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Figure 28: Each point in the shaded area carries a continuum of lines such that the associated affine standard cut produces C^1-curves without line segments.

Hence for each contact point (line) one can find a continuum of contact lines (points) such that either the associated projective or affine standard cutting procedure produces a C^1-curve without line segments.

Remark 13.2 *Dual to Remark 10.2 one has that the affine standard cut produces parabolas if it lies on the parabola through* $\mathbf{a}$ *and* $\mathbf{b}$ *with tangents* $\mathbf{ac}$ *and* $\mathbf{bc}$.

Addendum After writing the paper we became aware of unpublished notes by Bernd Mulansky with similar ideas as in the proof of Theorem 6.1.

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Since A_0 and A_1 map lines onto lines, they also describe maps on $\P^*$. These are the inverse dual maps $B_0 = (A_0^*)^{-1}$ and $B_1 = (A_1^*)^{-1}$ which map the ideal line onto itself and the lines ($\hat{=}$ points of $\P^*$) **ca**, **ab**, **bc** onto **da**, **ag**, **de** and **de**, **gb**, **be**, respectively. Thus the envelope of the standard curve is obtained by successive applications of B_0 and B_1 to **ca** and **bc**. In other words, the polar scheme is given by the standard triangle **ca**, **ab**, **bc** in $\P^*$, the contact line **g** and the contact point **de** while the ideal line of $\P$ assumes the role of **m**.

Next we use the coordinates of Section 12 and the polarity with respect to the circle around **m** through **a** and **b**. Then the ideal line and the lines **ca**, **ab**, **bc** are mapped onto the points **m**, **a**, **c**, **b**. Further, the points on the cubic given in Remark 12.1 correspond to the tangents of the curve

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \frac{1}{(t^2 - t + 1)^2} \begin{bmatrix} (1-t)^3(1+t) \\ t^3(2-t) \end{bmatrix} ,$$

while the points in the interior of the shaded area shown in Figure 24 correspond to cuts which do not intersect this curve. This is illustrated in Figure 27: If the affine standard cut intersects the standard triangle only in the shaded area, then there is a continuum of contact points on the cut such that the associated standard curve has no corners and line segments.

Figure 27: Each line intersecting the standard triangle only in the shaded area carries a continuum of contact points such that the associated affine standard cut produces C^1-curves without line segments.

Next consider the parabola of Figure 25. Its envelope corresponds to the ellipse through **m**, **a**, **b** with tangents **ac** and **cb**. It is shown in Figure 28 and in analogy to above we obtain the following: If the contact point lies in the shaded area of Figure 28 there is a continuum of contact lines through it such that affine applications of this standard cut produces no corners and line segments. Figure 28 also shows the cubic of Figure 25.

Remark 13.1 *The union of the shaded areas in Figures 24 and 27 as well as the union of the shaded areas in Figures 25 and 28 fills the standard triangle.*

Figure 25: Each line meeting the shaded area but not the line segment **ab** carries a continuum of contact points such that the associated standard ccut produces C^1-curves without line segments.

13 Affine standard cuts

The interpolation scheme in Section 10 could also be based on a standard cut which is mapped affinely onto the corners of $\mathcal{P}_0$. Then the extra points $\mathbf{m}_i$ and $\mathbf{m}$ are not needed and one can apply the Gregory-Qu result. However, such a scheme applied to the standard triangle falls under the Gregory-Qu condition, see Figure 3, only if the standard cut with the notation defined by Figure 26 is such that $0 < \alpha = \beta$ and

$$\xi + 2\eta < 1 \quad \text{and} \quad 2\xi + \eta < 1 \ , \quad \text{where} \quad \xi = \frac{\beta a}{a+b} \ , \quad \eta = \frac{\alpha b}{a+b} \ ,$$

i.e. the standard cut must be parallel to the line **ab** while the contact point can only lie in the interior of the dotted area shown in Figure 26, right.

Figure 26: Standard cut with notation (left) and the set of all feasible contact elements based on the Gregory-Qu condition (right).

However, as we will show in the sequel there are many more affine standard cuts producing C^1-curves. Hence the Gregory-Qu condition is too restrictive.

Consider the scheme given by the algorithm in Section10, where A_0 and A_1 are affine maps mapping **a**, **c**, **b** onto **a**, **d**, **g** and **g**, **e**, **b** respectively.

Thus for any given contact point (x, y) there exists a continuum of slopes $-\beta/\alpha$ such that the standard cut produces curves without line segments if and only if

$$x^3 + x^2y + xy^2 + y^3 - 3x^2 - 3xy - 3y^2 + 3x + 3y - 1 \geq 0 \ .$$

The region of all contact points satisfying this cubic inequality is shaded in Figure 24(right).

Figure 24: Each point in the shaded area carries a continuum of lines such that the associated standard cut produces C^1-curves without line segments.

Remark 12.1 *For later use we observe that the cubic boundary has the parametrization*

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2\alpha^2 - 2\alpha + 1} \begin{bmatrix} \alpha^3 \\ (1-\alpha)^3 \end{bmatrix} \ .$$

In order to get the region of all contact lines having a contact point such that the standard cut does not produce line segments, we transform the first two inequalities above into

$$\frac{(1-\beta)^2}{\alpha + \beta^2 - \alpha\beta} \leq \frac{x}{\alpha} \leq 1 - \frac{(1-\alpha)^2}{\alpha^2 + \beta - \alpha\beta} \ .$$

Note that these inequalities imply $0 \leq x/\alpha \leq 1$. Hence we drop the term x/α and transform the remaining inequality into $\alpha + \beta \geq 1$. Thus whenever $\alpha + \beta \geq 1$, one can find a continuum of contact points on the corresponding contact line such that the standard cut produces no line segments. The lines whose intercepts α and β sum to 1 envelope the parabola

$$x^2 + y^2 - 2xy - 2x - 2y + 1 = 0 \ .$$

This parabola forms the boundary of the shaded region shown in Figure 25. It means that exactly all contact lines which touch or go through the shaded area of Figure 25 but not through the line segment **ab** have a contact point **c** leading to a standard curve without line segments.

Remark 12.2 *The parabola above lies in the shaded area of Figure 24.*

Figure 23: Fixed points in the standard triangle.

12 Computing the fixed points

In order to compute the fixed points of A_0 and A_1 as given by the standard triangle shown in Figure 20 one may choose any suitable coordinate system: Let **c**, **a**, **b** and **m** be represented by the homogeneous coordinates

$$\begin{bmatrix}1\\0\\0\end{bmatrix}, \quad \begin{bmatrix}1\\1\\0\end{bmatrix}, \quad \begin{bmatrix}1\\0\\1\end{bmatrix}, \quad \text{and} \begin{bmatrix}1\\1\\1\end{bmatrix},$$

then the endpoints **d** and **e** of the cutting edge and the contact point **g** are represented by

$$\begin{bmatrix}1\\\alpha\\0\end{bmatrix}, \quad \begin{bmatrix}1\\0\\\beta\end{bmatrix}, \quad \text{and} \begin{bmatrix}1\\x\\y\end{bmatrix}$$

where $\alpha, \beta, x, y \in (0,1)$ and $x/\alpha + y/\beta = 1$, and the projective maps A_0 and A_1 have the matrix representations

$$A_0 \mathrel{\widehat{=}} \begin{bmatrix}1&1&1\\\alpha&1&x\\0&0&y\end{bmatrix} \begin{bmatrix}1-x&&\\&y-x-\alpha y+\alpha&\\&&1-\alpha\end{bmatrix} \begin{bmatrix}1&1&1\\0&1&0\\0&0&1\end{bmatrix}^{-1},$$

$$A_1 \mathrel{\widehat{=}} \begin{bmatrix}1&1&1\\0&0&x\\\beta&1&y\end{bmatrix} \begin{bmatrix}1-y&&\\&x-y-\beta x+\beta&\\&&1-\beta\end{bmatrix} \begin{bmatrix}1&1&1\\0&0&1\\0&1&0\end{bmatrix}^{-1}.$$

With the aid of a computer algebra system one can now easily show that A_0 and A_1 have no fixed points in the standard triangle except for **a** and **b** if and only if

$$\frac{1-y}{x} \cdot \frac{z}{1+z} \leq \frac{\beta}{\alpha} \leq \frac{y}{1-x} \cdot \frac{1+z}{z}, \quad \text{where } z = 1-x-y .$$

Figure 22: The standard left right corner and some projective image of it

respect to **a** and **b** and let x', y', z' be the affine scales of the projections of $\mathbf{a}(x + 2^{-1-n})$, $\mathbf{c}'$ and $\mathbf{a}(x + 2^{-n})$ with respect to the projections of $\mathbf{a}(x)$ and $\mathbf{a}(x + 2^{1-n})$. Since the $x_1 \ldots x_{n-1}$-corner is a projective image of the 01-corner under the map $A_{x_1} \ldots A_{x_{n-1}}$, one has

$$\begin{aligned} \xi &:= \mathrm{cr}[x, y|1, 0] = \mathrm{cr}[x', y'|1, 0] = \frac{x'-1}{x'}\frac{y'}{y'-1} \ , \\ \zeta &:= \mathrm{cr}[z, y|1, 0] = \mathrm{cr}[z', y'|1, 0] \ , \end{aligned}$$

where $\mathrm{cr}[a, b|c, d]$ denotes the cross ratio of a, b with respect to c, d. These two identities imply for $y' \geq y$ (or $y' \leq y$)

$$x' = \frac{y'}{y'(1-\xi)+\xi} \geq (\leq) \frac{y}{y(1-\xi)+\xi} = x$$

and analogously $z' \geq$ (or $\leq) z$. Hence $z' - x' \leq \max\{1 - x, z - 0\}$.

Therefore the central projection of any $x_1 \ldots x_n$01-corner onto the line **ab** is shorter than the projection of the $x_1 \ldots x_n$-corner by some uniform factor $q < 1$. This means that the end points of the corner sequence above converge to the same point. ■

Further assume that A_0 has a fixed point **f** besides **a** in the standard triangle illustrated in Figure 23. Then the line **mf** is fixed under A_0 but not pointwise since the intersection **d** with the line **ab** is mapped onto a different point **e**. Hence A_0 restricted onto **mf** can have at most two fixed points, namely **m** and the intersection with the fixed line **ac**. Thus if A_0 has a fixed point besides **a** in the standard triangle, it lies between on the edge **ac**. The analogous result holds for A_1.

11 Standard curves without line segments

Theorem 6.4 guarantees that the standard curve is C^1 but not necessarily the absence of line segments. In the following, we will investigate when a standard cut leads to a curve with or without line segments. A first result is the following theorem.

Theorem 11.1 *The standard curve $\mathcal{S}$ has no line segments if and only if the projective maps A_0 and A_1 as given by the standard cut, see Section 10, have no fixed points in the standard triangle besides* $\mathbf{a}$ *and* $\mathbf{b}$*, respectively.*

Proof

Assume that the map A_0 has a fixed point $\mathbf{f} \neq \mathbf{a}$ in the standard triangle. Then each $0 \ldots 0$-corner contains $\mathbf{a}$ and $\mathbf{f}$. As already mentioned in Section 10 the standard curve has no corners because of Theorem 6.4. Hence it follows from Corollary 5.3 that the $0 \ldots 0$-corners become arbitrarily flat after sufficiently many iterations. Thus $\mathcal{S}$ contains the line segment $\mathbf{af}$. Similarly the standard curve has a line segment ending at $\mathbf{b}$ if A_1 has a second fixed point besides $\mathbf{b}$ in the standard triangle.

In order to prove the converse we observe first that the construction of the standard curve implies, since the cuts are local, that no three contact points – denoted by $\mathbf{a}(\sum x_i 2^{-i})$ in Section 10 – are collinear. Therefore if we can show that the set of all contact points is dense in $\mathcal{S}$, then $\mathcal{S}$ has no line segment.

Now assume that A_0 and A_1 possess no fixed points besides $\mathbf{a}$ or $\mathbf{b}$, in the standard triangle, respectively, and let $\mathbf{q}$ be an arbitrary point on $\mathcal{S}$. We need to show that $\mathbf{q}$ is a limit point of the contact points on the left and also of the contact points on the right side of $\mathbf{q}$:

(1) Let $\mathbf{q} = \mathbf{a}$, then the $0 \ldots 0$-corners form a strictly monotone sequence of triangles $\mathbf{a}, A_0^n\mathbf{c}, A_0^n\mathbf{b}$. Hence the sequence $A_0^n\mathbf{b}$ cannot have two accumulation points and converges to a point $\mathbf{f}$ in the standard triangle. Since $\mathbf{f}$ must be a fixed point of A_0, it follows that $\mathbf{a} = \mathbf{f}$. (2) One can proceed similarly if $\mathbf{q} = \mathbf{b}$. (3) Let $\mathbf{q} = \mathbf{a}(\sum_{i=1}^m x_i 2^{-i}) \neq \mathbf{a}$ be a contact point of $\mathcal{S}$ where $x_m = 1$. Then the right endpoints of the $x_1 \ldots x_m 0 \ldots 0$-corners and the left endpoints of the $x_1 \ldots x_{m-1} 01 \ldots 1$-corners converge to $\mathbf{q}$:

$$\lim_{n\to\infty} A_{x_1} \ldots A_{x_m} A_0^n \mathbf{b} = A_{x_1} \ldots A_{x_m} \mathbf{a} = \mathbf{q} \ ,$$
$$\lim_{n\to\infty} A_{x_1} \ldots A_{x_{m-1}} A_0 A_1^n \mathbf{a} = A_{x_1} \ldots A_{x_{m-1}} A_0 \mathbf{b} = A_{x_1} \ldots A_{x_{m-1}} A_1 \mathbf{a} = \mathbf{q} \ .$$

(4) In all other cases there is a sequence x_n containing infinitely many pairs $x_n x_{n+1} = 01$ such that the $x_1 \ldots x_n$-corners contain $\mathbf{q}$.

Figure 22 shows on the right side in bold the left right corner of some $x_1 \ldots x_{n-1}$-corner $\mathbf{a}(x), \mathbf{c}', \mathbf{a}(x + 2^{1-n})$ where $x = \sum_{i=1}^{n-1} x_i 2^{-i}$. It is the projective image of the 01-corner also shown in bold on the left side.

Consider the central projection with $\mathbf{m}$ as center onto the line $\mathbf{ab}$. Let x, y, z be the affine scales of the projections of $\mathbf{a}(1/4)$, $\mathbf{c}$ and $\mathbf{a}(1/2)$ with

Then, from what is said above, we can derive the following algorithm to compute all corner end points after some n iterations.

Algorithm

Given: Standard corner as in Figure 20, number of iterations n

1. $\mathbf{a}(0) := \mathbf{a}, \quad \mathbf{a}(1) := \mathbf{b}$

2. For $l = 1, 2, \ldots, n$

For $i = 1, 3, 5, \ldots, 2^l - 1$

$$\mathbf{a}(\frac{i}{2^l}) := \begin{cases} A_0 \mathbf{a}(\frac{2i}{2^l}) & \text{if } 2i \leq 2^l \\ A_1 \mathbf{a}(\frac{2i}{2^l} - 1) & \text{if } 2i > 2^l \end{cases}$$

Figures 8 and 21 were obtained by this algorithm and subsequent projective maps. The figures show the starting sequence of contact elements and the standard cut which defines the fill-in scheme used to generate the interpolating curve which is also shown. Further we used $\mathbf{m}_i = \mathbf{a}_i - \mathbf{a}_{i+1} + \mathbf{a}_{i+2}$, for $i = 0, 2, 4, \ldots, m-2$. These points are suppressed in the figures. In Section 12 we present an explicit matrix representation for the maps A_0 and A_1.

Remark 10.1 *The algorithm above is formally identically to the matrix subdivision algorithm introduced and studied in [Micchelli & Prautzsch '87, Prautzsch '91]. However, here A_0, A_1 are not affine, but projective maps. Hence, the analysis there does not apply to our scheme here.*

Figure 21: Interpolant produced by the algorithm.

Remark 10.2 *If the standard cut with its contact point is a contact element of the conic through the points* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{m}$ *with the tangents* $\mathbf{ac}$ *and* $\mathbf{bc}$, *then the projective maps $\mathcal{A}_0$ and $\mathcal{A}_1$ map this conic onto itself since any conic is determined by three points and two tangents. Thus the cutting scheme above generalizes Haase's alias de Casteljau's rational algorithm for conics, see e.g.* [Boehm & Prautzsch '94, p 243].

vertex **c**, and the two contact points **a**, **b** of the standard corner and their images. Hence for each corner of $\mathcal{P}_n$ formed by two successive contact points and the vertex between one needs to specify the image of **m**. We choose for all corners of $\mathcal{P}_n$ between $\mathbf{a}_i$ and $\mathbf{a}_{i+1}$ the same image for **m** and denote it by $\mathbf{m}_i$. Since cross ratios are invariant under projective maps, the fill-in scheme above satisfies the conditions of Theorem 6.4 provided that **m** does not meet a supporting line of the polygon **ab** with end tangents **ac** and **bc** and similarly $\mathbf{m}_i$ does not meet a supporting line $\mathbf{a}_i\mathbf{a}_{i+1}$ with end tangents $\mathbf{a}_i\mathbf{c}_i$ and $\mathbf{a}_{i+1}\mathbf{c}_i$. Hence the limiting curve is a C^1-interpolant.

Figure 20: Applying a standard cut.

If one applies the fill-in scheme above to the standard corner itself, one obtains the **standard curve**. Mapping the standard curve to each corner of $\mathcal{P}_0$ gives the same result as if $\mathcal{P}_0$ is cut iteratively by the standard cut since every corner $\mathbf{a}_i\mathbf{c}_i\mathbf{a}_{i+1}\mathbf{m}_i$ is a projective image of the standard corner. Another option is to map the standard curve onto each corner of $\mathcal{P}_0$ by an affine map which would yield a different curve but eliminate the need for the extra points $\mathbf{m}_i$. This is discussed in Section 13.

In the following we describe a very simple algorithm to produce the standard curve: Applying the standard cut to the standard corner itself generates a left and a right corner. Further applications of the standard cut produce a left left, a left right, etc., corner. We will label these corners by sequences $x_1 \ldots x_n$ of binary digits $x_i \in \{0, 1\}$ where 0 stands for "left" and 1 for "right".

Let A_0 and A_1 be the **projective maps** which map the standard corner onto the left and right corner, respectively, i.e. A_0 and A_1 map $\mathbf{a}, \mathbf{c}, \mathbf{b}, \mathbf{m}$ onto $\mathbf{a}, \mathbf{d}, \mathbf{g}, \mathbf{m}$ and $\mathbf{g}, \mathbf{e}, \mathbf{b}, \mathbf{m}$ respectively. Since the cutting scheme under consideration is projectively invariant, the standard curve restricted to the left or right corner is a projective image of the entire curve under A_0 and A_1, respectively. Hence $A_{x_1} \ldots A_{x_n}$ maps the standard corner onto the $x_1 \ldots x_n$-corner. Note, e.g., that $A_0 A_1$ maps the standard corner via the right corner onto the left right corner.

Let $\mathbf{a}(x)$ be the straightforward 1-1 correspondence between the dyadics $x = \sum_{i=1}^n x_i 2^{-i}$ in $[0, 1]$ and the contact points on the standard curve, i.e. let $\mathbf{a}(1) = \mathbf{b}$ and $\mathbf{a}(\sum_{i=1}^n x_i 2^{-i})$ be the left end point of the $x_1 \ldots x_n$-corner.

by construction in $(0, 1)$. Thus we need to show that $g/h \geq$ constant > 0 for $(\alpha, \beta) \to (0, 0)$. With the law of sine it follows

$$\begin{aligned} \frac{g}{h} &= \frac{\sin \sigma\alpha \cdot \sin \tau\beta}{\sin(180 - \sigma\alpha - \tau\beta)} \frac{\sin(180 - \alpha - \beta)}{\sin \alpha \cdot \sin \beta} \\ &= \frac{\sin \sigma\alpha \cdot \sin \tau\beta}{\sin \alpha \cdot \sin \beta} \frac{\sin(\alpha + \beta)}{\sin(\sigma\alpha + \tau\beta)} . \end{aligned}$$

Hence if $(\alpha, \beta) \to (0, 0)$ the inferior and superior limits of the ratio g/h are (positive) values between σ and τ. This concludes the proof.

10 A convexity preserving interpolation scheme

As an application of Theorem 6.4 we describe a convexity preserving interpolation scheme where the shape of the curve can be modified by a very direct shape handle. The input for the scheme is a sequence of contact elements in the affine plane, i.e. line segments with one distinguished point $\mathbf{a}_i$ on each segment, $i = 0, \ldots, m$, as illustrated in Figure 19. Let $\mathbf{c}_i$ be the point where the lines through $\mathbf{a}_i$ and $\mathbf{a}_{i+1}$ intersect. Then $\mathbf{a}_0\mathbf{c}_0\mathbf{c}_1 \ldots \mathbf{c}_{m-1}\mathbf{a}_m$ and $\mathbf{a}_0\mathbf{a}_1 \ldots \mathbf{a}_m$ are two polygons $\mathcal{P}_0$ and $\mathcal{Q}_0$ as needed for a fill-in scheme where $\mathcal{P}_0$ circumscribes $\mathcal{Q}_0$.

Figure 19: Polygon to be interpolated.

Note that the tangent at $\mathbf{a}_i$ has to be chosen such that it intersects the tangents at $\mathbf{a}_{i-1}$ and $\mathbf{a}_{i+1}$ on opposite sides of $\mathbf{a}_i$. Thus if the tangents are not given properly, one must explicitly introduce inflection points with tangents —⊙— as further contact elements, see Figure 19.

The fill-in scheme is specified by a standard cut in a standard corner **acb** as illustrated in Figure 20. A user can choose the cutting line **de** and the contact point **g** arbitrarily inside the standard triangle. The standard cut is then mapped by projectivities onto each corner of $\mathcal{P}_0$ and further iterates $\mathcal{P}_n$. The images of the standard cut determine all cuts on every $\mathcal{P}_n$. The underlying projective maps are determined by the four points **m**, the corner

Here $\mathbf{m}$ is the ideal point of the vertical axis. Thus if one views the ideal point of the vertical axis as the ideal line of $\P^*$, the projective scales $\xi = \xi(\mathcal{G})$ and $\eta = \eta(\mathcal{S})$ become affine scales on the line $\mathbf{q}$ of $\P^*$ with respect to the affine systems $\mathcal{F}, \mathcal{R} - \mathcal{F}$ and $\mathcal{R}, \mathcal{F} - \mathcal{R}$. Hence the envelopes $\mathcal{Q}_n^\sharp$ determine a Gregory-Qu scheme and are covered by Theorem 7.1 (compare Figures 14 and 17).

In particular, one may run the scheme by Le Méhauté and Utreras with $\omega = 1/2$ and $\xi = \eta = 1/4$ fixed for the entire algorithm. Then one still obtains a C^1-interpolant. This particular scheme which is excluded by Le Méhauté Utreras and was proposed in [Carnicer '92]. The polar scheme is Chaikin's construction of piecewise quadratic splines [Chaikin '74]. Hence it follows from Remark 10.2 and 13.2 that Carnicer's interpolant consists of segments of hyperbolas and parabolas.

9 Choosing a good contact line

Using the terminology of Theorem 6.7 a fill-in scheme in a Euclidean plane is completely defined by a sequence of numbers σ and τ and the choices of the tangents. If the σ's and τ's satisfy the conditions of Theorem 6.7, the limiting curve $\mathcal{P}_\infty$ has no corners but it can have line segments. We show that for any sequence of σ's and τ's satisfying the conditions in Theorem 6.7 one can find tangents such that the cuts also satisfy the conditions of Theorem 6.1.

Figure 18 shows the triangle of Figures 9 and 11 again with further notation. It is the corner of some $\mathcal{P}_n$. We will see that there is some $\delta > 0$ such that for every $\mathcal{P}_n$ and any of its corners $\delta \leq g/h \leq 1 - \delta$ where g and h are given by Figure 18. Thus if the tangent at the new contact point $\odot$ is chosen to be parallel to the base of the triangle, the condition of Theorem 6.1 is satisfied. Other, although less simple choices, are also possible.

Figure 18: Choosing a tangent.

The ratio g/h depends continuously on α, β, σ and τ. Since corner cutting does not create corners which are sharper than the ones being cut, we have $180 - \alpha - \beta \geq \varphi > 0$ where φ is the interior angle defined by the two edges of $\mathcal{P}_0$ shown in Figure 10. Hence we consider g/h only over the compact domain $\varepsilon \leq \sigma, \tau \leq 1 - \varepsilon$, $0 \leq \alpha, \beta \leq 180 - \varphi$. For $\alpha, \beta > 0$ the ratio g/h is

Figure 15: Construction by Le Méhauté and Utreras.

This scheme is polar to a Gregory-Qu scheme in ¶*: Consider two consecutive edges $\mathcal{F}$ and $\mathcal{R}$ of Q_n and two consecutive edges $\mathcal{G}$ and $\mathcal{S}$ of Q_{n+1} with a common point $\mathbf{q}$ as illustrated in Figure 16. Let $\xi : 1 - \xi - \eta : \eta$ denote the ratios in which the slopes of $\mathcal{G}$ and $\mathcal{S}$ divide the slopes of $\mathcal{R}$ and $\mathcal{F}$. Then, obviously $\xi + \eta = 1 - \omega \in (0, 1/2)$ which means that (ξ, η) lies in the region shaded in Figure 17.

Figure 16: New formulation of the construction by Le Méhauté and Utreras.

Figure 17: The Le Méhauté-Utreras region.

Figures 15 and 16 show a special case of the situation shown in Figure 9.

greater than 1/3 produces a curve with corners, see [Gregory & Qu '96], but without line segments since Theorem 6.1 applies. Thus, the polar scheme satisfies the premises of Theorems 6.4 and 6.7 but not the Gregory-Qu condition in ¶*. Moreover one has:

Theorem 7.1 *Any local corner cutting scheme in an affine plane where all corners are cut simultaneously by local cuts produces a limiting curve without line segments if the accumulation points of the* (ξ, η) *lie in the closure of the triangle shaded in Figure 14 without its three vertices.*

Remark 7.2 *In particular, if* $\underline{\xi}$ *and* $\underline{\eta} > 0$ *and* $\overline{\xi} + \overline{\eta} < 1$*, then the conditions of Theorem 7.1 are satisfied.*

Proof

We introduce any scalar product in the affine plane, consider all pairs of two consecutive edges of $\mathcal{P}_n$ and denote the maximum length of these pairs by m_n. Then proceeding similarly as in the proof of Theorem 6.1 one shows that there is some $q < 1$ such that $m_{n+2} < q \cdot m_n$ for all n. Hence the theorem follows from Lemma 5.1. ■

Figure 14: The region of Theorem 7.1.

8 The interpolation schemes by Carnicer and Le Méhauté & Utreras

Very recently Le Méhauté and Utreras [1994] presented an interpolatory refinement scheme which produces a sequence of polygons Q_n in a Euclidean plane converging to a C^1-interpolant of Q_0. Le Méhauté and Utreras assume that the Q_n are graphs of piecewise linear functions. In order to construct Q_{n+1} from Q_n they choose at each vertex of Q_n a supporting line. Then they determine a new contact point $\odot$ between each pair of consecutive vertices $\bullet$ indirectly by the ratio $\omega : 1 - \omega$ as illustrated in Figure 15. For the entire scheme ω is a fixed number in $(1/2, 1)$, while the supporting lines at any fixed vertex can be different for all Q_n with this vertex.

$\mathbf{p}$, $\mathbf{q}$ and the vertex $\mathbf{r}$ between them. This triangle with a next cut $\mathbf{st}$ is illustrated in Figure 13.

Figure 13: Cutting a corner of $\mathcal{P}_n$.

Corner cutting under the Gregory-Qu condition means that for each edge one has

$$\text{contact point} = (1-c)\cdot\text{left end point} + c\cdot\text{right end point}$$

where $c > \varepsilon := \min\{\underline{\xi}, \underline{\eta}\}$. Hence the ratio $k : l$ of the distance k of $\mathbf{p}$ to $\mathbf{r}$ and the distance l of $\mathbf{q}$ to $\mathbf{r}$ lies between ε/R and R/ε. Furthermore, let s and t be such that $\mathbf{s} = (1-s)\mathbf{p} + s\mathbf{r}$ and $\mathbf{t} = (1-t)\mathbf{q} + t\mathbf{r}$. Then the Gregory-Qu conditions imply

$$\begin{array}{lll} s:(1-s) > & \varepsilon\cdot\lambda/3 : (1-\varepsilon)\lambda & > \varepsilon/3 : 1-\varepsilon/3 \\ s:(1-s) < & (1-\varepsilon)(1-2\varepsilon)\lambda : \varepsilon\lambda & < 1-\varepsilon : \varepsilon \end{array}$$

and analogous inequalitites for t. Thus s and t lie between $\varepsilon/3$ and $1-\varepsilon$. Now we can write the new contact point $\mathbf{c}$ on $\mathbf{st}$ as

$$\mathbf{c} = (1-c)\mathbf{s} + c\mathbf{t} = p\mathbf{p} + (1-p)((1-q)\mathbf{r} + q\mathbf{q}) \ ,$$

where $(1-q) : q = (s - cs + ct) : (c - ct) =: x$, which shows that $q = 1/(1+x)$ lies in a closed interval of $(0,1)$ depending only on ε. Further one has

$$\frac{\sin\sigma\alpha}{\sin\alpha} = \frac{\sin c\alpha}{\sin\beta}\cdot\frac{\sin\beta}{\sin\alpha} = \frac{(1-q)l}{f}\cdot\frac{k}{l} \ ,$$

cf. Figure 13. Since corner cutting under the Gregory-Qu condition produces C^1-curves, we can, without loss of generality, assume $\gamma \geq 90°$, see Corollary 5.3. Then we have $f < k + ql \leq k(1 + qR/\varepsilon)$. Thus $\sin\sigma\alpha/\sin\alpha$ is bounded away from 0 and 1 for all possible q. Therefore, σ must lie in some compact subinterval of $(0,1)$ and the result of Gregory and Qu follows from Theorem 6.7.

The converse, however, is not true since the Gregory-Qu condition restricts the cut, i.e. the choice of the new contact line $\mathbf{st}$, while there is no such restriction in Theorems 6.4 and 6.1. Furthermore, a cutting scheme where all corners are cut simultaneously and where all ξ's and all η's are equal and

Figure 11: Cutting a corner of $\mathcal{P}_n$.

Figure 12: The fill-in region of Dyn et al.

A special case of Theorem 6.7 was also studied in [Hejna '88]. There the lines $\mathcal{S}$, $\mathcal{T}$ are given by the bisectors of $\mathcal{P}$ and $\mathcal{R}$, and $\mathcal{Q}$ and $\mathcal{R}$, respectively. The new contact line is given by the bisector of the lines $\mathcal{S}$ and $\mathcal{T}$ as just described.

7 The Gregory-Qu result

The Gregory-Qu corner cutting scheme also satisfies the conditions of Theorems 6.4 and 6.7. In order to show this, we need a property first derived by Gregory and Qu. Let $\mathcal{P}_n$ be a sequence of polygons in the Euclidean plane. If, for all n, $\mathcal{P}_{n+1}$ arises from $\mathcal{P}_n$ by cutting all corners simultaneously such that $(\overline{\xi}, \overline{\eta})$ and $(\underline{\xi}, \underline{\eta})$ lie in the Gregory-Qu region shown in Figure 3, then there is a positive constant R not depending on n such that the lengths κ and λ of any two successive edges of any $\mathcal{P}_n$ satisfy $1/R \leq \kappa/\lambda \leq R$.

Now consider a triangle on $\mathcal{P}_n$ formed by two consecutive contact points

Remark 6.5 *The new contact line* **st** *in Figure 9 can be any line which does not intersect the secant* **pq**.

Remark 6.6 *The limiting curve $\mathcal{P}_\infty$ is tangent to $\mathcal{P}_0$ at the endpoints and has no corners there.*

As described before Corollary 5.3 we can take Theorem 6.1 to the elliptic plane: Theorem 6.1 constrains the ratios σ and τ, i.e. we have conditions of the form $a : b = \sigma \geq \varepsilon$ for certain lengths a and b. Now let α and β be the corresponding lengths on the sphere. Then there are, as we already observed, two positive constants k and K independent of a and b such that $\alpha \geq ka$ and $\beta \leq Kb$. Hence we obtain $\alpha : \beta \geq ka : Kb \geq \varepsilon k/K$, i.e. if we replace Euclidean by spherical distances, Theorem 6.1 is also valid on the elliptic plane.

Again we can use the polarity which maps points onto lines with the same coordinates. Then the conditions of Theorem 6.1 constrain ratios of angles for a fill-in scheme in the dual plane. As before Theorem 6.4 we express these conditions in terms of the polar-fill-in scheme in the primal (point) plane and bring them back to a Euclidean plane in the same way as we took the conditions for ratios of lengths to the elliptic plane. Then, finally, Theorem 6.1 assumes the following form:

Theorem 6.7 *Consider a sequence of polygon pairs $\mathcal{P}_n$, $\mathcal{Q}_n$ obtained by a fill-in scheme from* **acb**. *Assume that $\mathcal{Q}_0$ is a polygon in a Euclidean plane and let* **m** *be a point which does not meet any supporting line of $\mathcal{Q}_0$. Further suppose that*

- *all corners of every $\mathcal{P}_n$ are cut away eventually*
- *there is some $\delta > 0$ such that all corners of $\mathcal{Q}_{n+1}$ with end tangents $\mathcal{S}$ and $\mathcal{T}$ which replace an edge $\mathcal{R}$ of $\mathcal{Q}_n$ between two contact lines $\mathcal{P}$, $\mathcal{Q}$ satisfy $\sigma, \tau \in [\delta, 1-\delta]$ where $\sigma = angle\mathcal{RS} : angle\mathcal{RP}$ and $\tau = angle\mathcal{RT} : angle\mathcal{RQ}$ as illustrated in Figure 11.*

Then the limiting curve is C^1.

A weaker form of Theorem 6.7 was first proved in [Dyn et al. '91]. Figure 12 shows successive contact points • of $\mathcal{Q}_n$ connected by secants and a newly introduced contact point ⊙ which lies on $\mathcal{Q}_{n+1}$. The interpolatory refinement scheme of Dyn et al allows to build only sequences of polygons such that the interior angle γ is flatter than a certain average of φ and ψ. Their conditions imply that the maximum interior angle formed by the secants of each $\mathcal{Q}_{n+1}$ is flatter than some constant in $(0, 1)$ times the maximum interior angle of $\mathcal{Q}_n$.

This is not the case under the assumptions of Theorem 6.7. Consequently the fill-in region of the scheme by Dyn et al is smaller than the one presented here. Figure 12 shows their fill-in region, the shaded area, for this example.

which implies that the maximum distance between successive contact points becomes arbitrarily small for $n \to \infty$. ■

Figure 10: $\mathcal{P}_0$ and a corner of $\mathcal{P}_n$ with two further cuts.

Note that σ represents $\mathbf{s}$ with respect to the affine system $\mathbf{r}, \mathbf{p} - \mathbf{r}$. It is called the **affine scale** with respect to $\mathbf{r}$, $\mathbf{p}$. Moreover, $\mathbf{s}$ has the projective coordinates $1, \sigma$ with respect to the frame $\mathbf{r}$, $\mathbf{x}$; $\mathbf{p}$ where $\mathbf{x}$ is the ideal point of the line and $\mathbf{p}$ is chosen as the unit point. Thus σ is the **projective scale** with respect to $\mathbf{r}, \mathbf{x}; \mathbf{p}$ or equivalently the **cross ratio** of $\mathbf{s}, \mathbf{p}$ with respect to $\mathbf{r}, \mathbf{x}$, see e.g. [Boehm & Prautzsch '94, pp 210 and 227].

Next let the envelopes $\mathcal{Q}_n^\sharp$ and $\mathcal{P}_n^\sharp$ satisfy the premises of Theorem 6.1 where $\mathcal{Q}_n^\sharp$ circumscribes $\mathcal{P}_n^\sharp$. Then the limit of these envelopes has no line segments in $\P^*$ which implies that $\mathcal{P}_\infty = \mathcal{Q}_\infty = \lim \mathcal{Q}_n$ has no corners. Recall from Section 2 that $\mathcal{Q}_0^\sharp$ lies in an affine space of $\P^*$ where the ideal line is a point $\mathbf{m}$ of $\P$ which does not lie on any line of $\mathcal{Q}_0^\sharp$. Using Lemma 2.1, the conditions of Theorem 6.1 for the polar scheme can be expressed in terms of $\mathcal{P}_n$ and $\mathcal{Q}_n$. Then Theorem 6.1 takes on the following form:

Theorem 6.4 *Consider a sequence of polygon pairs $\mathcal{P}_n$, $\mathcal{Q}_n$ obtained by a fill-in scheme from* **acb**. *Assume that $\mathcal{Q}_0$ is a polygon in the projective plane and let* $\mathbf{m}$ *be a point which does not meet any supporting line of $\mathcal{Q}_0$. Further suppose that*

- *all corners of every $\mathcal{P}_n$ are cut away eventually*

- *there is some $\varepsilon > 0$ such that all corners of $\mathcal{Q}_{n+1}$ with end tangents $\mathcal{S}$ and $\mathcal{T}$ which replace an edge $\mathcal{R}$ of $\mathcal{Q}_n$ between two contact lines $\mathcal{P}$, $\mathcal{Q}$ satisfy $\sigma, \tau \in [\varepsilon, 1-\varepsilon]$ where σ and τ are the projective scales of $\mathcal{S}$ and $\mathcal{T}$ with respect to $\mathcal{R}$, $\mathcal{X}$; $\mathcal{P}$ or $\mathcal{R}$, $\mathcal{Y}$; $\mathcal{Q}$, respectively, and $\mathcal{X}$ and $\mathcal{Y}$ are the lines through* $\mathbf{m}$, *cf. Figure 9 for an illustration.*

Then the limiting curve $\mathcal{P}_\infty$ has no corners.

Figure 9: Cutting a corner of $\mathcal{P}_n$ and $\mathcal{Q}_n^\sharp$.

Remark 6.2 *The contact points can lie anywhere in the interior of the corresponding edge.*

Remark 6.3 *The limiting curve $\mathcal{P}_\infty$ interpolates $\mathcal{P}_0$ at the endpoints and has no line segments there.*

Proof
Without loss of generality, assume that **a**, **c**, **b** are the points $(-1,0),(0,1)$, and $(1,0)$ of the Euclidean plane $\mathbb{R}^2$, respectively, as illustrated in Figure 10. Because of Lemma 5.1 it suffices to show that the longest edge of the polygons $\mathcal{P}_n$ shrinks to zero. Note that for all n the slope of any edge of $\mathcal{P}_n$ lies between the slopes of the two edges of $\mathcal{P}_0$. Thus the length of any edge is bounded by the length of its orthogonal projection onto the line **ab** multiplied by $\sqrt{2}$.

Furthermore, if the projections of all contact points become dense in **ab** for $n \to \infty$, then the projections of the edges and thus the edges themselves become arbitrarily small. Hence we just need to consider two successive contact points of some $\mathcal{P}_n$. Figure 10 shows $\mathcal{P}_0$ and an arbitrary corner with two further cuts where $a, b, \ldots, f$ denote lengths of certain edges after their projections onto **ab**. Because of symmetry reasons we can restrict ourselves to the case $a \geq b$. Under this provision it is easy to see that

$$c > d \geq \varepsilon^2 a \geq \frac{\varepsilon^2}{2}(a+b)$$

and that

$$\begin{aligned} c &\leq e + (1-\varepsilon) f \\ &\leq e + (1-\varepsilon)(a-e+b) \\ &= \varepsilon e + (1-\varepsilon)(a+b) \\ &\leq \varepsilon(1-\varepsilon)a + (1-\varepsilon)(a+b) \\ &\leq (1-\varepsilon^2)(a+b) \ , \end{aligned}$$

Figure 7: Central projection from $\mathcal{E}$ onto the unit sphere.

Figure 8: C^1-interpolation on the sphere.

6 Main results

In this section we will apply Lemma 5.1 to derive simple fill-in strategies which produce curves without corners or line segments. A major result is the following.

Theorem 6.1 *Consider a sequence of polygon pairs $\mathcal{P}_n$, $\mathcal{Q}_n$ obtained by a fill-in scheme from* **acb**. *Assume that $\mathcal{P}_0$ is a polygon in an affine plane. Further suppose that*

- *all corners of every $\mathcal{P}_n$ are cut away eventually*
- *there is some $\varepsilon > 0$ such that all edges of $\mathcal{P}_{n+1}$ with endpoints* **s** *and* **t** *which cut a corner between two contact points* **p**, **q** *and a vertex* **r** *of $\mathcal{P}_n$ satisfy $\sigma, \tau \in [\varepsilon, 1-\varepsilon]$ where $\mathbf{s} = \mathbf{r} + \sigma(\mathbf{p} - \mathbf{r})$ and $\mathbf{t} = \mathbf{r} + \tau(\mathbf{q} - \mathbf{r})$, cf. Figure 9 for an illustration.*

Then the limiting curve $\mathcal{P}_\infty$ has no line segments.

For the converse assume that there are arbitrarily large indices n such that $\mathcal{P}_n$ has an edge which is longer than some positive ε. Then there is a sequence of long edges in the convex hull of $\mathcal{P}_0$ if the sequence is decreasing or in the convex hull of $\lim \mathcal{P}_n$ in case the sequence is increasing. Since under our definition the convex hull of a convex curve is compact, a subsequence of these long edges converges to a line segment of $\mathcal{P}_\infty$. ■

Remark 5.2 *If the maximum edge length of the* $\mathcal{P}_n$ *goes to zero, then the maximum edge length of the* $\mathcal{Q}_n$ *also goes to zero. Hence, in this case the polygons* $\mathcal{P}_n$ *and* $\mathcal{Q}_n$ *converge to the same curve.*

The pairs of antipodal points on the unit sphere in a Euclidean 3-space form a model of the projective plane. If one introduces angles between and distances on (geodesic) lines, one has the elliptic plane, see [Hilbert & Cohn-Vossen '32, pp 238-242]. Under the correlation (actually it is even a polarity) which relates points and lines with the same coordinates the angle of a corner equals the length of the dual line segment. Moreover, projecting a plane $\mathcal{E}$ which does not contain the origin onto the sphere with the origin as center of projection as illustrated in Figure 7 one has that every angle and length in $\mathcal{E}$ corresponds to an angle or length on the sphere, respectively. Comparing all angles or lengths in a compact subset of $\mathcal{E}$ with their images on the sphere, we obtain ratios which are bounded from above and below by positive constants. Hence we can take Lemma 5.1 to the elliptic plane, by polarization further to the dual plane, then by the polarity above to the elliptic point plane again, and finally back to a Euclidean plane. Then Lemma 5.1 assumes the following form:

Corollary 5.3 *[de Boor '90] Let* $\mathcal{P}_n$, $\mathcal{Q}_n$ *be a sequence of polygon pairs in a Euclidean plane obtained by a fill-in scheme from* **acb**. *If the sharpest corner of the polygons* $\mathcal{Q}_n$ *flattens out completely as* $n \to \infty$, *then* $\mathcal{Q}_\infty = \lim \mathcal{Q}_n$ *has no corners (and no endcorners). The converse holds for any monotone sequence of convex polygons* $\mathcal{Q}_n$ *converging to a convex curve without corners and end corners (Example 3.1 shows that end corners must not be disregarded).*

In particular, one should note that besides the above **all** results of this paper hold for spherical cutting schemes which produce a sequence of geodesic polygons on a sphere. For example the C^1-interpolant shown in Figure 8 was produced by the scheme presented in Section 10.

Remark 5.4 *The corners of the* $\mathcal{Q}_n$ *flatten out if and only if the corners of the* $\mathcal{P}_n$ *flatten out.*

and every new edge and restrict further cuts such that the already chosen contact points are not cut away. Then it follows that one can describe any local corner cutting scheme in this way.

The chosen contact points also form a polygon. Thus with the above extension a local corner cutting algorithm generates a sequence of polygon pairs $\mathcal{P}_n$, $\mathcal{Q}_n$ where $\mathcal{Q}_n$ connects the contact points on the edges of $\mathcal{P}_n$. We will call this extended procedure where we obtain a sequence of polygon pairs $\mathcal{P}_n$, $\mathcal{Q}_n$ as described above, a **fill-in scheme**, see Figure 6 for an illustration. Further we will say that $\mathcal{P}_n$ circumscribes $\mathcal{Q}_n$.

Let **a** and **b** be two contact points on successive edges of $\mathcal{P}_0$ as illustrated in Figure 6. In order to investigate the differentiability of the limiting curve between **a** and **b** one may truncate all $\mathcal{P}_n$ at **a** and **b** and assume that $\mathcal{P}_0$ consists of just the three vertices **a**, **c**, **b**, see Figure 6. Thus without loss of generality we will always assume in the sequel that

- the starting polygons $\mathcal{P}_0$ and $\mathcal{Q}_0$ for a fill-in scheme form a non-degenerate triangle **acb** and
- **ac** and **cb** determine the end tangents for all $\mathcal{P}_n$ and $\mathcal{Q}_n$.

Then the enveloping curves $\mathcal{Q}_n^\sharp$ and ${\mathcal{P}_n}^\sharp$ are again polygons which determine a fill-in scheme. We will call it the **polar fill-in-scheme**. Note that under polarization $\mathcal{P}_n$ and $\mathcal{Q}_n$ change their roles: $\mathcal{Q}_n^\sharp$ circumscribes ${\mathcal{P}_n}^\sharp$. However, note that under a correlation $\mathcal{P}_n$ and $\mathcal{Q}_n$ do not change their roles.

Remark 4.1 *In general $\mathcal{P}_n$ and $\mathcal{Q}_n$ converge to different curves. However, the conditions under which we study fill-in schemes in the sequel will always guarantee that the $\mathcal{P}_n$ and $\mathcal{Q}_n$ have the same limiting curve.*

5 The basic lemma

From Lemma 2.1, it follows that a fill-in scheme produces a C^1-curve if the polar scheme produces a curve without line segments and vice versa. The following simple lemma provides a sufficient and necessary condition.

Lemma 5.1 *Let $\mathcal{P}_n$, $\mathcal{Q}_n$ be a sequence of polygon pairs in a Euclidean plane obtained by a fill-in scheme from* **acb**. *If the maximum edge length of the polygons $\mathcal{P}_n$ goes to zero as $n \to \infty$, then $\mathcal{P}_\infty = \lim \mathcal{P}_n$ has no line segments. The converse holds for any monotone sequence of convex polygons $\mathcal{P}_n$ converging to a convex curve without line segments.*

Proof
Assume that $\mathcal{P}_\infty$ has a line segment $\mathcal{S}$. Under local corner cutting any three vertices of any $\mathcal{P}_n$ are not collinear and therefore for any n the line segment $\mathcal{S}$ lies in some triangle formed by two successive edges of $\mathcal{P}_n$. Hence not all edges become arbitrarily small.

monotonely decreasing, $\mathcal{H}_0 \supseteq \mathcal{H}_1 \supseteq \dots$, such that $\bigcap_{n=0}^{\infty} \mathcal{H}_n$ has a non-empty interior. Then we define $\lim \mathcal{C}_n$ as the boundary of $\bigcap_{n=0}^{\infty} \mathcal{H}_n$ without $\mathcal{D}$.

If the sets $\mathcal{H}_n$ are monotonely increasing such that $\bigcup_{n=0}^{\infty} \mathcal{H}_n$ is bounded, we define $\lim \mathcal{C}_n$ analogously as the boundary of $\bigcup_{n=0}^{\infty} \mathcal{H}_n$ without $\mathcal{D}$. Under polarization we obtain then monotonely decreasing sets, $\mathcal{H}_0^p \supseteq \mathcal{H}_1^p \supseteq \mathcal{H}_2^p \supseteq \dots$. From Section 2 it follows that $\cup \mathcal{H}_n$ is the polar set of $\cap \mathcal{H}_n^p$. Thus, Lemma 2.1 implies $(\lim \mathcal{C}_n)^p = \lim \mathcal{C}_n^p$. In both cases for in- and decreasing sequences $(\mathcal{H}_n)$ we define the end tangents of $\lim \mathcal{C}_n$ also by $\mathcal{D}$.

Example 3.1 *Even if all $\mathcal{C}_n$ do not start with a corner, the limiting curve may do so: Consider three affinely independent points* $\mathbf{a}$, $\mathbf{c}$, $\mathbf{b}$ *and the open polygons $\mathcal{C}_n$ with vertices* $\mathbf{a}, (\frac{n}{n+1}\mathbf{c} + \frac{1}{n+1}\mathbf{a}), (\frac{1}{n+1}\mathbf{c} + \frac{n}{n+1}\mathbf{b}), \mathbf{b}$ *without end corners. Then $\lim \mathcal{C}_n$ is the line segment* $\mathbf{ab}$ *with end tangents* $\mathbf{ac}$ *and* $\mathbf{cb}$ *as illustrated in Figure 5.*

Figure 5: An example.

Let the sets $\mathcal{H}_n$ from above be monotonely increasing.

4 Fill-in schemes

Let $\mathcal{P}_n$ be a sequence of polygons such that $\mathcal{P}_n$ is obtained from $\mathcal{P}_{n-1}$ by one or several successive local cuts. Then there is at least one point on each edge of every $\mathcal{P}_n$ which is never cut away. Here we will call such points **contact points**. They are marked by **solid dots** $\bullet$ in all the figures.

Figure 6: Contact points on $\mathcal{P}_0$ and $\mathcal{P}_1$ and the polygons $\mathcal{Q}_0$ and $\mathcal{Q}_1$.

A local corner cutting algorithm is completely described by a sequence of cuts. Here, however, we will **also** choose one contact point on each initial

Remark 2.2 *Consider a correlation (i.e. a projective map) between ¶ and ¶*. On viewing the image of the ideal line as the ideal line in ¶*, the correlation is an extended affine map. Hence the image of $\mathcal{C}$ under a correlation is again a convex curve. Moreover, the image of a corner or line segment is also a corner or line segment respectively.*

3 Open curves

So far we considered only closed curves. But it is easy to include also open curves: An open curve $\mathcal{C}$ in an affine plane A is called **convex** if there is a second curve $\mathcal{D}$ such that the union of $\mathcal{C}$ and $\mathcal{D}$ forms a closed convex curve.

Let $\mathcal{C}$ be an open convex curve and $\mathcal{C} \cup \mathcal{D}$ be a closed convex curve. Then the supporting lines of $\mathcal{C} \cup \mathcal{D}$ contacting $\mathcal{C}$ form an open convex curve $\mathcal{C}^\sharp$ in the dual plane. Let $\mathbf{a}$ and $\mathbf{b}$ be the end points of $\mathcal{C}$. If $\mathcal{C} \cup \mathcal{D}$ has a corner at $\mathbf{a}$ (or $\mathbf{b}$), then the envelope $\mathcal{C}^\sharp$ starts (or ends) with a line segment whose length depends on $\mathcal{D}$. We will say that $\mathcal{C}$ **starts or ends with a corner** if $\mathcal{C} \cup \mathcal{D}$ has a corner at $\mathbf{a}$ or $\mathbf{b}$.

Let $\mathcal{F}$ and $\mathcal{L}$ be the first and last point of $\mathcal{C}^\sharp$ in ¶*, then $\mathcal{C}^\sharp$ is completely determined by $\mathcal{C}$ and the lines $\mathcal{F}$ and $\mathcal{L}$ in ¶ (without need of $\mathcal{D}$). We will call the lines $\mathcal{F}$ and $\mathcal{L}$ the end tangents of $\mathcal{C}$. Moreover, $\mathcal{C}$ is the envelope of $\mathcal{C}^\sharp$ with end tangents $\mathbf{a}$ and $\mathbf{b}$ in ¶*. Figure 4 gives an illustration, where $\mathcal{C}$ with end tangents $\mathcal{F}$ and $\mathcal{L}$ has no end corners while $\mathcal{C}^\sharp$ with end tangents $\mathbf{a}$ and $\mathbf{b}$ does. The right side of Figure 4 is a dual image of the left side obtained by the polarity with respect to the circle $\mathcal{D}$, cf. e.g. [Brieskorn & Knörrer '86, pp 580-582] for a description of this construction.

Figure 4: An open convex curve and its envelope.

Further we recall, e.g. from [Schneider '93, Corollary 1.4.5], that a compact convex set is the convex hull of its boundary. We employ this relationship to define the **limit of a sequence of open convex curves** $\mathcal{C}_n$ with the same end points and end tangents: Assume that the end tangents are specified by some common arc $\mathcal{D}$ as described above where $\mathcal{D}$ is disjoint with all $\mathcal{C}_n$. Further let the convex hulls $\mathcal{H}_n$ of the convex curves $\mathcal{C}_n \cup \mathcal{D}$ be closed cone which is complementary to the cone $\mathbf{x}+\mathbb{R}(\mathbf{x}-\overset{\circ}{\mathcal{M}})$. If there is more than one supporting line at $\mathbf{x}$, we say that $\mathcal{M}$ and $\partial\mathcal{M}$ have a **corner** at $\mathbf{x}$.

Further, let ¶ be the projective extension of A. Then the **polar set** $\mathcal{M}^p$ is defined as the subset of the dual plane $¶^*$ formed by all lines in ¶ which do not intersect $\mathcal{M}$, see [Schneider '93, Section 1.6]. Since every point $\mathbf{p}\in A\backslash\mathcal{M}$ carries a line which does not meet $\mathcal{M}$ [cf. Schneider '93, Thm. 1.3.4], $\mathcal{M}$ consists of all points ($\widehat{=}$lines in $¶^*$) which do not meet $\mathcal{M}^p$. Hence $\mathcal{M}$ is the polar set of $\mathcal{M}^p$ [Schneider '93, Thm. 1.6.1] and $\mathcal{M}^p$ lies in an(y) affine plane (of $¶^*$ whose ideal line is a point of $\mathcal{M}$).

Note that the above **polarization reverses set operations**. Namely if $\mathcal{M}$ is a subset of another convex set $\mathcal{N}$ in A, then $\mathcal{M}^p$ contains $\mathcal{N}^p$ and if the union of some convex sets $\mathcal{M}_i$ in A with non-empty interior is convex, then this union is polar to the intersection of the polar sets $\mathcal{M}_i^p$, cf. also [Schneider '93, Theorem 1.6.2].

Next we observe that $\mathcal{M}^p$ is also convex. Namely every line in $¶^*$ (=pencil of lines in ¶), which intersects $\mathcal{M}^p$, does so in a connected line segment. This line segment forms a cone in $\mathcal{P}$ complementary to some cone $\mathbf{p}+\mathbb{R}(\mathcal{M}-\mathbf{p})$. Further, if $\mathcal{M}$ is bounded, $\mathcal{M}^p$ has a non-empty interior and is also bounded. This follows from the fact that the polar set of an open (closed) triangle of A is a closed (open) triangle in some affine plane of $¶^*$ and since a bounded set with non-empty interior is contained in a triangle and contains a triangle.

Let $\mathcal{M}$ be bounded. Then the closure of $\mathcal{M}$ can be obtained by intersecting all closed triangles (=intersection of three half spaces) containing $\mathcal{M}$, while the interior of $\mathcal{M}$ is formed by the union of all open triangles contained in $\mathcal{M}$. Hence the closure (interior) of $\mathcal{M}$ is the polar set of the interior (closure) of $\mathcal{M}^p$. This enables us to prove the following lemma.

Lemma 2.1 *Let $\mathcal{M}$ be a bounded convex set of the affine plane with non-empty interior. Then the supporting lines of $\mathcal{M}$ form the boundary of $\mathcal{M}^p$.*

Proof

A line $\mathcal{L}$ in ¶ supports $\mathcal{M}$ iff it meets the closure but not the interior of $\mathcal{M}$ or iff $\mathcal{L}$ as a point of $¶^*$ does not lie in the interior but in the closure of $\mathcal{M}^p$ which proves the lemma. ■

The lemma shows that the supporting lines of a convex curve $\mathcal{C}$ form a convex curve $\mathcal{C}^\sharp$ in the dual plane. It is the **envelope** of $\mathcal{C}$. Note that with the usual identification of ¶ and $¶^{**}$ we further obtain that $\mathcal{C}$ is the envelope of $\mathcal{C}^\sharp$.

There is another crucial consequence of the lemma above: Namely, the supporting lines $\mathcal{U}$ at some point $\mathbf{x}$ of $\mathcal{C}$ are the points $\mathcal{U}$ of a line segment of $\mathcal{C}^\sharp$ which lies on the line $\mathbf{x}$ in $¶^*$. Similarly, a line segment of $\mathcal{C}$ corresponds to a corner of $\mathcal{C}^\sharp$.

the sequence $(\mathcal{P}_n)$ converges to a C^1-curve, if the points $(\overline{\xi}, \overline{\eta})$ and $(\underline{\xi}, \underline{\eta})$ lie in the interior of the shaded region shown in Figure 3.

Figure 3: The Gregory-Qu region.

Not every corner cutting algorithm which produces C^1-curves satisfies the Gregory-Qu condition. In this paper we will present a necessary and sufficient condition for a local corner cutting scheme to produce C^1-curves. Our result also leads to a new proof of the result by Gregory and Qu. While the proofs in [Gregory & Qu '96] and [de Boor '90] are analytic, our proofs are geometric.

Furthermore we will consider the dual statement of our result. A corner cutting process is dual to an **interpolatory refinement scheme**, i.e. a scheme which produces a sequence of polygons $\mathcal{Q}_n$ such that for all n the vertices of $\mathcal{Q}_n$ are also vertices of $\mathcal{Q}_{n+1}$ as illustrated in Figure 6.

Finally we present a convexity preserving local interpolation scheme with geometric shape handles well suited for interactive geometric design. The method is very much related to the matrix subdivision scheme studied in, e.g., [Prautzsch '91].

2 Preliminaries

Here we recall some simple well-known geometric facts which are crucial for the entire paper.

Throughout we will work in an affine plane A over $\mathbb{R}$ equipped with the standard topology and extend A if needed by the ideal line. Often we will need a scalar product in A and thus consider a Euclidean plane instead.

In this paper we define a **closed convex curve** in an affine plane as the boundary of a bounded convex subset with non-empty interior. Note that a convex set without interior points is either a point, line, or a line segment, see e.g. [Limaye '81, p 330, Lemma 4].

For the rest of this section let $\mathcal{M}$ be an open or closed convex subset of A with non-empty interior $\mathring{\mathcal{M}}$ and let $\mathbf{x}$ be some boundary point of $\mathcal{M}$. Then the lines through $\mathbf{x}$ which do not intersect $\mathring{\mathcal{M}}$ are the **supporting lines** of $\mathcal{M}$ and its boundary $\partial\mathcal{M}$ at $\mathbf{x}$. Note that the supporting lines at $\mathbf{x}$ form a

1 Introduction

Many algorithms in Computer Aided Geometric Design are corner cutting algorithms like de Casteljau's, degree elevation, or knot insertion. Corner cutting is attractive since it provides simple geometric constructions of curves. It means to produce a sequence $(\mathcal{P}_n)$ of **polygons**, i.e. piecewise linear curves, such that $\mathcal{P}_{n+1}$ is obtained from $\mathcal{P}_n$ by **one** or **several** cuts as illustrated in Figure 1.

Figure 1: Corner cuts.

In this paper we assume that any three consecutive vertices of a polygon are distinct and consider only local cuts. Following de Boor [1990, Section 3] we will call a cut **local** if it removes exactly one vertex and adds two new ones. Thus if $\mathcal{P}_{n+1}$ is obtained from $\mathcal{P}_n$ by m local cuts, then $\mathcal{P}_{n+1}$ has m more vertices than $\mathcal{P}_n$ and each edge of $\mathcal{P}_n$ contains an edge of $\mathcal{P}_{n+1}$.

Local corner cutting has been studied by de Rham [1947, 1956], Gregory & Qu [1996], and de Boor [1990]: Let $(\mathcal{P}_n)$ be a sequence of polygons generated by local cuts such that all corners of $\mathcal{P}_n$ are cut simultaneously, i.e. $\mathcal{P}_{n+1}$ divides each edge of $\mathcal{P}_n$ into three segments as illustrated in Figure 2. Let the lengths of these 3 segments have the ratios $\xi : 1 - \xi - \eta : \eta$ where ξ and η may be different for each n and each edge of $\mathcal{P}_n$.

Figure 2: The Gregory-Qu scheme.

Further, let $\overline{\xi}, \overline{\eta}, \underline{\xi}, \underline{\eta}$ be the supremum and infimum of all ξ's and η's. Then Gregory & Qu and later de Boor, with a different proof, showed that

Abstract

In this paper we consider corner cutting and convexity preserving interpolatory refinement schemes in the plane and on the sphere. Using well-known facts from projective geometry we present a unified approach to such schemes and geometric derivations of simple conditions which guarantee that a scheme generates C^1-curves. Our results generalize all of the results known so far and provide the ground for a new convexity preserving C^1-interpolation scheme with a simple direct shape handle.

Keywords

Spherical and planar corner cutting, interpolatory refinement, convex curves, duality, matrix subdivision, C^1-interpolation, conics, Haase's algorithm

A geometric look at corner cutting

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