# BILOCAL *-AUTOMORPHISMS OF $B(H)$

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Abstract. In this note we show that the bilocal *-automorphisms of the $C^*$-algebra $B(H)$ of all bounded linear operators acting on a complex infinite dimensional separable Hilbert space $H$ are precisely the unital algebra *-endomorphisms of $B(H)$.

The study of local derivations of operator algebras has been initiated by Kadison [5] and Larson and Sourour [8]. A linear map $\Delta$ on an algebra is called a local derivation if for each point in the algebra the value of $\Delta$ at that point coincides with the value of a derivation at the point (the derivation may vary from point to point). In [5] and [8] results were presented which show that in certain settings local derivations are in fact derivations.

As for automorphisms of operator algebras, the first hint to the concept of local automorphisms appeared in [7] (see Some concluding remarks (5) in [7]). The definition is straightforward: a local automorphism of a given Banach algebra $\mathcal{A}$ is a linear map $\phi : \mathcal{A} \to \mathcal{A}$ with the property that for every $x \in \mathcal{A}$ there exists an (algebra) automorphism $\phi_x$ of $\mathcal{A}$ such that $\phi(x) = \phi_x(x)$. In the paper [8], Larson and Sourour proved that if $X$ is an infinite dimensional Banach space, then every surjective local automorphism of the algebra $B(X)$ of all bounded linear operators on $X$ is an automorphism. In [1], Brešar and Šemrl showed that in the case of an infinite dimensional separable Hilbert space $H$ the assumption of surjectivity can be relaxed, i.e., every local automorphism of $B(H)$ is an automorphism. Since then considerable attention has been paid to those concepts, and several results have been obtained and published concerning local derivations and local automorphisms. For some more details see Introduction 0.5 and Chapter 3 in [9].

In the paper [12] Zhu and Xiong introduced the interesting notion of bilocal derivations. If $X$ is a Banach space and $\mathcal{A}$ is a subalgebra of $B(X)$, then the linear map $\Theta : \mathcal{A} \to \mathcal{A}$ is called a bilocal derivation if for every $T \in \mathcal{A}$ and $x \in X$ there exists a derivation $\delta_{T,x} : \mathcal{A} \to \mathcal{A}$ such that $\Theta(T)x = \delta_{T,x}(T)x$. One of the main results in [12] tells us that if $\mathcal{A}$ is a Banach subalgebra of $B(X)$ which contains all finite rank operators and the identity,

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then every bilocal derivation of $\mathcal{A}$ is necessarily a derivation. This is a remarkable strengthening of the result obtained by Larson and Sourour in [8] for local derivations of $B(X)$.

Although there are a number of papers on local automorphisms of operator algebras in the literature, it is somewhat surprising that no one has addressed relating questions for transformations that could be called bilocal automorphism or bilocal *-automorphisms. The aim of this note is to point out that, unlike with derivations, the concept of bilocality when applied for *-automorphisms is in fact weak to provide results which would show that bilocal *-automorphisms are automatically *-automorphisms. Namely, we have the following characterization.

**Theorem 1.** *Let $H$ be a complex Hilbert space and $\phi : B(H) \to B(H)$ a linear transformation. If $H$ is infinite dimensional and separable, then the following two assertions are equivalent:*

(i) *For every $T \in B(H)$ and $x \in H$ there exists an algebra *-automorphism $\phi_{T,x}$ of $B(H)$ such that $\phi(T)x = \phi_{T,x}(T)x$;*

(ii) *$\phi$ is a unital algebra *-endomorphism of $B(H)$.*

*If $H$ is finite dimensional, then (i) holds if and only if $\phi$ is either an algebra *-automorphism or an algebra *-antiautomorphism of $B(H)$.*

*Proof.* It is well known that every algebra *-automorphism of $B(H)$ is implemented by a unitary operator, i.e., it is of the form $A \mapsto UAU^*$ with some unitary $U$ on $H$. Similarly, every algebra *-antiautomorphism of $B(H)$ is of the form $A \mapsto UA^{tr}U^*$, where $U$ is again unitary and $^{tr}$ denotes the transposition with respect to a given complete orthonormal system in $H$. See, e.g., Theorem A.8 in [9].

Assume that $H$ is infinite dimensional separable and $\phi : B(H) \to B(H)$ is a linear transformation which satisfies (i). By the bilocal property of $\phi$ one can easily check that $\phi$ is contractive, $\phi(I) = I$ and that $\phi(V)$ is an isometry for every isometry $V \in B(H)$.

Let $A \in B(H)$ be self-adjoint. For every $x \in H$ we have a unitary $U_{A,x} \in B(H)$ such that

$$\langle \phi(A)x, x\rangle = \langle U_{A,x}AU_{A,x}^*x, x\rangle = \langle AU_{A,x}^*x, U_{A,x}^*x\rangle \in \mathbb{R}.$$

This gives us that $\phi(A)$ is also self-adjoint. It is now clear that $\phi$ sends self-adjoint isometries into self-adjoint isometries. Those elements of $B(H)$ can be characterized as the operators of the form $2P - I$ with some projection (i.e., self-adjoint idempotent) $P \in B(H)$. Since $\phi(I) = I$, it then follows that $\phi$ sends projections to projections. However, it is a folk result by now that any bounded linear transformation on $B(H)$ (or even on some more general algebras) which preserves the projections is a so-called Jordan *-endomorphism. For $\phi$ this means that it satisfies $\phi(A)^2 = \phi(A^2)$ and $\phi(A)^* = \phi(A^*)$, $A \in B(H)$. This result can be found, for example, in Appendix in [9], see Theorem A.4. By an important theorem of Størmer, any Jordan *-homomorphism from a $C^*$-algebra into $B(H)$ can be decomposed as a direct sum of an algebra *-homomorphism and an algebra *-antihomomorphism (see, e.g., Theorem A.6 [9]). Using this result and the representation theory of $B(H)$ (see Section 10.4 in [6]) the structure of Jordan *-endomorphisms of $B(H)$ can be described completely (cf. Theorem A.11 [9]). Namely, it follows that we have (finite or countably infinite) collections $\{U_\alpha\}$ and $\{V_\beta\}$ of isometries on $H$ with pairwise orthogonal ranges such that

$$\phi(A) = \sum_\alpha U_\alpha A U_\alpha^* + \sum_\beta V_\beta A^{tr} V_\beta^*, \quad A \in B(H). \tag{1}$$

Since $\phi(I) = I$, we have $\sum_\alpha U_\alpha U_\alpha^* + \sum_\beta V_\beta V_\beta^* = I$. Now, plug a proper isometry $S$ into the place of $A$ above. Since $\phi(S)$ is also an isometry, we infer that the antihomomorphic part of $\phi$, i.e., the second sum in (1) is in fact missing. Therefore, we have

$$\phi(I) = I, \quad \phi(A) = \sum_\alpha U_\alpha A U_\alpha^*, \quad A \in B(H).$$

It follows that $\phi$ is a unital algebra *-endomorphism of $B(H)$.

Conversely, assume that $\phi$ is a unital algebra *-endomorphism. Just as above, $\phi$ can be written in the form

$$\phi(A) = \sum_\alpha U_\alpha A U_\alpha^*, \quad A \in B(H), \tag{2}$$

where the $U_\alpha$'s are isometries with pairwise orthogonal ranges and $\sum_\alpha U_\alpha U_\alpha^* = I$ (again, the collection $\{U_\alpha\}$ is necessarily countable).

Pick any operator $A \in B(H)$ and unit vector $x \in H$. We are looking for a unitary operator $U$ on $H$ (that may depend on $A$ and $x$) which satisfies $\phi(A)x = UAU^*x$. Apparently, the problem can be solved if and only if there is a vector $y \in H$ and a unitary operator $U$ such that

$$U^*x = y, \quad U^*\phi(A)x = Ay.$$

It is easy to see that we can find such a unitary $U \in B(H)$ precisely when there is a unit vector $y \in H$ such that

$$\|\phi(A)x\| = \|Ay\|, \quad \langle \phi(A)x, x\rangle = \langle Ay, y\rangle.$$

We use the decomposition (2) of $\phi$ to see that this is equivalent to the assertion that we can find a unit vector $y \in H$ such that

$$\sum_\alpha \|AU_\alpha^* x\|^2 = \sum_\alpha \|U_\alpha A U_\alpha^* x\|^2 = \|\sum_\alpha U_\alpha A U_\alpha^* x\|^2 = \|\phi(A)x\|^2 = \|Ay\|^2$$

and, similarly,

$$\sum_\alpha \langle AU_\alpha^* x, U_\alpha^* x\rangle = \langle \phi(A)x, x\rangle = \langle Ay, y\rangle.$$

These two equalities can be rewritten as

$$\sum_{\substack{\alpha \\ U_\alpha^* x \neq 0}} \|U_\alpha^* x\|^2 \langle A^* A(U_\alpha^* x/\|U_\alpha^* x\|), U_\alpha^* x/\|U_\alpha^* x\|\rangle =$$

$$\sum_\alpha \|AU_\alpha^* x\|^2 = \|\phi(A)x\|^2 = \|Ay\|^2 = \langle A^* Ay, y\rangle$$

and, similarly,

$$\sum_{\substack{\alpha \\ U_\alpha^* x \neq 0}} \|U_\alpha^* x\|^2 \langle A(U_\alpha^* x/\|U_\alpha^* x\|), U_\alpha^* x/\|U_\alpha^* x\|\rangle =$$

$$\sum_\alpha \langle AU_\alpha^* x, U_\alpha^* x\rangle = \langle \phi(A)x, x\rangle = \langle Ay, y\rangle.$$

We have

$$\sum_\alpha \|U_\alpha^* x\|^2 = \sum_\alpha \langle U_\alpha U_\alpha^* x, x\rangle = \langle x, x\rangle = 1,$$

therefore, the first lines in the preceeding two displayed formulae can be interpreted as countable (finite or countably infinite) convex combinations. At this point we refer to a remarkable result by Li, Poon and Sze [4] concerning the convexity of the Davis-Wielandt shell of an operator. For an $A \in B(H)$ this set is defined by

$$DW(A) = \{(\langle Ax, x\rangle, \langle A^* Ax, x\rangle) \,:\, x \in H, \|x\| = 1\}.$$

Theorem 3.3 in [4] tells us that if $\dim H \geq 3$, then $DW(A)$ is convex. We also need to recall a result by Rubin and Wesler [11] which asserts that in Euclidean spaces the convex sets are necessarily $\sigma$-convex in the sense that the sum of any convergent countably infinite convex combination of elements of a convex set necessarily belongs to the set in question. These two results and the above discussion shows that the original problem has a solution, i.e., for any operator $A \in B(H)$ and unit vector $x \in H$ we have a unitary operator $U$ on $H$ which satisfies $\phi(A)x = UAU^* x$. Consequently, any unital algebra *-endomorphism $\phi$ of $B(H)$ is a bilocal *-automorphism meaning that it satisfies (i).

Assume now that $H$ is of finite dimension and let $\phi$ be a linear transformation on $B(H)$ which has the property (i). Following the argument given in the infinite dimensional case above, we see that $\phi$ is a unital Jordan *-endomorphism. Since its kernel is a proper Jordan ideal of $B(H)$, in the present finite dimensional case we obtain that $\phi$ is injective and hence bijective. Therefore, $\phi$ is a Jordan *-automorphism. It is a consequence of a celebrated result of Herstein that any Jordan homomorphism from an algebra onto a prime algebra is necessarily an isomorphism or an antiisomorphism (see, e.g., Theorem A.7 in [9]). Therefore, we obtain that $\phi$ is either an algebra *-automorphism or an algebra *-antiautomorphism of $B(H)$.

Conversely, if $\phi$ is an algebra *-automorphism of $B(H)$, then it obviously has property (i). As for the case where $\phi$ is an algebra *-antiautomorphism we can easily see that it is sufficient to consider the transformation $\phi(A) = A^{tr}$, $A \in B(H)$ and verify that it has the property (i). The first part of the argument which has been employed in the infinite dimensional case to prove the implication (ii)$\Rightarrow$(i) shows that this particular map has the bilocal property (i) if and only if we have the inclusion $DW(A^{tr}) \subset DW(A)$ (and hence also the equality $DW(A^{tr}) = DW(A)$) for every $A \in B(H)$. But this fact has been proven in [4, Theorem 3.7] (we emphasize finite dimensionality, this relation does not hold in infinite dimension, see [4, Example 3.5]).

The proof of the theorem is complete. □

As we see above bilocal *-automorphisms of $B(H)$ are not necessarily *-automorphisms. One of the possibilities to obtain a result in which we do conclude that the local transformations under consideration are in fact automorphisms is to pose a more restrictive condition that could be called 3-locality. Indeed, we have the following corollary.

**Corollary 2.** *Let $H$ be a complex infinite dimensional separable Hilbert space. Assume $\phi : B(H) \to B(H)$ is a linear transformation with the property that for any operator $A \in B(H)$ and vectors $x, y \in H$ we have an algebra *-automorphism $\phi_{A,x,y}$ of $B(H)$ such that*

$$\phi(A)x = \phi_{A,x,y}(A)x, \quad \phi(A)y = \phi_{A,x,y}(A)y.$$

*Then $\phi$ is an algebra *-automorphism of $B(H)$.*

*Proof.* Since $\phi$ is also a bilocal *-automorphism, i.e., it has property (i), we know that it is necessarily a unital *-endomorphism of $B(H)$. Moreover, we assert that $\phi$ maps rank-one operators to rank-one operators. Indeed, selecting any rank-one operator $A \in B(H)$, for any $x, y \in H$ we have a unitary $U_{A,x,y} \in B(H)$ such that

$$\phi(A)x = U_{A,x,y}AU_{A,x,y}^*x, \quad \phi(A)y = U_{A,x,y}AU_{A,x,y}^*y.$$

It follows trivially that $\phi(A)x, \phi(A)y$ are linearly dependent for all pairs $x, y \in H$ of vectors which gives us that $\phi(A)$ is rank-one. Therefore, in the form (2) of $\phi$, on the right hand side in the sum we must have one and only one term. This means that $\phi$ is an algebra *-automorphism of $B(H)$ and the proof is complete. □

One may naturally ask what happens in the corollary if $H$ is finite dimensional. We leave it as an open problem. Observe that the only question that should be answered here is whether the transformation $A \mapsto A^{tr}$ has the above 3-local property or not. If the dimension is 2, then it has which is the consequence of the fact that every $2 \times 2$ complex matrix is unitarily similar to its transpose. This fact follows, for example, from the nice result of Murnaghan [10] stating that for $2 \times 2$ matrices, the triple $(\operatorname{Tr} X, \operatorname{Tr} X^2, \operatorname{Tr} X^*X)$ forms a complete set of unitary invariants. As for the $n \times n$ case with $n \geq 3$, there are matrices which are not unitarily similar to their transpose (see, [3], solution of Problem 159, and also [2], p. 13). Of course, this does not answer our question on the 3-locality of the transposition map raised above but shows that the question may be very nontrivial (also recall the result stating that the Davis-Wielandt shells of a matrix and its transpose coincide what we have used in the proof of the finite dimensional part of our theorem).

We conclude our note by proposing two problems. The first one is to consider bilocal *-automorphisms of subalgebras of $B(H)$ (for example, standard *-algebras of operators). As for the second one, we emphasize that in this paper we have dealt with transformations which in some sense locally belong to the group of algebra *-automorphisms rather than to the group of all algebra automorphisms of $B(H)$. To study related questions for this latter and larger group, to investigate bilocal automorphisms even bijective ones we find it an interesting problem that seems highly nontrivial and calls for essentially different ideas.

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