

# ON ZEROS OF RECIPROCAL POLYNOMIALS

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ABSTRACT. The purpose of this paper is to show that all zeros of the reciprocal polynomial

$$P_m(z) = \sum_{k=0}^m A_k z^k \quad (z \in \mathbb{C})$$

of degree  $m \geq 2$  with real coefficients  $A_k \in \mathbb{R}$  (i.e.  $A_m \neq 0$  and  $A_k = A_{m-k}$  for all  $k = 0, \dots, \lfloor \frac{m}{2} \rfloor$ ) are on the unit circle, provided that the "coefficient condition"

$$|A_m| \geq \sum_{k=1}^{m-1} |A_k - A_m|$$

is satisfied.

Moreover, if the "coefficient condition" holds, then all zeros  $e^{iu_j}$  ( $j = 1, 2, \dots, m$ ) can be arranged such that

$$\left| e^{i \frac{2\pi j}{m+1}} - e^{iu_j} \right| < \frac{\pi}{m+1} \quad (j = 1, \dots, m).$$

If  $m = 2n + 1$  is odd, then  $-1 = e^{iu_{n+1}}$  is always a zero, and all zeros of  $P_{2n+1}$  are single.

If  $m = 2n$  is even, if the "coefficient condition" holds with equality and if

$$\operatorname{sgn} A_{2n} = \operatorname{sgn} (-1)^{k+1} (A_k - A_{2n})$$

for all  $k = 1, 2, \dots, n$  with  $A_k - A_{2n} \neq 0$ , then  $u_n = u_{n+1} = \pi$ , the number  $-1 = e^{iu_n} = e^{iu_{n+1}}$  is a double zero of  $P_{2n}$ . Otherwise all zeros of  $P_{2n}$  are single.

## 1. INTRODUCTION

The Coxeter transformation was introduced in the representation theory of finite dimensional algebras (see [2]). The characteristic polynomial of the Coxeter transformation of an oriented graph whose underlying graph is a wild star is a Salem polynomial (see [3], [4]).

Allowing circles in the underlying graph, the spectral properties of the Coxeter transformations get much more complicated. These properties are related to polynomials of the form

$$l(z^m + z^{m-1} + \dots + z + 1) + (z^k + z^{m-k}) \quad (z \in \mathbb{C})$$

where  $m, k$  are fixed non-negative integers with  $m \geq 2$ ,  $1 \leq k \leq \lfloor \frac{m}{2} \rfloor$  and  $l$  is a fixed real number.

The zeros of the first expression  $l(z^m + z^{m-1} + \dots + z + 1)$  are

$$\epsilon_j = e^{i \frac{j}{m+1} 2\pi} \quad (j = 1, 2, \dots, m)$$

the  $(m + 1)$ st roots of unity except 1, *they are on the unit circle*. It is surprising that adding  $z^k + z^{m-k}$  to the first expression the polynomial obtained inherits this property. Moreover, *not just all zeros remain on the unit circle but they move away from  $\epsilon_j$  just a little* even if we add a

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linear combination  $\sum_{k=1}^{\lfloor \frac{m}{2} \rfloor} a_k(z^k + z^{m-k})$  to the expression  $l(z^m + z^{m-1} + \dots + z + 1)$ , *provided that  $|l|$  is large enough*. This leads to the main result of the paper: giving a sufficient condition for reciprocal polynomials to have all of their zeros on the unit circle and also giving the location of the zeros.

Our basic tool is a transformation of semi-reciprocal polynomials called the Chebyshev transformation. Although this transformation seems to be well known we could not find a suitable reference. In Section 2, based on [1], we summarize the properties of the Chebyshev transformation. In Section 3 we formulate our results and prove them. In Section 4 we discuss the necessity of our sufficient condition.

## 2. THE CHEBYSHEV TRANSFORMATION

A polynomial  $p$  of the form  $p(z) = \sum_{j=0}^m a_j z^j$  ( $z \in \mathbb{C}$ ) where  $a_j \in \mathbb{C}$  are given numbers with  $a_m \neq 0$ ,  $a_j = a_{m-j}$  ( $j = 0, \dots, \lfloor \frac{m}{2} \rfloor$ ) is called a *reciprocal polynomial of degree  $m$* .

We need a more general class of reciprocal polynomials (of even degree).

**Definition 1.** A polynomial  $p$  of the form

$$(1) \quad p(z) = \sum_{j=0}^{2n} a_j z^j \quad (z \in \mathbb{C})$$

where  $n \in \mathbb{N}$ ,  $a_0, \dots, a_{2n} \in \mathbb{R}$  and

$$(2) \quad a_j = a_{2n-j} \quad (j = 0, \dots, n-1)$$

is called a *real semi-reciprocal polynomial of degree at most  $2n$* . If  $a_{2n} \neq 0$  we call  $p$  a *real reciprocal polynomial of degree  $2n$* .

Denote by  $\mathcal{R}_{2n}$  the set of all real semi-reciprocal polynomials of degree at most  $2n$ .

If  $p \in \mathcal{R}_{2n}$ ,  $p \neq o$  ( $o$ =the zero polynomial), then there is an integer  $k$ ,  $0 \leq k \leq n$ , such that

$$(3) \quad a_{2n} = a_{2n-1} = \dots = a_{n+k+1} = 0 = a_{n-k-1} = \dots = a_0 \text{ but } a_{n+k} = a_{n-k} \neq 0.$$

Hence

$$(4) \quad p(z) = \sum_{j=0}^{2n} a_j z^j = z^n \left[ a_{n+k} \left( z^k + \frac{1}{z^k} \right) + \dots + a_{n+1} \left( z + \frac{1}{z} \right) + a_n \right].$$

Let  $T_j$  be the  $j$ th Chebyshev polynomial of the first kind, defined by

$$T_j(\cos x) = \cos jx \quad (j = 0, 1, \dots).$$

With  $z + \frac{1}{z} = x$  we have  $z^j + \frac{1}{z^j} = C_j(x)$  ( $j = 1, 2, \dots$ ) (see e.g. [6], p. 224) where

$$C_j(x) := 2T_j\left(\frac{x}{2}\right) \quad (x \in \mathbb{C}, j = 1, 2, \dots)$$

are the normalized Chebyshev polynomials of the first kind. For us it will be now more convenient to define  $C_0$  by

$$C_0(x) := T_0(x) \quad (x \in \mathbb{C}).$$

Hence, from (4)

$$(5) \quad p(z) = z^n \sum_{j=0}^k a_{n+j} C_j(x) = a_{n+k} z^n \prod_{j=1}^k (x - \alpha_j)$$

where  $\alpha_j \in \mathbb{C}$  ( $j = 1, \dots, k$ ) are the zeros of the polynomial  $\sum_{j=0}^k a_{n+j} C_j(x)$ . Equation (5) remains true in the case when  $k = 0$ , i.e.  $p(z) = a_n z^n$  if we agree that

$$(6) \quad \prod_{j=1}^0 b_j := 1.$$

Going back to the variable  $z$  we get that

$$p(z) = a_{n+k} z^{n-k} \prod_{j=1}^k z \left( z + \frac{1}{z} - \alpha_j \right) = a_{n+k} z^{n-k} \prod_{j=1}^k (z^2 - \alpha_j z + 1).$$

With this we have justified

**Proposition 1.** *Every non-zero polynomial  $p \in \mathcal{R}_{2n}$  has the decomposition*

$$(7) \quad p(z) = a_{n+k} z^{n-k} \prod_{j=1}^k (z^2 - \alpha_j z + 1)$$

where  $\alpha_1, \dots, \alpha_k \in \mathbb{C}$ ,  $a_{n+k} \neq 0$  for some  $k$  with  $0 \leq k \leq n$  and the convention (6) is adopted. If  $p \in \mathcal{R}_{2n}$  is a reciprocal polynomial of degree  $2n$ , then (7) holds with  $k = n$ .

**Definition 2.** *The Chebyshev transform of a non-zero polynomial  $p \in \mathcal{R}_{2n}$  having the decomposition (7) is defined by*

$$(8) \quad \mathcal{T}p(x) = a_{n+k} \prod_{j=1}^k (x - \alpha_j)$$

(with (6) adopted) while for the zero polynomial  $p = 0$  let

$$(9) \quad \mathcal{T}0(x) = 0.$$

It is clear that  $\mathcal{T}$  maps  $\mathcal{R}_{2n}$  into the set  $\mathcal{P}_n$  of all polynomials of degree  $\leq n$  with real coefficients.

**Proposition 2.** *The Chebyshev transform  $\mathcal{T}$  is an isomorphism of the (real) vector space  $\mathcal{R}_{2n}$  onto  $\mathcal{P}_n$ .*

*Proof.* (i)  $\mathcal{T}$  preserves the addition and the multiplication by a real constant. Using (5) and (3) (to include also the zero coefficients into the sum) we can write  $\mathcal{T}p$  into the form

$$\mathcal{T}p(x) = a_{n+k} \prod_{j=1}^k (x - \alpha_j) = \sum_{j=0}^k a_{n+j} C_j(x) = \sum_{j=0}^n a_{n+j} C_j(x)$$

and the last form of  $\mathcal{T}p$  is valid also for the zero polynomial. Taking now another  $q \in \mathcal{R}_{2n}$  with  $q(z) = \sum_{j=0}^{2n} b_j z^j$  ( $b_j = b_{2n-j}$  for  $j = 0, \dots, n-1$ ) and constants  $\alpha, \beta \in \mathbb{R}$  we have

$$(\alpha p + \beta q)(z) = \sum_{j=0}^{2n} (\alpha a_j + \beta b_j) z^j$$

thus

$$\begin{aligned} \mathcal{T}(\alpha p + \beta q)(x) &= \sum_{j=0}^n (\alpha a_{n+j} + \beta b_{n+j}) C_j(x) \\ &= \alpha \sum_{j=0}^n a_{n+j} C_j(x) + \beta \sum_{j=0}^n b_{n+j} C_j(x) = \alpha (\mathcal{T}p(x)) + \beta (\mathcal{T}q(x)). \end{aligned}$$

(ii)  $\mathcal{T}$  maps onto  $\mathcal{P}_n$ . Every polynomial  $\tilde{r} \in \mathcal{P}_n$  can uniquely be written as a (real) linear combination of  $C_0, C_1, \dots, C_n$  in the form  $\tilde{r}(x) = \sum_{j=0}^n A_{n+j} C_j(x)$  ( $A_{n+j} \in \mathbb{R}$ ). With  $r(z) :=$

$\sum_{j=0}^{2n} A_j z^j$  where  $A_j := A_{2n-j}$  for  $j = 0, \dots, n-1$  we have  $r \in \mathcal{R}_{2n}$  and

$$\mathcal{T}r = \tilde{r}$$

proving our claim.

(iii)  $\mathcal{T}$  is one-to-one. Namely, if  $\mathcal{T}p = \mathcal{T}q$  for  $p, q \in \mathcal{R}_{2n}$ , then  $\mathcal{T}p - \mathcal{T}q = \mathcal{T}(p - q) = o$  hence, by (8), (9)  $p - q = o$ ,  $p = q$ .  $\square$

**Lemma 1.** (i) Let  $p$  be a real reciprocal polynomial of degree  $2n$ . Then all zeros of  $p$  are on the unit circle if and only if all zeros of its Chebyshev transform  $\mathcal{T}p$  are in the closed interval  $[-2, 2]$ .

(ii) Moreover, if all zeros  $\alpha_j$  of  $\mathcal{T}p$  are in  $[-2, 2]$ , written as  $\alpha_j = 2 \cos u_j$  with  $u_j \in [0, \pi]$  ( $j = 1, 2, \dots, n$ ), then all zeros of  $p$  are given by

$$e^{\pm i u_j} \quad (j = 1, 2, \dots, n).$$

The multiplicity of  $\alpha_j \neq \pm 2$  is the same as the multiplicities of  $e^{i u_j}$  and  $e^{-i u_j}$  ( $j = 1, 2, \dots, n$ ) while in the case of  $\alpha_j = \pm 2$  the multiplicities of the corresponding zeros  $e^{i u_j} = \pm 1$  of  $p$  are doubled.

*Proof.* (i) *Necessity.* Suppose that all zeros of  $p$  are on the unit circle. They can be arranged in conjugate pairs  $(\beta_1, \bar{\beta}_1), (\beta_2, \bar{\beta}_2) \dots (\beta_n, \bar{\beta}_n)$ . By assumption  $|\beta_j|^2 = \beta_j \bar{\beta}_j = 1$ ,  $\bar{\beta}_j = \frac{1}{\beta_j}$  ( $j = 1, \dots, n$ ), hence

$$p(z) = a_{2n} \prod_{j=1}^n (z - \beta_j)(z - \bar{\beta}_j) = a_{2n} \prod_{j=1}^n (z^2 - (\beta_j + \bar{\beta}_j)z + 1)$$

and

$$\mathcal{T}p(x) = a_{2n} \prod_{j=1}^n (x - (\beta_j + \bar{\beta}_j)).$$

It is clear that  $|\beta_j + \bar{\beta}_j| = |2 \operatorname{Re}(\beta_j)| \leq 2|\beta_j| = 2$ .

(i) *Sufficiency.* Assume that the Chebyshev transform has the form

$$\mathcal{T}p(x) = a_{2n} \prod_{j=1}^n (x - \alpha_j)$$

where  $a_{2n} \neq 0$  and  $\alpha_j \in [-2, 2]$  ( $j = 1, \dots, n$ ). Then

$$p(z) = a_{2n} \prod_{j=1}^n (z^2 - \alpha_j z + 1).$$

Since  $\alpha_j \in [-2, 2]$  we have  $z^2 - \alpha_j z + 1 = (z - \beta_j)(z - \bar{\beta}_j)$  with  $\beta_j \bar{\beta}_j = 1 = |\beta_j|^2$  proving that all zeros  $\beta_1, \bar{\beta}_1, \beta_2, \bar{\beta}_2, \dots, \beta_n, \bar{\beta}_n$  of  $p$  are on the unit circle.

(ii) We have  $\alpha_j = 2 \cos u_j = \beta_j + \bar{\beta}_j$ . Writing  $\beta_j$  as  $e^{iq_j}$  (here we may suppose that  $0 \leq q_j \leq \pi$ ) we obtain that  $2 \cos u_j = e^{iq_j} + e^{-iq_j} = 2 \cos q_j$  hence  $u_j = q_j$  ( $j = 1, 2, \dots, n$ ). The statement concerning the multiplicities is obvious.  $\square$

### 3. RESULTS AND PROOFS

**Theorem 1.** *All zeros of the (real reciprocal) polynomial*

$$(10) \quad h_m(z) = l(z^m + z^{m-1} + \dots + z + 1) + \sum_{k=1}^{\lfloor \frac{m}{2} \rfloor} a_k (z^{m-k} + z^k) \quad (z \in \mathbb{C})$$

of degree  $m$  where  $l, a_1, \dots, a_{\lfloor \frac{m}{2} \rfloor} \in \mathbb{R}$ ,  $l \neq 0$ ,  $m \in \mathbb{N}$ ,  $m \geq 2$ , are on the unit circle if

$$(11) \quad |l| \geq 2 \sum_{k=1}^{\lfloor \frac{m}{2} \rfloor} |a_k|.$$

Moreover, if (11) is satisfied, then for even  $m = 2n$  all zeros of  $h_m$  can be given as

$$e^{iu_j}, e^{-iu_j} \quad (j = 1, 2, \dots, n)$$

where

$$\frac{j - \frac{1}{2}}{m + 1} 2\pi < u_j < \frac{j + \frac{1}{2}}{m + 1} 2\pi \quad (j = 1, 2, \dots, n - 1)$$

$$\frac{n - \frac{1}{2}}{m + 1} 2\pi < u_n \leq \pi.$$

In the last inequality  $u_n \leq \pi$ , we have equality if and only if

$$(12) \quad |l| = 2 \sum_{k=1}^{\lfloor \frac{m}{2} \rfloor} |a_k| \text{ and } \operatorname{sgn} l = \operatorname{sgn} (-1)^{k+1} a_k \text{ for all } k = 1, 2, \dots, n \text{ with } a_k \neq 0.$$

If (12) holds, then  $-1 = e^{i\pi} = e^{-i\pi}$  is a double zero of  $h_m$  and all other zeros are single. For odd  $m = 2n + 1$  all zeros of  $h_m$  are single, they can be given as

$$-1, e^{iu_j}, e^{-iu_j} \quad (j = 1, 2, \dots, n)$$

where

$$\frac{j - \frac{1}{2}}{m + 1} 2\pi < u_j < \frac{j + \frac{1}{2}}{m + 1} 2\pi \quad (j = 1, 2, \dots, n).$$

*Remark 1.* The statement concerning the location of the zeros of  $h_m$  can also be formulated as follows.

If (11) is satisfied, then all the zeros  $e^{iu_j}$  ( $j = 1, 2, \dots, m$ ) of  $h_m$  can be arranged such that

$$|\epsilon_j - e^{iu_j}| < \frac{\pi}{m + 1} \quad (j = 1, \dots, m)$$

where, as in the introduction,  $\epsilon_j$  are the  $(m + 1)$ st roots of unity, except 1.

Namely, for even  $m = 2n$ , let  $u_j$  ( $j = 1, 2, \dots, n$ ) be the same as in Theorem 1 and  $u_{n+j} := 2\pi - u_{n+1-j}$  ( $j = 1, 2, \dots, n$ ). If (12) does not hold, then all zeros of  $h_m$  are single. If (12) holds, then  $u_n = u_{n+1} = \pi$  and  $-1 = e^{iu_n} = e^{iu_{n+1}}$  is a double zero and all other zeros are single.

For odd  $m = 2n + 1$  let  $u_j$  ( $j = 1, 2, \dots, n$ ) be the same as in Theorem 1,  $u_{n+1} := \pi$  and  $u_{n+1+j} := 2\pi - u_{n+1-j}$  ( $j = 1, 2, \dots, n$ ). The number  $-1 = e^{iu_{n+1}}$  is always a zero and all zeros are single.

*Proof.* The basic idea of our proof is the following. Assume that (11) holds and let

$$x_j = 2 \cos \frac{j + \frac{1}{2}}{m + 1} 2\pi \quad (j = 0, \dots, \left\lfloor \frac{m}{2} \right\rfloor).$$

If  $m = 2n$  is an even number, we show that  $\operatorname{sgn} \mathcal{T}h_{2n}(x_j) = \operatorname{sgn} (-1)^j \operatorname{sgn} l$  ( $j = 0, 1, \dots, n-1$ ) and  $\mathcal{T}h_{2n}(x_n) = 0$  if (12) holds, otherwise  $\operatorname{sgn} \mathcal{T}h_{2n}(x_j) = \operatorname{sgn} (-1)^j \operatorname{sgn} l$  ( $j = 0, \dots, n$ ).

If  $m = 2n + 1$  is odd, then  $h_{2n+1}(z) = (z + 1)\bar{h}_{2n}(z)$  with a suitable reciprocal polynomial  $\bar{h}_{2n}$  from  $\mathcal{R}_{2n}$ . We show that  $\operatorname{sgn} \mathcal{T}\bar{h}_{2n}(x_j) = \operatorname{sgn} l \operatorname{sgn} (-1)^j$  ( $j = 0, 1, \dots, n$ ).

Applying Lemma 1 completes the proof.

**Case 1:**  $m = 2n$ .

With the notation  $v_j(z) = z^j + z^{j-1} + \dots + 1 = \frac{z^{j+1} - 1}{z - 1}$ ,  $e_j(z) = z^j$ ,  $w_j(z) = z^{j+1}$  ( $j = 0, 1, \dots$ ) we have

$$h_{2n}(z) = lv_{2n}(z) + \sum_{k=1}^n a_k e_k(z) \cdot w_{2n-2k}(z),$$

$$\mathcal{T}h_{2n}(x) = l\mathcal{T}v_{2n}(x) + \sum_{k=1}^n a_k \mathcal{T}(e_k \cdot w_{2n-2k})(x).$$

The zeros of  $v_{2n}$  are the  $(2n + 1)$ st roots of unity, except 1:  $e^{\frac{2j\pi i}{2n+1}}$  ( $j = 1, 2, \dots, 2n$ ). They can be arranged into conjugate pairs:  $\left(e^{\frac{2j\pi i}{2n+1}}, e^{\frac{2(2n+1-j)\pi i}{2n+1}}\right) = \left(e^{\frac{2j\pi i}{2n+1}}, e^{-\frac{2j\pi i}{2n+1}}\right)$  ( $j = 1, \dots, n$ ), thus

$$\begin{aligned} v_{2n}(z) &= \prod_{j=1}^{2n} \left(z - e^{\frac{2j\pi i}{2n+1}}\right) = \prod_{j=1}^n \left(z - e^{\frac{2j\pi i}{2n+1}}\right) \left(z - e^{-\frac{2j\pi i}{2n+1}}\right) \\ &= \prod_{j=1}^n \left(z^2 - 2 \cos \frac{2j\pi}{2n+1} z + 1\right), \end{aligned}$$

$$\mathcal{T}v_{2n}(x) = \prod_{j=1}^n \left(x - 2 \cos \frac{2j\pi}{2n+1}\right).$$

Similarly, for each  $0 \leq k \leq n$  the zeros of  $w_{2n-2k}$  are the  $(2n - 2k)$ st roots of  $-1$ :  $e^{\frac{(2j-1)\pi i}{2n-2k}}$  ( $j = 1, \dots, 2n - 2k$ ). They can be arranged into conjugate pairs

$$\left(e^{\frac{(2j-1)\pi i}{2n-2k}}, e^{\frac{(2(2n-2k+1-j)-1)\pi i}{2n-2k}}\right) = \left(e^{\frac{(2j-1)\pi i}{2n-2k}}, e^{-\frac{(2j-1)\pi i}{2n-2k}}\right) \quad (j = 1, \dots, n - k).$$

Therefore

$$w_{2n-2k}(z) = \prod_{j=1}^{2n-2k} \left(z - e^{\frac{(2j-1)\pi i}{2n-2k}}\right) = \prod_{j=1}^{n-k} \left(z^2 - 2 \cos \frac{(2j-1)\pi}{2n-2k} z + 1\right),$$

$$\mathcal{T}(e_k w_{2n-2k})(x) = \prod_{j=1}^{n-k} \left(x - 2 \cos \frac{(2j-1)\pi}{2n-2k}\right).$$

Denote by  $U_n$  the  $n$ th Chebyshev polynomial of the second kind (see for example in [6]), defined by

$$U_n(\cos x) = \frac{\sin(n+1)x}{\sin x} \quad (n = 0, 1, \dots).$$

We claim that

$$(13) \quad \mathcal{T}v_{2n}(x) = U_n\left(\frac{x}{2}\right) + U_{n-1}\left(\frac{x}{2}\right),$$

$$(14) \quad \mathcal{T}(e_k \cdot w_{2n-2k})(x) = 2U_{n-k}\left(\frac{x}{2}\right).$$

To justify the first identity we note that

$$(15) \quad U_n(\cos y) + U_{n-1}(\cos y) = \frac{\sin(n+1)y + \sin ny}{\sin y} = 2 \frac{\sin \frac{(2n+1)y}{2} \cos \frac{y}{2}}{\sin y} = \frac{\sin \frac{(2n+1)y}{2}}{\sin \frac{y}{2}}.$$

The right hand side is zero if and only if  $y = \frac{2j\pi}{2n+1}$  ( $j \in \mathbb{Z} \setminus \{0\}$ ) hence all zeros of  $U_n\left(\frac{x}{2}\right) + U_{n-1}\left(\frac{x}{2}\right)$  are  $2 \cos \frac{2j\pi}{2n+1}$  ( $j = 1, \dots, n$ ). Since both sides of (13) are monics which have the same zeros, they are identical.

The zeros of  $T_p$  can be calculated easily from their definition, for  $p \in \mathbb{N}$  they are

$$\cos \frac{(2j-1)\pi}{2p} \quad (j = 1, \dots, p).$$

Thus for  $k < n$  the zeros of the monic  $2T_{n-k}(\frac{x}{2})$  are  $2 \cos \frac{(2j-1)\pi}{2n-2k}$  ( $j = 1, \dots, n-k$ ). They are the same as the zeros of  $\mathcal{T}(e_k \cdot w_{2n-2k})$ , hence (14) holds. It also holds for  $k = n$  since then both sides of (14) are equal to 2.

Next we evaluate  $\mathcal{T}h_{2n}$  at the points

$$x_j = 2 \cos \frac{j + \frac{1}{2}}{m+1} 2\pi \quad (j = 0, \dots, n)$$

of the interval  $[-2, 2]$ . Since  $x_j = 2 \cos y_j$  with  $y_j = \frac{j + \frac{1}{2}}{2n+1} 2\pi$  we have by (13), (14)

$$\begin{aligned} \mathcal{T}h_{2n}(x_j) &= l \left( U_n \left( \frac{x_j}{2} \right) + U_{n-1} \left( \frac{x_j}{2} \right) \right) + \sum_{k=1}^n 2a_k T_{n-k} \left( \frac{x_j}{2} \right) \\ &= 2 \left[ \frac{\frac{l}{2} \sin \frac{2n+1}{2} y_j}{\sin \frac{1}{2} y_j} + \sum_{k=1}^n a_k \cos(n-k)y_j \right] = 2 \left[ \frac{\frac{l}{2} (-1)^j}{\sin \frac{y_j}{2}} + \sum_{k=1}^n a_k \cos(n-k)y_j \right]. \end{aligned}$$

If  $j = 0, 1, \dots, n-1$ , then  $0 < \sin \frac{y_j}{2} < 1$ ,  $\sum_{k=1}^n |a_k \cos(n-k)y_j| \leq \sum_{k=1}^n |a_k|$  and by (11) the sign of the expression in the bracket is  $(-1)^j \operatorname{sgn} l$ .

If  $j = n$ , then  $y_n = \pi$  and the expression in the bracket is

$$\frac{l}{2} (-1)^n + \sum_{k=1}^n a_k (-1)^{n-k} = (-1)^n \left( \frac{l}{2} + \sum_{k=1}^n a_k (-1)^k \right).$$

Its sign is  $(-1)^n \operatorname{sgn} l$  if in (11) strict inequality holds or if in (11) we have equality and at least for one  $k$  ( $1 \leq k \leq n$ ) we have  $\operatorname{sgn} l = \operatorname{sgn} (-1)^k a_k$ . If we have equality in (11) and  $\operatorname{sgn} l = \operatorname{sgn} (-1)^{k+1} a_k$  for all  $k = 1, \dots, n$  such that  $a_k \neq 0$ , then the expression in the bracket is zero.

Thus either  $\operatorname{sgn} \mathcal{T}h_{2n}(x_j) = \operatorname{sgn} (-1)^j \operatorname{sgn} l$  ( $j = 0, \dots, n$ ) or  $\operatorname{sgn} \mathcal{T}h_{2n}(x_j) = \operatorname{sgn} (-1)^j \operatorname{sgn} l$  ( $j = 0, 1, \dots, n-1$ ) and  $\mathcal{T}h_{2n}(x_n) = 0$ . In both cases  $\mathcal{T}h_{2n}$  has  $n$  distinct zeros in the interval  $[-2, 2]$ . Writing these in the form  $2 \cos u_j$  with  $0 \leq u_1 \leq u_2 \leq \dots \leq u_n \leq \pi$  and applying Lemma 1 we can complete the proof in the first case.

**Case 2:**  $m = 2n + 1$ .

We have  $h_{2n+1}(z) = (z+1)\bar{h}_{2n}(z)$  with

$$\bar{h}_{2n}(z) = l\bar{v}_{2n}(z) + \sum_{k=1}^n a_k z^k \bar{w}_{2n-2k}(z)$$

where

$$\begin{aligned} \bar{v}_{2n}(z) &= z^{2n} + z^{2n-2} + \dots + z^2 + 1 = v_n(z^2), \\ \bar{w}_{2n-2k}(z) &= \frac{w_{2n+1-2k}(z)}{z+1} = \frac{z^{2n+1-2k} + 1}{z+1}. \end{aligned}$$



Using the factorization of  $v_n$  we get

$$\bar{v}_{2n}(z) = v_n(z^2) = \prod_{j=1}^n \left( z^2 - e^{\frac{2j\pi i}{n+1}} \right) = \prod_{j=1}^n \left( z - e^{\frac{j\pi i}{n+1}} \right) \left( z - e^{\frac{j\pi i}{n+1} - \pi i} \right).$$

Arranging the zeros of  $\bar{v}_{2n}$  into conjugate pairs  $\left( e^{\frac{j\pi i}{n+1}}, e^{-\frac{j\pi i}{n+1}} \right)$  ( $j = 1, \dots, n$ ) we have

$$\bar{v}_{2n}(z) = \prod_{j=1}^n \left( z - e^{\frac{j\pi i}{n+1}} \right) \left( z - e^{-\frac{j\pi i}{n+1}} \right) = \prod_{j=1}^n \left( z^2 - 2 \cos \frac{2j\pi}{2n+1} z + 1 \right)$$

therefore

$$\mathcal{T}\bar{v}_{2n}(x) = \prod_{j=1}^n \left( x - 2 \cos \frac{j\pi}{n+1} \right).$$

We can easily calculate the zeros of  $\bar{w}_{2n-2k}$  (we omit this elementary calculation) and obtain the factorization

$$\bar{w}_{2n-2k}(z) = \prod_{j=1}^{n-k} \left( z - e^{\frac{(2j-1)\pi i}{2n-2k+1}} \right) \left( z - e^{-\frac{(2j-1)\pi i}{2n-2k+1}} \right) = \prod_{j=1}^{n-k} \left( z^2 - 2 \cos \frac{(2j-1)\pi}{2n-2k+1} z + 1 \right)$$

therefore

$$\mathcal{T}(e_k \cdot \bar{w}_{2n-2k})(x) = \prod_{j=1}^{n-k} \left( x - 2 \cos \frac{(2j-1)\pi}{2n-2k+1} \right).$$

Next we show that

$$(16) \quad \mathcal{T}\bar{v}_{2n}(x) = U_n \left( \frac{x}{2} \right),$$

$$(17) \quad \mathcal{T}(e_k \cdot \bar{w}_{2n-2k})(x) = U_{n-k} \left( \frac{x}{2} \right) - U_{n-k-1} \left( \frac{x}{2} \right).$$

where we have to adopt the convention

$$(18) \quad U_{-1}(x) = 0 \quad (x \in \mathbb{C}).$$

The first identity follows from the fact that the zeros of both sides are the same. To justify the second we note that

$$\begin{aligned} U_{n-k}(\cos y) - U_{n-k-1}(\cos y) &= \frac{\sin(n-k+1)y - \sin(n-k)y}{\sin y} \\ &= \frac{2 \cos \frac{(2n-2k+1)y}{2} \sin \frac{y}{2}}{\sin y} = \frac{\cos \frac{(2n-2k+1)y}{2}}{\cos \frac{y}{2}} \end{aligned}$$

for all  $k = 0, \dots, n$  provided that the convention (18) is adopted.

If  $k = n$ , then both sides of (17) are equal to 1 thus (17) holds. For  $k < n$  the right hand side of (17) is zero if and only if  $y = \frac{(2j-1)\pi}{2n-2k+1}$  ( $j \in \mathbb{Z}$ ) hence all zeros of  $U_{n-k} \left( \frac{x}{2} \right) - U_{n-k-1} \left( \frac{x}{2} \right)$  are  $2 \cos \frac{(2j-1)\pi}{2n-2k+1}$  ( $j = 1, \dots, n-k$ ), they are the same as the zeros of  $\mathcal{T}(e_k \cdot \bar{w}_{2n-2k})$  proving (17).

By the linearity of the Chebyshev transform and by (16),(17) we have

$$\mathcal{T}\bar{h}_{2n}(x) = l\mathcal{T}\bar{v}_{2n}(x) + \sum_{k=1}^n a_k \mathcal{T}(e_k \cdot \bar{w}_{2n-2k})(x) = l U_n\left(\frac{x}{2}\right) + \sum_{k=1}^n a_k \left[ U_{n-k}\left(\frac{x}{2}\right) - U_{n-k-1}\left(\frac{x}{2}\right) \right].$$

Next we evaluate  $\mathcal{T}\bar{h}_{2n}$  at the points

$$\bar{x}_j = x_j = 2 \cos \frac{j + \frac{1}{2}}{2n+2} 2\pi \quad (j = 0, \dots, n)$$

of the interval  $[-2, 2]$ . Since  $\bar{x}_j = 2 \cos \bar{y}_j$  with  $\bar{y}_j = \frac{j + \frac{1}{2}}{2n+2} 2\pi$  we have

$$\begin{aligned} \mathcal{T}\bar{h}_{2n}(\bar{x}_j) &= 2 \left[ \frac{l \sin(n+1)\bar{y}_j}{2 \sin \bar{y}_j} + \frac{\sum_{k=1}^n a_k \cos \frac{2n-2k+1}{2} \bar{y}_j}{2 \cos \frac{\bar{y}_j}{2}} \right] \\ &= 2 \left[ \frac{l (-1)^j}{2 \sin \bar{y}_j} + \sum_{k=1}^n a_k \frac{\cos \frac{2n-2k+1}{2} \bar{y}_j}{2 \cos \frac{\bar{y}_j}{2}} \right] = 2 \frac{\frac{l}{2} (-1)^j + \sum_{k=1}^n a_k \sin \frac{\bar{y}_j}{2} \cos \frac{2n-2k+1}{2} \bar{y}_j}{\sin \bar{y}_j}. \end{aligned}$$

Since  $\bar{y}_j \in ]0, \pi[$  we have  $\sin \bar{y}_j > 0$ ,  $0 < \sin \frac{\bar{y}_j}{2} < 1$ ,  $|\cos \frac{2n-2k+1}{2} \bar{y}_j| \leq 1$  for all  $k = 1, \dots, n$  therefore the sign of the expression in the bracket is  $\text{sgn } l \text{sgn } (-1)^j$ . Thus  $\text{sgn } (\mathcal{T}\bar{h}_{2n}(x_j)) = \text{sgn } l \text{sgn } (-1)^j$  ( $j = 0, 1, \dots, n$ ) proving that  $\mathcal{T}\bar{h}_{2n}$  has  $n$  different zeros in  $[-2, 2]$ . Writing these zeros in the form  $2 \cos u_j$  with  $0 \leq u_1 \leq u_2 \leq \dots \leq u_n \leq \pi$  and applying Lemma 1 the proof is completed in the second case as well.  $\square$

We can formulate Theorem 1 in a more symmetric way. This formulation explains, in a certain way, the appearance of the factor 2 in (11).

**Theorem 2.** *All zeros of the reciprocal polynomial*

$$(19) \quad P_m(z) = \sum_{k=0}^m A_k z^k \quad (z \in \mathbb{C})$$

of degree  $m \geq 2$  with real coefficients  $A_k \in \mathbb{R}$  (i.e.  $A_m \neq 0$  and  $A_k = A_{m-k}$  for all  $k = 0, \dots, [\frac{m}{2}]$ ) are on the unit circle, provided that

$$(20) \quad |A_m| \geq \sum_{k=1}^{m-1} |A_k - A_m|.$$

If (20) holds, then all zeros  $e^{iu_j}$  ( $j = 1, 2, \dots, m$ ) of  $P_m$  can be arranged such that

$$|\epsilon_j - e^{iu_j}| < \frac{\pi}{m+1} \quad (j = 1, \dots, m).$$

If  $m = 2n + 1$  is odd, then  $-1 = e^{iu_{n+1}}$  is always a zero and all zeros of  $P_m$  are single.  
 If  $m = 2n$  is even

$$(21) \quad \begin{cases} |A_{2n}| = \sum_{k=1}^{2n-1} |A_k - A_{2n}| & \text{and} \\ \operatorname{sgn} A_{2n} = \operatorname{sgn} (-1)^{k+1} (A_k - A_{2n}) & \text{for all } k = 1, 2, \dots, n \text{ with } A_k - A_{2n} \neq 0 \end{cases}$$

holds, then  $u_n = u_{n+1} = \pi$ , the number  $-1 = e^{iu_n} = e^{iu_{n+1}}$  is a double zero of  $P_m$  and all other zeros are single. Otherwise (i.e. if  $m = 2n$ , (21) does not hold) all zeros of  $P_m$  are single.

*Proof.* Comparing the coefficients of  $z^j$  in  $h_m$  and  $P_m$  we see that for even  $m = 2n$

$$A_{2n} = A_0 = l, A_{2n-1} = A_1 = l + a_1, \dots, A_{n+1} = A_{n-1} = l + a_{n-1}, A_n = l + 2a_n$$

thus  $l = A_{2n}$ ,  $a_k = A_{2n-k} - A_{2n} = A_k - A_{2n}$  for  $k = 1, 2, \dots, n-1$  and  $2a_n = A_n - A_{2n}$ . Therefore the condition (11)

$$|l| \geq 2 \sum_{k=1}^n |a_k|$$

can be written as

$$|A_{2n}| \geq 2 \sum_{k=1}^{n-1} |A_k - A_{2n}| + |A_n - A_{2n}| = \sum_{k=1}^{2n-1} |A_k - A_{2n}|$$

which is the same as (20).

For odd  $m = 2n + 1$  the comparison of the coefficients gives that

$$A_{2n+1} = A_0 = l, A_{2n} = A_1 = l + a_1, \dots, A_{n+1} = A_n = l + a_n$$

thus  $l = A_{2n+1}$ ,  $a_k = A_{2n+1-k} - A_{2n+1} = A_k - A_{2n+1}$  for  $k = 1, 2, \dots, n$  and (11) can be written as

$$|A_{2n+1}| \geq 2 \sum_{k=1}^n |A_k - A_{2n+1}| = \sum_{k=1}^n (|A_k - A_{2n+1}| + |A_{2n+1-k} - A_{2n+1}|) = \sum_{k=1}^{2n} |A_k - A_{2n+1}|$$

proving (20). The statement concerning the location of the zeros follows from Remark 1.  $\square$

#### 4. NECESSARY AND SUFFICIENT CONDITIONS

If the degree  $m$  of  $P_m$  is small we can easily obtain necessary and sufficient conditions for all zeros of  $P_m$  to be on the unit circle.

If  $m = 2$ , then  $P_2(z) = A_2 z^2 + A_1 z + A_2 = z \left( A_2 \left( z + \frac{1}{z} \right) + A_1 \right)$  hence  $\mathcal{T}P_2(x) = A_2 x + A_1$ . The only zero of  $\mathcal{T}P_2$  is in  $[-2, 2]$  if and only if

$$(22) \quad |A_2| \geq \frac{1}{2} |A_1|.$$

This is the criteria for  $P_2$  to have all zeros on the unit circle.

If  $m = 3$ , then  $P_3(z) = A_3 z^3 + A_2 z^2 + A_2 z + A_3 = (z + 1) (A_3 z^2 + (A_2 - A_3) z + A_3)$ . By (22) the zeros of  $P_3$  are on the unit circle if and only if

$$(23) \quad |A_3| \geq \frac{1}{2} |A_2 - A_3|.$$

If  $m = 4$ , then  $P_4(z) = A_4z^4 + A_3z^3 + A_2z^2 + A_3z + A_4 = z^2 \left( A_4 \left( z^2 + \frac{1}{z^2} \right) + A_3 \left( z + \frac{1}{z} \right) + A_2 \right)$  hence with  $x = z + \frac{1}{z}$  we get that  $\mathcal{T}P_4(x) = A_4(x^2 - 2) + A_3x + A_2$ . By Lemma 1 all zeros of  $P_4$  are on the unit circle if and only if the discriminant of  $\mathcal{T}P_4$  is non-negative :

$$(24) \quad A_3^2 - 4A_4(A_2 - 2A_4) \geq 0$$

and

$$(25) \quad -2 \leq x_1, \quad x_2 \leq 2$$

hold where  $x_1 \leq x_2$  are the real zeros of  $\mathcal{T}P_4$ . A simple calculation shows that (24) and (25) are equivalent to

$$(26) \quad 2\sqrt{\max\{A_2A_4 - 2A_4^2, 0\}} \leq |A_3| \leq \min\{4|A_4|, |A_4| + \frac{1}{2}A_2 \operatorname{sgn} A_4\}.$$

This is the criterion for  $P_4$  to have all of its zeros on the unit circle.

For  $m = 2$  (22) holds if and only if

$$A_1 \in [-2|A_2|, 2|A_2|]$$

while (20) gives only the smaller interval

$$A_1 \in [A_2 - |A_2|, A_2 + |A_2|].$$

This shows that (20) for  $m = 2$  is not necessary. The situation is similar for  $m = 3$ .

For  $m = 4$  the necessary and sufficient condition (26) is non-linear in the coefficients, while our sufficient condition (20) is linear for all  $m \geq 2$ . In some special cases we get necessary and sufficient conditions.

**Corollary 1.** *All zeros of the polynomial*

$$l(z^m + z^{m-1} + \dots + z + 1) + (z^k + z^{m-k}) \quad (z \in \mathbb{C})$$

where  $m, k$  are fixed non-negative integers with  $m \geq 2$ ,  $1 \leq k \leq \lfloor \frac{m}{2} \rfloor$  and  $l$  is a fixed positive number, are on the unit circle for all  $m \geq 2$  and for all  $k = 1, 2, \dots, \lfloor \frac{m}{2} \rfloor$  if and only if

$$l \geq 2.$$

Namely, taking  $m = 2$ ,  $k = 1$  by (22) all zeros of the resulting polynomial  $lz^2 + (l+2)z + l$  are on the unit circle if and only if  $l \notin (-\frac{2}{3}, 2)$  therefore  $l \geq 2$ . On the other hand if  $l \geq 2$ , then by Theorem 1 all zeros of the polynomial  $l(z^m + z^{m-1} + \dots + z + 1) + (z^k + z^{m-k})$  are on the unit circle.

*Remark 2.* A preliminary version of some parts of this paper was reported in [5].

A. Schinzel [7] generalized Theorem 2 to the case of self-inversive polynomials over  $\mathbb{C}$ , i.e. polynomials  $P_m(z) = \sum_{k=0}^m A_k z^k$  for which  $A_k \in \mathbb{C}$ ,  $A_m \neq 0$ ,  $\epsilon \bar{A}_k = A_{m-k}$  for all  $k = 0, \dots, m$  with a fixed  $\epsilon \in \mathbb{C}$ ,  $|\epsilon| = 1$ . He proved that all zeros of  $P_m$  are on the unit circle, provided that

$$|A_m| \geq \inf \sum_{k=0}^m |cA_k - d^{m-k}A_m|,$$

where the infimum is taken over all  $c, d \in \mathbb{C}$  and  $|d| = 1$ .

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