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Neutron Spectra in Heterogeneous Assemblies, a Summary of the Work of the Karlsruhe Group

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1. Introduction

The investigation of neutron spectra in nuclear reactors and assemblies is of considerable interest. The real precise prediction of the neutron balance in a thermal reactor depends on the knowledge of the neutron spectra particularly at lower energies. Strongly dependent is especially the long time behaviour of a thermal reactor where U^{238} is largely converted into Pu^{239} with its important fission resonance at 0.3 eV.

The spectra are mostly used to produce properly averaged cross section data for a more simple multigroup calculation. But the interaction of slow neutrons with the scattering atoms of, say, hydrogen is also of interest for pure physical reasons. Therefore the theory of neutron thermalization is a link between reactor theory and pure physics.

There are two main influences on neutron spectra. The first is the mechanism of the single process, where a neutron is scattered by a moderating atom. The second is the superposed net diffusion process which transports the slow neutrons to the absorbing materials like fuel rods etc. The second process, of course, is of interest only, if the reactor is a heterogeneous one. But this is true for almost all existing reactors. The investigation of the first influence suggests the picture of an infinite homogeneous assembly where no net diffusion process takes place and the interest is focussed on the single scattering process. The investigation of the influence of the heterogeneity suggests a picture where the single scattering process is as simple as possible but typical and the heterogeneity is idealized into the δ function type sink and source model, which allows for a simple mathematical description.

The investigation of the influence of the single scattering process on neutron spectra has found a widespread interest during the last years because of the more physical background. The Karlsruhe group, however, has also emphasized the study of the influence of heterogeneity on neutron spectra. This paper summarizes the work on this second aspect.

2. The heterogeneous reactor model

H. Hurwitz jr., M.S. Nelkin and G.J. Habetler derive in their early paper [1] the equations for space dependent neutron spectra. They start from the Boltzmann equation, use the spherical harmonics method and arrive finally at an equation of the diffusion type.

The equation is the following one:

(1) $$(\Sigma_a - D\Delta)\, \phi(\vec{r},E) = L(\phi(\vec{r},E))$$

(2) $$L(\phi(\vec{r},E)) = \int dE'\, \Sigma_o(E' \rightarrow E)\, \phi(\vec{r},E') - \Sigma_S(E) \cdot \phi(\vec{r},E) = -\frac{\partial q}{\partial E}$$

(3) $$\Sigma_S(E) = \int \Sigma_o(E \rightarrow E')\, dE'$$

(4) $$D = \frac{1}{3\,\Sigma_S}$$

Σ_a is the absorption cross section, in principle space and energy dependent
q is the slowing down density defined by (2)
Δ is the Laplace Operator
$\vec{r}$ is the spatial vector
E is the energy of the neutron
$\Sigma_o(E' \rightarrow E)$ is the energy transport cross section which gives in case of a scattering process the neutron from E' to E.
The other quantities are defined in the equations (2) - (4).

We deal now with (1) and explain the Meetz model of heterogeneity [2]. In the simplest case we have a one dimensional model, $\vec{r}$ is to be replaced by the simple coordinate x. We consider a finite reactor configuration of thickness 2l with

(5) $$N_o = 2N + 1$$

that is, with an odd number of fuel elements of thickness $2a$ and spacing d (fig. 1).

We further require that there shall be moderator to such an extent that the reactor consists of N_o Wigner Seitz cells only, that is

(6) $$2l = N_o d$$

If we describe the fuel plates by δ functions, this model leads to the following equation:

(7) $$D \frac{\partial^2 \phi}{\partial x^2} + L(\phi) + f_- = 0$$

where

(8) $$f_- = - 2a \, \Sigma_a(\varepsilon) \sum_{k=-N}^{+N} \delta(x-kd) \, \phi(kd,\varepsilon)$$

We use here and in the following the normalized energy

(9) $$\varepsilon = \frac{E}{kT}$$

where k is the Boltzmann constant and T the temperature of the moderator.

$\Sigma_a(\varepsilon)$ is now an effective cross section adapted to the absorption strength in the real, finite plates. In the case of no self shielding $\Sigma_a(E)$ is simply the absorption cross section of the fuel: In case of such a self shielding the value and energy dependence of $\Sigma_a(\varepsilon)$ must be taken from a transport calculation inside the fuel. In the simple but artificial case of the validity of a diffusion equation

$$\frac{d^2\phi}{dx^2} - K^2 \phi = 0$$

we have

$$\Sigma_a(\varepsilon) = (\Sigma_a)_{fuel} \cdot \frac{\sinh K a}{K a}$$

The boundary conditions which determine together with (7) the problem are the following:

(10) $$\phi(\pm 1,\mathcal{E}) = 0$$

(11) $$q(x,0) = 0$$

(12) $$q(x,\mathcal{E}_f) = \nu \cdot 2a \cdot \sum_{k=-N}^{+N} \delta(x-kd) \cdot \int_0^{\mathcal{E}_f} \phi(kd,\tilde{\mathcal{E}})\ \Sigma_f(\tilde{\mathcal{E}})\ d\tilde{\mathcal{E}}$$

In (12) we assume that all fission neutrons appear at a discrete energy $\mathcal{E}_f$. Σ_f is the fission cross section, ν the number of neutrons per fission.
We now introduce two abbreviations:

(13) $$\frac{D}{\xi\Sigma_S} = x_o^{\ 2}$$

(14) $$\frac{2a\ \Sigma_a(\mathcal{E})}{d\ \xi\Sigma_S} = p(\mathcal{E})$$

ξ is the logarithmic energy decrement for scattering down processes at high energy that is: a constant value.
We now make the following ansatz:

(15) $$\phi = \sqrt{l} \cdot \sum_{p=o}^{\infty} \varphi_p(\mathcal{E})\ v_p(x)$$

v_p are the orthonormalized eigenfunctions of the Δ operator and the boundary conditions (9):

(16) $$v_p = \frac{1}{\sqrt{l}} \cos \alpha_p x$$

(17) $$\alpha_p = \frac{\pi}{2l} (2p + 1)$$

Inserting this into (7) leads to the following equation

$$(18) \qquad - x_o^2 \, \alpha_q^2 \, \varphi_q + \frac{L(\varphi_q)}{\xi \Sigma_S} - p(\mathcal{E}) \sum_p \varphi_p(\mathcal{E}) \; d \left[v_p, v_q \right] = 0$$

$d \left[v_p, v_q \right]$ is Meetz's matrix for heterogeneous plate type assemblies

$$(19) \qquad d \left[v_p, v_q \right] = \sum_{k=-N}^{+N} d \cdot v_p(kd) \; v_q(kd)$$

This matrix has a "one" in the diagonal terms:

$$(20) \qquad \begin{aligned} 1 + p + q &= N_o n \qquad && n = 1,2,3 \\ p - q &= N_o m \qquad && m = 0, \pm 1, \pm 2, \pm 3 \end{aligned}$$

All other terms are zero. (20) implies the fact, that the system (18) splits into N+1 different sets of equations, where only the quantities

$$\varphi_{N_o m+s} \qquad\qquad \varphi_{N_o m-s-1}$$

appear.

We obey the fact that

$$\alpha^2_{N_o m-s-1} = \alpha^2_{N_o m+s}$$

and rearrange the indices:

$$(21) \qquad \varphi_{N_o m-s-1} \longrightarrow \varphi_{-(N_o m+s)}$$

(21) implies that now m goes from $-\infty$ to $+\infty$ and covers then all index constellations in question.

Therefore we have

$$(22) \qquad -x_o^2\,\alpha^2_{N_o r+s}\,\varphi_{N_o r+s} + \frac{1}{\xi\Sigma_S} L(\varphi_{N_o r+s}) - p(\mathcal{E}) \sum_{m=-\infty}^{+\infty} \varphi_{N_o m+s} = 0$$

$$r = -\infty \ldots . +\infty$$
$$s = 0,1,2\ldots .N$$

The boundary condition (12) obeys the same symmetry. The exitation strength of the q mode shall be C_q. We have

$$(23) \qquad C_q = \int_{-1}^{+1} q(x,\mathcal{E}_F)\, v_q\, dx = \sqrt{1}\,\frac{2a\,\nu}{d} \sum_p d\left[v_p\, v_q\right] \chi_p$$

$$\chi_p = \int_o^{\mathcal{E}_F} \varphi_p(\tilde{\mathcal{E}})\,\Sigma_f(\tilde{\mathcal{E}})\, d\tilde{\mathcal{E}}$$

We have the same matrix $d\left[v_p, v_q\right]$ and the same conclusions, therefore it obeys the same frequency selection

$$(24) \qquad q = N_o r+s$$

That means

$$(25) \qquad C_s = \sqrt{1}\;\frac{2a\,\nu}{d} \sum_{m=-\infty}^{+\infty} \chi_{N_o m+s} \qquad \text{for all values of } r$$

By means of this boundary condition our problem becomes an eigenvalue problem with ν as eigenvalue. Therefore only one of the N+1 sets of equations can have a non trivial solution. From (16) one concludes that the term cos $\alpha_o x$ must be among the components. Therefore only the set s = 0 is here of interest because the reactor is regular and undisturbed (see [2]).

The frequency selection (24) has an immediate consequence for our ansatz (15).

It happens that

$$\cos \alpha_{N_o m} \, kd = \cos \alpha_o \, kd$$

and

$$\cos \alpha_{N_o m} \left(kd + \frac{d}{2}\right) = \cos \alpha_o \left(kd + \frac{d}{2}\right) \cdot (-1)^m$$

Applying this result to (15) it follows that at the fuel plates (x=kd) and in the middle of the moderator volume ($x=kd + \frac{d}{2}$) the spectra are the same for all the different Wigner Seitz cells. This is the analog to the fact that in the homogeneous case the problem is separable that means, the spectrum there is space independent. L. Dresner [3] has shown that this result is generally true, the only assumptions to be made are the following:

The fuel arrangement in the reactor must be periodical and the outer boundary must be such that the image method can be applied. This immediately shows that this method fails in the case of a cylindrical core boundary, it can be true there only approximately.

3. A principal classification of neutron spectra

The classification in question is a principal one. But it can be demonstrated in the easiest way, if the heavy gas model is choosen for the scattering of the neutron by the moderator. There we have [1]

$$\left.\begin{aligned}
&(26) \qquad L(\phi) = \xi \Sigma_S\left(\mathcal{E}\frac{\partial^2\phi}{\partial\mathcal{E}^2} + \mathcal{E}\frac{\partial\phi}{\partial\mathcal{E}} + \phi\right) + 0\left(\left(\frac{m}{M}\right)^2\right) \\
&(27) \qquad q(\phi) = -\xi\Sigma_S\left(\mathcal{E}\frac{\partial\phi}{\partial\mathcal{E}} + (\mathcal{E}-1)\,\phi\right) + 0\left(\left(\frac{m}{M}\right)^2\right)
\end{aligned}\right\} \quad \text{if } \mathcal{E} \gg \frac{m}{M}$$

m is the neutron mass and M the mass of the moderator atom. In order to be consistent we must put:

$$\xi = 2\,\frac{m}{M}$$

$$\Sigma_S = \Sigma_S^{\ o} = \text{constant}$$

(26) inserted into (7) gives an elliptical differential equation. This is true in principle for all energies. One can recognize the character of the possible solutions of (7) if one asks for the solutions of $L(\phi) = 0$ (for example in the sense of a perturbation ansatz).

We have two solutions. First:

$$(28) \qquad \phi = \phi_1 = \mathcal{E}\cdot e^{-\mathcal{E}}$$

(28) is a Maxwell distribution and refers in it's character entirely to the elliptic character of the equation (7), (26). The general implication is that we have a spectrum which refers to an equilibrium or near equilibrium state. Spectra of this type obey the following rule:

$$(29) \qquad q(\mathcal{E}) \sim 0(e^{-\mathcal{E}}) \qquad \text{for large values of } \mathcal{E}$$

Spectra of this type shall be called type A spectra.
Besides of (28) there is another solution for which we can give the asymptotic expansion

$$(30) \qquad \phi = \phi_2 = \frac{1}{\varepsilon} \sum_{n=1}^{N} n! \left(\frac{1}{\varepsilon}\right)^{n-1} + O(e^{-\xi}) \quad N = 1,2,3...$$

For values sufficiently large ϕ_2 approximates the solution of the equation

$$(31) \qquad \bar{L}(\phi) = \xi \Sigma_S \left(\varepsilon \frac{\partial \phi}{\partial \varepsilon} + \phi\right) = 0$$

But the equation (7), (31) is of a completely different character. It is of parabolic nature and describes slowing down processes (Ausgleichsprobleme). Spectra of this type shall be called type B spectra.

The reactor spectra are of the mixed type in the region of interest and this is the reason why the mathematical treatment is so difficult:
Mathematical tools are developed for either the type B or the type A solution, in the first for example the Laplace transformation, in the second for example the expansion into eigenfunctions of $L(\phi) = 0$ with $q(\varepsilon) = 0$ for large values of ε, because these eigenfunctions belong to the type A.

Reactor spectra for low energies are close to type A, for high energies close to type B and the change takes place gradually and has no turning point which is physically meaningful.

4. Slowing down spectra in a plate type reactor (type B)

In the following we will concentrate on the heavy gas model which leads to (26) and (27) although the procedure presented here is not restricted to that model.

This chapter refers to a first paper of the author [4]. The details can be seen there.

It is assumed there that only the slowing down case is of interest, where L given by (26) can be replaced by $\bar{L}$ (31). In that case our problem is as follows:

$$(32) \qquad - x_o^2 \, \alpha_{N_o r}^2 \, \varphi_{N_o r} + \varepsilon \frac{\partial \phi}{\partial \varepsilon} + \phi - p(\varepsilon) \sum_{m=-\infty}^{+\infty} \varphi_{N_o m} = 0$$

and according to (25)

$$\varphi_{N_o r}(\varepsilon_F) = \sqrt{\Gamma} \cdot \frac{2av}{d \cdot \xi \, \Sigma_S} \sum_{m=-\infty}^{+\infty} \chi_{N_o m}$$

but this means

$$(33) \qquad \varphi_{N_o r}(\varepsilon_F) = S_o = \text{constant} \neq f(r)$$

In addition we assume a simple absorption law

$$(34) \qquad p(\varepsilon) = p_o \cdot \frac{1}{\sqrt{\varepsilon/\varepsilon_F}}$$

In order to obtain the solution we make the following ansatz, where ß is a constant still to be determined.

$$(35) \qquad \varphi_{N_o r}(\varepsilon) = \sum_{v=o}^{\infty} \frac{\overset{v}{f}_{N_o r}}{\varepsilon^{1-\beta+\frac{v}{2}}}$$

(35) leads to the equation

$$(36) \qquad (\alpha_{N_o r}^2 \, x_o^2 - \beta + \tfrac{v}{2}) \, \overset{v}{f}_{N_o r} = - p_o \sum_{m=-\infty}^{+\infty} \overset{v-1}{f}_{N_o m}$$

From here we conclude, that ß has to be one of the values $\alpha_{N_o r}^{2}\, x_o^{2}$ in order to let $\overset{o}{f}_{N_o r} \neq 0$. For very high energies that is for the fission energy $\mathcal{E}_F$ the term $\overset{o}{f}_{N_o r}$ in (35) alone survives and represents the mode for which (33) is to be applied. So the absorption $p(\mathcal{E})$ at lower energies intermingles the modes, a pure mode at $\mathcal{E} = \mathcal{E}_F$ splits into all other modes. The second conclusion is that in (36) the term on the right is independent of r. But this means that the way in which $\overset{v}{f}_{N_o r}$ depends on r is given in (36). Using the abbreviation

$$(37) \qquad \overset{v}{F} = \sum_{m=-\infty}^{+\infty} \overset{v}{f}_{N_o m}$$

we obtain

$$(38) \qquad \overset{v}{f}_{N_o r} = \frac{-p_o \overset{v-1}{F}}{\alpha_{N_o r}^{2}\, x_o^{2} - ß + \frac{v}{2}}$$

Summation over r gives the final formula for obtaining the $\overset{v}{F}$ and from this by means of (38) the $\overset{v}{f}_{N_o r}$.

$$(39) \qquad \overset{v}{F} = -p_o \cdot \sum_{r} \frac{1}{\alpha_{N_o r}^{2}\, x_o^{2} - ß + \frac{v}{2}} \cdot \overset{v-1}{F}$$

The Fourier series (15) becomes a real result only in the case where all the summation can be performed. Using (35) in (15) leads to the more general series (40), which can be summed up by means of the residue calculus [2]:

$$(40) \qquad \sum_{m=-\infty}^{+\infty} \frac{\cos \alpha_{N_o m}\, x}{\alpha_{N_o m}^{2}\, x_o^{2} + s} = h(s) \cdot F(x,s)$$

$$(40a) \qquad h(s) = \frac{d}{2x_o \sqrt{s}} \; \frac{\sinh \frac{d}{x_o} \sqrt{s}}{\left(\cosh \frac{d}{x_o} \sqrt{s} - \cos \frac{\pi}{N_o}\right)}$$

$$(40b) \qquad F(x,s) = \frac{\sinh \frac{x'}{x_o} \sqrt{s}\, \cos \frac{k+1}{N_o} \pi - \sinh \frac{x'-d}{x_o} \sqrt{s}\, \cos \frac{k}{N_o} \pi}{\sinh \frac{d}{x_o} \sqrt{s}}$$

$$x = x' + kd \qquad k = 0, \pm 1, \pm 2, \dots, \pm N$$

$$0 \leq x' \leq d$$

From here on the calculation is straight forward and leads to the following result:

$$\phi(\mathcal{E}) = \frac{S_o}{\mathcal{E}} \sum_{m=-\infty}^{+\infty} \left(\frac{\mathcal{E}}{\mathcal{E}_F}\right)^{\alpha_{N_o m}^2 x_o^2} A_m$$

(41) $$A_m = \sum_{v=o}^{\infty} (-1)^v \left(\frac{\mathcal{E}}{\mathcal{E}_F}\right)^{-\frac{v}{2}} p_o^v \cdot B_v^{(m)} \cdot F(x, -\alpha_{N_o m}^2 x_o^2 + \frac{v}{2})$$

$$B_v^{(m)} = \prod_{\sigma=1}^{v} h(-\alpha_{N_o m}^2 x_o^2 + \frac{\sigma}{2})$$

(41) is convergent for all values of $\mathcal{E} < \mathcal{E}_F$ and is the solution of our slowing down problem. It has two significant features:
For sufficiently large values of $\mathcal{E}$ only the first term of the series A_m is to be taken into account and a simple series remains which describes the smoothing out process of the δ functions acting as sources at $\mathcal{E} = \mathcal{E}_F$. The smoothing out process is of the Θ function type. On the other hand:
for sufficiently small values of $\mathcal{E}$ only the term A_o is significant in the main series. We have again a simple series which describes the flux peaks between the fuel elements in its energy and space dependence. There is a region in between where only one term in both series is significant:

(42) $$\phi \sim \frac{S_o}{\mathcal{E}^{1 - \alpha_o^2 x_o^2}} \cdot \cos \alpha_o x$$

The smoothing out process of the source δ function produced the ground mode $\cos \alpha_o x$ and the absorption of the fuel plates is not yet significant. For reasonable absorption strengths of the fuel plates this pure ground mode takes over somewhere in the neighborhood of $\frac{\mathcal{E}_F}{\mathcal{E}} \sim 100$, that means $\mathcal{E} \approx 20$ keV if heavy water is used.

The main conclusion we draw from this is the following:
It is possible (at least for heavy water and graphite) to separate the smoothing out process of the heterogeneous fission neutron distribution from the formation of flux peaks in the moderator at low energies.
Details of this calculation can be seen in paper [4].

5. Equilibrium spectra in a plate type reactor (type A)

As indicated in chapter 4 we will use the heavy gas model.
The physical conditions for spectra of the equilibrium type are given in experiments which use pulsed neutrons. Sometime after the shot which injects the neutrons into a non multiplying assembly there are no more fast neutrons to be slowed down and the existing neutrons, although decaying in time, have a sort of equilibrium with the moderator. This type of experiment is investigated in a second paper of the author together with L. Dresner [5].

In the analysis of the pulsed neutron experiments we have to add a term in (7) which cares for the time dependence.
We have:

$$D \frac{\partial^2 \phi}{\partial x^2} + L(\phi) - \Sigma_a^{MoD} \cdot \phi + f_- = \frac{1}{v} \frac{\partial \phi}{\partial t} \tag{43}$$

v is the neutron velocity and Σ_a^{MoD} allows for an absorption in the moderator. Please note that we have not yet taken into account an absorption in the moderator up to now.
We make the following ansatz:

$$\phi = e^{-\lambda t} \sqrt{1} \sum_{p=o}^{\infty} \varphi_p(\mathcal{E})\ v_p(x) \tag{44}$$

with the same notation as in (15). Some time after the pulse the frequency selection (24) must hold and we obtain the following equations for the $\varphi_p(\mathcal{E})$ with $p = N_o r$

(45)

$$- x_o^2\ \alpha_{N_o r}^2\ \varphi_{N_o r} + \mathcal{E} \frac{\partial^2 \varphi_{N_o r}}{\partial \mathcal{E}^2} + \mathcal{E} \frac{\partial \varphi_{N_o r}}{\partial \mathcal{E}} + \varphi_{N_o r} + \mathcal{E}^{-\frac{1}{2}} \left(\alpha\, \varphi_{N_o r} - p_t \sum_{m=-\infty}^{+\infty} \varphi_{N_o m}\right) = 0$$

$$\alpha = \frac{\lambda - \Sigma_a^{MoD}\ v}{\xi \Sigma_s \cdot \left(\frac{2T}{m}\right)^{\frac{1}{2}}} \tag{46}$$

(46a) $$p_t = \frac{2a\ \Sigma_a(\mathcal{E}=1)}{d\,\xi\,\Sigma_S} \cdot \frac{1}{\sqrt{\varepsilon}}$$

Additionally we have:

(47)
$$\begin{aligned} \phi(\pm 1) &= 0 \\ q(x,0) &= 0 \\ q(x,\varepsilon) &= 0 \qquad \text{for large values of } \mathcal{E}\,. \end{aligned}$$

We now make use of the fact that we look for spectra of the type A. We expand the $\varphi_{N_o r}(\varepsilon)$ into eigenfunctions of the heavy gas operator $\frac{1}{\xi\Sigma_S}$ L. These eigenfunctions are as follows:

(48) $$\frac{1}{\xi\,\Sigma_S} L(\omega_p) = -\,\sigma_p\,\omega_p$$

$$q(\omega_p) = 0 \qquad\qquad \text{for } \varepsilon = 0 \text{ and } \varepsilon = \infty$$

For the case of the heavy gas model we have

(49) $$\omega_p = \varepsilon e^{-\varepsilon} L_p^{(1)}(\varepsilon)$$

(50) $$\sigma_p = p \quad ; \qquad p = 0,1,2,3\ldots$$

$L_p^{(1)}(\varepsilon)$ are the Laguerre polynomials of order one, which we use in the normalized form

(51) $$L_p^{(1)}(\varepsilon) = \sqrt{p+1}\ \sum_{\mu=o}^{p} (-1)^{\mu} \frac{p!}{\mu!(\mu+1)!(p-\mu)!}\ \varepsilon^{\mu}$$

The operator is not self adjoint. The adjoints of (49) are:

(52) $$\omega_k^{+} = L_k^{(1)}(\varepsilon)$$

We now consider the following expansion:

(53)
$$\varphi_{N_o r} = \sum_{p=o}^{\infty} a_{r,p}\,\omega_p(\mathcal{E})$$

Inserting (53) into (45) leads to a cumbersome but straight forward calculation, where the matrix elements V_{pq} are to be considered.

(54)

$$V_{pq} = \int_o^{\infty} \omega_p(\mathcal{E})\, \mathcal{E}^{-\frac{1}{2}}\, \omega_q^{+}(\mathcal{E})\, d\mathcal{E} = \frac{1}{\pi\sqrt{(p+1)(q+1)}} \sum_{l=o}^{p} \frac{\Gamma(p-l+\frac{1}{2})\,\Gamma(q-l+\frac{1}{2})\cdot\Gamma(l+\frac{3}{2})}{(p-l)!\,(q-l)!\,l!}$$

It should be emphasized that the formalism outlined in [5] does not use in principle a special model for the thermalization. However, if the heavy gas model is used all expressions can be given explicitly as in (54).

The problem (45) - (47) is a homogeneous one and constitutes an eigenvalue problem for α. Because of the two dimensions in question $(x,\mathcal{E})$ α has a two dimensional set of possible values, one degree of freedom refers tothe space dependence the other to the energy dependence. Note that only in case of type A spectra the eigenvalue of the problem is double indexed. For type B spectra the condition (12) establishes a source which makes the problem not homogeneous in the $\mathcal{E}$ direction, we have only a one dimensional set for the α's there.

The calculation following the insertion of (53) in (43) is cumbersome although straight forward. Under certain conditions it is enough to consider in (53) only two terms p=0 and p=1. If, for example, the structure of the diffusion cooling effect in heterogeneous non multiplying assemblies is the point of interest, it is enough to have these two terms, because they can indicate a spectrum cooling or heating. The result of this investigation of the diffusion cooling effect is given in [5]. But the several expansions are not driven to the latest stage there. If one does so, the result is as simple as follows:

(55) $$\lambda = (\Sigma_a^{MoD}\, v) +$$

$$\alpha_o^{\,2}\, D\, \bar{v} - \alpha_o^{\,4}\, D^2 \frac{\bar{v}}{\xi \Sigma_S} \cdot \frac{1}{8}$$

$$+ \frac{2a}{d}\, \Sigma_a v - \left[\frac{2a}{d}\, (\Sigma_a v)\right]^2 \frac{1}{\xi \Sigma_S \bar{v}} \left(\frac{1}{8}\, k\, \left(\frac{d}{x_o}\right) + \frac{1}{12}\, \frac{d^2}{x_o^{\,2}} +\right.$$

$$\left. 0 \left((\alpha_o^{\,2} \frac{D}{\Sigma_S})^{\mu} \cdot (\frac{2a}{d}\, \Sigma_a v)^{\nu} \right) \right. \quad ; \qquad \mu + \nu \geq 3$$

We used the following abbreviations:

(56) $$k\left(\frac{d}{x_o}\right) = \frac{\frac{x_o}{d} \sinh \frac{d}{x_o}}{\frac{2x_o^{\,2}}{d^2} \left(\cosh \frac{d}{x_o} - 1\right)} - 1 = \frac{d^2}{12\, x_o^{\,2}} + \ldots\ldots$$

$\bar{v}$ is the Maxwellian average velocity of a neutron gas at temperature T;

(57) $$\bar{v} = \left(\frac{8T}{\pi\, m}\right)^{\frac{1}{2}}$$

It is also assumed, that not only $\alpha_o^{\,2}\, x_o^{\,2}$ but also $\alpha_o^{\,2}\, d^2 \ll 1$.

From the investigation of the diffusion coolingeffect in homogeneous assemblies, which is presented in [5], one can argue that at least in the term $\alpha_o^{\,4} \frac{1}{8}$ has to be replaced by $\frac{1}{6}$ if not only the first two terms in (53) but all of them are taken into account.

The discussion of (55) is interesting:
There are three first order terms. The first is the contribution to the decay constant coming from the homogeneous absorption of the moderator, the second is the leakage term $\alpha_o^{\,2}\, D\, \bar{v}$ and the third comes from the homogenized heterogeneous absorption in the plates $\frac{2a}{d}\, \Sigma_a v$. In the second order to which the formalism is given here there are two diffusion cooling effects. The first is the well known

leakage effect $\alpha_o^4 D^2$ due the diffusion of the neutrons, the second refers to the diffusion process into the lumped absorbers. This second diffusion cooling effect disappears if $\frac{d}{x_o} \rightarrow 0$ because this implies that for a neutron the assembly is no longer heterogeneous.

It should be mentioned that the energy dependent Milne problem was investigated for type A spectra by E. Kiefhaber [6] in his master's thesis. Instead of the heavy gas operator the Nelkin kernel was used in (2) and (43). The mathematical procedure was again the expansion into eigenfunctions of the operator L as in (53). But the eigenfunctions are here no longer simply the Laguerre functions. It is necessary to express the eigenfunctions in itself as an expansion into Laguerre functions. The experimental group of K.-H. Beckurts ([12], [13], [14], [15]) did a lot of measurements using pulsed neutrons in light water and it was very useful and successful to have this comparison.

. Reactor spectra in the thermal range, type A and type B

As pointed out in chapter 3, the neutron spectrum is an actual thermal reactor is not purely a type A or type B spectrum and this establishes the mathematical difficulty. In the case of a plate type reactor a solution of the spectrum problem was given by the author of this paper [7]. But it was necessary to restrict the application of the method outlined there to small absorption strengths of the plates, because it was not possible to perform the summation of all Fourier series. However, it was possible to give the asymptotic expansion of the problem in question, which goes parallel to the procedure given in chapter 4 of this paper and there it was possible to perform all the Fourier summations. It is an asymptotic solution only because of the essential singularity at ∞ of the heavy gas operator (26). From here on the way to handle the problem was rather obvious:

Suppose that $\overline{\varnothing}$ is a function with the following features

$$\overline{\varnothing}(\mathcal{E}) = \varnothing_{as}(\mathcal{E}) \qquad \mathcal{E} \geq \mathcal{E}_o$$

$$\overline{\varnothing}(\mathcal{E}) = P(\mathcal{E}) \qquad \mathcal{E} \leq \mathcal{E}_o$$

$\varnothing_{as}(\mathcal{E})$ is the asymptotic expansion valid for large values of $\mathcal{E}$ and $P(\mathcal{E})$ is an arbitray but smooth function which fulfills the boundary condition at $\mathcal{E} = 0$ and fits smoothly into $\varnothing_{as}$ at $\mathcal{E} = \mathcal{E}_o$. Then the function

$$\varnothing_{Diff}(\mathcal{E}) = \varnothing(\mathcal{E}) - \overline{\varnothing}(\mathcal{E})$$

where $\varnothing(\mathcal{E})$ is the solution of the problem, is clearly a function which represents a spectrum of type A. But this means that we can apply for $\varnothing_{Diff}(\mathcal{E})$ an expansion into eigenfunctions of the operator L as given in (53). As long as there is no absorption in the moderator, strong enough to influence the neutron spectra there, all Fourier summations can be performed. And this is true even for the two dimensional case. H. Kunze used this procedure in his master's thesis [8]. The method shall be described here a little bit in more detail. We look into the most simple case of a homogeneous infinite medium. Here we have:

(58) $$\frac{d^2\phi}{d\mathcal{E}^2} + \mathcal{E}\,\frac{d\phi}{d\mathcal{E}} + \phi - p_t \cdot \mathcal{E}^{\frac{1}{2}}\,\phi = 0$$

In the homogeneous case we have to put 2a=d, (14) therefore tends to the following:

(59) $$p_t = \frac{\Sigma_a(\mathcal{E} = 1)}{\xi\,\Sigma_S}$$

We have the following boundary conditions:

(60) $$q(0) = 0$$

$$q(\mathcal{E}) = \text{constant at high energies}$$

In the first step we look into the asymptotic expansion

(61) $$\phi_{as}^{(1)}(\mathcal{E}) = \sum_{v=0}^{N} \frac{g_v}{\mathcal{E}^{1+\frac{v}{2}}}$$

Inserting (61) into (58) we obtain

(62) $$g_{v+2} = \frac{2}{v+2}\left((1+\frac{v}{2})\;(2+\frac{v}{2})\;g_v - p_t\;g_{v+1}\right)$$

that is

$$g_1 = -\,2\,p_t\,g_o \quad ; \qquad g_2 = 2\,(1+p_t^{\,2})\,g_o \quad ; \quad \ldots .$$

Besides of (61) there is a second asymptotic expansion:

(63) $$\phi_{as}^{(2)}(\mathcal{E}) = \mathcal{E}\,e^{-\mathcal{E}}\sum_{v=0}^{N} \frac{g_v}{\mathcal{E}^{\frac{v}{2}}}$$

$\phi_{as}^{(1)}$ and $\phi_{as}^{(2)}$ are the asymptotic expansion of $\phi^{(1)}$ and $\phi^{(2)}$, functions which have logarithmic singularities at $\mathcal{E} = 0$. We look into the linear combination which cancels the logarithmic terms at $\mathcal{E} = 0$

(64) $$\emptyset(\mathcal{E}) = \emptyset^{(1)}(\mathcal{E}) + \lambda\, \emptyset^{(2)}(\mathcal{E})$$

Because of the contribution $\emptyset^{(1)}(\mathcal{E})$ $\emptyset$ does not belong to the Hilbert space of the eigenfunctions of L, (49).

It is possible to expand the wanted, regular function $\emptyset(\mathcal{E})$ at $\mathcal{E} = 0$ into a power series of $\mathcal{E}^{\frac{1}{2}}$.

(65) $$\emptyset = \sum_{v=o}^{\infty} a_v\, \mathcal{E}^{1+\frac{v}{2}}$$

We now consider an auxiliary function $\overline{\emptyset}(\mathcal{E})$ of the following properties:

(66) $$\overset{N}{\overline{\emptyset}}(\mathcal{E}) = \emptyset^{(1)}_{as}(\mathcal{E}) \qquad \text{for} \qquad \mathcal{E} \geq \mathcal{E}_o$$

$$\overset{N}{\overline{\emptyset}}(\mathcal{E}) = a\mathcal{E} + b\,\mathcal{E}^{\frac{3}{2}} + c\,\mathcal{E}^2 + d\,\mathcal{E}^{\frac{5}{2}} \qquad \text{for } \mathcal{E} \leq \mathcal{E}_o$$

The notation $\overset{N}{\overline{\emptyset}}$ indicates that the asymptotic expansion (61) shall be taken up to v=N. The form of $\overset{N}{\overline{\emptyset}}$ for $\mathcal{E} \leq \mathcal{E}_o$ is suggested by (65). a, b, c and d are determined by the conditions that $\overline{\emptyset}$ and its first three derivatives at $\mathcal{E} = \mathcal{E}_o$ shall be continuous.

We now define a quantity

(67) $$S = -\left[\mathcal{E}\,\frac{d^2\,\overset{N}{\overline{\emptyset}}(\mathcal{E})}{d\mathcal{E}^2} + \mathcal{E}\,\frac{d\,\overset{N}{\overline{\emptyset}}(\mathcal{E})}{d\,\mathcal{E}} + \overset{N}{\overline{\emptyset}}(\mathcal{E}) - p_t \cdot \mathcal{E}^{-\frac{1}{2}}\,\overset{N}{\overline{\emptyset}}(\mathcal{E})\right] ;\ \mathcal{E} \leq \mathcal{E}_o$$

For values $\mathcal{E} \geq \mathcal{E}_o$ S has the order $O\left(\frac{1}{\mathcal{E}^{1+\frac{N+1}{2}}}\right)$.

As an approximation to this we define, that

$$S = 0 \qquad \mathcal{E} \geq \mathcal{E}_o \tag{68}$$

We put now:

$$\phi_{Diff} = \phi - \overline{\phi}^{N} \tag{69}$$

Then we obtain

$$\mathcal{E}\,\frac{d^2\,\phi_{Diff}}{d\,\mathcal{E}^2} + \mathcal{E}\,\frac{d\,\phi_{Diff}}{d\,\mathcal{E}} + \phi_{Diff} - p_t \cdot \mathcal{E}^{-\frac{1}{2}}\,\phi_{Diff} = S(\mathcal{E}) \tag{70}$$

The function ϕ_{Diff} now lies in the Hilbert space of the eigenfunctions of the operator L. We therefore put

$$\phi_{Diff} = \sum_{p=o}^{\infty} a_p\,\omega_p(\mathcal{E}) \tag{71}$$

ω_p is given in (49). Using (71) in (70) we obtain

$$-p \cdot a_p - p_t \sum_{q=o}^{\infty} V_{pq}\, a_q = C_p \tag{72}$$

$$C_p = \int_o^{\infty} \omega_p^{+}\, S(\mathcal{E})\, d\mathcal{E} \tag{73}$$

This equation set determines the a_p and we find in the order p_o the result:

$$\phi(\mathcal{E}) = \overline{\phi}^{N}(\mathcal{E}) + \sum_{p=o}^{p_o} a_p\,\omega_p(\mathcal{E}) \tag{74}$$

This method has been applied to the example $p_t = 0.25$.
We have chosen $\mathcal{E}_o = 16$, $N = 10$ and $p_o = 5$.

Fig. 2 shows the result of this calculation here. There is complete agreement with the solution of Hurwitz, Nelkin and Habetler [1]. The figure shows $\mathcal{E}\emptyset(\mathcal{E})$ in its dependence on $\sqrt{\mathcal{E}}$. Furthermore $\mathcal{E}\,\overset{10}{\bar{\emptyset}}(\mathcal{E})$ has been drawn, too. $\overset{N}{\bar{\emptyset}}(\mathcal{E})$ represents qualitativly the neutrons which are slowed down, where as $\Sigma\, a_p\, \omega_p$ represents qualitativly the neutrons being in a sort of equilibrium with the moderator.

The significance of this method outlined here is, that it allows to handle the problem of two dimensional heterogeneous spectra. For this case we use the following model:

A reactor of rectangular shape is considered, for reasons of simplicity it shall be quadratic, that means, the reactor has now N_o^2 rods of diameter 2a and consists of an integer number of Wigner Seitz cells. The absorption in the moderator is again neglected.

In the chapter 4 we saw that it is possible in most of the cases to separate the smooth out process of the fission neutrons into the ground mode from the formation of the flux peaks between the fuel rods at lower energies. We therefore concentrate our attentinn on the thermal and epithermal part of the spectrum and do not care for the slowing down part at higher energies. This is possible because we ask for the shape of the spectra only and we do not ask for a proper criticality conditinn.

Therefore we have the following problem:

(75)

$$x_o^2 \left(\frac{\partial^2\emptyset}{\partial x^2} + \frac{\partial^2\emptyset}{\partial y^2}\right) + \frac{1}{\xi\Sigma_S} L(\emptyset) - p_t \cdot \mathcal{E}^{-\frac{1}{2}} \sum_{k=-N}^{+N} \sum_{l=-N}^{+N} d^2 \delta(x-kd)\,\delta(y-ld) \cdot \emptyset^R(kd, ld, \mathcal{E}) = 0$$

There we have redefined p_t in the following way:

$$p_t = \frac{\Sigma_a(\mathcal{E}=1)\cdot \pi a^2}{\xi\,\Sigma_S\, d^2} \qquad (76)$$

Σ_a is again an effective cross section as described in chapter 2.

Additionally we have

(77) $$\emptyset(\pm 1,y,\mathcal{E}) = 0 \quad ; \quad \emptyset(x,\pm 1,\mathcal{E}) = 0$$

(78) $$q(x,y,0) = 0$$

(79) $$q(x,y,\mathcal{E}) = \text{const} \cdot \cos \alpha_o x \cdot \cos \alpha_o y \text{ for large values of } \mathcal{E}$$

In (75) $\emptyset^R(kd,ld, \ )$ is not the flux at x=kd, y=kd which would be the analog to the one dimensional case. This analog does not work because there is a logarithmic singularity of the flux at x=kd, y=kd. Therefore $\emptyset^R$ is the flux average over the surface of the rod approximated by the following expression:

(80)

$$\emptyset^R(kd,ld,\mathcal{E}) = \frac{\emptyset(kd,ld+a,\mathcal{E}) + \emptyset(kd,ld,-a,\mathcal{E}) + \emptyset(kd+a,ld,\mathcal{E}) + \emptyset(kd-a,ld,\mathcal{E})}{4}$$

We now make again a Fourier ansatz. For the same symmetry reasons as in chapter 2 we have a frequency selection. Making use of it we have

(81) $$\emptyset(x,y,\mathcal{E}) = \sum_{m=-\infty}^{+\infty} \sum_{r=-\infty}^{+\infty} \varphi_{N_or,\, N_om}(\mathcal{E}) \cdot \cos \alpha_{N_or}\, x \cdot \cos \alpha_{N_om}\, y$$

Inserting (81) in (75) and using (80) we obtain the following:

(82)

$$\mathcal{E} \cdot \frac{d^2\varphi_{N_or,\, N_os}}{d\,\mathcal{E}^2} + \mathcal{E}\frac{d\varphi_{N_or,\, N_os}}{d\,\mathcal{E}} + (1- (\alpha_{N_or}^2+\alpha_{N_os}^2)\, x_o^2)\varphi_{N_or,\, N_os} -$$

$$-p_t \cdot \mathcal{E}^{-\frac{1}{2}} \sum_{m,t=-\infty}^{+\infty} \varphi_{N_om,\, N_ot} \cdot \cos^{R_o}(\alpha_{N_om}\, x) \cdot \cos^{R_o}(\alpha_{N_ot}\, y) = 0$$

The index R_o of the cos in the sum means that the average is to be taken at the central fuel element [2]. The reduction of taking the average at the central fuel element only is a consequence of the special symmetry of the reactor considered here.

We now apply the same calculus to (82) as in the homogeneous case and obtain the result in the following form:

(83)

$$\phi(x,y,\varepsilon) = \sum_{r,s=-\infty}^{+\infty} \left\{ \bar{\varphi}^{N}_{N_o r,\, N_o s}(\varepsilon) + \sum_{p=o}^{p_o} a_{N_o r,\, N_o s,\, p} \cdot \omega_p(\varepsilon) \right\} \cos \alpha_{N_o r} x \cdot \cos \alpha_{N_o s}\, y$$

$$\bar{\varphi} = \varphi_{as} \quad , \qquad \varepsilon > \varepsilon_o$$

$$\bar{\varphi} = \text{Polyn.}, \qquad 0 < \varepsilon < \varepsilon_o$$

All the coefficients of the above Fourier series can be written as linear combinations of terms of the form

(84) $$\frac{1}{\gamma + (\alpha_{N_o r}^2 + \alpha_{N_o s}^2)\, x_o^2}$$

Thus evaluating (83) in order to obtain the final result a Fourier series of the type

(85) $$\sum_{r,s=-\infty}^{+\infty} \frac{\cos \alpha_{N_o r}\, x \cdot \cos \alpha_{N_o s}\, y}{\gamma + (\alpha_{N_o r}^2 + \alpha_{N_o s}^2)\, x_o^2}$$

has to be evaluated.

But more than that: to establish the asymptotic solution analogous to (61) and (35) and the equation system for the $a_{N_o r,\, N_o s,\, p}$ analogous to (72) made it already necessary to evaluate series of the type (85). This was done by K. Meetz [2]. One summation can be carried out by means of the calculus of residues, see (40). The remaining single series converges rather rapidly inside the moderator, but its convergence is very poor at the boundary of the fuel rods, because of the logarithmic singularities of the flux in the lattice points of the reactor. But using a known Fourier series with the same logarithmic singularities and looking for the difference between (85) and these known series one can improve the convergence considerably and therefore it is possible to evaluate (85). For details see either [2] or [8].

The method explained here has been applied to twelve types of reactors in the limit $N_0=\infty$. Besides the neutron spectra themselves the values of the η factor and the thermal utilization have been calculated. In determining the latter quantity a $\frac{1}{v}$ -absorption law in the moderator has been assumed. The mean values of the absorption and fission cross sections of the fuel elements were taken in the energy interval $0 \leq \varepsilon \leq 16$; in the moderator the spatial average of the absorption cross section has been calculated, too, by means of the spectrum determined neglecting the neutron absorption in the moderator.

The parameter of the twelve reactors considered here are given in table 1.

Table 1

Reactor	Moderator		d [cm]	a [cm]	p %	ρ [g/cm^3]	p_t	x_o/d	p_t^M
I	D_2O	(40°C)	10	1.6	0.7115	18.4	0.1046	0.186	$3.33 \cdot 10^{-4}$
II		"	16.8	"	"	"	0.03705	0.1108	"
III		"	20	"	"	"	0.02614	0.093	"
IV		"	10	"	2	"	0.2274	0.186	"
V		"	16.8	"	"	"	0.08058	0.1108	"
VI		"	20	"	"	"	0.05686	0.093	"
VII	Graphite		10	1.2	0.7115	"	0.1791	0.3925	$4.34 \cdot 10^{-4}$
VIII	"	(300°C)	20	"	"	"	0.04479	0.1962	"
IX		"	30	"	"	"	0.01991	0.1309	"
X		"	10	"	2	10	0.212	0.3925	"
XI		"	20	"	"	"	0.05299	0.1962	"
XII		"	30	"	"	"	0.02356	0.1309	"

$a, d, p_t \frac{x_o}{d}$ have the meaning defined in the text above, $p_t^M = \frac{\Sigma_a^M(\mathcal{E} = 1)}{2\mu\, \Sigma_S^o}$ is the absorption parameter of the moderator, ρ is the density of the fuel. In the calculation of the f-factor the fuel was assumed to consist of p % U^{235} and (100-p)% U^{238}.

The values p = 0.7115; ρ = 18.4 g/cm^3 correspond to those of natural uranium; p = 2 and ρ = 10 g/cm^3 are the data of uranium oxide enriched up to 2% U^{235}. The macroscopic absorption cross section of natural uranium at the moderator temperature of 40°C was assumed to be $\Sigma_a(\mathcal{E} = 1) = 0.3467$ cm^{-1}. At a fuel enrichment of 2% we have Σ_a $(\mathcal{E} = 1) = 0.7541$ cm^{-1} for metallic uranium at 40°C moderator temperature and $\Sigma_a(\mathcal{E} = 1) = 0.3034$ cm^{-1} for enriched uranium oxide at 300°C moderator temperature. As absorption cross section in heavy water has been used

$$\Sigma_a^M(\mathcal{E} = 1) = 7.736 \cdot 10^{-5} \text{ cm}^{-1} \text{ at } 40^\circ\text{C and graphite of } 300^\circ\text{C}$$

$$\Sigma_a^M(\mathcal{E} = 1) = 2.58 \cdot 10^{-4} \text{ cm}^{-1}$$

Finally the Sachs-Teller-mass has been used for heavy water.

Figures 3 - 14 show the neutron spectra in the reactor I - XII. The neutron flux ϕ is plotted versus the energy $\mathcal{E}$ in double logarithmic scale. There is drawn the flux $\phi^{R_o}(\mathcal{E})$ at the boundary of the fuel rods and the spectrum

$\phi(\frac{d}{2}, \frac{d}{2}, \mathcal{E})$ in the middle of the moderator, respectively.

For comparison a Maxwellian distribution has been fitted to the spectrum in the moderator at low energies. In the low energy range all the spectra have Maxwellian character, the maximum being usually shifted. One recognizes clearly the change from the Maxwellian part of the spectrum to the characteristic $\frac{1}{\mathcal{E}}$ slowing down spectrum. A decrease of the lattice parameter d results in a simultaneous, almost equal increase of the neutron temperature in fuel and moderator. The neutron temperature is here defined as the temperature of a Maxwellian distribution fitted to the given spectrum in its maximum. An increase of the absorption cross section of the fuel at unchanged geometry to the contrary has as consequence a displacement of the maximum of the spectrum in the fuel, but almost none in the moderator.

Figures 3 - 14 show that the spectra in fuel and moderator are far from being identical in the epithermal region. In an infinite reactor ($N_0 = \infty$) they tend to meet asymptotically, of course. But this becomes true only at very high energies. H.C. Honeck [9] made in the numerical treatment of the problem in the Wigner Seitz cell the assumption, that the spectra in moderator and fuel are identical above $\varepsilon = 9$. This assumption was necessary because of the limited machine capacity. It seems to us to become dubious at least for strong absorption. According to our experience a small inaccuracy in the calculation of the high energy part of the spectrum results in serious deviations from the rigorous solution in the thermal energy range.

Table 2

Reactor	f	$f^{1\,gr}_{ws}$	$\bar{\sigma}^5_f$ [barn]	$\bar{\sigma}^5_a$	$\bar{\sigma}^8_a$	η
I	0.99648	0.99606	415.96	491.09	2.037	1.30965
II	0.98750	0.98655	446.82	526.56	2.171	1.31500
III	0.98125	0.97996	452.11	532.65	2.194	1.31585
IV	0.99807	0.99764	373.24	441.24	1.853	1.70659
V	0.99217	0.99117	422.12	497.82	2.067	1.71586
VI	0.98786	0.98649	430.90	507.89	2.105	1.71731
VII	0.97453	0.97215	271.48	322.74	1.427	1.27012
VIII	0.89185	0.88192	311.89	370.06	1.606	1.28157
IX	0.77493	0.75494	320.73	380.41	1.645	1.28373
X	0.97801	0.97554	263.40	313.27	1.392	1.68171
XI	0.90395	0.89361	307.96	365.49	1.589	1.69160
XII	0.79586	0.77458	317.20	377.20	1.634	1.69351

Maxwellian distr. at moderator-temperature

Reactor	$\bar{\sigma}^5_f$	$\bar{\sigma}^5_a$	$\bar{\sigma}^8_a$	η
I	481.44	566.94	2.316	1.32037
II	"	"	"	"
III	"	"	"	"
IV	"	"	"	1.72304
V	"	"	"	"
VI	"	"	"	"
VII	341.01	403.86	1.730	1.29026
VIII	"	"	"	"
IX	"	"	"	"
X	"	"	"	1.69962
XI	"	"	"	"
XII	"	"	"	"

Maxwellian distr. with displaced temperature

Reactor	T^U_n [°C]	$\bar{\sigma}^5_f$	$\bar{\sigma}^5_a$	$\bar{\sigma}^8_a$	η
I	74	453.78	534.45	2.203	1.31578
II	65	460.74	542.62	2.231	1.31700
III	56	468.11	551.26	2.261	1.31823
IV	134	414.12	488.16	2.040	1.71469
V	109	429.69	506.29	2.104	1.71703
VI	105	433.59	510.85	2.120	1.71757
VII	443	301.10	358.68	1.556	1.27680
VIII	372	319.22	379.16	1.636	1.28306
IX	357	323.21	383.67	1.653	1.28440
X	472	294.68	351.42	1.527	1.68371
XI	386	315.32	374.74	1.619	1.69121
XII	346	326.56	387.47	1.668	1.69502

Table 2 gives the values of the thermal utilization f calculated for our twelve reactors I - XII. For comparison these quantities have been determined in a one group diffusion calculation for the corresponding Wigner Seitz cells.

An estimate of the improvement in accuracy to be expected from a rigorous consideration of the energy dependence of the neutron density can be obtained from a comparison of the f-factor values in table 2. The difference between these quantities, determined by our method and the one group approximation, respectively, is negligeably small for D_2O-moderated reactors with natural uranium fuel elements (this is true only for a $\frac{1}{v}$ -absorption law). But this is not the case in reactors with enriched fuel elements, especially in graphite moderated reactors. In any case the accuracy of the monoenergetic, transport-theoretical calculation can be improved essentially, if the absorption cross sections used are averaged over the neutron spectra determined in the diffusion approximation.

In table 2 the values of the η factor determined by means of the neutron spectra shown in figures 3 - 14 and those calculated from Maxwellian distribution at moderator temperature and the temperature T_n^U of the neutrons at the boundary of the fuel rods are also given for comparison. Finally the mean values of the fission and absorption cross section of U^{238} are given as obtained from the three types of spectra just specified. As expected the mean values of these cross sections, obtained from Maxwellian distributions, are too large, because these functions decrease so rapidly, that the epithermal cross section values have no influence on the mean value. This has consequence, that the η factors determined with Maxwellian distributions are too large.

7. The representation of reactor spectra (type A and type B) by simple sets of functions

The method outlined in chapter 6 made use of the concept of an auxiliary function $\bar{\phi}^N$ which represents the reactor spectra at high energies. In the case of chapter 6 a special energy $\mathcal{E}_o$ was chosen where the asymptotic solution and a polynomial expression (66) were tied togehter. This has the taste of being artificial and reminds of the concept of a "cut off" although it has nothing to do with it.

There are several possibilities to improve this procedure. One possibility is the following:
Suppose that in (74) one wants to consider an approximation of the degree p_o. Then for $\bar{\phi}^N$ the following unique representation is possible

$$(86) \qquad \bar{\phi}^N \rightarrow \bar{\phi}^{p_o} = \mathcal{E}^{\alpha_o^2 x_o^2 - 1} \left[\sum_{v=o}^{p_o+1} a_v \, \mathcal{E}^{-\frac{v}{2}} \right] e^{-\frac{\gamma}{\mathcal{E}}}$$

γ is chosen arbitrarily and it comes out that for example $\gamma=3$ is, for heavy gas model spectra, a reasonable choice in a sense to be described below. The a_v are now chosen in such a way, that for large values of $\mathcal{E}$ the asymptotic representation up to the $p_o + 1$ degree is fitted. For small values of $\mathcal{E}$ the function ϕ_{p_o} tends strongly toward zero. If one applies analog to (67) the full differential operator to it, the source term S has the following order:

$$(87) \qquad S(\mathcal{E}) = O\left(\frac{1}{\mathcal{E}^{-\alpha+1+\frac{p_o+2}{2}}} e^{-\frac{\gamma}{\mathcal{E}}} \right)$$

Because of (87) the first p_o+1 moments C_p are convergent:

$$(88) \qquad C_p = \int_o^\infty S(\mathcal{E}) \cdot \omega_p^+(\mathcal{E}) \, d\mathcal{E} = \text{finite for } p \leq p_o$$

Now $\gamma=3$ makes the first p_o+1 values C_p not too large, the source term S is still behaving smoothly. Recent investigations have shown, that $p_o=1$, that is two terms $\omega_p(\mathcal{E})$ in (74), is already a good way to represent reactor spectra with an error of about 5% provided that not highly enriched fuels are used. Now, two terms in (74) and three terms in (86) give the chance to do all calculations fully analytically, so that there is a way to make the whole question of reactor spectra an easy thing again. But it should be emphasized that there are also other ways to keep the function $\overline{\phi}^N$ simple and unique.

8. Iterated multigroup method

A completely different approach to the problem in question has been given by K. Meetz, K. Ott and S. Sanatani [10]. It does not claim, however, to be a systematic method, because it is partially based on intuitive arguments.

Let us recall that we have, roughly speaking, two categories of problems concerned with either the spatial distribution or the energy spectrum of neutrons in a heterogeneous assembly. In an analysis of the spatial distribution in the moderator one might well use diffusion theory, if the scattering mean free path is small compared with the fuel rod distance d. Introduction of point singularities instead of the boundary conditions at the fuel surface makes the solution of the diffusion problem in a regular lattice an easy task. This has been outlined in detail in Sect. II for a one-dimensional reactor model. The knowledge of the flux on the fuel rod surface is also sufficient for a good estimation fo fuel reaction rates, if the absorption length of the fuel is large compared with the rod diameter. Hence, it is reasonable to keep the singularity method for the spatial distribution in any approach to the spectral problem.

There a similarly simple and satisfactory entry does not seem to exist. The use of eigenfunctions of the monatomic gas scattering kernel has its difficulties, as we have seen in the previous section. It may, therefore, be worthwhile to try a multigroup method. It was felt, however, by the authors of [10] that one should improve the quality of such a method by incorporating an iteration procedure.

Let us briefly outline the way this has been done in [10] for the case of an infinite homogeneous medium. The basic idea is to use the n flux mean values $\bar{\phi}_i$, obtained from the solution of the multigroup equations for a n group theory, for an improvement of the flux distribution that has been used in calculating the group constants. To do this in a systematic way, one may choose a set of spectral functions $\phi_i(a_1^{(i)} \dots a_{n_i}^{(i)}; \mathcal{E})$, which describe the flux in the i-th interval and depend on arbitrary constants $a_1 \dots a_{n_i}$ besides the energy $\mathcal{E}$: ($\mathcal{E} = E/KT$). The choice of the ϕ_i is determined by physical arguments. For groups in the slowing down region one may use the asymptotic expansion

(89) $$\phi(\mathcal{E}) = \frac{a_1}{\mathcal{E}} + \frac{a_2}{\mathcal{E}^{3/2}} + \frac{a_3}{\mathcal{E}^2} + \ldots$$

while in the thermal region the shape is Maxwellian:

(90) $$\phi(\mathcal{E}) = \mathcal{E}e^{-\mathcal{E}}(a_1' + a_2'\mathcal{E} + a_3'\mathcal{E}^2 + \ldots)$$

In the epithermal region a combination of (89) and (90) may be taken as the best description. Now the constants $a_{..}^{(i)}$ are objects of the iteration:
In zero order one starts with a convenient set of constants ${}^0a_{..}^{(i)}$ for the calculation of zero order group constants. The resulting first order mean values $\bar{\phi}_i^{(1)}$ are then required from the functions $\phi_i(..{}^1a_{...}^{(i)}:, \mathcal{E})$ in combination with a sufficient number of continuity conditions at the group boundaries. First order constants ${}^1a_{..}^{(i)}$ are obtained by solving the corresponding linear equations and first order group constants from the functions $\phi_i(\ldots{}^1a_{....}^{(i)}.., \mathcal{E})$ etc.

The results of this procedure have been compared with the numerical solution of Hurwitz, Nelkin and Habetler [1] for the heavy gas model in heavy water. The agreement is quite good for both values of the absorption parameter $4\,p_t = \Delta = 4\,\Sigma_a/\xi\Sigma_s(\Delta = 0{,}1; \Delta = 1)$. However, there is a characteristic difference, namely a minimum in $\mathcal{E}\,\phi(\mathcal{E})$ in the epithermal region just above the Maxwell peak. This is probably due to the different scattering kernels used: heavy gas approximation in [1] and Wigner-Wilkins kernel in [10] . It may be mentioned that Corngold's correction of the heavy gas model [11] points in the same direction.

There is no difficulty to combine the multigroup method with the singularity approach for the spatial distribution. As an example, the one dimensional model described in Sect. II has been studied in [10] . As has been mentioned in Sect. II the flux spectra on the fuel boundaries and in the center between the fuel plates are independent of the cell position. It is reasonable to take the spectra at these positions as representative for the neutron spectrum in a heterogeneous assembly. This has the further advantage that the corresponding multigroup constants are likewise independent of the cell position. Spectra calculated this way are in very good agreement with those obtained by Kunze [8] in his more systematic but more tedious approach.

Due to its extreme simplicity and the satisfactory results the iterated multi-group method promises to be useful for practical calculations, although it is certainly unsatisfactory from a more systematic point of view.

9. Final remark

This summary presents the work of the theoretical group of Karlsruhe on the subject of neutron spectra. The goal was to investigate how strong the influence of the heterogeneity is and to predict the thermal and epithermal part of reactor spectra in order to obtain by this properly weighted thermal neutron cross sections.

One successful first application of these spectra was the calculation of the critical experiment of the Karlsruhe reactor FR 2. It came out that 54 fuel elements were predicted and criticality has reached with 52 elements.

It should be mentioned that it is felt that these procedure are not restricted to either the heavy gas model or the diffusion theory. These simple models were chosen only to develope in the most simple cases the general methods.

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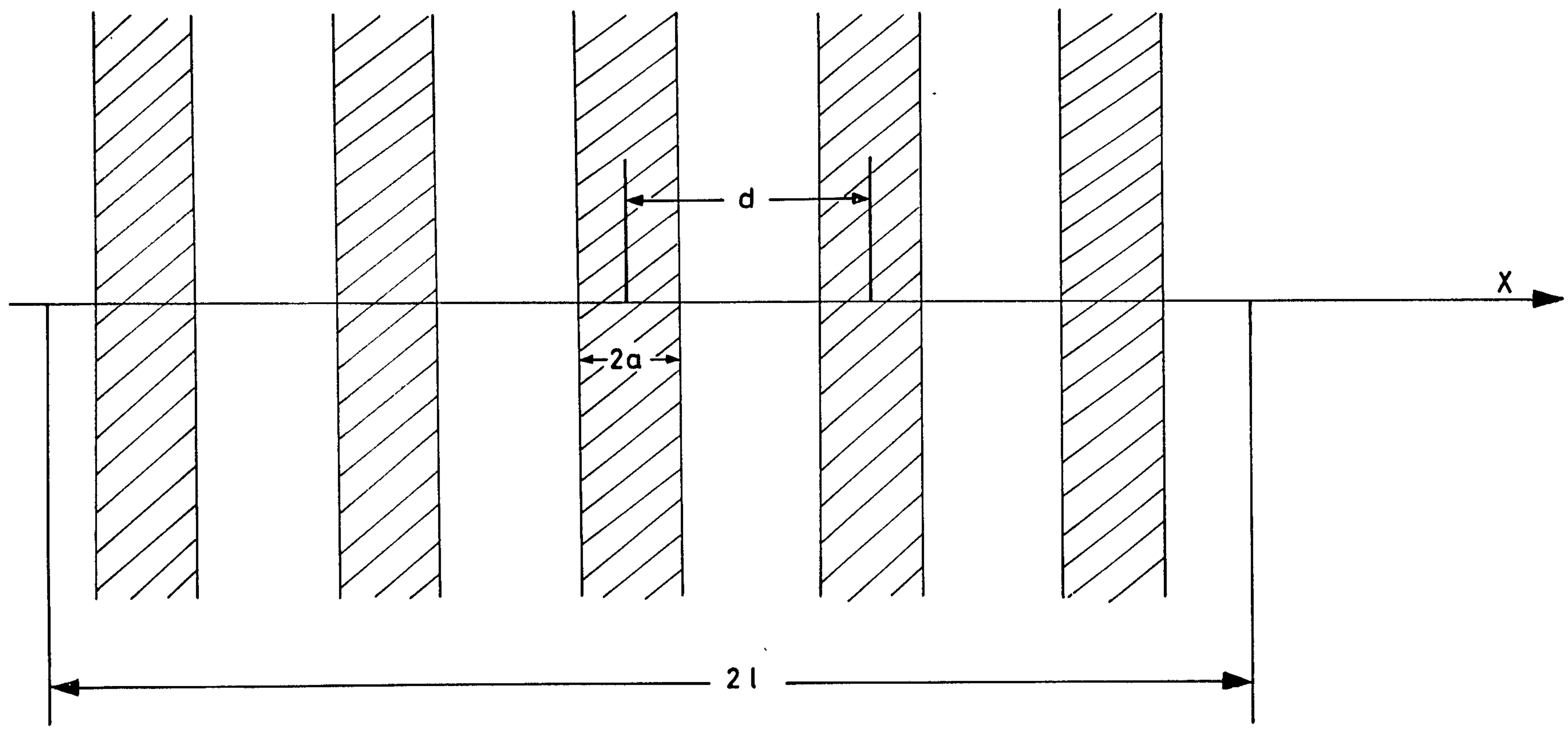
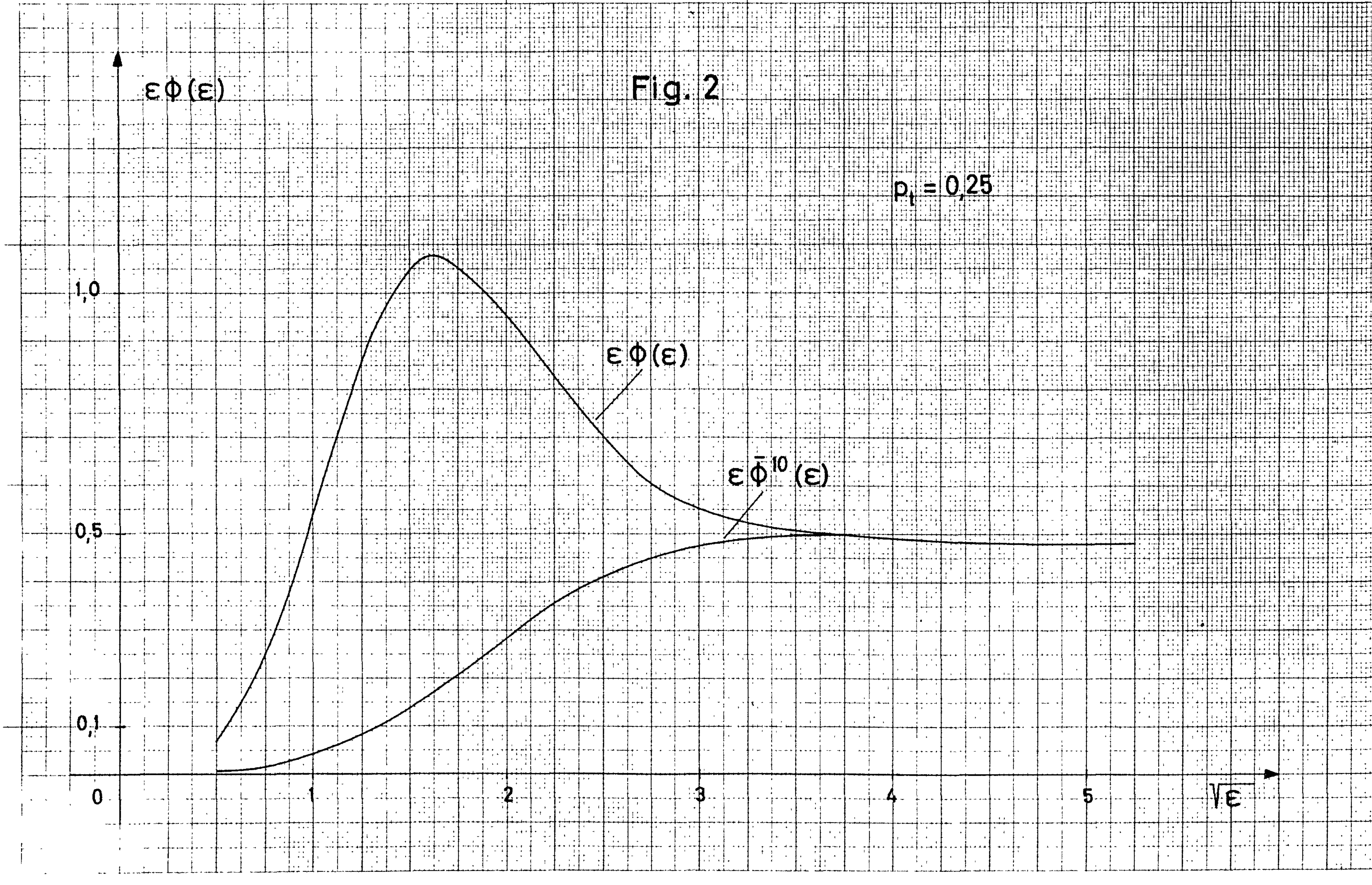


Fig. 1

Fig. 2



$\varepsilon\phi(\varepsilon)$
$p_t = 0,25$
1,0
0,5
0,1
0
1
2
3
4
5
$\sqrt{\varepsilon}$
$\varepsilon\phi(\varepsilon)$
$\varepsilon\bar{\phi}^{10}(\varepsilon)$

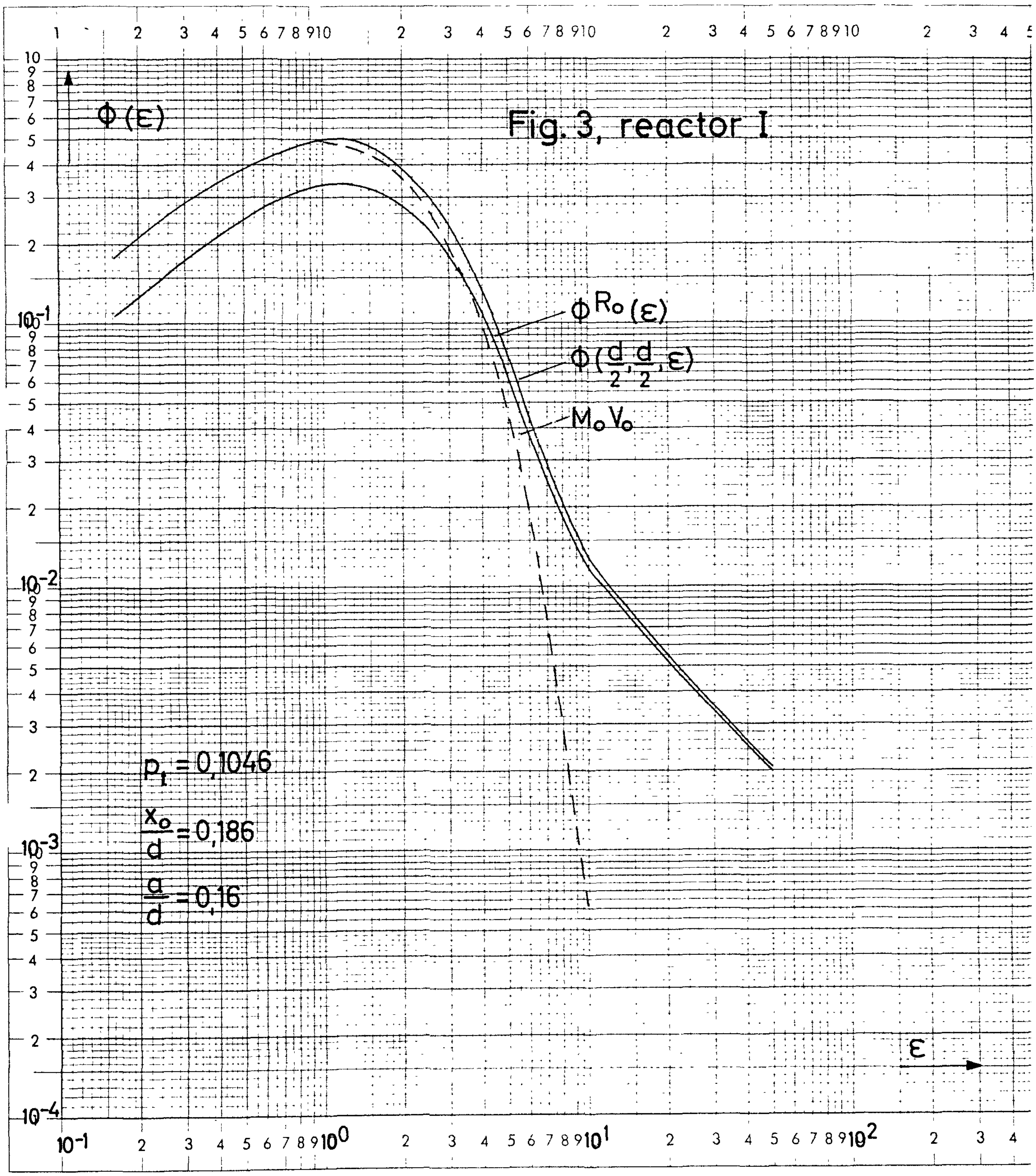


Fig. 3, reactor I
$\Phi(\varepsilon)$
$\Phi^{R_0}(\varepsilon)$
$\Phi(\frac{d}{2},\frac{d}{2},\varepsilon)$
$M_0 V_0$
$p_1 = 0,1046$
$\frac{x_0}{d} = 0,186$
$\frac{a}{d} = 0,16$
ε
10^{-1}
10^{0}
10^{1}
10^{2}
10^{-2}
10^{-3}
10^{-4}

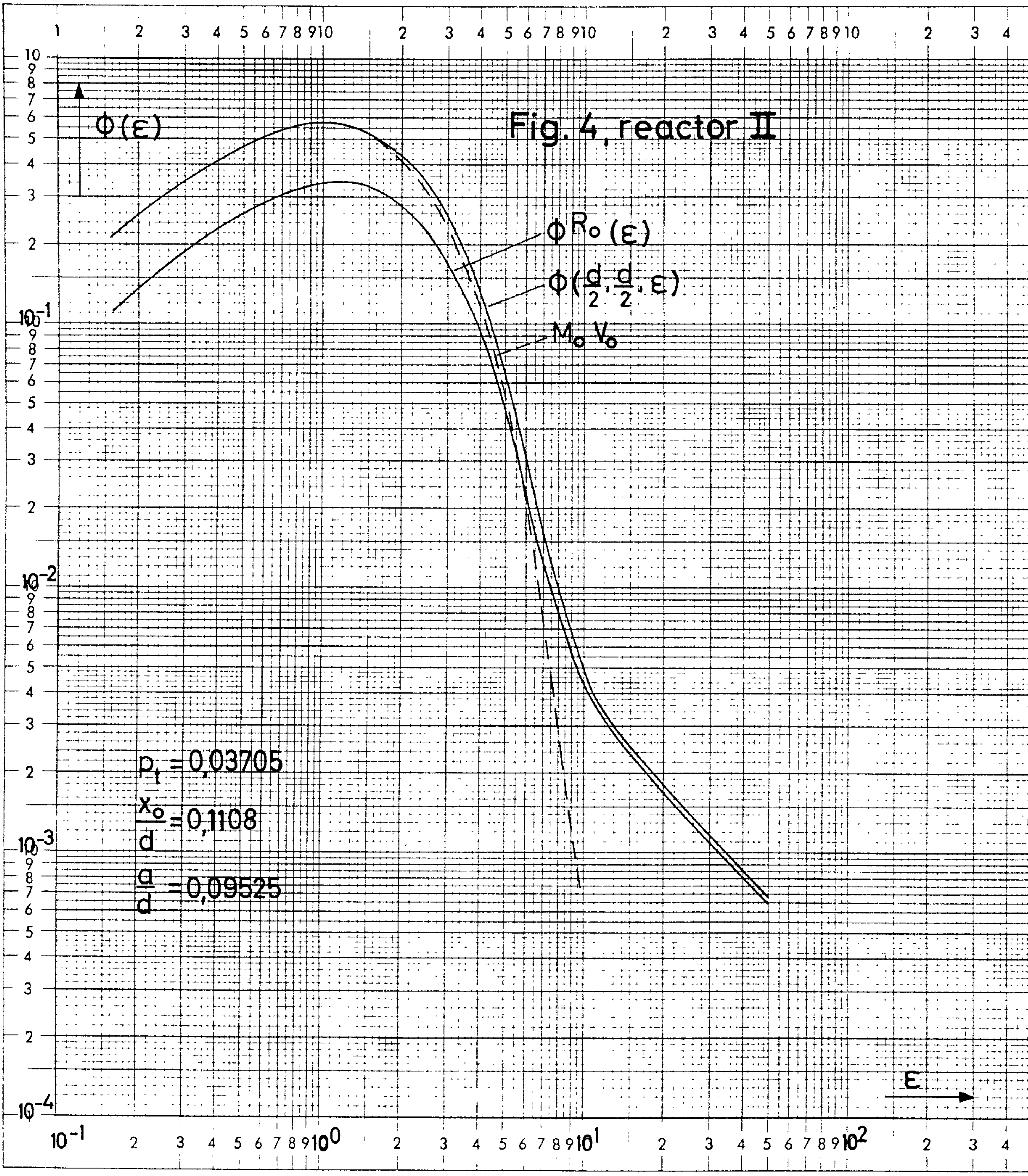
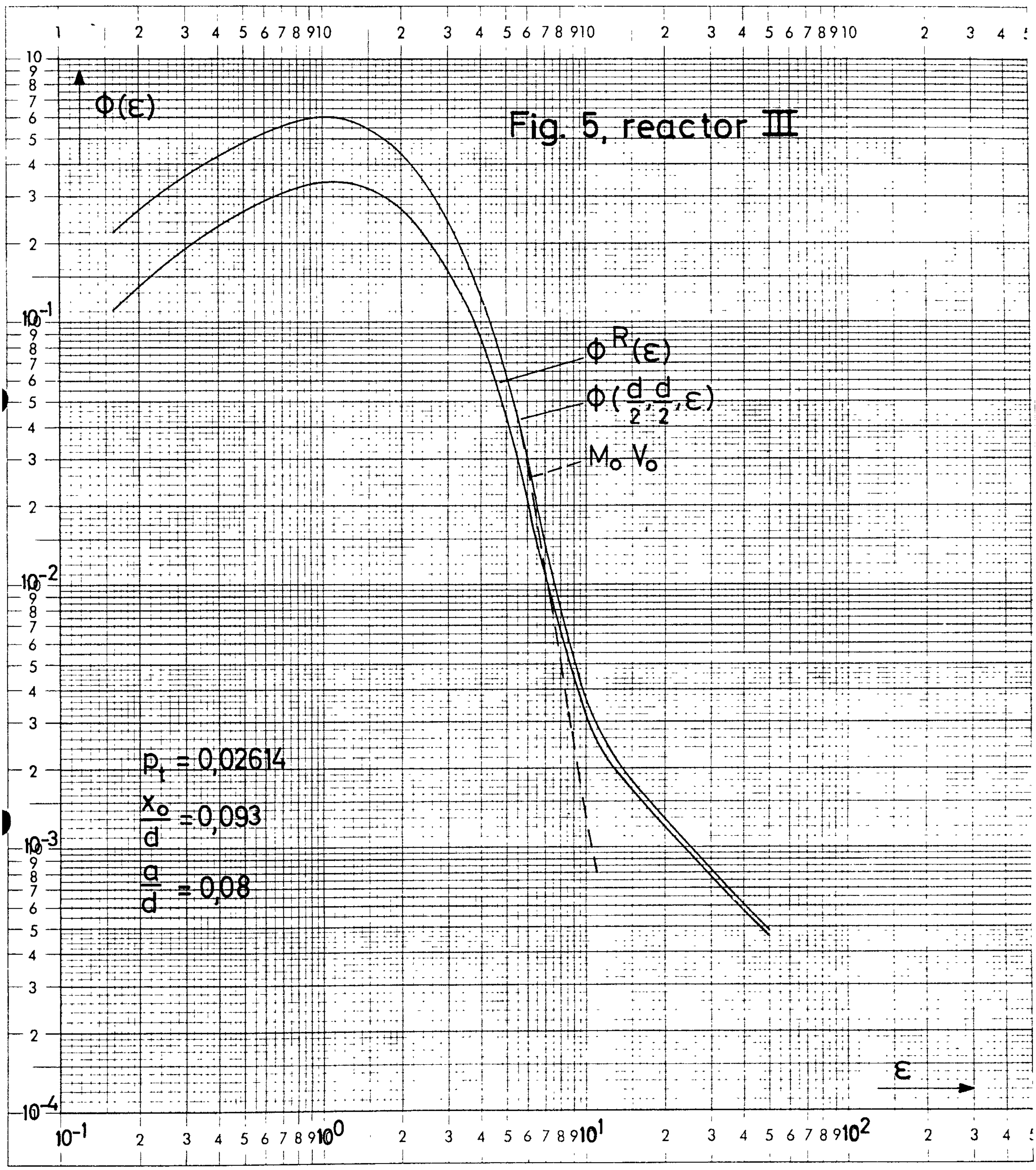


Fig. 4, reactor II
$\phi(\varepsilon)$
$\phi^{R_o}(\varepsilon)$
$\phi(\frac{d}{2},\frac{d}{2},\varepsilon)$
$M_o V_o$
$p_t = 0{,}03705$
$\frac{x_o}{d} = 0{,}1108$
$\frac{a}{d} = 0{,}09525$
ε
10^{-1}
10^{0}
10^{1}
10^{2}
10^{-2}
10^{-3}
10^{-4}



Φ(ε)
Fig. 5, reactor III
$\Phi^R(\varepsilon)$
$\Phi(\frac{d}{2},\frac{d}{2},\varepsilon)$
$M_o V_o$
$p_t = 0,02614$
$\frac{x_o}{d} = 0,093$
$\frac{a}{d} = 0,08$
ε

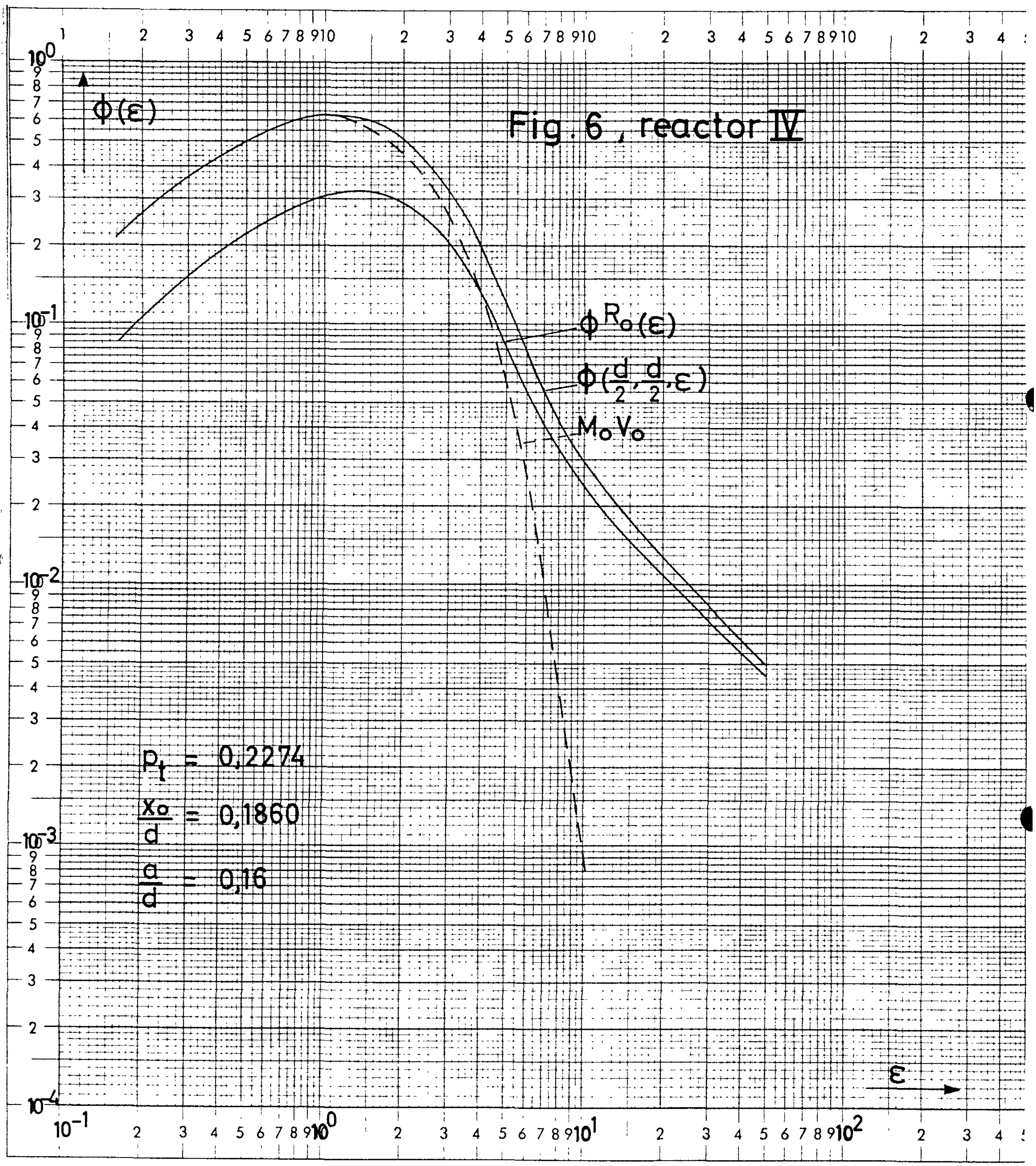


Fig. 6, reactor IV
$\phi(\varepsilon)$
$\phi^{R_o}(\varepsilon)$
$\phi(\frac{d}{2},\frac{d}{2},\varepsilon)$
M_oV_o
p_t = 0,2274
$\frac{x_o}{d}$ = 0,1860
$\frac{a}{d}$ = 0,16
ε

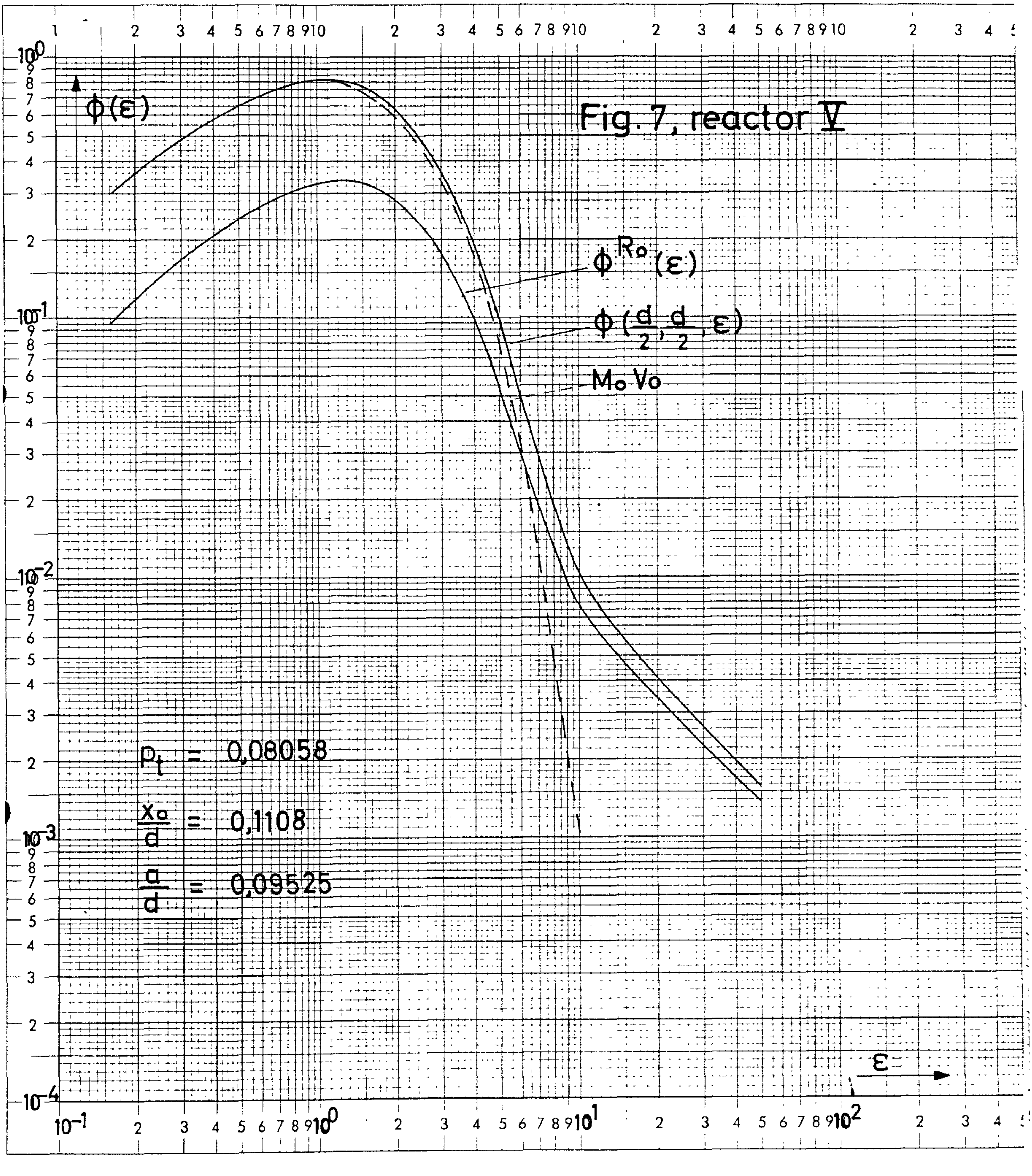


Fig. 7, reactor V
φ(ε)
φ^{Ro}(ε)
φ(d/2, d/2, ε)
MoVo
p_t = 0,08058
x_o/d = 0,1108
a/d = 0,09525
ε
10^0
10^-1
10^-2
10^-3
10^-4
10^0
10^1
10^2

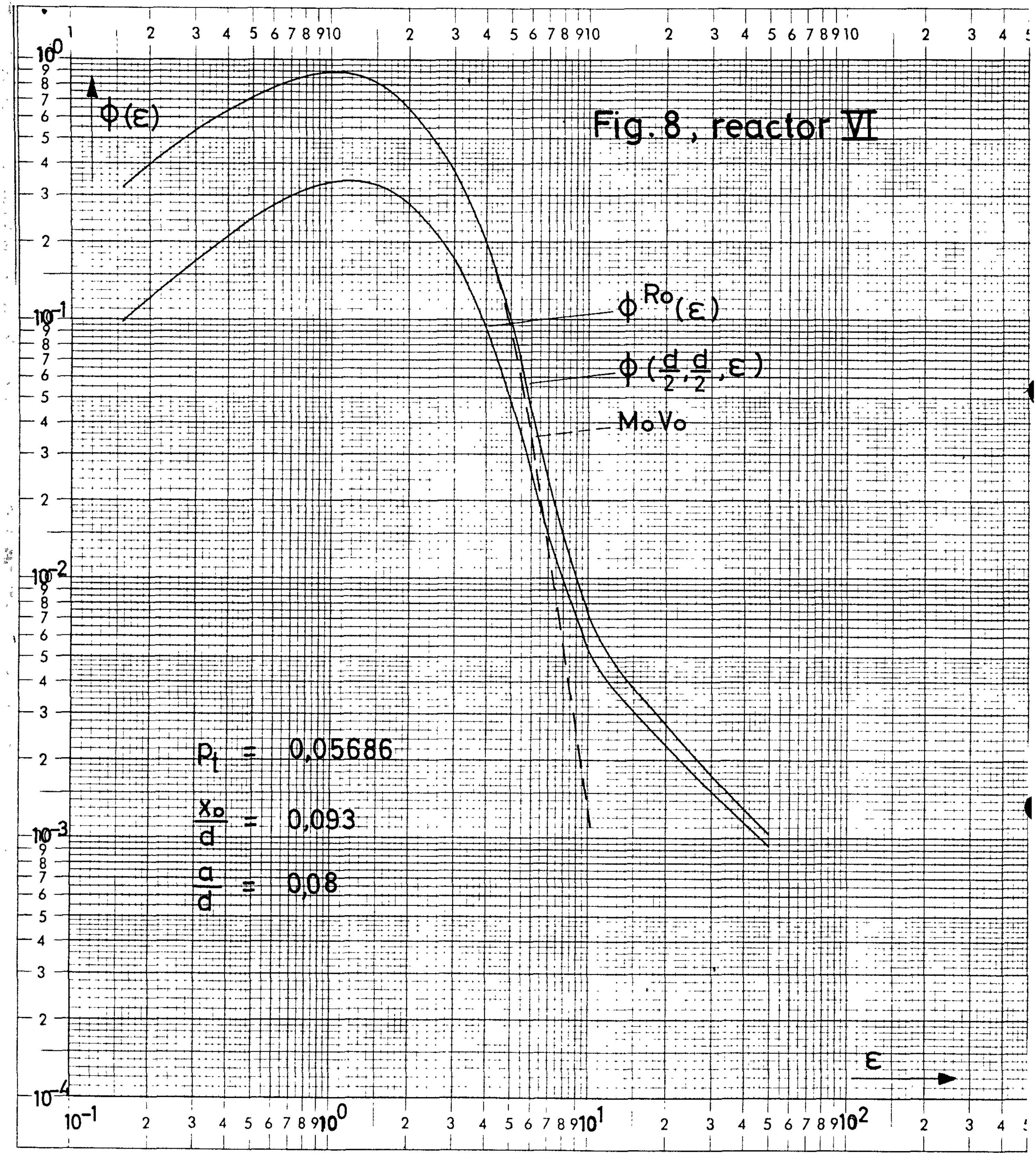


Fig. 8, reactor VI
$\phi(\varepsilon)$
$\phi^{Ro}(\varepsilon)$
$\phi(\frac{d}{2},\frac{d}{2},\varepsilon)$
MoVo
p_t = 0,05686
$\frac{x_o}{d}$ = 0,093
$\frac{a}{d}$ = 0,08
ε

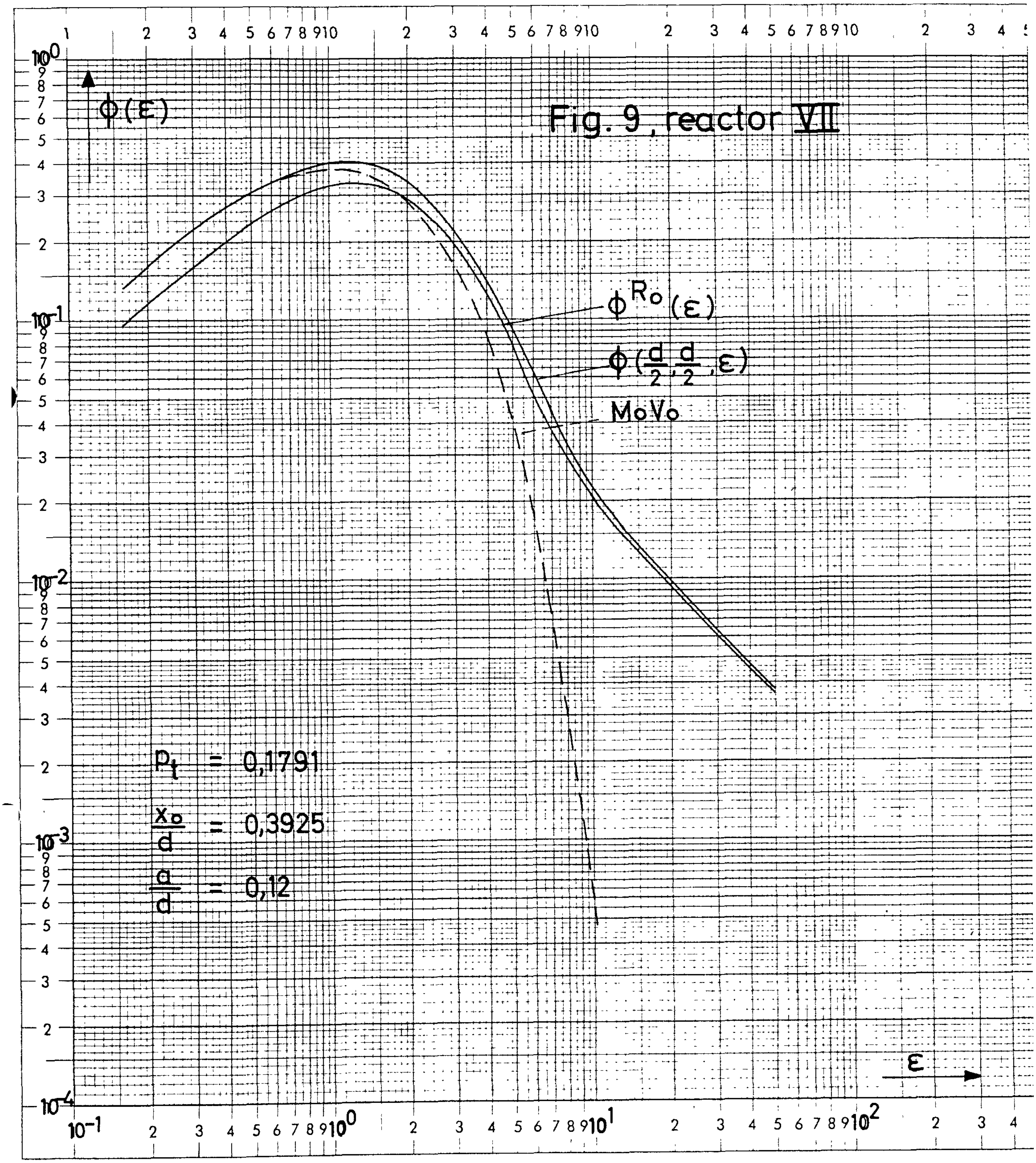


Fig. 9, reactor VII
$\phi(\varepsilon)$
$\phi^{R_0}(\varepsilon)$
$\phi(\frac{d}{2},\frac{d}{2},\varepsilon)$
MoVo
p_t = 0,1791
$\frac{x_0}{d}$ = 0,3925
$\frac{a}{d}$ = 0,12
ε

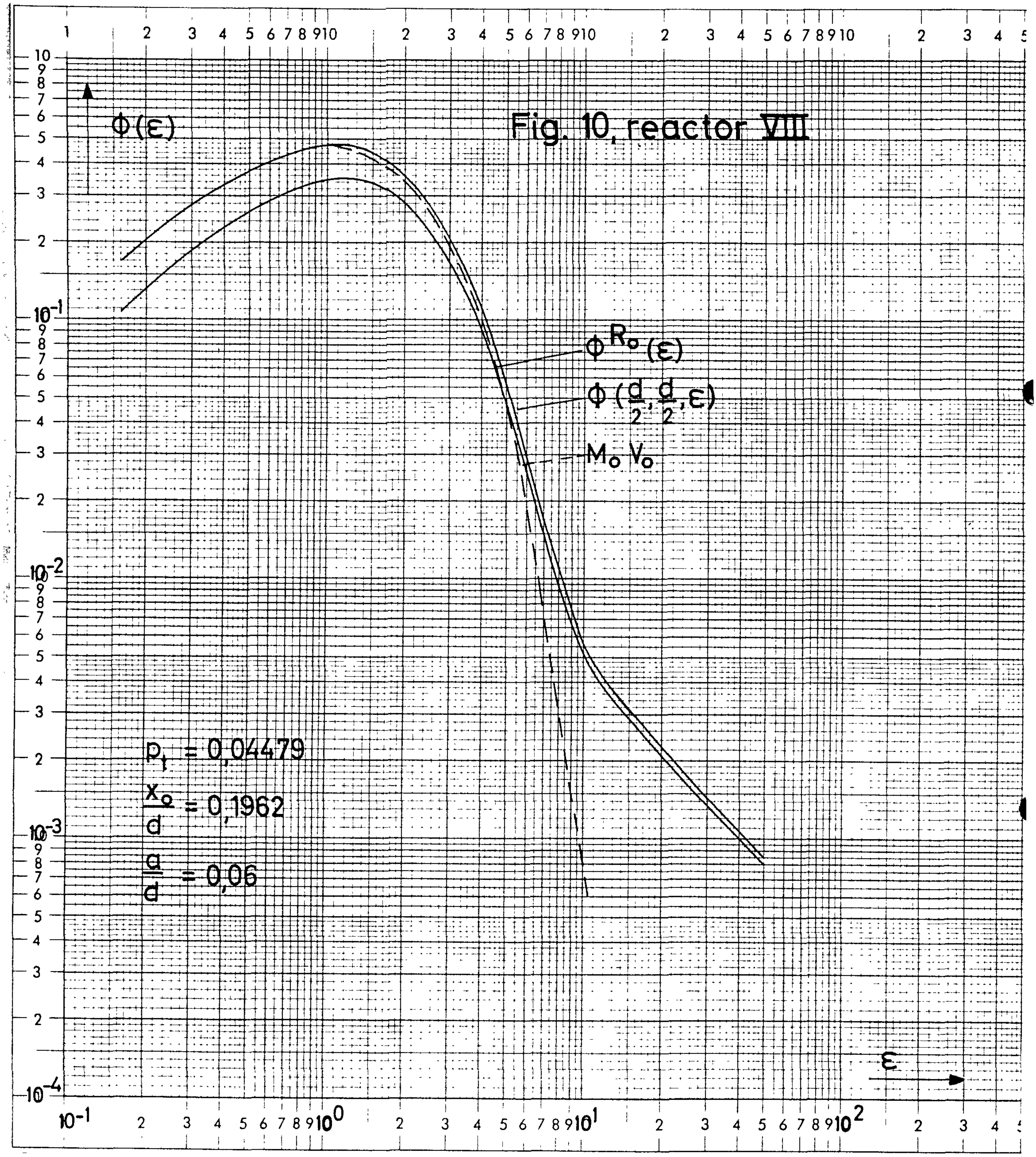


Fig. 10, reactor VIII
$\Phi(\varepsilon)$
$\Phi^{R_o}(\varepsilon)$
$\Phi(\frac{d}{2},\frac{d}{2},\varepsilon)$
$M_o V_o$
$p_1 = 0{,}04479$
$\frac{x_o}{d} = 0{,}1962$
$\frac{a}{d} = 0{,}06$
ε

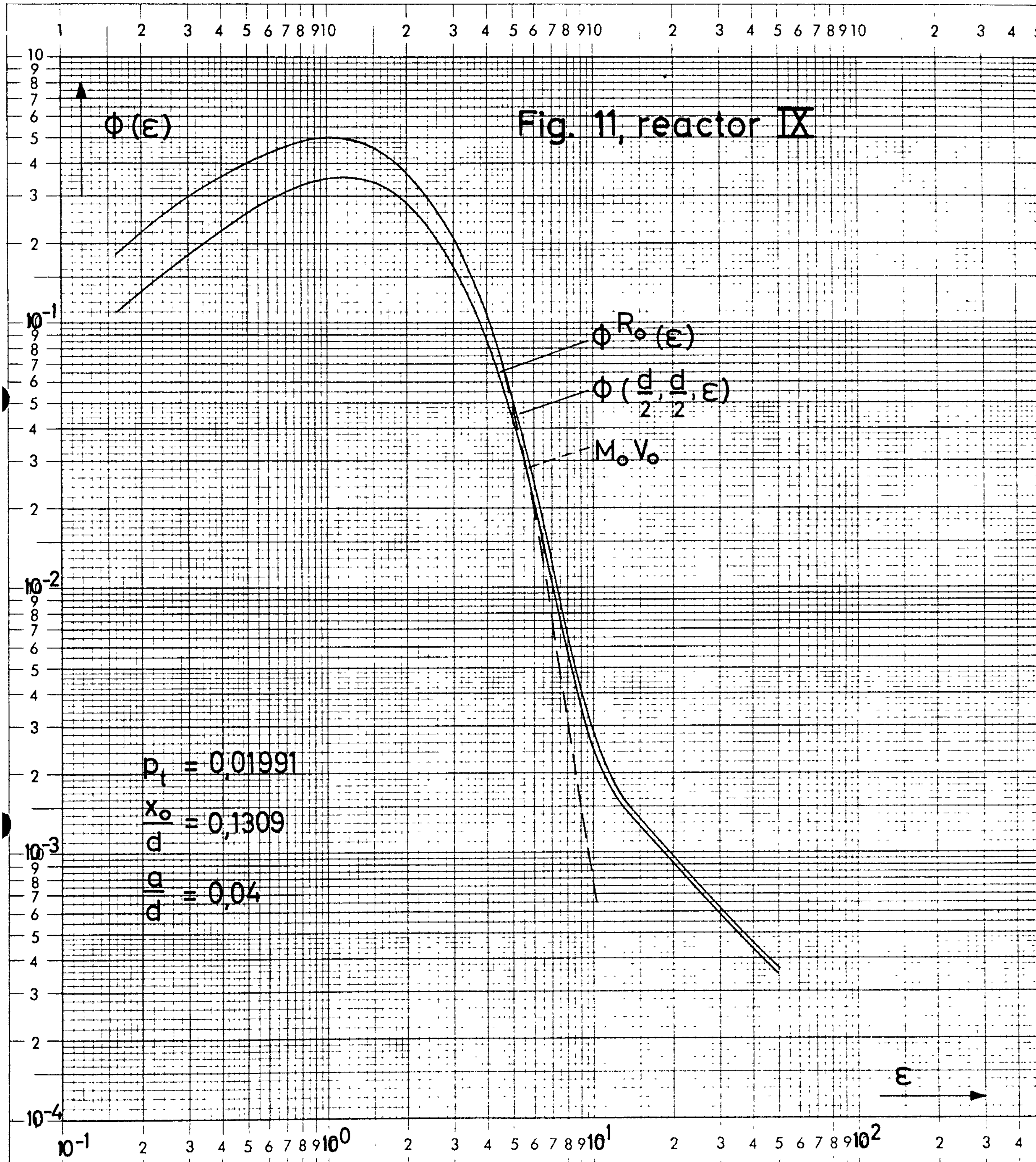


Fig. 11, reactor IX
$\phi(\varepsilon)$
$\phi^{R_o}(\varepsilon)$
$\phi(\frac{d}{2},\frac{d}{2},\varepsilon)$
$M_o V_o$
$p_t = 0{,}01991$
$\frac{x_o}{d} = 0{,}1309$
$\frac{a}{d} = 0{,}04$
ε

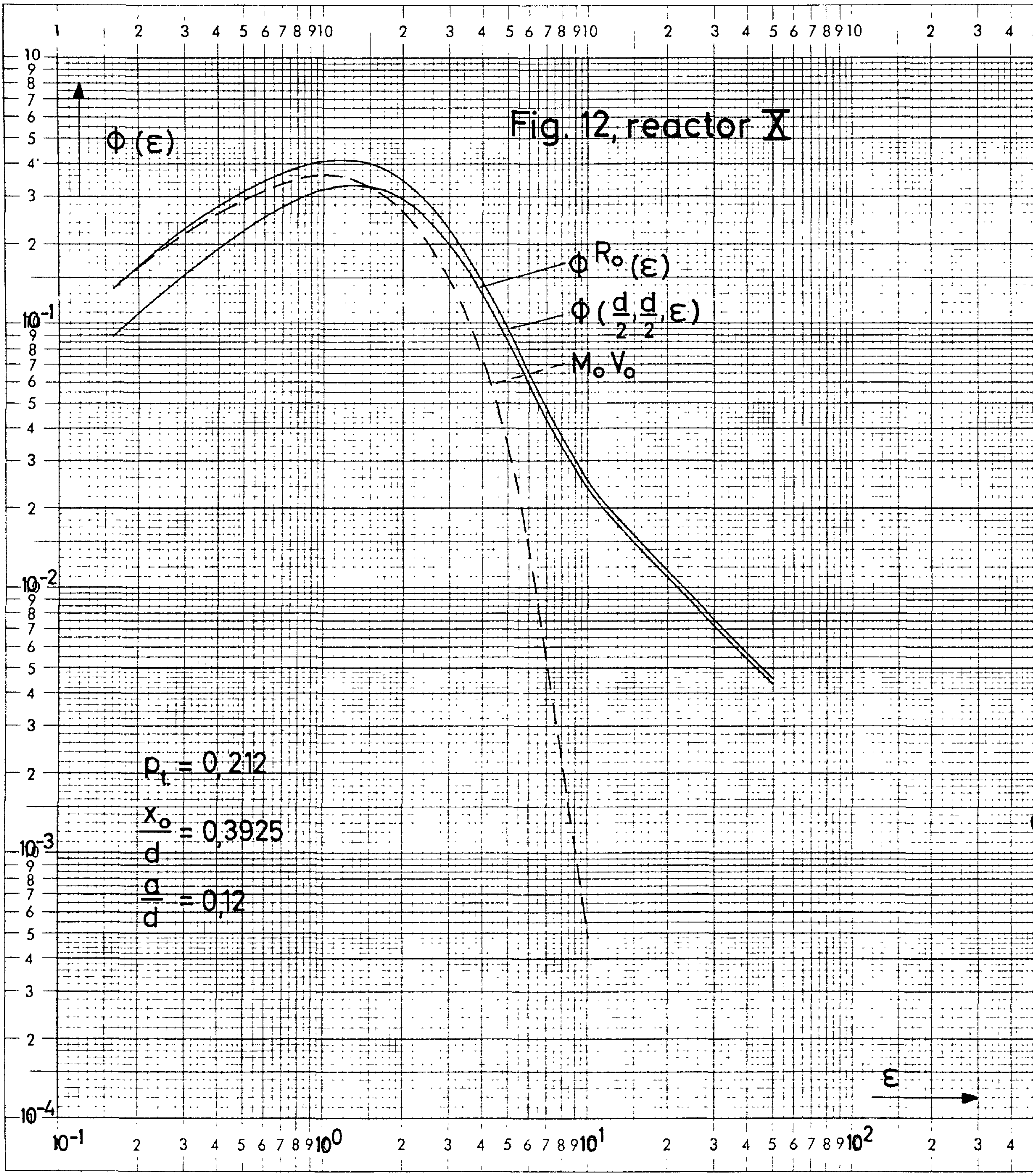


Fig. 12, reactor X
Φ (ε)
$\phi^{R_o}(\varepsilon)$
$\phi(\frac{d}{2},\frac{d}{2},\varepsilon)$
$M_o V_o$
$p_t = 0{,}212$
$\frac{x_o}{d} = 0{,}3925$
$\frac{a}{d} = 0{,}12$
ε

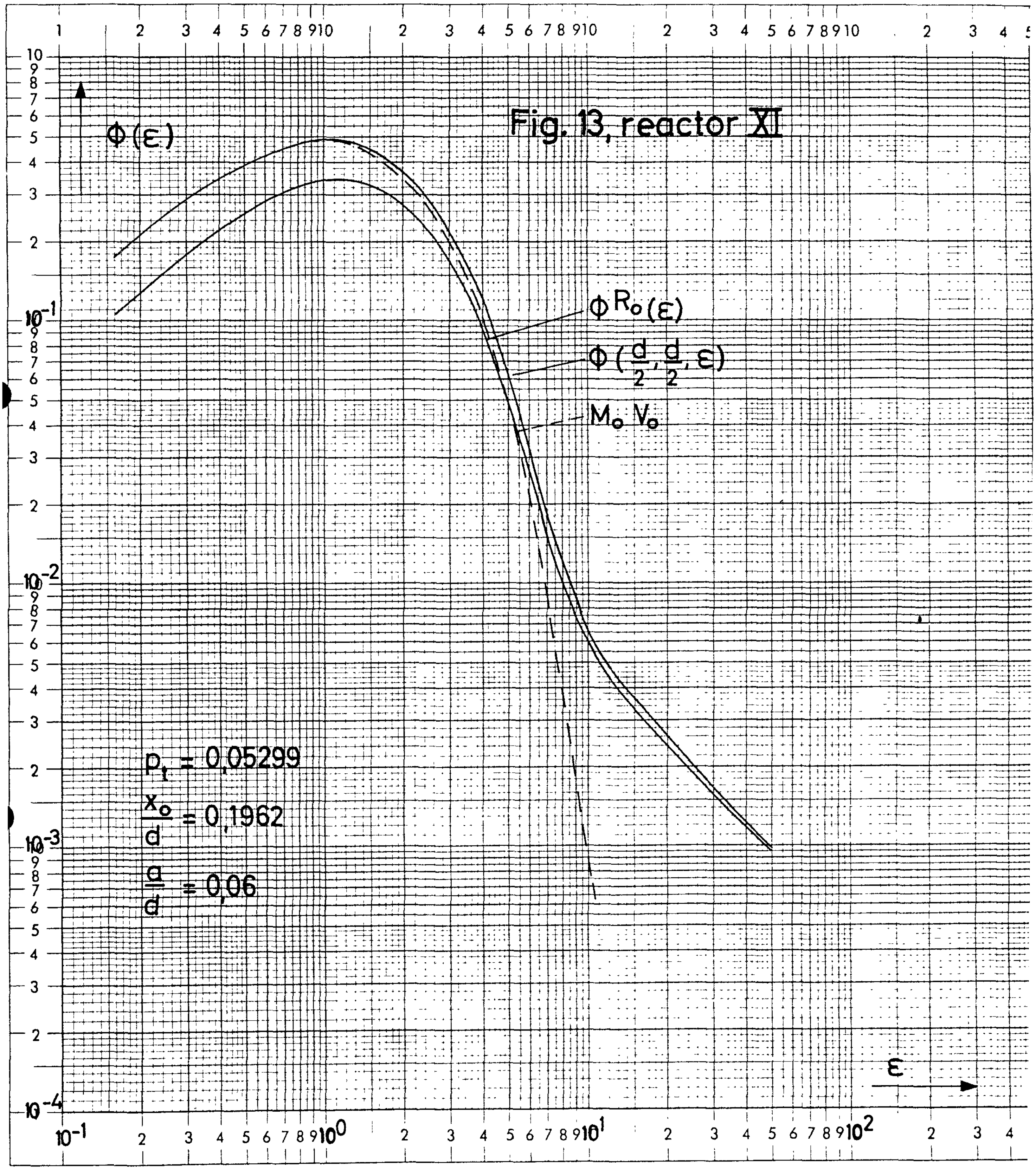


Fig. 13, reactor XI
$\Phi(\varepsilon)$
$\Phi^{R_0}(\varepsilon)$
$\Phi(\frac{d}{2},\frac{d}{2},\varepsilon)$
$M_0 V_0$
$p_t = 0{,}05299$
$\frac{x_0}{d} = 0{,}1962$
$\frac{a}{d} = 0{,}06$
ε

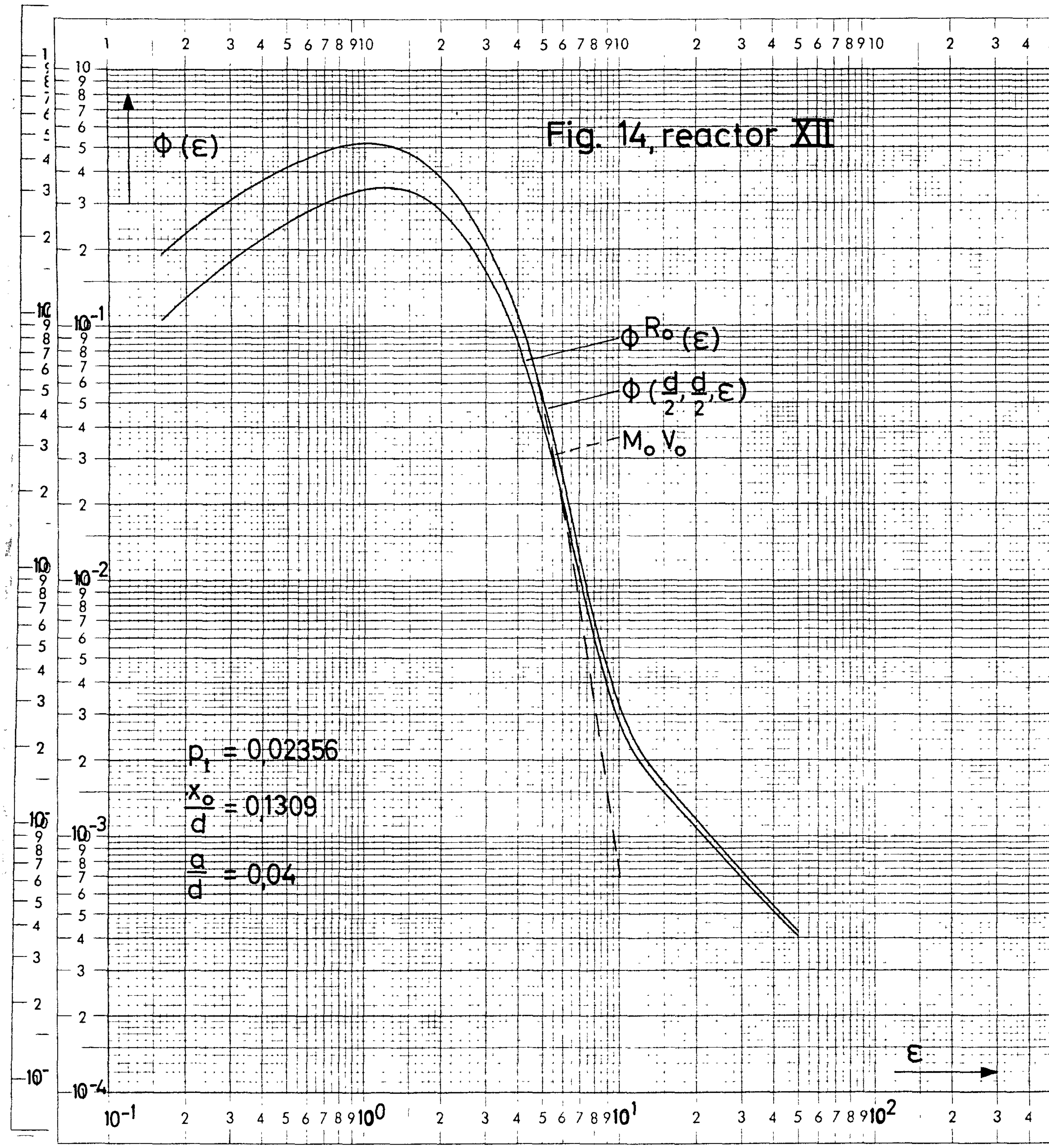


Fig. 14, reactor XII
$\phi(\varepsilon)$
$\phi^{R_0}(\varepsilon)$
$\phi(\frac{d}{2},\frac{d}{2},\varepsilon)$
$M_0 V_0$
$p_t = 0,02356$
$\frac{x_0}{d} = 0,1309$
$\frac{a}{d} = 0,04$
ε