RUIN PROBABILITIES FOR ERLANG(2) RISK PROCESSES[1]

David C M Dickson, The University of Melbourne

and

Christian Hipp, The University of Karlsruhe

Abstract

In this paper we consider a risk process in which claim inter-arrival times have an Erlang(2) distribution. We consider the infinite time survival probability as a compound geometric random variable and give expressions from which both the survival probability from initial surplus zero and the ladder height distribution can be calculated. We consider explicit solutions for the survival/ruin probability in the case where the individual claim amount distribution is phase-type, and show how the survival/ruin probability can be calculated for other individual claim amount distributions.

Keywords: Erlang process, ruin probability, phase-type distribution, recursive calculation.

1. Introduction

In this paper we shall consider a risk process where claims occur as an Erlang process. In particular we shall assume that the times between claims (and the time until the first claim) form a sequence of independent and identically distributed random variables, denoted $\{T_i\}_{i=1}^{\infty}$, with density function

$$k(t) = \beta^2 t e^{-\beta t} \qquad \text{for } t > 0,$$

i.e. an Erlang$(2, \beta)$ distribution.

Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of independent and identically distributed random variables, where X_i denotes the amount of the i-th claim. Let $F(x)$ be the distribution, and, when it exists, let $f(x)$ denote the density function of X_i. We denote the mean individual claim amount by m_1. Let c denote the insurer's premium income per unit time. We will assume that

$$cE(T_i) > E(X_i)$$

[1]This research was conducted during the authors' visits to the Laboratory of Actuarial Mathematics in the first half of 1997.

for all i.

The probability of ultimate ruin from initial surplus u for this risk process is defined as

$$\psi(u) = \Pr\left(u + \sum_{i=1}^{n}(cT_i - X_i) < 0 \quad \text{for some } n,\ n = 1, 2, 3, ...\right)$$

and let $\delta(u) = 1 - \psi(u)$ denote the survival probability. If the moment generating function of X_i exists, then the adjustment coefficient for this risk process is the unique positive number R such that

$$E[\exp\{-cRT_i\}]E[\exp\{RX_i\}] = 1 \tag{1.1}$$

Our purpose in this paper is to find both analytical and numerical solutions for ruin/survival probabilities. In section 2 we set out some basic formulae relating to $\delta(u)$. In section 3 we show how these formulae can be used to find $\delta(0)$, and hence how we can give a compound geometric representation of $\delta(u)$. In section 4, we discuss the case of phase-type distributions. Finally, in section 5, we show how the compound geometric representation of $\delta(u)$ can be used to compute bounds and approximations for ruin probabilities in situations where an analytical solution for $\delta(u)$ does not exist.

2. Preliminaries

First, let us adapt some results from Dickson (1997). Starting from

$$\delta(u) = \int_0^\infty k(t) \int_0^{u+ct} \delta(u + ct - x)dF(x)dt$$

we have

$$c^2 \frac{d^2}{du^2}\delta(u) - 2\beta c \frac{d}{du}\delta(u) + \beta^2\delta(u) = \beta^2 \int_0^u \delta(u - x)dF(x) \tag{2.1}$$

Further, defining

$$\delta^*(s) = \int_0^\infty e^{-su}\delta(u)du \qquad \text{and} \qquad f^*(s) = \int_0^\infty e^{-sx}dF(x)$$

to be the Laplace transforms of $\delta(u)$ and $F(x)$ respectively, we have

$$\delta^*(s) = \frac{c^2 s\delta(0) + \beta^2 m_1 - 2\beta c}{c^2 s^2 - 2\beta cs + \beta^2(1 - f^*(s))}$$

Finally, defining L to be the maximum aggregate loss, so that $\delta(u) = \Pr(L \leq u)$, the Laplace transform of L, denoted $\phi^*(s)$, is

$$\phi^*(s) = E[e^{-sL}] = \frac{c^2 s^2 \delta(0) + (\beta^2 m_1 - 2\beta c)s}{c^2 s^2 - 2\beta c s + \beta^2 (1 - f^*(s))} \tag{2.2}$$

Note that just as in the classical risk model, L has a compound geometric distribution, so we can also write

$$\phi^*(s) = \frac{\delta(0)}{1 - \psi(0) r^*(s)} \tag{2.3}$$

where

$$r^*(s) = \int_0^\infty e^{-sx} dR(x)$$

and where $R(x)$ is the ladder height distribution. For convenience, we introduce the distribution $Q(x)$ defined by

$$Q(x) = \frac{1}{m_1} \int_0^x (1 - F(x)) dx$$

with Laplace transform

$$q^*(s) = \frac{1 - f^*(s)}{m_1 s}.$$

Note that $Q(x)$ is the ladder height distribution associated with the classical risk model.

3. Main Results

In this section we derive a formula for $\delta(0)$ and for the ladder height distribution. We start from formula (2.2) which can be written as

$$\phi^*(s) = \frac{c^2 s \delta(0) + \beta^2 m_1 - 2\beta c}{c^2 s - 2\beta c + \beta^2 m_1 q^*(s)} \tag{3.1}$$

>From $\beta^2 m_1 - 2\beta c < 0$ we see that the numerator

$$c^2 s \delta(0) + \beta^2 m_1 - 2\beta c$$

will have a positive zero at $s_0 = (2\beta c - \beta^2 m_1)/(c^2 \delta(0))$. Since $\phi^*(s)$ is positive for all $s > 0$, s_0 must also be a solution of the equation

$$I(s) = c^2 s - 2\beta c + \beta^2 m_1 q^*(s) = 0. \tag{3.2}$$

Notice that (3.2) is the defining equation for the adjustment coefficient for the problem, given in (1.1). However, also in the case without an adjustment coefficient, (3.2) will have a positive solution, and this solution will be unique. To see this, notice that

$$\begin{aligned} I(0) &= -2\beta c + \beta^2 m_1 < 0, \\ \lim_{s\to\infty} I(s) &= \infty \ , \text{ and} \\ I''(s) &= \beta^2 \int x^2 e^{-sx}(1-F(x))dx > 0. \end{aligned}$$

So if $I'(0) \geq 0$, the function $I(s)$ will be increasing on $(0, \infty)$, and if $I'(0) < 0$ then the function $I(s)$ will be decreasing up to some point, and it will be increasing from this point on, so in both cases the solution to (3.2) exists, and it is unique. Hence $\delta(0)$ can be identified:

$$\delta(0) = \frac{2\beta c - \beta^2 m_1}{c^2 s_0} \ . \tag{3.3}$$

We can now write (3.1) as

$$\phi^*(s) = \frac{c^2(s-s_0)\delta(0)}{c^2(s-s_0) + \beta^2 m_1[q^*(s) - q^*(s_0)]},$$

and hence

$$\begin{aligned} r^*(s) &= \frac{\beta^2 m_1[q^*(s) - q^*(s_0)]}{c^2(1-\delta(0))(s_0 - s)} \\ &= \gamma m_1 \frac{q^*(s) - q^*(s_0)}{s_0 - s} \end{aligned}$$

with

$$\gamma = \frac{\beta^2}{c^2 - 2\beta c/s_0 + \beta^2 m_1/s_0} \ .$$

In order to find the distribution with this Laplace Transform, we write $r^*(s)$ as

$$\begin{aligned} r^*(s) &= \gamma \frac{1}{s_0 - s} \int_0^\infty (e^{-sx} - e^{-s_0 x})(1-F(x))dx \\ &= \gamma \frac{1}{s_0 - s} \int_0^\infty e^{-s_0 x}(e^{-(s-s_0)x} - 1)(1-F(x))dx \\ &= \gamma \int_0^\infty e^{-s_0 x} \int_0^x e^{-(s-s_0)y} dy (1-F(x))dx \\ &= \gamma \int_0^\infty e^{-(s-s_0)y} \int_y^\infty e^{-s_0 x}(1-F(x))dxdy \\ &= \gamma \int_0^\infty e^{-sy} G(y)dy \end{aligned}$$

where

$$G(y) = \int_y^\infty e^{-s_0(x-y)}(1 - F(x))dx, \; y > 0.$$

Hence the Lebesgue density of the ladder height distribution, denoted $r(y)$, is given by

$$r(y) = \frac{\beta^2}{c^2 - 2\beta c/s_0 + \beta^2 m_1/s_0} G(y), \; y > 0. \tag{3.4}$$

Example 1. *If F is an exponential distribution, $F(x) = 1 - \exp(-\alpha x)$, we have*

$$G(y) = e^{s_0 y} \int_y^\infty e^{-(\alpha+s_0)x} dx = \frac{1}{\alpha + s_0} e^{-\alpha y}$$

and so the density of $R(x)$ is proportional to $\exp(-\alpha y)$, $y > 0$, which yields $R(x) = F(x)$ in this simple case. In this case, s_0 is the unique positive solution of

$$c^2 s^2 + \left(\alpha c^2 - 2\beta c\right) s + \left(\beta^2 - 2\alpha\beta c\right) = 0$$

which exists because of

$$\beta^2 - 2\alpha\beta c < 0.$$

4. Phase-type Distributions

Phase-type distributions, introduced by Neuts (1975), are defined via a continuous time homogeneous Markov chain $X(t)$ on a finite state space $\{0, 1, ..., I\}$, $I \geq 1$. The state 0 is absorbing, and the Markov chain is assumed to be irreducible. Then the random variable

$$T = \inf\{t \geq 0 : X(t) = 0\}$$

is finite almost everywhere, and its distribution is called a *phase-type distribution* with parameters $\pi^* = (\pi_0, ..., \pi_I)$ – the starting distribution - and $B = (b_{i,j})_{i,j=1,...,I}$ - the infinitesimal operator defined by

$$\mathbb{P}\{X(t+h) = j | X(t) = i\} = b_{i,j}h + o(h), \; h \to 0, \; i \neq j, \; i, j = 1, ..., I$$
$$\mathbb{P}\{X(t+h) = i | X(t) = i\} = 1 - b_{i,i}h + o(h), \; h \to 0, i = 1, ..., I.$$

If $\pi_0 = 0$ then $\mathbb{P}\{T = 0\} = 0$, and in this case the distribution is called proper phase-type, and we shall use the notation $\pi = (\pi_1, ..., \pi_I)$. Examples of phase-type distributions are convolutions and mixtures of exponential distributions. Convolutions or mixtures of phase-type distributions are also phase-type. For further general properties of these distributions see Neuts (1977) and Asmussen (1987); the more general and very useful notation of matrix-exponential distributions can be found in Asmussen and Bladt (1996). Applications of phase-type distributions to queueing and ruin theory are given in Asmussen and Rolski (1991) and in Asmussen (1992).

For phase-type claim size distributions, the infinite time survival probability can be given in explicit form, for exponential as well as for Erlang(2,β) inter-arrival times. Let us first recall the situation of the classical compound Poisson model. Let the inter-arrival times T_i have an exponential distribution with parameter β. Then the ruin probability is non-trivial only if

$$\psi_c(0) = \beta m_1/c < 1, \tag{4.1}$$

where ψ_c denotes the ruin probability in the classical risk model. If the claim size distribution has a proper phase-type distribution with parameters π and B, then

$$m_1 = -\pi' \left(B'\right)^{-1} e \tag{4.2}$$

and

$$\psi_c(u) = \psi_c(0)\widehat{\pi}' \exp(u\widehat{B})e. \tag{4.3}$$

Here, $e = (1, ..., 1)$ is the $I-$vector with all its entries equal to 1, and

$$\widehat{\pi} = -\frac{1}{m_1}\left(B'\right)^{-1}\pi, \tag{4.4}$$

$$\widehat{B} = B + \psi(0)b\widehat{\pi}. \tag{4.5}$$

The $I \times I-$matrix $b\widehat{\pi}$ has entries $b_i\widehat{\pi}_j$, $i, j = 1, ..., I$, where b_i is defined via

$$\mathbb{P}\{X(t+h) = 0|X(t) = i\} = b_i h + o(h),\ h \to 0,\ \ i = 1, ..., I.$$

The matrix exponential $\exp(sA)$ can be computed, e.g., with Maple V, Release 3 or 4, in the *linalg* package. There, for example, with

$$A = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$$

the command `exponential(A,t);` gives the result

$$\begin{pmatrix} e^{-t} & te^{-t} \\ 0 & e^{-t} \end{pmatrix}.$$

Example 2. *When F is the Erlang$(2,1)$ distribution, $\beta = 1$ and $c = 4$ we have $I = 2$, $\pi = (1,0)$, and*

$$B = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}.$$

Then $\psi_c(0) = 1/2$, $\widehat{\pi} = (1/2, 1/2)$,

$$\widehat{B} = \begin{pmatrix} -1 & 1 \\ 1/4 & -3/4 \end{pmatrix},$$

and formula (4.3) yields the following result:

$$\psi_c(u) = 0.55317\, exp(-0.35961\, u) - 0.05317 exp(-1.39039\, u).$$

Example 3. *If we instead take a mixture of two Erlang(2) distributions, say the symmetric mixture of Erlang(2,1) and Erlang(2,2),* $\beta = 1$, $c = 4$, *then* $I = 4$,

$$\begin{aligned} B &= \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & -2 \end{pmatrix}, \ \pi = \begin{pmatrix} 1/2 \\ 0 \\ 1/2 \\ 0 \end{pmatrix}, \\ \widehat{B} &= \begin{pmatrix} -1 & 1 & 0 & 0 \\ 1/8 & -7/8 & 1/16 & 1/16 \\ 0 & 0 & -2 & 2 \\ 1/4 & 1/4 & 1/8 & -15/8 \end{pmatrix}, \ \widehat{\pi} = \begin{pmatrix} 1/3 \\ 1/3 \\ 1/6 \\ 1/6 \end{pmatrix}, \end{aligned}$$

and the ruin probability equals

$$\begin{aligned} \psi_c(u) &= 0.40026 \exp(-0.51949\, u) - 0.04764 \exp(-2.43637\, u) \\ &\quad +0.02238 \exp(-1.39707\, u) \cos(0.15311 u) \\ &\quad -0.21635 \exp(-1.39707\, u) \sin(0.15311\, u). \end{aligned}$$

Let us now consider the case of an Erlang(2,β) inter-arrival time distribution. If the claim size has an arbitrary proper phase-type distribution with parameters (π, B), then the Laplace transform of Q is given by

$$q^*(s) = \frac{1}{m_1} \pi' (s\, Id - B)^{-1} e,$$

where $m_1 = -\pi' B^{-1} e$ and Id is the $I \times I-$identity matrix. The defining equation for s_0 is

$$c^2 s - 2\beta c + \beta^2 \pi' (s\, Id - B)^{-1} e = 0, \tag{4.6}$$

which gives us

$$\begin{aligned} \psi(0) &= \frac{c^2 s_0 - 2\beta c + \beta^2 m_1}{c^2 s_0} \qquad (4.7) \\ &= \frac{\beta^2}{c^2 s_0} \left(\pi' (s_0\, Id - B)^{-1} e - m_1 \right). \end{aligned}$$

The ladder height distribution is again phase-type with parameters $(\widetilde{\pi}, B)$, where

$$\widetilde{\pi} = \rho(B')^{-1}\left(B' - s_0\,Id\right)^{-1}\pi. \tag{4.8}$$

and where the parameter ρ has to be chosen such that $\widetilde{\pi}$ is a stochastic vector. To derive this from

$$G(y) = \int_y^\infty e^{-s_0(x-y)}(1 - F(x))dx,$$

notice that

$$\begin{aligned} 1 - F(x) &= \pi' \exp(xB)e, \\ e^{-s_0 x}\,(1 - F(x)) &= \pi' \exp(x(B - s_0\,Id))e, \\ \int_y^\infty e^{-s_0 x}\,(1 - F(x))\,dx &= \pi'(B - s_0\,Id)^{-1} \exp(y(B - s_0\,Id))e, \\ e^{-s_0 y}\int_y^\infty e^{-s_0 x}\,(1 - F(x))\,dx &= \pi'(B - s_0\,Id)^{-1} \exp(yB)e, \end{aligned}$$

and, finally, the tail probabilities of $R(x)$ are proportional to

$$\int_u^\infty e^{-s_0 y}\int_y^\infty e^{-s_0 x}\,(1 - F(x))\,dxdy = \pi'(B - s_0\,Id)^{-1}B^{-1}\exp(uB)e,$$

The ruin probability is again the tail probability of an (improper) phase-type distribution in the form of (4.3),

$$\psi(u) = \psi(0)\widetilde{\pi}' \exp(u\widehat{B})e$$

with $\widehat{B}$ defined as in (4.5).

Example 4. *When F is an Erlang$(2,1)$ distribution, $\beta = 1$ and $c = 4$ we have again $I = 2$, $\pi = (1,0)$, and*

$$B = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}.$$

The number s_0 is the positive solution of

$$\frac{1}{s+1} + \frac{1}{(s+1)^2} = 8 - 16s$$

which gives

$$s_0 = -\tfrac{3}{8} + \tfrac{1}{8}\sqrt{41} = 0.425391.$$

According to (4.8), the vector $\widetilde{\pi}$ *is given by*

$$\widetilde{\pi} = \frac{1}{2(9+\sqrt{41})} \begin{pmatrix} 5+\sqrt{41} \\ 13+\sqrt{41} \end{pmatrix} = \begin{pmatrix} 0.37016 \\ 0.62984 \end{pmatrix},$$

and the ruin probability is

$$\psi(u) = 0.17269 \exp(-0.75\,u) - 0.05424 \exp(-1.17539\,u).$$

Example 5. *When* F *is the symmetric mixture of Erlang(2,1) and Erlang(2,2),* $\beta = 1$ *and* $c = 4$*, we obtain*

$$s_0 = 0.43992,$$
$$\widetilde{\pi} = \begin{pmatrix} 0.58840 \\ 0.24115 \\ 0.18645 \\ 0.08399 \end{pmatrix},$$
$$\psi(0) = 0.07653,$$

and the ruin probability is

$$\psi(u) = 0.11071 \exp(-0.82942\,u) + 0.04578 \exp(-1.82904\,u) \\ -0.05093 \exp(-1.13370\,u) - 0.02902 \exp(-2.14776\,u).$$

5. Calculation of Ruin Probabilities

Since the ruin probability $\psi(u)$ is the tail probability of a compound geometric random variable, we can compute bounds for $\psi(u)$ using "Method 1" of Dufresne and Gerber (1989). All that we require to apply this method is the survival probability $\delta(0)$ and the ladder height distribution $R(x)$. An accurate way of approximating $\psi(u)$ is to average the upper and lower bounds. (See Dickson *et al* (1995).)

In this section we give two numerical examples. In each case, claims arrive as an Erlang(2,2) process, the individual claim amount distribution has mean 1 and the premium income per unit time is 1.1. Further, in each case, we rescaled the surplus process by a factor of 100 in order to perform calculations. (See Dufresne and Gerber (1989) or Dickson *et al* (1995).)

Example 6. *Let the individual claim amount distribution be Erlang(2,2). Then from Dickson (1997) we have*

$$\psi(u) = 0.8841 \exp(-0.1818u) - 0.0109 \exp(-2.7892u)$$

and hence we can compare exact and approximate values. Table 1 shows some bounds for, exact values of, and approximations to $\psi(u)$.

u	Lower bound	Exact value	Approximation	Upper bound
0	0.87322	0.87322	0.87322	0.87322
5	0.35423	0.35619	0.35620	0.35817
10	0.14189	0.14350	0.14351	0.14513
15	0.05684	0.05782	0.05783	0.05881
20	0.02277	0.02329	0.02330	0.02383
25	0.00912	0.00938	0.00939	0.00966

Table 1: Ruin probabilities for Erlang(2,2) claims

We can see from Table 1 that this method of approximating $\psi(u)$ *produces excellent approximations.*
Figure 1 shows (exact) values of $\psi(u)$. *For interest, we have also plotted values of the ruin probability when claims occur as a Poisson process with parameter 1. This figure shows that ruin probabilities are smaller when claims occur as an Erlang(2,2) process, as we would expect due to the smaller variance of claim inter-arrival times compared with exponential inter-arrival times under the Poisson process. Figure 2 shows a comparison of the ladder height distributions for these two models. The ladder height distribution when we have Erlang inter-arrival times is always below the corresponding distribution for the classical risk model.*

Example 7. *Let the individual claim amount distribution be Pareto(2,1). In this case we cannot solve explicitly for* $\psi(u)$, *nor for* s_0. *Recalling that we have rescaled the surplus process by a factor of 100,* s_0 *is found as the root of*

$$1.21s - 0.044 + 4\int_0^\infty \frac{e^{-sy}}{(100+y)^2}dy = 0$$

We can solve numerically to find $s_0 = 0.02916$ *and hence* $\delta(0) = 0.113364$.
We can find the ladder height distribution by integrating (3.4) giving

$$R(x) = 1 - s_0^{-1}\left(m_1\gamma(1 - Q(x)) - r(x)\right)$$

with $r(x)$ *given here by*

$$r(x) = \gamma\int_x^\infty e^{-s_0(x-y)}\frac{100^2}{(100+y)^2}dy$$

We can easily compute $r(x)$ numerically. In our application, we truncated the upper limit of integration in such a way that the value of the discarded part of the integral was at most 10^{-20}. We observe in passing that in this example the dominant term in $R(x)$ as $x \to \infty$ is $Q(x)$ so that the behaviour of the ladder height density as $x \to \infty$ is essentially the same under the Erlang(2,2) claims arrival model as under the classical risk model.
Table 2 shows some bounds and approximations to $\psi(u)$.

u	*Lower bound*	*Approximation*	*Upper bound*
0	*0.88664*	*0.88664*	*0.88664*
5	*0.69965*	*0.69993*	*0.70021*
10	*0.60187*	*0.60218*	*0.60249*
15	*0.53082*	*0.53113*	*0.53144*
20	*0.47495*	*0.47525*	*0.47555*
25	*0.42923*	*0.42952*	*0.42981*

Table 2: Ruin probabilities for Pareto(2,1) claims

Figure 3 shows the same comparison as Figure 1. Once again we note that the probability of ruin is smaller in the case of Erlang(2,2) inter-arrival times. Figure 4 shows a comparison of the two ladder height distributions. Unlike in the previous example, the ladder height distribution when we have Erlang inter-arrival times is below the corresponding distribution for the classical risk model, at least for values we have calculated. Unlike in the previous example, it is not possible to verify this analytically.

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