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# A NEW CONSTRUCTION OF SALEM POLYNOMIALS

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Abstract. An earlier result of the author on the zeros of reciprocal polynomials is applied to give a new construction of Salem numbers.

Résumé. Cet article applique un résultat précédent de l'auteur sur les zéros des polynomes réciproques pour répondre à la question de la construction des nouveaux nombres Salem.

1. **Definitions and preliminary results.** A monic reciprocal polynomial with integer coefficients having exactly one zero (of multiplicity 1) outside the unit circle is called a *Salem polynomial* [2].

A *Salem number* is a real algebraic integer $\alpha > 1$ of degree $\geq 4$, all of whose conjugates, apart from $\alpha$ and $\alpha^{-1}$, lie on the unit circle.

Here we give a new construction of Salem polynomials and therefore (in case of their real zeros are positive) a construction of Salem numbers. This is based on the following result of the author [4].

*All zeros of the (real reciprocal) polynomial*

$$u_m(z) = u_{(m,l,\mathbf{a})}(z) = l(z^m + z^{m-1} + \cdots + z + 1) + \sum_{k=1}^{[\frac{m}{2}]} a_k(z^{m-k} + z^k) \tag{1}$$

*of degree* $m$ *where* $l \in \mathbb{R}$, $l \neq 0$, $m \in \mathbb{N}$, $m \geq 2$, $\mathbf{a} = (a_1, \dots, a_{[\frac{m}{2}]}) \in \mathbb{R}^{[\frac{m}{2}]}$ *are on the unit circle if*

$$|l| \geq 2 \sum_{k=1}^{[\frac{m}{2}]} |a_k|. \tag{2}$$

Write $v_m(z) := z^m + z^{m-1} + \cdots + z + 1$. It has been proved in [2] that for any sequence of positive integers $m_1, m_2, \dots, m_s$, $s \geq 3$, the polynomial

$$g(z) = (z+1) \prod_{i=1}^{s} v_{m_i}(z) - z \sum_{k=1}^{s} \Big( v_{m_k - 1}(z) \prod_{i=1, i\neq k}^{s} v_{m_i}(z) \Big) \tag{3}$$

is either cyclotomic or Salem.

The polynomial (3) is cyclotomic only if $s = 3$ and $(m_1+1)^{-1} + (m_2+1)^{-1} + (m_3+1)^{-1} \geq 1$ or $s = 4$ and $m_1 = m_2 = m_3 = m_4 = 1$.

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To determine the location of zeros of reciprocal polynomials we apply Chebyshev transformation. This transformation projects the unit circle on the real interval $[-2, 2]$.

A polynomial $p$ of the form $p(z) = \sum_{j=0}^{2n} a_j z^j$ $(z \in \mathbb{C})$ where $n \in \mathbb{N}$, $a_0, \dots, a_{2n} \in \mathbb{R}$ and $a_j = a_{2n-j}$ $(j = 0, \dots, n-1)$ is called a *real semi-reciprocal polynomial* of degree at most $2n$. If $a_{2n} \neq 0$ we call $p$ a *real reciprocal polynomial* of degree $2n$. Denote by $\mathcal{R}_{2n}$ the set of all real semi-reciprocal polynomials of degree at most $2n$.

If $p \in \mathcal{R}_{2n}$, $p \neq o$ ($o$ = the zero polynomial), then there is an integer $k$, $0 \le k \le n$, such that $a_{2n} = a_{2n-1} = \dots = a_{n+k+1} = 0 = a_{n-k-1} = \dots = a_0$ but $a_{n+k} = a_{n-k} \neq 0$ hence

$$p(z) = z^n \Big[a_{n+k}\Big(z^k + \frac{1}{z^k}\Big) + \dots + a_{n+1}\Big(z + \frac{1}{z}\Big) + a_n\Big]. \tag{4}$$

Let $T_j$ be the $j$-th Chebyshev polynomial of the first kind, and $C_j$ be the $j$-th normalized Chebyshev polynomial of the first kind defined by $C_0(x) = T_0(x)$, $C_j(x) = 2T_j(\frac{x}{2})$ for $j > 0$.

With $z + \frac{1}{z} = x$, we have $z^j + \frac{1}{z^j} = C_j(x)$ $(j = 1, 2, \dots)$, and hence by (4)

(5)

$$p(z) = z^n \sum_{j=0}^{k} a_{n+j} C_j(x) = a_{n+k} z^n \prod_{j=1}^{k} (x - \alpha_j) = a_{n+k} z^{n-k} \prod_{j=1}^{k} (z^2 - \alpha_j z + 1),$$

where $\alpha_j \in \mathbb{C}$ $(j = 1, \dots, k)$ are the zeros of the polynomial $\sum_{j=0}^{k} a_{n+j} T_j(x)$.

The *Chebyshev transform* of a non-zero polynomial $p \in \mathcal{R}_{2n}$ having the factorization (5) is defined by

$$\mathcal{T}p(x) = a_{n+k} \prod_{j=1}^{k} (x - \alpha_j),$$

while for the zero polynomial $p$ we put $\mathcal{T}p(x) = 0$.

It is clear that $\mathcal{T}$ maps $\mathcal{R}_{2n}$ into the set $\mathcal{P}_n$ of all polynomials of degree $\le n$ with real coefficients. In fact, we have the following:

PROPOSITION 1. *The Chebyshev transform $\mathcal{T}$ is an isomorphism between the (real) vector spaces $\mathcal{R}_{2n}$ and $\mathcal{P}_n$.*

Our basic tool is the following lemma (*cf.* [2]).

LEMMA 1. *Let $f(z)$ be a monic integral reciprocal polynomial and let $\bar{f}(z)$ be defined by*

$$\bar{f}(z) = \begin{cases} f(z) & \text{if the degree of } f \text{ is } 2n, \\ f(z)(z+1) & \text{if the degree of } f \text{ is } 2n-1. \end{cases}$$

*Then $f(z)$ is a Salem polynomial if and only if the Chebyshev transform $\mathcal{T}\bar{f}(x)$ of $\bar{f}(z)$ has $n-1$ zeros in the interval $[-2, 2]$.*

**2. The construction.** Write $\tilde{v}_{2n}(z) = \frac{v_{2n+1}(z)}{z+1}$. Similarly, if $u_m(z)$ is a reciprocal polynomial of the form (1) and $m = 2n+1$, we write $\tilde{u}_{2n}(z) = \frac{u_{2n+1}(z)}{z+1}$.

LEMMA 2.

(i) *The values of the polynomial* $\mathcal{T}u_{2n}(x)$ *alternate sign at the zeros of* $\mathcal{T}\tilde{v}_{2n}(x)$.
(ii) *The values of the polynomial* $\mathcal{T}\tilde{u}_{2n-2}(x)$ *alternate sign at the zeros of* $\mathcal{T}v_{2n}(x)$.

PROOF. It is easy to see that $\mathcal{T}\tilde{v}_{2n}(x) = U_n(\frac{x}{2}) = \prod_{j=0}^{n}(x-\beta_j)$ and $\mathcal{T}v_{2n}(x) = U_n(\frac{x}{2}) + U_{n-1}(\frac{x}{2}) = \prod_{j=1}^{n}(x-\gamma_j)$ where $\beta_j = 2\cos\frac{2j\pi}{2n+2}$, $\gamma_j = 2\cos\frac{2j\pi}{2n+1}$ for $j = 1,2,\ldots,n$ and $U_n$ is the $n$-th Chebyshev polynomial of the second kind (for the details see [4]).

In [4], we found that

$$\mathcal{T}u_{2n}(x) = l\Big[U_n\Big(\frac{x}{2}\Big) + U_{n-1}\Big(\frac{x}{2}\Big)\Big] + \sum_{k=1}^{n} 2a_k T_{n-k}\Big(\frac{x}{2}\Big), \tag{6}$$

$$\mathcal{T}\tilde{u}_{2n-2}(x) = lU_{n-1}\Big(\frac{x}{2}\Big) + \sum_{k=1}^{n} a_k\Big[U_{n-k-1}\Big(\frac{x}{2}\Big) - U_{n-k-2}\Big(\frac{x}{2}\Big)\Big]. \tag{7}$$

Substituting the corresponding zeros into (6), (7) we have, after some calculations, that

$$\mathcal{T}u_{2n}(\beta_j) = 2\Big(\frac{l}{2}(-1)^{j+1} + \sum_{k=1}^{n} a_k \cos\frac{2j(n-k)\pi}{2n+2}\Big) \tag{8}$$

$$\mathcal{T}\tilde{u}_{2n-2}(\gamma_j) = 2\frac{\frac{l}{2}(-1)^{j+1} + \sum_{k=1}^{n} a_k\cos\frac{2j(2n-2k-1)\pi}{2(2n+1)}}{2\cos\frac{j\pi}{2n+1}}.$$

By the condition (2), we get that $\operatorname{sgn}\mathcal{T}\tilde{u}_{2n-2}(\gamma_j) = \operatorname{sgn} l \operatorname{sgn}(-1)^{j+1}$ and $\operatorname{sgn}\mathcal{T}u_{2n}(\beta_j) = \operatorname{sgn} l\operatorname{sgn}(-1)^{j+1}$ proving Lemma 2. ∎

THEOREM 1. *Let* $m_1, m_2, \ldots, m_s$ *and* $l_1, l_2, \ldots, l_s$ *be arbitrary positive integers and for* $i = 1, 2, \ldots, s$; $m_i \geq 2$ *let* $\mathbf{a}_i = (a_{i1}, \ldots, a_{i[\frac{m_i}{2}]})$ *be vectors with non-negative integer components such that*

$$l_i \geq \sigma_i \quad \textit{where} \quad \sigma_i = 2\sum_{k=1}^{[\frac{m_i}{2}]} a_{ik} \tag{9}$$

*holds while for* $m_i = 1$ *let* $\mathbf{a_i}$ *be the zero vector. Further let*

$$u_{m_i}(z) := u_{(m_i,l_i,\mathbf{a}_i)}(z) = l_i(z^{m_i} + z^{m_i-1} + \cdots + z + 1) + \sum_{k=1}^{[\frac{m_i}{2}]} a_{ik}(z^{m_i-k} + z^k) \quad \text{if } m_i \geq 2$$

$$u_{m_i}(z) := u_{(m_i,l_i,\mathbf{a}_i)}(z) = l_i(z+1) \quad \text{if } m_i = 1.$$

*Define the polynomials*

$$f(z) = f_{(m_1,\ldots,m_s),(l_1,\ldots,l_s),(\mathbf{a}_1,\ldots,\mathbf{a}_s)}(z) = (z+1)\prod_{i=1}^{s} v_{m_i}(z) - z\sum_{k=1}^{s}\Big(u_{m_k-1}(z)\prod_{i=1,i\neq k}^{s} v_{m_i}(z)\Big) \tag{10}$$

*where*

$$u_{m_i-1}(z) := u_{(m_i-1,l_i,\mathbf{a}_i)}(z) \quad \text{if } m_i \geq 2$$

$$u_{m_i-1}(z) := l_i \quad \text{if } m_i = 1.$$

*Then* $f(x)$ *is a Salem polynomial unless one of the following conditions hold, in which case the polynomial is cyclotomic:*

(a) $s = 1$, $l_1 = 1$ *or* $l_1 = 2$ *and* $m_1 \geq 1$
(b) $s = 2$, $l_1 = l_2 = 1$, $m_1 \geq 1$ *and* $m_2 \geq 1$
$s = 2$, $l_1 = 2$ *or* $l_1 = 3$, $l_2 = 1$ *and* $m_1 = m_2 = 1$
$s = 2$, $l_1 = 2$, $l_2 = 1$ *and* $m_1 = 1$, $m_2 = 2$
$s = 2$, $l_1 = l_2 = 2$ *and* $m_1 = m_2 = 1$
(c) $s = 3$, $l_1 = l_2 = l_3 = 1$ *and* $(m_1+1)^{-1} + (m_2+1)^{-1} + (m_3+1)^{-1} \geq 1$
$s = 3$, $l_1 = 2$, $l_2 = l_3 = 1$ *and* $m_1 = m_2 = m_3 = 1$
(d) $s = 4$, $l_1 = l_2 = l_3 = l_4 = 1$ *and* $m_1 = m_2 = m_3 = m_4 = 1$.

Proof. We may assume that $m_1, m_2, \ldots, m_s$ are arranged such that all odd $m_i$'s are listed first, *i.e.*, $m_i = 2n_i+1$, $n_i \geq 0$ for $1 \leq i \leq r$ and $m_i = 2n_i$, $n_i \geq 1$ if $r+1 \leq i \leq s$ for some $0 \leq r \leq s$. Taking out the factor $z+1$ corresponding to the zero $-1$ of $f(z)$, we have

$$\begin{aligned} f(z) = {} & (z+1)^{r+1}\prod_{i=1}^{r}\tilde{v}_{2n_i}\prod_{i=r+1}^{s} v_{2n_i} \\ & - z(z+1)^{r-1}\sum_{k=1}^{r}\Big(u_{2n_k}\prod_{i=1,i\neq k}^{r}\tilde{v}_{2n_i}\prod_{i=r+1}^{s} v_{2n_i}\Big) \\ & - z(z+1)^{r+1}\sum_{k=r+1}^{s}\Big(\tilde{u}_{2n_k-2}\prod_{i=1}^{r}\tilde{v}_{2n_i}\prod_{i=r+1,i\neq k}^{s} v_{2n_i}\Big) \\ = {} & (z+1)^{r-1}\big((z+1)^2 f_1(z) - z f_2(z) - z(z+1)^2 f_3(z)\big) \end{aligned}$$

with suitable polynomials $f_1, f_2, f_3$.

Write $d = \sum_{i=1}^{s} n_i$ then the degree of $f$ is $N = \sum_{i=1}^{s} m_i + 1 = 2d + r + 1$. Let

$$F(x) = \begin{cases} \frac{\mathcal{T}f(x)}{(x+2)^{\frac{r-1}{2}}} = (x+2)\mathcal{T}f_1(x) - \mathcal{T}f_2(x) - (x+2)\mathcal{T}f_3(x) & \text{if } r \text{ is odd} \\ \frac{\mathcal{T}\bar{f}(x)}{(x+2)^{\frac{r}{2}}} = (x+2)\mathcal{T}f_1(x) - \mathcal{T}f_2(x) - (x+2)\mathcal{T}f_3(x) & \text{if } r \text{ is even.} \end{cases}$$

Applying Proposition 1 we can rewrite $F$ as

$$F(x) = (x+2)\prod_{i=1}^{r}\mathcal{T}\tilde{v}_{2n_i}\prod_{i=r+1}^{s}\mathcal{T}v_{2n_i} - \sum_{k=1}^{r}\Big(\mathcal{T}u_{2n_k}\prod_{i=1,i\neq k}^{r}\mathcal{T}\tilde{v}_{2n_i}\prod_{i=r+1}^{s}\mathcal{T}v_{2n_i}\Big)$$
$$- (x+2)\sum_{k=r+1}^{s}\Big(\mathcal{T}\tilde{u}_{2n_k-2}\prod_{i=1}^{r}\mathcal{T}\tilde{v}_{2n_i}\prod_{i=r+1,i\neq k}^{s}\mathcal{T}v_{2n_i}\Big).$$

By Lemma 1 we have to show that $\mathcal{T}\bar{f}$ has $[\frac{N+1}{2}]-1$ zeros in $[-2,2]$, or that $F$ has $[\frac{N+1}{2}]-1-[\frac{r}{2}] = d$ zeros in the interval $[-2,+2]$ apart from the exceptional cases (a)–(d) when each of the $d+1$ zeros of $F$ are in $[-2,+2]$.

To determine the number of zeros of $\mathcal{T}\bar{f}$ we use:

Lemma 3. *Let $\Phi_i$, $\Psi_i$, $h_i$ $(i = 1,2,\dots,s)$ and $h_0$ be polynomials with real coefficients such that $h_0, h_1, \dots, h_s$ are positive on the interval $(a,b]$, $h_0(a) = 0$ and for each $i = 1,2,\dots,s$*

- (A) *all zeros of $\Phi_i$ are single, and $\Phi_i(b) > 0$,*
- (B) *$\Psi_i$ alternates sign at the zeros of $\Phi_i$, and $\Psi_i$ is positive at the largest zero of $\Phi_i$,*
- (C) *for each $i = 1,2,\dots,s$ such that $h_i(a) \neq 0$ and $\Psi_i(a) \neq 0$; $\Psi_i$ alternates sign on $\{a\} \cup \{$zeros of $\Phi_i\}$, and if $\Phi_i$ has no zeros on $(a,b]$, then $\Psi_i > 0$.*

*Let $M$ be the number of zeros of the product $\Phi_1 \cdots \Phi_s$ on $(a,b)$, counted with multiplicity. Then the function*

$$\Phi = h_0\prod_{i=1}^{s}\Phi_i - \sum_{k=1}^{s}\Big(h_k\Psi_k\prod_{i=1,i\neq k}^{s}\Phi_i\Big)$$

*has $M$ zeros in $[a,b]$.*

Lemma 3 can be proved following the arguments of Proposition 4.1, Corollaries 1, 2 of [2], pp. 246–249. Condition (7) of Corollary 1 was changed to (C) to accomodate our situation, this again causes just a small deviation from the arguments of [2].

In the sequel let $\Phi = F$, $h_0 = x+2$,

$$\Phi_i = \begin{cases}\mathcal{T}\tilde{v}_{2n_i} \\ \mathcal{T}v_{2n_i}\end{cases} \qquad h_i = \begin{cases}1 \\ x+2\end{cases} \qquad \Psi_i = \begin{cases}\mathcal{T}u_{2n_i} & \text{if } 1 \le i \le r \\ \mathcal{T}\tilde{u}_{2n_i-2} & \text{if } r+1 \le i \le s.\end{cases}$$

The number of zeros (counted with multiplicity) of the product $\Phi_1 \cdots \Phi_s$ is easily seen to be $\sum_{i=1}^{s} n_i = d$. All these zeros are in $(-2, 2)$, denote by $\alpha_d$ the largest one.

Choose $[a, b]$ to be $[-2, \alpha_d + \epsilon]$ where $0 < \epsilon < 2 - \alpha_d$. We show that the conditions of Lemma 3 are satisfied. Obviously $h_i$ are positive on $(-2, \alpha_d + \epsilon]$ and $h_0(-2) = 0$.

(A) All zeros of $\mathcal{T}v_{2n_i}$ and $\mathcal{T}\tilde{v}_{2n_i}$ in $(-2, \alpha_d + \epsilon)$ are single. Since $\mathcal{T}v_{2n_i}(2) = U_{n_i}(1) + U_{n_i-1}(1) = 2n_i + 1 > 0$, $\mathcal{T}\tilde{v}_{2n_i}(2) = U_{n_i}(1) = n_i + 1 > 0$ and $\mathcal{T}v_{2n_i}$, $\mathcal{T}\tilde{v}_{2n_i}$ have no zeros in $[\alpha_d + \epsilon, 2]$ they are positive at the point $\alpha_d + \epsilon$.

(B) is ensured by Lemma 2, and also by Lemma 2 we have $\operatorname{sgn} \mathcal{T}u_{2n_i} = 1 > 0$ at the largest zero of $\mathcal{T}\tilde{v}_{2n_i}$ and $\operatorname{sgn} \mathcal{T}\tilde{u}_{2n_i-2} = 1 > 0$ at the largest zero of $\mathcal{T}v_{2n_i}$.

To show that (C) is also satisfied we remark first that $\Psi_i(-2) = \mathcal{T}u_{2n_i}(-2) = 0$ $(1 \le i \le r)$ if and only if $x+2$ is a factor of it, *i.e.*, if and only if $-1$ is a double zero of $u_{2n_i}$, and this holds if only if (see [4, Theorem 1])

(11)
$$l_i = 2\sum_{k=1}^{n_i} a_{ik} \quad \text{and} \quad \operatorname{sgn}(-1)^{k+1} = 1 \quad \text{for all } k = 1, \dots, n_i \text{ for which } a_{ik} \neq 0.$$

Next, we remark that $h_i(-2) \neq 0$ and $\Psi_i(-2) \neq 0$ holds if only if $1 \le i \le r$ and (11) is *not satisfied.*

Let us fix a subscript $i$ with this property.

Case 1. $\Phi_i = \mathcal{T}\tilde{v}_{2n_i}$ has no zeros on $(-2, \alpha_d + \epsilon]$. This holds if and only if $m_i = 1$, $n_i = 0$. Then we have $\Psi_i = \mathcal{T}u_{2n_i} = l_i > 0$ as required by (C).

Case 2. $m_i > 1$. By 2. $\Psi_i$ alternates sign at the zeros of $\Phi_i$. By Lemma 1 the smallest zero of $\Phi_i$ is $\beta_{n_i} = 2\cos\frac{2n_i\pi}{2n_i+2}$. To show that (C) holds in this case it is enough to prove that

$$\operatorname{sgn} \Psi_i(-2) \neq \operatorname{sgn} \Psi_i(\beta_{n_i}). \tag{12}$$

From (8) $\operatorname{sgn} \Psi_i(\beta_{n_i}) = \operatorname{sgn} \mathcal{T}u_{2n_i}(\beta_{n_i}) = (-1)^{n_i+1}$, and by (6)

$$\begin{aligned}\Psi_i(-2) = \mathcal{T}u_{2n_i}(-2) &= l_i[U_{n_i}(-1) + U_{n_i-1}(-1)] + \sum_{k=1}^{n_i} 2a_{ik}T_{n_i-k}(-1) \\ &= l_i(-1)^{n_i} + \sum_{k=1}^{n_i} 2a_{ik}(-1)^{n_i-k} = (-1)^{n_i}\Big(l_i + 2\sum_{k=1}^{n_i} a_{ik}(-1)^k\Big)\end{aligned}$$

and the expression in the last parenthesis is positive by $l_i > 0$ and by the assumption that (11) does not hold, proving that (12) is satisfied.

Since $\epsilon > 0$ was arbitrary we have proved that $F$ has $d$ zeros in the interval $[-2, \alpha_d]$. The polynomial $F$ is of degree $d+1$ its $d+1$-th zero can only be outside this interval.

The exceptional cases of the statement are the cases, when $F$ has $d+1$ zeros in the interval $[-2,2]$, *i.e.*, when $F(2) \geq 0$ since $\operatorname{sgn} F(\alpha_d) = -1$. A simple calculation shows that

$$F(2) = 4\prod_{i=1}^{r}(n_i+1)\prod_{i=r+1}^{s}(2n_i+1)\Bigg[1 - \sum_{k=1}^{r}\frac{(2n_k+1)l_k + 2\sum_{i=1}^{n_k} a_{ik}}{4(n_k+1)} - \sum_{k=r+1}^{s}\frac{n_k l_k + \sum_{i=1}^{n_k} a_{ik}}{2n_k+1}\Bigg].$$

The first sum in the bracket is $\geq \sum_{k=1}^{r} l_k/4$ the second $\geq \sum_{k=r+1}^{s} l_k/3$. Using this, one can prove that if $s \geq 4$ then $F(2) \geq 0$ implies that $l_1 = \cdots = l_s = 1$. However, $s \geq 5$ and $l_1 = \cdots = l_s = 1$ is not possible since then two sums are at least $s/4 > 1$ making $F(2) \geq 0$ impossible. This is how we get the exceptional case (d). Similarly, if $s = 3$ then $F(2) \geq 0$ implies $l_k \leq 2$ $(k = 1,2,3)$, if $s = 2$ then $F(2) \geq 0$ implies that $l_k \leq 3$ $(k = 1,2)$, finally if $s = 1$ then $F(2) \geq 0$ implies that $l_1 \leq 4$. For a given $s$ $(= 3,2,1)$ we list the possible values of $l_1,\ldots,l_s$ and using the condition $F(2) \geq 0$ we can select the suitable cases (a)–(c). ■

Remark 1. We remark that the cases $s \geq 3$ and $l_k = 1$ $(k = 1,\ldots,s)$ are known for example from [5]. In this way our result gives a new method for the description of trees with maximum eigenvalue $\leq 2$.

Remark 2. The sets defined in [5], [3] and [6] are proper subsets of the set defined by polynomials of Theorem 1. For example, the Salem number $\simeq 1.673324849$ defined by the polynomial

$$\begin{aligned} &z^{14} - z^{12} - z^{11} - z^{10} - z^9 - 2z^8 - 3z^7 - 2z^6 - z^5 - z^3 - z^2 + 1 \\ &\quad = (z+1)v_1(z)v_{12}(z) - z\big(2v_{11}(z) + z^5 + z^6\big)(z+1) - zv_{12}(z) \end{aligned}$$

which is of the form (9) of Theorem 1, is not in the previously known sets. This holds since previous sets were defined by help of graphs. Due to the discrete nature of graphs there are "gaps" in these sets. The Salem number above is in such a gap.

Remark 3. In some cases the polynomials of the form (9) are Salem polynomials even if $u_{m_i-1}$ do not satisfy condition (8) but their zeros are on the unit circle. For example let $s = 1$, $m_1 = 16$, $l_1 = 2$ and

$$u_{15}(z) = 2v_{15}(z) + z^5 + z^6 + z^9 + z^{10}; \quad f_1(z) = (z+1)v_{16}(z) - zu_{15}(z).$$

Then $f_1(z) = (z+1)\Phi_4(z)\Phi_{12}(z)g_1(z)$ where $g_1(z)$ is the Salem polynomial of the smallest known Salem number.

This holds since the Chebyshev transform of $f_1(z)/(z+1)$ has 7 zeros on the interval $[-2,2]$. The polynomial $g_1$ is an irreducible factor of many Salem polynomials of the form

$$(z+1)v_{m-1}(z) - z\big(2v_{m-2}(z) + z^i + z^j + z^{m-2-i} + z^{m-2-j}\big), \tag{13}$$

for example if $m = 20$, $i = 3$, $j = 6$; $m = 21$, $i = 2$, $j = 9$; $m = 25$, $i = 1$, $j = 10$. Also, the majority of the small Salem numbers listed in parameter set for the Salem polynomials of the 8 smallest Salem numbers

| # | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $m$ | 17 | 19 | 17 | 16 | 15 | 19 | 13 | 25 |
| $i$ | 5 | 3 | 3 | 3 | 3 | 1 | 4 | 0 |
| $j$ | 6 | 5 | 5 | 6 | 5 | 7 | 5 | 8. |

It may be of interest to show that the construction defined in Theorem 1 yields all small Salem numbers.

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