CHARACTERIZATIONS OF SOME CLASSES OF QUASI-HEREDITARY ALGEBRAS

Piroska Lakatos

Abstract. Inspired by the work of Mirollo and Vilonen [MV] describing the categories of perverse sheaves as module categories over certain finite dimensional algebras, Dlab and Ringel introduced [DR2] an explicit recursive construction of these algebras in terms of the algebras $A(\gamma)$. In particular, they characterized the quasi-hereditary algebras of Cline-Parshall-Scott [PS] and constructed them in this way. The present paper provides a characterization of lean, shallow and replete algebras in terms of this recursive process. A detailed presentation of the results will appear elsewhere.

Let D be a division $K-$algebra, C a basic K-algebra, ${}_DS_C$ and ${}_CT_D$ finite-dimensional bimodules with K acting centrally. Let $\gamma:\ {}_CT_D \otimes_D S_C \to {}_CC_C$, be a bimodule homomorphism whose image lies in $\operatorname{rad} C$. Let $B = D \ltimes ({}_DS_C \otimes_C T_D)$ the "split" $K-$algebra with the coordinate-wise addition and multiplication given by

$$(d_1, s_1 \otimes t_1)(d_2, s_2 \otimes t_2) = \big(d_1d_2, d_1s_2 \otimes t_2 + s_1 \otimes t_1d_2 + s_1\gamma(t_1 \otimes s_2) \otimes t_2\big).$$

Clearly, B is a local $K-$algebra with $\operatorname{rad} B = S_C \otimes_C T$. It follows that S has the structure of a B-C-bimodule by $(d, s \otimes t) \cdot s' = ds' + s\gamma(t \otimes s')$ and T the structure of a C-B-bimodule by $t' \cdot (d, s \otimes t) = t'd + \gamma(t' \otimes s)t$. In [DR2], the 2×2 matrix $A = \begin{pmatrix} B & S \\ T & C \end{pmatrix}$ with multiplication given by

$$\begin{pmatrix} b & s \\ t & c \end{pmatrix}\begin{pmatrix} b' & s' \\ t' & c' \end{pmatrix} = \begin{pmatrix} bb' + (0, s \otimes t') & b \cdot s' + sc' \\ t \cdot b' + ct' & \gamma(t \otimes s') + cc' \end{pmatrix},$$

is shown to be a $A(\gamma)$ ring, viz. the quotient of the tensor algebra over the $(C \times D) - (C \times D)-$bimodule $T \otimes S$ by the ideal generated by the elements $t \otimes s - \gamma(t \otimes s)$.

Let $e_1 = \begin{pmatrix} (1,0) & 0 \\ 0 & 0 \end{pmatrix}$ and $\mathbf{e}_C = (e_2, e_3, \dots, e_n)$ be a complete sequence of primitive orthogonal idempotents of C so that $\sum_{i=1}^n e_i = 1 : A_A = \bigoplus_{i=1}^{n} e_iA$. Write $\mathbf{e} = \mathbf{e}_A = (e_1, e_2, \dots, e_n)$, $\varepsilon_i = e_i + e_{i+1} + \dots + e_n$ for $1 \le i \le n$ and $\varepsilon_{n+1} = 0$.

Let $\Delta_C(i)$ and $\Delta^o_C(i)$ be the right and left standard modules of $(C, \mathbf{e}_C)$, respectively (see [D] for the basic definitions and notation). Dlab and Ringel

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have shown in [DR2] that $(A, \mathbf{e})$ is quasi-hereditary if and only if $(C, \mathbf{e}_C)$ is quasi-hereditary and S_C and ${}_CT$ have Δ_C-filtration and Δ^o_C- filtration, respectively; in fact, they have shown that all basic quasi-hereditary algebras over a perfect field K can be obtained by iterating this construction, starting with a division K-algebra C. In the present note, we are going to characterize lean algebras, as well as shallow and replete quasi-hereditary algebras A in terms of properties of C, ${}_DS_C$, ${}_CT_D$ and the homomorphism γ.

Recall that $(A, \mathbf{e})$ is *lean* (see [ADL]) if

$$e_i(\operatorname{rad} A)^2 e_j = e_i(\operatorname{rad} A)\varepsilon_m(\operatorname{rad} A)e_j \text{ for all } 1 \le \mathrm{i}, \mathrm{j} \le \mathrm{n} \text{ and } \mathrm{m} = \min\{\mathrm{i}, \mathrm{j}\}.$$

Equivalently, $(A, \mathbf{e})$ is lean if and only if $(C, \mathbf{e}_C)$ is lean and for $i, j \ge 2$, the products $e_iAe_1Ae_j$ $(\subseteq e_i(\operatorname{rad} A)^2 e_j)$ belong to $(\operatorname{rad} C)^2$. Since these products generate the image of γ, we have the following statement.

Proposition 1. *The algebra* $(A, \mathbf{e})$ *is lean if and only if* $(C, \mathbf{e}_C)$ *is lean and* $\operatorname{Im}\gamma \subseteq (\operatorname{rad} C)^2$.

Denoting the *standard* right and left modules of A by $\Delta(i) = \Delta_A(i)$ and $\Delta^o(i) = \Delta^o_A(i)$, respectively, $(A, \mathbf{e})$ said to be $quasi-hereditary$ if

$$\dim_K A = \sum_{i=1}^{n} (1/d_i) \dim_K \Delta(i) \dim_K \Delta^o(i), \qquad (*)$$

where $d_i = \dim_K \operatorname{End} S(i)$; here, and in what follows, $S(i)$, $P(i)$ and $V(i)$ denote the simple right $A-$module, $P(i) \simeq e_iA$ its projective cover and $V(i)$ the kernel of the canonical epimorphism $P(i) \to \Delta(i)$. The equality $(*)$ is equivalent to the fact that $\operatorname{End}\Delta(i) \simeq \operatorname{End} S(i)$ for all $1 \le i \le n$ and that the regular representation A_A has a Δ-filtration (which has been the original definition of Cline-Parshall-Scott; see also [DR1]). Indeed, this follows from the following series of statements $(A) - (C)$ (cf.[D2]):

(A) For every $A-$module X, $[X : S(i)] = (1/d_i) \dim_K Xe_i$; thus

$$[A_A : S(n)] = (1/d_n) \dim_K Ae_n = (1/d_n) \dim_K \Delta^o(n).$$

(B) Always, $\dim_K Ae_nA \le (1/d_n) \dim_K \Delta(n) \dim_K \Delta^o(n)$.

(C) The equivality $\dim_K Ae_nA = (1/d_n) \dim_K \Delta(n) \dim_K \Delta^o(n)$ holds if and only if $\operatorname{End}_A \Delta(n) = \operatorname{End}_A S(n)$ and $Ae_nA \simeq \bigoplus_{\text{finite}} \Delta(n)$.

The quasi-hereditary algebra $(A, \mathbf{e})$ is called *shallow* if all $\operatorname{rad}\Delta(i)$ and $\operatorname{rad}\Delta^o(i)$ $(1 \le i \le n)$ are semi-simple. This is equivalent to the fact (see [ADL]) that

$$e_i(\operatorname{rad} A)^2 Ae_j = e_i(\operatorname{rad} A)\varepsilon_M(\operatorname{rad} A)e_j \text{ for all } 1 \le i, j \le n \text{ and } M = \operatorname{Max}\{i, j\}.$$

As a consequence, both Δ_C-filtration of S_C and Δ_C^o-filtration of ${}_CT$ are in this case *top* filtrations (see [ADL]), and $(C, \mathbf{e}_C)$ is a shallow quasi-hereditary algebra. The following example shows that these properties are not sufficient for $(A, \mathbf{e})$ to be shallow.

EXAMPLE. Let $(A, \mathbf{e})$ be the path algebra of the quiver $2 \longrightarrow 1 \longrightarrow 3$; then $(A, \mathbf{e})$ is quasi-hereditary (in fact, hereditary). Since $e_2(\operatorname{rad} A)^2 e_3 \neq 0$ and $e_2(\operatorname{rad} A)e_3(\operatorname{rad} A)e_3 = 0$, $(A, \mathbf{e})$ is not shallow. However, $C = \varepsilon_2 A \varepsilon_2$ is shallow quasi-hereditary algebra and both S_C and ${}_CT$ are simple C-modules. The key missing property is leanness.

PROPOSITION 2. *The algebra* $(A, \mathbf{e})$ *is a shallow if and only if* $(C, \mathbf{e}_C)$ *is a shallow quasi-hereditary algebra,* S_C *has a top* Δ_C *-filtration,* ${}_CT$ *has a top* Δ_C^o *-filtration and* $\operatorname{Im} \gamma \subseteq (\operatorname{rad} C)^2$.

Proof (of sufficiency). We need only to show that, under the conditions for C, , S_C, ${}_CT$ and γ,

$$e_i(\operatorname{rad} A^2)e_j \subset e_i(\operatorname{rad} A)\varepsilon_M(\operatorname{rad} A)e_j \text{ for all } 1 \leq \mathrm{i,j} \leq \mathrm{n} \text{ and } \mathrm{M} = \mathrm{Max}\{\mathrm{i,j}\}.$$

First, let $i, j \geq 2$. Then, in view of the fact that $(C, \mathbf{e}_C)$ is shallow,

$$e_i(\operatorname{rad} A)\varepsilon_M(\operatorname{rad} A)e_j = e_i(\operatorname{rad} C)\varepsilon_M(\operatorname{rad} C)e_j = e_i(\operatorname{rad} C)^2 e_j,$$

which, in turn equals to $e_i(\operatorname{rad} A)\varepsilon_2(\operatorname{rad} A)e_j$. Since, by Proposition 1, $(A, \mathbf{e})$ is lean, we get

$$e_i(\operatorname{rad} A)^2 e_j = e_i(\operatorname{rad} A)\varepsilon_m(\operatorname{rad} A)e_j \subset e_i(\operatorname{rad} A)\varepsilon_2(\operatorname{rad} A)e_j,$$

as required.

If $i = 1, j \geq 2$, the inclusion $e_1(\operatorname{rad} A)^2 e_j \subset e_1(\operatorname{rad} A)\varepsilon_j(\operatorname{rad} A)e_j$ follows from the fact that the Δ-filtration of $\operatorname{rad} P(1)$, induced by the top Δ_C-filtration of S_C (which exists by [DR2]), is a top filtration. A similar argument can be applied to $i \geq 2, j = 1$.

Recall that the quasi-hereditary algebra $(A, \mathbf{e})$ is said to be *replete* if all $V(i) = e_i A \varepsilon_{i+1} A$ are projective, top submodules of $\operatorname{rad} P(i) = e_i(\operatorname{rad} A)$, and all $V^o(i) = A\varepsilon_{i+1} A e_i$ are projective, top submodules of $\operatorname{rad} P^o(i) = (\operatorname{rad} A)e_i$ (see [ADL]). If $(A, \mathbf{e})$ is a replete quasi-hereditary algebra, then $(C, \mathbf{e}_C)$ is a replete quasi-hereditary algebra and both S_C and ${}_CT$ are projective C-modules. Again, these conditions alone do not imply that $(A, \mathbf{e})$ is replete. In the above Example, $(C, \mathbf{e}_C)$ is replete and both S_C and ${}_CT$ are simple (projective) C-modules, but $(A, \mathbf{e})$ is not replete.The missing property is leanness again.

Proposition 3. *The algebra* $(A, \mathbf{e})$ *is a replete quasi-hereditary algebra if and only if* $(C, \mathbf{e}_C)$ *is a replete quasi-hereditary algebra* S_C *and* ${}_CT$ *are projective* $C-$*modules and* $\operatorname{Im}\gamma \subseteq (\operatorname{rad} C)^2$.

Proof (of sufficiency) We want to show that $V(i)$ is a projective, top submodule of $\operatorname{rad} P(i)$. Consider first $i \geq 2$. Since $V_C(i) = e_i(\operatorname{rad} A)\varepsilon_{i+1}A\varepsilon_2$ is projective, top C-submodule of the C-module $\operatorname{rad} P_C(i) = e_i(\operatorname{rad} A)\varepsilon_2$, $V(i) = V_C(i) \otimes_C \varepsilon_2 A$ is a projective A-module.
Moreover, since A is lean by Proposition 1, we have the equality

$$e_i(\operatorname{rad} A)^2 e_{i+1} = e_i(\operatorname{rad} A)\varepsilon_2(\operatorname{rad} A)\varepsilon_{i+1},$$

and thus can identify the top of $V(i)$ in the top of $\operatorname{rad} P(i)$ with the top of $V_C(i)$. This yields a top embedding of $V(i)$ in $\operatorname{rad} P(i)$. Furthermore, since S_C is projective C-module, $V(1) = \operatorname{rad} P(1) = S_C \otimes_C \varepsilon_2 A$ is projective A-module (trivially embedded in $\operatorname{rad} P(1)$ as a top submodule).

One can use similar arguments to the deal with the left A-modules $V^o(i)$.

Let us conclude our brief note with the remark that similar characterizations of the left and right medial algebras (for a definition, see [ADL]) can be made combining the conditions of Propositions 2 and 3.

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Institute of Mathematics and Informatics,
Lajos Kossuth University
P.O.Box 12, 4010 Debrecen, Hungary
E-mail address: lapi@math.klte.hu