Equilibrium of masonry vaults and open stairs

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SUMMARY. In this paper the unilateral model for masonry is applied to explain the equilibrium of vaults. In particular, the present study is concerned with the application of the safe theorem of limit analysis to spiral vaults, that is, curved constructions modeled as continuous unilateral bodies. On allowing for singular stresses in the form of line or surface Dirac deltas, statically admissible stress fields concentrated on surfaces (and on their folds) lying inside the masonry, are considered. The unilateral restrictions require that the membrane surface lies in between the extrados and intrados surfaces of the vault and that the stress function, representing the stress, be concave. Such a constraint is, in general, not satisfied on a given shape for given loads: in such a case, the shape has to be modified to fit the constraint. A particular application, namely the double spiral stair of Sanfelice' palace in Naples, is considered.

1 INTRODUCTION

The present work is concerned with the application of the safe theorem of Limit Analysis to masonry spiral stairs treated as unilateral masonry-like structures. The case study of the double spiral stair of *Palazzo Sanfelice* in Naples, is analysed.

The unilateral model for masonry, which, though in a mathematically unconscious way, has been around since the nineteenth century (see Moseley [1]), was first rationally introduced by Heyman in [2] and divulgated and extended in Italy, thanks to the effort of Salvatore Di Pasquale [3] and other members of the Italian school of Structural Mechanics, such as G.Romano and M.Romano [4], Baratta [5], Del Piero [6], Como [7], Angelillo [8].

The unilateral model, which appears as the clue of structural interpretation behind the design of the great Architecture masterpieces of the past, in the form first proposed by Heymjan, is based on three crude assumptions:

1. The material does not bear the slightest tensile load (unilateral assumption) and then can detach, at zero energy, along any internal interface (fracture line).

2. The material is indefinitely resistant to compression: that is the mean compressive stresses are so low that crushing of the material is not an issue.

3. There is no possibility of sliding along a fracture line (infinite friction).

One of the main consequences of these simplifying assumptions, is that the machinery of limit analysis can be applied to these kind of unilateral structures, specifically, by considering the static and kinematic theorems on the basis of admissible stress and strain fields (see [15]). Here we focus on the static theorem, the main tool being represented by statically admissible singular stress fields, in the form of surface and line Dirac deltas applied on material surfaces and lines. The support of these Dirac deltas can be interpreted an internal membranes/arches, by means of which the load is balanced with admissible internal stresses, so that the safety of the structure can be assessed: if such structures can be created inside the masonry and are compressed, then the masonry body is safe.

The use of singular stress fields for the problem of equilibrium of masonry-like, No–Tension materials in 2d, has been recently introduced by Lucchesi et al. in [9] and applied to a reinforcement problem by De Faveri et al. in [10]. Here we propose a simple method for assessing the safety of vaults and domes, based on a stress function formulation, from which both the shape of the structure and the force field inside it, are easily generated. The use of folded stress functions for describing singular stress fields in elastic bodies was first introduced in Fraternali et al [11].

In the present study, on allowing for singular stresses in the form of line and surface Dirac deltas, statically admissible stress fields concentrated on surfaces (and on their folds) lying inside the masonry, are considered. Such surface and line structures are unilateral membranes/arches, whose geometry is described a la Monge, and their equilibrium can be formulated in Pucher form. It is assumed that the load applied to the vault is carried by such a (possibly folded) membrane structure *S*. The geometry of the membrane *S* (i.e. of the support of the singularities) is not fixed, in the sense that it can be displaced and distorted, provided that one keeps it inside the masonry.

In the case of pure parallel loading, the problem of equilibrium of the membrane *S*, is reduced to a singular partial differential equation of the second order where the shape *f* and the stress function *F* appear symmetrically. The unilateral restrictions require that the membrane surface lies in between the extrados and intrados surfaces of the vault and that the stress function be concave.

Such a constraint is, in general, not satisfied on a given shape for given loads: in such a case, the shape has to be modified to fit the constraint. In a sense, the unilateral assumption renders the membrane an underdetermined structure that must adapt its shape in order to satisfy the unilateral restrictions.

2 ANALYSIS

2.1 *Geometry*

The relevant geometrical dimensions of a vault are the geometry of the intrados and extrados surfaces, its thickness *t*, the geometry of the fillings, and for vaults supported on arches rising from piers, the form and dimension of the enlargements (countersinks) of the piers at their tops.

It is assumed that the load applied to the vault is carried by a membrane structure *S* of thickness *t'*. The geometry of the membrane *S* is not fixed, in the sense that we can displace and distort it, provided that we keep it inside the masonry. Notice that the surface *S* that we consider is continuous but not necessarily smooth.

The shell surface *S* carrying the stress is defined *a la Monge,* by considering as parameters the x_1, x_2 components of the points of *S*, and defining the third component as $x_3 = f(x_1, x_2)$, f being a continuous function of its arguments. The orthonormal triad associated to the Cartesian reference system is denoted with $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

The couple $\{x_1, x_2\}$ belongs to a connected 2d domain Ω, called the planform of the membrane *S*. Ω is a plane domain whose boundary $\partial\Omega$ is composed of a finite number of closed, curves of finite perimeter, endowed (a.e.) of a unit outer normal **n**, contained in the plane orthogonal to the axis x_3. A 3d view of such surface structures, which could be fitted inside the masonry (and possibly the filling) in two typical cases, is depicted in Figure 1.

By denoting $\{\cdot, \cdot, \cdot\}$ the components of vector quantities with respect to the Cartesian frame, and **x** the position vectors of points on the surface *S*, the parametric description of the surface *S* is

$$\mathbf{x} = \{x_1, x_2, f(x_1, x_2)\}\ ,\ \ \{x_1, x_2\} \in \Omega\ . \tag{1}$$

The *natural* or *covariant* base vectors tangent to *S*, associated to the curvilinear parameters $\{x_1, x_2\}$, are

$$\mathbf{a}_1 = \{1, 0, f_{,1}\}\ ,\ \mathbf{a}_2 = \{0, 1, f_{,2}\} \tag{2}$$

where a comma followed by an index, say α, stands for differentiation with respect to x_α. The unit normal to $\mathcal{S}$ (also called **m** in what follows) is defined as

$$\mathbf{m} := \mathbf{a}_3 = \tfrac{1}{J}\{-f_{,1}, -f_{,2}, 1\} \tag{3}$$

$J = \sqrt{1 + f_{,1}^{\,2} + f_{,2}^{\,2}}$, being the *Jacobian* determinant, that is the ratio between the differential surface area on $\mathcal{S}$ and its projection on the planform. Finally, the *reciprocal* or *contravariant* base vectors are

$$\mathbf{a}^1 = \frac{1}{J^2}\{1 + f_{,2}^{\,2}, -f_{,1} f_{,2}, f_{,1}\} \quad , \quad \mathbf{a}^2 = \frac{1}{J^2}\{-f_{,1} f_{,2}, 1 + f_{,1}^{\,2}\ f_{,2}\}. \tag{4}$$

2.2 *Membrane equilibrium in Pucher form*

The generalized membrane stress on S, is defined by the (surface) stress tensor **T**. In the covariant base defined in (1), by adopting summation convention over repeated Greek indices: $\alpha, \beta, \gamma ... = 1, 2$, it can be represented as follows

$$\mathbf{T} = T^{\alpha\beta}\, \mathbf{a}_\alpha \otimes \mathbf{a}_\beta \ , \tag{5}$$

$T^{\alpha\beta}$ being the contravariant components of **T**. Notice that the basis $\{\mathbf{a}_1, \mathbf{a}_2\}$ is neither unit nor orthogonal, then though the contravariant components are useful and convenient, they are *non-physical* components of stress, and a bit of algebra is needed to transform them into Cartesian stress components.

Membrane equilibrium is dictated by the condition that the divergence of the generalized surface stress **T** balances the load $\mathbf{q} = \{\mathrm{q}_1, \mathrm{q}_2, \mathrm{q}_3\}$, defined per unit surface area on $\mathcal{S}$. Such a vector/differential condition can be written, explicitly, as follows

$$\frac{\partial}{\partial x_\gamma}\left(T^{\alpha\beta}\, \mathbf{a}_\alpha \otimes \mathbf{a}_\beta\right)\mathbf{a}^\gamma + \mathbf{q} = \mathbf{0} \ . \tag{6}$$

The most simple way of thinking about membrane equilibrium of a thin shell under a load $\mathbf{q} = \{\mathrm{q}_1, \mathrm{q}_2, \mathrm{q}_3\}$ defined per unit projected area (that is $\mathbf{p} = J\,\mathbf{q}$), is to adopt *Pucher*'s approach (see [2]). With Pucher's transformation the generalized contravariant stress components $T^{\alpha\beta}$ on the surface are transformed into projected stress components $S_{\alpha\beta} = J\, T^{\alpha\beta}$, in the planform ($J$ being the Jacobian of the coordinate transformation and, as already remarked, the ratio between the surface differential areas on the surface and on the planform). Indeed, by projecting equation (6) into the three non-coplanar directions $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{m} = \mathbf{a}_3\}$, after some algebra, one obtains

$$S_{11,1} + S_{12,2} + p_1 = 0 \ , \ S_{21,1} + S_{22,2} + p_2 = 0 \ , \ S_{\alpha\beta}\, f_{,\alpha\beta} - p_\gamma\, f_{,\gamma} + p_3 = 0 \ . \tag{7}$$

The equilibrium equations, in terms of such projected stresses, in the plane of the planform, are identical to those of the plane problem and, in the case of pure vertical loading (say **p**={0 ,0 ,-p}), may be solved with the use of an *Airy stress function F* in the form

$$S_{11} = F_{,22} \ , \ S_{22} = F_{,11} \ , \ S_{12} = S_{21} = -F_{,12} \ , \tag{8}$$

F being a function of x_1, x_2, usually assumed as smooth, and that here we consider only continuous.
For vertical loading, transverse equilibrium is ruled by the balance of the vertical component of the force per unit projected area, with the scalar product of the matrix of the Pucher stress components times the matrix of the covariant components of the Hessian of the function f (see equation (7)[3]). In terms of the

Airy's solution, for the case of pure vertical loading, such transverse equilibrium equation can be rewritten in the form

$$F_{,22}\ f_{,11} + F_{,11}\ f_{,22} - 2F_{,12}\ f_{,12} = p\ . \quad (9)$$

2.3 *Unilateral membranes*

The model vault that we consider is composed of rigid No-Tension material in the sense of Heyman, that is the following material restrictions are imposed on the generalized stress **T** and on the strain **E**: the stress **T** is negative semi-definite (10), the strain **E** is positive semi-definite (11), and the stress **T** does no work for the corresponding strain **E** (12)

$$\mathbf{T} \in Sym^{-}\ ,\ (10) \qquad \mathbf{E} \in Sym^{+}\ ,\ (11) \qquad \mathbf{T}\cdot\mathbf{E} = 0\ ,\ (12)$$

The first to consider membrane equilibrium with Pucher's transformation for NT materials were Angelillo and Fortunato in [13]. As it is shown in [13], if the membrane is made of NT material, in order that the surface stress tensor be negative semi-definite also the matrix of the projected stresses must be negative semi-definite. In terms of the stress function *F*, this condition can be translated into the two conditions

$$F_{,11} + F_{,22} \le 0\ ,\ F_{,11}F_{,22} - F_{,12}{}^{2} \ge 0, \quad (13)$$

that is *F* must be concave.

2.4 *Singular stress in the membrane*

If *F* is only continuous, it may have folds, and, along these folds, the projected stress (and hence the actual stress) is a line Dirac delta with support along the projection Γ of the fold on the planform . The *Hessian* **H** of *F* is singular transversely to Γ (namely it has a uniaxial singular part directed as the normal to Γ); the intensity of the concentrated second derivative component in the direction of the normal **h** to Γ, is represented by the jump of the directional derivative of *F* in the direction of **h**, called F_h. Therefore, the singular part of the Hessian **H** of *F* can be written as

$$\mathbf{H}_s = \delta(\Gamma)\, F_h \mathbf{h} \otimes \mathbf{h}\ , \quad (14)$$

$\delta(\Gamma)$ being the unit line Dirac delta applied on Γ.

Based on relations (8), the singular part of the corresponding projected stress is also a line Dirac delta defined on Γ, uniaxial in the direction of the unit tangent **k** to Γ:

$$\mathbf{S}_s = \delta(\Gamma)\, F_h \mathbf{k} \otimes \mathbf{k}\ . \quad (15)$$

Notice that the concavity of F_h implies the concavity of the fold, that is of the kink across the curve where the jump of slope takes place, such a curve being the restriction of *F* to the supporting curve Γ. Then F_h is negative and the corresponding projected singular stress concentrated on Γ is compressive.

2.5 *Equilibrium on the NT membrane in terms of F*

Based on the Airy solution (8), in the case of pure vertical loading, the problem of equilibrium for the unilateral membrane $\mathcal{S}$ reduces to the form:

Find a concave stress function F satisfying equation (9), with the boundary conditions

$$F = g \quad \text{or} \quad \frac{dF}{dn} = h\ , \quad \text{on } \partial\Omega\ , \quad (16)$$

where g and h are the contact internal moment and the contact internal shear force along a 1d beam structure, having the same form of the projected boundary $\partial\Omega$, and loaded by the projected tractions acting at the boundary.

2.6 *Membrane equilibrium in polar coordinates*

In many cases of interest, such as those concerning domes and spiral stairs based on a circular planform, it is convenient to adopt a Monge description of the surface $\mathcal{S}$ based on polar coordinates $\{\theta^1=r, \theta^2=\theta\}$, defined over the planform of the membrane, as follows

$$\mathbf{x} = \{r\cos\theta, r\sin\theta, f(r,\theta)\} \quad , \quad \{x_1, x_2\} \in \Omega \ . \tag{17}$$

The simplest way of thinking about membrane equilibrium of a thin shell under purely vertical loading (defined per unit projected area) is to adopt Pucher's approach, as described above.

With Pucher transformation the generalized stresses on the surface are transformed into projected stresses in the planform. In the covariant base (tangent to the planform) associated to the curvilinear reference $\{\theta^1=r, \theta^2=\theta\}$,

$$\mathbf{b}_1 = \{\cos\theta, \sin\theta\} \ , \ \mathbf{b}_2 = \{-r\sin\theta, r\cos\theta\} \ , \tag{18}$$

and in the variable Cartesian base

$$\mathbf{k}_1 = \{\cos\theta, \sin\theta\} \ , \ \mathbf{k}_2 = \{-\sin\theta, \cos\theta\} \ , \tag{19}$$

the projected stress **S** can be represented as follows

$$\mathbf{S} = S^{\alpha\beta}\, \mathbf{b}_\alpha \otimes \mathbf{b}_\beta = \sigma_{\alpha\beta}\, \mathbf{k}_\alpha \otimes \mathbf{k}_\beta \ , \tag{20}$$

where $S^{\alpha\beta}$, $\sigma_{11} = \sigma_{rr}$, $\sigma_{22} = \sigma_{\theta\theta}$, $\sigma_{12} = \sigma_{21} = \sigma_{\theta r}$, denote the contravariant components and the physical components of the projected stress in the polar reference $\{r,\theta\}$.

The equilibrium equations for such projected stresses in the plane of the planform are identical to those of the plane problem and, on considering the general case of distributed loading defined per unit projected area

$$\mathbf{p} = p^1\,\mathbf{b}_1 + p^2\,\mathbf{b}_2 - p\,\mathbf{e}_3 = p_r\,\mathbf{k}_1 + p_\theta\,\mathbf{k}_2 - p\,\mathbf{e}_3 \ , \tag{21}$$

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$$S^{11}{}_{/1} + S^{12}{}_{/2} + p^1 = 0 \ , \ S^{21}{}_{/1} + S^{22}{}_{/2} + p^2 = 0 \ , \tag{22}$$

where "./α" stands for covariant derivative with respect to θ^α. In the case of pure vertical loading such equations may be solved with the use of an Airy stress function *F*, namely:

$$S^{11} = \left(\frac{1}{r}F_{,1} + \frac{1}{r^2}F_{,22}\right) \ , \ S^{22} = \frac{1}{r^2}F_{,11} \ , \ S^{12} = -\frac{1}{r}\left(\frac{1}{r}F_{,2}\right)_{,1} \ , \tag{23}$$

where ".,α" stands for differentiation with respect to θ^α.

The forces acting transversely to the surface $\mathcal{S}$ (defined per unit projected area of the planform) are balanced by the scalar product of the matrix of the contravariant components of the projected stress times the matrix of the covariant components of the curvature:

$$S^{\alpha\beta} f_{/\alpha\beta} - p^{\gamma} f_{,\gamma} - p = 0 \ . \tag{24}$$

The above equilibrium equations (22), (24), can be rewritten in terms of physical stress components $\sigma_{11} = \sigma_{rr}$, $\sigma_{22} = \sigma_{\theta\theta}$, $\sigma_{12} = \sigma_{21} = \sigma_{\theta r}$ on considering that, for the case at hand:

$$S^{11} = \sigma_{rr} \quad , \quad S^{22} = \frac{1}{r^2}\sigma_{\theta\theta} \quad , \quad S^{12} = \frac{1}{r}\sigma_{r\theta} \quad , \tag{25}$$

and

$$f_{,11} = f_{,11} \quad , \quad f_{,22} = r^2\left(\frac{1}{r} f_{,1} + \frac{1}{r^2} f_{,22}\right) \quad , \quad f_{,12} = r\left(\frac{1}{r} F_{,2}\right)_{,1} \ . \tag{26}$$

By substituting into the above equations, after some algebra, one obtains

$$\begin{gathered}\frac{1}{r}\frac{\partial(r\,\sigma_{rr})}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{r\theta}}{\partial\theta} - \frac{\sigma_{\theta\theta}}{r} + p_r = 0 \ , \quad \frac{1}{r^2}\frac{\partial(r^2\,\sigma_{\theta r})}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta} + p_\theta = 0 \ , \\ \sigma_{rr}\frac{\partial^2 f}{\partial r^2} + \sigma_{\theta\theta}\left(\frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2}\right) + 2\sigma_{r\theta}\frac{\partial\left(\frac{1}{r}\frac{\partial f}{\partial\theta}\right)}{\partial r} - p_r\frac{\partial f}{\partial r} - p_\theta\frac{1}{r}\frac{\partial f}{\partial\theta} = p \ .\end{gathered} \tag{27}$$

In the case of pure vertical loading directed downward, considering the Airy solution, that is

$$\sigma_{rr} = \left(\frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial\theta^2}\right) \ , \quad \sigma_{\theta\theta} = \frac{\partial^2 F}{\partial r^2} \quad , \quad \sigma_{r\theta} = -\frac{\partial\left(\frac{1}{r}\frac{\partial F}{\partial\theta}\right)}{\partial r} \quad , \tag{28}$$

the equation of transverse equilibrium (27)[3] can be expressed in polar coordinates, in terms of the function *F*, as follows

$$\left(\frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial\theta^2}\right)\frac{\partial^2 f}{\partial r^2} + \frac{\partial^2 F}{\partial r^2}\left(\frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2}\right) - \frac{\partial\left(\frac{1}{r}\frac{\partial F}{\partial\theta}\right)}{\partial r}\frac{\partial\left(\frac{1}{r}\frac{\partial f}{\partial\theta}\right)}{\partial r} = p \ . \tag{29}$$

We recall that, if the membrane is made of NT material, the projected stresses must be negative semi-definite and the corresponding Airy stress function must be concave. A number of equilibrium solutions for domes, in the case of axial symmetry, can be found in a recent paper on masonry vaults by Angelillo et al. [13] and for some special spiral stairs in [14], [15].

3 A CASE STUDY

3.1. *Geometric and historical description of the staircase*

There are many spiral stairs more or less known realized by Ferdinando Sanfelice, worthy of mention, among these, is the staircase carved into the tuff at Palazzo Santoro [16], whose shape replicates that of a cone (Figure 1a). In San Felice's buildings, bold shapes are often enclosed even in little spaces and in the service areas, such as the service staircase in Figure 2b), which was in the home residence of the architect at the end of seventeenth century.

The work focuses on "structural understanding" of one of these stairs, taken as example, that is one of the two stairs of Sanfelice's Palace at Sanità district, that the famous architect bought and renovated for his family (Figures 1c, d).

The double circular staircase, which connects only the mezzanine and the first floor of the building (because the other floors are accessible from the adjacent staircase, with the aim that the part in exclusive use of the family, remains isolated from the rest of the building) is supported by a cruise vault. Climbing on it, the impression is of loosing the way, and then finding the destination, without a clear feeling of what path has been traveled: two ramps depart from the opposite sides of the hall, crossing between them, and then finding themselves in opposite direction. All in an implosion of spaces, complex even to be photographed [17].

Figure 1: Palazzo Santoro, via Salita Capodimonte 10, Napoli [16]; b) small spiral staircase enclosed in a little room of the Sanfelice's Palace at Sanità district (Naples, Italy); c) The staircase in Sanfelice's Palace at Sanità district (Naples, Italy) object of the study [17]; d) the support vault.

3.2. *Analytical development: the helical stair by Ferdinando Sanfelice*

The model above described has been applied to evaluate the Pucher stresses for a masonry helicoidal stair in Naples, Rione Sanità, designed by Ferdinando Sanfelice. Two distinct stress function $F_1(\rho, \vartheta)$, $F_2(\rho, \vartheta)$ have been examined as solution of the equilibrium problem.

The helical vault of the stair has a circular cross section. Its extrados and intrados are respectively the graphs of the functions:

$$\begin{aligned} \mathbf{y}_{\mathrm{intr}} &= \rho\cos\vartheta\,\mathbf{e_1} + \rho\,\mathrm{sen}\vartheta\,\mathbf{e_2} + \left(\sqrt{R_s^2 - (-R_p - R_s + \rho)^2}\right)\mathbf{e_3} \\ \mathbf{y}_{\mathrm{extr}} &= \rho\cos\vartheta\,\mathbf{e_1} + \rho\,\mathrm{sen}\vartheta\,\mathbf{e_2} + \left(\sqrt{\left(R_s + s\right)^2 - (-R_p - R_s + \rho)^2}\right)\mathbf{e_3} \end{aligned} \tag{30}$$

While the equation of the membrane surface contained into the vault is given by:

$$f = \rho\cos\vartheta\,\mathbf{e_1} + \rho\,\mathrm{sen}\vartheta\,\mathbf{e_2} + \left(\sqrt{\frac{1}{R_s^2}\left[R_s^2\,\delta_1 - \left(R_p^2 - 2R_p(R_s - \rho) + 2R_s\rho - \rho^2\right)(R_s - \delta_1 + \delta_2)\right]}\right)\mathbf{e_3} \tag{31}$$

where ρ and ϑ are respectively the plane angle and the plane radius with respect to the center of the stair, R_p the internal radius of the stair (0,25 m), R_s the half span of the cross section of the vault (0,9 m), δ_1 (0.35 m) and δ_2 (0.12 m) the distances from intrados at springing and key respectively.

A picture of the above membrane surface is reported in Figure 2, together with the stress function chosen:

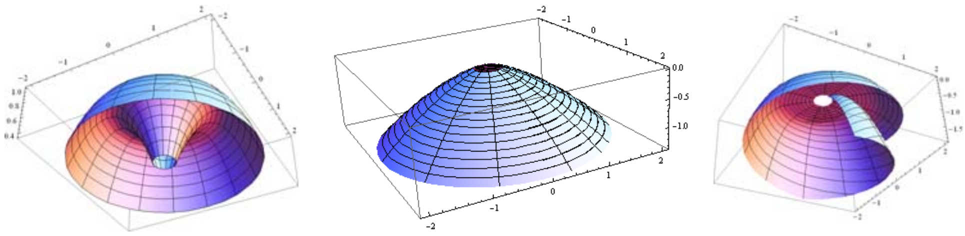

Figure 2: Membrane surface (left) and stress function F_1 (center) and F_2 (right)

The real geometry of the problem suggest a non-axisymmetric representation of the above functions modifying the analytical expressions in (31) adding the term $\frac{H\vartheta}{2\pi}$ to the third component, where H is the inter-floor height (5.7 m).

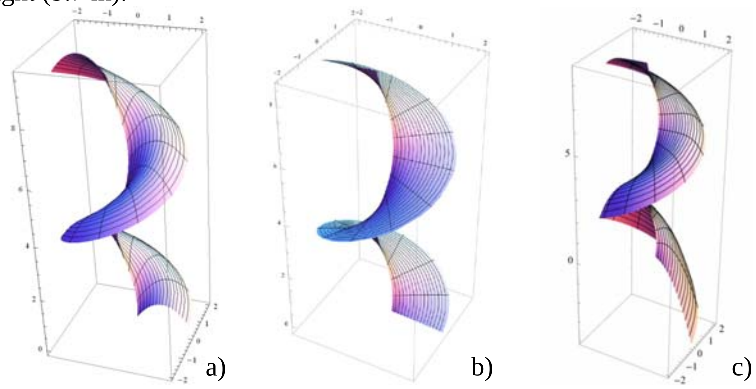

Figure 3: Membrane surface (a) and stress function F_1 (b) and F_2 (c)

Two different stress function in case of constant load, eliminating singularity, can be expressed respectively by:

$$F_1 = \rho\cos\vartheta\,\mathbf{e_1} + \rho\,\mathrm{sen}\vartheta\,\mathbf{e_2} + \left(\frac{-pR_s^2(R_p - R_s)\rho - \frac{1}{2}pR_s^2\rho^2}{4(R_s - \delta_1 + \delta_2)} - \frac{H\vartheta}{2\pi} \right)\mathbf{e_3} \tag{32}$$

$$F_2 = \rho\cos\vartheta\,\mathbf{e_1} + \rho\,\mathrm{sen}\vartheta\,\mathbf{e_2} + \left(\frac{2\pi(R_s - \delta_1 + \delta_2)\rho^2\ e^{\frac{HR_s^2\vartheta}{2\pi(R_s-\delta_1+\delta_2)\rho^2}}}{HR_s^2} + \frac{pR_s^2(R_p - R_s)\rho - \frac{1}{2}pR_s^2\rho^2}{4(R_s - \delta_1 + \delta_2)} \right)\mathbf{e_3} \tag{33}$$

That in the case of the geometrical parameters in (2) and for unit load become respectively, see Figure 3 b) and 3 c):

$$F_1 = \rho\cos\vartheta\ \mathbf{e_1} + \rho\,\mathrm{sen}\vartheta\ \mathbf{e_2} + \left(-0.034757\rho - 0.015112\rho^2 \right)\mathbf{e_3} \tag{34}$$

$$F_2 = \rho\cos\vartheta\ \mathbf{e_1} + \rho\,\mathrm{sen}\vartheta\ \mathbf{e_2} + \left[-0.91172\ \rho^2\ e^{\frac{-1.096749}{\rho^2}} + \left(-0.034757\rho - 0.015112\rho^2\right) \right]\mathbf{e_3} \tag{35}$$

The Pucher stresses for the F_1 stress function have the form:

$$S_{\rho\rho}\left(F_1\right)=\frac{p\,R_s^2\,(R_p+R_s+\rho)}{4\left(R_s-\delta_1+\delta_2\right)\rho}\qquad S_{\vartheta\vartheta}\,(F_1)=\frac{p\,R_s^2}{4\left(R_s-\delta_1+\delta_2\right)}\qquad S_{\rho\vartheta}\,(F_1)=\frac{H}{2\pi\,\rho^2} \tag{36}$$

while for the F_2 stress function are:

$$\begin{aligned}
S_{\rho\rho}\left(F_2\right)&=-\frac{p\,R_s^2\left(R_p+R_s+\rho\right)}{4\rho\left(R_s-\delta_1+\delta_2\right)}-e^{\frac{-HR_s^2\vartheta}{2\pi(R_s-\delta_1+\delta_2)\rho^2}}\,\frac{H^2R_s^4+4\pi HR_s^2\left(R_s-\delta_1+\delta_2\right)\vartheta\rho^2+8\pi^2\rho^4\left(R_s-\delta_1+\delta_2\right)^2}{2H\pi\,R_s^2\,\rho^4\left(R_s-\delta_1+\delta_2\right)}\\
S_{\vartheta\vartheta}\left(F_2\right)&=-\frac{p\,R_s^2\left(R_p+R_s+\rho\right)}{4\left(R_s-\delta_1+\delta_2\right)}-2\,e^{\frac{-HR_s^2\vartheta}{2\pi(R_s-\delta_1+\delta_2)\rho^2}}\,\frac{H^2R_s^4\vartheta^2+\pi HR_s^2\left(R_s-\delta_1+\delta_2\right)\vartheta\rho^2+2\pi^2\rho^4\left(R_s-\delta_1+\delta_2\right)^2}{H\pi\,R_s^2\left(R_s-\delta_1+\delta_2\right)\rho^4}\\
S_{\rho\vartheta}\left(F_2\right)&=e^{-\frac{HR_s^2\vartheta}{2\pi(R_s-\delta_1+\delta_2)\rho^2}}\,\frac{-H\,R_s^2\vartheta+\pi\rho^2\left(R_s-\delta_1+\delta_2\right)^2}{\pi\left(R_s-\delta_1+\delta_2\right)\rho^4}
\end{aligned} \tag{37}$$

The level curves and 3D plots of the Pucher stresses $F_1(\rho,\vartheta)$ and $F_2(\rho,\vartheta)$ are reported in the following pictures:

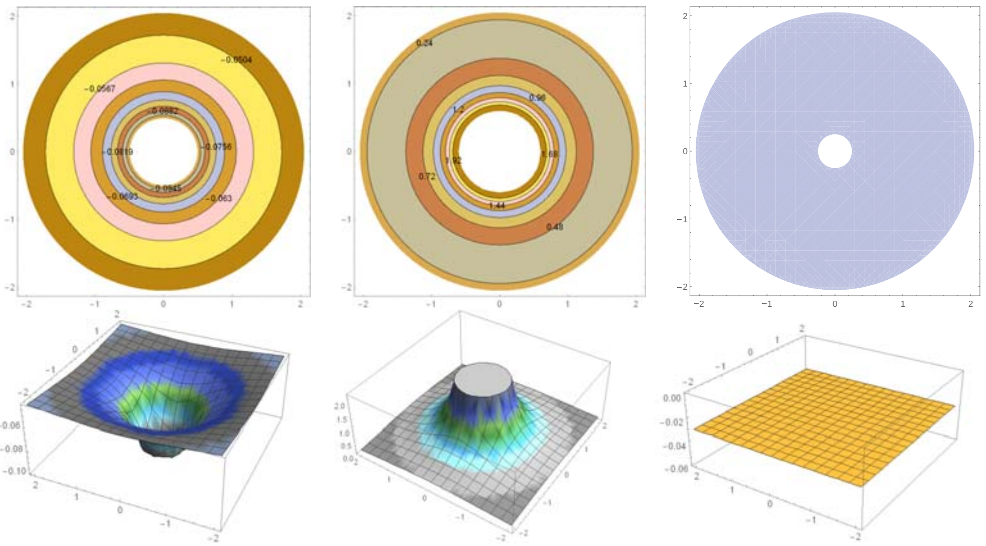

Figure 4: Pucher stresses $S_{\rho\rho}$ (left), $S_{\rho\vartheta}$ (center) and $S_{\vartheta\vartheta}$ (right) for the stress function $F_1(\rho,\vartheta)$

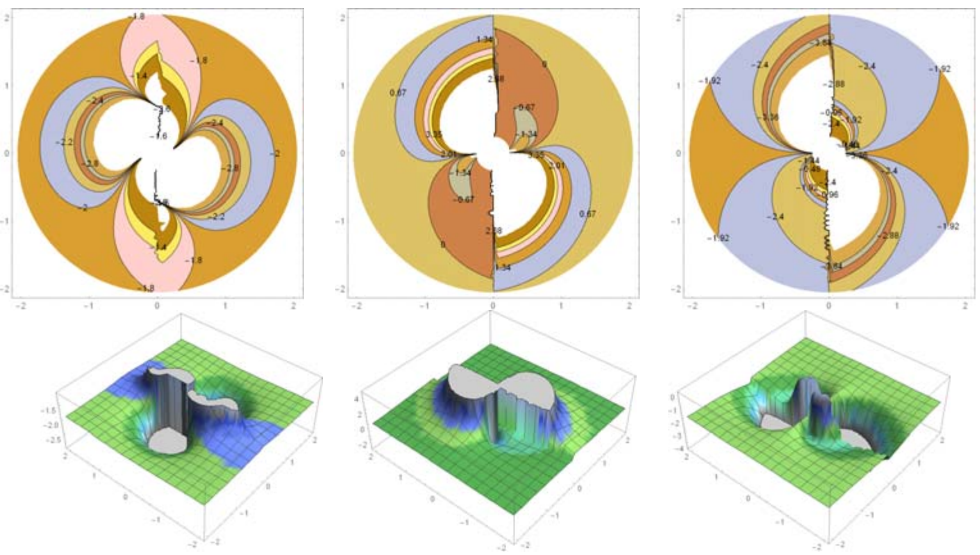

Figure 5: Pucher stresses $S_{\rho\rho}$ (left), $S_{\rho\vartheta}$ (center) and $S_{\vartheta\vartheta}$ (right) for the stress function $F_2(\rho,\vartheta)$

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