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THE FUNDAMENTAL SOLUTIONS OF MODERATELY THICK LAMINATED ANISOTROPIC SHALLOW SHELLS

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Abstract—In this paper, the partial differential equations governing moderately thick laminated anisotropic shallow shells are transferred into a set of ordinary differential equations by using the method of plane wave decomposition. With the aid of the Hörmander operator method, these ordinary differential equations are reduced to a tenth order ordinary differential equation. The fundamental solutions of moderately thick laminated anisotropic shallow shells are presented in a definite integral form. The numerical computation of the fundamental solutions is discussed in detail. Some computational formulations have been given.

1. INTRODUCTION

As is well known, fundamental solutions play an important role in boundary element methods which are widely used in analysis of plates, shells and some structures. The great progress has been made in the boundary element methods of isotropic body. However, to authors' knowledge, there are the few research results of boundary element methods in anisotropic plates and shells. In general, it is difficult to obtain the fundamental solutions of anisotropic body in closed form. Thus man has to use numerical fundamental solutions in boundary element methods. Lukasiewicz [1] has presented an approximate fundamental solution of orthotropic thin plates by Fourier transform technique. Wang and Huang [2] and Wang [3] have presented the fundamental solutions of orthotropic thick plates and orthotropic thin shells in a definite integral form by the Hörmander operator method [4] and the method of plane wave decomposition [5]. Wang and Huang [2, 6] have analyzed moderately thick orthotropic plates with boundary element methods.

With the increasing use of fibre-reinforced composite material, the structures of laminated plates and shells are widely used in engineering. Recent research shows that transverse shear deformation effects are more pronounced in anisotropic plates and shells than in isotropic plates and shells. In this paper, the fundamental solutions of laminated anisotropic shallow shells including shear transverse deformation have been presented by the use of the Hörmander method [4] and the method of plane wave decomposition [5]. The computation of the fundamental solutions is discussed in detail.

2. BASIC EQUATIONS

In this section, the basic equations of moderately thick laminated anisotropic shallow shells are reviewed.

The relations between the generalized displacements and the strains are

$$
\begin{aligned}
\epsilon_1 &= \frac{\partial u}{\partial x} + wk_1 & \epsilon_2 &= \frac{\partial v}{\partial y} + wk_2 \\
\epsilon_6 &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} & \kappa_1 &= \frac{\partial \psi_x}{\partial x} \\
\kappa_2 &= \frac{\partial \psi_y}{\partial y} & \kappa_6 &= \frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} \\
\epsilon_4 &= \frac{\partial w}{\partial y} + \psi_y & \epsilon_5 &= \frac{\partial w}{\partial x} + \psi_x
\end{aligned}
\tag{1}
$$

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in which u, v and w indicate the displacements of the midplane of the shells, ψ_x and ψ_y represent the rotations of the shells about x and y coordinate axes respectively. k_1 and k_2 are principal curvatures of shallow shells in x_1 and x_2 directions respectively.

The relations between the stress resultants and the strains are

$$N_i = A_{ij}\epsilon_j + B_{ij}\kappa_j \qquad M_i = B_{ij}\epsilon_j + D_{ij}\kappa_j$$

$$(A_{ij}, B_{ij}, D_{ij}) = \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} Q_{ij}^{(k)}(1, z, z^2)\,\mathrm{d}z \quad (i, j = 1, 2, 6) \tag{2}$$

$$Q_1 = C_{45}\epsilon_4 + C_{55}\epsilon_5 \qquad Q_2 = C_{44}\epsilon_4 + C_{45}\epsilon_5$$

$$C_{ij} = \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} Q_{ij}^{(k)} K_i K_j\,\mathrm{d}z \quad (i, j = 4, 5) \tag{3}$$

in which h_k is the vertical distance from the midplane, $z = 0$, to the upper surface of the kth lamina. K_4 and K_5 are the shear correction factors [7, 8]. $Q_{ij}^{(k)}$ are plane stress reduced stiffness coefficients of the kth lamina [7, 8].

The equilibrium equations of the shells are

$$\frac{\partial N_1}{\partial x} + \frac{\partial N_6}{\partial y} + q_x = 0$$

$$\frac{\partial N_6}{\partial x} + \frac{\partial N_2}{\partial y} + q_y = 0$$

$$\frac{\partial Q_1}{\partial x} + \frac{\partial Q_2}{\partial y} - N_1 k_1 - N_2 k_2 + q_z = 0$$

$$\frac{\partial M_1}{\partial x} + \frac{\partial M_6}{\partial y} - Q_1 + m_x = 0$$

$$\frac{\partial M_6}{\partial x} + \frac{\partial M_2}{\partial y} - Q_2 + m_y = 0. \tag{4}$$

Substituting equations (1), (2) and (3) into (4), we obtain the following differential equations using the generalized displacements as basic unknowns.

$$\Delta_{ij}^* U_j + P_i = 0 \quad (i, j = 1, 2, 3, 4, 5) \tag{5}$$

where U_j represents the displacements of the shallow shells in the direction of x_1, x_2 and x_3, and the rotations in the direction of x_1 and x_2, i.e. U_j indicates u, v, w, ψ_x and ψ_y. P_i represents the generalized loads, i.e. P_i indicates q_x, q_y, q_z, m_x and m_y respectively. Δ_{ij}^* is the differential operators which can be found in Ref. [8].

3. FUNDAMENTAL SOLUTIONS

According to the definition of fundamental solutions, the fundamental solutions of moderately thick laminated anisotropic shallow shells are a set of special solutions of equations (5) under the action of a set of unit point load, i.e.

$$\Delta_{ij}^* U_{kj}^*(\zeta, \mathbf{x}) = -\delta(\zeta, \mathbf{x})\delta_{ki} \tag{6}$$

in which $\delta(\zeta, \mathbf{x})$ is the Dirac δ function, ζ and $\mathbf{x}$ represent the coordinates of the source point and a field point respectively, $U_{kj}^*(\zeta, \mathbf{x})$ represents the generalized displacements in the j direction at the field point $\mathbf{x}$ of an infinite shell when a unit point load is applied at the k

direction of the source point ζ. Equation (6) is a set of partial differential equations. It is difficult to obtain its solutions directly. In this paper, we first use the method of plane wave decomposition to transfer equation (6) into a set of ordinary differential equations. We expand $U^*_{kj}(\zeta, \mathbf{x})$ and $\delta(\zeta, \mathbf{x})$ into a plane wave.

$$\delta(\zeta, \mathbf{x}) = -\frac{1}{4\pi^2}\int_0^{2\pi} |\rho|^{-2}\,\mathrm{d}\theta, \qquad U^*_{kj}(\zeta, \mathbf{x}) = \int_0^{2\pi} \tilde{U}^*_{kj}(\rho)\,\mathrm{d}\theta \tag{7}$$

in which $\rho = \omega_1(x_1 - \xi) + \omega_2(x_2 - \eta)$, (ω_1, ω_2) are coordinates of a point on the unit circle, i.e. $\omega_1 = \cos\theta$, $\omega_2 = \sin\theta$, (x_1, x_2) *and* (ξ, η) are coordinates of a field point and the source point respectively. $\phi(\rho)$ is a function depending only on ρ. Substituting equation (7) into equation (6), and considering differential relationship $\dfrac{\partial}{\partial x_\alpha} = \omega_\alpha \dfrac{\mathrm{d}}{\mathrm{d}\rho}$, we have

$$\tilde{\Delta}^*_{ij}\tilde{U}^*_{kj}(\rho) = \frac{1}{4\pi^2}|\rho|^{-2}\,\delta_{ki} \tag{8}$$

in which

$$\tilde{\Delta}^*_{ij} = \begin{bmatrix} a_{11}\dfrac{\mathrm{d}^2}{\mathrm{d}\rho^2} & a_{12}\dfrac{\mathrm{d}^2}{\mathrm{d}\rho^2} & a_{13}\dfrac{\mathrm{d}}{\mathrm{d}\rho} & a_{14}\dfrac{\mathrm{d}^2}{\mathrm{d}\rho^2} & a_{15}\dfrac{\mathrm{d}^2}{\mathrm{d}\rho^2} \\ a_{12}\dfrac{\mathrm{d}^2}{\mathrm{d}\rho^2} & a_{22}\dfrac{\mathrm{d}^2}{\mathrm{d}\rho^2} & a_{23}\dfrac{\mathrm{d}}{\mathrm{d}\rho} & a_{24}\dfrac{\mathrm{d}^2}{\mathrm{d}\rho^2} & a_{25}\dfrac{\mathrm{d}^2}{\mathrm{d}\rho^2} \\ -a_{13}\dfrac{\mathrm{d}}{\mathrm{d}\rho} & -a_{23}\dfrac{\mathrm{d}}{\mathrm{d}\rho} & a_{33}\dfrac{\mathrm{d}^2}{\mathrm{d}\rho^2} - a_k & a_{34}\dfrac{\mathrm{d}}{\mathrm{d}\rho} & a_{35}\dfrac{\mathrm{d}}{\mathrm{d}\rho} \\ a_{14}\dfrac{\mathrm{d}^2}{\mathrm{d}\rho^2} & a_{24}\dfrac{\mathrm{d}^2}{\mathrm{d}\rho^2} & -a_{34}\dfrac{\mathrm{d}}{\mathrm{d}\rho} & a_{44}\dfrac{\mathrm{d}^2}{\mathrm{d}\rho^2} - C_{55} & a_{45}\dfrac{\mathrm{d}^2}{\mathrm{d}\rho^2} - C_{45} \\ a_{15}\dfrac{\mathrm{d}^2}{\mathrm{d}\rho^2} & a_{25}\dfrac{\mathrm{d}^2}{\mathrm{d}\rho^2} & -a_{35}\dfrac{\mathrm{d}}{\mathrm{d}\rho} & a_{45}\dfrac{\mathrm{d}^2}{\mathrm{d}\rho^2} - C_{45} & a_{55}\dfrac{\mathrm{d}^2}{\mathrm{d}\rho^2} - C_{44} \end{bmatrix} \tag{9}$$

$$\begin{aligned}
&a_{11} = A_{11}\omega_1^2 + 2A_{16}\omega_1\omega_2 + A_{66}\omega_2^2 && a_{22} = A_{66}\omega_1^2 + 2A_{26}\omega_1\omega_2 + A_{22}\omega_2^2\\
&a_{12} = A_{16}\omega_1^2 + (A_{12} + A_{66})\omega_1\omega_2 + A_{26}\omega_2^2 && a_{14} = B_{11}\omega_1^2 + 2B_{16}\omega_1\omega_2 + B_{66}\omega_2^2\\
&a_{15} = B_{16}\omega_1^2 + (B_{12} + B_{66})\omega_1\omega_2 + B_{26}\omega_2^2 && a_{25} = B_{66}\omega_1^2 + 2B_{26}\omega_1\omega_2 + B_{22}\omega_2^2\\
&a_{33} = C_{55}\omega_1^2 + 2C_{45}\omega_1\omega_2 + C_{44}\omega_2^2 && a_{44} = D_{11}\omega_1^2 + 2D_{16}\omega_1\omega_2 + D_{66}\omega_2^2\\
&a_{45} = D_{16}\omega_1^2 + (D_{12} + D_{66})\omega_1\omega_2 + D_{26}\omega_2^2 && a_{55} = D_{66}\omega_1^2 + 2D_{26}\omega_1\omega_2 + D_{22}\omega_2^2\\
&a_k = A_{11}k_1^2 + 2A_{12}k_1k_2 + A_{22}k_2^2 && a_{24} = a_{15}\\
&a_{13} = (A_{11}k_1 + A_{12}k_2)\omega_1 + (A_{16}k_1 + A_{26}k_2)\omega_2\\
&a_{23} = (A_{16}k_1 + A_{26}k_2)\omega_1 + (A_{12}k_1 + A_{22}k_2)\omega_2\\
&a_{34} = (C_{55} - k_1B_{11} - k_2B_{12})\omega_1 + (C_{45} - k_1B_{16} - k_2B_{26})\omega_2\\
&a_{35} = (C_{45} - k_1B_{16} - k_2B_{26})\omega_1 + (C_{44} - k_1B_{12} - k_2B_{22})\omega_2.
\end{aligned} \tag{10}$$

By the use of the Hörmander operator method, the solutions of equation (8) can be represented in the following from:

$$\tilde{U}^*_{kj}(\rho) = {}^{\mathrm{co}}\tilde{\Delta}^*_{jk}\phi(\rho) \tag{11}$$

where $\phi(\rho)$ is an unknown scalar function depending only on ρ, and ${}^{\mathrm{co}}\tilde{\Delta}^*$ is the cofactor matrix of $\tilde{\Delta}^*$. Thus the fundamental solutions of the generalized displacements for moderately thick laminated anisotropic shallow shells are as follows.

$$
\begin{aligned}
\tilde{U}^*_{\alpha\beta}(\rho) &= b_{\alpha\beta}D^8\phi(\rho) + d_{\alpha\beta}D^6\phi(\rho) + e_{\alpha\beta}D^4\phi(\rho) + f_{\alpha\beta}D^2\phi(\rho) \\
\tilde{U}^*_{\alpha 3}(\rho) &= -\tilde{U}^*_{3\alpha}(\rho) = b_\alpha D^7\phi(\rho) + d_\alpha D^5\phi(\rho) + e_\alpha D^3\phi(\rho) \\
\tilde{U}^*_{\alpha 3+\beta}(\rho) &= \tilde{U}^*_{3+\beta\alpha}(\rho) = g_{\alpha\beta}D^8\phi(\rho) + h_{\alpha\beta}D^6\phi(\rho) + s_{\alpha\beta}D^4\phi(\rho) \\
\tilde{U}^*_{33}(\rho) &= AD^8\phi(\rho) + BD^6\phi(\rho) + CD^4\phi(\rho) \\
\tilde{U}^*_{33+\alpha}(\rho) &= -\tilde{U}^*_{3+\alpha 3}(\rho) = f_\alpha D^7\phi(\rho) + c_\alpha D^5\phi(\rho) \\
\tilde{U}^*_{\alpha+3\beta+3}(\rho) &= q_{\alpha\beta}D^8\phi(\rho) + r_{\alpha\beta}D^6\phi(\rho) + t_{\alpha\beta}D^4\phi(\rho)
\end{aligned} \tag{12}
$$

in which $D^k\phi(\rho) = \dfrac{\mathrm{d}^k\phi(\rho)}{\mathrm{d}\rho^k}$, $(k = 1, 2, \ldots)$, $\alpha, b = 1, 2$.

$$
\begin{aligned}
b_{11} &= a_{33}(a_{22}\alpha_1 + a_{24}\alpha_2 + a_{25}\alpha_3) \\
b_{22} &= a_{33}(a_{11}\alpha_1 + a_{14}\alpha_{18} + a_{15}\alpha_{19}) \\
b_{12} &= b_{21} = -a_{33}(a_{12}\alpha_1 + a_{24}\alpha_{18} + a_{25}\alpha_{19}) \\
d_{11} &= a_{22}(a_{33}\alpha_5 - a_{55}\alpha_4 + a_{35}\alpha_8 + a_{45}\alpha_6 + a_{34}\alpha_7) + a_{23}(a_{23}\alpha_1 + a_{24}\alpha_7 + a_{25}\alpha_8) \\
&\quad + a_{24}(a_{23}\alpha_7 + a_{35}\alpha_9 - a_{25}\alpha_6 + a_{24}\alpha_{10}) + a_{25}(a_{23}\alpha_8 + a_{34}\alpha_{11} + a_{25}\alpha_4 - a_{24}\alpha_6) \\
d_{22} &= a_{11}(a_{33}\alpha_5 - a_{55}\alpha_4 + a_{35}\alpha_8 + a_{45}\alpha_6 + a_{34}\alpha_7) + a_{13}(a_{13}\alpha_1 + a_{14}\alpha_7 + a_{15}\alpha_8) \\
&\quad + a_{14}(a_{13}\alpha_7 + a_{35}\alpha_{20} - a_{15}\alpha_6 + a_{14}\alpha_{10}) + a_{15}(a_{13}\alpha_8 - a_{34}\alpha_{20} + a_{15}\alpha_4 - a_{14}\alpha_6) \\
d_{12} &= d_{21} = -a_{12}(a_{33}\alpha_5 - a_{55}\alpha_4 + a_{35}\alpha_8 + a_{45}\alpha_6 + a_{34}\alpha_7) - a_{23}(a_{13}\alpha_1 + a_{14}\alpha_7 + a_{15}\alpha_8) \\
&\quad - a_{24}(a_{13}\alpha_7 + a_{35}\alpha_{20} - a_{15}\alpha_6 + a_{14}\alpha_{10}) - a_{25}(a_{13}\alpha_8 - a_{34}\alpha_{20} + a_{15}\alpha_4 - a_{14}\alpha_6) \\
e_{11} &= a_{22}(C_{44}\alpha_4 + a_k\alpha_{28} + a_{34}\alpha_{12} + a_{35}\alpha_{13} - C_{45}\alpha_6) + a_{23}(a_{24}\alpha_{12} + a_{25}\alpha_{13} - a_{23}(\alpha_{28} - \alpha_5)) \\
&\quad + a_{24}(a_{23}\alpha_{12} + a_k\alpha_{14}) + a_{25}(a_k\alpha_{15} + a_{23}\alpha_{13}) \\
e_{22} &= a_{11}(C_{44}\alpha_4 + a_k\alpha_{28} + a_{34}\alpha_{12} + a_{35}\alpha_{13} - C_{45}\alpha_6) + a_{13}(a_{14}\alpha_{12} + a_{15}\alpha_{13} - a_{13}(\alpha_{28} - \alpha_5)) \\
&\quad + a_{14}(a_{13}\alpha_{12} + a_k\alpha_{21}) + a_{15}(a_k\alpha_{22} + a_{13}\alpha_{13}) \\
e_{12} &= e_{21} = -a_{12}(C_{44}\alpha_4 + a_k\alpha_{28} + a_{34}\alpha_{12} + a_{35}\alpha_{13} - C_{45}\alpha_6) - a_{23}(a_{14}\alpha_{12} + a_{15}\alpha_{13} \\
&\quad - a_{13}(\alpha_{28} - \alpha_5)) - a_{24}(a_{13}\alpha_{12} + a_k\alpha_{21}) - a_{25}(a_k\alpha_{22} + a_{13}\alpha_{13}) \\
f_{11} &= \alpha_{16}\alpha_{17} \qquad f_{22} = \alpha_{23}\alpha_{17} \qquad f_{12} = f_{21} = -\alpha_{24}\alpha_{17} \\
b_1 &= -a_{12}(a_{23}\alpha_1 + a_{24}\alpha_7 + a_{25}\alpha_8) + a_{22}(a_{13}\alpha_1 + a_{14}\alpha_7 + a_{15}\alpha_8) \\
&\quad + a_{24}(a_{13}\alpha_2 + a_{35}\alpha_{25} - a_{23}\alpha_{18}) - a_{25}(-a_{13}\alpha_3 + a_{34}\alpha_{25} + a_{23}\alpha_{19}) \\
b_2 &= a_{11}(a_{23}\alpha_1 + a_{24}\alpha_7 + a_{25}\alpha_8) - a_{12}(a_{13}\alpha_1 + a_{14}\alpha_7 + a_{15}\alpha_8) \\
&\quad - a_{14}(a_{13}\alpha_2 + a_{35}\alpha_{25} - a_{23}\alpha_{18}) + a_{15}(-a_{13}\alpha_3 + a_{34}\alpha_{25} + a_{23}\alpha_{19}) \\
d_1 &= a_{12}(a_{23}(\alpha_{28} - \alpha_5) - a_{24}\alpha_{12} - a_{25}\alpha_{13}) - a_{22}(a_{13}(\alpha_{28} - \alpha_5) - a_{14}\alpha_{12} - a_{15}\alpha_{13}) \\
&\quad + a_{24}(a_{23}\alpha_{21} - a_{13}\alpha_{14}) - a_{25}(a_{13}\alpha_{15} - a_{23}\alpha_{22}) \\
d_2 &= -a_{11}(a_{23}(\alpha_{28} - \alpha_5) - a_{24}\alpha_{12} - a_{25}\alpha_{13}) + a_{12}(a_{13}(\alpha_{28} - \alpha_5) - a_{14}\alpha_{12} - a_{15}\alpha_{13}) \\
&\quad - a_{14}(a_{23}\alpha_{21} - a_{13}\alpha_{14}) + a_{15}(a_{13}\alpha_{15} - a_{23}\alpha_{22}) \\
e_1 &= \alpha_{26}\alpha_{17} \qquad e_2 = \alpha_{27}\alpha_{17} \\
g_{11} &= a_{33}(-a_{12}\alpha_2 + a_{22}\alpha_{18} + a_{25}\alpha_{25})
\end{aligned}
$$

$$g_{12} = a_{33}(-a_{12}\alpha_3 + a_{22}\alpha_{19} - a_{24}\alpha_{25})$$

$$g_{21} = a_{33}(a_{11}\alpha_2 - a_{12}\alpha_{18} - a_{15}\alpha_{25})$$

$$g_{22} = a_{33}(a_{11}\alpha_3 - a_{12}\alpha_{19} + a_{14}\alpha_{25})$$

$$h_{11} = -a_{12}(a_{23}\alpha_7 + a_{24}\alpha_{10} + a_{35}\alpha_9 - a_{25}\alpha_6) + a_{22}(a_{13}\alpha_7 + a_{14}\alpha_{10} + a_{35}\alpha_{20} - a_{15}\alpha_6)$$
$$- a_{23}(a_{13}\alpha_2 - a_{23}\alpha_{18} + a_{35}\alpha_{25}) + a_{25}(a_{13}\alpha_{11} + a_{23}\alpha_{20} - a_k\alpha_{25})$$

$$h_{21} = a_{11}(a_{23}\alpha_7 + a_{24}\alpha_{10} + a_{35}\alpha_9 - a_{25}\alpha_6) - a_{12}(a_{13}\alpha_7 + a_{14}\alpha_{10} + a_{35}\alpha_{20} - a_{15}\alpha_6)$$
$$+ a_{13}(a_{13}\alpha_2 - a_{23}\alpha_{18} + a_{35}\alpha_{25}) - a_{15}(a_{13}\alpha_{11} + a_{23}\alpha_{20} - a_k\alpha_{25})$$

$$h_{12} = a_{12}(-a_{23}\alpha_8 + a_{24}\alpha_6 - a_{34}\alpha_{11} - a_{25}\alpha_4) - a_{22}(-a_{13}\alpha_8 + a_{14}\alpha_6 + a_{34}\alpha_{20} - a_{15}\alpha_4)$$
$$+ a_{23}(-a_{13}\alpha_3 + a_{34}\alpha_{25} + a_{23}\alpha_{19}) - a_{24}(a_{13}\alpha_{11} + a_{23}\alpha_{20} - a_k\alpha_{25})$$

$$h_{22} = -a_{11}(-a_{23}\alpha_8 + a_{24}\alpha_6 - a_{34}\alpha_{11} - a_{25}\alpha_4) + a_{12}(-a_{13}\alpha_8 + a_{14}\alpha_6 + a_{34}\alpha_{20} - a_{15}\alpha_4)$$
$$- a_{13}(-a_{13}\alpha_3 + a_{34}\alpha_{25} + a_{23}\alpha_{19}) + a_{14}(a_{13}\alpha_{11} + a_{23}\alpha_{20} - a_k\alpha_{25})$$

$$s_{11} = -a_{12}(a_{23}\alpha_{12} + a_k\alpha_{14}) + a_{22}(a_{13}\alpha_{12} + a_k\alpha_{21}) + a_{23}(a_{13}\alpha_{14} - a_{23}\alpha_{21})$$

$$s_{21} = a_{11}(a_{23}\alpha_{12} + a_k\alpha_{14}) - a_{12}(a_{13}\alpha_{12} + a_k\alpha_{21}) - a_{13}(a_{13}\alpha_{14} - a_{23}\alpha_{21})$$

$$s_{12} = -a_{12}(a_{23}\alpha_{13} + a_k\alpha_{15}) + a_{22}(a_{13}\alpha_{13} + a_k\alpha_{22}) - a_{23}(a_{23}\alpha_{22} - a_{13}\alpha_{15})$$

$$s_{22} = a_{11}(a_{23}\alpha_{13} + a_k\alpha_{15}) - a_{12}(a_{13}\alpha_{13} + a_k\alpha_{22}) + a_{13}(a_{23}\alpha_{22} - a_{13}\alpha_{15})$$

$$A = a_{11}(a_{22}\alpha_1 + a_{24}\alpha_2 + a_{25}\alpha_3) - a_{12}(a_{12}\alpha_1 + a_{25}\alpha_{19} + a_{24}\alpha_{18})$$
$$+ a_{14}(a_{22}\alpha_{18} + a_{25}\alpha_{25} - a_{12}\alpha_2) - a_{15}(a_{12}\alpha_3 - a_{22}\alpha_{19} - a_{24}\alpha_{25})$$

$$B = -a_{11}(a_{22}(\alpha_{28} - \alpha_5) + a_{24}\alpha_{14} + a_{25}\alpha_{15}) + a_{12}(a_{12}(\alpha_{28} - \alpha_5) + a_{14}\alpha_{14} + a_{15}\alpha_{15})$$
$$+ a_{14}(a_{12}\alpha_{14} - a_{22}\alpha_{21}) - a_{15}(a_{22}\alpha_{22} - a_{12}\alpha_{15})$$

$$C = \alpha_{29}\alpha_{17}$$

$$f_1 = a_{11}(a_{22}\alpha_7 - a_{23}\alpha_2 - a_{25}\alpha_9) - a_{12}(a_{12}\alpha_7 - a_{23}\alpha_{18} - a_{25}\alpha_{20})$$
$$+ a_{13}(a_{12}\alpha_2 - a_{22}\alpha_{18} - a_{25}\alpha_{25}) - a_{15}(a_{22}\alpha_{20} - a_{12}\alpha_9 - a_{23}\alpha_{25})$$

$$f_2 = a_{11}(a_{22}\alpha_8 - a_{23}\alpha_3 + a_{24}\alpha_9) - a_{12}(a_{12}\alpha_8 - a_{23}\alpha_{19} + a_{24}\alpha_{20})$$
$$+ a_{13}(a_{12}\alpha_3 - a_{22}\alpha_{19} + a_{24}\alpha_{25}) - a_{14}(a_{12}\alpha_9 - a_{22}\alpha_{20} + a_{23}\alpha_{25})$$

$$c_1 = a_{11}(a_{23}\alpha_{14} + a_{22}\alpha_{12}) - a_{12}(a_{12}\alpha_{12} + a_{23}\alpha_{21}) + a_{13}(a_{22}\alpha_{21} - a_{12}\alpha_{14})$$

$$c_2 = a_{11}(a_{23}\alpha_{15} + a_{22}\alpha_{13}) - a_{12}(a_{12}\alpha_{13} + a_{23}\alpha_{22}) + a_{13}(a_{22}\alpha_{22} - a_{12}\alpha_{15})$$

$$q_{11} = a_{33}(a_{25}\alpha_{30} + a_{15}\alpha_{31} - a_{55}\alpha_{29}) \qquad q_{22} = a_{33}(a_{25}\alpha_{32} + a_{14}\alpha_{33} - a_{44}\alpha_{29})$$

$$q_{12} = q_{21} = a_{33}(a_{45}\alpha_{29} - a_{24}\alpha_{30} - a_{14}\alpha_{31})$$

$$r_{11} = a_{55}(a_{12}\alpha_{24} - a_{11}\alpha_{16} - a_{13}\alpha_{26}) - a_{35}(a_{25}\alpha_{27} + a_{35}\alpha_{29} + a_{15}\alpha_{26})$$
$$+ a_{25}(a_{25}\alpha_{23} - a_{15}\alpha_{24} - a_{35}\alpha_{27}) - a_{15}(a_{35}\alpha_{26} + a_{25}\alpha_{24} - a_{15}\alpha_{16}) + a_{33}C_{44}\alpha_{29}$$

$$r_{22} = a_{44}(a_{12}\alpha_{24} - a_{11}\alpha_{16} - a_{13}\alpha_{26}) - a_{34}(a_{24}\alpha_{27} + a_{14}\alpha_{26} + a_{34}\alpha_{29})$$
$$+ a_{24}(a_{24}\alpha_{23} - a_{14}\alpha_{24} - a_{34}\alpha_{27}) - a_{14}(a_{34}\alpha_{26} + a_{24}\alpha_{24} - a_{14}\alpha_{16}) + a_{33}C_{55}\alpha_{29}$$

$$r_{12} = r_{21} = -a_{45}(a_{12}\alpha_{24} - a_{11}\alpha_{16} - a_{13}\alpha_{26}) + a_{35}(a_{24}\alpha_{27} + a_{34}\alpha_{29} + a_{14}\alpha_{26})$$
$$- a_{25}(a_{24}\alpha_{23} - a_{14}\alpha_{24} - a_{34}\alpha_{27}) + a_{15}(a_{34}\alpha_{26} + a_{24}\alpha_{24} - a_{14}\alpha_{16}) - a_{33}C_{45}\alpha_{29}$$

$$t_{11} = C_{44}(a_{11}\alpha_{16} - a_{12}\alpha_{24} + a_{13}\alpha_{26})$$
$$t_{22} = C_{55}(a_{11}\alpha_{16} - a_{12}\alpha_{24} + a_{13}\alpha_{26})$$
$$t_{12} = t_{21} = -C_{45}(a_{11}\alpha_{16} - a_{12}\alpha_{24} + a_{13}\alpha_{26})$$

$$\alpha_1 = a_{44}a_{55} - a_{45}^2 \quad \alpha_2 = a_{25}a_{45} - a_{24}a_{55} \quad \alpha_3 = a_{24}a_{45} - a_{25}a_{44}$$
$$\alpha_4 = a_{33}C_{55} + a_k a_{44} \quad \alpha_5 = a_{45}C_{45} - a_{44}C_{44} \quad \alpha_6 = a_{33}C_{45} + a_k a_{45}$$
$$\alpha_7 = a_{34}a_{55} - a_{35}a_{45} \quad \alpha_8 = a_{35}a_{44} - a_{34}a_{45} \quad \alpha_9 = a_{34}a_{25} - a_{24}a_{35}$$
$$\alpha_{10} = a_{33}C_{44} + a_k a_{55} \quad \alpha_{11} = a_{24}a_{35} - a_{25}a_{34} \quad \alpha_{12} = a_{35}C_{45} - a_{34}C_{44}$$
$$\alpha_{13} = a_{34}C_{45} - a_{35}C_{55} \quad \alpha_{14} = a_{25}C_{45} - a_{24}C_{44} \quad \alpha_{15} = a_{24}C_{45} - a_{25}C_{55}$$
$$\alpha_{16} = a_{22}a_k - a_{23}^2 \quad \alpha_{17} = C_{45}^2 - C_{44}C_{55} \quad \alpha_{18} = a_{15}a_{45} - a_{14}a_{55}$$
$$\alpha_{19} = a_{14}a_{45} - a_{15}a_{44} \quad \alpha_{20} = a_{15}a_{34} - a_{14}a_{35} \quad \alpha_{21} = a_{15}C_{45} - a_{14}C_{44}$$
$$\alpha_{22} = a_{14}C_{45} - a_{15}C_{55} \quad \alpha_{23} = a_{11}a_k - a_{13}^2 \quad \alpha_{24} = a_{12}a_k - a_{23}a_{13}$$
$$\alpha_{25} = a_{14}a_{25} - a_{15}a_{24} \quad \alpha_{26} = a_{12}a_{23} - a_{22}a_{13} \quad \alpha_{27} = a_{12}a_{13} - a_{11}a_{23}$$
$$\alpha_{28} = a_{55}C_{55} - C_{45}a_{45} \quad \alpha_{29} = a_{12}^2 - a_{11}a_{22} \quad \alpha_{30} = a_{15}a_{12} - a_{11}a_{25}$$
$$\alpha_{31} = a_{12}a_{25} - a_{15}a_{22} \quad \alpha_{32} = a_{12}a_{14} - a_{11}a_{24} \quad \alpha_{33} = a_{12}a_{24} - a_{14}a_{22}. \tag{13}$$

Substituting equation (11) into equation (8), we obtain a tenth order ordinary differential equation, which can be written in the following form.

$$\frac{d^4}{d\rho^4}\left(\frac{d^6}{d\rho^6} + a_1\frac{d^4}{d\rho^4} + a_2\frac{d^2}{d\rho^2} + a_3\right)\phi(\rho) = \frac{1}{4\pi^2\,|\rho|^2 A_1} \tag{14}$$

in which

$$A_1 = a_{11}b_{11} + a_{12}b_{12} + a_{14}g_{11} + a_{15}g_{12}$$
$$A_2 = a_{11}d_{11} + a_{12}d_{12} + a_{13}b_1 + a_{14}h_{11} + a_{15}h_{12}$$
$$A_3 = a_{11}e_{11} + a_{12}e_{12} + a_{13}d_1 + a_{14}s_{11} + a_{15}s_{12}$$
$$A_4 = a_{11}f_{11} + a_{12}f_{12} + a_{13}e_1$$
$$a_1 = \frac{A_2}{A_1} \qquad a_2 = \frac{A_3}{A_1} \qquad a_3 = \frac{A_4}{A_1}. \tag{15}$$

Integrating equation (14) for four times, deleting the constants of integration [3, 9] and by use of the method of variation of arbitrary parameters, unknown function $\phi(\rho)$ can be obtained, which is of the form:

$$\phi(\rho)=2\sum_{i=1}^{3}\lambda_i(p_i^2\rho^2\ln|\rho|+2\ln|\rho|+3)$$
$$+2\sum_{i=1}^{3}\lambda_i\left(e^{p_i\rho}\int_{\rho}^{\infty}\frac{e^{-p_i\sigma}}{\sigma}d\sigma-e^{-p_i\rho}\int_{-\infty}^{\rho}\frac{e^{p_i\sigma}}{\sigma}d\sigma\right) \tag{16}$$

where

$$\lambda_1=\frac{1}{16\pi^2A_1(p_1^2-p_2^2)(p_1^2-p_3^2)p_1^4},\qquad \lambda_2=\frac{1}{16\pi^2A_1(p_2^2-p_1^2)(p_2^2-p_3^2)p_2^4}$$

$$\lambda_3=\frac{1}{16\pi^2A_1(p_3^2-p_1^2)(p_3^2-p_2^2)p_3^4},\qquad p_1=\left(\eta_1+\eta_2-\frac{1}{3}a_1\right)^{1/2}$$

$$p_2=\frac{1}{\sqrt{2}}(\sqrt{r+\xi_1}+i\sqrt{r-\xi_2}),\qquad p_3=\frac{1}{\sqrt{2}}(\sqrt{r+\xi_1}-i\sqrt{r-\xi_2})$$

$$r=(\xi_1^2+\xi_2^2)^{1/2},\qquad \xi_1=-\frac{1}{2}(\eta_1+\eta_2)-\frac{1}{3}a_1$$

$$\xi_2=\frac{\sqrt{3}}{2}(\eta_1-\eta_2),\qquad \eta_1=\left(-\frac{q}{2}+\sqrt{b}\right)^{1/3}$$

$$\eta_2=\left(-\frac{q}{2}-\sqrt{b}\right)^{1/3},\qquad q=\left(\frac{2}{27}a_1^3-\frac{1}{3}a_1a_2+a_3\right)$$

$$b=\frac{1}{4}a_3^2+\frac{1}{27}(a_2^3+a_1^3a_3)-\frac{1}{108}a_1^2a_2^2-\frac{1}{6}a_1a_2a_3.$$

In order to obtain the fundamental solutions, it is necessary to calculate the function $\phi(\rho)$ and its derivatives. For simplicity, we introduce the following two functions.

$$A_i(p_i\rho)=e^{p_i\rho}\int_{\rho}^{\infty}\frac{e^{-p_i\sigma}}{\sigma}d\sigma-e^{-p_i\rho}\int_{-\infty}^{\rho}\frac{e^{p_i\sigma}}{\sigma}d\sigma$$
$$B_i(p_i\rho)=e^{p_i\rho}\int_{\rho}^{\infty}\frac{e^{-p_i\sigma}}{\sigma}d\sigma+e^{-p_i\rho}\int_{-\infty}^{\rho}\frac{e^{p_i\sigma}}{\sigma}d\sigma\quad(i=1,2,3). \tag{17}$$

The above two functions have the following differential relations:

$$\frac{dA_i(p_i\rho)}{d\rho}=p_iB_i(p_i\rho)-\frac{2}{\rho},\qquad \frac{dB_i(p_i\rho)}{d\rho}=p_iA_i(p_i\rho). \tag{18}$$

Using the expression of exponential integral [10], we have

$$A_1(p_1\rho)=e^{p_1\rho}E_1(p_1\rho)-e^{-p_1\rho}E_i(p_1\rho)$$
$$B_1(p_1\rho)=e^{p_1\rho}E_1(p_1\rho)+e^{-p_1\rho}E_i(p_1\rho)\quad\text{for}\quad\rho>0$$
$$A_1(p_1\rho)=-e^{|p_1\rho|}E_1(|p_1\rho|)+e^{|p_1\rho|}E_1(|p_1\rho|)$$
$$B_1(p_1\rho)=-e^{|p_1\rho|}E_1(|p_1\rho|)-e^{|p_1\rho|}E_1(|p_1\rho|)\quad\text{for}\quad\rho<0 \tag{19}$$

$$A_2(p_2\rho)=e^{p_2\rho}\left(E_1(p_2\rho)-\frac{\pi}{2}i(1-\operatorname{sgn}\rho)\right)-e^{-p_2\rho}\left(-E_1(-p_2\rho)+\frac{\pi}{2}i(1+\operatorname{sgn}\rho)\right)$$

$$B_2(p_2\rho)=e^{p_2\rho}\left(E_1(p_2\rho)-\frac{\pi}{2}i(1-\operatorname{sgn}\rho)\right)+e^{-p_2\rho}\left(-E_1(-p_2\rho)+\frac{\pi}{2}i(1+\operatorname{sgn}\rho)\right). \tag{20}$$

Because of $\bar{p}_2\rho = p_3$, we have $A_3(p_3\rho) = A_3(\bar{p}_2\rho)$ and $B_3(p_3\rho) = B_3(\bar{p}_2\rho)$. According to the property of complex variable function, there are the following relations.

$$A_3(p_3\rho) = \overline{A_2(p_2\rho)}, \qquad B_3(p_3\rho) = \overline{B_2(p_2\rho)}. \tag{21}$$

As long as $A_2(p_2\rho)$ and $B_2(p_2\rho)$ are obtained, we can calculate $A_3(p_2\rho)$ and $B_3(p_2\rho)$ by use of the above equations. In what follows, we only consider the calculation of $A_1(p_2\rho)$, $B_1(p_2\rho)$, $A_2(p_2\rho)$ and $B_2(p_2\rho)$. Using the series expression of the exponential function and exponential integral [10], we have

$$A_1(p_1\rho) = -2\sum_{n=0}^{\infty}\frac{(p_1\rho)^{2n}}{(2n)!}(\gamma + \ln|p_1\rho| - F(2n+1))$$

$$B_1(p_1\rho) = -2\sum_{n=1}^{\infty}\frac{(p_1\rho)^{2n-1}}{(2n-1)!}(\gamma + \ln|p_1\rho| - F(2n)) \tag{22}$$

$$A_2(p_2\rho) = -2\sum_{n=0}^{\infty}\frac{(p_2\rho)^{2n}}{(2n)!}\left(\gamma + \ln|p_2\rho| + i\left(\frac{\pi}{2} - \alpha\right) - F(2n+1)\right)$$

$$B_2(p_2\rho) = -2\sum_{n=1}^{\infty}\frac{(p_2\rho)^{2n-1}}{(2n-1)!}\left(\gamma + \ln|p_2\rho| + i\left(\frac{\pi}{2} - \alpha\right) - F(2n)\right) \tag{23}$$

where $\gamma(=0.57721\cdots)$ is the Euler constant.

$$\alpha = \arctan\left|\frac{\sqrt{r+\xi_1}}{\sqrt{r-\xi_1}}\right|, \qquad F(m+1) = \sum_{s=1}^{m}\frac{1}{s}, \qquad F(1) = 0. \tag{24}$$

Using equations (11) and (12), we obtain $\phi(\rho)$ and its derivatives as follows:

$$\phi(\rho) = 2\sum_{i=1}^{3}\lambda_i(p_i^2\rho^2\ln|\rho| + (2\ln|\rho| + 3) + A_i(p_i\rho))$$

$$D\phi(\rho) = 2\sum_{i=1}^{3}\lambda_i(2p_i^2\rho\ln|\rho| + p_i^2\rho + p_iB_i(p_i\rho))$$

$$D^2\phi(\rho) = 2\sum_{i=1}^{3}\lambda_i(2p_i^2\ln|\rho| + 3p_i^2 + p_i^2A_i(p_i\rho))$$

$$D^3\phi(\rho) = 2\sum_{i=1}^{3}\lambda_ip_i^3B_i(p_i\rho), \qquad D^4\phi(\rho) = 2\sum_{i=1}^{3}\lambda_ip_i^4A_i(p_i\rho)$$

$$D^5\phi(\rho) = 2\sum_{i=1}^{3}\lambda_ip_i^5B_i(p_i\rho), \qquad D^6\phi(\rho) = 2\sum_{i=1}^{3}\lambda_ip_i^6A_i(p_i\rho)$$

$$D^7\phi(\rho) = 2\sum_{i=1}^{3}\lambda_ip_i^7B_i(p_i\rho), \qquad D^8\phi(\rho) = 2\sum_{i=1}^{3}\lambda_ip_i^8A_i(p_i\rho)$$

$$D^9\phi(\rho) = 2\sum_{i=1}^{3}\lambda_ip_i^9B_i(p_i\rho) - \frac{1}{4\pi^2A_1\rho}$$

$$D^{10}\phi(\rho) = 2\sum_{i=1}^{3}\lambda_ip_i^{10}A_i(p_i\rho) + \frac{1}{4\pi^2A_1\rho^2}. \tag{25}$$

In the procedure of obtaining equations (25), we have used the following relations:

$$\sum_{i=1}^{3}\lambda_ip_i^4 = 0, \qquad \sum_{i=1}^{3}\lambda_ip_i^6 = 0, \qquad \sum_{i=1}^{3}\lambda_ip_i^8 = \frac{1}{16\pi^2A_1}. \tag{26}$$

The computation of $\phi(\rho)$ and its derivatives is reduced to the calculation of $A_1(p_i\rho)$, $B_1(p_i\rho)$, $A_2(p_i\rho)$ and $B_2(p_i\rho)$. Taking a few terms of equations (22) and (23), we can obtain quite results for $A_1(p_i\rho)$, etc. when $|p_i\rho|$ is small. Using the asymptotic representation of exponential integral E_1 and E_i in Refs [10–12] and combining equations (19) and (20), we can calculate $A_1(p_i\rho)$, etc. when $|p_i\rho|$ is large. Using the above described method, we can obtain a good result for the calculation of $A_1(p_i\rho)$, $B_1(p_i\rho)$, $A_2(p_i\rho)$ and $B_2(p_i\rho)$. In the calculation of the fundamental solutions, we have to deal with the following integral:

$$I_1 = \int_0^{2\pi} F_1(\theta) \frac{\mathrm{d}^k \phi(\rho)}{\mathrm{d}\rho^k}, \qquad (k = 1, 2, 3, \ldots, 9). \tag{27}$$

In the range between 0 and 2π, the integrand has two points which make $\rho = 0$. We first determine values θ_0 which make $\rho = 0$ and then split $(0, 2\pi]$ into four intervals. As the integrand is a periodic function, the four intervals can be written in the following form:

$$\left(\theta_0, \theta_0 + \frac{\pi}{2}\right), \left(\theta_0 + \frac{\pi}{2}, \theta_0 + \pi\right)$$

$$\left(\theta_0 + \pi, \theta_0 + \frac{3\pi}{2}\right), \left(\theta_0 + \frac{3\pi}{2}, \theta_0 + 2\pi\right).$$

We can calculate the integral I_1 on each interval with any numerical integral method. The value θ_0 can be determined by the following equation:

$$\theta_0 = \arctan\left(-\frac{x - \xi}{y - \eta}\right). \tag{28}$$

Up to now, we have obtained the computational formulation of the fundamental solutions of the generalized displacements. Substituting equations (7) and (12) into equation (2) and using equations (1) and (3), we can obtain the fundamental solutions of the generalized forces for moderately thick laminated anisotropic shallow shells.

4. CONCLUSIONS

The fundamental solutions of moderately thick laminated anisotropic shallow shells have been presented in a definite integral form. They can be used to analyze the distribution of stresses and displacements in the neighborhood of the singular point which a concentrated force is applied at. They can also be taken as the kernel function of boundary integral equation method and used to analyze the static and dynamic problems of the shells. The boundary element analysis of moderately thick laminated anisotropic shallow shells will be given in successive paper.

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