STOCHASTIC ORDERS GENERATED BY INTEGRALS: A UNIFIED STUDY.

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Abstract

We consider stochastic orders of the following type: Let $\mathfrak{F}$ be a class of functions and let P and Q be probability measures. Then define $P \leq_{\mathfrak{F}} Q$, if $\int f\ dP \leq \int f\ dQ$ for all f in $\mathfrak{F}$. Marshall (1991) posed the problem of characterizing the maximal cone of functions generating such an ordering. We solve this problem by using methods from functional analysis. Another purpose of this paper is to derive properties of such *integral stochastic orders* from conditions satisfied by the generating class of functions. The results are illustrated by several examples. Moreover, we show that the likelihood ratio order is closed with respect to weak convergence, though it is not generated by integrals.

INTEGRAL STOCHASTIC ORDERS; CLOSURE PROPERTIES; MAXIMAL GENERATOR; LIKELIHOOD RATIO ORDER.

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1 Introduction.

Stochastic orders are an important tool in many areas of probability and statistics. For a comprehensive treatment of this subject including a variety of applications we refer to the books of Shaked and Shanthikumar (1994) or Szekli (1995). A survey of the recent literature is given in the bibliography of Mosler and Scarsini (1994).

Many of the stochastic orders, which are in common use, are defined as follows. Let $(S, \mathcal{A})$ be some measure space, and let $\mathfrak{F}$ be some class of measurable functions $f : S \to \mathbb{R}$. Then a relation $\leq_{\mathfrak{F}}$ is defined on the set of all probability measures (p.m.) on $(S, \mathcal{A})$ by

$$P \leq_{\mathfrak{F}} Q \quad \text{if} \quad \int f\ dP \leq \int f\ dQ \quad \text{for all } f \in \mathfrak{F},$$

such that the integrals exist. Whitt (1986) introduced the notion *integral stochastic order* for these relations. He observed that a relation defined in this

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way needs not to be transitive. He proposed to avoid this problem by restriction to sets of functions and/or probability measures, such that all integrals exist.

One purpose of this paper is to find such an appropriate restriction. This is carried out in section 2 by introducing a weight function b and by restriction to b-bounded functions and to the set $\mathbb{P}_b$ of p.m.'s for which $\int b\ dP$ exists. In this setting it is possible to develop a unified theory of integral stochastic orders. We introduce a duality of a function space and a space of signed measures with integration as bilinear mapping. Thus we can use well known results from functional analysis for our investigations. By applying the bipolar theorem, we get as the main result of section 3 the characterization of the maximal generating class of functions, that induces a given integral stochastic order. This solves a problem posed by Marshall (1991).

Stoyan (1983) defined important closure and preservation properties of stochastic orders. In section 4 we show how these properties arise directly from conditions satisfied by the underlying class of functions $\mathfrak{F}$. Most of the closure properties, that can be found in Shaked and Shanthikumar (1994) can easily be deduced from these results.

This is illustrated by several examples in section 5. We also give an example of a stochastic order, which is not induced by integrals, namely the *likelihood ratio order*. Nevertheless this order has interesting closure properties. In Theorem 5.8 we show that it is closed with respect to weak convergence. This result seems to be new.

2 Preliminaries.

First we want to make some remarks about our notation. Sets of functions are mostly denoted by capital *fraktur* letters as $\mathfrak{F}, \mathfrak{V}, \mathfrak{R}, \mathfrak{B}$, ..., whereas we use *calligraphic* letters as $\mathcal{A}, \mathcal{B}$, ... for σ-algebras. Sets of (signed) measures are denoted by letters in *blackboard* like $\mathbb{M}$, $\mathbb{P}$,... . We hope that these arrangements increase the legibility of our paper.

Let $(S, \mathcal{S})$ be a measure space and let $b : S \to [1, \infty)$ be a measurable function, called *weight function*. We consider the set $\mathfrak{B}_b$ of measurable functions $f : S \to \mathbb{R}$, for which

$$\|f\|_b := \sup_{s \in S} \frac{|f(s)|}{b(s)} < \infty.$$

For a signed measure μ on $\mathcal{A}$ we denote the positive and negative variation by μ^+ resp. μ^-. As usual $|\mu| := \mu^+ + \mu^-$ is the total variation. Integrals are mostly written in the functional form $\mu(f) := \int f\ d\mu := \int f\ d\mu^+ \Leftrightarrow \int f\ d\mu^-$. Notice that $\mu(f)$ exists and is finite if and only if $\mu^+(|f|) + \mu^-(|f|) < \infty$.

The set of all signed measures μ on $\mathcal{A}$ with $|\mu|(b) = \mu^+(b) + \mu^-(b) < \infty$ is denoted by $\mathbb{M}_b$. We write $\mathbb{P}$ for the set of all probability measures (p.m.) on $\mathcal{A}$, and $\mathbb{P}_b := \mathbb{P} \cap \mathbb{M}_b$ is the restriction of $\mathbb{M}_b$ to $\mathbb{P}$. $\mathbb{P}_b$ is nonvoid as it contains all p.m.'s with finite support. $\mathbb{M}_b^N$ is the set of all signed measures with $\mu(S) = 0$. Notice that the difference of two p.m.'s lies in $\mathbb{M}_b^N$ and that every measure in $\mathbb{M}_b^N$ is a multiple of such a difference, i.e. $\mathbb{M}_b^N$ is the linear span of $\mathbb{P}_b \Leftrightarrow \mathbb{P}_b$.

For the formulation of our first lemmas we need some notions from functional analysis, which can be found e.g. in Choquet (1969), §22.

A pair (E, F) of vector spaces is said to be in *duality*, if there is a bilinear mapping $\langle \cdot, \cdot \rangle : E \times F \to \mathbb{R}$. The duality is said to be *strict*, if for each $0 \neq x \in E$ there is a $y \in F$ with $\langle x, y \rangle \neq 0$ and if for each $0 \neq y \in F$ there is an $x \in E$ with $\langle x, y \rangle \neq 0$.

Lemma 2.1 *$\mathbb{M}_b$ and $\mathfrak{B}_b$ are in strict duality under the bilinear mapping*

$$\begin{array}{rcl} \langle \cdot, \cdot \rangle : \mathbb{M}_b \times \mathfrak{B}_b \to \mathbb{R} & & \\ \langle \mu, f \rangle & := & \mu(f) \end{array} . \tag{2.1}$$

Proof. Obviously $\mathfrak{B}_b$ and $\mathbb{M}_b$ are vector spaces. For $f \in \mathfrak{B}_b$ we have $|f| \leq \|f\|_b \cdot b$, and hence

$$|\mu(f)| \;\leq\; \mu^+(|f|) + \mu^-(|f|) \leq \|f\|_b \cdot \Big(\mu^+(b) + \mu^-(b)\Big) < \infty$$

for $\mu \in \mathbb{M}_b$. Thus the mapping $\langle \cdot, \cdot \rangle$ is well defined. It remains to show the strictness of the duality.
(i) $\mathfrak{B}_b$ contains the indicator functions of all sets $A \in \mathcal{A}$, as $b \geq 1$. Therefore $\mu(f) = 0$ for all $f \in \mathfrak{B}_b$ implies $\mu(A) = 0$ for all $A \in \mathcal{A}$, and thus $\mu \equiv 0$.
(ii) $\mathbb{M}_b$ contains all one point measures δ_s, $s \in S$. Hence $\mu(f) = 0$ for all $\mu \in \mathbb{M}_b$ implies $\delta_s(f) = f(s) = 0$ for all $s \in S$ and consequently $f \equiv 0$. □

Remark. In part (i) of the proof we needed the requirement $b \geq 1$ for the weight function. Sometimes there is a naturally given weight function b', which only fulfils $b' \geq 0$. Then we can use $b := b' + 1$, leading to $\mathbb{M}_b = \mathbb{M}_{b'}$ and $\mathfrak{B}_{b'} \subset \mathfrak{B}_b$, i.e. the measure space remains the same and even more functions can be handled.

Unfortunately the duality $(\mathbb{M}_b^N, \mathfrak{B}_b)$ is not strict, as $\mu(f) = 0$ for all $\mu \in \mathbb{M}_b^N$ only implies f constant. But strict duality can be obtained by identifying functions which differ only by a constant. Formally, we define an equivalence relation $f \sim g$ if $f \Leftrightarrow g$ is constant. Denoting the corresponding quotient space by $\mathfrak{B}_{b/\sim}$ we get the following lemma.

Lemma 2.2 *$\mathbb{M}_b^N$ and $\mathfrak{B}_{b/\sim}$ are vector spaces in strict duality under the bilinear mapping (2.1).*

A crucial role in our further investigations plays the bipolar theorem for convex cones. Therefore we introduce the notion of polars. We follow the notation of Choquet (1969).

The *polar* M° of a set $M \subset E$ (in the duality (E, F)) is defined by

$$M^\circ := \{y \in F : \langle x, y\rangle \geq -1 \text{ for all } x \in M\}.$$

The polar of a set $N \subset F$ is defined analogously.

As usual a subset K of a vector space is called a *cone*, if $x \in K$ implies $\alpha x \in K$ for all $\alpha \geq 0$. We define the *dual cone* of an arbitrary set $M \subset E$ by

$$M^+ := \{y \in F : \langle x, y\rangle \geq 0 \text{ for all } x \in M\}.$$

It is easy to see that M^+ is a convex cone.

Now the bipolar theorem for convex cones has the following form.

Theorem 2.3 *(Corollary 22.10 in Choquet (1969)).*
Suppose E and F are in strict duality and $X \subset E$ is a convex cone. Then $X^{\circ\circ}$ is the $\sigma(E, F)$-closure of X and $X^\circ = X^+$.

3 Main results.

Order relations for probability distributions are often defined as follows:
Let $\mathfrak{F}$ be a class of real-valued functions. Then

$$P \preceq Q \quad \text{if} \quad \int f \, dP \leq \int f \, dQ \tag{3.1}$$

for all $f \in \mathfrak{F}$, for which the integrals exist. This definition can be found e.g. in Stoyan (1983), Def. 1.1.2, or Marshall (1991), p. 231.

Such an “order relation” needs not be transitive, as the following example shows. If $\mathfrak{F}$ contains only the identity and Q denotes the Cauchy distribution (which has no expectation value), then $\delta_1 \preceq Q$ and $Q \preceq \delta_0$, but $\delta_1 \not\preceq \delta_0$. However, if all integrals exist, this flaw can not appear.

In order to ensure this, we assume the existence of a weight function b. Then by Lemma 2.1 $\int f \, dP$ exists for all $f \in \mathfrak{B}_b$ and $P \in \mathbb{P}_b$. Therefore we can define an order relation $\leq_{\mathfrak{F}}$ on $\mathbb{P}_b$ via (3.1) for $\mathfrak{F} \subset \mathfrak{B}_b$. Since the right hand side of (3.1) can be rewritten as $\int f \, d(Q - P) \geq 0$, the relation $\leq_{\mathfrak{F}}$ can be defined as follows.

Definition 3.1 *For $\mathfrak{F} \subset \mathfrak{B}_b$ let the (binary) relation $\leq_{\mathfrak{F}}$ be defined on $\mathbb{P}_b$ by*

$$P \leq_{\mathfrak{F}} Q \quad :\Leftrightarrow \quad Q - P \in \mathfrak{F}^+ \;\Leftrightarrow\; P(f) \leq Q(f) \quad \forall f \in \mathfrak{F}. \tag{3.2}$$

We denote such a relation as an *integral stochastic order* and $\mathfrak{F}$ is called *generator*.

Directly from the definition we get the following result.

Lemma 3.2 *For arbitrary $\mathfrak{F} \subset \mathfrak{B}_b$ we have:*
a) $\leq_{\mathfrak{F}}$ is transitive und reflexive, i.e. a pre-order.
b) $\leq_{\mathfrak{F}}$ is a (partial) order, if and only if $\mathfrak{F}$ is separating in $\mathbb{P}_b$, i.e. if it holds:

$$P, Q \in \mathbb{P}_b,\ P(f) = Q(f)\ \forall f \in \mathfrak{F} \quad \Rightarrow \quad P = Q. \tag{3.3}$$

Proof. a) is an easy consequence of the fact that $\mathfrak{F}^+$ is a convex cone.
b) The antisymmetry of $\leq_{\mathfrak{F}}$ is equivalent to property (3.3). Thus a) implies the assertion. □

There may be different classes of functions, which generate the same stochastic order. For checking $P \leq_{\mathfrak{F}} Q$ it is desirable to have "small" generators, whereas "large" generators are interesting for applications. Marshall (1991) posed the question of characterizing the largest generator (which he denoted by *stochastic completion*). We will solve this problem now in our setting.

Definition 3.3 *Let b be a weight function and let $\mathfrak{F} \subset \mathfrak{B}_b$ be an arbitrary generator of an order $\leq_{\mathfrak{F}}$ on $\mathbb{P}_b$.*
a) The set

$$\mathfrak{R}_{\mathfrak{F}} := \{f \in \mathfrak{B}_b : P, Q \in \mathbb{P}_b,\ P \leq_{\mathfrak{F}} Q \ \Rightarrow\ P(f) \leq Q(f)\}$$

is called **maximal generator** *of $\leq_{\mathfrak{F}}$ (in $\mathfrak{B}_b$).*
b) We denote by $\tilde{\mathfrak{R}}_{\mathfrak{F}}$ the set of all measurable functions $f : S \to \mathbb{R}$ (not necessarily in $\mathfrak{B}_b$) with the property:
$P, Q \in \mathbb{P}_b$ and $P \leq_{\mathfrak{F}} Q$ imply $P(f) \leq Q(f)$, if the integrals exist.
$\tilde{\mathfrak{R}}_{\mathfrak{F}}$ is called **extended maximal generator***.*

Remarks: 1. $\mathfrak{R}_{\mathfrak{F}} = \{f \in \mathfrak{B}_b : \mu(f) \geq 0 \text{ for all } \mu \in \mathfrak{F}^+ \cap \mathbb{M}_b^N\}$.
2. $\mathfrak{R}_{\mathfrak{F}} = \tilde{\mathfrak{R}}_{\mathfrak{F}} \cap \mathfrak{B}_b$.
3. Obviously $\mathfrak{F} \subset \mathfrak{R}_{\mathfrak{F}}$. Though in general $\mathfrak{F}$ is a proper subset of $\mathfrak{R}_{\mathfrak{F}}$, they both generate the same order on $\mathbb{P}_b$.
4. If $\mathfrak{V}$ is any generator of $\leq_{\mathfrak{F}}$, then $\mathfrak{V} \subset \mathfrak{R}_{\mathfrak{F}}$. Thus $\mathfrak{R}_{\mathfrak{F}}$ is not only a *maximal* element in the set of all generators of $\leq_{\mathfrak{F}}$, but even the *greatest* element.
5. $\mathfrak{R}_{\mathfrak{F}}$ and $\tilde{\mathfrak{R}}_{\mathfrak{F}}$ implicitly depend on the weight function b.

The following properties can be deduced directly from the definitions.

Lemma 3.4 *Let $\mathfrak{F} \subset \mathfrak{V} \subset \mathfrak{B}_b$ and $P, Q \in \mathbb{P}_b$.*

a) $P \leq_{\mathfrak{V}} Q \ \Rightarrow\ P \leq_{\mathfrak{F}} Q$.

b) $\mathfrak{R}_{\mathfrak{F}} \subset \mathfrak{R}_{\mathfrak{V}}$ *and* $\tilde{\mathfrak{R}}_{\mathfrak{F}} \subset \tilde{\mathfrak{R}}_{\mathfrak{V}}$*.*

c) If $\mathfrak{V} \subset \mathfrak{R}_{\mathfrak{F}}$*, then* $\leq_{\mathfrak{V}}$ *and* $\leq_{\mathfrak{F}}$ *are identical.*

The next two results are counterparts to Proposition 3.3 and 3.5 in Marshall (1991).

Theorem 3.5 *Let* $\mathfrak{F}$ *be an arbitrary generator of an order* $\leq_{\mathfrak{F}}$ *. Then:*

a) $\mathfrak{R}_{\mathfrak{F}}$ *contains the convex cone spanned by* $\mathfrak{F}$*;*

b) $\mathfrak{R}_{\mathfrak{F}}$ *contains all constant functions;*

c) If the sequence $(f_n)_{n\in\mathbb{N}} \subset \mathfrak{R}_{\mathfrak{F}}$ *converges uniformly to* f*, then* $f \in \mathfrak{R}_{\mathfrak{F}}$*.*

Proof. a) Let $f_1, ..., f_n \in \mathfrak{F}$ and $a_1, ..., a_n \in \mathbb{R}_{>0}$. Since $\mathfrak{B}_b$ is a vector space, we have $f := \sum a_i f_i \in \mathfrak{B}_b$. Now, if $P, Q \in \mathbb{P}_b$ with $P \leq_{\mathfrak{F}} Q$, then by definition $P(f_i) \leq Q(f_i)$ and thus

$$P(f) = \sum_i a_i\ P(f_i)\ \leq\ \sum_i a_i\ Q(f_i) = Q(f).$$

Hence $f \in \mathfrak{R}_{\mathfrak{F}}$.

b) is trivial.

c) Let $(f_i) \subset \mathfrak{R}_{\mathfrak{F}}$ be a sequence with $\|f_i - f\|_\infty \to 0$ for some $f : S \to \mathbb{R}$. Then for every $\varepsilon > 0$ there is a $n \in \mathbb{N}$ with $\|f_n - f\|_\infty \leq \varepsilon$. Thus, as $b \geq 1$, $|f(x)| \leq |f_n(x)| + \varepsilon \leq (\|f_n\|_b + \varepsilon) \cdot b(x)$. Consequently $f \in \mathfrak{B}_b$. Therefore, if $P \leq_{\mathfrak{F}} Q$, we have

$$P(f) = P(\lim f_n) = \lim P(f_n) \leq \lim Q(f_n) = Q(f),$$

establishing $f \in \mathfrak{R}_{\mathfrak{F}}$. □

Theorem 3.6 *Let* $(\Omega, \mathcal{A}, \rho)$ *be a* σ*-finite measure space and let* $f : \Omega \times S \to \mathbb{R}$ *be a* $\mathcal{A} \otimes \mathcal{S}$*-measurable function, which fulfils the following assumptions:*

(i) $f(\omega, \cdot) \in \mathfrak{F}$ *for all* $\omega \in \Omega$*;*

(ii) There exists a ρ*-integrable function* $c : \Omega \to \mathbb{R}_{\geq 0}$ *with* $|f(\omega, s)| \leq c(\omega) \cdot b(s)$ *for all* $\omega \in \Omega,\ s \in S$*.*

Then $g(\cdot) := \int f(\omega, \cdot)\rho(d\omega)$ *exists and belongs zu* $\mathfrak{R}_{\mathfrak{F}}$*.*

Proof. Since $|f(\omega, x)| \leq c(\omega) \cdot b(x)$ we have for all $\mu \in \mathbb{M}_b$:

$$\int\!\!\int |f(\omega, x)|\ \rho(d\omega)|\mu|(dx) \leq \int c(\omega)\rho(d\omega) \cdot \int b(x)|\mu|(dx) < \infty. \qquad (3.4)$$

Specializing $\mu = \delta_s,\ s \in S$, we can infer the existence of

$$g(s) = \int f(\omega, s)\rho(d\omega).$$

Now (3.4) and (ii) imply $\|g\|_b \le \int c\, d\rho < \infty$. Hence $g \in \mathfrak{B}_b$ and we can apply Fubini's theorem. Thus for $P, Q \in \mathbb{P}_b$ with $P \le_{\mathfrak{F}} Q$ we have

$$\begin{aligned} \int g(x)P(dx) &= \iint f(\omega,x)\rho(d\omega)P(dx) = \iint f(\omega,x)P(dx)\rho(d\omega) \\ &\le \iint f(\omega,x)Q(dx)\rho(d\omega) = \int g(x)Q(dx). \end{aligned}$$

This yields $g \in \mathfrak{R}_{\mathfrak{F}}$. □

Now we are ready for our main result.

Theorem 3.7 *$\mathfrak{R}_{\mathfrak{F}}$ is the $\sigma(\mathfrak{B}_b, \mathbb{M}_b)$-closure of the convex cone, which is spanned by $\mathfrak{F}$ and the constant functions.*

Proof. First we consider the strict duality $(\mathbb{M}_b^N, \mathfrak{B}_{b/\sim})$. As $\mathbb{M}_b^N$ is the linear span of $\mathbb{P}_b - \mathbb{P}_b$, the definition of $\mathfrak{R}_{\mathfrak{F}}$ implies

$$f \in \mathfrak{R}_{\mathfrak{F}/\sim} \;\Leftrightarrow\; \mu(f) \ge 0 \;\; \forall \mu \in (\mathfrak{F}_{/\sim})^+ \;\Leftrightarrow\; f \in ((\mathfrak{F}_{/\sim})^+)^+. \tag{3.5}$$

From Theorem 3.5 we deduce that $\mathfrak{R}_{\mathfrak{F}/\sim}$ contains the convex cone spanned by $\mathfrak{F}_{/\sim}$. Now a look at Lemma 3.4 c) shows that we can assume without loss of generality $\mathfrak{F}_{/\sim}$ to be a convex cone. But by Lemma 2.2 $\mathbb{M}_b^N$ and $\mathfrak{B}_{b/\sim}$ are in strict duality, and thus Theorem 2.3 and (3.5) imply that $\mathfrak{R}_{\mathfrak{F}/\sim}$ is the $\sigma(\mathfrak{B}_{b/\sim}, \mathbb{M}_b^N)$-closure of $\mathfrak{F}_{/\sim}$. Now the result follows immediately from the definition of the equivalence relation $\sim$. □

Theorem 3.7 is rather of theoretical nature. As the $\sigma(\mathfrak{B}_b, \mathbb{M}_b)$-topology is hard to handle, it is not very useful for applications. In our next result, however, we give a sufficient condition for $\mathfrak{F} = \mathfrak{R}_{\mathfrak{F}}$, which is very easy to check.

Corollary 3.8 *If $\mathfrak{F} \subset \mathfrak{V} \subset \mathfrak{R}_{\mathfrak{F}}$, and $\mathfrak{V}$ is a convex cone containing the constant functions and closed under pointwise convergence, then $\mathfrak{V} = \mathfrak{R}_{\mathfrak{F}}$.*

Proof. It is enough to show that $\mathfrak{V}$ is closed with respect to the topology $\sigma(\mathfrak{B}_b, \mathbb{M}_b)$. Since $\mathbb{M}_b$ includes all one point measures, the $\sigma(\mathfrak{B}_b, \mathbb{M}_b)$-topology is finer than the topology of pointwise convergence. Hence each set, which is closed under pointwise convergence, is also closed with respect to $\sigma(\mathfrak{B}_b, \mathbb{M}_b)$. □

In general $\tilde{\mathfrak{R}}_{\mathfrak{F}}$ is larger than $\mathfrak{R}_{\mathfrak{F}}$. For example, if $S = \mathbb{R}$, $b \equiv 1$ and $\mathfrak{R}_{\mathfrak{F}}$ is the set of all bounded increasing functions, then $\tilde{\mathfrak{R}}_{\mathfrak{F}}$ is the set of all increasing functions. This is an easy consequence of the following theorem.

Theorem 3.9 *Let $(f_n) \subset \mathfrak{R}_{\mathfrak{F}}$ be a monotone sequence, converging to a real-valued function f (not necessarily in $\mathfrak{B}_b$). Then $f \in \tilde{\mathfrak{R}}_{\mathfrak{F}}$.*

Proof. For probability measures $P, Q \in \mathbb{P}_b$ with $P \leq_{\mathfrak{F}} Q$ and $P(|f|), Q(|f|)$ finite we have to show that $P(f) \leq Q(f)$ holds. But if $f_n \in \mathfrak{R}_{\mathfrak{F}}$, then $P(f_n)$ and $Q(f_n)$ are finite and $P(f_n) \leq Q(f_n)$ holds. Now the monotone convergence theorem implies $P(f) = \lim P(f_n) \leq \lim Q(f_n) = Q(f)$. □

In contrast to maximal generators there are in general no *minimal* generators, see 3.11. Nevertheless there are often "small" generators, which give rise to the same order (and are better suited for checking $P \leq_{\mathfrak{F}} Q$). In the next theorem we use the following notation:
A subset B of a cone K in an arbitrary vector space is called a *base*, if for every $x \in K$ with $x \neq 0$ there is exactly one $y \in B$ and one $\alpha \in \mathbb{R}_{>0}$ with $x = \alpha y$.

Theorem 3.10
a) If $\mathfrak{F}$ is a convex cone, then each base of $\mathfrak{F}$ also generates $\leq_{\mathfrak{F}}$.

b) Each set, which is dense in $\mathfrak{F}$ with respect to uniform convergence, is a generator of $\leq_{\mathfrak{F}}$.

Proof. a) If $\mathfrak{V}$ is a base of $\mathfrak{F}$, then Theorem 3.5 implies that $\mathfrak{R}_{\mathfrak{V}}$ contains the convex cone, which is spanned by $\mathfrak{V}$. Thus $\mathfrak{V} \subset \mathfrak{F} \subset \mathfrak{R}_{\mathfrak{V}}$. Now from Lemma 3.4 c) we can infer that $\leq_{\mathfrak{F}}$ and $\leq_{\mathfrak{V}}$ are identical.

b) follows like a) from 3.4 c) and 3.5 b). □

In the proof of the next theorem we use some well known properties of the usual stochastic order $\leq_{st}$, which is introduced in section 5.

Theorem 3.11 *In case $S = \mathbb{R}$ there is no minimal generator for $\leq_{st}$.*

Proof. Suppose $\mathfrak{F}_1$ is a minimal generator. Due to 3.10 a) and 3.5 b) we can assume without loss of generality

$$\inf f = 0 \quad \text{and} \quad \sup f = 1 \tag{3.6}$$

for all $f \in \mathfrak{F}_1$.

We denote by $\mathfrak{W}$ the set of all increasing functions, for which (3.6) holds, and let $\mathfrak{E}$ be the set of extreme points of $\mathfrak{W}$. Obviously $\mathfrak{E}$ is the set of all functions, which assume only the values 0 and 1, and which have exactly one switch from 0 to 1. As each $f \in \mathfrak{W}$ can be approximated uniformly by step functions in $\mathfrak{W}$, it is easy to see that $\mathfrak{W}$ is the (weak) closure of the convex hull of $\mathfrak{E}$. Thus, by minimality of $\mathfrak{F}_1$, Theorem 3.7 implies $\mathfrak{F}_1 \subset \mathfrak{E}$.

If we denote by A the set of switching points of functions in $\mathfrak{F}_1$, then A must be dense in $\mathbb{R}$. Otherwise, if there would be $a < b$ with $A \cap [a,b] = \emptyset$, then we would have $f(a) = f(b)$ for all $f \in \mathfrak{F}_1$ and thus $\delta_b \leq_{st} \delta_a$, a contradiction.

On the other hand, for every $A' \subset \mathbb{R}$, which is dense in $\mathbb{R}$, the corresponding set of increasing step functions with one switch from 0 to 1 is a generator of $\leq_{st}$. But if we delete an arbitrary element a_0 from a dense set A, then $\tilde{A} := A \backslash \{a_0\}$ is still dense in $\mathbb{R}$. Therefore, if we delete a function f from $\mathfrak{F}_1$, then $\tilde{\mathfrak{F}} := \mathfrak{F}_1 \backslash \{f\}$ still generates $\leq_{st}$. Thus $\mathfrak{F}_1$ can not be minimal. □

4 Closure Properties.

In this section we define some interesting properties of (integral) stochastic orders and show, how they arise directly as consequences of corresponding properties of the generator $\mathfrak{F}$. Most of these properties have been defined by Stoyan (1983). Some of them can also be found in Marshall (1991).

Sometimes it is more convenient to formulate the results for random variables (r.v.) instead of p.m.'s. Therefore we write $X \leq_{\mathfrak{F}} Y$, if $P_X \leq_{\mathfrak{F}} P_Y$ holds for the accompanying distributions.

Definition 4.1 *Let S be some ordered metric vector space and let $b : S \to [1,\infty)$ be some weight function. Let $\leq_{\mathfrak{F}}$ be some (pre-)order on $\mathbb{P}_b$. Then $\leq_{\mathfrak{F}}$ has*
a) Property **(R)**, *if $a \leq b$ implies $\delta_a \leq_{\mathfrak{F}} \delta_b$;*
b) Property **(E)**, *if $X \leq_{\mathfrak{F}} Y$ implies $EX \leq EY$ (assumed the expectations exist);*
c) Property **(M)**, *if $X \leq_{\mathfrak{F}} Y$ implies $aX \leq_{\mathfrak{F}} aY$ for all $a \geq 0$;*
d) Property **(T)**, *if $X \leq_{\mathfrak{F}} X + a$ holds for all r.v. X and all positive a;*
e) Property **(C)**, *if $P_1 \leq_{\mathfrak{F}} P_2$ implies $P_1 * Q \leq_{\mathfrak{F}} P_2 * Q$ for all p.m. Q;*
f) Property **(W)**, *if $\leq_{\mathfrak{F}}$ is closed with respect to weak convergence, i.e. if $P_n \leq_{\mathfrak{F}} Q_n$ holds for all $n \in \mathbb{N}$ and the sequences $(P_n), (Q_n)$ converge weakly to P resp. Q, then $P \leq_{\mathfrak{F}} Q$.*

We will show that all these properties can be traced back to properties of the generator $\mathfrak{F}$ respectively $\mathfrak{R}_{\mathfrak{F}}$. Part e) of the following theorem can be found in similar form in Stoyan (1983), Prop. 1.1.2. and the if-part of f) is contained in Proposition 3.13 of Marshall (1991). As usual, we denote the set of all bounded continuous functions by $\mathfrak{C}_b$.

Theorem 4.2 *a) Property (R) holds, if and only if all functions in $\mathfrak{F}$ are increasing.*

b) Property (E) holds, if and only if $\tilde{\mathfrak{R}}_{\mathfrak{F}}$ *contains all increasing linear functions.*
c) Property (M) holds, if and only if $\mathfrak{R}_{\mathfrak{F}}$ *is scale invariant, i.e.* $f \in \mathfrak{R}_{\mathfrak{F}}$ *and* $a > 0$ *implies* $f_a \in \mathfrak{R}_{\mathfrak{F}}$*, where* $f_a(x) := f(ax)$.
d) Property (T) holds, if and only if all functions in $\mathfrak{F}$ *are increasing.*
e) Property (C) holds, if and only if $\mathfrak{R}_{\mathfrak{F}}$ *is invariant under translations.*
f) Property (W) holds, if and only if there is a generator $\mathfrak{V} \subset \mathfrak{C}_b$ *for* $\leq_{\mathfrak{F}}$.

Proof. Part a) - e) and the if-part of f) are easy to check. Therefore we only prove the only-if-part of f). Suppose $\leq_{\mathfrak{F}}$ has property (W) and there is no generator $\mathfrak{V} \subset \mathfrak{C}_b$. Then define $\mathfrak{F}_1 := \mathfrak{R}_{\mathfrak{F}} \cap \mathfrak{C}_b$,

$$\mathbb{F} := \{\mu \in \mathbb{M}_b^N : \mu \in \mathfrak{F}^+\} \quad \text{and} \quad \mathbb{F}_1 := \{\mu \in \mathbb{M}_b^N : \mu \in \mathfrak{F}_1^+\}.$$

We endow $\mathbb{M}_b^N$ with the weak topology $\sigma(\mathbb{M}_b^N, \mathfrak{C}_b)$. Then, by assumption, $\mathbb{F}$ is a closed convex cone and a proper subset of $\mathbb{F}_1$. Hence, by a separation theorem for closed convex sets (Choquet (1969), Theorem 21.12), for $\mu \in \mathbb{F}_1 \backslash \mathbb{F}$ there is a continuous linear functional L on $\mathbb{M}_b^N$ with

$$L(\mu) \;<\; \inf_{\nu \in \mathbb{F}} L(\nu) =: \alpha. \tag{4.1}$$

Since $\mathbb{F}$ is a convex cone, $\alpha = 0$. Now, by Proposition 22.4 in Choquet (1969) we can represent L as a function $f \in \mathfrak{C}_b$ with $L(\mu) = \mu(f)$. Thus (4.1) implies $\inf_{\nu \in \mathbb{F}} \nu(f) = 0$ and $\mu(f) < 0$. Hence $f \in \mathbb{F}^+ = \mathfrak{R}_{\mathfrak{F}}$ and $f \notin \mathfrak{F}_1$, a contradiction. □

5 Examples.

A. The usual stochastic order $\leq_{st}$.

The best known stochastic order relation is the *usual stochastic dominance* $\leq_{st}$. This order can already be found in Lehmann (1955), and has been studied in a very general setting by Kamae, Krengel and O'Brien (1977). They assume S to be a Polish space, endowed with a closed partial order. Then they define $P \leq_{st} Q$ if $\int f\, dP \leq \int f\, dQ$ for all measurable bounded increasing functions f.

We give an equivalent definition in our setting. Let S be an ordered Polish space and choose $b \equiv 1$ as weight function. This means that $\mathbb{P}_b$ is the set of all p.m.'s and $\mathfrak{B}_b$ is the set of all bounded measurable functions. Now define $\leq_{st}$ to be the order induced by the generator $\mathfrak{F}_{st}$, where $\mathfrak{F}_{st}$ is the set of all indicator functions 1_A, such that A is a measurable increasing set.

Remark: Mosler and Scarsini (1991) claim that all increasing sets in a partially ordered Polish space are (Borel-)measurable. This statement is wrong, as is easily seen by the following example in $\mathbb{R}^2$. Let $M \subset \mathbb{R}_{>0}$ be a nonmeasurable set and define

$$A := \{(x, y) \in \mathbb{R}^2 : x > 0,\ y > 1/x\} \cup \{(x, 1/x) : x \in M\}.$$

Then A is a nonmeasurable increasing set.

Our definition is consistent with that given by Kamae, Krengel and O'Brien (1977). This is a consequence of part a) of the following theorem. A proof of it can be found in Kamae, Krengel and O'Brien (1977) and Stoyan (1983), but most of the results can be deduced easily from 3.7, 3.8 and 4.2.

Theorem 5.1 *For the order $\leq_{st}$ the following statements hold:*

a) $\mathfrak{R}_{\mathfrak{F}}$ is the set of all measurable increasing bounded functions.

b) $\tilde{\mathfrak{R}}_{\mathfrak{F}}$ is the set of all measurable increasing functions.

c) The order $\leq_{st}$ has the properties (R), (E), (M), (T), (C) and (W).

Remark. By Theorem 4.2 (W) holds if and only if there is a generator $\mathfrak{V}$ of $\leq_{st}$, which consists of bounded continuous functions.

If S is a Polish vector space, endowed with a vector space ordering $\leq$ and a metric d, (with d and $\leq$ invariant under translations), then a generator $\mathfrak{V} \subset \mathfrak{C}_b$ can be defined as follows. For A a closed increasing set and $n \in \mathbb{N}$ define

$$f_{A,n}(x) := \max\{0, 1 \Leftrightarrow n \cdot d(A, x)\}. \tag{5.1}$$

Then the set $\mathfrak{V}$ of all these functions generates $\leq_{st}$.

This can be proved as follows. By Theorem 1 (vi) of Kamae, Krengel and O'Brien (1977) the set $\mathfrak{F}$ of all indicator functions 1_A of closed increasing sets A is a generator of $\leq_{st}$. Now, for fixed A and $n \to \infty$ the sequence $(f_{A,n})_{n \in \mathbb{N}}$ converges monotonely to 1_A. Thus we have $\mathfrak{F} \subset \mathfrak{R}_{\mathfrak{V}}$. Hence 3.4 implies the assertion, if we can verify that $f_{A,n}$ is increasing, bounded and continuous. Since boundedness and continuity are obvious, only monotonicity remains to be shown. But for $x \leq y$ we have $a \leq a + y \Leftrightarrow x$ for all $a \in A$, as $\leq$ is invariant under translations. Hence, since A is an increasing set, $A' := \{a + y \Leftrightarrow x,\ a \in A\} \subset A$. As d is invariant under translations, this implies

$$d(x, A) \ = \ \inf_{a \in A} d(x, a) \ = \ \inf_{a \in A} d(x + y \Leftrightarrow x, a + y \Leftrightarrow x) \ = \ d(y, A') \ \geq \ d(y, A).$$

Thus $f_{A,n}(x) \leq f_{A,n}(y)$.

It seems intuitively clear that the function $f_{A,n}$ is increasing. But without the assumption, that d and $\leq$ are invariant under translations, this needs not to be true, as is seen from the following counterexample.

Let $S = \mathbb{R}^2$ and define the ordering $\leq$ by the convex cone $C = \{(x,x) : x \geq 0\}$. The metric d shall be defined by

$$d((x_1,x_2),(y_1,y_2)) \;=\; |x_1^3 \Leftrightarrow y_1^3| + |x_2^3 \Leftrightarrow y_2^3|.$$

Then d induces the euclidian topology and the ordering $\leq$ is closed. If $A := \{(x,x) : x \geq 1\}$, then A is increasing and closed, but $x \to d(x,A)$ is not decreasing (and thus $f_{A,n}$ is not increasing). Let $x = (1,0)$ and $y = (2,1)$. Then $x \leq y$ and $d(x,A) = 1 < 7 = d(y,A)$. Moreover, $x \to d(x,A)$ is even strictly increasing on the set $I := \{(x_1,x_2) : x_1, x_2 \geq 1; x_1 \neq x_2\}$.

Nevertheless, if A is a closed increasing set in an arbitrary Polish space, which is endowed with a closed partial order, then the indicator function 1_A is the pointwise limit of a decreasing sequence of continuous increasing functions. This follows from Theorem 5 in the Appendix of Nachbin (1965).

B. Convex and increasing convex order.

In case $S = \mathbb{R}^n$ the convex order $\leq_{cx}$ and the increasing convex order $\leq_{icx}$ are well known, see e.g. Shaked and Shanthikumar (1994). In this case it is necessary to use a nontrivial weight function. This is seen by the following example for $S = \mathbb{R}$. If Q denotes Cauchy's distribution (which has no expectation value), and f is a convex function, then $\int f\, dQ$ exists if and only if f is constant. Therefore we use the weight function $b(s) = 1 + \|s\|$. Thus $\leq_{cx}$ and $\leq_{icx}$ are defined for random variables, for which $E\boldsymbol{X}$ exists.

Theorem 5.2 *The relations $\leq_{icx}$ and $\leq_{cx}$ are partial orders.*

Proof. It suffices to show that $\leq_{icx}$ is antisymmetric. Suppose $\boldsymbol{X} \leq_{icx} \boldsymbol{Y}$ and $\boldsymbol{Y} \leq_{icx} \boldsymbol{X}$ and let $F_{\boldsymbol{X}}$ resp. $F_{\boldsymbol{Y}}$ be the corresponding distribution functions. For $\boldsymbol{t} \in \mathbb{R}^n$ define

$$\phi_{\boldsymbol{t}}(\boldsymbol{x}) := \max_{i=1}^{n}(x_i \Leftrightarrow t_i)^+$$

and let $\Psi_{\boldsymbol{X}}(\boldsymbol{t}) := E\phi_{\boldsymbol{t}}(\boldsymbol{X})$. Since $\phi_{\boldsymbol{t}}$ is increasing and convex, we can infer $\Psi_{\boldsymbol{X}}(\boldsymbol{t}) = \Psi_{\boldsymbol{Y}}(\boldsymbol{t})$ for all $\boldsymbol{t} \in \mathbb{R}^n$. Since for $\boldsymbol{e} := (1,1,...,1)$ it holds

$$\lim_{\varepsilon \to 0} \frac{\Psi_{\boldsymbol{X}}(\boldsymbol{t} + \varepsilon \boldsymbol{e}) \Leftrightarrow \Psi_{\boldsymbol{X}}(\boldsymbol{t})}{\varepsilon} \;=\; F_{\boldsymbol{X}}(\boldsymbol{t}) \Leftrightarrow 1,$$

we get $F_{\boldsymbol{X}}(\boldsymbol{t}) = F_{\boldsymbol{Y}}(\boldsymbol{t})$ for all $\boldsymbol{t} \in \mathbb{R}^n$. □

Theorem 5.3
a) For $\leq_{cx}$ the following holds:

(i) $\mathfrak{R}_{\mathfrak{F}}$ *is the set of all convex functions in* $\mathfrak{B}_b$*.*

(ii) $\tilde{\mathfrak{R}}_{\mathfrak{F}}$ *is the set of all convex functions.*

(iii) $\leq_{cx}$ *has the properties (E), (M) and (C), but the properties (R), (T) and (W) do not hold.*

b) For $\leq_{icx}$ *the following holds:*

(i) $\mathfrak{R}_{\mathfrak{F}}$ *is the set of all increasing convex functions in* $\mathfrak{B}_b$*.*

(ii) $\tilde{\mathfrak{R}}_{\mathfrak{F}}$ *is the set of all increasing convex functions.*

(iii) $\leq_{icx}$ *has the properties (R), (E), (M), (T) and (C), but the property (W) does not hold.*

Proof. The Theorem easily follows from 3.7, 3.8 and 4.2.

C. Ordering by Laplace transform and by moment generating functions.

The following order can be found in Chapter 3.B of Shaked and Shanthikumar (1994).

Definition 5.4
Let $S = \mathbb{R}_{\geq 0}$, $b \equiv 1$ *and* $\mathfrak{F}_{Lt}$ *be the class of functions of the form* $f(s) = -e^{-\alpha s}$, $\alpha > 0$. *The order induced by* $\mathfrak{F}_{Lt}$ *is called* **Laplace transform order** *and is denoted by* $\leq_{Lt}$*.*

To characterize the maximal generator of $\leq_{Lt}$ we need the following notion. A function $f : \mathbb{R}_{>0} \to \mathbb{R}$ is said to be *completely monotone*, if all derivatives exist and satisfy $(-1)^n f^{(n)}(x) \geq 0$ for all $n \in \mathbb{N}_0$ and $x > 0$.

The following theorem can be found in similar form in Shaked and Shanthikumar (1994), Th. 3.B.1, 3.B.2 and 3.B.4. The characterization of $\tilde{\mathfrak{R}}_{\mathfrak{F}}$ is originally due to Reuter and Riederich (1981), but their proof is very complicated and lengthy. Therefore we show, how it can be deduced from section 3.

Theorem 5.5 *For the order* $\leq_{Lt}$ *the following holds:*

(i) $\mathfrak{R}_{\mathfrak{F}}$ *is the set of all bounded functions, that have a completely monotone derivative.*

(ii) $\tilde{\mathfrak{R}}_{\mathfrak{F}}$ *is the set of all functions, that have a completely monotone derivative.*

(iii) $\leq_{Lt}$ has the properties (R), (E), (M), (T), (C) and (W).

Proof of (i). Let $\mathfrak{V}$ be the set of all functions, that have a completely monotone derivative. By a celebrated theorem of Bernstein, a completely monotone function f is the Laplace transform of some measure μ, i.e. $f(s) = \int e^{-sx}\ \mu(dx)$. Therefore, if $F \in \mathfrak{V}$ with $f = F'$, then $F(s) = \int(\Leftrightarrow e^{-sx}/x)\ \mu(dx) + c$, where $c \in \mathbb{R}$ is some constant. Hence Theorem 3.6 implies $\mathfrak{V} \subset \mathfrak{R}_{\mathfrak{F}}$. On the other hand, $\mathfrak{V}$ is a convex cone, which contains the constant functions. Hence the assertion follows from Corollary 3.8, if we can show that $\mathfrak{V}$ is closed with respect to pointwise convergence. But it is well known that completely monotone functions can be characterized by the alternating signs of their finite differences, cf. Reuter and Riederich (1981), Theorem 1 (2). From this characterization it can easily be deduced that $f \in \mathfrak{V}$, if and only if

$$(\Leftrightarrow 1)^n \Delta_{h_0} \Delta_{h_1} ... \Delta_{h_n} f(t) \geq 0$$

for all $t, h_0, h_1, ..., h_n \in \mathbb{R}_{>0}$, $n \in \mathbb{N}_0$. But this pointwise characterization implies that $\mathfrak{V}$ is closed with respect to pointwise convergence. □

Integer-valued random variables are very often analyzed by utilization of the moment generating function $t \to E(t^X)$. Thus it is natural to introduce a *moment generating function order* $\leq_{mgf}$, which is induced by the class of functions of the form $s \to t^s$, $t \in (0, 1)$. But this order is equivalent to $\leq_{Lt}$, as $t^s = \Leftrightarrow e^{-\alpha s}$ with $\alpha := \Leftrightarrow \log t$.

D. Further examples.

A plenty of further examples of integral stochastic orders can be found in Shaked and Shanthikumar (1994). For each of these orders they state some "closure properties". Most of them can easily be deduced from Theorem 4.3. Our property (C) e.g. can be found in Shaked and Shanthikumar (1994) for several orders in Theorem 2.A.6, 3.A.5, 3.B.4, 3.B.10, 3.B.13, 5.A.4 etc.. But property (C) also holds for some more orders, for example for $\leq_{ccx}$, $\leq_{iccx}$, $\leq_{uo-cx}$ and for the supermodular ordering $\leq_{sm}$, which has been utilized by Szekli et. al. (1994). This follows immediately from 4.2 e), since their generating classes of functions are invariant under translations.

E. A counterexample: The likelihood ratio order $\leq_{lr}$.

We don't want to conceal that there are some important stochastic orders, which are not induced by integrals. One of them is the likelihood ratio order $\leq_{lr}$, cf. section 1.C in Shaked and Shanthikumar (1994).

Definition 5.6
The real random variables X, Y are said to be in **likelihood ratio order**

($X \leq_{lr} Y$), if they have densities f and g with respect to some σ-finite measure μ, such that

$$f(y)g(x) \ \leq \ f(x)g(y) \quad \textit{for all } x \leq y. \tag{5.2}$$

Remarks: 1. Inequality (5.2) can be rewritten in the form

$$f(y)g(x) \ \leq \ f(x \wedge y)g(x \vee y) \quad \text{for all } x, y \in \mathbb{R}.$$

where $\wedge$ and $\vee$ are the usual lattice operators. In this form, the definition can easily be extended to vector valued random variables. This *multivariate likelihood ratio order* can be found in Karlin and Rinott (1980).

2. An equivalent definition of $\leq_{lr}$ can be given as follows:

$$P \leq_{lr} Q \ \text{ if } \ P(J)Q(I) \leq P(I)Q(J) \ \text{ for all intervals } I, J \text{ with } I \leq J,$$

where $I \leq J$ means that $x \in I$ and $y \in J$ imply $x \leq y$.
From the proof of Theorem 1.C.2 in Shaked and Shanthikumar (1994) one can deduce that this is equivalent to the definition given above.

Proposition 5.7 *The order $\leq_{lr}$ is not an integral stochastic order.*

Proof. Assume $\mathfrak{R}_{\mathfrak{F}}$ is the maximal generator of $\leq_{lr}$. Since $\leq_{lr}$ has property (R), Theorem 4.2 a) implies that all functions in $\mathfrak{R}_{\mathfrak{F}}$ are increasing. On the other hand, $\mathfrak{R}_{\mathfrak{F}}$ must include all bounded increasing functions as $\leq_{lr}$ is stronger than $\leq_{st}$. Hence $\mathfrak{R}_{\mathfrak{F}}$ is the set of all bounded increasing functions, and thus $\leq_{st}$ and $\leq_{lr}$ must be identical. But it is well known that this is not true. □

Nevertheless $\leq_{lr}$ shares most of the properties introduced in Definition 4.1. It is well known that $\leq_{lr}$ has the properties (R), (E) and (M). It is also well known that property (C) does not hold, see e.g. Shanthikumar and Yao (1986). But it seems to be unknown so far that $\leq_{lr}$ has property (W).

Theorem 5.8 *The likelihood ratio order $\leq_{lr}$ has property (W), i.e. it is closed with respect to weak convergence.*

Proof. Assume $P_n \leq_{lr} Q_n$ for all $n \in \mathbb{N}$ and that $(P_n), (Q_n)$ converge weakly to P resp. Q. By Corollary 2.1 in Karlin and Rinott (1980) for arbitrary $A, B \in \mathcal{B}$

$$P_n(A)\ Q_n(B) \leq P_n(A \wedge B)\ Q_n(A \vee B),$$

where $A \wedge B := \{a \wedge b : a \in A, b \in B\}$ and $A \vee B$ is defined analogously.

Let $I_x(h)$ denote the closed interval $[x - h, x + h]$. Then for arbitrary $x, y \in \mathbb{R}$ we have $I_x(h) \wedge I_y(h) = I_{x \wedge y}(h)$ and $I_x(h) \vee I_y(h) = I_{x \vee y}(h)$. Hence

$$P_n(I_y(h))\ Q_n(I_x(h)) \leq P_n(I_{x \wedge y}(h))\ Q_n(I_{x \vee y}(h)). \tag{5.3}$$

Now let μ be a σ-finite measure with $\mu(I_x(h)) > 0$ for all x and h, which dominates P and Q. Such a measure exists, as you can add Lebesgue's measure to an arbitrary dominating measure without affecting the dominance property.

By the portmanteau theorem (cf. Dudley (1989), Theorem 11.1.1), inequality (5.3) implies

$$P(I_y(h))\ Q(I_x(h)) \leq P(I_{x\wedge y}(h))\ Q(I_{x\vee y}(h)),$$

if the intervals are μ-continuity sets.

Next we construct a sequence $h_n \downarrow 0$, such that the intervals $I_x(h_n)$ are μ-continuity sets for μ-almost all $x \in \mathbb{R}$. Denote by M the countable set of atoms of μ. Then the set $M \Leftrightarrow M$ is also countable. Now select $h_n \in (M \Leftrightarrow M)^c$ with $h_n \downarrow 0$. Then $I_x(h_n)$ is a μ-continuity set for all $x \in M$. Furthermore, the set of all $x \in M^c$, for which $I_x(h_n)$ is not a μ-continuity set, is at most countable. Hence $I_x(h_n)$ is a μ-continuity set for μ-almost all $x \in \mathbb{R}$.

Consequently, for μ-almost all $x, y \in \mathbb{R}$,

$$\frac{P(I_y(h_n))}{\mu(I_y(h_n))}\ \frac{Q(I_x(h_n))}{\mu(I_x(h_n))} \leq \frac{P(I_{x\wedge y}(h_n))}{\mu(I_{x\wedge y}(h_n))}\ \frac{Q(I_{x\vee y}(h_n))}{\mu(I_{x\vee y}(h_n))},$$

as $\{x, y\} = \{x \wedge y, x \vee y\}$.

Now the general differentiation theorem for measures (Wheeden and Zygmund (1977), Theorem 10.49 and Corollary 10.50) yields

$$f(y)\ g(x) \leq f(x \wedge y)\ g(x \vee y) \text{ for } \mu\text{-almost all } x, y \in \mathbb{R},$$

where f and g are μ-densities of P resp. Q. Hence $P \leq_{lr} Q$. □

There are several other orders, which are stronger than $\leq_{st}$. Examples are the *hazard rate order* $\leq_{hr}$, the *reversed hazard rate order* $\leq_{rh}$ and the *shifted likelihood ratio order* $\leq_{lr\uparrow}$. With exactly the same proof as for Proposition 5.7 it follows that these orders are no integral stochastic orders.

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