Topological Aspects of Numberings

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Abstract

We investigate connections between the syntactic and semantic distance of programs on an abstract, recursion theoretic level. For a certain rather restrictive notion of interdependency of the two kinds of distances, there remain only few and "unnatural" numberings allowing for such close relationship. Weakening the requirements leads to the discovery of universal metrics such that for an arbitrary recursively enumerable family of functions a numbering compatible with such a metric can uniformly be constructed. We conclude our considerations with some implications on learning theory.

1 Introduction

In any acceptable numbering, or *Gödel numbering*, of the partial recursive functions, any non-trivial extensional property of the programs is at least as hard as the halting problem [13]. For example, functional equivalence, the *equality problem*, is Π^0_2-hard. On the other hand there exist one-one numberings, or *Friedberg numberings* [6], leaving the equality problem as a trivial task. If both the functions and the programs are considered as points in two respective topological spaces then choosing these topologies to be discrete and choosing the numberings as Friedberg leads to the (trivial) continuity of the indexing $e \mapsto \psi_e$: Similar programs denote similar functions and vice versa, as similarity degenerates to equality in this case. One might ask what happens if less trivial topologies are considered. We will investigate this question for the topology of pointwise convergence defined by some specific distance measure D on functions and suitable recursive metrics on the programs. Regarded questions, then, are of the kind: Which numberings of which families of functions are compatible with which distance measuring on programs, in which particular sense of being "compatible"?

In Section 2, the modeling of the situation, in particular the measure D (together with a modification) and its relationship to some distance measure c on programs, are presented and justified. In Section 3, a particularly strong kind of mutual dependency is considered, which at least for total functions implies that as well the numbering as, conversely,

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all "choice functions" (which to recursive functions assign corresponding programs) are continuous if regarded as mappings between the two respective spaces. This amounts to a rather extreme type of numbering, resembling (but not coinciding with) Friedberg numberings. A weaker, more realistic notion of relatedness is investigated from Section 4 onwards: Similar programs must denote similar functions and for similar functions *there exist* similar programs. One main result, then, says that there exist *universal metrics* c on programs: Any numbering of a family F of functions can recursively be transformed into a numbering of F compatible with c. Questions about the opposite direction, "from numberings to metrics", are considered in Section 5. If the numbering possesses a padding function then some corresponding c can be constructed, but for certain "natural" given distance measures it might occur that no compatible numbering can possess a padding function. Section 6 contains connections to learning theory.

Notation and fundamentals Our notation mainly follows the books by Odifreddi [12] and Soare [17].

$I\!N$ is the set of natural numbers including 0, Q the set of rationals. $A \subseteq I\!N$ is an *initial segment* iff $x \in A$ and $y < x$ imply $y \in A$, for all x, y. $f, g, \ldots$ denote total functions $I\!N \to I\!N$ (if not stated otherwise), $\mu, \nu, \ldots$ partial functions $I\!N \to I\!N$. $dom(\mu)$ $= \{x : \mu(x)\downarrow\}$ is the domain of μ. $\mu \sqsubseteq \nu$ iff for all x, if $\mu(x)\downarrow$ then $\nu(x)\downarrow$ and $\mu(x) = \nu(x)$. Notions $\mu(x)\uparrow$, $\mu(x)\downarrow = \nu(x)\downarrow$, $\mu(x)\downarrow \neq \nu(x)\downarrow$ etc. are as usual. PREC is the set of partial recursive functions $I\!N \to I\!N$, REC the set of (total) recursive functions $I\!N \to I\!N$. A partial function with finite initial segment as domain is called a *string*. Strings are denoted by letters σ, τ. They are often written in a way as, e.g., $010, 0^8, 5^3 1^7$. A similar notation is used for total, almost everywhere constant functions, e.g., $0 2 0^2 1^5 0^\infty$.

$\varphi, \psi, \vartheta, \phi \ldots$ denote partial functions $I\!N \times I\!N \to I\!N$. As usual, ψ_e is the function $x \mapsto \psi(e, x)$. For any family F of partial functions $I\!N \to I\!N$, F is *contained in* ψ iff $F \subseteq \{\psi_e : e \in I\!N\}$, and is *enumerated by* ψ iff $F = \{\psi_e : e \in I\!N\}$. ψ is *dense* iff for each string σ there exists e such ψ_e is total and extends σ. ψ is *one-one* iff, for all e, e', $\psi_e = \psi_{e'}$ implies $e = e'$.

ψ is called a *numbering* (of $\{\psi_e : e \in I\!N\}$) iff ψ is partial recursive, and the indices $e \in I\!N$ are called *programs* w.r.t. ψ. A numbering φ is *acceptable* iff for any numbering ψ there is $g \in REC$ such that $\psi_e = \varphi_{g(e)}$ for all e. For $e, t \in I\!N$, $\psi_{e,t}$ is defined by

$$\psi_{e,t}(x) = \begin{cases} y & \text{if } \psi_e(x)\downarrow = y,\ x \le t, \text{ and } \psi_e(x) \text{ is} \\ & \text{computed from } x \text{ in at most } t \text{ steps,} \\ \uparrow & \text{otherwise.} \end{cases}$$

Notation $\mu_t(x)$ for $\mu : I\!N \to I\!N$ is also used. As usual, we fix some "standard" acceptable numbering φ. $K = \{e : \varphi_e(e)\downarrow\}$ is the halting problem of φ.

A *distance measure* on $I\!N$ is a total function $c : I\!N \times I\!N \to Q$ satisfying $c(i, j) \ge 0$, $c(i, j) = c(j, i)$, and $c(i, i) = 0$ for all i, j. A *topological space* is defined by its set of points, P, and collection G of open sets. The distance function c generates a topolgy G on set of points $I\!N$ iff

(a) for all $x \in I\!N$ and $r \in Q$, $r > 0$, there is an open set $O \in G$ satisfying $x \in O$ and $O \subseteq \{y : c(x,y) < r\}$ and

(b) for all $O \in G$ and $x \in O$ there is an $r > 0$ such that $y \in O$ for all y with $c(x,y) < r$.

A *metric* (on $I\!N$) is a distance measure c satisfying

$$c(i,j) = 0 \;\Leftrightarrow\; i = j \text{ and}$$
$$c(i,j) \leq c(i,k) + c(k,j) \qquad \text{(triangle inequality).}$$

A metric always generates some topology, but there are distance functions that do not. E.g., choose $c(i,j) = 0$ if $|i-j| \leq 2$ and $c(i,j) = 1$ otherwise.

In Section 2 below, distance measures and topologies will also be regarded on the partial functions $I\!N \to I\!N$.

2 Topology of functions and programs

In all what follows we will use a fixed metric D for the distance of total functions, but extending D to partial functions will lead to two different possibilities, each of them being pursued in the further investigation. D will now be introduced and justified.

There are various ways to measure distances between total functions. Rogers [15], §15.2, and Odifreddi [12], V.3, consider the metric (the *Baire metric*)

$$D(f,g) = \begin{cases} 0 & \text{if } f = g, \\ max\{\frac{1}{x+1} : f(x) \neq g(x)\} & \text{otherwise} \end{cases}$$

that is, only the first argument where f and g disagree counts and the particular values taken are ignored. We will adopt this choice, our justification being as follows.

- Total functions can be regarded as points in the product space $I\!N^\infty$. If with each factor $I\!N$ its discrete topology is associated then D generates the corresponding product topology for the space of total functions $I\!N \to I\!N$.

- D models pointwise convergence, i.e.,

$$\text{for all } f \text{ and sequences } (f_k), (\forall x)[f(x) = \lim_k f_k(x)] \text{ iff } \lim_k D(f_k, f) = 0.$$

- One might possibly prefer a distance measure D_1 not taking into account the particular arguments, so that, e.g., all functions $0^n 1 0^\infty$ would have the same distance from 0^∞. This would obviously be in conflict with pointwise convergence. Furthermore, if – as appears natural – one additionally wanted monotonicity, i.e., $D_1(f,f') \geq D_1(g,g')$ whenever $|f(x) - f'(x)| \geq |g(x) - g'(x)|$ for all x, then the consequence would be a constant minimal distance $r > 0$ between any two non-equal functions, namely, $r = D_1(0^\infty, 10^\infty)$. Such a metric would be topologically equivalent to the discrete one and thus unable to express any non-trivial notion of convergence. Thus, for our metric D, while the topology generated by D is invariant under "permutation of the rows", that is, under replacing every function f by $f \circ \pi$ for some permutation π, D itself is not and must use different weights for the respective places where functions disagree.

- One might argue that only differing at infinitely many places should cause nonzero distances between functions, in accordance with a standard level of abstraction in recursion theory. But there are disadvantages of such a measure D_2. It does not generate the product topology on $I\!N^\infty$ and therefore does not allow for pointwise convergence. Also, such D_2 hardly appears feasible w.r.t. the process of computing. Thirdly, D_2 would not be compatible with general recursive operators: For Φ with $\Phi(f) = (f(0))^\infty$ we would obtain $D_2(0^\infty, 10^\infty) = 0$, but $D_2(\Phi(0^\infty), \Phi(10^\infty)) = D_2(0^\infty, 1^\infty) > 0$.
- Metrics such as $D_3(f,g) = 2^{-x-1}$ for the first x with $f(x) \neq g(x)$ are equivalent to D in all respects important in the following. This would also be the case if more places where functions disagree were taken into account, as, e.g., for $D_4(f,g) = \sum_{\{x:f(x)\neq g(x)\}} 2^{-x-1}$, or if even the function values were considered. Using such metrics would only complicate our proofs without noticeable benefits.

Next, regard partial functions. There are two philosophies we have to choose between. If $\mu(x)\uparrow$ then this might be interpreted as (**a**) $\mu(x)$ taking a special value "undefined", or (**b**) $\mu(x)$ representing some unknown value which "might come up later in a worst possible way". Philosophy (**a**) corresponds to identifying partial functions with their graphs (considering these "being given") while (**b**) regards them as never known at the undefined places. Thus, if $\mu(x)\uparrow$ and $\nu(x)\uparrow$ then according to (**a**) μ and ν agree at x, but according to (**b**) they disagree at x in a strongest possible way.

We will prefer using approach (**b**), but, as turns out, approach (**a**) will unavoidably come in at certain points, see Definition 2.3, Remarks 2.6 below. Beside more intuitive reasons for favoring (**b**), our main mathematical one is that when regarding the approximations μ_s, ν_s to μ, ν, respectively, then $D(\mu,\nu)$ is approximated from above by the distances $D(\mu_s,\nu_s)$ in the (**b**)-setting while the approximation might oscillate infinitely often between upper and lower bounds in case (**a**). Thus, our preferred way to extend D follows the idea that *a partial function represents the collection of all total (not necessarily computable) ones extending it:*

$$D(\mu,\nu) = max\{D(f,g) : f \text{ extends } \mu \wedge g \text{ extends } \nu\}.$$

But additionally, a measure D^* that captures view (**a**) will be regarded.

Definition 2.1 For partial functions $\mu, \nu : I\!N \to I\!N$:

$$D(\mu,\nu) = max\{\frac{1}{x+1} : \mu(x)\uparrow \;\vee\; \nu(x)\uparrow \;\vee\; \mu(x)\downarrow \neq \nu(x)\downarrow\}$$

where $max(\emptyset) = 0$. Furthermore

$$D^*(\mu,\nu) = max\left\{\frac{1}{x+1} : (\mu(x)\downarrow \wedge \nu(x)\uparrow) \;\vee\; (\mu(x)\uparrow \wedge \nu(x)\downarrow) \;\vee\; \mu(x)\downarrow\neq \nu(x)\downarrow\right\}$$

As a consequence, a function μ is properly partial iff $D(\mu,\mu) > 0$, i.e., properly partial functions are those *having a positive self-distance*, thus expressing partiality as the "inner uncertainty" in assigning values to arguments. It follows that D ist not a metric, not even a distance measure. All axioms of a metric are satisfied with the exception that $D(\mu,\mu) = 0$ is weakened to

$$D(\mu,\nu) = 0 \text{ iff } \mu,\nu \text{ are total and } \mu = \nu.$$

Of course, D is a metric on total functions. As for D^*, $D^*(\mu,\nu) = 0$ iff $\mu = \nu$, and D^* is a metric. The next proposition is immediate from Definition 2.1.

Proposition 2.2 (1) $D^*(\mu,\nu) \leq D(\mu,\nu)$ for all μ,ν, with equality iff $(\forall x)\left[\frac{1}{x+1} > D^*(\mu,\nu) \;\Rightarrow \mu(x)\downarrow \vee \nu(x)\downarrow\right]$
(2) If $dom(\mu)$, $dom(\nu)$ are initial segments then $\mu = \nu$ or $D^*(\mu,\nu) = D(\mu,\nu)$. In particular, $D^*(f,g) = D(f,g)$ for total f,g.
(3) $D(\mu,\nu) \leq \max\{D(\mu,\rho), D(\rho,\nu)\}$, with equality if $D(\mu,\rho) \neq D(\rho,\nu)$. The corresponding is true for D^*.

Let us now regard distance measures c on $\mathbb{N}$ and their possible compatibility with D, D^*, w.r.t. some underlying numbering. We aim at c being computable. Observe the following facts.

(1) Let ψ be a one-one numbering of PREC and let $c(i,i) = 0$, $c(i,j) = 1$ if $i \neq j$. Then c is a recursive metric on $\mathbb{N}$ generating the discrete topology. c behaves well in yielding similar (i.e., in this case, equal) programs to compute similar functions (up to their possible self-distance). The price we have to pay for this desirable property is high: As ψ is one-one, for usual operators building computable functions from simpler ones there are no corresponding computable functions on programs, representing them on the syntactic level.

(2) Consider the "natural" distance function on programs (represented as bit strings), which is their Hamming distance divided by the maximum of their lengths. Then for the acceptable numberings φ coming from interpreting usual programming languages such as Basic or C there are arbitrarily similar programs i,j with $D(\varphi_i,\varphi_j) = D^*(\varphi_i,\varphi_j) = 1$: Just expand two small programs computing the constant functions 0 and 1, respectively, by large identical parts that will never be executed. Small changes of a program can cause drastic changes of the computed function.

In our central definitions, we will not only consider the desired continuity of "naming" functions by programs, but also that one in the opposite direction. $\mathbb{Q}^+$ is the set of non-negative rationals. A *scaling* is a total function $h : \mathbb{Q}^+ \to \mathbb{Q}^+$ which is strictly increasing, continuous from above at 0 and satisfies $h(0) = 0$.

Definition 2.3 For a numbering ψ and a distance measure c on $I\!N$, the following axioms are regarded.

(A) $\quad (\forall i,j)[D(\psi_i,\psi_j) \;\leq\; max\{c(i,j), D(\psi_i,\psi_i), D(\psi_j,\psi_j)\}]$

(B) $\quad (\forall i,j)(\exists k)[\psi_j = \psi_k \;\wedge\; c(i,k) \leq D(\psi_i,\psi_j)]$

(C) $\quad (\exists \text{ scaling } h)(\forall i,j)[c(i,j) \leq h(D(\psi_i,\psi_j))]$

(A*) $\quad (\forall i,j)[D^*(\psi_i,\psi_j) \;\leq\; c(i,j)]$

(B*), (C*) are as (B), (C), respectively, with D replaced with D^*. Note that this is also the case with (A), (A*), as $D^*(\psi_i,\psi_i) = D^*(\psi_j,\psi_j) = 0$.

ψ, c *match* if (A), (B), (C) are satisfied. Expressions "ψ matches c" and "c matches ψ" will also be used. ψ, c are *compatible* if (A), (B) hold.

If h, say in (C), is recursive then by replacing the (hopefully) recursive c by $c', c'(i,j) = h^{-1}(c(i,j))$, will lead to the stronger form $(\forall i,j)[c'(i,j) \;\leq\; D(\psi_i,\psi_j)]$; hence the interesting case is that c can be found to be recursive while h can not.

Facts 2.4 (A*) implies (A), (B*) implies (B), and (C*) implies (C). If ψ is total then (A) and (A*), (B) and (B*), (C) and (C*) are equivalent, respectively. If ψ is such that all $dom(\psi_i)$ are initial segments then (B) and (B*) are equivalent. If ψ is one-one and all $dom(\psi_i)$ are initial segments then (C) and (C*) are equivalent. If ψ is one-one and ψ, c satisfy (B) then ψ, c satisfy (C) with computable scaling $h(q) = q$; the corresponding is true for (B*), (C*), respectively.

Proof Immediate by definitions and by Proposition 2.2. Observe that $c(i,i) = 0$, as c is assumed to be a distance measure. ■

Remark 2.5 One might try to use the axiom $(\forall i,j)[D(\psi_i,\psi_j) \leq c(i,j)]$, which is stronger than (A*). But as soon as partial functions come in, this is in conflict with c being a distance measure, namely, with $c(i,i) = 0$. Furthermore, if c is an arbitrary recursive function satisfying $(\forall i,j)[D(\psi_i,\psi_j) \leq c(i,j)]$ then $c(i,i) = 0$ would imply that ψ_i is total, hence if additionally (C) was satisfied then one could decide the totality of functions in ψ by testing whether $c(i,i) = 0$. Thus that overly strong axiom would restrict our considerations to trivial cases.

Proposition 2.6

(A) is equivalent to

$$(\forall i,j,x)\left[\psi_i(x)\downarrow \neq \psi_j(x)\downarrow \wedge (\forall y < x)[\psi_i(y)\downarrow = \psi_j(y)\downarrow] \;\Rightarrow\; \tfrac{1}{x+1} \;\leq\; c(i,j)\right]$$

(A*) is equivalent to

$$(\forall i,j,x)\left[(\psi_i(x)\downarrow \wedge \psi_j(x)\uparrow) \vee (\psi_i(x)\uparrow \wedge (\psi_j(x)\downarrow) \vee \psi_i(x)\downarrow \neq \psi_j(x)\downarrow) \Rightarrow \tfrac{1}{x+1} \leq c(i,j)\right]$$

Proof For the first statement, abbreviate $\psi_i(x) \downarrow \neq \psi_j(x) \downarrow \wedge (\forall y \leq x)[\psi_i(y) \downarrow = \psi_j(y) \downarrow]$ by $P_{ij}(x)$. We prove that, for all i, j, $D(\psi_i, \psi_j) \leq max\{c(i,j), D(\psi_i, \psi_i), D(\psi_j, \psi_j)\}$ iff $(\forall x)\left[P_{ij}(x) \Rightarrow \frac{1}{x+1} \leq c(i,j)\right]$. This can easily be done by distinguishing the cases

(a) $D(\psi_i, \psi_j) = 0$,

(b) $0 \neq D(\psi_i, \psi_j) = \frac{1}{a+1}$ and $\psi_i(a) \uparrow \vee \psi_j(a) \uparrow$,

(c) $0 \neq D(\psi_i, \psi_j) = \frac{1}{a+1}$ and $\psi_i(a) \downarrow \wedge \psi_j(a) \downarrow$.

The second statement is immediate from Definition 2.3. ■

Remarks 2.7 (Continuity) Both (A) and (A*) state that similar programs compute similar functions, with regard to distance functions (D) and (D*), respectively. Conversely, (B), (C), (B*), (C*) are conditions of the kind that the similarity of functions implies the similarity of programs computing them: Either for all programs (axioms (C), (C*)) or for appropriate ones (axioms (B), (B*)).

D^* is a metric, hence there is the unique topology G^* on the partial functions from $\mathbb{N}$ to $\mathbb{N}$ which is generated by D^*. For any numbering ψ, let G^*_ψ be the induced topology on $\{\psi_e : e \in \mathbb{N}\}$, and let c be such that a topology H_c is generated by c. Then one verifies:

1. If ψ, c satisfy (A*) then the mapping $e \mapsto \psi_e$ is continuous w.r.t. topologies H_c, G^*_ψ. This would still be true if (A*) was replaced by the weaker axiom ($\exists$ scaling h) $(\forall i,j)[D^*(\psi_i, \psi_j) \leq h(c(i,j))]$. If ψ is total and ψ, c satisfy (A) then $e \mapsto \psi_e$ is continuous with respect to H_c, G^*_ψ.

2. If ψ, c satisfy (C*) then every "program selecting" function $S : \{\psi_e : e \in \mathbb{N}\} \to \mathbb{N}$, i.e., $\psi_{S(\psi_e)} = \psi_e$ for all e, is continuous w.r.t. G^*_ψ, H_c. Moreover, the mapping $e \mapsto \psi_e$ is open, i.e., images of open sets are open. By Facts 2.4, these facts hold true if ψ, c satisfy (C) and

    - ψ is total or
    - ψ is one-one and all ψ_i have initial segments as domains.

The question arises whether there is some topology G on partial functions such that, for all ψ, c, (A) implies that $e \mapsto \psi_e$ is continuous w.r.t. topologies H_c, G_ψ. Such a topology would uniformly capture philosophy (b) on partiality in such a way that axiom (A) guarantees that approximating a program and approximating the meant function commute. But G does not exist, as the next proposition shows. For sake of simplicity, we will identify ψ and the mapping $e \mapsto \psi_e$ in the following.

Proposition 2.8 For any numbering ψ and distance measure c which generates a topolgy, there are $\tilde{\psi}, \tilde{c}$ with the same properties such that

- if ψ, c satisfy any one of axioms (A), (B), (C) then so do $\tilde{\psi}, \tilde{c}$;

- the only topology on $\{\tilde{\psi}_e : e \in I\!N\}$ allowing $\tilde{\psi}$ to be continuous is the "indiscrete" topology $\{\emptyset, \{\tilde{\psi}_e : e \in I\!N\}\}$.

Corollary 2.9 There is a numbering $\tilde{\psi}$ of all partial recursive functions and a recursive distance measure $\tilde{c}$ such that $\tilde{\psi}, \tilde{c}$ match, $\tilde{c}$ generates a topology, and $\{\emptyset, \text{PREC}\}$ is the only topology for which $\tilde{\psi}$ is continuous.

Proof of corollary We anticipate a result in Section 3 (Theorem 3.2): There exist a numbering ψ of PREC and a recursive distance measure c such that ψ, c match and c generates a topology. Then use $\tilde{\psi}, \tilde{c}$ from Proposition 2.8. ■

Proof of Proposition 2.8 For ψ, c given, define $\tilde{\psi}, \tilde{c}$ as follows.

$$\tilde{\psi}_{2i} = \psi_i, \tilde{\psi}_{2i+1}(x) \uparrow \text{ for all } x;$$
$$\tilde{c}(2i+a, 2j+b) = c(i,j) \text{ for } a, b \in \{0,1\}.$$

In particular, $\tilde{c}(2i+1, 2i+a) = 0$ for $a \in \{0,1\}$. For indices $2i, 2j$ we have immediately from the definition that any one of axioms (A), (B), (C) for ψ, c implies the corresponding one for $\tilde{\psi}, \tilde{c}$. For indices $2i+1, 2j+b$, (A), (B), (C) hold by definition, as $D(\tilde{\psi}_{2i+1}, \tilde{\psi}_{2j+b}) = D(\tilde{\psi}_{2i+1}, \tilde{\psi}_{2i+1}) = 1$. Furthermore, $\tilde{c}$ generates a topology if c does.

Assume that $\tilde{c}$ generates the topology H on $I\!N$ and G is any topology on $\{\tilde{\psi}_e : e \in I\!N\}$ such that ψ is continuous. We have for all i, j:

$$(*) \qquad \tilde{c}(i,j) = 0 \Rightarrow (\forall O \in G)[\tilde{\psi}_i \in O \Leftrightarrow \tilde{\psi}_j \in O].$$

This is true because, by definition of continuity, $O \in G$ implies $\{e : \tilde{\psi}_e \in O\} \in H$ and second, as H is generated by $\tilde{c}$, $\{e : \tilde{\psi}_e \in O\}$ must with any i contain all j such that $\tilde{c}(i,j) < r$, for an appropriate $r > 0$. See condition (b) in the definition of generating a topology, Section 1.

It follows that for all i, j, a, b, O:

$$\begin{array}{lll}
\tilde{\psi}_{2i+a} \in O & \Leftrightarrow \tilde{c}(2i+1, 2i+a) = 0, & \text{as } c(i,i) = 0 \\
 & \Leftrightarrow \tilde{\psi}_{2i+1} \in O & \text{by } (*) \\
 & \Leftrightarrow \tilde{\psi}_{2j+1} \in O, & \text{as } \tilde{\psi}_{2i+1} = \tilde{\psi}_{2j+1} \\
 & \Leftrightarrow \tilde{c}(2j+1, 2j+b) = 0 & \\
 & \Leftrightarrow \tilde{\psi}_{2j+b} \in O. &
\end{array}$$

Hence $O = \text{PREC}$ if $O \neq \emptyset$. ■

By Corollary 2.9 there is no non-trivial topology on PREC such that (A) for ψ, c implies the continuity of ψ. In order to better understand the situation, observe that in the definition of D and of axioms (A), (B), (C) the behaviour of a partial function μ at places beyond $min\{x : \mu(x) \uparrow\}$ is totally ignored. Define the total function f to *weakly extend* μ if $(\forall x)[\frac{1}{x+1} > D(\mu, \mu) \Rightarrow \mu(x) = f(x)]$. Then

$$
\begin{aligned}
D(\mu,\nu) &= max\{D(f,g) : \mu \sqsubseteq f \text{ and } \nu \sqsubseteq g\} \\
&= max\{D(f,g) : f \text{ weakly extends } \mu \text{ and } g \text{ weakly extends } \nu\}.
\end{aligned}
$$

An important advantage of this definition is that every partial recursive function possesses a total recursive weak extension. As a consequence, for a partial recursive μ, it will not make a difference in our subsequent considerations whether *all total recursive* weak extensions of μ are regarded or *all total* weak extensions. Thus for the rest of this section, let $I\!F$ be either REC or the set of all total functions from $I\!N$ to $I\!N$. We ask whether (A) implies the continuity of ψ or something similar, in an appropriate setting using weak extensions and our given topology on $I\!F$. There are different ways one might try.

(1) Each ψ_i represents some unique "unknown" weak extension $f_i \in I\!F$ and ψ is taken to be continuous if the mapping $i \mapsto f_i$ is. But unfortunately there exist ψ, c such that c generates a topology, ψ, c match and no family $\{f_i : i \in I\!N\}$ where each f_i weakly extends ψ_i is such that the mapping $i \mapsto f_i$ is continuous (Theorem 2.10).

(2) Each ψ_i represents the set $[\psi_i] = \{f : f \in I\!F, f \text{ weakly extends } \psi_i\}$. Nöbeling [11] provides a notion of continuity for mappings from the points of a topological space to sets of points of another one. In our setting, his definition says that for any open set $O' \subseteq I\!F$ the set $\{i : (\forall f \in I\!F)[f \text{ weakly extends } \psi_i \Rightarrow f \in O']\}$ is open. But trying to apply this definition again does not work, see Examples 2.11 below.

(3) It remains to try to replace continuity by an appropriate weaker requirement. This at last will lead to a positive result (Theorem 2.13).

Theorem 2.10 There is a numbering ψ and recursive distance measure c such that ψ, c match, c generates a topology, and no mapping $i \mapsto f_i$ from $I\!N$ to $I\!F$ that satisfies $(\forall i)$ $[f_i \text{ weakly extends } \psi_i]$ is continuous.

Proof Let $q_0, q_1, \ldots$ be a computable enumeration of $\{q : q \in Q\!\!\!\!\!\!\;Q, 0 \leq q \leq 1\}$, where $q_0 = 0$, $q_1 = 1$. Define $\psi_0(x) = 0, \psi_1(x) = 1, \psi_i(x)\uparrow$ for $i \geq 2$ and all x, and $c(i,j) = |q_i - q_j|$ for all i, j. It is easy to verify that c generates a topology and ψ, c match.

Let $i \mapsto f_i$ be as described in the theorem. Then $f_0 = \psi_0 = 0^\infty$ and $f_1 = \psi_1 = 1^\infty$. Since the rational numbers are dense, there are for every $\epsilon > 0$ indices i, j such that $f_i(0) = 0, f_j(0) \neq 0$, and $q_j - \epsilon < q_i < q_j$. It follows that $c(i,j) < \epsilon$ while $D(f_i, f_j) = 1$. Such a mapping is not continuous. ■

Examples 2.11 (a) The numbering $\tilde{\psi}$ and distance measure $\tilde{c}$ from Corollary 2.9 match and $\tilde{c}$ generates a topology. But the mapping which to $i \in I\!N$ assigns $[\tilde{\psi}_i] = \{f : f \in I\!F$, f weakly extends $\tilde{\psi}_i\}$ is not continuous in the sense of Nöbelings definition. (b) A simpler example, though not covering all of PREC, is $\psi_i = 0^i, c(i,j) = 0$ for all i, j.

Proof (a) Let O' be any non-trivial open subset of $I\!F$. Then there is an index $2i$ such that $[\tilde{\psi}_{2i}] \subseteq O', \tilde{\psi}_{2i+1}(x)\uparrow$ for all x, and $\tilde{c}(2i, 2i+1) = 0$. If the mapping $e \mapsto [\tilde{\psi}_e]$ were

continuous then $U = \{e : (\forall f \in \mathbb{F})[f \text{ weakly extends } \tilde{\psi}_e \Rightarrow f \in O']\}$ would be open. For U, we have $2i \in U$ and $[\tilde{\psi}_e] \subseteq O'$ for all $e \in U$. U cannot contain $2i+1$, as otherwise $O' = \mathbb{F}$ would follow. Thus $2i \in U$, $2i+1 \notin U$, $\tilde{c}(2i, 2i+1) = 0$, which contradicts to U being open in the topology generated by $\tilde{c}$.
(b) Use the same argument. ■

In the following definition, $\mathbb{F}, [\ldots]$, and the topology on $\mathbb{F}$ are as above, ψ is a partial function from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$.

Definition 2.12 Let a topology on $\mathbb{N}$ be given. ψ is *weakly continuous* iff for every open set $O' \subseteq \mathbb{F}$ there is an open set $O \subseteq \mathbb{N}$ such that for all $i, [\psi_i] \subseteq O'$ implies $i \in O$ and $[\psi_i] \subseteq \mathbb{F} \setminus O'$ implies $i \notin O$.

Obviously, continuity of the mapping $i \mapsto [\psi_i]$ in Noebeling's sense implies weak continuity. Weak continuity reflects the "inner uncertainty" of partial functions in such a way that it might be impossible to discern by open sets of indices whether or not $[\psi_i] \subseteq O'$, namely, if $\psi_i(x) \uparrow$ for too small x (so that $[\psi_i] \not\subseteq O'$ and $[\psi_i \cap O' \neq \emptyset]$).

Theorem 2.13 If c generates a topology on $\mathbb{N}$ and ψ, c satisfy axiom (A) then ψ is weakly continuous with respect to that topology.

Proof Given O', define O to be the open kernel of $\{i : [\psi_i] \cap O' \neq \emptyset\}$. O is by definition open and $i \notin O$ for any i with $[\psi_i] \cap O' = \emptyset$. It remains to show that $[\psi_i] \subseteq O'$ implies $i \in O$.

If $[\psi_i] \subseteq O'$ then there is an x such that

$$(*) \qquad (\forall y < x)[\psi_i(y)\downarrow] \wedge (\forall f \in \mathbb{F})[(\forall y < x)[f(y) = \psi_i(y)] \Rightarrow f \in O'].$$

If ψ_i is partial then take as x the first number where ψ_i is undefined, and if ψ_i is total then x exists because O' is open. We show that, for all j, $c(i,j) < \frac{1}{x+1}$ implies $[\psi_j] \cap O' \neq \emptyset$. It then follows that there is an open ball around i consisting of indices in $\{e : [\psi_e \cap O'] \neq \emptyset\}$, i.e., $i \in O$.

Assume $[\psi_j] \cap O' = \emptyset$. By $(*)$, no total function coinciding with ψ_i on all $y < x$ can weakly extend ψ_j. Hence there exists $x' < x$ such that $\psi_j(y)\downarrow$ for all $y \leq x'$, and $\psi_j(x') \neq \psi_i(x')$. It follows from (A) in the form of Proposition 2.6 that $c(i,j) \geq \frac{1}{x'+1} > \frac{1}{x+1}$. ■

We conclude this section by examples that motivate our later results.

Examples 2.14 (a) Let ψ be total and one-one. (Remember that any r.e. family of total recursive functions can be numbered one-one [9].) Then $c(i,j) = D(\psi_i, \psi_j)$ is a recursive metric satisfying all axioms, with recursive scaling $h(q) = q$ in (C) and (C*).
(b) Let ψ be such that all ψ_i are strings, and define $c(i,j) = 0$ for all i, j. Then (B), (B*), (C), (C*) are satisfied trivially. (A) holds iff there is one partial or total recursive function extending all ψ_i. (A*) holds iff $\psi_i = \psi_j$ for all i, j.

Example 2.15 The following will be a standard construction in subsequent proofs, using the fact that $D(\psi_i, \psi_j)$ is approximated from above by the distances $D(\psi_{i,t}, \psi_{j,t})$. For any recursive $s : I\!N \times I\!N \to I\!N$, $c(i,j) = D(\psi_{i,s(i,j)}, \psi_{j,s(i,j)})$ is a recursive function. c satisfies $D(\psi_i, \psi_j) \leq c(i,j)$, hence (A*) and (A) hold. But c, as it stands, is not a distance measure, since $c(i,i) > 0$ in general. Thus the case $i = j$ will have to be dealt with separately.

3 Matching

As example 2.14(a) has shown, matching can be easily achieved for appropriate numberings of "poor" families of functions, such as r.e. families of total recursive functions. What about rich families, i.e., those containing REC? We will see that matching, even (A*), (B*), (C*), is still possible, but a high price of becoming "unnatural" has to be paid.

Theorem 3.1 There exist a numbering ψ containing all total recursive functions and a recursive distance measure c such that (A*), (B*), (C*) are satisfied with a K-recursive scaling h. Furthermore, c generates a topology on $I\!N$.

Proof Let ϕ be a one-one numbering containing REC such that all ϕ_e have initial segments as domains. Define $U = \{(e,s) : e < s \wedge (\forall y \leq e)[\phi_{e,s}(y)\downarrow]\}$ and let $(e_0, s_0), (e_1, s_1), \ldots$ be an enumeration of U. ψ and c are defined by

$$\begin{array}{rcll} \psi_i & = & \phi_{e_i} & \\ c(i,j) & = & \begin{cases} 0 & \text{if } e_i = e_j \\ D(\phi_{e_i, s_i+s_j}, \phi_{e_j, s_i+s_j}) & \text{otherwise.} \end{cases} \end{array}$$

If ϕ_e is total then $(e,s) \in U$ for almost all s, and there is a pair (e_i, s_i) with $e_i = e$, hence $\psi_i = \phi_{e_i} = \phi_e$. It follows that ψ contains REC. We show that ψ, c satisfy axioms (A*), (B*), (C*).

(A*): If $e_i = e_j$ then $D^*(\psi_i, \psi_j) = D^*(\phi_{e_i}, \phi_{e_j}) = 0$. Otherwise $D^*(\psi_i, \psi_j) \leq D(\psi_i, \psi_j) = D(\phi_{e_i}, \phi_{e_j}) \leq D(\phi_{e_i, s_i+s_j}, \phi_{e_j, s_i+s_j}) = c(i,j)$. (See example 2.8.)

(B*): If $\psi_i = \psi_j$ then (B*) is satisfied with $k = i$, as $c(i,i) = 0$. If $\psi_i \neq \psi_j$ then $D^*(\psi_i, \psi_j) = D(\psi_i, \psi_j)$ by Proposition 2.2. There is a stage s such that $D(\psi_i, \psi_j) = D(\phi_{e_i}, \phi_{e_j}) = D(\phi_{e_i,t}, \phi_{e_j,t})$ for all $t > s$ and there is some k with $e_k = e_j$ and $s_k > s$. Then $\psi_k = \psi_j$ and $c(i,k) = D(\phi_{e_i, s_i+s_k}, \phi_{e_k, s_i+s_k}) = D(\psi_i, \psi_j) = D^*(\psi_i, \psi_j)$.

(C*): For every x let $s(x)$ be the first stage $t \geq x$ such that $\phi_{i,t}(y)\downarrow$ for all $i, y \leq x$ with $y \in dom(\phi_i)$ and let $r(x) = min\{D(\phi_e, \phi_{e'}) : e < e' \leq s(x)\}$. s is clearly K-recursive. So is r, as $D(\phi_e, \phi_{e'}) > 0$ and $D(\phi_e, \phi_{e'}) = \lim_t D(\phi_{e,t}, \phi_{e',t})$, the latter sequence taking values in $\{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ and converging after finitely many steps. For r, we have $r(x) > 0$ and $r(x) \geq r(x+1)$ for all x. Furthermore, $\lim_x r(x) = 0$. Defining h by

$$\begin{array}{rcll} h(0) & = & 0 & \\ h(q) & = & max\left\{\frac{1}{x+1} : r(x) \leq q\right\} + q & \text{for } q > 0 \end{array}$$

yields a K-recursive scaling.

Assume by way of contradiction that $c(i,j) > h(D^*(\psi_i, \psi_j))$. For $x = \frac{1}{c(i,j)} - 1$ there is at least one $k \in \{i,j\}$ such that $\phi_{e_k, s_i+s_j}(x) \uparrow$ while $\phi_{e_i}(x) \downarrow = \phi_{e_j}(x) \downarrow$. Then $e_k \leq x$ and $s_i + s_j \leq s(x)$, so it follows that $e_i, e_j \leq s(x)$. From $c(i,j) > 0$ we have $e_i \neq e_j$, say $e_i < e_j$. Then $r(x) \leq D(\phi_{e_i}, \phi_{e_j})$ and (with Proposition 2.2) $h(D^*(\psi_i, \psi_j)) = h(D(\psi_i, \psi_j)) = h(D(\phi_{e_i}, \phi_{e_j})) \geq \frac{1}{x+1} = c(i,j)$ in contradiction to the assumption.

It remains to show that c generates a topology. D^*, as a metric, generates the topology G on $\{\psi_i : i \in I\!N\}$ where $O' \in G$ iff $(\forall i)[\psi_i \in O' \Rightarrow (\exists r > 0)(\forall j)[D^*(\psi_i, \psi_j) < r \Rightarrow \psi_j \in O']]$. It follows by axioms (A*), (C*) that c generates the topology H with $O \in H \Leftrightarrow (\exists O' \in G)[O = \{i : \psi_i \in O'\}]$. One verifies easily that from the conditions (a), (b) that define generating a topology (see Section 1), (a) holds because of (C*) and (b) because of (A*). ■

Theorem 3.2 There exist a numbering ψ of all partial recursive functions and a recursive distance measure c such that ψ, c match, with K-recursive scaling in axiom (C). Moreover, (A*) holds. c generates a topology on $I\!N$.

Proof We use ϕ, ψ, U, c as in the proof of Theorem 3.1. Let φ be any (e.g., acceptable) numbering of PREC. Define θ by

$$\theta_i(x) = \begin{cases} y & \text{if } \psi_i(x) \downarrow = y+1 \text{ and } \psi_i(z) \downarrow > 0 \text{ for all } z < x \\ \varphi_e(x) & \text{if there is } x' < x \text{ with } \psi_i(x') \downarrow = 0, \psi_i(x'+1) \downarrow = e \\ & \text{and } \psi_i(z) \downarrow > 0 \text{ for all } z < x' \\ \uparrow & \text{otherwise} \end{cases}$$

It follows from the construction that $D^*(\theta_i, \theta_j) \leq D^*(\psi_i, \psi_j) \leq D(\psi_i, \psi_j) \leq D(\theta_i, \theta_j)$ for all i, j, in particular $\theta_i = \theta_j$ whenever $\psi_i = \psi_j$.

Given any φ_e, there exists the total recursive function f with

$$f(x) = \begin{cases} \varphi_e(x) + 1 & \text{if } (\forall z \leq x)[\varphi_e(z) \downarrow] \\ 0 & \text{if } \varphi_e(x) \uparrow \text{ and } (\forall z < x)[\varphi_e(z) \downarrow] \\ e & \text{otherwise, i.e., } (\exists z < x)[\varphi_e(z) \uparrow] \end{cases}$$

As in the proof of Theorem 3.1, ψ contains REC, hence there is an index i with $\psi_i = f$. By definition of θ, θ_i decodes ψ_i in such a way that $\theta_i = \varphi_e$. It follows that θ enumerates all partial recursive functions. We show that axioms (A*), (B), (C) are satisfied.

(A*): As in the proof of Theorem 3.1, $D^*(\psi_i, \psi_j) \leq c(i,j)$, hence $D^*(\theta_i, \theta_j) \leq D^*(\psi_i, \psi_j) \leq c(i,j)$.

(B), (C) immediately follow from the corresponding parts in the proof of Theorem 3.1 together with $D^*(\psi_i, \psi_j) \leq D(\psi_i, \psi_j) \leq D(\theta_i, \theta_j)$. As for (B), recall that $\psi_j = \psi_k$ implies $\theta_j = \theta_k$.

As has been shown in the proof of Theorem 3.1, c generates a topology. ■

We turn now to necessary conditions for matching.

Proposition 3.3 If ψ satisfies (A), (C) with a recursive c then the equality problem for total functions in ψ is decidable. If ψ satisfies (A*), (C*) with recursive c then the equality problem of ψ is decidable.

Proof For total functions ψ_i, ψ_j we have

$$\begin{array}{lll} \psi_i \neq \psi_j & \Rightarrow & c(i,j) \geq D(\psi_i, \psi_j) > 0 \text{ by (A)} \\ \psi_i = \psi_j & \Rightarrow & c(i,j) \leq h(D(\psi_i, \psi_j)) = 0 \text{ by (C)}, \end{array}$$

i.e., $\psi_i = \psi_j \Leftrightarrow c(i,j) = 0$. Analogously, for arbitrary ψ_i, ψ_j,

$$\begin{array}{lll} \psi_i \neq \psi_j & \Rightarrow & c(i,j) \geq D^*(\psi_i, \psi_j) > 0 \text{ by } (A^*) \\ \psi_i = \psi_j & \Rightarrow & c(i,j) \leq h(D^*(\psi_i, \psi_j)) = 0 \text{ by } (C^*). \end{array}$$

So, again, $\psi_i = \psi_j \Leftrightarrow c(i,j) = 0$. ∎

For an acceptable numbering, the complement of the halting problem is m-reducible to equality of total functions. Hence we have

Corollary 3.4 No acceptable numbering satisfies (A), (C) with recursive c.

In this connection, recall Remarks 2.5 on the requirement $(\forall i,j)[D(\psi_i, \psi_j) \leq c(i,j)]$.

By Proposition 3.3, if ψ matches some recursive c then ψ is similar to a one-one numbering, and if even (A*), (C*) hold then ψ can be recursively translated into a one-one numbering of the same functions. But if rich enough, ψ cannot be one-one itself, as the following Proposition shows.

Proposition 3.5 Let ψ be a numbering and c recursive. If ψ, c satisfy axioms (A), (C) with recursive scaling h then ψ cannot contain all total recursive functions. Hence if ψ is one-one and ψ, c are compatible then ψ cannot contain all total recursive functions.

Proof Define $\alpha(i,j) = min\{x : h(\frac{1}{x+1}) \leq c(i,j)\}$ if $c(i,j) > 0$ and $\alpha(i,j) \uparrow$ if $c(i,j) = 0$. α is partial recursive with recursive domain $\{(i,j) : c(i,j) > 0\}$. By (C), if $c(i,j) > 0$ then

$(*) \quad \alpha(i,j)\downarrow \ \wedge \ (\exists x \leq \alpha(i,j))[\psi_i(x)\uparrow \ \vee \ \psi_j(x)\uparrow \ \vee \ \psi_i(x)\downarrow \neq \varphi_j(x)\downarrow]$.

Assume that ψ contains REC. Then ψ is dense. Define strings $\sigma_s, s = 0, 1, \ldots$ as follows. σ_0 is the empty string.

σ_{s+1}: Find some i such that $\sigma_s \sqsubseteq \psi_i, (\forall x \leq \alpha(i,s) + s)[\psi_i(x)\downarrow]$ and
(a) $c(i,s) > 0$ or
(b) $c(i,s) = 0 \wedge (\exists j)[c(j,s) = 0 \wedge (\exists x)[\psi_i(x)\downarrow \neq \psi_j(x)\downarrow$
$\wedge (\forall y < x)[\psi_i(y)\downarrow = \psi_j(y)\downarrow]]]$
Define $\sigma_{s+1}(x) = \psi_i(x)$ if $\sigma_s(x)\downarrow \vee \ x \leq \alpha(i,s) + s$ and ψ otherwise.

Each stage $s+1$ terminates. As ψ is dense, there are infinitely many i with $\sigma_s \sqsubseteq \psi_i$ and $(\forall k \leq \alpha(i,s)+s)[\psi_i(x)\downarrow]$. If among them no one with $c(i,s)>0$ is found then, again by density, for any one with $c(i,s)=0$ there are j,x satisfying $c(j,s)=0,\ \ \psi_i(x)\downarrow \neq \psi_j(x)\downarrow$, and $(\forall y<x)[\psi_i(y)\downarrow = \psi_j(y)\downarrow]$.

The strings σ_s grow in an unbounded way and coincide where defined, hence $f = lim_s\sigma_s$ is total recursive. For each s, we obtain the following for the i found in stage $s+1$. If $c(i,s)>0$ then, by $(*)$, $(\exists x \leq \alpha(i,s))[\psi_s(x)\uparrow \ \vee\ \psi_i(x)\downarrow = \sigma_s(x)\downarrow \neq \psi_s(x)]$, since $\psi_i(x)\downarrow$ for all $x \leq \alpha(i,s)$. If $c(i,s)=0$ then for the j,x found according to case (b) in the algorithm we have $c(j,s)=0$, $D(\psi_i,\psi_i)<\frac{1}{x+1}$, $D(\psi_j,\psi_j)<\frac{1}{x+1}$, $D(\psi_i,\psi_j)\geq\frac{1}{x+1}$. Assuming $\psi_s(y)\downarrow$ for all $y\leq x$ we obtain $D(\psi_s,\psi_s)<\frac{1}{x+1}$ and, by (A), $D(\psi_i,\psi_s)<\frac{1}{x+1}$, $D(\psi_j,\psi_s)<\frac{1}{x+1}$, hence $D(\psi_i,\psi_j)\leq max\{D(\psi_i,\psi_s),D(\psi_j,\psi_s)\}<\frac{1}{x+1}$ by Proposition 2.2(3), a contradiction. Thus, ψ_s cannot be total if $c(i,s)=0$.

In all, writing $i(s)$ for the i obtained in stage $s+1$, we obtain that for all s, ψ_s is not total or $\alpha(i(s),s)\downarrow\ \wedge\ (\exists x\leq\alpha(i(s),s))[\psi_s(x)\uparrow\ \vee\ \psi_{i(s)}(x)\downarrow = \sigma_{s+1}(x)\downarrow\neq\psi_s(x)\downarrow]$ and it follows that f is not enumerated by ψ. The second statement of the proposition follows by Facts 2.4. ∎

By a *pseudometric* we mean a distance measure c that satisfies the triangle inequality (but may fail to satisfy $c(i,j)=0 \Rightarrow i=j$).

Proposition 3.6 If the numbering ψ matches some recursive pseudometric c then ψ does not contain all total recursive functions.

Proof Let ψ_j be total. We have by (B) that for every i there is k with $\psi_j=\psi_k$ and $c(i,k)\leq D(\psi_i,\psi_j)$. By (C), $c(k,j)\leq h(D(\psi_k,\psi_j))=h(0)=0$ and the triangle inequality yields $c(i,j)\leq c(i,k)+c(k,j)=c(i,k)\leq D(\psi_i,\psi_j)$. Thus we have

$$(\forall i,j)[\psi_j \text{ total} \quad\Rightarrow\quad c(i,j)\leq D(\psi_i,\psi_j)].$$

Let α be defined by $\alpha(i,j)=min\{x:\frac{1}{x+1}\leq c(i,j)\}$ if $c(i,j)>0$ and $\alpha(i,j)\uparrow$ otherwise. Then it follows that for all i,j

$(*)$ if $c(i,j)>0$ then ψ_j is not total or $(\exists x\leq\alpha(i,j))[\psi_i(x)\uparrow \vee \psi_i(x)\downarrow\neq\psi_j(x)\downarrow]$.

Assume that ψ contains REC. Strings σ_s are defined in a similar way as above.

σ_{s+1}: Find some i such that $c(i,s)>0, \sigma_s\sqsubseteq\psi_i$, and $(\forall x\leq\alpha(i,s)+s)[\psi_i(x)\downarrow]$.
Define $\sigma_{s+1}(x)=\psi_i(x)$ if $x\leq\alpha(i,s)+s\ \vee\ \sigma_s(x)\downarrow$.

In each stage $s+1$, the search for i terminates: Let i_0,i_1 be indices of σ_s0^∞, σ_s1^∞, respectively. Then by (A) and the triangle inequality, $0<c(i_0,i_1)\leq c(i_0,s)+c(i_1,s)$, hence $c(i_0,s)>0$ or $c(i_1,s)>0$, and for the chosen $i\in\{i_0,i_1\}$, $\alpha(i,s)\downarrow$, $\psi_i(x)=\sigma_s(x)$ if $\sigma_s(x)\downarrow$, and $(\forall x\leq\alpha(i,s)+s)[\psi_i(x)\downarrow]$ as ψ_i is total.

We conclude by $(*)$ that for all s

$$\psi_s \text{ is not total or } (\exists x\leq\alpha(i(s)))[\psi_{i(s)}(x)\uparrow\ \vee\ \psi_{i(s)}(x)\downarrow\neq\psi_s(x)\downarrow]$$

for the $i = i(s)$ found in step $s + 1$. It follows that $f = lim_s \sigma_s$ is not contained in ψ. ■

We will obtain results stronger than Propositions 3.5, 3.6 in Section 6 (Theorems 6.1, 6.3, 6.4).

Proposition 3.7 (1) Let ψ match the pseudometric c. Then $c(i, j) = D(\psi_i, \psi_j)$ whenever ψ_i, ψ_j are total. (2) Let ψ and the pseudometric c satisfy (A*), (B*), (C*). Then $c(i, j) = D^*(\psi_i, \psi_j)$ for all i, j.

Proof (1) For total ψ_i, ψ_j, $D(\psi_i, \psi_j) \leq c(i, j)$ by (A). On the other hand, by (B), there is k with $\psi_k = \psi_j$ and $c(i, k) \leq D(\psi_i, \psi_j)$. By (C), $c(j, k) \leq h(D(\psi_j, \psi_k)) = h(0) = 0$, i.e., $c(i, k) = 0$. The triangle inequality yields $c(i, j) \leq c(i, k) + c(j, k) = c(i, k) \leq D(\psi_i, \psi_j)$. (2) Analogously with D replaced by D^*, and arbitrary ψ_i, ψ_j. ■

We summarize our necessary conditions on matching in the following

Theorem 3.8 Let the numbering ψ match some recursive distance measure c. Then

- The equality problem for total functions is decidable in ψ.
- ψ is not acceptable.
- If (C) is satisfied with computable scaling h then ψ does not contain all total recursive functions.
- If ψ is one-one then ψ does not contain all total recursive functions.
- If c is a pseudometric then ψ does not contain all total recursive functions. Furthermore, $c(i, j) = D(\psi_i, \psi_j)$ if ψ_i, ψ_j are total, hence $D(\psi_i, \psi_j)$ can be computed from i, j for total ψ_i, ψ_j.

In Example 2.14, a rather restricted class of numberings ψ was presented with the property that $D(\psi_i, \psi_j)$ was computable from i, j and defining $c(i, j) = D(\psi_i, \psi_j)$ was possible, so that all axioms were satisfied. A less trivial one is given by the following

Example 3.9 Let ϕ have the properties that (1) all $dom(\phi_e)$ are initial segments and (2) $\{(e, \sigma) : \sigma \sqsubseteq \phi_e\}$ is recursive. Then there is a one-one numbering ψ enumerating the same functions as ϕ such that $D(\psi_i, \psi_j)$ can be computed from i, j if $i \neq j$, hence by defining $c(i, j) = D(\psi_i, \psi_j)$ if $i \neq j$ and $c(i, i) = 0$ we obtain a metric that satisfies all our axioms.

Proof With $A = \{(e, \sigma) : \sigma \sqsubseteq \phi_e\}$, define $E = \{e : (\forall e' < e)(\exists \sigma)[A(e, \sigma) \neq A(e', \sigma)]\}$, and let $e_0, e_1, \ldots$ be an enumeration of E. Set $\psi_i = \phi_{e_i}$. ψ is one-one, as all $dom(\phi_e)$ are initial segments. If $i \neq j$ then one can effectively find some σ with $A(e_i, \sigma) \neq A(e_j, \sigma)$, and the least prefix τ of σ such that $A(e_i, \tau) \neq A(e_j, \tau)$. Then $D(\psi_i, \psi_j) = \frac{1}{|\tau|}$ and c can be defined as described above. ■

Observe that the transition from ϕ to ψ in this example is necessary. E.g., ϕ with $\phi_e(x) = 1$ if $e \in K_x$ and 0 otherwise enumerates a class of total recursive functions. But the equality problem of ϕ is not decidable, hence by Proposition 3.3 there is no recursive c that matches ϕ.

4 Universal Metrics

As section 3 has shown, matching a recursive distance function – and, in fact, satisfying axioms (A) and (C) – is only possible for a severely restricted class of numberings. This reminds us of Rice's theorem. The situation changes if we switch to compatibility. Our main result in this section will be the existence of universal metrics.

Definition 4.1 A distance measure c^* on $I\!N$ is called *universal* iff every numbering ϕ can be recursively translated into a numbering ψ such that ϕ, ψ enumerate the same functions, and ψ, c^* satisfy axioms (A*), (B). A *universal metric* is a universal distance measure which is a metric.

For a universal c^* and for ψ as in the definition, ψ, c^* are compatible. Also, by (A*), ψ (i.e., the mapping $e \mapsto \psi_e$) is continuous with regard to the topologies generated by c^*, D^*, respectively (see Remarks 2.7).

Our following construction will be based on an arbitrary total recursive, one-one, and dense numbering. Families of functions that can be numbered this way are, e.g., the polynomials or the primitive recursive functions.

Theorem 4.2 There are recursive universal metrics. A particular one can be constructed from any total recursive, one-one, and dense numbering p by defining $c^*(i,j) = D(p_i, p_j)$. Moreover, from any numbering ϕ the corresponding numbering ψ as well as a translation from ϕ to ψ can be obtained uniformly in an index of ϕ.

Proof We show that c^* as defined in the theorem has the required properties.

(1) c^* is total recursive
If $i = j$ then $c^*(i,j) = 0$. If $i \neq j$ then $p_i \neq p_j$, $c^*(i,j) > 0$. The first x with $p_i(x) \neq p_j(x)$ can be found effectively, hence $c^*(i,j) = \frac{1}{x+1}$ can be computed from i, j.

(2) Construction of ψ.
Let ϕ be given. ψ as demanded is constructed in stages s, together with the sets I_s of those indices i for which ψ_i has already been defined. Odd stages are used to ensure each ψ_i being defined, and the even ones to guarantee that every ϕ_e is equal to some ψ_i and, furthermore, that (B) holds. We will use a total recursive enumeration *pair* of all pairs of indices such that *pair* takes every pair as a value infinitely often.

Stage 0: $\psi_0 = \phi_0, I_0 = \{0\}$

Stage $2s+1$: Let i be the first index such that ψ_i has not been defined up to stage $2s$, and let k be the first index in I_{2s} such that

$$(\forall k' \in I_{2s})[c^*(i,k) \leq c^*(i,k')].$$

Define: $\psi_i = \psi_k, I_{2s+1} = I_{2s} \cup \{i\}$

Stage $2s+2$: For $pair(s) = (e,k)$: If $k \notin I_{2s+1}$ then do nothing. Otherwise let i be the first index not in I_{2s+1} satisfying

$$c^*(i,k) = D(\phi_{e,2s+1}, \psi_{k,2s+1}) \text{ and}$$
$$(\forall k' \in I_{2s+1})[c^*(i,k') \geq D(\phi_{e,2s+1}, \psi_{k,2s+1})].$$

Define $\psi_i = \phi_e, I_{2s+2} = I_{2s+1} \cup \{i\}$

(3) Each stage terminates.
This is immediate for stages 0 and $2s+1$, and for the case $k \notin I_{2s+1}$ in stage $2s+2$. Otherwise, assume $k \in I_{2s+1}$ in stage $2s+2$. As $\phi_{e,2s+1}, \psi_{k,2s+1}$ are finite functions, $D(\phi_{e,2s+1}, \psi_{k,2s+1}) = \frac{1}{x+1} > 0$ for some unique x. As p is dense, there exists an i such that $p_i(y) = p_k(y)$ for all $y < x$ and $p_i(x) \neq p_{k'}(x)$ for all $k' \in I_{2s+1}$. i satisfies $c^*(i,k) = \frac{1}{x+1} = D(\phi_{e,2s+1}, \psi_{k,2s+1})$ and $c^*(i,k') \geq D(\phi_{e,2s+1}, \psi_{k,2s+1})$ for all $k' \in I_{2s+1}$. It follows that the search for an appropriate i is successful and stage $2s+2$ terminates.

(4) It is immediate from the construction that c^* is a metric, that ψ and ϕ enumerate the same functions, and that ϕ can be translated into ψ by a total recursive function. Also, an index for ψ and likewise for a translating function can be uniformly obtained from an index of ϕ.

(5) ψ, c^* satisfy (A*).
Assume this is false. Then there is a first stage t such that, for some indices i,j, ψ_i, ψ_j have been defined at stage t and $D^*(\psi_i, \psi_j) > c^*(i,j)$. $t = 0$ is impossible, hence $t = 2s+a$, $a \in \{1,2\}$. Without loss of generality, ψ_i is newly defined at stage $2s+a$. Because of $D^*(\psi_i, \psi_j) > 0$ we have $\psi_i \neq \psi_j$, hence $j \in I_{2s+a-1}$.

If $a = 1$ then $\psi_i = \psi_k$ for some $k \in I_{2s}$. By the minimality of $2s+1$, $D^*(\psi_i, \psi_j) = D^*(\psi_k, \psi_j) \leq c^*(k,j)$, hence $c^*(i,j) < c^*(k,j)$ and, by Proposition 2.2 (3), $c^*(i,k) = max\{c^*(i,j), c^*(k,j)\} = c^*(k,j)$, hence $c^*(i,j) < c^*(i,k)$. But this contradicts to the choice of k as taking minimal value $c^*(i,k)$ among all $c^*(i,k')$, $k' \in I_{2s}$.

Otherwise $a = 2$. The algorithm chooses $k \in I_{2s+1}$ with $c^*(i,k) = D(\phi_{e,2s+1}, \psi_{k,2s+1}) \geq D(\psi_i, \psi_k) \geq D^*(\psi_i, \psi_k)$ and $c^*(i,k') \geq c^*(i,k)$ for all $k' \in I_{2s+1}$. As $j \in I_{2s+1}$, $c^*(i,j) \geq c^*(i,k)$. By the assumption, $D^*(\psi_i, \psi_j) > c^*(i,j) = max\{c^*(i,j), c^*(i,k)\} \geq c^*(j,k)$ (Proposition 2.2 (3)). We have

(a) $c^*(j,k) < D^*(\psi_i, \psi_j)$.

Furthermore, $D^*(\psi_i, \psi_k) \leq c^*(i,k) \leq c^*(i,j) < D^*(\psi_i, \psi_j)$, so that, by Proposition 2.2 (3), $D^*(\psi_j, \psi_k) = max\{D^*(\psi_i, \psi_j), D^*(\psi_i, \psi_k)\} = D^*(\psi_i, \psi_j)$, and together with (a):

(b) $c^*(j,k) < D^*(\psi_j, \psi_k)$.

On the other hand it follows from the minimality of stage $2s+2$ and from $j,k \in I_{2s+1}$ that $D^*(\psi_j, \psi_k) \leq c^*(j,k)$.

(6) ψ, c^* satisfy (B).

For the sake of clarity, rewrite (B) as

$$(\forall k,j)(\exists i)[\psi_i = \psi_j \ \wedge \ c^*(k,i) \leq D(\psi_k, \psi_j)].$$

If $D(\psi_k, \psi_j) = 0$ then $\psi_k = \psi_j$, hence (B) is satisfied with $i = k$.

Otherwise $D(\psi_k, \psi_j) > 0$. Let e be such that $\phi_e = \psi_j$. Then $k \in I_{2s+1}$ and $D(\psi_{k,2s+1}, \phi_{e,2s+1}) = D(\psi_k, \phi_e)$ for almost all s. For infinitely many among these s, $pair(s) = (e,k)$. The i chosen in any corresponding stage $2s+2$ satisfies $\psi_i = \phi_e = \psi_j$ and $c^*(k,i) = D(\psi_k, \phi_e) = D(\psi_k, \psi_j)$. ■

What are necessary conditions on a metric to be universal? One such property is stated in the next theorem.

Theorem 4.3 With regard to a universal metric c^*, all indices are accumulation points.

Proof Let c^* be universal and $i \in \mathbb{N}, r \in \mathbb{Q}, r > 0$ be given. Let ϕ be any total, one-one, and dense numbering, and ψ constructed to ϕ as in Theorem 4.2. By the density of ϕ and ψ, there exists j such that $0 < D(\psi_i, \psi_j) < r$. By (B) there is k with $\psi_j = \psi_k$ and $c^*(i,k) \leq D(\psi_i, \psi_j)$. By (A) and as ψ is total, $c^*(i,k) \geq D(\psi_i, \psi_k) = D(\psi_i, \psi_j)$, thus $c^*(i,k) = D(\psi_i, \psi_j)$. It follows that $0 < c^*(i,k) < r$, i.e., k is in the r-neighbourhood of i. ■

For purposes of further illustration we regard some simple metrics and ask what numberings can be compatible with them. It turns out that conditions of density of the metric space alone do not suffice to imply universality of the metric. We list several more or less dense metrics, only the last of them being universal. In particular, the rationals with their natural metric – though a standard example of a dense space – do only allow for a quite limited class of compatible numberings.

Examples 4.4
(a) Let $q_0, q_1, \ldots$ be a one-one, recursive enumeration of the non-negative rationals and define $c(i,j) = min\{1, |q_i - q_j|\}$. Then any numbering ψ satisfying axioms (A*), (B) together with c enumerates one single function, i.e., $\psi_i = \psi_j$ for all i, j.

Our further examples are based on some recursive "alphabet" $\Sigma \subseteq \mathbb{N}$ and total one-one numbering p of $\Sigma^* 0^\infty$. For specific choices of Σ, p, and c defined by them, we obtain results on any ψ satisfying (A*), (B) together with c as follows.

(b) $\Sigma = \{1\}, c(i,j) = D(p_i, p_j)$. Then $\psi_0, \psi_1, \ldots$ converges to a single partial recursive function.

(c) $\Sigma = \{0,1\}, c(i,j) = D(p_i, p_j)$. Then some partial recursive μ *majorizes* ψ in the sense that $(\forall i,x)[\psi_i(x)\downarrow \ \Rightarrow \ \mu(x)\downarrow \ \wedge \ \psi_i(x) \leq \mu(x)]$. In particular, if $(\forall x)(\exists i)[\psi_i(x)\downarrow]$ then ψ is majorized by a total recursive function.

(d) $\Sigma = \mathbb{N}, c(i,j) = D(p_i, p_j)$. Then there is no restriction to $F = \{\psi_i : i \in \mathbb{N}\}$, c is universal.

Proofs (a): Assume by way of contradiction that F contains two different functions. Then there are x, y such that the sets

$$I = \{i : \psi_i(x) \downarrow = y\}\,,\; J = \{j : \psi_j(x) \uparrow \;\vee\; \psi_j(x) \downarrow \neq y\}$$

are both non-empty, $I \cap J = \emptyset$, $I \cup J = I\!N$. For all $i \in I, j \in J$ we obtain $c(i,j) \geq \frac{1}{x+1}$ by (A*). Thus the sets $\{q_i : i \in I\}$, $\{q_j : j \in J\}$ must have a positive distance from each other and there exists some q_k not contained in their union. This contradicts to $I \cup J = I\!N$.

(b): It suffices to regard the special case $p_i = 1^i 0^\infty$. Then $c(i,j) = \frac{1}{1+min\{i,j\}}$ whenever $i \neq j$. Define $f(x) = \psi_{x+1}(x)$. Then for all $i \geq x+1$, $c(i, i+1) < \frac{1}{x+1}$ and thus, by (A*), $(\psi_i(x) \uparrow \wedge \psi_{i+1}(x) \uparrow) \vee \psi_i(x) \downarrow = \psi_{i+1}(x) \downarrow$. Hence $(\psi_i(x) \uparrow \wedge f(x) \uparrow) \vee \psi_i(x) \downarrow = f(x) \downarrow$ for all x and all $i \geq x+1$, i.e., ψ converges pointwise to f.

(c) For any x let $g(x)$ be the first index such that $(\forall \sigma \in \{0,1\}^{x+1})(\exists i \leq g(x))[\sigma \sqsubseteq p_i]$. We show

$(*)$ $(\forall x)(\forall j)(\exists i \leq g(x))[(\psi_i(x) \uparrow \wedge \psi_j(x) \uparrow) \vee \psi_i(x) \downarrow = \psi_j(x) \downarrow]$.

This is clear if $\psi_j \in \{\psi_0, \ldots, \psi_{g(x)}\}$. Assume $\psi_j \notin \{\psi_0, \ldots, \psi_{g(x)}\}$. For p_j, there is some $\sigma \in \{0,1\}^{x+1}$ such that p_j extends σ. Define $i := min\{k : \sigma \sqsubseteq p_k\}$ for this σ. Then $i \leq g(x)$ by choice of g. Axiom (A*) implies $(\psi_i(y) \uparrow \wedge \psi_j(y) \uparrow) \vee \psi_i(y) \downarrow = \psi_j(y) \downarrow$ for all y with $\frac{1}{y+1} > c(i,j) = D(p_i, p_j)$, hence for all $y \leq x$ (as $D(p_i, p_j) \leq \frac{1}{x+2}$). Thus $(*)$ follows.

Defining μ by $\mu(x) = \begin{cases} max\{\psi_i(x) : i \leq g(x) \wedge \psi_i(x) \downarrow\} & \text{if such } i \text{ exists} \\ \uparrow & \text{otherwise} \end{cases}$

we obtain a partial recursive function majorizing all ψ_i.

(d) In this case, p is dense and the assertion follows by Theorem 4.2. ■

To refine our results we present two additional constructions: First, of some c and F such that a K-recursive enumeration of F is compatible with c, but no (computable) numbering is; and second, of some F and a family c_e of metrics so that for each e some numbering ψ_e compatible with c_e exists, but such ψ_e cannot be obtained uniformly in e.

Theorem 4.5 There is a metric c such that the constant functions $\{0^\infty, 1^\infty, \ldots\}$ possess a K-recursive enumeration compatible with c, in fact, one that satisfies axioms $(\forall i,j)[D(\psi_i, \psi_j) \leq c(i,j)]$ and (B), but no (computable) numbering of them is compatible with c.

Proof Let S be a simple set. Our metric c will have the following properties:

(a) If $i \notin S$ and $j \neq i$ then $c(i,j) = 1$

(b) If $i, j \in S$ and $r > 0$ then there are an $n \in I\!N$ and a sequence $k_0, k_1, \ldots, k_n, k_{n+1} \in S$ such that $i = k_0, j = k_{n+1}$ and $c(k_m, k_{m+1}) \leq r$ for $m = 0, \ldots, n$.

From (a), (b), the assertion follows:
Let $a_0, a_1, \ldots$ be a K-recursive enumeration, without repetition, of the complement of S. Define the K-recursive enumeration ψ of the constant functions by

$$\psi_i = \begin{cases} 0^\infty & \text{if } i \in S \\ (n+1)^\infty & \text{if } i = a_n \,. \end{cases}$$

If $\psi_i \neq \psi_j$ then at least one of i, j is not in S and $c(i,j) = 1$ by (a). Furthermore, $D(\psi_i, \psi_j) = 1$ in this case. Hence $c(i,j) = D(\psi_i, \psi_j)$ whenever $\psi_i \neq \psi_j$, so that $D(\psi_i, \psi_j) \leq c(i,j)$ and (B) follow. (Remember that here, as all ψ_i are total, (A) or (A*) is $(\forall i,j)[D(\psi_i,\psi_j) \leq c(i,j)]$.) Assume that some (computable) numbering ϕ of the constant functions, compatible with c, would exist. Then $\phi_i = \phi_j$ for all $i, j \in S$: By (b), there is a chain $i = k_0, k_1, \ldots, k_n, k_{n+1} = j$ such that $c(k_m, k_{m+1}) < 1$ for $m = 0, \ldots, n$; by (A), then, $D(\phi_{k_m}, \phi_{k_{m+1}}) = 0$, since the distance between constant functions is either 0 or 1; hence $\phi_i = \phi_j$. For $i \in S$, then, the infinite r.e. set $\{j : \phi_j(0) \neq \phi_i(0)\}$ is a subset of the complement of S, in contradiction to the assumption that S is simple.

Construction of c:
Let R be some infinite recursive subset of S. Define ν to be partial recursive and one-one with $dom(\nu) = S \setminus R$, $range(\nu) = \mathbb{N}$. ν can be extended to a computable and one-one partial function μ from $\mathbb{N}$ to $\mathbb{Q}$ such that $dom(\mu) = S$, $range(\mu) = \{q : q \in \mathbb{Q}, q \geq 0\}$, and for all $x \in S$, $x \in R$ implies $\mu(x) \notin \mathbb{N}$. Define

$$c(i,j) = \begin{cases} 0 & \text{if } i = j \\ min\{1, |\mu(i) - \mu(j)|\} & \text{if } i, j \in S \text{ and } i \neq j \\ 1 & \text{otherwise.} \end{cases}$$

c is a metric: The triangle inequality is satisfied. So is the condition $c(i,j) = 0 \Leftrightarrow i = j$: If $i = j$ then $c(i,j) = 0$ by definition, and if $i \neq j$ then, by definition of μ, $c(i,j) = |\mu(i) - \mu(j)| > 0$ or $c(i,j) = 1$.

c is recursive: If $i = j$ then $c(i,j) = 0$. If $i \neq j$ and $i, j \notin R$ then $c(i,j) = 1$: This is by definition the case if $i \notin S$ or $j \notin S$, and for $i, j \in S$ we obtain $\mu(i) \in \mathbb{N}$, $\mu(j) \in \mathbb{N}$, $\mu(i) \neq \mu(j)$, hence $min\{1, |\mu(i) - \mu(j)|\} = 1$. If $i \in R, j \notin R$ then compute $n \in \mathbb{N}$ with $n < \mu(i) < n+1$. There are exactly two indices k, k' such that $\mu(k) = n$ and $\mu(k') = n+1$. Then $c(i,k) = \mu(i) - n$, $c(i,k') = n+1-\mu(i)$, so that $c(i,j)$ can be computed by comparing j with k and k'. Last, if $i, j \in R$ then $\mu(i)\downarrow$, $\mu(j)\downarrow$, and $min\{1, |\mu(i) - \mu(j)|\}$ can be directly evaluated.

Proof of (a), (b):
(a) is immediate from the definition of c. As for (b), let i, j, r be given. There is a chain of non-negative rationals $q_0, \ldots, q_{n+1}$ such that $q_0 = i, q_{n+1} = j$, and $|q_m - q_{m+1}| < r$ for $m = 0, \ldots, n$. As $range(\mu)$ contains all non-negative rationals as elements, $q_m = \mu(k_m)$ for some k_m, for each $m = 0, \ldots, n+1$. μ is one-one, hence $k_0 = i, k_{n+1} = j$. Moreover $c(k_m, k_{m+1}) = min\{1, |q_m - q_{m+1}|\} < r$. ■

Theorem 4.6 There is a uniformly recursive family of metrics c_e such that, for each e, the class of all constant functions possesses a recursive numbering ψ^e compatible with c_e, but it is impossible to obtain such numberings uniformly in e.

Proof The proof of Theorem 4.5 is modified in such a way that the new requirements are met. Define sets

$$\begin{aligned} R &= \{4n : n \in \mathbb{N}\} \\ S_e &= \{2n : n \in \mathbb{N}\} \cup \{s : e \in K \setminus K_s\}. \end{aligned}$$

Thus in addition to the even numbers, S_e contains those odd numbers s of stages in the enumeration of K for which $K_s(e) \neq K(e)$. The sets S_e are r.e. uniformly in e. Defining c_e as in the proof of Theorem 4.5 but with S_e instead of S, and repeating the argument from there, we conclude that, for all e and any numbering ϕ^e compatible with c_e, $\phi^e_i = \phi^e_j$ for all $i, j \in S_e$.

Let a_e be the first stage s such that $K_s(e) = K(e)$. Then, for each e, the numbering ψ^e of the constant functions, defined by

$$\psi^e_i = \begin{cases} j^\infty & \text{if } i = 2(a_e + j) + 1 \\ 0^\infty & \text{otherwise} \end{cases}$$

is compatible with c_e.

It remains to show that for no family of numberings ϕ^e with these properties it is possible to uniformly compute an index of ϕ^e from e. Assume the contrary for $\{\phi^e : e \in \mathbb{N}\}$. Then g with $g(e) = min\{k : \phi^e_k(0) \neq \phi^e_{k+1}(0)\}$ is recursive. As $\phi^e_i = \phi^e_j$ for all $i, j \in S_e$, $g(e)$ or $g(e) + 1$ is not in S_e. Thus by definition of the sets S_e we obtain that $K_{g(e)+1}(e) = K(e)$ for all e, so that K would be recursive. ■

5 From Numberings to Metrics

The previous section dealt with the question: Given a metric c, which numberings are compatible with c? In particular c can be universal or, on the other hand, allow only for quite special numberings. We will now consider the reverse question, namely, look for metrics compatible with given numberings. As will turn out, acceptable numberings always possess compatible recursive metrics. We need the following topological result.

Fact 5.1 Let $T = (\mathbb{N}, E)$ be a tree with recursive set $E \subseteq \mathbb{N} \times \mathbb{N}$ of edges and let c be any computable function from E to $\{q : q \in \mathbb{Q}, 0 < q \leq 1\}$. Then c can be extended to a recursive metric on $\mathbb{N}$.

Proof Begin by defining $c(j, i) = c(i, j)$ whenever $(i, j) \in E$ and $c(i, i) = 0$ for all i. If $i \neq j$ then identify the unique sequence $a_0, a_1, \ldots, a_{n+1}$ of nodes which without repetition connects i to j, i.e.,

$$\begin{aligned} &a_0 = i, a_{n+1} = j \\ &(\forall m \in \{0, \ldots, n\})[(a_m, a_{m+1}) \in E \ \vee \ (a_{m+1}, a_m) \in E] \\ &(\forall m, m' \in \{0, \ldots, n+1\})[m \neq m' \ \Rightarrow \ a_m \neq a_{m'}]. \end{aligned}$$

Define $c'(i, j) = c(a_0, a_1) + c(a_1, a_2) + \ldots + c(a_n, a_{n+1})$ and $c(i, j) = min\{c'(i, j), 1\}$.

c is recursive and extends the original c on the edges. $c(i, j) = c(j, i)$ as the respective sequences from i to j and from j to i differ from each other only by the ordering of their elements. $c(i,j) = 0 \Leftrightarrow i = j$ is immediate. As for the triangle inequality, consider the two sequences that connect i to k and k to j, respectively. Then the desired sequence connecting i to j is obtained by concatenating the two former ones and removing all loops. It follows that $c'(i,j) \leq c'(i,k) + c'(k,j)$, and an easy calculation yields $c(i,j) = min\{1, c'(i,j)\} \leq min\{1, c'(i,k)\} + min\{1, c'(k,j)\} = c(i,k) + c(k,j)$. ■

Theorem 5.2 Every numbering that possesses a padding function is compatible with a suitable recursive metric, in fact, axioms (A*), (B) can be satisfied. In particular, every acceptable numbering is compatible with a recursive metric.

Proof Using some standard pairing function, we may assume that our padding function h is a "double padding" which strictly increases w.r.t. compressed triples: $h : \mathbb{N}^3 \to \mathbb{N}, \langle e,a,b\rangle < h(e,a,b)$, $\langle e,a,b\rangle < \langle e',a',b'\rangle$ implies $h(e,a,b) < h(e',a',b')$. Thus, $\psi_{h(e,a,b)} = \psi_e$ and $range(h)$ is recursive. There exists the total recursive function g with

$$g(j) = \begin{cases} j & \text{if } j \notin range(h) \\ e & \text{if } j = h(e,a,b) \text{ (for any } a,b). \end{cases}$$

Define

$$E = \{(0,j) : j \notin range(h), j \neq 0\} \cup \{(i, h(e,i,b)) : e,i,b \in \mathbb{N}\}.$$

E is recursive and, by choice of h, determines a tree with root 0 and set of nodes $\mathbb{N}$. Observe that for every i and e there are infinitely many j with $(i,j) \in E$ and $\psi_j = \psi_e$. Define c on E by

$$c(i,j) = D(\psi_{i,j}, \psi_{g(j),j})$$

and extend c to a recursive metric on $\mathbb{N} \times \mathbb{N}$ according to Fact 5.1. We show that ψ, c satisfy the requirements in the theorem.

Axiom (A*): (A*) holds trivially if $i = j$, so assume $i \neq j$. Consider the unique sequence $i = a_0, a_1, \ldots, a_{n+1} = j$ as in the proof of Fact 5.1. For each m, $0 \leq m \leq n$, we have $D(\psi_{a_m,a_{m+1}}, \psi_{g(a_{m+1}),a_{m+1}}) \geq D(\psi_{a_m}, \psi_{g(a_{m+1})}) = D(\psi_{a_m}, \psi_{a_{m+1}})$. By the triangle inequality for D and as $D(\psi_i, \psi_j) \leq 1$:

$$\begin{aligned} D^*(\psi_i, \psi_j) &\leq D(\psi_i, \psi_j) \\ &\leq min\{1, \sum_{m=0}^{n} D(\psi_{a_m}, \psi_{a_{m+1}})\} \\ &\leq min\{1, \sum_{m=0}^{n} D(\psi_{a_m,a_{m+1}}, \psi_{g(a_{m+1}),a_{m+1}})\} \\ &= min\{1, \sum_{m=0}^{n} c(a_m, a_{m+1})\} \\ &= c(i,j). \end{aligned}$$

Axiom (B): If $D(\psi_i, \psi_j) = 0$ then $\psi_i = \psi_j$, and $k = i$ is the desired index. If $D(\psi_i, \psi_j) > 0$ then there is a stage s such that $D(\psi_{i,t}, \psi_{j,t}) = D(\psi_i, \psi_j)$ for all $t > s$. Choose b such that $h(j, i, b) > s$, and $k = h(j, i, b)$. Then $\psi_k = \psi_j$ and $c(i, k) = D(\psi_{i,k}, \psi_{g(k),k}) = D(\psi_{i,k}, \psi_{j,k}) = D(\psi_i, \psi_j)$. ■

Observe that from Theorem 5.2 we obtain again the result (Theorem 4.2) that an arbitrary numbering ϕ can be effectively translated into some ψ enumerating the same family of functions, such that there exists a recursive metric which with ψ satisfies (A*), (B). Just define $\psi_{\langle e,i \rangle} = \phi_e$, where $\langle \ldots \rangle$ is a standard pairing function, and apply Theorem 5.2.

The following example shows that possessing a padding function is by far not a necessary condition for a numbering to be compatible with an appropriate recursive metric, the contrary can be true in a rather radical way.

Example 5.3 Consider $F = \{0^i 1^\infty : i \in I\!N\}$ and the recursive metric c where $c(i, j) = 0$ if $i = j$ and $\frac{1}{min\{i,j\}+1}$ otherwise. Then for ψ with $\psi_i = 0^i 1^\infty$ we have $c(i, j) = D(\psi_i, \psi_j) = D^*(\psi_i, \psi_j)$, so that all axioms in Definition 2.3 hold. But ψ is the only numbering of F which with c satisfies (A). Hence c forces any compatible numbering of F to be one-one and forbids having a padding function.

Proof $c(i, j) = D(\psi_i, \psi_j) = D^*(\psi_i, \psi_j)$ is immediate. Let ϕ be any numbering of $\{0^i 1^\infty : i \in I\!N\}$ which is different from ψ and let m be minimal with the property that there exists $i \neq m$ with $\phi_i = 0^m 1^\infty$. Then by the minimality of m, $i > m$ for any such i. There exist $n > m$ and $j > m$ with $\phi_j = 0^n 1^\infty$. Then $min\{i, j\} > m$ and $c(i, j) = \frac{1}{min\{i,j\}+1} < \frac{1}{m+1} = D(0^m 1^\infty, 0^n 1^\infty) = D(\phi_i, \phi_j)$. As all functions in ϕ are total, it follows that ϕ, c do not satisfy axiom (A). ■

ψ in this example, being total, is a "poor" numbering. Remember Proposition 3.5: A one-one numbering containing all total recursive functions cannot be compatible with a recursive distance measure.

6 Connections to Learning Theory

There are connections between numberings and Gold-style learning theory [3], [4], [7]. In particular, Wiehagen and Jung [18], [19] gave characterizations of the basic notions of learning theory in terms of numberings. Kummer [8] characterized in terms of numberings those enumerable classes of total functions which are co-learnable.

The fundamental notion of learning in the limit ("explanatory learning", or just "Ex") goes back to Gold [7]: A machine M *learns* a class F of total recursive functions iff for all $f \in F$, the function $n \mapsto M(f(0)f(1)\ldots f(n))$ assigns to almost all n the same number e, which is an index of f with respect to some given acceptable numbering φ. Note that M can be chosen total without losing any learning power, just by postponing any new guess and repeating the old one until the computation in question has terminated. M

finitely learns F iff for all $f \in F$ there is m such that $M(f(0)\dots f(n)) = ?$ for $n < m$ and $M(f(0)\dots f(m)) = e$, where e is an index of f.

Theorem 6.1 If ψ, c satisfy axioms (A) and (C) via a recursive scaling h then the class of all total functions in ψ is Ex-learnable. In particular, ψ cannot contain all total recursive functions. As a consequence, if ψ is one-one and ψ, c are compatible then the class of total functions in ψ is Ex-learnable, ψ cannot contain all total recursive functions.

Proof For any i let $g(i)$ be a program for the function

$\varphi_{g(i)}(x)$: Search for some j such that $c(i,j) = 0$ and $(\forall z \leq x)[\psi_j(z)\downarrow]$; output $\psi_j(x)$.

Let α be defined by $\alpha(i,j) = min\{x : h(\frac{1}{x+1}) < c(i,j)\}$ if $c(i,j) > 0$, and $\alpha(i,j)\uparrow$ otherwise. Given some total recursive function as input, the learner produces a list R of "removed indices". For any current stage of R, $g(i)$ is put out, where $i = min\{x : x \notin R\}$. The machine then searches for some input string σ already present and for j, such that

(a) $c(i,j) > 0, \sigma(x)\downarrow$ for all $x \leq \alpha(i,j)$, and $(\forall x \leq \alpha(i,j))[\psi_j(x)\downarrow = \sigma(x)]$ or
(b) there is some $x \in dom(\sigma)$ such that $\varphi_{g(i)}(x)\downarrow \neq \sigma(x)$,

and when successful, includes i in R.

Let f be the input function. Then no index of f can be listed in R: If i is listed in R according to case (a) then there is ψ_j with $c(i,j) > 0$ and $(\forall x \leq \alpha(i,j))[\psi_j(x)\downarrow = f(x)]$. By (C), $h(D(\psi_i, \psi_j)) \geq c(i,j) > 0$, hence there is an $x \leq \alpha(i,j)$ such that $\psi_i(x)\uparrow \vee\ \psi_i(x)\downarrow \neq f(x)$. If i is included in R by (b) then there are j, x with $c(i,j) = 0$, $\psi_j(z)\downarrow$ for all $z \leq x$ and $\psi_j(x) \neq f(x)$. It follows by (A) that either $\psi_i(z)\downarrow = \psi_j(z)$ for all $z \leq x$ and hence $\psi_i(x)\downarrow \neq f(x)$, or $\psi_i(z)\uparrow$ for some $z \leq x$. Thus, in both cases $\psi_i \neq f$.

If f is contained in ψ then the output sequence will converge to some $g(i)$ where $i \notin R$, $\varphi_{g(i)}(x) = f(x)$ for all $x \in dom(\varphi_{g(i)})$. Furthermore, $c(i,j) = 0$ for all indices j of f. For any such j and any x, $\varphi_{g(i)}$ on argument x will output $\psi_j(x)$ if computing $\varphi_{g(i)}$ has not terminated (by some j') beforehand, hence $\varphi_{g(i)}$ is total. In all, the learner converges to an index of f. It follows that $\{\psi_e : e \in I\!N, \psi_e \text{ total}\}$ is Ex-learnable.

The last part of theorem follows by Proposition 2.4. ■

As Gold [7] has shown, such a class cannot contain all total recursive functions. But remember Theorems 3.1, 3.2: ψ can contain REC (and, in fact, can enumerate all of PREC and matching can be achieved) if the scaling in (C) is allowed to be K-recursive. As Theorem 6.3 will show, even this becomes impossible if c is required to be a pseudometric. But first, we demonstrate that the scaling being recursive is in fact a critical property.

Example 6.2 The class $F = \{f : f \in \text{REC}, range(f) \text{ finite}, \varphi_{max(range(f))} = f\}$ is Ex-learnable, but for no numbering ψ containing F and no recursive distance function c, (A) and (C) are satisfied with recursive scaling h.

Proof The algorithm mapping σ to $max\{\sigma(x) : \sigma(x)\downarrow\}$ is clearly an Ex-learner for F. Assume by way of contradiction that ψ, c satisfy (A), (C) with recursive h and that ψ

contains F. We will uniformly in i construct functions $\varphi_{g(i)}$ such that $(\forall i)[\varphi_{g(i)} \in F]$ and $(\forall i, s)[\varphi_{g(i)} \neq \psi_s]$. By the recursion theorem, for e with $\varphi_e = \varphi_{g(e)}, \varphi_e \in F$ and φ_e is not in ψ.

Given i, $g(i)$ is such that $\varphi_{g(i)} = \lim_s \sigma_s$ where the strings σ_s are defined as follows.

$\sigma_0 = max\{2, i\}$

σ_{s+1} :
With $dom(\sigma_s) = \{0, \ldots, z\}$, define $x := min\{y : y > z \wedge h(\frac{1}{y+1}) < \frac{1}{z+2}\}$. Search for j, j' such that

$$\begin{array}{lll} (\forall y \leq z)[\psi_j(y)\downarrow & = \sigma_s(y) & \wedge \ \ \psi_{j'}(y)\downarrow = \sigma_s(y)] \\ (\forall y)[z < y \leq x & \Rightarrow & \psi_j(y)\downarrow = 0] \\ (\forall y)[z < y \leq x & \Rightarrow & \psi_{j'}(y)\downarrow = 1]. \end{array}$$

Then define

$$\sigma_{s+1} = \begin{cases} \psi_j(0)\ldots\psi_j(x) & \text{if } c(s,j) \geq c(s,j') \\ \psi_{j'}(0)\ldots\psi_{j'}(x) & \text{if } c(s,j) < c(s,j'). \end{cases}$$

The search terminates: There are $k, k' > max\{2, i\}$ such that $\varphi_k = \sigma_s 0^{x-z} k^\infty$, $\varphi_{k'} = \sigma_s 1^{x-z} k^\infty$. Then $\varphi_k, \varphi_{k'} \in F$ and, by the assumption, there are j, j' so that $\psi_j, \psi_{j'}$ extend $\sigma_s 0^{x-z}$, $\sigma_s 1^{x-z}$, respectively.

Obviously, $\varphi_{g(i)} = \lim_s \sigma_s$ exists and is total. We show that $(\forall s)(\exists y)[\sigma_s(y)\downarrow \ \wedge \ (\psi_s(y)\uparrow \ \vee \ \psi_s(y)\downarrow \neq \sigma_s(y))]$, so that $\varphi_{g(i)} \neq \psi_s$ for all s follows. Regarding the definition of σ_{s+1} above, we only discuss the case $c(s,j) \geq c(s,j')$, the other one being analogous.

Case 1: $c(s,j) \geq \frac{1}{z+2}$. By (C), $h(D(\psi_s, \psi_j)) \geq c(s,j)$, hence $D(\psi_s, \psi_j) > \frac{1}{x+1}$. Since $\psi_j(y)\downarrow$ for all $y \leq x$, $\psi_s(y)\uparrow \vee \psi_s(y)\downarrow \neq \psi_j(y)\downarrow$ for some $y \leq x$. *Case 2*: $c(s,j) < \frac{1}{z+2}$. Assuming $\psi_s(z+1)\downarrow$ leads to $\psi_s(z+1) \neq \psi_j(z+1) \vee \psi_s(z+1) \neq \psi_{j'}(z+1)$. If, e.g., $\psi_s(z+1) \neq \psi_j(z+1)$ then, by construction, $D(s, \psi_j) = \frac{1}{z+2}$, $D(\psi_s, \psi_s) \leq \frac{1}{z+3}$, $D(\psi_j, \psi_j) \leq \frac{1}{x+2} \leq \frac{1}{z+3}$, $c(s,j) < \frac{1}{z+2}$, in contradiction to (A). We proceed analogously for $\psi_s(z+1) \neq \psi_{j'}(z+1)$. It follows that $\psi_s(z+1)\uparrow$. ■

We now prove the theorem announced above.

Theorem 6.3 Let ψ be a numbering and c a recursive pseudometric such that ψ, c match. Then the class of all total functions in ψ is Ex-learnable.

Proof Let g be defined as in the proof of theorem 6.1: To compute $\varphi_{g(i)}(x)$, search for k with $c(i,k) = 0$ and $(\forall z \leq x)[\psi_k(z)\downarrow]$; output $\psi_k(x)$. The learner M proceeds in a similar way as there, enumerating a list R of removed indices. For any current stage of R, $g(i)$ is put out, where $i = min\{x : x \notin R\}$. i is included in R if an input string σ and an index j are found such that $\psi_j(x)\downarrow = \sigma(x)$ for all $x \in dom(\sigma)$ and $c(i,j) > \frac{1}{|\sigma|+1}$.

If $f = \psi_i$ then i is never removed: Assume there were σ, j with $\psi_j(x)\downarrow = \sigma(x) = \psi_i(x)$ for all $x \in dom(\sigma)$ and $c(i,j) > \frac{1}{|\sigma|+1}$. By (B), there is k with $\psi_k = \psi_i$ and $c(k,j) \leq D(\psi_j, \psi_i)$. By (C), $c(i,k) = 0$. By the triangle inequality, $c(i,j) \leq c(i,k) + c(k,j) = c(k,j) \leq$

$D(\psi_i, \psi_j)$. As $\psi_i(x) \downarrow = \psi_j(x) \downarrow$ for all $x \leq |\sigma|$, $D(\psi_i, \psi_j) \leq \frac{1}{|\sigma|+1}$. This contradicts to $c(i,j) > \frac{1}{|\sigma|+1}$.

If $c(i,j) > 0$ for some j with $\psi_j = f$ then i is at some time removed.

Assume that the process has converged to $i \notin R$ and that $\psi_j = f$. Then $c(i,j) = 0$. We show that $\varphi_{g(i)} = f$. By construction, the amalgamation $\varphi_{g(i)}$ is total. For any x, the k found in the computation of $\varphi_{g(i)}(x)$ satisfies $c(i,k) = 0$ and $\psi_k(z) \downarrow$ for all $z \leq x$. Hence $D(\psi_k, \psi_k) < \frac{1}{x+1}$, $c(j,k) \leq c(i,k) + c(i,j) = 0$, and by (A), $D(\psi_j, \psi_k) \leq max\{c(j,k), D(\psi_j, \psi_j), D(\psi_k, \psi_k)\} = D(\psi_k, \psi_k) < \frac{1}{x+1}$. It follows that $\psi_j(x) = \psi_k(x) = \varphi_{g(i)}(x)$. ■

The converse of Theorem 6.3 is false, as will follow from Theorem 6.4 below. Here, a class F of total recursive functions is called *reliably Ex-learnable* iff it is Ex-learnable via a machine M which for any total input function f either converges to a program of f or else diverges, i.e., changes its hypotheses infinitely often.

Theorem 6.4 Let ψ be a dense numbering and c a recursive pseudometric such that ψ, c match. Then the class of total functions in ψ is reliably Ex-learnable.

Corollary 6.5 There are Ex-learnable classes F of total recursive functions such that no numbering ψ which contains F can match any recursive pseudometric c.

Proof of Corollary There exist dense classes F of total recursive functions which are Ex-learnable, but not reliably Ex-learnable [3], [10], [20]. ■

Proof of Theorem 6.4 A modification ϕ of ψ is defined as follows. To compute $\phi_i(x)$, search for some j such that $c(i,j) < \frac{1}{2x+2}$, $(\forall y < x)[\phi_i(y) \downarrow = \psi_j(y) \downarrow]$, and $(\forall y < 2x+2)[\psi_j(y) \downarrow]$; put $\phi_i(x) = \psi_j(x)$. Every ϕ_i has an initial segment as its domain, furthermore ϕ_i is total if ψ_i is.

If ψ_i is total then $\phi_i = \psi_i$: By way of contradiction, let x be minimal with $\phi_i(x) \neq \psi_i(x)$. We have $\phi_i(x) = \psi_j(x)$ for some j with $c(i,j) < \frac{1}{2x+2}$, $(\forall y < x)[\psi_j(y) \downarrow = \phi_i(y) = \psi_i(y)]$, and $(\forall y < 2x+2)[\psi_j(y) \downarrow]$. As ψ is dense, ψ contains a total function g such that $g(y) = \psi_j(y)$ for all $y < 2x+2$. By (B) there is an index k with $\psi_k = g$ and $c(j,k) \leq D(\psi_j, g) < \frac{1}{2x+2}$. Together with $c(i,j) < \frac{1}{2x+2}$ we conclude $c(i,k) < \frac{1}{x+1}$ by the triangle inequality, hence $D(\psi_i, \psi_k) < \frac{1}{x+1}$ by (A). It follows that $\psi_i(x) = \psi_k(x) = \psi_j(x)$ in contradiction to the assumption.

The learning machine again proceeds by collecting forbidden indices in the list R. For any current stage of R, $h(i)$ is put out, where $i = min\{x : x \notin R\}$ and h is a one-one recursive translation of ϕ into φ. i is included in R if for some input string σ already present

(a) $(\exists x)[\phi_i(x) \downarrow \neq \sigma(x) \downarrow]$ or
(b) $(\exists x)(\exists j)[c(i,j) \geq \frac{1}{2x+2} \wedge (\forall y < 2x+2)[\psi_i(y) \downarrow = \sigma(y) \downarrow]]$.

Let f be the input function.

If $f = \psi_i$ then i is never included in R. i can certainly not be included in R by (a), as $\psi_i = \phi_i$. Assume i were included in R by (b). For the j found, there is k with $\psi_k = \psi_i$ and $c(k,j) \leq D(\psi_i, \psi_j)$, by (B). Furthermore, $c(i,k) = 0$ by (C), so that $c(i,j) \leq c(i,k) + c(k,j) \leq D(\psi_i, \psi_j)$ follows, hence $D(\psi_i, \psi_j) \geq \frac{1}{2x+2}$ for the x found in (b). This contradicts to $(\forall y < 2x+2)[\psi_i(y)\downarrow = \sigma(y)\downarrow = f(y)]$ in (b).

It remains to show that each i with $\phi_i \neq f$ will at some time be removed. This is immediate if $(\exists x)[\phi_i(x)\downarrow \neq f(x)]$, by (a). Otherwise, ϕ_i is partial, $dom(\phi_i) = \{y : y < x\}$ for some x, and $(\forall y < x)[\phi_i(y)\downarrow = f(y)]$. By definition of $\phi_i(x)$, there is no j such that $c(i,j) < \frac{1}{2x+2}$, $(\forall y < x)[\phi_i(y) = \psi_j(y)\downarrow]$ and $(\forall y < 2x+2)[\psi_j(y)\downarrow]$. But as ψ is dense, there exists j with $(\forall y < 2x+2)[\psi_j(y)\downarrow = f(y)]$. This j also satisfies $(\forall y < x)[\psi_j(y)\downarrow = \phi_i(y)]$. It follows that $c(i,j) \geq \frac{1}{2x+2}$, since otherwise $\phi_i(x)$ would be defined. Thus, i is included in R by (b).

In all, for any input f, the learning algorithm either converges to the program $h(i)$ for $\phi_i = f$ or else diverges, as all indices i are successively removed and h is one-one. ■

In our next theorem, we consider recursively enumerable families of total recursive functions. It follows from Kummer [8] and from our characterization in part (**a**) below that, for such a family F, every numbering of F possesses a compatible pseudometric iff F is co-learnable with respect to these numberings. Here, F is *co-learnable* w.r.t. the numbering ψ of F if, for all $f \in F$, there is e such that $\{M(f(0)\dots f(n)) : n \in I\!N\} = (I\!N \cup \{?\}) \setminus \{e\}$ and $\psi_e = f$. Furthermore (Kummer [8]), F is finitely learnable iff F is *effectively discrete*, i.e., uniformly in an index of a numbering, from any numbering ψ of F and any i a string σ can be effectively determined such that $\sigma \sqsubseteq \psi_i \;\wedge\; (\forall j)[\sigma \sqsubseteq \psi_j \;\Rightarrow\; \psi_j = \psi_i]$. An equivalent definition for F being effectively discrete is that there is a recursively enumerable set $\{\delta_{h(i)} : i \in I\!N\}$ of strings (where $h \in R_1$ and δ is a canonical numbering of the finite functions) such that $(\forall f \in F)(\exists i)[\delta_{h(i)} \sqsubseteq f \;\wedge\; (\forall g \in F)[\delta_{h(i)} \sqsubseteq g \;\Rightarrow\; f = g]]$, and $(\forall f, g \in F)(\forall i)[(\delta_{h(i)} \sqsubseteq f \;\wedge\; \delta_{h(i)} \sqsubseteq g) \;\Rightarrow\; f = g]$. Using a result by Selivanov [16], Freivalds, Kinber, and Wiehagen [5] proved that a family F of total recursive functions is effectively discrete iff the equality problem for functions in F is uniformly decidable w.r.t. numberings of F. Theorem 6.6(**b**) extends this characterization.

Theorem 6.6 Let F be a recursively enumerable class of total recursive functions.

(**a**) The following statements are equivalent:

- Every numbering of F has a recursive equality problem.
- For every numbering ψ of F the pseudometric d_ψ, where $d_\psi(i,j) = D(\psi_i, \psi_j)$, is computable.
- For every numbering of F, there is a compatible distance measure.

(**b**) The following statements are equivalent:

- F is effectively discrete.
- The equality problem of numberings of F is uniformly recursive with respect to the numberings.

– The pseudometrics d_ψ are uniformly computable w.r.t. indices of numberings ψ of F.

– For every numbering of F, there uniformly exists a compatible recursive distance measure.

Proof (**a**) If the equality problem of ψ is recursive then d_ψ is a recursive pseudometric and satisfies all axioms in Definition 2.3, as all ψ_i are total. It remains to show that, conversely, if some ψ enumerating F possesses a recursive compatible distance measure then the equality problem of ψ must be recursive. This will be done in two steps: We show that (1) if ψ possesses such c then F is discrete (with respect to D) and (2) it then follows that equality in ψ is recursive.

Ad (1). Assume that F is not discrete and w.l.o.g. let 0^∞ be an accumulation point. Let B be any r.e., non-recursive set. We will construct a numbering ψ of F such that, for all i, j,

$$(*) \quad i,j \in B \wedge i \neq j \;\Rightarrow\; \psi_i \neq \psi_j \text{ and}$$
$$i \notin B \;\Leftrightarrow\; \psi_i = 0^\infty.$$

Then assume that there would exist a recursive distance measure c compatible with ψ. By (A), (B), $(\forall i,j)[i \neq j \wedge j \in B \;\Rightarrow\; c(i,j) = D(\psi_i, \psi_j)]$, as j is the only index for ψ_j. It follows that B would be recursive: Fix some $i \notin B$. Given j, if $c(i,j) = 0$ then $D(\psi_i, \psi_j) = 0$, $j \notin B$. If $c(i,j) > 0$ then test whether $(\exists x)[\frac{1}{x+1} \geq c(i,j) \wedge \psi_j(x) > 0]$. If so then $\psi_j \neq 0^\infty, j \in B$. If not then $c(i,j) > D(\psi_i, \psi_j)$, hence $j \notin B$.

Construction of ψ. We may assume that F is given by a one-one numbering ϕ, and $\phi_0 = 0^\infty$. Let B be represented as $B = \bigcup_s B_s$ where B_0 is recursive and infinite, and $B_s \subseteq B_{s+1}$. In our construction, indices e for ϕ will become marked as "used" and (some) indices i for ψ as "defined".

Initialization: $\psi_i(x) = 0$ for all i, x with $i \notin B_x$; mark 0 as "used". (Thus $\psi_i = 0^\infty = \phi_0$ is readily defined if $i \notin B$, without i being marked as "defined".)

Enumerate B. For any $i \in B$ appearing, if i is not marked as "defined" so far, look for the first e such that e is not marked "used" and ϕ_e extends the initialization of ψ_i. (No proper extending is necessary if $i \in B_0$.) ϕ_e exists since 0^∞ is an accumulation point. Define $\psi_i = \phi_e$, mark i as "defined" and e as "used". Furthermore, for all $e' < e$ that are not marked "used", find elements $i' \in B_0$ not already marked "defined" and define in a one-one manner $\psi_{i'} = \phi_{e'}$; mark those i' as "defined" and e' as "used". (This is to guarantee that every ϕ_e becomes contained in ψ.)

By construction, ψ satisfies $(*)$.

Ad (2). We make use of a result by M. Kummer (Lemma 8 in [8]). A set A is said to be a *weak representative* for ψ if

(I) A is recursively enumerable,

(II) $(\forall i)(\exists j)[j \notin A \wedge \psi_j = \psi_i]$,

(III) $(\forall i)$[if there are infinitely many j with $\psi_j = \psi_i$ then at least one of them is in A].

Kummer's result is that if ψ has a weak representative then the equality problem of ψ is recursive. We show that from our assumptions on ψ above the existence of a weak representative follows.

Define A to be the set of all indices i such that

$$(\forall j < i)(\exists x)[\psi_j(x) \neq \psi_i(x)] \text{ and}$$
$$(\exists j > i)[c(i,j) = 0 \vee (c(i,j) > 0 \wedge (\forall x \leq \tfrac{1}{c(i,j)})[\psi_i(x) = \psi_j(x)])].$$

A is obviously r.e. As for (II), take some i. If there is $j \neq i$ with $\psi_j = \psi_i$ then because of the first condition, one of i, j is not in A. Otherwise i is the only index of ψ_i and by (A), (B), $c(i,j) = D(\psi_i, \psi_j)$ for all j. In particular, for all $j > i$, $c(i,j) = D(\psi_i, \psi_j) > 0$, and $\psi_i(x) \neq \psi_j(x)$ for $x = \frac{1}{c(i,j)} - 1$, so that i does not satisfy the second condition, hence $i \notin A$. In both cases, (II) holds. At last to show (III), consider any i, j such that $j > i$ and $\psi_i = \psi_j$. If $c(i,j) = 0$ then i is enumerated in A. If $c(i,j) > 0$ then $(\forall x \leq \frac{1}{c(i,j)})[\psi_i(x) = \psi_j(x)]$ and i is also enumerated in A.

(**b**) If F is effectively discrete then clearly the equality problem of numberings of F is uniformly decidable in an index of the numbering. Also, analogously to part (**a**), it follows from the uniform recursiveness of equality in F that the pseudometrics d_ψ can be computed uniformly in an index of ψ. Each d_ψ together with ψ satisfies all axioms in Definition 2.3. It remains to show that if there uniformly exist compatible distance measures for the numberings of F then F is effectively discrete.

Let $\{\phi^e : e \in \mathbb{N}\}$ (that is, the corresponding universal function) be an acceptable numbering of the two-place p.r. functions. By assumption, there is a partial recursive γ such that if ϕ^e is a numbering of F then $\gamma(e)\downarrow$ and $c_e = \phi^{\gamma(e)}$ is the characteristic function of equality in ϕ^e. The function h appearing in the definition of effective discreteness is obtained from a fixed, one-one numbering ψ of F. We define a recursive function f enumerating a family $\phi^{f(i)}$, $i \in \mathbb{N}$, of numberings. $\phi^{f(i)}_{j+1} = \psi_j$ for all i, j. $\phi^{f(i)}_0$ is defined separately where, by the recursion theorem, we may use f in the definition.

$\phi^{f(i)}_0(x) = \psi_i(x)$ if $c_{f(i),x}(0, i+1)\uparrow$ or $0 < c_{f(i),x}(0, i+1)\downarrow \leq \frac{1}{x+1}$. If this is true for all x then $\phi^{f(i)}_0 = \psi_i$. Otherwise there is the first $x = s_i$ such that $c_{f(i),s_i}(0, i+1)\downarrow$ and $c_{f(i)}(0, i+1) = 0$ or $c_{f(i)}(0, i+1) > \frac{1}{s_i+1}$. Look for the first $j \neq i$ such that $\psi_j(x) = \psi_i(x)$ for all $x \leq s_i$. If such a j is found, define $\phi^{f(i)}_0(x) = \psi_j(x)$ for all $x \geq s_i$, otherwise $\phi^{f(i)}_0(x)\uparrow$ for all $x \geq s_i$.

No $\phi^{f(i)}_0$ can be total. Assume that $\phi^{f(i)}_0$ would be total because s_i has never been found. Then $\phi^{f(i)}_0 = \psi_i$. It follows that $\phi^{f(i)}$ is a numbering of F, hence $c_{f(i)}(0, i+1)\downarrow$ and $0 < c_{f(i)}(0, i+1) \leq \frac{1}{x+1}$ for all x, which is impossible. Second, assume that s_i has been found and $\phi^{f(i)}_0$ would be total. Then $\phi^{f(i)}_0 = \psi_j$ for some $j \neq i$, again $\phi^{f(i)}$ enumerates F and $c_{f(i)}(0, i+1)\downarrow$. $c_{f(i)}(0, i+1) = 0$ is impossible by (A), as $\psi_j \neq \psi_i$. Hence $c_{f(i)}(0, i+1) > \frac{1}{s_i+1}$ and, by choice of ψ_j, $c_{f(i)}(0, i+1) > D(\phi^{f(i)}_0, \phi^{f(i)}_{i+1})$. By (B) there would exist some k with $\phi^{f(i)}_k = \phi^{f(i)}_{i+1}$ and $c_{f(i)}(0, k) \leq D(\phi^{f(i)}_0, \phi^{f(i)}_{i+1})$, which is impossible as ψ is one-one and $c_{f(i)}(0, i+1) > D(\phi^{f(i)}_0, \phi^{f(i)}_{i+1})$.

Thus, for all i, $\phi^{f(i)}_0$ is a string. Given i, the algorithm to find s_i terminates and $h(i)$ is defined to be the canonical index of $\phi^{f(i)}_0$. ∎

The measures d_ψ in Theorem 6.4 are pseudometrics. It is not possible, in general, to find metrics compatible with the numberings ψ. Consider ψ with $\psi_0 = \psi_1 = 0^\infty$ and $\psi_i = 0^i 1^\infty$ for $i > 1$. Then for any pseudometric c compatible with ψ, $c(i,j) = \frac{1}{j+1}$ for $i \leq 1$ and $j > 1$. It follows by the triangle inequality that $c(0,1) \leq \frac{2}{j+1}$ for all j, that is, that $c(0,1) = 0$.

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