathbb{R}^d\to\mathbb{R}$ be a trigonometric polynomial of order G such that*

$$p(x)=\sum_{\substack{j\in(\mathbb{Z}\backslash\{0\})^d,\\ ||j||_\infty\leq G}} a_j\cos(2\pi\langle j,x\rangle)+b_j\sin(2\pi\langle j,x\rangle).$$

Assume that for

$$\sigma_N^2:=\int_{[0,1)^d}\left(\sum_{i=1}^N p(M_nx)\right)^2 dx$$

there exists $\Sigma_{p,M_n}>0$ with

$$\lim_{N\to\infty}\frac{\sigma_N^2}{N}=\Sigma_{p,M_n}. \tag{3.5.3}$$

Then we have

$$\limsup_{N\to\infty}\frac{\left|\sum_{k=1}^N p(M_nx)\right|}{\sqrt{2N\log(\log(N))}}=\sqrt{\Sigma_{p,M_n}}\quad\text{a.e.} \tag{3.5.4}$$

The proofs of both Theorems are similar to those of Theorem 3.1.1 resp. Theorem 3.1.2. Therefore we only briefly discuss the differences. We have to restrict ourselves to functions $f(x)=\sum_{j\in\mathbb{Z}^d\backslash\{0\}} a_j\cos(2\pi\langle j,x\rangle)+b_j\sin(2\pi\langle j,x\rangle)$ with $a_j,b_j=0$ for $j\in\mathbb{Z}^d$ with $j_i=0$ for at least one $i\in\{1,\dots,d\}$. Consider $f(x)=\cos(2\pi\langle(1,0)^T,x\rangle)$. Furthermore let $(M_n)_{n\geq 1}$ be some lacunary sequence of 2×2-matrices satisfying the weak Hadamard gap condition (1.0.5) such that $((M_n)_{1,1})_{n\geq 1}$ regarded as a sequence of integers does not satisfy (1.0.4). The system $(f(M_n x))_{n\geq 1}$ easily translates into some one-dimensional function system with some non-lacunary sequence $(M_n)_{n\geq 1}$ and thus the Central Limit Theorem does not hold in general. Besides, observe that Lemma 2.0.3 does not apply in the case of lacunary sequences which only satisfy the weak Hadamard gap condition (ref255) since $||M_{n+k}^T j||_\infty \geq q^k ||M_N^T||_\infty$ for some $k \geq \log_q(||j||)_\infty$ and all $j \in \mathbb{Z}^d \backslash \{0\}$ is not necessarily satisfied. Instead we use the following

Lemma 3.5.3 *Let $(M_n)_{n\geq 1}$ be a lacunary sequence of non-singular $d \times d$-matrices satisfying the weak Hadamard gap condition (1.0.5). Let f be a periodic functions of mean zero with $V_{HK}(f) \leq 1$. Assume that the Fourier series exists and converge to f. Then there is some absolute constant $C > 0$ which only depends on q such that*

$$\left| \int_{[0,1)^d} \left(\sum_{n=1}^{N} f(M_n x) \right)^2 dx \right| \leq CN$$

for all $n \in \mathbb{N}$.

Proof. For notational convenience we assume $f(x) = \sum_{j\in(\mathbb{Z}\backslash\{0\})^d} a_j \cos(2\pi\langle j, x\rangle)$. The general case is similar. We have

$$\begin{aligned} &\left| \int_{[0,1)^d} \left(\sum_{n=1}^{N} f(M_n x) \right)^2 dx \right| \\ &\leq \sum_{j,j'\in(\mathbb{Z}\backslash\{0\})^d} \sum_{n,n'=1}^{N} \left| \int_{[0,1)^d} a_j \cos(2\pi\langle M_n j, x\rangle) a_{j'}(2\pi\langle M_n j, x\rangle)\, dx \right| \\ &\leq \sum_{j,j'\in(\mathbb{Z}\backslash\{0\})^d} \sum_{n,n'=1}^{N} \frac{|a_j a_{j'}|}{2} (\mathbf{1}_{M_n j + M_{n'} j' = 0} + \mathbf{1}_{M_n j - M_{n'} j' = 0}) \\ &\leq 2 \sum_{j,j'\in(\mathbb{Z}\backslash\{0\})^d} \sum_{1\leq n\leq n'\leq N} |a_j a_{j'}| \mathbf{1}_{|j_i| = \frac{(M_{n'})_{i,i}}{(M_n)_{i,i}} |j_i'| \forall i}. \end{aligned}$$

By Lemma 2.0.1 we have $|a_j| \leq \prod_{i=1}^d (2\pi |j_i|)^{-1}$. Therefore we conclude

$$\begin{aligned} &\left| \int_{[0,1)^d} \left(\sum_{n=1}^{N} f(M_n x) \right)^2 dx \right| \\ &\leq 2 \sum_{j,j'\in(\mathbb{Z}\backslash\{0\})^d} \sum_{1\leq n\leq n'\leq N} |a_{j'}| \prod_{i=1}^{d} (2\pi|j_i|)^{-1} \mathbf{1}_{|j_i| = \frac{(M_{n'})_{i,i}}{(M_n)_{i,i}} |j_i'| \forall i}. \end{aligned}$$

Since for any $n \in \mathbb{N}$ there is some $i \in \{1, \ldots, d\}$ with $(M_{n+1})_{i,i} \geq q(M_n)_{i,i}$ and $(M_{n+1})_{i',i'} \geq (M_n)_{i',i'}$ for $i' \neq i$ it is easy to see that

$$\prod_{i=1}^{d} |j_i|^{-1} \leq q^{n-n'} \prod_{i=1}^{d} |j_i'|^{-1}$$

if $|j_i| = \frac{(M_{n'})_{i,i}}{(M_n)_{i,i}}|j_i'|$ for all $i \in \{1,\dots,d\}$. Thus we get

$$\left|\int_{[0,1)^d}\left(\sum_{n=1}^{N} f(M_n x)\right)^2 dx\right| \le 2\sum_{1\le n\le n'\le N} q^{n-n'} \sum_{j'\in(\mathbb{Z}\backslash\{0\})^d} \frac{1}{(2\pi)^d}|a_{j'}|\prod_{i=1}^{d}|j_i'|^{-1}.$$

With an application of Cauchy-Schwarz inequality we obtain

$$\begin{aligned}\left|\int_{[0,1)^d}\left(\sum_{n=1}^{N} f(M_n x)\right)^2 dx\right| & \\ &\le 2\sum_{1\le n\le n'\le N} q^{n-n'}\left(\sum_{j'\in(\mathbb{Z}\backslash\{0\})^d} a_{j'}^2\right)^{1/2}\frac{1}{(2\pi)^d}\left(\sum_{j'\in(\mathbb{Z}\backslash\{0\})^d}|j_i'|^{-2}\right)^{1/2}\\ &\le C||f||_2 N\end{aligned}$$

for some constant which only depends on q and therefore the Lemma is proved.

Since we assume

$$f(x) = \sum_{j\in(\mathbb{Z}\backslash\{0\})^d} a_j\cos(2\pi\langle j,x\rangle) + b_j\sin(2\pi\langle j,x\rangle)$$

we ensure that $||M_n^T j||_\infty \ge ||M_n^T||_\infty$ for all j and therefore the conclusion of (3.2.5) remains valid in the case of lacunary sequences satisfying only the weak Hadamard gap condition. Furthermore Lemma 2.0.4 in general is not true for lacunary sequences which satisfy only the weak Hadamard gap condition. This Lemma was used in the proof of Lemma 2.0.5 to estimate $|V_{2m}(x)|$ in (2.0.18). Since p is a trigonometric polynomial of order G we see by $V_{HK}(p) \le 1$ and Lemma 2.0.1 that $\sum_{||j||_\infty\le G}|a_j| + |b_j| \le c$ for some absolute constant $c > 0$ which depends on d and G. With $L(N,G) \le cN$ we easily obtain $|V_{2m}(x)| \le cN$ for some constant which depends on q, d and G. Repeating the further calculation we finally observe

$$\int_{[0,1)^d}\left(\sum_{n=1}^{N} p(M_n x)\right)^4 dx \le cN^2 \tag{3.5.5}$$

for some constant $c > 0$ which depends on q, d and G. Moreover we used Lemma 2.0.4 to estimate $||R_k||_\infty$ in (3.2.27) resp. $||S_k||_\infty$ in (3.2.28). With a similar argumentation as above we see that $||R_k||_\infty, ||S_k||_\infty \le c(1+\log_q(G+1))|\Delta_k|$ for some absolute constant $c > 0$ which only depends on q, d and G. Instead of (3.2.36) further calculation yield

$$\mathbb{E}\left[(V_{K+1} - s_{K+1}^2)^2\right] \le o(K^{2(1+\eta)})$$

where the implied constant only depends on q, d and G. Together with (3.5.5) we repeat the further calculation and finally observe (3.2.37) which concludes the proof of Theorem

3.5.1. To prove Theorem 3.5.2 we observe that a similar argumentation as above yields

$$||V_K - s_K^2||_2 \leq c\frac{K^{1+\eta}}{(\log(K))^{(1+\varepsilon)/2}}$$

for some constant $c > 0$ which depends on q, d and G. The remainder of this Theorem remains the same except for (3.3.20) where we applied Lemma 3.3.2. Observe that for proving Lemma 3.3.2 we need changes similar to those in the proof of Lemma 2.0.5, i.e. we have to show $|V_{2m}(x)| \leq c||p||_2 P$ for some constant $c > 0$ which only depends on q, d and G where V_{2m} is defined in (3.3.7). By definition V_{2m} is a trigonometric polynomial such that any frequency vector u is of the form $u = M_n^T j \pm M_{n'}^T j'$ with $||M_n^T j \pm M_{n'}^T j'||_\infty < ||M_{m'}^T||_\infty$. It is easy to see that for such u we have

$$|(M_n^T)_{i,i} j_i \pm (M_{n'}^T)_{i,i} j_i'| < ||M_{m'}^T||_\infty \quad \text{for all } i \in \{1, \ldots, d\}.$$

Without loss of generality we assume $n' \geq n$. Therefore for any $i \in \{1, \ldots, d\}$ we get

$$\left| j_i \pm \frac{(M_{n'}^T)_{i,i}}{(M_n^T)_{i,i}} j_i' \right| < \frac{||M_{m'}^T||_\infty}{(M_n^T)_{i,i}}.$$

Denote the number of $l \in \{n+1, \ldots, n'\}$ with $(M_l)_{i,i} \geq q||M_{l-1}^T||_\infty$ by $r_{i,n,n'}$. Assume $r_{i,m',n} \geq 1$. For $r_{i,n,n'} \geq 1$ and we have

$$\left| j_i \pm \frac{(M_{n'}^T)_{i,i}}{(M_n^T)_{i,i}} j_i' \right| < 1$$

and thus we obtain $|j_i| > q^{r_{i,n,n'}}|j_i'| - 1$. Since $|j_i| \geq 1$ it is easy to see that there is some constant $C > 0$ such that $|j_i| \geq Cq^{r_{i,n,n'}}$ for all $i \in \{1, \ldots, d\}$ with $r_{i',m,n} \geq 1$. For $r_{i,m',n} = 0$ using a similar argumentation we have $|j_i| \geq C'q^{r_{i,n,n'}-1} \geq Cq^{r_{i,n,n'}}$ for some constants $C, C' > 0$ depending only on q. Furthermore there is some constant $C > 0$ such that $\prod_{i=1}^d |j_i|^{-1} \leq Cq^{n-n'} \prod_{i=1}^d |j_i'|^{-1}$. By Lemma 2.0.1 and Cauchy-Schwarz inequality we see

$$\begin{aligned}
|V_{2m}(x)| &\leq \sum_{n,n'} \sum_{j,j'} |a_j a_{j'}| + |a_j b_{j'}| + |b_j a_{j'}| + |b_j b_{j'}| \\
&\leq C \sum_{n \leq n'} q^{n-n'} \sum_{j,j'} (|a_{j'}| + |b_{j'}|) \prod_{i=1}^d \frac{1}{2\pi|j_i'|} \\
&\leq C \sum_{n \leq n'} q^{n-n'} (2G)^d \sum_{j'} (|a_{j'}| + |b_{j'}|) \prod_{i=1}^d \frac{1}{2\pi|j_i'|} \\
&\leq C \sum_{n \leq n'} q^{n-n'} (2G)^d \left(\sum_{j'} a_{j'}^2 + b_{j'}^2 \right)^{1/2} \left(\sum_{j'} \prod_{i=1}^d \frac{1}{2\pi|j_i'|} \right)^{1/2} \\
&\leq C||p||_2 P
\end{aligned}$$

for some constant $C > 0$ which only depends on q, d and G and therefore Theorem 3.5.2 is proved.

4 Double infinite matrices and the inverse of the discrepancy

In this chapter we discuss a double infinite matrix such that each $N \times d$-projection defines a sequence of N points in $[0,1)^d$ which is constructed partly by a randomized Halton sequence and partly by a certain lacunary sequence. We show that with large probability the star-discrepancy of this sequences is bounded by $C\sqrt{d}/\sqrt{N}$ for some universal constant C independent of N and d which so far is up to some constant the best known upper bound for the star-discrepancy of a point set such that the number of points is "small" in comparison with the dimension. The main result is presented in section 4.1. In section 4.2 we review some tools for proving the main theorem, especially the maximal Bernstein inequality and δ-bracketing covers. In section 4.3 we give an upper bound on the star-discrepancy of a subsequence of a randomized Halton sequence which is induced by a certain modulo class. Finally in section 4.4 we prove the main theorem.

4.1 Main result

Let $x = (x_1, \dots, x_d) \in [0,1)^d$. For $i \in \{1, \dots, d\}$ and some integer $p_i \geq 2$ we define the p_i-adic decomposition of x_i by $x_i = \sum_{j=0}^{\infty} \alpha(j,i) p_i^{-j-1}$ where $\alpha(j,i) \in \{0, \dots, p_i - 1\}$ for all $i \in \{1, \dots, d\}$ and $j \geq 0$. Now set

$$T_{p_i}(x_i) = \frac{\alpha(m,i)+1}{p_i^{m+1}} + \sum_{j>m} \frac{\alpha(j,i)}{p_i^{j+1}}$$

where $m = \min\{j : \alpha(j,i) \neq p_i - 1\}$. Furthermore for a collection of pairwise coprime odd integers $p = (p_1, \dots, p_d)$ set $T_p(x) = (T_{p_1}(x_1), \dots, T_{p_d}(x_d))$. Observe that for $x_0 = 0$ the sequence $(x_n)_{n \geq 1}$ with $x_n = T_p(x_{n-1})$ for $n \geq 1$ defines a Halton sequence. Therefore for some uniformly distributed $x_0 \in [0,1)^d$ and pairwise coprime odd integers $p_1, \dots, p_d$ we call this sequence a randomized Halton sequence.

Theorem 4.1.1 *Let $(x_{1,i})_{i \geq 1}$ be a sequence of independent random variables which are uniformly distributed in $[0,1)$ and let $(p_i)_{i \geq 1}$ be the sequence of all odd prime numbers. For all integers $n \geq 1$ and $i \geq 1$ define*

$$x_{n+1,i} = \begin{cases} T_{p_i}(x_{n,i}), & \text{if } i = 1 \text{ or } 2^{12 \cdot 2^i} < n - 1, \\ \langle 2^{\lceil \log_2(i) \rceil + 1} x_{n,i} \rangle, & \text{if } i \geq 2 \text{ and } 2^{12 \cdot 2^i} \geq n - 1. \end{cases} \tag{4.1.1}$$

Then for any $\varepsilon > 0$ the probability, that for any integers $N \geq 1$ and $d \geq 1$ the set of points $P = \{(x_{1,1}, \ldots, x_{1,d}), \ldots, (x_{N,1}, \ldots, x_{N,d})\} \subset [0,1)^d$ satisfies

$$D_N^*(P) \leq (2576 + 357 \log(\varepsilon^{-1})) \frac{\sqrt{d}}{\sqrt{N}}, \tag{4.1.2}$$

is at least $1 - \varepsilon$.

4.2 Auxiliary lemmata

Lemma 4.2.1 (Maximal Bernstein inequality, [26, Lemma 2.2]) *For some integer $N \geq 1$ let $Z_1, \ldots, Z_N$ be a sequence of i.i.d. random variables with mean zero and variance $\sigma^2 > 0$ such that $|Z_1| \leq 1$. Then for any $t > 0$ we have*

$$\mathbb{P}\left(\max_{M \in \{1,\ldots,N\}} \left| \sum_{n=1}^{M} Z_n \right| > t \right) \leq 2 \exp\left(-\frac{t^2}{2N\sigma^2 + 2t/3} \right). \tag{4.2.1}$$

For integers $N \geq 1$ and $d \geq 1$ and an N-element set of d-dimensional points

$$\{(x_{1,1}, \ldots, x_{1,d}), \ldots, (x_{N,1}, \ldots, x_{N,d})\}$$

denote the star-discrepancy by $D_N^d(x_{n,i})$. Furthermore for an integer $0 \leq M < N$ write $D_{M,N}^d(x_{n,i})$ for the star-discrepancy of the $N - M$-element point set

$$\{(x_{M+1,1}, \ldots, x_{M+1,d}), \ldots, (x_{N,1}, \ldots, x_{N,d})\}.$$

Lemma 4.2.2 ([21, Proposition 3.16]) *Let $0 \leq M < N$ be integers. Then for points $y_1, \ldots, y_N \in [0,1)^d$ we have*

$$D_N^d(y_1, \ldots, y_N) \leq \frac{M D_M^d(y_1, \ldots, y_M)}{N} + \frac{(N-M) D_{M,N}^d(y_{M+1}, \ldots, y_N)}{N}$$

and

$$D_{M,N}^d(y_{M+1}, \ldots, y_N) \leq \frac{N D_N^d(y_1, \ldots, y_N)}{N-M} + \frac{M D_M^d(y_1, \ldots, y_M)}{N-M}.$$

Let $v, w \in [0,1)^d$. We write $v \leq w$ if $v_i \leq w_i$ for all $i \in \{1, \ldots, d\}$. For some $\delta > 0$ a set Δ of elements in $[0,1)^d \times [0,1)^d$ is called a δ-bracketing cover if for every $x \in [0,1)^d$ there exists $(v,w) \in \Delta$ with $v \leq x \leq w$ and $\lambda(\overline{[v,w)}) \leq \delta$ for $\overline{[v,w)} = [0,w) \setminus [0,v)$. The following Lemma gives an upper bound on the cardinality of a δ-bracketing cover.

Lemma 4.2.3 ([32, Theorem 1.15]) *For any $d \geq 1$ and $\delta > 0$ there exists some δ-bracketing cover Δ with*

$$|\Delta| \leq \frac{1}{2}(2e)^d(\delta^{-1} + 1)^d.$$

Corollary 4.2.4 *For any integers $d \geq 1$ and $h \geq 1$ there exists some 2^{-h}-bracketing cover Δ with*

$$|\Delta| \leq \frac{1}{2}(2e)^d(2^{h+2}+1)^d$$

such that for any $(v,w) \in \Delta$ and any $i \in \{1,\dots,d\}$ we have

$$\begin{aligned} v_i &= 2^{-(\lceil \log_2(i) \rceil + 1)(h+1)} a_i, \\ w_i &= 2^{-(\lceil \log_2(i) \rceil + 1)(h+2)} b_i \end{aligned}$$

for some integers $a_i \in \{0,1,\dots,2^{(\lceil \log_2(i) \rceil + 1)(h+1)}\}$ and $b_i \in \{0,1,\dots,2^{(\lceil \log_2(i) \rceil + 1)(h+2)}\}$.

Proof. Let Δ be some $2^{-(h+2)}$-bracketing cover of $[0,1)^d$. By Lemma 4.2.3 we have

$$|\Delta| \leq \frac{1}{2}(2e)^d(2^{-(h+2)}+1)^d.$$

For $(v,w) \in \Delta$ and $i \in \{1,\dots,d\}$ define

$$\begin{aligned} y_{v,i} &= \max\left\{2^{-(\lceil \log_2(i) \rceil + 1)(h+1)} a_i \leq v_i : a_i \in \mathbb{Z}\right\}, \\ z_{w,i} &= \min\left\{2^{-(\lceil \log_2(i) \rceil + 1)(h+2)} b_i \geq w_i : b_i \in \mathbb{Z}\right\}. \end{aligned}$$

For $y_v = (y_{v,i})_{i \in \{1,\dots,d\}} \in [0,1)^d$ we obtain

$$\lambda(\overline{[y_v, v)}) \leq \sum_{i=1}^{d} 2^{-(\lceil \log_2(i) \rceil + 1)(h+1)} \leq 2^{-(h+1)} \sum_{i=1}^{d} i^{-(h+1)} \leq 2^{-(h+1)}.$$

Analogously for $z_w = (z_{w,i})_{i \in \{1,\dots,d\}} \in [0,1)^d$ we have

$$\lambda(\overline{[z, z_w)}) \leq 2^{-(h+2)}.$$

Thus we get

$$\lambda(\overline{[y_v, z_w)}) \leq \lambda(\overline{[y_v, v)}) + \lambda(\overline{[v, w)}) + \lambda(\overline{[w, z_w)}) \leq 2^{-h}.$$

Set $\tilde{\Delta} = \{(y_v, z_w) : (v,w) \in \Delta\}$. Since Δ is a $2^{-(h+2)}$-bracketing cover for any $x \in [0,1)^d$ there exists $(v,w) \in \Delta$ and $(y_v, z_w) \in \tilde{\Delta}$ with $y_v \leq v \leq x \leq w \leq z_w$. Therefore $\tilde{\Delta}$ is a 2^{-h}-bracketing cover and the conclusion of the proof follows by $|\tilde{\Delta}| \leq |\Delta|$.

4.3 Randomized Halton sequences

Note that we assume that the integers $p_1, \dots, p_d$ are odd since we later need sequences such that not only $(x_n)_{n \geq 1}$ is a low-discrepancy sequence but also subsequences $(x_{n_l})_{l \geq 1}$ where the elements n_l belong to one particular modulo class with modulo 2^κ for some integer κ have sufficiently small discrepancy. This shall be ensured by the following

Lemma 4.3.1 *For some integer $d \geq 2$ let $(x_n)_{n\geq 1}$ be a randomized Halton sequence in $[0,1)^d$ constructed by the first d odd primes. Let $N \geq 2^{12\cdot 2^d}$ be the number of points. For some integers $1 \leq \kappa \leq \log_2(8\log_2(N))$ and $\gamma \in \{0,\ldots,2^\kappa - 1\}$ set*

$$N_{\kappa,\gamma} = \{n : n \in \{1,\ldots,N\}, n \equiv \gamma \pmod{2^\kappa}\}$$

and define $P_{N,\kappa,\gamma} = \{x_n : n \in N_{\kappa,\gamma}\}$. Then the star-discrepancy of $P_{N,\kappa,\gamma}$ satisfies

$$D^*_{N,\kappa,\gamma}(\{x_n : n \in N_{\kappa,\gamma}\}) = D^*_{|N_{\kappa,\gamma}|}(P_{N,\kappa,\gamma}) \leq \frac{\sqrt{d}}{\sqrt{|N_{\kappa,\gamma}|}}. \tag{4.3.1}$$

Proof. The proof of this Lemma which is an application of the Chinese Remainder Theorem is mainly based on the proofs of [44, Theorem 3.6] and [3, Corollary 1]. For some x_0 let the p_i-adic decomposition of $x_{n,i} = (T^n_{p_i}(x_0))_i$ be given by

$$x_{n,i} = \sum_{j=0}^{\infty} \alpha(j,i,x_0,n) p_i^{-j-1}$$

for suitable integers $\alpha(j,i,x_0,n) \in \{0,1,\ldots,p_i - 1\}$. Observe that for any $i \in \{1,\ldots,d\}$ there exists at most one $N_i \in \{1,\ldots,N\}$ such that

$$\begin{aligned}\alpha(j,i,x_0,1) = \alpha(j,i,x_0,2) = \cdots = \alpha(j,i,x_0,N_i)& \\ \neq \alpha(j,i,x_0,N_i+1) = \cdots = \alpha(j,i,x_0,N)&\end{aligned}$$

for all $j \geq \lceil \log_{p_i}(N) \rceil$. Let $\pi : \{1,\ldots,d\} \to \{1,\ldots,d\}$ be a permutation satisfying $N_{\pi(i)} \geq N_{\pi(j)}$ if $i \geq j$ and set $\pi(0) = 0$ and $\pi(d+1) = N$. Therefore there exist constants $g_{m,i}$ for any $m \in \{1,\ldots,d+1\}$ and $i \in \{1,\ldots,d\}$ such that

$$\sum_{j=\lceil \log_{p_i}(N) \rceil}^{\infty} \alpha(j,i,x_0,n) {p_i}^{-j-1} = g_{m,i} \tag{4.3.2}$$

for all $n \in \{N_{\pi(m-1)} + 1, \ldots, N_{\pi(m)}\}$. Now fix some set $\mathcal{N} = \{N_{\pi(m-1)} + 1, \ldots, N_{\pi(m)}\}$ and define

$$\Psi_i(x_n) = \Psi_i \left(\sum_{j=0}^{\lceil \log_{p_i}(N) \rceil - 1} \alpha(j,i,x_0,n) {p_i}^{-j-1} + g_{m,i} \right) = \sum_{j=0}^{\lceil \log_{p_i}(N) \rceil - 1} \alpha(j,i,x_0,n) {p_i}^{j}.$$

Set $n_l = 2^\kappa(l-1)+\gamma$ for any integer $l \geq 1$ and some $\gamma \in \{0,\ldots,2^\kappa - 1\}$. By definition of T_p it is easy to see that for $n, n+1 \in \mathcal{N}$ and $i \in \{1,\ldots,d\}$ we have $\Psi_i(x_{n+1}) = \Psi_i(x_n) + 1$. Thus we obtain $\Psi_i(x_{n_{l+1}}) = \Psi_i(x_{n_l}) + 2^\kappa$ for $n_{l+1}, n_l \in \mathcal{N}$. We now shall show the following version of the Chinese Remainder Theorem:
Let $\beta_1,\ldots,\beta_d$ and $s_1,\ldots,s_d$ be positive numbers, then there exists some integer β such that any solution of

$$\begin{aligned} \Psi_1(x_{n_l}) &\equiv \beta_1 \pmod{p_1^{s_1}} \\ &\vdots \\ \Psi_d(x_{n_l}) &\equiv \beta_d \pmod{p_d^{s_d}} \end{aligned} \tag{4.3.3}$$

satisfies

$$l \equiv \beta \qquad (\mathrm{mod}\ p_1^{s_1} p_2^{s_2} \cdots p_d^{s_d}). \tag{4.3.4}$$

Observe that we have $\Psi_i(x_{n_{l'}}) \equiv \Psi_i(x_{n_l}) \pmod{p_i^{s_i}}$ only if $2^\kappa(l'-l) \equiv 0 \pmod{p_i^{s_i}}$. Since 2^κ and $p_i^{s_i}$ are coprime we have $(l'-l)|p_i^{s_i}$. Now for any $i \in \{1,\dots,d\}$ define the map $\Xi_i : \mathbb{Z} \to \mathbb{Z}/p_i^{s_i}\mathbb{Z}$ with $\Xi_i(l) = \Psi_i(x_{n_l}) + p_i^{s_i}\mathbb{Z}$. Observe that Ξ_i is periodic, i.e. $\Xi_i(l) = \Xi_i(l + \alpha p_i^{s_i})$ for any integer α, and $\Xi_i|_{\{1,\dots,p_i^{s_i}\}}$ is bijective, i.e. $\Xi_i(l) \neq \Xi_i(l')$ for $l, l' \in \{1,\dots,p_i^{s_i}\}$ with $l \neq l'$. Since the system (4.3.3) only has a solution if $\Xi_i(l) \equiv \beta_i \pmod{p_i^{s_i}}$ for $i \in \{1,\dots,d\}$ we see that are integers $\alpha_1,\dots,\alpha_d$ such any solution satisfies $l \equiv \alpha_i \pmod{p_i^{s_i}}$ for $i \in \{1,\dots,d\}$. By classical Chinese Remainder Theorem there exists some integer β such for any solution l we conclude (4.3.4). Thus among $\prod_{i=1}^d p_i^{s_i}$ consecutive numbers of the sequence $(n_l)_{n\geq 1}$ there is exactly one l such that $\Psi_i(x_{n_l}) = a_i$ for any collection of numbers $a_i \in \{0,\dots,p_i^{s_i}-1\}$ with $i \in \{1,\dots,d\}$. Let B be any box of the form

$$B = \prod_{i=1}^d [a_i p_i^{-s_i}, (a_i+1)p_i^{-s_i})$$

with $a_i \in \{0,\dots,p_i^{s_i}-1\}$ and $i \in \{1,\dots,d\}$. Observe that $x \in B$ if for $i \in \{1,\dots,d\}$ the first s_i digits in the p_i-adic decomposition of x_i are uniquely defined, i.e. we have $\sum_{j=0}^{s_i-1} \alpha(j,i) p_i^{-j-1} = a_i$ for all

$$x_i = \sum_{j=0}^{\infty} \alpha(j,i) p_i^{-j-1} \in B_i = [a_i p_i^{-s_i}, (a_i+1)p_i^{-s_i}).$$

By definition of the Ψ_i this is equivalent to $\Psi_i(x) \equiv \sum_{j=0}^{s_i-1} \alpha(j,i) p_i^j \pmod{p_i^{s_i}}$ for all $i \in \{1,\dots,d\}$. Thus there is exactly one l with $x_{n_l} \in B$ among $\prod_{i=1}^d p_i^{s_i}$ consecutive numbers of the sequence $(x_{n_l})_{l\geq 1}$. Therefore we obtain

$$\left| \left\{ l : x_{n_l} \in B, l \in \left\{ L+1, \dots, L + \prod_{i=1}^d p_i^{s_i} \right\} \subset \mathcal{N} \right\} \right| = 1. \tag{4.3.5}$$

Now for $i \in \{1,\dots,d\}$ and integers $r_i \geq 0$ let

$$\mathcal{C}_i(r_i) = \left\{ [0, c_i p_i^{-r_i}) : c_i \in \{0,\dots,p_i^{r_i}-1\} \right\}$$

be a family of intervals. Furthermore set

$$\mathcal{A}_i(r_i) = \left\{ [a_i p_i^{-s_i}, (a_i+1)p_i^{-s_i}) : a_i \in \{0, 1 \dots, p_i^{s_i}-1\}, s_i \in \{0,\dots,r_i\} \right\}.$$

For integers $r_1,\dots,r_d \geq 0$ let

$$\mathcal{B}(r_1,\dots,r_d) = \left\{ B = \prod_{i=1}^d B_i : B_i \in \mathcal{C}_i(r_i) \cup \mathcal{A}_i(r_i) \text{ for any } i \in \{1,\dots,d\} \right\}$$

be a collection of boxes.

For any box $B \subseteq [0,1)^d$ and any set $\mathcal{M}$ of positive integers set

$$\mathcal{D}_{\mathcal{M}}(B) = \left| \sum_{n \in \mathcal{M}} (\mathbf{1}_B(x_n) - \lambda(B)) \right| .$$

Furthermore for integers $\kappa, \gamma \geq 0$ with $\gamma \in \{0, \ldots, 2^\kappa - 1\}$ and any set $\mathcal{N}$ as defined above let

$$\mathcal{N}_{\kappa,\gamma} = \{n \in \mathcal{N} : n \equiv \gamma \pmod{2^\kappa}\} .$$

Now we shall show that for any set of integers $r_1, \ldots, r_d \geq 0$ and any box

$$B = \prod_{i=1}^{d} B_i \in \mathcal{B}(r_1, \ldots, r_d)$$

we have

$$\mathcal{D}_{\mathcal{N}_{\kappa,\gamma}}(B) \leq \prod_{\substack{i \in \{1,\ldots,d\}, \\ B_i \notin \mathcal{A}_i(r_i)}} \left(\frac{p_i - 1}{2} r_i + 1 \right) . \tag{4.3.6}$$

We are going to prove this inequality by induction on the number k of indices i such that $B_i \notin \mathcal{A}_i(r_i)$. Thus we first assume $k = 0$. We have $B = \prod_{i=1}^{d} [a_i p_i^{-s_i}, (a_i + 1) p_i^{-s_i})$ for suitable integers $s_1, \ldots, s_d$ and $a_1, \ldots, a_d$. By (4.3.5) we obtain

$$\left\lfloor |\mathcal{N}_{\kappa,\gamma}| \prod_{i=1}^{d} p_i^{-s_i} \right\rfloor \leq \sum_{n \in \mathcal{N}_{\kappa,\gamma}} \mathbf{1}_B(x_n) \leq \left\lceil |\mathcal{N}_{\kappa,\gamma}| \prod_{i=1}^{d} p_i^{-s_i} \right\rceil .$$

Since $\sum_{n \in \mathcal{N}_{\kappa,\gamma}} \lambda(B) = |\mathcal{N}_{\kappa,\gamma}| \prod_{i=1}^{d} p_i^{-s_i}$ we conclude $\mathcal{D}_{\mathcal{N}_{\kappa,\gamma}}(B) \leq 1$ for $k = 0$. Now assume that (4.3.6) has been proved for $|\{i : B_i \notin \mathcal{A}_i(r_i)\}| = k - 1$. Consider some box $B \in \mathcal{B}(r_1, \ldots, r_d)$ with $|\{i : B_i \notin \mathcal{A}_i(r_i)\}| = k$. Without loss of generality we may assume $B_i \notin \mathcal{A}_i(r_i)$ for $i \in \{1, \ldots, k\}$ and $B_i \in \mathcal{A}_i(r_i)$ for $i \in \{k+1, \ldots, d\}$. Then we have $B_k = [0, c_k p_k^{-r_k})$ for some integer c_k with $c_k \in \{0, \ldots, p_k^{r_k} - 1\}$. We get $c_k p_k^{-r_k} = \sum_{j=1}^{r_k} e_j p_k^{-j}$ for integers e_j with $e_j \in \{0, \ldots, p_k - 1\}$ for $1 \leq j \leq r_k$. Therefore the interval B_k can be decomposed into e_1 intervals of length p_k^{-1}, e_2 intervals of length p_k^{-2} and so on. Set $e = \sum_{j=1}^{r_k} e_j$. Then

$$B_k = \bigcup_{t=1}^{e} E_t$$

for pairwise disjoint $E_t \in \mathcal{A}_k(r_k)$ with $t \in \{1, \ldots, e\}$. Thus we obtain

$$B = \bigcup_{t=1}^{e} (B_1 \times \cdots \times B_{k-1} \times E_t \times B_{k+1} \times \cdots \times B_d).$$

By induction hypothesis we observe

$$\begin{aligned}\mathcal{D}_{\mathcal{N}_{\kappa,\gamma}}(B) &\leq \sum_{t=1}^{e} \mathcal{D}_{\mathcal{N}_{\kappa,\gamma}}\left(B_1 \times \cdots \times B_{k-1} \times E_t \times B_{k+1} \times \cdots \times B_d\right) \\ &\leq e \prod_{i=1}^{k-1} \left(\frac{p_i - 1}{2} r_i + 1\right).\end{aligned} \tag{4.3.7}$$

Furthermore set $F = [c_k p_k^{-r_k}, 1) = [0,1) \backslash B_k$. We have

$$\begin{aligned}\mathcal{D}_{\mathcal{N}_{\kappa,\gamma}}(B) \leq\ & \mathcal{D}_{\mathcal{N}_{\kappa,\gamma}}\left(B_1 \times \cdots \times B_{k-1} \times [0,1) \times B_{k+1} \times \cdots \times B_d\right) \\ &+ \mathcal{D}_{\mathcal{N}_{\kappa,\gamma}}\left(B_1 \times \cdots \times B_{k-1} \times F \times B_{k+1} \times \cdots \times B_d\right).\end{aligned}$$

Thus we get

$$\begin{aligned}\mathcal{D}_{\mathcal{N}_{\kappa,\gamma}}(B) \leq\ & \prod_{i=1}^{k-1} \left(\frac{p_i - 1}{2} r_i + 1\right) \\ &+ \mathcal{D}_{\mathcal{N}_{\kappa,\gamma}}\left(B_1 \times \cdots \times B_{k-1} \times F \times B_{k+1} \times \cdots \times B_d\right).\end{aligned}$$

Observe that F can be decomposed into $1 + \sum_{j=1}^{r_k} p_k - 1 - e_j = (p_k - 1) r_k - e + 1$ intervals in $\mathcal{A}_k(r_k)$. Thus we get

$$\mathcal{D}_{\mathcal{N}_{\kappa,\gamma}}(B) \leq ((p_k - 1) r_k - e + 2) \prod_{i=1}^{k-1} \left(\frac{p_i - 1}{2} r_i + 1\right).$$

By (4.3.7) we have

$$\mathcal{D}_{\mathcal{N}_{\kappa,\gamma}}(B) \leq \frac{e + (p_k - 1) r_k - e + 2}{2} \prod_{i=1}^{k-1} \left(\frac{b_i - 1}{2} r_i + 1\right).$$

Hence (4.3.6) is proved for any k. Now let $J = \prod_{i=1}^{d} [0, v_i) \subset [0,1)^d$ be some arbitrary box. For any $i \in \{1, \ldots, d\}$ set $r_i = \lceil \log_{p_i}(N) \rceil$ and furthermore let c_i be the integer such that $c_i p_i^{-r_i} \leq v_i < (c_i + 1) p_i^{-r_i}$. Take some

$$\mathcal{N}_{\kappa,\gamma,m} = \{n \in \{N_{\pi(m-1)} + 1, \ldots, N_{\pi(m)}\}, n \equiv \gamma \pmod{2^\kappa}\}.$$

By definition of $\mathcal{N}_{\kappa,\gamma,m}$ for any $i \in \{1, \ldots, d\}$ we have $x_{n,i} = z_{n,i} p_i^{-r_i} + g_{m,i}$ for some integer $z_{n,i}$ depending on $n \in \mathcal{N}_{\kappa,\gamma,m}$ and $0 \leq g_{m,i} < p_i^{-r_i}$ independent of n. For $v_i - c_i p_i^{-r_i} \leq g_{m,i}$ set $v_i' = c_i p_i^{-r_i}$, otherwise set $v_i' = (c_i + 1) p_i^{-r_i}$ and let $B_m = \prod_{i=1}^{d} [0, v_i')$. It is easy to see that

$$\sum_{n \in \mathcal{N}_{\kappa,\gamma,m}} \mathbf{1}_J(x_n) = \sum_{n \in \mathcal{N}_{\kappa,\gamma,m}} \mathbf{1}_{B_m}(x_n).$$

Thus by (4.3.6) we get

$$\begin{aligned}
\mathcal{D}_{N_{\kappa,\gamma}}(J) &\leq \sum_{m=1}^{d+1} \mathcal{D}_{\mathcal{N}_{\kappa,\gamma,m}}(J) \\
&\leq \sum_{m=1}^{d+1} \left| \sum_{n\in\mathcal{N}_{\kappa,\gamma,m}} (\mathbf{1}_J(x_n) - \lambda(J)) \right| \\
&\leq \sum_{m=1}^{d+1} \left| \sum_{n\in\mathcal{N}_{\kappa,\gamma,m}} (\mathbf{1}_{B_m}(x_n) - \lambda(B_m)) \right| + \sum_{m=1}^{d+1} |\mathcal{N}_{\kappa,\gamma,m}| \cdot |\lambda(J) - \lambda(B_m)| \\
&\leq \sum_{m=1}^{d+1} \mathcal{D}_{\mathcal{N}_{\kappa,\gamma,m}}(B_m) + |N_{\kappa,\gamma}| \sum_{i=1}^{d} p_i^{-r_i} \\
&\leq (d+1) \prod_{i=1}^{d} \left(\frac{p_i - 1}{2} r_i + 1 \right) + |N_{\kappa,\gamma}| \sum_{i=1}^{d} p_i^{-r_i}.
\end{aligned}$$

Since $r_i = \lceil \log_{p_i}(N) \rceil$ we observe

$$\begin{aligned}
D^*_{|N_{\kappa,\gamma}|}(P_{N,\kappa,\gamma}) &= \sup_J \frac{\mathcal{D}_{N_{\kappa,\gamma}}(J)}{|N_{\kappa,\gamma}|} \\
&\leq \frac{d}{N} + \frac{1}{|N_{\kappa,\gamma}|} \prod_{i=1}^{d} \frac{i+1}{i} \left(\frac{p_i - 1}{2\log(p_i)} \log(N) + \frac{p_i + 1}{2} \right).
\end{aligned} \tag{4.3.8}$$

Next we shall show

$$\frac{i+1}{i} \left(\frac{p_i - 1}{2\log(p_i)} \log(N) + \frac{p_i + 1}{2} \right) \leq (i+1)\log(N). \tag{4.3.9}$$

for all $i \in \{1, \dots, d\}$. This is easy to see for $i \leq 4$. It is well-known that for $i \geq 5$ we have $i \leq p_i \leq 1 + 7/4 \cdot i \log(i)$ (see, e.g. [15, Theorem 8.8.4]). Therefore we get

$$\begin{aligned}
\frac{i+1}{i} \left(\frac{p_i - 1}{2\log(p_i)} \log(N) + \frac{p_i + 1}{2} \right) &\leq \frac{i+1}{i} \left(\frac{7/4 \cdot i \log(i)}{2\log(i)} \log(N) + 2i\log(i) \right) \\
&\leq \frac{i+1}{i} \left(\frac{7}{8} i \log(N) + 2i \log(i) \right) \\
&\leq (i+1)\log(N).
\end{aligned}$$

Thus (4.3.9) is proved. Together with (4.3.8) we have

$$D^*_{|N_{\kappa,\gamma}|}(P_{N,\kappa,\gamma}) \leq \frac{d}{N} + \frac{(d+1)!(\log(N))^d}{|N_{\kappa,\gamma}|}. \tag{4.3.10}$$

It remains to show

$$\frac{\sqrt{d}}{\sqrt{|N_{\kappa,\gamma}|}} + \frac{(d+1)!(\log(N))^d}{\sqrt{d|N_{\kappa,\gamma}|}} \leq 1. \tag{4.3.11}$$

Then the statement of the Lemma follows by (4.3.10) and

$$
\begin{aligned}
D^*_{|N_{\kappa,\gamma}|}(P_{N,\kappa,\gamma}) &\leq \frac{d}{|N_{\kappa,\gamma}|} + \frac{(d+1)!(\log(N))^d}{|N_{\kappa,\gamma}|} \\
&\leq \frac{\sqrt{d}}{\sqrt{|N_{\kappa,\gamma}|}} \left(\frac{\sqrt{d}}{\sqrt{|N_{\kappa,\gamma}|}} + \frac{(d+1)!(\log(N))^d}{\sqrt{d|N_{\kappa,\gamma}|}} \right) \\
&\leq \frac{\sqrt{d}}{\sqrt{|N_{\kappa,\gamma}|}}.
\end{aligned}
$$

In order to show (4.3.11) we estimate the second term and observe

$$
\frac{(d+1)!(\log(N))^d}{\sqrt{d|N_{\kappa,\gamma}|}} \leq \frac{(d+1)!(\log(N))^d}{\sqrt{2^{-\kappa-1}dN}} \leq \frac{4}{\sqrt{\log(2)}} \frac{(d+1)!(\log(N))^{d+1/2}}{\sqrt{dN}} \tag{4.3.12}
$$

where we used $\kappa \leq \log_2(8\log_2(N))$ for the second inequality. Since for any fixed d the derivative of $(\log(N))^{d+1/2}/\sqrt{N}$ is negative for $N \geq e^{2(d+1/2)}$ it is enough to restrict ourselves to the case $N = 2^{12\cdot 2^d} > e^{2(d+1/2)}$. Therefore we first shall show

$$
\frac{4}{\sqrt{\log(2)}} \frac{(d+1)!}{\sqrt{d}} \leq \frac{1}{2}(\log(N))^{d-1/2} \tag{4.3.13}
$$

resp. equivalently

$$
\frac{8(d+1)!}{12^{d-1/2}(\log(2))^d\sqrt{d}} \leq 2^{d^2-d/2}. \tag{4.3.14}
$$

This shall be done by induction. It can easily be verified that (4.3.14) is true for $d = 2$. Thus we may assume that (4.3.14) holds for some integer $d \geq 2$. We get

$$
\begin{aligned}
\frac{8((d+1)+1)!}{12^{(d+1)-1/2}(\log(2))^{(d+1)}\sqrt{d+1}} &\leq (d+2)\frac{8(d+1)!}{12^{d-1/2}(\log(2))^d\sqrt{d}} \\
&\leq 2^{2d+1/2}\frac{8(d+1)!}{12^{d-1/2}(\log(2))^d\sqrt{d}} \\
&\leq 2^{2d+1/2} \cdot 2^{d^2-d/2} \\
&\leq 2^{(d+1)^2-(d+1)/2}
\end{aligned}
$$

and therefore we have (4.3.13) for any integer $d \geq 2$. For $d \geq 2$ we get

$$
(12 \cdot 2^d)^{2d} \leq 2^{2d^2+2\log_2(12)d} \leq 2^{6\cdot 2^d} \leq \sqrt{N}.
$$

Hence $(\log(N))^{2d}/\sqrt{N} \leq 1$ immediately follows. With $\sqrt{d}/\sqrt{|N_{\kappa,\gamma}|} \leq 1/2$ and (4.3.13) we observe (4.3.11) which finally concludes the proof.

4.4 Proof of Theorem 4.1.1

The proof of this Theorem is mainly based on [3]. For some integers $N \geq 1$ and $d \geq 1$ we simply write

$$D_N^d(x_{n,i}) = D_N^d((x_{1,1}, \ldots, x_{1,d}), \ldots, (x_{N,1}, \ldots, x_{N,d})).$$

For all integers $m \geq 1$ and $d \geq 1$ we define

$$\mathcal{F}_{m,d,\varepsilon} = \left\{ \max_{M \in \{2^m+1,\ldots,2^{m+1}\}} M D_M^d(x_{n,i}) \geq C_{m,d,\varepsilon} \sqrt{d} \sqrt{2^{m+1}} \right\}$$

with

$$C_{m,d,\varepsilon} = \begin{cases} 1819 + 252 \log(\varepsilon^{-1}), & \text{if } 12 \cdot 2^d > m, \\ 1821 + 252 \log(\varepsilon^{-1}), & \text{if } 12 \cdot 2^d \leq m. \end{cases}$$

We shall show that

$$\mathbb{P}\left(\bigcup_{d \geq 1} \bigcup_{m \geq 1} \mathcal{F}_{m,d,\varepsilon} \right) \leq \varepsilon. \tag{4.4.1}$$

Therefore on the complement of $\cup_{d \geq 1} \cup_{m \geq 1} \mathcal{F}_{m,d,\varepsilon}$ which has measure bounded from below by $1 - \varepsilon$ for any integer $N \geq 1$ and $d \geq 1$ we have

$$N D_N^d(x_{n,i}) \leq (1821 + 252 \log(\varepsilon^{-1})) \sqrt{d} \sqrt{2N} \leq (2576 + 357 \log(\varepsilon^{-1})) \sqrt{d} \sqrt{N}$$

which concludes the proof. By (4.3.8) which also holds in the case $d = 1$ and $N \geq 3$ it is easy to see that for $d = 1$ and $m \geq 1$ we observe

$$\mathbb{P}(\mathcal{F}_{m,d,\varepsilon}) = 0. \tag{4.4.2}$$

Therefore we may assume $d \geq 2$. We now claim

$$\mathbb{P}\left(\bigcup_{d \geq 2} \bigcup_{m \geq 1} \mathcal{F}_{m,d.\varepsilon} \right) = \mathbb{P}\left(\bigcup_{d \geq 2} \bigcup_{m \in \{1,\ldots,12 \cdot 2^d - 1\}} \mathcal{F}_{m,d.\varepsilon} \right). \tag{4.4.3}$$

Let $d \geq 2$ be given and assume $m \geq 12 \cdot 2^d$. Furthermore set $\mu = 12 \cdot 2^d$. By Lemma 4.2.2 for $M \in \{2^m + 1, \ldots, 2^{m+1}\}$ we obtain

$$M D_M^d(x_{n,i}) \leq 2^\mu D_{2^\mu}^d(x_{n,i}) + (M - 2^\mu) D_{2^\mu, M}^d(x_{n,i}). \tag{4.4.4}$$

Now observe that since $(x_{2^\mu,1}, \ldots, x_{2^\mu,d})$ is uniformly distributed the points

$$\{(x_{2^\mu+1,1}, \ldots, x_{2^\mu+1,d}), \ldots, (x_{M,1}, \ldots, x_{M,d})\}$$

are elements of a randomized Halton sequence denoted by $(q_n)_{n\geq 1}$. Therefore by Lemma 4.3.1 and another application of Lemma 4.2.2 we have

$$\begin{aligned} D^d_{2^\mu,M}(x_{n,i}) = D^d_{2^\mu,M}(q_{n,i}) &\leq \frac{2^\mu D^d_{2^\mu}(q_{n,i}) + MD^d_M(q_{n,i})}{M-2^\mu} \\ &\leq \frac{2^\mu\sqrt{d}/\sqrt{2^\mu} + M\sqrt{d}/\sqrt{M}}{M-2^\mu} \\ &< \frac{2\sqrt{d}\sqrt{2^{m+1}}}{M-2^\mu}. \end{aligned} \tag{4.4.5}$$

Together with (4.4.4) we get

$$\begin{aligned} \mathcal{F}_{m,d,\varepsilon}\backslash\mathcal{F}_{\mu-1,d,\varepsilon} \subset &\left\{\max_{M\in\{2^m+1,\ldots,2^{m+1}\}} MD^d_M(x_{n,i}) \geq (1821+252\log(\varepsilon^{-1}))\sqrt{d}\sqrt{2^{m+1}}\right\} \Big\backslash \\ &\left\{2^\mu D^d_{2^\mu}(x_{n,i}) \geq (1819+252\log(\varepsilon^{-1}))\sqrt{d}\sqrt{2^\mu}\right\} \\ \subset &\left\{\max_{M\in\{2^m+1,\ldots,2^{m+1}\}} (M-2^\mu)D^d_{2^\mu,M}(x_{n,i}) > 2\sqrt{d}\sqrt{2^{m+1}}\right\} \\ = &\emptyset. \end{aligned}$$

Therefore for $m \geq \mu$ we have $\mathbb{P}(\mathcal{F}_{m,d,\varepsilon}\backslash\mathcal{F}_{\mu-1,d,\varepsilon}) = 0$ and (4.4.3) follows immediately. Thus we may assume $d \geq 2$ and $m \in \{1,\ldots,12\cdot 2^d - 1\}$ now. Furthermore we may assume

$$\frac{\sqrt{d}}{\sqrt{2^{m+1}}} \leq \frac{1}{64} \tag{4.4.6}$$

since otherwise $\mathcal{F}_{m,d,\varepsilon} = \emptyset$. Let $\tilde{k}(m) = \max\{k \geq 1 : 12\cdot 2^k \leq m\}$ and for $m \geq 48$ set $L_m = 2^{12\cdot 2^{\tilde{k}(m)}}$ resp. for $m < 48$ set $L_m = 0$. Moreover we define the sets

$$\begin{aligned} \mathcal{G}_{m,d,\varepsilon} &= \begin{cases} \left\{L_m D^d_{L_m}(x_{n,i}) \geq (910+126\log(\varepsilon^{-1}))\sqrt{d}\sqrt{L_m}\right\}, & \text{if } L_m > 0, \\ \emptyset, & \text{if } L_m = 0, \end{cases} \\ \mathcal{H}_{m,d,\varepsilon} &= \left\{\max_{L_m+1\leq M\leq 2^{m+1}} (M-L_m)D^d_{L_m,M}(x_{n,i}) \geq (909+126\log(\varepsilon^{-1}))\sqrt{d}\sqrt{2^{m+1}}\right\}. \end{aligned}$$

Now we claim

$$\mathcal{F}_{m,d,\varepsilon} \subseteq \mathcal{G}_{m,d,\varepsilon} \cup \mathcal{H}_{m,d,\varepsilon} \tag{4.4.7}$$

for all $d \geq 2$ and $m \in \{1,\ldots,12\cdot 2^d - 1\}$. Since this trivially holds for $L_m = 0$ we may assume $L_m \geq 1$ and therefore we have $m \geq 48$ and $\tilde{k}(m) \geq 2$. By Lemma 4.2.2 for the complement of $\mathcal{G}_{m,d,\varepsilon} \cup \mathcal{H}_{m,d,\varepsilon}$ we observe

$$\begin{aligned} \max_{L_m+1\leq M\leq 2^{m+1}} MD^d_M(x_{n,i}) &\leq \max_{L_m+1\leq M\leq 2^{m+1}} (L_m D^d_{L_m}(x_{n,i}) + (M-L_m)D^d_{L_m,M}(x_{n,i})) \\ &\leq (910+126\log(\varepsilon^{-1}))\sqrt{d}\sqrt{L_m} \\ &\quad +(909+126\log(\varepsilon^{-1}))\sqrt{d}\sqrt{2^{m+1}} \\ &\leq (1819+252\log(\varepsilon^{-1}))\sqrt{d}\sqrt{2^{m+1}}. \end{aligned}$$

Thus we have (4.4.7). Now for any $d \geq 2$ we shall show

$$\bigcup_{m\in\{1,\dots,12\cdot 2^d-1\}} \mathcal{F}_{m,d,\varepsilon} \subseteq \bigcup_{m\in\{1,\dots,12\cdot 2^d-1\}} \mathcal{H}_{m,d,\varepsilon}. \tag{4.4.8}$$

For any $k \geq 2$ by definition of L_m we have

$$\mathcal{G}_{12\cdot 2^k,m,\varepsilon} = \mathcal{G}_{12\cdot 2^k+1,m,\varepsilon} = \dots = \mathcal{G}_{12\cdot 2^{k+1}-1,m,\varepsilon}.$$

Therefore by (4.4.7) we obtain

$$\bigcup_{m\in\{1,\dots,12\cdot 2^d-1\}} \mathcal{F}_{m,d,\varepsilon} \subseteq \bigcup_{k\in\{2,\dots,d-1\}} \mathcal{G}_{12\cdot 2^k,d,\varepsilon} \cup \bigcup_{m\in\{1,\dots,12\cdot 2^d\}} \mathcal{H}_{m,d,\varepsilon}. \tag{4.4.9}$$

For $k = 2$ we have $m = 12 \cdot 2^2 = 48$ and $L_m = 2^{48}$. With $L_{47} = 0$ we get

$$\begin{aligned}
\mathcal{G}_{48,d,\varepsilon} &\subseteq \left\{2^{48} D^d_{2^{48}}(x_{n,i}) \geq (910 + 126\log(\varepsilon^{-1}))\sqrt{d}\sqrt{2^{48}}\right\} \\
&\subseteq \left\{L_{47} + (2^{48} - L_{47}) D^d_{L_{47},2^{48}}(x_{n,i}) \geq (910 + 126\log(\varepsilon^{-1}))\sqrt{d}\sqrt{2^{48}}\right\} \\
&\subseteq \left\{(2^{48} - L_{47}) D^d_{L_{47},2^{48}}(x_{n,i}) \geq (909 + 126\log(\varepsilon^{-1}))\sqrt{d}\sqrt{2^{48}}\right\} \\
&\subseteq \mathcal{H}_{47,d,\varepsilon}
\end{aligned}$$

where the second line follows by Lemma 4.2.2. For $k \geq 3$ and $m = 12 \cdot 2^k$ we have $\tilde{k}(m) = k$ and $L_m = 2^{12\cdot 2^k}$. Moreover we obtain $\tilde{k}(m-1) = k-1$ and furthermore $L_{m-1} = 2^{12\cdot 2^{k-1}} = \sqrt{L_m}$. Thus we have

$$\begin{aligned}
\mathcal{G}_{m,d,\varepsilon} &\subseteq \left\{L_m D^d_{L_m}(x_{n,i}) \geq (910 + 126\log(\varepsilon^{-1}))\sqrt{d}\sqrt{L_m}\right\} \\
&\subseteq \left\{L_{m-1} + (L_m - L_{m-1}) D^d_{L_{m-1},L_m}(x_{n,i}) \geq (910 + 126\log(\varepsilon^{-1}))\sqrt{d}\sqrt{L_m}\right\} \\
&\subseteq \left\{(L_m - L_{m-1}) D^d_{L_{m-1},L_m}(x_{n,i}) \geq (909 + 126\log(\varepsilon^{-1}))\sqrt{d}\sqrt{L_m}\right\} \\
&\subseteq \mathcal{H}_{m-1,d,\varepsilon}.
\end{aligned}$$

Together with (4.4.9) we observe (4.4.8). Thus by (4.4.1), (4.4.2) and (4.4.3) the Theorem is proved if we show

$$\sum_{d\geq 2} \sum_{m\in\{1,\dots,12\cdot 2^d-1\}} \mathbb{P}(\mathcal{H}_{m,d,\varepsilon}) \leq \varepsilon. \tag{4.4.10}$$

Now we shall prove

$$\mathbb{P}(\mathcal{H}_{m,d,\varepsilon}) \leq \frac{\varepsilon}{6 \cdot 2^{2d}} \tag{4.4.11}$$

for all $d \geq 2$ and $m \in \{1, \dots, 12 \cdot 2^d - 1\}$. Then (4.4.10) follows by

$$\sum_{d\geq 2} \sum_{m\in\{1,\dots,12\cdot 2^d-1\}} \mathbb{P}(\mathcal{H}_{m,d,\varepsilon}) \leq \sum_{d\geq 2} 12 \cdot 2^d \cdot \frac{\varepsilon}{6} \cdot 2^{-2d} = \varepsilon.$$

To prove (4.4.11) let $d \geq 2$ and $m \in \{1, \ldots, 12 \cdot 2^d - 1\}$ be fixed now. To estimate $D^d_{L_m,M}(x_{n,i})$ we define a finite system of subsets of $[0,1)^d$ with the help of δ-bracketing covers such that $[0,x)$ for any $x \in [0,1)^d$ can be approximated well enough by a union of this sets. Set

$$H = \left\lceil \frac{m+1}{2} - \frac{\log_2(d)}{2} - 2 \right\rceil . \tag{4.4.12}$$

As a consequence for any $h \in \{0, \ldots, H\}$ we have

$$\sqrt{d}\sqrt{2^{m+1}} \leq 2^{m-h}. \tag{4.4.13}$$

For any $h \in \{1, \ldots, H\}$ let Δ_h be an 2^{-h}-bracketing cover of $[0,1)^d$. By Corollary 4.2.4 we may assume

$$|\Delta_h| \leq \frac{1}{2}(2e)^d (2^{h+2}+1)^d. \tag{4.4.14}$$

For any $x \in [0,1)^d$ we now define a finite sequence of points $\beta_h(x)$ for $h \in \{0, \ldots, H+1\}$ in the following manner. Let $(v,w) \in \Delta_H$ such that $v \leq x \leq w$. We set $\beta_{H+1}(x) = w$ and $\beta_H(x) = v$. The points $\beta_1(x), \ldots, \beta_{H-1}(x)$ are defined by induction. Thus assume that for some $h \in \{1, \ldots, H-1\}$ the point $\beta_{h+1}(x)$ is already defined. Let $(v,w) \in \Delta_h$ with $v \leq \beta_{h+1}(x) \leq w$ and set $\beta_h(x) = v$. Moreover set $\beta_0(x) = 0$. Therefore we observe

$$0 = \beta_0(x) \leq \beta_1(x) \leq \cdots \leq \beta_H(x) \leq x \leq \beta_{H+1}(x) \leq 1.$$

For $h \in \{0, \ldots, H-1\}$ we have $(\beta_h(x), w) \in \Delta_h$ for some point $w \in [0,1)^d$. Furthermore we have $(\beta_H(x), \beta_{H+1}(x)) \in \Delta_H$. Then by Corollary 4.2.4 for $h \in \{0, \ldots, H+1\}$ and $i \in \{1, \ldots, d\}$ there exist integers $a_{h,i} \in \{0, \ldots, 2^{(\lceil \log_2(i)+1 \rceil)(h+1)}\}$ such that

$$(\beta_h(x))_i = 2^{-(\lceil \log_2(i)+1 \rceil)(h+1)} a_{h,i}. \tag{4.4.15}$$

For $h \in \{0, \ldots, H\}$ set $K_h(x) = \overline{[\beta_h(x), \beta_{h+1}(x))}$. Note that the sets $K_h(x)$ are pairwise disjoint and satisfy

$$\bigcup_{h=0}^{H-1} K_h(x) \subseteq [0,x) \subseteq \bigcup_{h=0}^{H} K_h(x) \tag{4.4.16}$$

By definition $\beta_h(x) \leq \beta_{h+1}(x) \leq w$ for some $w \in [0,1)^d$ with $(\beta_h(x), w) \in \Delta_h$ and hence

$$\lambda(K_h(x)) \leq \lambda\left(\overline{[\beta_h(x), w)}\right) \leq 2^{-h} \tag{4.4.17}$$

for any $h \in \{0, \ldots, H\}$. Now define

$$S_h = \left\{ \overline{[\beta_h(x), \beta_{h+1}(x))} : x \in [0,1)^d \right\}.$$

Observe that we may define the points β_h such that $\beta_h(x) = \beta_h(y)$ for $x, y \in [0,1)^d$ with $\beta_{h+1}(x) = \beta_{h+1}(y)$. Therefore by Corollary 4.2.4 we have

$$|S_h| = \left| \left\{ \beta_{h+1}(x) : x \in [0,1)^d \right\} \right| \leq |\Delta_{h+1}| \leq \frac{1}{2}(2e)^d (\sqrt{5})^{(h+3)d} \tag{4.4.18}$$

for any integer $h \in \{0, \dots, H\}$. For $m \geq 48$ we set $s = \tilde{k}(m)$. Otherwise we set $s = 1$. Let now $n \in \{L_m + 1, \dots, 2^{m+1}\}$ be an integer. For $m < 48$ we have $s = 1$ and therefore we obtain $x_{n,i} = T_{p_i}(x_{n-1,i})$ for $i \leq s$ by definition while for $i > s$ we get

$$n \leq 2^{m+1} \leq 2^{48} \leq 2^{12 \cdot 2^i}.$$

Thus for $i \geq 2$ we have $x_{n,i} = \langle 2^{\lceil \log_2(i) \rceil + 1} x_{n-1,i} \rangle$. For $m \geq 48$ and $i \leq s = \tilde{k}(m)$ we get

$$n > L_m = 2^{12 \cdot 2^{\tilde{k}(m)}} \geq 2^{12 \cdot 2^i}$$

and thus we obtain $x_{n,i} = T_{p_i}(x_{n-1,i})$. Furthermore for $i > s = \tilde{k}(m)$ we observe

$$n \leq 2^{m+1} \leq 2^{12 \cdot 2^i}$$

and we obtain $x_{n,i} = \langle 2^{\lceil \log_2(i) \rceil + 1} x_{n-1,i} \rangle$. We see that in the sequence

$$\left\{ (x_{L_m,1}, \dots, x_{L_m,d}), \dots, (x_{2^{m+1},1}, \dots, x_{2^{m+1},d}) \right\}$$

the first s coordinates form a randomized Halton sequence while the sequence formed by the remaining coordinates is a sequence of fractional parts of the product of some initial value and elements of a lacunary sequence. Hence for any $M \in \{2^m + 1, \dots, 2^{m+1}\}$ by Lemma 4.2.2 and 4.3.1 we have

$$\begin{aligned} (M - L_m) D^s_{L_m,M}(x_{n,i}) &\leq L_m D^s_{L_m}(x_{n,i}) + M D^s_M(x_{n,i}) \\ &\leq \sqrt{s}\sqrt{L_m} + \sqrt{s}\sqrt{M} \leq 2\sqrt{s}\sqrt{M}. \end{aligned} \tag{4.4.19}$$

For some $h \in \{1, \dots, H+1\}$ and a point $y \in [0,1)^d$ the point $\beta_h(y)$ can be written as $(u_h(y), v_h(y))$ for $u_h(y) \in [0,1)^s$ and $v_h(y) \in [0,1)^{d-s}$. Moreover set $U_h(y) = [0, u_h(y))$ and $V_h(y) = [0, v_h(y))$. Thus we have $U_h(y) \times V_h(y) = [0, \beta_h(y))$. Observe that any set $K_h(y) \in S_h$ may be written as

$$\begin{aligned} K_h(y) &= \overline{[\beta_h(y), \beta_{h+1}(y))} \\ &= ((U_{h+1}(y) \backslash U_h(y)) \times V_{h+1}(y)) \cup (U_h(y) \times (V_{h+1}(y) \backslash V_h(y))). \end{aligned}$$

For $h = 0$ we simply have $K_0(y) = U_1(y) \times V_1(y)$. Furthermore set $U_0(y) = V_0(y) = \emptyset$. Thus by (4.4.17) we observe

$$\lambda(U_{h+1}(y) \backslash U_h(y)) \cdot \lambda(V_{h+1}(y)) + \lambda(U_h(y)) \cdot \lambda(V_{h+1}(y) \backslash V_h(y)) \leq \lambda(K_h(y)) \leq 2^{-h} \tag{4.4.20}$$

for $h \in \{0, \dots, H\}$. Now let $y \in [0,1)^d$ be some arbitrary fixed point. Note that hereafter we skip the point y in the notation of the points β_h and the sets K_h resp. U_h and V_h to simplify notations. Furthermore let $L_m + 1 \leq M \leq 2^{m+1}$ be an integer. For simplicity we write $q_n = (x_{n,1}, \ldots, x_{n,s})$ and $r_n = (x_{n,s+1}, \ldots, x_{n,d})$. Then by (4.4.16) we have

$$\begin{aligned}\sum_{n=L_m+1}^{M} \mathbf{1}_{[0,y)}(x_n) &\geq \sum_{n=L_m+1}^{M} \mathbf{1}_{\beta_H}(x_n) \\ &= \sum_{n=L_m+1}^{M} \mathbf{1}_{U_H}(q_n) \cdot \mathbf{1}_{V_H}(r_n) \\ &= \sum_{n=L_m+1}^{M} \mathbf{1}_{U_1}(q_n) \cdot \mathbf{1}_{V_1}(r_n) \\ &\quad + \sum_{h=1}^{H-1} \sum_{n=L_m+1}^{M} \left(\mathbf{1}_{U_{h+1}\setminus U_h}(q_n) \cdot \mathbf{1}_{V_{h+1}}(r_n) + \mathbf{1}_{U_h}(q_n) \cdot \mathbf{1}_{V_{h+1}\setminus V_h}(r_n)\right). \end{aligned} \tag{4.4.21}$$

Analogously we also get

$$\begin{aligned}\sum_{n=L_m+1}^{M} \mathbf{1}_{[0,y)}(x_n) &\leq \sum_{n=L_m+1}^{M} \mathbf{1}_{U_1}(q_n) \cdot \mathbf{1}_{V_1}(r_n) \\ &\quad + \sum_{h=1}^{H} \sum_{n=L_m+1}^{M} \left(\mathbf{1}_{U_{h+1}\setminus U_h}(q_n) \cdot \mathbf{1}_{V_{h+1}}(r_n) + \mathbf{1}_{U_h}(q_n) \cdot \mathbf{1}_{V_{h+1}\setminus V_h}(r_n)\right). \end{aligned} \tag{4.4.22}$$

By using maximal Bernstein inequality we now shall give a lower bound on the probability that the system of inequalities

$$\max_{L_m+1\leq M\leq 2^{m+1}} \left| \sum_{h=L_m+1}^{M} \mathbf{1}_{U_{h+1}\setminus U_h}(q_n)\mathbf{1}_{V_{h+1}}(r_n) - \mathbf{1}_{U_{h+1}\setminus U_h}(q_n)\lambda(V_{h+1}) \right| > t, \tag{4.4.23}$$

$$\max_{L_m+1\leq M\leq 2^{m+1}} \left| \sum_{h=L_m+1}^{M} \mathbf{1}_{U_h}(q_n)\mathbf{1}_{V_{h+1}\setminus V_h}(r_n) - \mathbf{1}_{U_h}(q_n)\lambda(V_{h+1}\setminus V_h) \right| > t, \tag{4.4.24}$$

$$\max_{L_m+1\leq M\leq 2^{m+1}} \left| \sum_{h=L_m+1}^{M} \mathbf{1}_{U_1}(q_n)\mathbf{1}_{V_1}(r_n) - \mathbf{1}_{U_1}(q_n)\lambda(V_1) \right| > t \tag{4.4.25}$$

holds for all sets U_h, U_{h+1}, V_h and V_{h+1} with $h \in \{1, \ldots, H\}$ and some $t > 0$ to specified later. Set $\kappa = \kappa_h = \lceil \log_2(h+2) \rceil$. By Lemma 4.3.1 and (4.4.19) for any $h \in \{1, \ldots, H\}$ we have

$$\begin{aligned}\sum_{n=L_m+1}^{2^{m+1}} \mathbf{1}_{U_{h+1}\setminus U_h}(q_n) &\leq \sum_{\substack{M\in\{L_m+1,\ldots,2^{m+1}\},\\ M\equiv\gamma \pmod{2^\kappa}}} \lambda(U_{h+1}\setminus U_h) + \sqrt{s} \cdot \sqrt{\frac{M-L_m}{2^\kappa} + 1} \\ &\leq \left(2^{m+1-\kappa} + 1\right) \lambda(U_{h+1}\setminus U_h) + \sqrt{s} \cdot \sqrt{2^{m+1-\kappa} + 1} \end{aligned} \tag{4.4.26}$$

and

$$\sum_{n=L_m+1}^{2^{m+1}} \mathbf{1}_{U_h}(q_n) \le \left(2^{m+1-\kappa}+1\right)\lambda(U_{h+1}\backslash U_h) + \sqrt{s}\cdot\sqrt{2^{m+1-\kappa}+1}. \tag{4.4.27}$$

Now let $h \in \{0,\dots,H\}$ be fixed and set $\mathcal{A}_h = V_{h+1}\backslash V_h$ resp. $\mathcal{A}_h = V_{h+1}$. Furthermore define by $f_{\mathcal{A}_h}(x) = f_{\mathcal{A}_h}(r_n) = \mathbf{1}_{\mathcal{A}_h}(x) - \lambda(\mathcal{A}_h)$ a real-valued function on $[0,1)^{d-s}$. We now shall show that for any system of indices $n_1,\dots,n_k$ with $n_{l+1}-n_l \ge h+2$ for all $l \in \{1,\dots,k-1\}$ the random variables $f_{\mathcal{A}_h}(r_n)$ are stochastically independent, i.e.

$$\mathbb{P}\left(f_{\mathcal{A}_h}(r_{n_1}) = c_1,\dots,f_{\mathcal{A}_h}(r_{n_k}) = c_k\right) = \prod_{l=1}^{k} \mathbb{P}\left(f_{\mathcal{A}_h}(r_{n_l}) = c_l\right). \tag{4.4.28}$$

We only prove the case $k=2$. The general case follows by induction. By (4.4.15) the set $\mathcal{A}_h$ is a union of axis-parallel boxes such that each corner of any box is of the form

$$\left(2^{-(\lceil\log_2(s+1)\rceil+1)(h+2)}a_{s+1},\dots,2^{-(\lceil\log_2(d)\rceil+1)(h+2)}a_d\right) \tag{4.4.29}$$

such that $a_i \in \{0,1,\dots,2^{(\lceil\log_2(d)\rceil+1)(h+2)}\}$ for any $i \in \{s+1,\dots,d\}$. Furthermore let $n, n' \in \{L_m+1,\dots,M\}$ be two indices with $n'-n \ge h+2$. We define a decomposition of $[0,1)^{d-s}$ by

$$\Sigma = \left\{ \prod_{i=s+1}^{d} \left[2^{-(\lceil\log_2(i)\rceil+1)n'}a_i, 2^{-(\lceil\log_2(i)\rceil+1)n'}(a_i+1)\right) : \right.$$
$$\left. a_i \in \left\{0,1,\dots,2^{(\lceil\log_2(i)\rceil+1)n'}-1\right\}, i \in \{s+1,\dots,d\}\right\}.$$

Note that by (4.4.29) the function $f_{\mathcal{A}_h}$ is constant on any box $\mathcal{B} \in \Sigma$. For some $c_1 \in \mathbb{R}$ define

$$\Sigma_{c_1} = \{\mathcal{B} \in \Sigma : f_{\mathcal{A}_h}(r_n) = c_1 \text{ for all } r_1 = (r_{1,s+1},\dots,r_{1,d}) \in \mathcal{B}\}.$$

Since $x_{n',i} = 2^{(\lceil\log_2(i)\rceil+1)(n'-1)}x_{1,i}$ for all $i \in \{s+1,\dots,d\}$ we have $f_{\mathcal{A}_h}(r_{n'}) = f_{\mathcal{A}_h}(r'_{n'})$ where $r'_{n'} = (x'_{n',s+1},\dots,x'_{n',d})$ with $x'_{n,i} = 2^{(\lceil\log_2(i)\rceil+1)(n'-1)}x'_{1,i}$ is an instance of the matrix for some initial value $r'_1 = (x'_{1,s+1},\dots,x'_{1,d})$ with $x'_{1,i} = x_{1,i}+2^{-(\lceil\log_2(i)\rceil+1)(n'-1)}a_i$ and $a_i \in \{0,1,\dots,2^{(\lceil\log_2(i)\rceil+1)(n'-1)}-1\}$ for all $i \in \{s+1,\dots,d\}$. Therefore for any $c_2 \in \mathbb{R}$ and any $\mathcal{B},\mathcal{B}' \in \Sigma$ we have

$$\mathbb{P}\left(f_{\mathcal{A}_h}(r_{n'}) = c_2 | r_1 \in \mathcal{B}\right) = \mathbb{P}\left(f_{\mathcal{A}_h}(r_{n'}) = c_2 | r_1 \in \mathcal{B}'\right).$$

Hence for any $c_2 \in \mathbb{R}$ and any $\mathcal{B} \in \Sigma$ we get

$$\begin{aligned}
\mathbb{P}\left(f_{\mathcal{A}_h}(r_{n'}) = c_2\right) &= \sum_{\mathcal{B}' \in \Sigma} \mathbb{P}\left(f_{\mathcal{A}_h}(r_{n'}) = c_2 | r_1 \in \mathcal{B}'\right)\mathbb{P}(r_1 \in \mathcal{B}') \\
&= \mathbb{P}\left(f_{\mathcal{A}_h}(r_{n'}) = c_2 | r_1 \in \mathcal{B}\right)\sum_{\mathcal{B}' \in \Sigma}\mathbb{P}(r_1 \in \mathcal{B}') \\
&= \mathbb{P}\left(f_{\mathcal{A}_h}(r_{n'}) = c_2 | r_1 \in \mathcal{B}\right).
\end{aligned}$$

Moreover for any $c_1, c_2 \in \mathbb{R}$ we obtain

$$\begin{aligned}
\mathbb{P}\left(f_{\mathcal{A}_h}(r_{n'}) = c_2 | f_{\mathcal{A}_h}(r_n) = c_1\right) \\
&= \frac{\mathbb{P}\left(f_{\mathcal{A}_h}(r_{n'}) = c_2, f_{\mathcal{A}_h}(r_n) = c_1\right)}{\mathbb{P}\left(f_{\mathcal{A}_h}(r_n) = c_1\right)} \\
&= \frac{\sum_{\mathcal{B}\in\Sigma} \mathbb{P}\left(f_{\mathcal{A}_h}(r_{n'}) = c_2, f_{\mathcal{A}_h}(r_n) = c_1 | r_1 \in \mathcal{B}\right) \mathbb{P}(r_1 \in \mathcal{B})}{\mathbb{P}\left(f_{\mathcal{A}_h}(r_n) = c_1\right)} \\
&= \sum_{\mathcal{B}\in\Sigma_{c_1}} \mathbb{P}\left(f_{\mathcal{A}_h}(r_{n'}) = c_2 | r_1 \in \mathcal{B}\right) \frac{\mathbb{P}(r_1 \in \mathcal{B})}{\mathbb{P}\left(f_{\mathcal{A}_h}(r_n) = c_1\right)} \\
&= \mathbb{P}\left(f_{\mathcal{A}_h}(r_{n'}) = c_2\right).
\end{aligned}$$

Thus (4.4.28) is proved. Furthermore set

$$\begin{aligned}
Q(L_m, M, \gamma) &= \{n \in \{L_m + 1, \ldots, M\} : q_n \in U_{h+1} \backslash U_h, n \equiv \gamma \pmod{2^\kappa}\}, \\
Q'(L_m, M, \gamma) &= \{n \in \{L_m + 1, \ldots, M\} : q_n \in U_h, n \equiv \gamma \pmod{2^\kappa}\}.
\end{aligned}$$

Then for $h \in \{1, \ldots, H\}$ by Lemma 4.2.1 we have

$$\begin{aligned}
\mathbb{P}\left(\max_{M\in\{L_m+1,\ldots,N\}} \left| \sum_{n=L_m+1}^{N} \mathbf{1}_{U_{h+1}\backslash U_h}(q_n) \cdot \mathbf{1}_{V_{h+1}}(r_n) - \mathbf{1}_{U_{h+1}\backslash U_h}(q_n)\lambda(V_{h+1})\right| > t\right) \\
&\leq \sum_{\gamma=1}^{2^\kappa} \mathbb{P}\left(\max_{n\in\{L_m+1,\ldots,M\}} \left| \sum_{n\in Q(L_m,M,\gamma)} \mathbf{1}_{V_{h+1}}(r_n) - \lambda(V_{h+1})\right| > \frac{t}{2^\kappa}\right) \\
&\leq 2\sum_{\gamma=1}^{2^\kappa} \exp\left(-\frac{t^2/2^{2\kappa}}{2\left(\sum_{n\in Q(L_m,M,\gamma)} 1\right)\lambda(V_{h+1})(1-\lambda(V_{h+1})) + 2t/(3\cdot 2^\kappa)}\right).
\end{aligned}$$

Thus by (4.4.26) we obtain

$$\begin{aligned}
\mathbb{P}\left(\max_{M\in\{L_m+1,\ldots,N\}} \left| \sum_{n=L_m+1}^{N} \mathbf{1}_{U_{h+1}\backslash U_h}(q_n) \cdot \mathbf{1}_{V_{h+1}}(r_n) - \mathbf{1}_{U_{h+1}\backslash U_h}(q_n)\lambda(V_{h+1})\right| > t\right) \\
\leq 2^{\kappa+1} \exp\left(-\frac{t^2/2^{1.5\kappa}}{2^{m+3}\lambda(U_{h+1}\backslash U_h)\lambda(V_{h+1}) + 2\sqrt{2}\cdot\sqrt{s}\cdot\sqrt{2^{m+1}}\lambda(V_{h+1}) + 2t/3}\right).
\end{aligned}$$

Furthermore (4.4.13) and (4.4.20) yield

$$\begin{aligned}
\mathbb{P}\left(\max_{M\in\{L_m+1,\ldots,N\}} \left| \sum_{n=L_m+1}^{N} \mathbf{1}_{U_{h+1}\backslash U_h}(q_n) \cdot \mathbf{1}_{V_{h+1}}(r_n) - \mathbf{1}_{U_{h+1}\backslash U_h}(q_n)\lambda(V_{h+1})\right| > t\right) \\
\leq 2^{\kappa+1} \exp\left(-\frac{t^2/2^{1.5\kappa}}{(8+2\sqrt{2})\cdot 2^{m-h} + 2t/3}\right). \quad (4.4.30)
\end{aligned}$$

Similarly using (4.4.27) we get

$$\mathbb{P}\left(\max_{M\in\{L_m+1,\dots,N\}}\left|\sum_{n=L_m+1}^{N}\mathbf{1}_{U_h}(q_n)\cdot\mathbf{1}_{V_{h+1}\setminus V_h}(r_n)-\mathbf{1}_{U_h}(q_n)\lambda(V_{h+1}\setminus V_h)\right|>t\right)$$
$$\leq 2^{\kappa+1}\exp\left(-\frac{t^2/2^{1.5\kappa}}{(8+2\sqrt{2})\cdot 2^{m-h}+2t/3}\right). \qquad (4.4.31)$$

Now set $t=C_1\sqrt{d}\sqrt{2^{m+1}}\sqrt{h\cdot 2^{1.5\kappa-h}}$ for some constant $C_1>0$ to specified later. Observe that by (4.4.13) we have $t\leq 2^{m-h+1}C_1$.
Therefore by (4.4.30) we get

$$\mathbb{P}\left(\max_{M\in\{L_m+1,\dots,N\}}\left|\sum_{n=L_m+1}^{N}\mathbf{1}_{U_{h+1}\setminus U_h}(q_n)\cdot\mathbf{1}_{V_{h+1}}(r_n)-\mathbf{1}_{U_{h+1}\setminus U_h}(q_n)\lambda(V_{h+1})\right|>t\right)$$
$$\begin{aligned}&\leq 4(h+2)\exp\left(-\frac{\left(C_1\sqrt{d}\sqrt{2^{m+1}}\sqrt{h\cdot 2^{1.5\kappa-h}}\right)^2}{2^{1.5\kappa}(8+2\sqrt{2}+2C_1)2^{m-h}}\right)\\&\leq 4\exp\left(-\left(\frac{2C_1^2}{8+2\sqrt{2}+2C_1}-1\right)hd\right)\end{aligned} \qquad (4.4.32)$$

where the last line follows by $(h+2)\leq e^{hd}$ for $d\geq 2$. Similarly using (4.4.31) we have

$$\mathbb{P}\left(\max_{M\in\{L_m+1,\dots,N\}}\left|\sum_{n=L_m+1}^{N}\mathbf{1}_{U_h}(q_n)\cdot\mathbf{1}_{V_{h+1}\setminus V_h}(r_n)-\mathbf{1}_{U_h}(q_n)\lambda(V_{h+1}\setminus V_h)\right|>t\right)$$
$$\leq 4\exp\left(-\left(\frac{2C_1^2}{8+2\sqrt{2}+2C_1}-1\right)hd\right). \qquad (4.4.33)$$

For $h=0$ set $t=C_2\sqrt{d}\sqrt{2^{m+1}}$ for some constant $C_2>0$ to be specified later. Thus by using a similar argumentation as above we get

$$\mathbb{P}\left(\max_{M\in\{L_m+1,\dots,N\}}\left|\sum_{n=L_m+1}^{M}\mathbf{1}_{U_1}(q_n)\mathbf{1}_{V_1}(r_n)-\mathbf{1}_{U_1}(q_n)\lambda(V_1)\right|>t\right)$$
$$\leq 4\exp\left(-\frac{t^2/2}{(8+2\sqrt{2})\cdot 2^m+2t/3}\right)\leq 4\exp\left(-\frac{C_2^2}{8+2\sqrt{2}+2/3\cdot C_2}d\right). \qquad (4.4.34)$$

Define

$$C_3=\frac{2C_1^2}{8+2\sqrt{2}+2C_1}-1,\quad C_4=\frac{C_2^2}{8+2\sqrt{2}+2/3\cdot C_2}. \qquad (4.4.35)$$

Observe that by (4.4.18) and sufficiently large constants C_1, C_2 resp. C_3, C_4 the system of inequalities (4.4.23), (4.4.24) and (4.4.25) hold on a set of measure which is bounded from below by

$$1 - \frac{1}{2}(2e)^d(\sqrt{5})^{3d} \cdot 4e^{-C_4 d} - (2e)^d \sum_{h=1}^{H} (\sqrt{5})^{(h+3)d} \cdot 4e^{-C_3 d} \geq 1 - \frac{\varepsilon}{6 \cdot 2^{2d}}. \qquad (4.4.36)$$

Now we shall find some constants C_1 and C_2 such that (4.4.36) is true. It is easy to see that for $C_3 \geq 2.7$ we have

$$4(50e)^d \sum_{h=1}^{H} (\sqrt{5})^{(h-1)d} e^{-C_3 hd} = 4(50e)^d e^{-C_3 d} \sum_{h=1}^{H} (\sqrt{5} e^{-C_3})^{(h-1)d} \leq 4.1(50e)^d e^{-C_3 d}.$$

Therefore we can estimate the left-hand side of (4.4.36) by

$$\begin{aligned} 1 - \frac{1}{2}(2e)^d(\sqrt{5})^{3d} \cdot 4e^{-C_4 d} - (2e)^d \sum_{h=1}^{H} (\sqrt{5})^{(h+3)d} \cdot 4e^{-C_3 d} \\ \geq 1 - \left(2/(\sqrt{5})^d + 4.1\right)(50e)^d e^{-\min(C_3, C_4) d} \\ \geq 1 - 4.5 \cdot e^{(1+\log(50) - \min(C_3, C_4))d}. \end{aligned}$$

Thus (4.4.36) holds if

$$-\log(4.5) + (\min(C_3, C_4) - 1 - \log(50))\, d \geq \log(\varepsilon^{-1}) + \log(6) + 2d.$$

By $d \geq 2$ it can easily be shown that (4.4.36) is true for

$$\min(C_3, C_4) \geq 7.947 + \frac{\log(\varepsilon^{-1})}{2}. \qquad (4.4.37)$$

By (4.4.35) this holds for

$$C_2 \geq \frac{15.894 + \log(\varepsilon^{-1})}{6} + \sqrt{86.054 + (4 + \sqrt{2}) \log(\varepsilon^{-1}) + \left(\frac{15.894 + \log(\varepsilon^{-1})}{6}\right)^2}$$

and because of $\sqrt{A+B} \leq \sqrt{A} + \sqrt{B}$ we may choose

$$C_2 = 14.575 + 5.748 \log(\varepsilon^{-1}). \qquad (4.4.38)$$

Similarly (4.4.37) holds for

$$C_1 \geq \frac{17.894 + \log(\varepsilon^{-1})}{4} + \sqrt{48.441 + 2.708 \log(\varepsilon^{-1}) + \left(\frac{17.894 + \log(\varepsilon^{-1})}{4}\right)^2}.$$

Thus we may take

$$C_1 = 15.907 + 2.146 \log(\varepsilon^{-1}). \qquad (4.4.39)$$

Therefore for any $M \in \{L_m + 1, \dots, 2^{m+1}\}$ by using (4.4.22), (4.4.32), (4.4.33), (4.4.34), (4.4.35), (4.4.36), (4.4.38) and (4.4.39) we get

$$\begin{aligned}
&\sum_{n=L_m+1}^{M} \mathbf{1}_{[0,y)}(x_n) \\
&\qquad \leq \sum_{n=L_m+1}^{M} \mathbf{1}_{U_1}(q_n) \cdot \mathbf{1}_{V_1}(r_n) \\
&\qquad\quad + \sum_{h=1}^{H} \sum_{n=L_m+1}^{M} \left(\mathbf{1}_{U_{h+1}\setminus U_h}(q_n) \cdot \mathbf{1}_{V_{h+1}}(r_n) + \mathbf{1}_{U_h}(q_n) \cdot \mathbf{1}_{V_{h+1}\setminus V_h}(r_n)\right) \\
&\qquad \leq \sum_{n=L_m+1}^{M} \lambda(U_{H+1})\lambda(V_{H+1}) + \sum_{n=L_m+1}^{M} \lambda(V_{H+1}) \left(\mathbf{1}_{U_{H+1}}(q_n) - \lambda(U_{H+1})\right) \\
&\qquad\quad + 2(15.907 + 2.146 \log(\varepsilon^{-1}))\sqrt{d}\sqrt{2^{m+1}} \sum_{h=1}^{H} \sqrt{h \cdot 2^{1.5(1+\log_2(h+2))-h}} \\
&\qquad\quad + (14.575 + 5.748 \cdot \log(\varepsilon^{-1}))\sqrt{d}\sqrt{2^{m+1}}
\end{aligned}$$

with probability at least $1 - \varepsilon/6 \cdot 2^{-2d}$. Thus with

$$\sum_{h=1}^{H} \sqrt{h \cdot 2^{1.5(1+\log_2(h+2))-h}} \leq 27.917$$

we obtain

$$\begin{aligned}
&\sum_{n=L_m+1}^{M} \mathbf{1}_{[0,y)}(x_n) \\
&\qquad \leq \sum_{n=L_m+1}^{M} \lambda(U_{H+1})\lambda(V_{H+1}) + \sum_{n=L_m+1}^{M} \lambda(V_{H+1}) \left(\mathbf{1}_{U_{H+1}}(q_n) - \lambda(U_{H+1})\right) \\
&\qquad\quad + (902.726 + 125.568 \log(\varepsilon^{-1}))\sqrt{d}\sqrt{2^{m+1}}.
\end{aligned}$$

By (4.4.12) and (4.4.19) we have

$$\begin{aligned}
&\sum_{n=L_m+1}^{M} \mathbf{1}_{[0,y)}(x_n) \\
&\leq (M - L_m)\lambda([0, \beta_{H+1}(x)) + 2\sqrt{d}\sqrt{2^{m+1}} + (902.726 + 125.568 \log(\varepsilon^{-1}))\sqrt{d}\sqrt{2^{m+1}} \\
&\leq (M - L_m)(\lambda([0, y)) + 2^{-H}) + (904.726 + 125.568 \log(\varepsilon^{-1}))\sqrt{d}\sqrt{2^{m+1}} \\
&\leq (M - L_m)\lambda([0, y)) + (908.726 + 125.568 \log(\varepsilon^{-1}))\sqrt{d}\sqrt{2^{m+1}}
\end{aligned} \tag{4.4.40}$$

on a set with probability at least $1-\varepsilon/6\cdot 2^{-2d}$. Similarly by using (4.4.21) instead of (4.4.22) we obtain

$$\begin{aligned}\sum_{n=L_m+1}^{M}&\mathbf{1}_{[0,y)}(x_n)\\ &\geq \sum_{n=L_m+1}^{M}\mathbf{1}_{U_1}(q_n)\cdot\mathbf{1}_{V_1}(r_n)\\ &\quad+\sum_{h=1}^{H-1}\sum_{n=L_m+1}^{M}\left(\mathbf{1}_{U_{h+1}\setminus U_h}(q_n)\cdot\mathbf{1}_{V_{h+1}}(r_n)+\mathbf{1}_{U_h}(q_n)\cdot\mathbf{1}_{V_{h+1}\setminus V_h}(r_n)\right)\\ &\geq (M-L_m)\lambda([0,y))-(908.726+125.568\log(\varepsilon^{-1}))\sqrt{d}\sqrt{2^{m+1}}\end{aligned}\tag{4.4.41}$$

on the same set of probability bounded from below by $1-\varepsilon/6\cdot 2^{-2d}$. Therefore we have proved (4.4.11) which finally concludes the proof of the Theorem.

5 Random Matrix ensembles with correlated entries

In this chapter we discuss a random matrix ensemble where the entries are taken from a multivariate lacunary system and show that the mean spectral distribution of this ensemble converges weakly to the semicircle law. In section 5.1 we present the main result which is proved in section 5.2. Furthermore in section 5.3 we give examples.

5.1 Main result

Theorem 5.1.1 *Let $(M_{n,1})_{n\geq 1}$ and $(M_{n,2})_{n\geq 0}$ be two integer-valued lacunary sequences satisfying the Hadamard gap condition (1.0.4) such that there exist $C>0$ and $\varepsilon>0$ with*

$$\sum_{\substack{n,n'\in\{1,\dots,2N\},\\ n>n'}}\ \sum_{j,j'\in\{1,\dots,N^K\}} (jj')^{-1}\mathbf{1}_{|jM_{n,1}-j'M_{n',1}|<1/2\cdot q^{-CN^{1-\varepsilon}}M_{n',1}} = o(N) \tag{5.1.1}$$

for any $K\in\mathbb{N}$. Furthermore let $f:\mathbb{R}^2\to\mathbb{R}$ be some function of finite total variation in the sense of Hardy and Krause satisfying

$$f(x+z)=f(x) \text{ for all } x\in[0,1)^2, z\in\mathbb{Z}^2, \quad \int_{[0,1)^2} f(x)\,dx=0, \quad \int_{[0,1)^2} f^2(x)\,dx=1. \tag{5.1.2}$$

Additionally assume that f satisfies

$$\int_{[0,1)} f(x_1,y_2)\,dx_1 = \int_{[0,1)} f(y_1,x_2)\,dx_2 = 0 \tag{5.1.3}$$

for any fixed $y_1,y_2\in[0,1)$. Let the Fourier series of f exist and converge to f. Now define a symmetric random matrix ensemble $(X_N)_{N\geq 1}$ by setting $X_N=(X_{n,n'})_{1\leq n,n'\leq N}$ with

$$X_{n,n'} = \frac{1}{\sqrt{N}} f(M_{n+n',1}x_1, M_{|n-n'|,2}x_2) \tag{5.1.4}$$

for any $n,n'\in\{1,\dots,N\}$ and $N\in\mathbb{N}$. Then the mean empirical distribution of the eigenvalues converges almost surely weakly to the semicircle law, i.e.

$$\lim_{N\to\infty}\mathbb{E}\left[\frac{1}{N}\operatorname{Tr}\left(X_N^K\right)\right] = \begin{cases}\frac{K!}{(K/2)!(K/2+1)!}, & K \text{ even},\\ 0, & K \text{ odd}.\end{cases} \tag{5.1.5}$$

5.2 Proof of Theorem 5.1.1

For $K, N \in \mathbb{N}$ let $p = p_{N^K}$ be the N^Kth Fejér mean of f as defined in (2.0.2) and set $r = r_{N^K} = f - p$. By Lemma 2.0.2 we have $||r||_2^2 \leq CN^{-K}$ for some absolute constant $C > 0$. We define the random matrix $(\tilde{X}_{n,n'})_{1\leq n,n'\leq N}$ by

$$\tilde{X}_{n,n'} = \frac{1}{\sqrt{N}} p(M_{n+n',1}x_1, M_{|n-n'|,2}x_2) \tag{5.2.1}$$

for any $n, n' \in \{1, \dots, N\}$. Now we claim

$$\lim_{N\to\infty} \mathbb{E}\left[\frac{1}{N}\operatorname{Tr}\left(X_N^K\right)\right] = \lim_{N\to\infty} \mathbb{E}\left[\frac{1}{N}\operatorname{Tr}\left(\tilde{X}_N^K\right)\right]. \tag{5.2.2}$$

It is easy to see that

$$\begin{aligned}
\mathbb{E}\left[\frac{1}{N}\operatorname{Tr}\left(X_N^K\right)\right] &= \mathbb{E}\left[\frac{1}{N}\sum_{n_1,\dots,n_K=1}^{N}\prod_{k=1}^{K} X_{n_k,n_{k+1}}\right] \\
&= \mathbb{E}\left[\frac{1}{N^{K/2+1}}\sum_{n_1,\dots,n_K=1}^{N}\prod_{k=1}^{K} f(M_{n_k+n_{k+1},1}x_1, M_{|n_k-n_{k+1}|,2}x_2)\right]
\end{aligned}$$

where $n_1 = n_{K+1}$. By decomposing $f = p + r$ we observe

$$\begin{aligned}
&\mathbb{E}\left[\frac{1}{N}\operatorname{Tr}\left(X_N^K\right)\right] \\
&=\frac{1}{N^{K/2+1}}\sum_{n_1,\dots,n_K=1}^{N}\mathbb{E}\left[\prod_{k=1}^{K} p(M_{n_k+n_{k+1},1}x_1, M_{|n_k-n_{k+1}|,2}x_2)\right] \\
&\quad+\frac{1}{N^{K/2+1}}\sum_{n_1,\dots,n_K=1}^{N}\sum_{\substack{J\subseteq\{1,\dots,K\},\\ J\neq\emptyset}}\mathbb{E}\left[\prod_{k\in J} r(M_{n_k+n_{k+1},1}x_1, M_{|n_k-n_{k+1}|,2}x_2)\right. \\
&\qquad\qquad\left.\cdot\prod_{k\notin J} p(M_{n_k+n_{k+1},1}x_1, M_{|n_k-n_{k+1}|,2}x_2)\right].
\end{aligned} \tag{5.2.3}$$

Since any set J is nonempty there exists some $k_J \in J$ for each J. Thus by Cauchy-Schwarz inequality we get

$$\begin{aligned}
&\left|\mathbb{E}\left[\prod_{k\in J} r(M_{n_k+n_{k+1},1}x_1, M_{|n_k-n_{k+1}|,2}x_2)\prod_{k\notin J} p(M_{n_k+n_{k+1},1}x_1, M_{|n_k-n_{k+1}|,2}x_2)\right]\right| \\
\leq&\mathbb{E}\left[r^2(M_{n_{k_J}+n_{k_J+1},1}x_1, M_{|n_{k_J}-n_{k_J+1}|,2}x_2)\right]^{1/2} \\
&\cdot\mathbb{E}\left[\prod_{k\in J\setminus\{k_J\}} r^2(M_{n_k+n_{k+1},1}x_1, M_{|n_k-n_{k+1}|,2}x_2)\prod_{k\notin J} p^2(M_{n_k+n_{k+1},1}x_1, M_{|n_k-n_{k+1}|,2}x_2)\right]^{1/2}
\end{aligned}$$

and therefore we obtain

$$\left|\mathbb{E}\left[\prod_{k\in J} r(M_{n_k+n_{k+1},1}x_1, M_{|n_k-n_{k+1}|,2}x_2)\prod_{k\notin J} p(M_{n_k+n_{k+1},1}x_1, M_{|n_k-n_{k+1}|,2}x_2)\right]\right|$$

$$\leq CN^{-K/2}\left(\prod_{k\in J\setminus\{k_J\}}||r||_\infty^2\right)^{1/2}\left(\prod_{k\notin J}||p||_\infty^2\right)^{1/2}$$

$$\leq CN^{-K/2}||f||_\infty^{K-1}$$

for some constant $C>0$ which only depends on K. Plugging this into (5.2.3) we have

$$\begin{aligned}\mathbb{E}\left[\frac{1}{N}\operatorname{Tr}\left(X_N^K\right)\right] &= \frac{1}{N^{K/2+1}}\sum_{n_1,\dots,n_K=1}^{N}\mathbb{E}\left[\prod_{k=1}^{K} p(M_{n_k+n_{k+1},1}x_1, M_{|n_k-n_{k+1}|,2}x_2)\right]\\ &+\frac{1}{N^{K/2+1}}\sum_{n_1,\dots,n_K=1}^{N}\sum_{\substack{J\subseteq\{1,\dots,K\},\\ J\neq\emptyset}} CN^{-K/2}||f||_\infty^{K-1}.\end{aligned}$$

Hence (5.2.2) follows immediately. Thus it is enough to study the asymptotic behaviour of $\mathbb{E}[1/N\cdot\operatorname{Tr}(\tilde{X}_N^K)]$. We have

$$f(x)=\sum_{j\in(\mathbb{Z}\setminus\{0\})^2} a_j\cos(2\pi\langle j,x\rangle)+b_j\sin(2\pi\langle j,x\rangle) \tag{5.2.4}$$

for suitable numbers a_j, b_j where without loss of generality we may assume $b_j=0$ for all $j\in(\mathbb{Z}\setminus\{0\})^2$. The general case is similar. Hence by definition we obtain

$$\begin{aligned}&\mathbb{E}\left[\frac{1}{N}\operatorname{Tr}\left(\tilde{X}_N^K\right)\right]\\ &=\frac{1}{N^{K/2+1}}\sum_{n_1,\dots,n_K\in\{1,\dots,N\}}\mathbb{E}\left[\prod_{k=1}^{K} p(M_{n_k+n_{k+1},1}x_1, M_{|n_k-n_{k+1}|,2}x_2)\right]\\ &=\sum_{\delta_1,\dots,\delta_K\in\{0,1\}^K}\ \sum_{\substack{j_1,\dots,j_K\in(\mathbb{Z}\setminus\{0\})^2,\\ ||j_k||_\infty\leq N^K\forall k}}\ \sum_{n_1,\dots,n_K\in\{1,\dots,N\}}\frac{1}{2^K}\frac{1}{N^{K/2+1}}\prod_{k=1}^{K}a'_{j_k}\\ &\cdot\mathbb{E}\left[\cos\left(2\pi\left(\sum_{k=1}^{K}(-1)^{\delta_k}j_{k,1}M_{n_k+n_{k+1},1}x_1+\sum_{k=1}^{K}(-1)^{\delta_k}j_{k,2}M_{|n_k-n_{k+1}|,2}x_2\right)\right)\right].\end{aligned} \tag{5.2.5}$$

Now we decompose the system of solutions for $\sum_{k=1}^{K}(-1)^{\delta_k}j_{k,1}M_{n_k+n_{k+1},1}=0$ and $\sum_{k=1}^{K}(-1)^{\delta_k}j_{k,2}M_{|n_k-n_{k+1}|,2}=0$ into different sets. Therefore we rearrange the sequence $(M_{n_k+n_{k+1},1})_{k\in\{1,\dots,K\}}$ resp. $(M_{|n_k-n_{k+1}|,2})_{k\in\{1,\dots,K\}}$ in decreasing order. Let π_1 and π_2

be permutations of $\{1,\ldots,K\}$ such that we have $n_{\pi_1(k)}+n_{\pi_1(k)+1}\geq n_{\pi_1(k')}+n_{\pi_1(k')+1}$ resp. $|n_{\pi_2(k)}-n_{\pi_2(k)+1}|\geq|n_{\pi_2(k')}-n_{\pi_2(k')+1}|$ for any $k\leq k'$. For $i\in\{1,2\}$ we define sequences $(l_{k,i})_{1\in\{1,\ldots,K\}}$ by setting $l_{k,1}=n_{\pi_1(k)}+n_{\pi_1(k)+1}$ and $l_{k,2}=|n_{\pi_2(k)}-n_{\pi_2(k)+1}|$ for any $k\in\{1,\ldots,K\}$. For any two sets $J_1,J_2\subset\{1,\ldots,K\}$ let the set $\mathcal{A}_{J_1,J_2}$ consist of any solution $\delta_1,\ldots,\delta_K,j_1,\ldots,j_K,n_1,\ldots,n_K$ such that for any $i\in\{1,2\}$ we have

$$\left|\sum_{k=1}^{H}(-1)^{\delta_{\pi_i(k)}}j_{\pi_i(k),i}M_{l_{k,i},i}\right|<\frac{1}{2}M_{l_{H,i},i} \tag{5.2.6}$$

if and only if $H\in J_i$. If $H\notin J_i$ for some $H\in\{1,\ldots,K\}$ then we have

$$\left|\sum_{k=H+1}^{K}(-1)^{\delta_{\pi_i(k)}}j_{\pi_i(k),i}M_{l_{k,i},i}\right|=\left|\sum_{k=1}^{H}(-1)^{\delta_{\pi_i(k)}}j_{\pi_i(k),i}M_{l_{k,i},i}\right|\geq\frac{1}{2}M_{l_{H,i},i}.$$

Thus we get

$$KN^KM_{l_{H+1,i},i}\geq\frac{1}{2}M_{l_{H,i},i}.$$

Simple calculation shows

$$l_{H,i}-l_{H+1,i}\leq\log_q(2K)+K\log_q(N). \tag{5.2.7}$$

Now let $H\in J_i$ for some $H\geq 2$. We have

$$\begin{aligned}\left|\sum_{k=1}^{H-1}(-1)^{\delta_{\pi_i(k)}}j_{\pi_i(k),i}M_{l_{k,i},i}\right| &\geq \left|j_{\pi_i(H),i}M_{l_{H,i},i}\right|-\left|\sum_{k=1}^{H}(-1)^{\delta_{\pi_i(k)}}j_{\pi_i(k),i}M_{l_{k,i},i}\right|\\ &> \frac{1}{2}M_{l_{H,i},i}.\end{aligned}$$

Assume that there exists another solution $\delta'_1,\ldots,\delta'_K,j'_{1,i},\ldots,j'_{K,i},n'_1,\ldots,n'_K$ in $\mathcal{A}_{J_1,J_2}$ with $\delta'_{\pi_i(k)}=\delta_{\pi_i(k)},j'_{\pi_i(k),i}=j_{\pi_i(k),i}$ and $l'_{k,i}=l_{k,i}$ for $k\in\{1,\ldots,H-1\}$ where any $l'_{k,i}$ is defined analogously to $l_{k,i}$. Without loss of generality we may assume that $l_{H,i}\geq l'_{H,i}$. Then similar calculation as above shows

$$l_{H,i}-l'_{H,i}<\log_q(2K)+K\log_q(N). \tag{5.2.8}$$

Now we are going to estimate

$$\sum_{\substack{\delta_1,\ldots,\delta_K,j_1,\ldots,j_K,\\ n_1,\ldots,n_K\in\mathcal{A}_{J_1,J_2}}}\frac{1}{2^K}\frac{1}{N^{K/2+1}}\prod_{k=1}^{K}a'_{j_k} \tag{5.2.9}$$

for any particular pair of sets J_1,J_2. Therefore at first we encounter all possible choices for $l_{1,i},\ldots,l_{K,i}$ and $i\in\{1,2\}$. Observe that by (5.2.7) and (5.2.8) the number of choices for $l_{k,i}$ is bounded by $C\log(N)$ for some constant $C>0$ independent of N if $k-1\notin J_i$ or $k\in J_i$, otherwise it is bounded by $2N$. Now we assume that J_1 or J_2 is not $\{2,4,\ldots,K\}$

for an even integer K. Then the total number of choices for $l_{1,i}, \ldots, l_{K,i}$ is bounded by $CN^{\lfloor K/2 \rfloor -1} \log(N)^{\lceil K/2 \rceil +1}$ for some constant $C > 0$ independent of N. Furthermore there are at most N choices for n_1 and together with $l_{1,i}, \ldots, l_{K,i}$ this uniquely determines any n_k for $k \in \{2, \ldots, K\}$. Thus the total number of choices for $n_1, \ldots, n_K$ is bounded by $CN^{\lfloor K/2 \rfloor} \log(N)^{\lceil K/2 \rceil +1}$. By Lemma 2.0.1 we have

$$\left| \sum_{\substack{\delta_1,\ldots,\delta_K, \\ j_1,\ldots,j_K}} \frac{1}{2^K} \prod_{k=1}^{K} a'_{j_k} \right| \leq C \log(N)^K \tag{5.2.10}$$

for some constant $C > 0$ independent of N. We conclude

$$\left| \sum_{\substack{\delta_1,\ldots,\delta_K, j_1,\ldots,j_K, \\ n_1,\ldots,n_K \in \mathcal{A}_{J_1,J_2}}} \frac{1}{2^K} \frac{1}{N^{K/2+1}} \prod_{k=1}^{K} a'_{j_k} \right| = o(1) \tag{5.2.11}$$

if J_1 or J_2 is not $\{2, 4, \ldots, K\}$ for an even integer K. Thus from now on we may assume

$$J_1 = J_2 = \{2, 4, \ldots, K\} \tag{5.2.12}$$

and we simply write $\mathcal{A}_{\{2,4,\ldots,K\}} = \mathcal{A}_{J_1,J_2}$. We further may assume that for any $C > 0$ and $\varepsilon > 0$ we have

$$l_{k,i} \geq l_{k+1,i} + 2CN^{1-\varepsilon} \tag{5.2.13}$$

for any $k \in \{2, 4, \ldots, K-2\}$ and sufficiently large N where C denotes the constant used in (5.1.1) since repeating the above argumentation reveals that all other solutions may be neglected. Observe that π_1 and π_2 define decompositions Δ_{π_1} and Δ_{π_2} of $\{1, \ldots, K\}$ into pairs such that for $i \in \{1, 2\}$ any pair $\{k, k'\} \in \Delta_{\pi_i}$ satisfies $\pi_i(k) + 1 = \pi_i(k')$ with $\pi_i(k)$ odd. We now claim that all solutions $\delta_1, \ldots, \delta_K$, $j_1, \ldots, j_K$, $n_1, \ldots, n_K$ such that $\Delta_{\pi_1} \neq \Delta_{\pi_2}$ may be neglected. Therefore we define a decomposition Δ of $\{1, \ldots, K\}$ such that for any k, k' in different subsets we have $\{k, k'\} \notin \Delta_{\pi_i}$ for any $i \in \{1, 2\}$. Observe that for $\Delta_{\pi_1} \neq \Delta_{\pi_2}$ there are at most $K/2 - 1$ subsets. Now we encounter all solutions. The number of possibilities to decompose $\{1, \ldots, K\}$ into pairs in two different ways is independent of N. Thus we may assume that Δ is fixed. We determine the n_k with $k \in \{1, \ldots, K\}$ in increasing order of the index k. There are N choices for n_1. Furthermore any choice of n_k and n_{k+1} uniquely defines $l_{\pi_i^{-1}(k),i}$ for $i \in \{1, 2\}$. If $\{1, \ldots, k-1\} \cap \mathcal{S} \neq \emptyset$ with $k \in \mathcal{S}$ for $\mathcal{S} \in \Delta$ then by (5.2.7) and (5.2.12) the number of choices for n_{k+1} is bounded by $C_1 \log(N)^{C_2}$ for some constants $C_1, C_2 > 0$ independent of N. Otherwise the number of choices is bounded by N. Hence the total number of choices for $n_1, \ldots, n_K$ is bounded by $C_1 N^{|\Delta|+1} \log(N)^{C_2}$ for some constants $C_1, C_2 > 0$

independent of N. Together with (5.2.10) we obtain

$$\left| \sum_{\substack{\delta_1,\dots,\delta_K,j_1,\dots,j_K,\\ n_1,\dots,n_K\in\mathcal{A}_{\{2,4,\dots,K\}},\\ \Delta_{\pi_1}\neq\Delta_{\pi_2}}} \frac{1}{2^K}\frac{1}{N^{K/2+1}}\prod_{k=1}^{K} a'_{j_k} \right| = o(1). \tag{5.2.14}$$

Thus we restrict ourselves to the case $\Delta_{\pi_1} = \Delta_{\pi_2}$. Now we claim that pair partitions which are not non-crossing may be neglected. Thus we encounter all pair partitions with $\{k_1, k_3\}, \{k_2, k_4\} \in \Delta$ for some $k_1 < k_2 < k_3 < k_4$. We begin by counting the choices for $n_{k_4+1}, \dots, n_K$, $n_{K+1} = n_1, \dots, n_{k_2}$. By (5.2.7) the number of choices for $l_{k_3,1}$ and $l_{k_3,2}$ is bounded by $C\log(N)$ with some constant $C > 0$ independent of N for any fixed n_{k_1} and n_{k_1+1}. Consequently, the number of choices for n_{k_3} is bounded by $C\log(N)^2$ for some constants $C > 0$ independent of N. Therefore we now determine $n_{k_3-1}, \dots, n_{k_2+1}$ in decreasing order and $n_{k_3+1}, \dots, n_{k_4}$ in increasing order. Thus l_{k_2} and l_{k_4} already are uniquely defined by n_1 and $(l_{k,1})_{k\in\{1,\dots,K\}\setminus\{k_2,k_4\}}$. Then using a similar argumentation as above we have

$$\left| \sum_{\substack{\delta_1,\dots,\delta_K,j_1,\dots,j_K,\\ n_1,\dots,n_K\in\mathcal{A}_{\{2,4,\dots,K\}},\\ \Delta\notin\mathcal{D}}} \frac{1}{2^K}\frac{1}{N^{K/2+1}}\prod_{k=1}^{K} a'_{j_k} \right| = o(1) \tag{5.2.15}$$

where $\mathcal{D}$ denotes the set of all non-crossing pair partitions of $\{1, \dots, K\}$. In the final step of the proof we further show that all solutions with $l_{k-1,1} \neq l_{k,1}$ for even integers k may be neglected as well. Therefore we first prove that for $\delta_1, \delta_2 \in \{0, 1\}$ and any constant $R \in \mathbb{R}$ we have

$$\sum_{0<|j_{k-1,1}|,|j_{k,1}|\leq N^K}\ \sum_{0\leq l_{k,1}\leq l_{k-1,1}\leq 2N} |j_{k-1,1}|^{-1}|j_{k,1}|^{-1} = \mathcal{O}(N) \tag{5.2.16}$$

where the sum is taken over all solutions with

$$\left|(-1)^{\delta_1} j_{k-1,1} M_{l_{k-1,1},1} + (-1)^{\delta_2} j_{k,1} M_{l_{k,1},1} - R\right| < \frac{1}{2} M_{l_{k,1},1}. \tag{5.2.17}$$

With $R = \alpha_{l_{k-1,1}} M_{l_{k-1,1},1}$ this condition is equivalent to

$$\left| \left((-1)^{\delta_1} j_{k-1,i} - \alpha_{l_{k-1,1}}\right) \frac{M_{l_{k-1,1},1}}{M_{l_{k,1},1}} + (-1)^{\delta_2} j_{k,1} \right| < \frac{1}{2}. \tag{5.2.18}$$

By Lemma 2.0.4 for each choice of $l_{k-1,1}$,$l_{k,1}$ and $j_{k-1,1}$ there exists at most one choice for $j_{k,1} = j_{k,1}(j_{k-1,1})$. Thus we estimate the left-hand side of (5.2.16) by

$$
\begin{aligned}
&\sum_{0\le l_{k,1}\le l_{k-1,1}\le 2N}\ \sum_{0<|j_{k-1,1}|,|j_{k,1}|\le N^K} |j_{k-1,1}|^{-1}|j_{k,1}|^{-1} \\
&\le \sum_{0\le l_{k,1}\le l_{k-1,1}\le 2N}\ \sum_{\substack{\lceil \alpha_{l_{k-1,1}}\rceil+1\le(-1)^{\delta_1}j_{k-1,1}\le N^K,\\ j_{k-1,1}\neq 0}} |j_{k-1,1}|^{-1}|j_{k,1}(j_{k-1,1})|^{-1} \\
&\quad+ \sum_{0\le l_{k,1}\le l_{k-1,1}\le 2N}\ \sum_{\substack{-N^K\le(-1)^{\delta_1}j_{k-1,1}\le\lfloor \alpha_{l_{k-1,1}}\rfloor-1,\\ j_{k-1,1}\neq 0}} |j_{k-1,1}|^{-1}|j_{k,1}(j_{k-1,1})|^{-1} \qquad (5.2.19)\\
&\quad+ \sum_{0\le l_{k,1}\le l_{k-1,1}\le 2N}\ \sum_{\lfloor \alpha_{l_{k-1,1}}\rfloor\le(-1)^{\delta_1}j_{k-1,1}\le\lceil \alpha_{l_{k-1,1}}\rceil} |j_{k-1,1}|^{-1}|j_{k,1}(j_{k-1,1})|^{-1}.
\end{aligned}
$$

There exists at most two $j_{k-1,1}$ such that $|(-1)^{\delta_1}j_{k-1,1} - \alpha_{l_{k-1,1}}| < 1$. Observe that in this case we have $\lfloor \alpha_{l_{k-1,1},1}\rfloor \le (-1)^{\delta_1}j_{k-1,1} \le \lceil \alpha_{l_{k-1,1},1}\rceil$. Therefore in any other case we have $|(-1)^{\delta_1}j_{k-1,1} - \alpha_{l_{k-1,1}}| \ge z$ for some $z \ge 1$. By (5.2.18) we get

$$|j_{k,1}(j_{k-1,1})| \ge \min(zq^{l_{k-1,1}-l_{k,1}} - 1/2, 1) \ge \min(q^{l_{k-1,1}-l_{k,1}} + z - 3/2, 1).$$

Therefore using Cauchy-Schwarz inequality we obtain

$$
\begin{aligned}
&\sum_{0\le l_{k,1}\le l_{k-1,1}\le 2N}\ \sum_{0<|j_{k-1,1}|,|j_{k,1}|\le N^K} |j_{k-1,1}|^{-1}|j_{k,1}|^{-1} \\
&\le \sum_{0\le l_{k,1}\le l_{k-1,1}\le 2N} 2\left(\sum_{(-1)^{\delta_1}j_{k,1}=\min(q^{l_{k-1,1}-l_{k,1}}-1/2,1)}^{\infty} j_{k,1}^{-2}\right)^{1/2} \\
&\quad\cdot\left(\left(\sum_{(-1)^{\delta_1}j_{k-1,1}=\lceil \alpha_{l_{k-1,1}}\rceil+1}^{N^K} j_{k-1,1}^{-2}\right)^{1/2} + \left(\sum_{(-1)^{\delta_1}j_{k-1,1}=-N^K}^{\lfloor \alpha_{l_{k-1,1}}\rfloor-1} j_{k-1,1}^{-2}\right)^{1/2}\right) \qquad (5.2.20)\\
&\quad+ \sum_{0\le l_{k,1}\le l_{k-1,1}\le 2N}\ \sum_{\lfloor \alpha_{l_{k-1,1}}\rfloor\le(-1)^{\delta_1}j_{k-1,1}\le\lceil \alpha_{l_{k-1,1}}\rceil} |j_{k-1,1}|^{-1}|j_{k,1}(j_{k-1,1})|^{-1} \\
&\le \sum_{0\le l_{k,1}\le l_{k-1,1}\le 2N} C\max(q^{l_{k-1,1}-l_{k,1}} - 3/2, 1)^{-1/2} \\
&\quad+ \sum_{0\le l_{k,1}\le l_{k-1,1}\le 2N}\ \sum_{\lfloor \alpha_{l_{k-1,1}}\rfloor\le(-1)^{\delta_1}j_{k-1,1}\le\lceil \alpha_{l_{k-1,1}}\rceil} |j_{k-1,1}|^{-1}|j_{k,1}(j_{k-1,1})|^{-1}.
\end{aligned}
$$

Since the first term is bounded by CN for some constant $C > 0$ which only depends on q it only remains to bound the second term. Let $l_{k-1,1}$ be some fixed integer. Now let $\tilde{l}_{k-1,1}$ be the largest integer such that (5.2.18) holds with $\tilde{l}_{k-1,1} = l_{k,1}$ for some arbitrary

$l_{k,1}$. With $|(-1)^{\delta_2} j_{k,1}| \geq 1$ we get

$$\left|(-1)^{\delta_1} j_{k-1,1} - \alpha_{l_{k-1,1}}\right| \frac{M_{l_{k-1,1},1}}{M_{\tilde{l}_{k-1,1}}} > 1/2.$$

For some $l_{k,1} \leq \tilde{l}_{k-1,1} - \log_q(2) - 1$ we obtain

$$\left|(-1)^{\delta_1} j_{k-1,1} - \alpha_{l_{k-1,1}}\right| \frac{M_{l_{k-1,1},1}}{M_{l_{k,1},1}} > \frac{1}{2} \frac{M_{\tilde{l}_{k-1,1},1}}{M_{l_{k,1},1}} \geq q^{\tilde{l}_{k-1,1} - l_{k,1}}$$

and therefore $|j_{k,1}| > 1/2 \cdot q^{\tilde{l}_{k-1,1} - l_{k,1}}$. Using a similar argumentation as above we have

$$\begin{aligned} \sum_{0 \leq l_{k,1} \leq l_{k-1,1} \leq 2N} \sum_{\lfloor \alpha_{l_{k-1,1}} \rfloor \leq (-1)^{\delta_1} j_{k-1,1} \leq \lceil \alpha_{l_{k-1,1}} \rceil} & |j_{k-1,1}|^{-1} |j_{k,1}(j_{k-1,1})|^{-1} \\ & \leq 2(\log_q(2) + 1)(2N+1) + C \sum_{l=0}^{2N} \sum_{l'=0}^{l} q^{l'-l} \\ & \leq CN \end{aligned}$$

for some constant $C > 0$ which only depends on q. Plugging this into (5.2.20) we get (5.2.16). With Lemma 2.0.1 we get

$$\begin{aligned} & \left| \sum_{\substack{\delta_1,\ldots,\delta_K, j_1,\ldots,j_K, \\ n_1,\ldots,n_K \in \mathcal{A}_{\{2,4,\ldots,K\}}, \\ l_{k-1,1} \neq l_{k,1} \text{ for some even } k, \\ \Delta \in \mathcal{D}}} \frac{1}{2^K} \frac{1}{N^{K/2+1}} \prod_{k=1}^{K} a'_{j_k} \right| \\ & \leq \sum_{\delta_1,\ldots,\delta_K \in \{0,1\}} \sum_{n_1,\ldots,n_K \in \{1,\ldots,N\}} C \\ & \quad \cdot \sum_{0 < |j_{1,1}|,\ldots,|j_{K,1}| \leq N^K} \prod_{k=1}^{K} |j_{k,1}|^{-1} \mathbf{1}_{\substack{|\sum_{k=1}^{H} (-1)^{\delta_{\pi_1(k)}} j_{\pi_1(k),1} M_{l_{k,1},1}| < 1/2 \cdot M_{l_{H,1},1} \\ \text{if and only if } H \text{ is even}}} \\ & \quad \cdot \sum_{0 < |j_{1,2}|,\ldots,|j_{K,2}| \leq N^K} \prod_{k=1}^{K} |j_{k,2}|^{-1} \mathbf{1}_{\substack{|\sum_{k=1}^{H} (-1)^{\delta_{\pi_2(k)}} j_{\pi_2(k),2} M_{l_{k,2},2}| < 1/2 \cdot M_{l_{H,2},2} \\ \text{if and only if } H \text{ is even}}} \end{aligned} \tag{5.2.21}$$

for some constant $C > 0$ independent of N. For any fixed $\delta_1, \ldots, \delta_K$, $n_1, \ldots, n_K$ and $j_{\pi_2(k),2}$ for $\pi_2(k) \leq \pi_2(H-2)$ there exists at most one $j_{\pi_2(H),2}$ for any $j_{\pi_2(H-1),2}$ and vice versa such that

$$\left| \sum_{k=1}^{H} (-1)^{\delta_{\pi_2(k)}} j_{\pi_2(k),2} M_{l_{k,2},2} \right| < \frac{1}{2} M_{l_{H,2},2}$$

holds. By Cauchy-Schwarz inequality we obtain

$$\sum_{0<|j_{1,2}|,\dots,|j_{K,2}|\le N^K} \prod_{k=1}^{K} |j_{k,2}|^{-1} \mathbf{1}_{\substack{|\sum_{k=1}^{H}(-1)^{\delta_{\pi_2(k)}} j_{\pi_2(k),2} M_{l_{k,2},2}| < 1/2\cdot M_{l_{H,2},2} \\ \text{if and only if } H \text{ is even}}} \le \left(2\sum_{j=1}^{\infty} j^{-2}\right)^{K/2} \le C \quad (5.2.22)$$

for some constant $C>0$ independent of N. Let H be the smallest even positive integer such that $l_{H-1,1} \neq l_{H,1}$. We use condition (5.1.1) and (5.2.13) for $k=H$ and apply (5.2.16) with $R = \sum_{m=1}^{k-2}(-1)^{\delta_{\pi_1(m)}} j_{\pi_1(m),1} M_{l_{m,1},1}$ for any other even integer k. By (5.2.21) and (5.2.22) we conclude

$$\left| \sum_{\substack{\delta_1,\dots,\delta_K, j_1,\dots,j_K, \\ n_1,\dots,n_K \in \mathcal{A}_{\{2,4,\dots,K\}}, \\ l_{H-1,1}\neq l_{H,1}, \\ \Delta\in\mathcal{D}}} \frac{1}{2^K} \frac{1}{N^{K/2+1}} \prod_{k=1}^{K} a'_{j_k} \right| = o(1) \qquad (5.2.23)$$

since $K\cdot N^K \cdot q^{-2CN^{1-\varepsilon}} \le 1/2\cdot q^{CN^{1-\varepsilon}}$ for sufficiently large N. Therefore we may assume $l_{k',1} = l_{k,1}$ for any $\{k,k'\}\in\Delta$.

We now show that we may also assume $l_{k',2} = l_{k,2}$ for $\{k,k'\}\in\Delta$. This immediately follows by proving $n_{k+1} = n_{k'}$ resp. $n_{k'+1} = n_k$ for any $\mathcal{S} = \{k,k'\}\in\Delta$ which we are going to show by induction. For $K=2$ this is trivial. For $K\ge 4$ there exists some $\{H-1,H\}\in\Delta$ since Δ is non-crossing. Furthermore by $l_{\pi_1^{-1}(H-1),1} = l_{\pi_1^{-1}(H),1}$ we have $n_{H-1} = n_{H+1}$. Set $\tilde{k} = k$ for $k\in\{1,\dots,H-1\}$ and $\tilde{k} = k-2$ for $k\in\{H+1,\dots,K-1\}$. Thus it may be reduced to the case $K-2$ and the conclusion follows by induction hypothesis. Observe that there are N choices for n_1 and for $k\in\{2,\dots,K\}$ the number of choices for n_{k+1} is either 1, in the case $\{k',k\}\in\Delta$ for some $k'\in\{1,\dots,k-1\}$, or N otherwise. Thus we have

$$\# n_1,\dots,n_K = N^{K/2+1}. \qquad (5.2.24)$$

Now let $\{k,k'\}\in\Delta$. With (5.2.6) it is easy to see that $(-1)^{\delta_{k'}} j_{k',i} + (-1)^{\delta_k} j_{k,i} = 0$ for any $i\in\{1,2\}$ resp. $j_{k'} = \pm j_k$. Hence we get

$$\begin{aligned} a'_{j_{k'}} &= \prod_{i=1}^{2} \frac{N^K+1-|j_{k,i}|}{N^K+1} \int_{[0,1)^2} f(x)\cos(2\pi\langle j_{k'},x\rangle)\,dx \\ &= \prod_{i=1}^{2} \frac{N^K+1-|j_{k,i}|}{N^K+1} \int_{[0,1)^2} f(x)\cos(2\pi\langle j_k,x\rangle)\,dx = a'_{j_k}. \quad (5.2.25) \end{aligned}$$

Therefore we obtain

$$\sum_{\substack{\delta_1,\dots,\delta_K,j_1,\dots,j_K,\\ n_1,\dots,n_K\in\mathcal{A}_{\{2,4,\dots,K\}},\\ l_{k-1,1}=l_{k,1}\forall k\in\{2,4,\dots,K\},\\ \Delta\in\mathcal{D}}} \frac{1}{2^K}\frac{1}{N^{K/2+1}}\prod_{k=1}^{K} a'_{j_k} = |\mathcal{D}|\left(\sum_{\substack{j\in(\mathbb{Z}\setminus\{0\})^2,\\ ||j||_\infty\le N^K}} a'^2_j\right)^{K/2}. \tag{5.2.26}$$

By (5.1.2) and $||f-p||_2^2 \le CN^{-K}$ we have $1 \ge \sum_{j\in(\mathbb{Z}\setminus\{0\})^2} a'^2_j \ge 1-CN^{-K}$. With (5.2.2), (5.2.5), (5.2.11), (5.2.14), (5.2.15) and (5.2.23) we get (5.1.5) which finally concludes the proof.

5.3 Examples

5.3.1 Superlacunary sequences

Consider a random matrix ensemble where the first generating lacunary sequence is superlacunary, i.e. let $(M_{n,1})_{n\ge1}$ be some sequence of positive integers such that

$$q_n = \frac{M_{n+1,1}}{M_{n,1}} \to \infty$$

for $n\to\infty$. In order to prove weak convergence to the semicircle law we have to verify (5.1.1). Repeating the proof of (5.2.16) with $R=0$ we obtain by applying Cauchy-Schwarz inequality

$$\begin{aligned}
&\sum_{\substack{n,n'\in\{1,\dots,2N\},\\ n>n'}}\sum_{j,j'\in\{1,\dots,N^K\}} (jj')^{-1}\mathbf{1}_{|jM_{n,1}-j'M_{n',1}|<1/2\cdot q^{-CN^{1-\varepsilon}}M_{n',1}}\\
&\le \sum_{\substack{n,n'\in\{1,\dots,2N\},\\ n>n'}}\sum_{j,j'\in\{1,\dots,N^K\}} (jj')^{-1}\mathbf{1}_{|jM_{n,1}/M_{n',1}-j'|<1/2}\\
&\le \sum_{\substack{n,n'\in\{1,\dots,2N\},\\ n>n'}}\left(\sum_{j=1}^{\infty} j^{-2}\right)^{1/2}\left(\sum_{j'=\max(q_{n'}^{n-n'}-1/2,1)}^{\infty}(j')^{-2}\right)^{1/2}\\
&\le \sum_{\substack{n,n'\in\{1,\dots,2N\},\\ n>n'}} C\min\left((q_{n'}^{n-n'}-3/2)^{-1/2},2\right)
\end{aligned}$$

for some constants $C > 0$ independent of N. With $\tilde{n} = \min\{n : q_n \geq 2\}$ we have $(q_{n'} - 3/2)^{-1/2} \leq 2(q_{n'})^{-1/2} \leq 2$ for $n' \geq \tilde{n}$. Thus we get

$$\sum_{\substack{n,n' \in \{1,\ldots,2N\}, \\ n>n'}} \sum_{j,j' \in \{1,\ldots,N^K\}} (jj')^{-1} \mathbf{1}_{|jM_{n,1} - j'M_{n',1}| < 1/2 \cdot q^{-CN^{1-\varepsilon}} M_{n',1}}$$
$$\leq C\tilde{n} + \sum_{n=1}^{2N} \sum_{k=1}^{n-1} C q_{n-1}^{-k/2}.$$

Since $q_n^{-1} \to 0$ for $n \to \infty$ we easily verify (5.1.1).

5.3.2 The sequence 2^n

We now are going to show that besides the Hadamard gap condition (1.0.4) further conditions on the generating lacunary sequences are necessary. Therefore we prove that for the sequences $M_{n,1} = M_{n,2} = 2^n$ which do not satisfy (5.1.1) the mean empirical eigenvalue distribution does not converge to the semicircle law in general. Here we consider $f(x_1, x_2) = 1/\sqrt{2} \cdot (\cos(2\pi(x_1 + x_2)) + \cos(4\pi(x_1 + x_2)))$. We restrict ourselves to the case $K = 4$ and repeat the proof of Theorem 5.1.1 until (5.2.15). Observe that there are precisely two non-crossing pair partitions of $\{1, 2, 3, 4\}$ which are $\{\{1, 2\}, \{3, 4\}\}$ resp. $\{\{1, 4\}, \{2, 3\}\}$. Thus $\mathbb{E}[1/N \cdot X_N^4]$ is the sum over solutions $\delta_1, \ldots, \delta_4, j_1, \ldots, j_4$ and $n_1, \ldots, n_4$ such that

$$(-1)^{\delta_1} j_{1,1} M_{n_1+n_2,1} + (-1)^{\delta_2} j_{2,1} M_{n_2+n_3,1} = (-1)^{\delta_3} j_{3,1} M_{n_3+n_4,1} + (-1)^{\delta_4} j_{4,1} M_{n_4+n_1,1} = 0$$

or

$$(-1)^{\delta_1} j_{1,1} M_{n_1+n_2,1} + (-1)^{\delta_4} j_{4,1} M_{n_4+n_1,1} = (-1)^{\delta_2} j_{2,1} M_{n_2+n_3,1} + (-1)^{\delta_3} j_{3,1} M_{n_3+n_4,1} = 0.$$

Without loss of generality by (5.2.13) we may assume that both cases are distinct. Hereafter we only focus on the first case. The second case is similar. We further get

$$\begin{aligned} (j_{2,1}, j_{2,2}) &= (-1)^{\delta_1+\delta_2} (2^{n_1-n_3} j_{1,1}, 2^{|n_1-n_2|-|n_2-n_3|} j_{1,2}), \\ (j_{4,1}, j_{4,2}) &= (-1)^{\delta_3+\delta_4} (2^{n_3-n_1} j_{3,1}, 2^{|n_3-n_4|-|n_4-n_1|} j_{3,2}). \end{aligned}$$

Therefore we have

$$\begin{aligned} \lim_{N\to\infty} \frac{1}{N} \mathbb{E}[X_N^4] &= \frac{2}{N^3} \sum_{n_1,n_3=1}^{N} \left(\sum_{n=1}^{N} \mathbb{E}\left[f\left(2^{n_3-n_1 \vee 0} x_1, 2^{|n-n_3|-|n_1-n| \vee 0} x_2\right) \right.\right. \\ &\qquad \left.\left. \cdot f\left(2^{n_1-n_3 \vee 0} x_1, 2^{|n_1-n|-|n-n_3| \vee 0} x_2\right)\right]\right)^2. \end{aligned}$$

For $n_1 = n_3$ we easily observe

$$\frac{2}{N^3} \sum_{n_1=1}^{N} \left(\sum_{n=1}^{N} \mathbb{E}[f^2] \right)^2 = 2 = C_2.$$

Thus it is enough to show

$$\sum_{n=1}^{N} \mathbb{E}\left[f\left(2^{n_3-n_1\vee 0}x_1, 2^{|n-n_3|-|n_1-n|\vee 0}x_2\right) f\left(2^{n_1-n_3\vee 0}x_1, 2^{|n_1-n|-|n-n_3|\vee 0}x_2\right)\right] \neq 0 \tag{5.3.1}$$

for some $n_1 \neq n_3$ in order to prove that the mean empirical eigenvalue distribution does not converge to the semicircle law. Now choose some $n_1 = n_3 + 1 > 1$. Then we have

$$|n_1 - n| - |n - n_3| = \begin{cases} 1, & n \leq n_3, \\ -1, & n \geq n_1. \end{cases}$$

Plugging this into the left-hand side of (5.3.1) and using the definition of f yields

$$\sum_{n=1}^{n_3} \mathbb{E}\left[f(x_1, x_2) f\left(2x_1, 2x_2\right)\right] + \sum_{n=n_1}^{N} \mathbb{E}\left[f(x_1, 2x_2) f\left(2x_1, x_2\right)\right] = \frac{n_3}{2} > 0.$$

Therefore we obtain

$$\lim_{N\to\infty} \frac{1}{N}\mathbb{E}[X_N^4] \neq 2.$$

By assumption on f we trivially have

$$\lim_{N\to\infty} \frac{1}{N}\mathbb{E}[X_N^2] = \mathbb{E}[f^2] = 1.$$

We conclude that the limiting mean empirical eigenvalue distribution does not have the density $1/R\pi \cdot \sqrt{R^2 - x^2} \cdot \mathbf{1}_{x^2 \leq R^2}$ for any $R > 0$ and hence it is not a semicircle law.

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