Alexander Shnirman[1,2], Gerd Schön[1] and Ziv Hermon[1]
[1] *Institut für Theoretische Festkörperphysik, Universität Karlsruhe, D-76128 Karlsruhe, Germany.*
[2] *School of Physics and Astronomy, Tel Aviv University, 69978 Tel Aviv, Israel.*

Low-capacitance Josephson junction arrays in the parameter range where single charges can be controlled are suggested as possible physical realizations of the elements which have been considered in the context of quantum computers. We discuss single and multiple quantum bit systems. The systems are controlled by applied gate voltages, which also allow the necessary manipulation of the quantum states. We estimate that the phase coherence time is sufficiently long for experimental demonstration of the principles of quantum computation.

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The issue of quantum computation has attracted much attention recently [1]. Quantum algorithms can perform certain types of calculations much faster than classical computers [2]. The basic concepts of quantum computation are quantum operations (gates) on quantum bits (qubits) and registers (arrays of qubits). A qubit can be a two-level system which can be prepared in arbitrary superpositions of its two eigenstates, usually denoted as $|0\rangle$ and $|1\rangle$. Quantum computation requires "quantum state engineering", i. e. the controlled preparation and manipulation of these quantum states. For quantum registers, "entangled" many-qubits states (like the EPR state of two spins) have to be constructed as well. This necessitates a coupling between different qubits. A serious limitation is the requirement that the phase coherence time is sufficiently long to allow the coherent quantum manipulations. Several physical systems have been proposed as qubits, the most advanced so far appears to be a chain with trapped ions [3,4].

In this Letter we propose an alternative system, composed of low-capacitance Josephson junctions. The coherent tunneling of Cooper pairs mixes different charge states. By controlling the gate voltages we can control the strength of the mixing. The physics of coherent Cooper-pair tunneling in this system has been established before [5–7]. The algorithms of quantum computation introduce new, well-defined rules. Their realization in experiments creates a new challenge. We consider first an ideal one-bit system, and describe the possible ways of constructing quantum states. Then we focus on a two-bit system, where we propose a controllable coupling and discuss the construction of two-bit states. Finally, we include the coupling to a realistic external electrodynamic environment which limits the phase coherence time.

The ideal system which we propose as a qubit is shown in Fig. 1a (with $R = 0$ and $L = 0$). It consists of two small superconducting grains connected by a tunnel junction with capacitance C_J and Josephson coupling energy E_J. An ideal voltage source is connected to the system via two external capacitors, C. We assume that Δ is the largest energy in the problem. At low temperatures quasiparticle tunneling is suppressed. It is further well established, from the study of parity effects [6,8,9], that below a crossover temperature, T^*, the superconducting state is either totally paired (when the number of electrons is even) or it has exactly one quasi-particle (when the number of electrons is odd). The crossover temperature is $T^* \approx \Delta/\ln N_\mathrm{eff}$, where N_eff is the number of electrons in the system near the Fermi energy. Typical values for Aluminum are in the range of 100 ... 200 mK. In the following we require that the total number of electrons in both grains is even. This condition is naturally satisfied for 50% of the qubits. If only one of the islands has an unpaired excitation it can escape to normal parts of the system - if such a channel is provided - since the gap energy Δ is gained in such a process [9].

Possible quantum states of the system are then characterized by the numbers of extra Cooper pairs on the up and down islands, n_u and n_d. Due to the external capacitors C, the total number $N \equiv n_\mathrm{u} + n_\mathrm{d}$ is fixed. Hence the set of basis states is parameterized by the number of Cooper pairs on one island or the difference $n \equiv (n_\mathrm{u} - n_\mathrm{d})/2$. The Hamiltonian of this system is

$$H = \frac{(n - CV/2)^2}{C + 2C_\mathrm{J}} - E_\mathrm{J} \cos\Theta \ , \tag{1}$$

where Θ is the conjugate to the variable n. To shorten notations we use units where $2e = 1$, $\hbar = 1$, except where it helps to keep results transparent. We consider systems where the charging energy of the internal capacitor $E_{C_\mathrm{J}} = (2e)^2/2C_\mathrm{J}$ is much larger than E_J. In this regime, for most values of the external voltage V, the energies of the states are dominated by the charging part of (1). However, for those values of V where the charging energies of two neighboring states $|n\rangle$ and $|n+1\rangle$ are nearly degenerate, the Josephson coupling becomes relevant. The eigenstates are now superpositions of $|n\rangle$ and $|n+1\rangle$ with a minimum energy gap E_J between both.

We concentrate on a voltage interval where only two adjacent charge states play a role. Then it is convenient to rewrite (1) in a spin-$\frac{1}{2}$ language:

$$H = \frac{CV}{2(C + 2C_\mathrm{J})}\sigma_z + \frac{E_\mathrm{J}}{2}\sigma_x \ , \tag{2}$$

where $|\uparrow\rangle \equiv |n\rangle$ and $|\downarrow\rangle \equiv |n+1\rangle$. Using this language we propose a few one-bit operations. If one chooses the operating point (i. e. the voltage) sufficiently far away from the degeneracy, the eigenstates are just $|\downarrow\rangle$ and $|\uparrow\rangle$. Then, switching the system suddenly to the degeneracy point for a time Δt and suddenly back, we can perform one of the basic one-bit operations - a spin flip:

$$U_{\text{flip}}(\Delta t) = \begin{pmatrix} \cos(E_{\text{J}}\Delta t/2) & i\sin(E_{\text{J}}\Delta t/2) \\ i\sin(E_{\text{J}}\Delta t/2) & \cos(E_{\text{J}}\Delta t/2) \end{pmatrix} \tag{3}$$

We got rid of time-dependent phases by working in the interaction picture, where the zero-order Hamiltonian is the one at the operation point. To estimate the time-width Δt of the voltage pulse needed for a total spin flip (the operation time), we note that a typical experimental value of E_{J} is of order 1K. It cannot be chosen much smaller, since the condition $k_{\text{B}}T \ll E_{\text{J}}$ must be satisfied. Therefore the operation time is very short: $\Delta t \approx 10^{-10}$s.

An alternative way to perform a coherent spin flip is probably more easy to realize: The system is pushed adiabatically to the degeneracy point, and an ac voltage with frequency $E_{\text{J}}/\hbar$ is applied. The process is analogous to the paramagnetic resonance (here the constant magnetic field component is in the x-direction, while the oscillating one is in the z-direction). The time-width of the ac pulse needed for the total spin flip depends on its amplitude, therefore it can be chosen much longer than 10^{-10}s.

To perform two-bit operations which result in entangled states, one has to couple the qubits in a controlled way. The ideal situation, where the coupling can be switched on and off, appears difficult to realize in microscopic and mesoscopic systems. Instead we suggest a system with a weak constant coupling between the qubits. By tuning the energy gaps of the individual qubits we can change the effective strength of the coupling. We propose to couple two qubits using an inductance as shown in Fig.1b (with $R = 0$). For $L = 0$ the system reduces to two uncoupled qubits, while for $L = \infty$ the Coulomb interaction couples both strongly. The values of L which are suitable for our purposes will be specified later. The Hamiltonian describing this system is

$$H = \sum_{i=1,2} \left\{ \frac{(n_i - V_i C_{\text{t}})^2}{2C_{\text{J}}} - E_{\text{J}} \cos\Theta_i \right\} + \frac{q^2}{2(2C_{\text{t}})} + \frac{\phi^2}{2L} - \frac{(n_1+n_2)q}{2C_{\text{J}}} - \frac{C_{\text{t}}}{4C_{\text{J}}^2}(n_1-n_2)^2 \ . \tag{4}$$

Here q denotes the total charge on the external capacitors of both qubits, ϕ is its conjugate variable, and $C_{\text{t}}^{-1} = C_{\text{J}}^{-1} + 2C^{-1}$. The (q,ϕ) oscillator produces an effective mean-field coupling between the qubits for frequencies smaller than $\omega_{\text{LC}}^{(2)} = 1/\sqrt{2C_{\text{t}}L}$. In order to have this coupling in a wide enough voltage range around the degeneracy point, we demand

$$A \equiv \frac{\hbar\omega_{LC}^{(2)}}{E_{\text{J}}} \gg 1 \ . \tag{5}$$

To obtain the mean-field coupling of the qubits we eliminate the variables q and ϕ. For this purpose we first perform a canonical transformation $\tilde{q} = q - \frac{(n_1+n_2)C_{\text{t}}}{C_{\text{J}}}$, $\tilde{\Theta}_i = \Theta_i + \frac{C_{\text{t}}}{C_{\text{J}}}\phi$, ($\phi$ and n_i unchanged), which leads to the new Hamiltonian (we omit the tildes):

$$H = \sum_{i=1,2} \left\{ \frac{(n_i - CV_i/2)^2}{C+2C_{\text{J}}} - E_{\text{J}} \cos\left(\Theta_i - \frac{C_{\text{t}}}{C_{\text{J}}}\phi\right) \right\} + \frac{q^2}{2(2C_{\text{t}})} + \frac{\phi^2}{2L} \ . \tag{6}$$

We assume that the fluctuations of ϕ are weak

$$(C_{\text{t}}/C_{\text{J}})\sqrt{\langle\phi^2\rangle} \ll 2\pi \ . \tag{7}$$

Otherwise the Josephson tunneling terms in the Hamiltonian (6) are washed out. (Below we will show this in a more rigorous way.) Assuming (7), we expand the $E_{\text{J}}\cos(..)$ terms of (6) in powers of ϕ and neglect powers higher than linear. Then we can trace out the variables q and ϕ. As a result we obtain an effective Hamiltonian, consisting of two one-bit Hamiltonians (1) and a coupling term: $H_{\text{coup}} = E_L\left[\sin\Theta_1 + \sin\Theta_2\right]^2$, where

$$E_L = 2\pi^2 \frac{C_{\text{t}}^2}{C_{\text{J}}^2}\frac{E_{\text{J}}^2 L}{\Phi_0^2} \ , \tag{8}$$

and $\Phi_0 \equiv h/2e$ is the flux quantum. In the spin-$\frac{1}{2}$ language we get

$$H_{\text{coup}} = -(E_L/4)\left(\sigma_y^{(1)} + \sigma_y^{(2)}\right)^2 \ . \tag{9}$$

This term provides the required weak coupling if it is small, i.e. if $E_L \ll E_{C_{\text{J}}}$.

The mixed term in (9) is important in certain situations. If the voltages V_1 and V_2 are such that both qubits are out of degeneracy, to a good approximation, the eigenstates of the two-bit system without coupling are $|\downarrow\downarrow\rangle$, $|\downarrow\uparrow\rangle$, $|\uparrow\downarrow\rangle$ and $|\uparrow\uparrow\rangle$. In a general situation, these states are separated by energies which are larger or much larger than E_{J} or E_L. Therefore, the effect of the coupling is small. If, however, a pair of these state is degenerate, the coupling may lift the degeneracy, changing the eigenstates drastically. For example, if $V_1 = -V_2$, the states $|\downarrow\uparrow\rangle$ and $|\uparrow\downarrow\rangle$ are degenerate. In this case the correct eigenstates are: $\frac{1}{\sqrt{2}}(|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle)$ and $\frac{1}{\sqrt{2}}(|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle)$ with the energy splitting E_L between them.

Now we propose a way to perform two-bit operations which result in entangled states. For this we choose the operating points for the qubits at different voltages, switch suddenly the voltages to be equal for a time Δt and switch suddenly back. The result is a "generalized" spin-flip, which may be described in the basis $\{|\downarrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\uparrow\uparrow\rangle\}$ by a matrix:

$$U^{(2)}_{\rm flip}(\Delta t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\left(\frac{E_L \Delta t}{2}\right) & i\sin\left(\frac{E_L \Delta t}{2}\right) & 0 \\ 0 & i\sin\left(\frac{E_L \Delta t}{2}\right) & \cos\left(\frac{E_L \Delta t}{2}\right) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \quad (10)$$

Instead of applying very short voltage pulses, one can push the system adiabatically to the degeneracy point ($V_1 = -V_2$), and apply an ac voltage pulse in the symmetric channel $V_1 + V_2 = A\exp(iE_L t)$.

The idealized picture outlined above has to be extended to account for possible dissipation mechanisms which cause decoherence and energy relaxation. In this Letter we focus on the effect of ohmic dissipation in the circuit, which originates mostly from the voltage sources (the quasi-particle tunneling is strongly suppressed at $T \le T^*$ [8,9]). We also consider the effect of LC resonances in the circuit. The system is shown in Fig. 1a, including the inductance L explicitly since the LC oscillatory mode plays an important role in the two-bit system. The Hamiltonian of the system is

$$H = \frac{(n - VC_{\rm t})^2}{2C_{\rm J}} - E_{\rm J}\cos\Theta + \frac{q^2}{2C_{\rm t}} + \frac{\phi^2}{2L} - \frac{nq}{C_{\rm J}} + \sum_j \left[\frac{p_j^2}{2m_j} + \frac{m_j\omega_j^2}{2}\left(x_j - \frac{\lambda_j}{m_j\omega_j^2}q\right)^2\right] , \quad (11)$$

with $\frac{\pi}{2}\sum_j \frac{\lambda_j^2}{m_j\omega_j}\delta(\omega - \omega_j) = R\omega$.

First, we estimate the energy relaxation time, τ_r, due to the ohmic dissipation. We assume that the system is prepared away from the degeneracy point in one of its eigenstates ($|n\rangle$ or $|n+1\rangle$). To apply the standard Golden Rule results for the transition rate, we perform two consecutive canonical transformations:

$$\tilde\phi = \phi + \sum_j \frac{\lambda_j}{m_j\omega_j^2}p_j , \quad \tilde x_j = x_j - \frac{\lambda_j}{m_j\omega_j^2}q , \quad (12)$$

(q and p_j unchanged), and $\bar q = \tilde q - \frac{nC_{\rm t}}{C_{\rm J}}$, $\bar\Theta = \Theta + \frac{C_{\rm t}}{C_{\rm J}}\tilde\phi$, ($\tilde\phi$ and n unchanged). Then, the part of the Hamiltonian connecting the states $|n\rangle$ and $|n+1\rangle$ is $H_{\rm t} = \frac{E_{\rm J}}{2}\exp(i\bar\Theta)\exp\left(-i\frac{C_{\rm t}}{C_{\rm J}}\tilde\phi\right)$ + h.c., and the transition rate from $|n\rangle$ to $|n+1\rangle$ is given by [10,11]:

$$\Gamma(\Delta E) = \frac{\pi}{2\hbar}E_{\rm J}^2 P(\Delta E) , \quad (13)$$

$$P(\Delta E) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dt\ \exp\left[4\frac{C_{\rm t}^2}{C_{\rm J}^2}K(t) + \frac{i}{\hbar}\Delta E t\right] , \quad (14)$$

$$K(t) \equiv \langle[\tilde\phi(t) - \tilde\phi(0)]\tilde\phi(0)\rangle = 2\int_0^\infty \frac{d\omega}{\omega}\frac{Re Z_{\rm t}(\omega)}{R_K} \times \left[\coth\left(\frac{\hbar\omega}{2k_B T}\right)[\cos(\omega t) - 1] - i\sin(\omega t)\right] . \quad (15)$$

Here $Z_{\rm t}^{-1} = i\omega C_{\rm t} + (R + i\omega L)^{-1}$ and ΔE is the energy gap between the two states. The qualitative behavior of the system is controlled by the dimensionless conductance $g = R_{\rm K}/4R$ ($R_K \equiv h/e^2$ is the quantum resistance). In our system the controlling parameter is renormalized. From (14) one can observe that $\tilde g = (C_{\rm J}^2/C_{\rm t}^2)g$ is the relevant parameter. Thus, choosing the external capacitances, C, smaller than the internal one, $C_{\rm J}$, we can reduce the effect of the dissipation. Physically, this means that the fluctuations produced by the resistor are screened by the small capacitors, and have little effect on the junction.

To be more concrete, we exploit the asymptotic formula for $P(\Delta E)$ [11]

$$P(\Delta E) = \frac{\exp(-2\gamma/\tilde g)}{\Gamma(2/\tilde g)}\frac{1}{\Delta E}\left[\frac{\pi}{\tilde g}\frac{\Delta E}{E_{C_{\rm t}}}\right]^{2/\tilde g} , \quad (16)$$

where $\Gamma(..)$ is the Gamma function. For large values of $\tilde g$ we obtain:

$$\tau_r \equiv \frac{1}{\Gamma(\Delta E)} \approx \tau_{\rm op}\frac{\tilde g}{2\pi^2}\frac{\Delta E}{E_{\rm J}} , \quad (17)$$

where $\tau_{\rm op} \approx h/E_{\rm J}$ is the operation time (see (3)).

At the degeneracy point the system is equivalent to the two-level model with a weak ohmic dissipation, which has been studied extensively [12]. It is well known that when $\tilde g \gg 1$ coherent oscillations take place. These oscillations make the spin-flip operation (3) possible. The decay time of the coherent oscillations is given by

$$\tau_d \approx \frac{\tilde g}{2\pi^2}\frac{h}{E_{\rm J}} = \frac{\tilde g}{2\pi^2}\tau_{op} , \quad (18)$$

and the energy gap $E_{\rm J}$ is slightly renormalized: $E_{\rm J} \to E_{\rm J}(E_{\rm J}/\hbar\omega_c)^{1/(g-1)}$. The physical cut-off ω_c is usually a system-dependent property. For a pure ohmic dissipation caused by a metallic resistor it may be as high as the Drude frequency. However, when additional capacitances and inductances are present in the circuit, the cut-off is lowered to the characteristic LC frequencies.

As indicated above, the LC phase fluctuations can wash out the Josephson coupling. To see this, we begin with the Hamiltonian (11) and trace out the bath variables and the oscillatory mode variables - ϕ, q. The partition function reads:

$$Z = \sum_{n_0}\int_{n_0}^{n_0} DnD\Theta\exp\left\{\int_0^\beta d\tau\left[i\Theta\dot n - \frac{(n - VC_{\rm t})^2}{2C_{\rm J}} + E_{\rm J}\cos\Theta\right] - \int_0^\beta\int_0^\beta d\tau d\tau'\frac{1}{2}G(\tau - \tau')n(\tau)n(\tau')\right\} , \quad (19)$$

where $G(\omega_n) = -(C_{\rm t}/C_{\rm J}^2)\left(1 + C_{\rm t}L\omega_n^2 + C_{\rm t}R|\omega_n|\right)^{-1}$. Below we show that in the relevant parameters' range the following inequality holds

$$1/(C_{\rm t}R) \gg 1/\sqrt{LC_{\rm t}} \gg R/L \ . \qquad (20)$$

Therefore, the natural cutoff for $G(\omega_n)$ is $\omega_c = \omega_{LC}^{(1)} = 1/\sqrt{LC_{\rm t}}$. We approximate $G(\omega_n) \approx -\frac{C_{\rm t}}{C_{\rm J}^2}(1 - C_{\rm t}L\omega_n^2 - C_{\rm t}R|\omega_n|)$ for $\omega_n < \omega_c$, and $G(\omega_n) = 0$ otherwise. We focus on the inductive (second) term of $G(\omega_n)$, and apply the standard charge representation technique [13]. Expanding $\exp\left[\int_0^\beta d\tau E_{\rm J}\cos(\Theta)\right]$ in powers of $E_{\rm J}$ and integrating over Θ term by term, one obtains a path integral over integer charge paths with instantaneous "jumps" between the different values of n. Each "jump" contributes a multiplicative factor of $E_{\rm J}/2\hbar$ to the weight of the path. The inductive term contributes another multiplicative factor for each "jump", so that $E_{\rm J}$ is renormalized as: $E_{\rm J} \to E_{\rm J}\,\exp\left(-\frac{2LC_{\rm t}^2\omega_c}{\pi R_K C_{\rm J}^2}\right)$. One can immediately observe that the condition that $E_{\rm J}$ is not renormalized to zero coincides with the small fluctuations condition (7). We emphasize that the phase fluctuations which may wash out the Josephson coupling are related to the "weakly fluctuating" phase ϕ, rather than to the "strongly fluctuating" $\tilde{\phi}$ (see (12)). Thus the effects of the inductance and the dissipation are well separated in this regime. One arrives at another way of viewing this separation by noting that the LC phase fluctuations are fast, therefore they effectively wash out the slower processes (like Josephson tunneling). These are the fast small fluctuations of ϕ that are responsible for the two-bit coupling (9). On the other hand, the phase fluctuations caused by the resistor are large only at low frequencies. In [14] we have extended the present arguments and showed that also the two-bit coupling is stable under the influence of the dissipation.

Several conditions have been assumed in this Letter in order to arrive at the controlled manipulation of qubits. Here we repeat these conditions and discuss the appropriate range of parameters. We start with $E_{\rm J} \approx 1\,{\rm K}$ as a suitable experimental condition. To satisfy $E_{\rm J} \ll E_{C_{\rm J}}$ we take $C_{\rm J} \approx 10^{-16}\,{\rm F}$, which is an experimentally accessible value. As we would like A to be large (5), it seems that L and C_t should be as small as possible. However, the two-bit coupling energy, E_L (8), should be larger than the temperature of the experiment. Assuming a reasonable working temperature of $20\,{\rm mK}$, we demand $E_L \approx 0.1\,{\rm K}$. From (5) and (8) we get $C_{\rm t} = E_L C_{\rm J}^2 A^2/e^2$. To have a wide enough operation voltage interval we take $A \approx 10$, and obtain $C_{\rm t} \approx 10^{-17} - 10^{-16}\,{\rm F}$ and $L \approx 10^{-8} - 10^{-7}{\rm H}$. Thus the renormalization of g is of the order of 10, and $\tau_r/\tau_{\rm op} \approx 10^2 - 10^3$ (assuming the realistic value $R \approx 100\,\Omega$) (17). Finally we observe that in this range of parameters the inequalities (7) and (20) are always satisfied. We conclude that the quantum manipulations we have discussed in this Letter can be tested experimentally using the currently available lithographic and cryogenic techniques. Application of the Josephson junction system as an element of a quantum computer is a more subtle issue, demanding either the fabrication of junctions with $C_{\rm J} < 10^{-16}\,{\rm F}$, or a further reduction of the working temperature.

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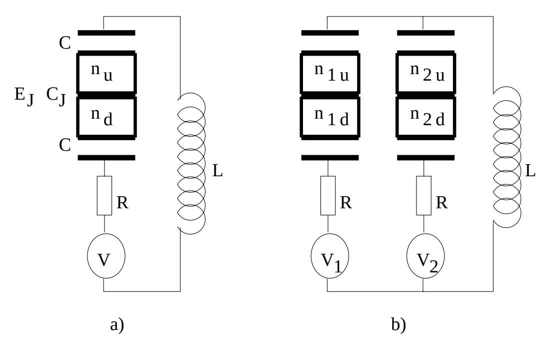

FIG. 1. a) one-qubit system; b) two-qubit system.