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Neutron Noise in a Reactor with an External Control Loop*

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Abstract. The noise in the neutron detector signal from a reactor with an external control loop is theoretically investigated. Delayed neutrons are included. A practical formula for the noise power spectral density contains 4 contributions: (1) White detector noise, (2) noise from an external reactivity perturbation, (3) noise of the same origin (i. e. branching processes) as that observed in zero-power experiments, (4) noise produced in the control loop by the neutron detection process. A rigorous stochastic treatment shows, that this formula is a very good approximation, when the neutron population is high and its fluctuations are sufficiently small.

1. Introduction

The output of a neutron detector in a reactor at steady state is the sum of its mean value and statistical fluctuations. The analysis of this noise, by different experimental, theoretically related methods, is an established technique for obtaining or supporting information on the reactor's dynamic performance [1, 2].

The frequency analysis of the current from a neutron-sensitive ionization chamber determines the power spectral density of the noise contained in the detector signal. This technique has been treated quite early in respect to zero-power reactors [3—5] and is convenient for power reactors, when the rate of neutron detections becomes very high.

Apart from a trivial contribution, i. e. white detector noise, the noise in the neutron detector signal reflects the fluctuations in the neutron population.

In a reactor at full power the neutron population fluctuates primarily because of the statistical modulation of the reactivity, by random variations of the coolant inlet temperature or velocity, formation of bubbles in the coolant etc. These effects constitute an external, autonomous reactivity perturbation $\varrho_0(t)$, which drives the reactor with its external and internal feedback loops.

The other extreme is a subcritical reactor with a spontaneous fission neutron source. In such zero-power assemblies reactivity perturbations are quite unimportant. The neutron fluctuations are, then, due only to the subcritical neutron multiplication being a stochastic branching process.

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So far, only this second case has got a satisfactory theoretical treatment, although obviously the effects observed in zero-power experiments should be present to some extent also in reactors at any power-level. However, it seems unrealistic to aim at a stochastic treatment of a reactor and retain all complications of space- and energy-dependence, external and internal feedbacks, mechanisms perturbing the reactivity etc.

We investigate here the power spectral density (PSD) of the neutron detector signal obtained from an externally perturbed point reactor with a single external control loop. The results will be directly applicable for long-time low-power experiments with a reactor under automatic control. The frequency characteristics of the control loop shall be quite general, and delayed neutrons are duly considered.

In section 3 a practical formula (3.11) is developed and discussed. In the rest of the paper an attempt is made to confirm this formula by a rigorous stochastic treatment. This confirmation could not be obtained quite generally. But we can show that (3.11) is at least a very good approximation, if the neutron population is sufficiently high and the fluctuations are small.

2. Reactor Model, Definitions

We consider a reactor with a control loop (Fig. 1); ϱ_0 signifies a reactivity perturbation (input), ϱ_c is the control reactivity (feedback); then $\varrho = \varrho_0 - \varrho_c$ is the net reactivity. Let $N(t)$ be the neutron population at time t and N_0 a constant demand value, and assume

$$|N(t) - N_0| \ll N_0 \quad \text{and} \quad |\varrho(t)| \ll 1, \tag{2.1}$$

as usual. In a deterministic model $N(t)$ is a linear function of the net reactivity,

$$\delta N = N(t) - N_0 = N_0 \int_{-\infty}^{\infty} h_0(\vartheta)\, \varrho(t-\vartheta)\, d\vartheta. \tag{2.2}$$

$h_0(t)$, which is zero for $t<0$, has a Fourier transform $H_0(\omega)$, i. e. the frequency response function of the reactor without feedback. With l = prompt neutron generation time, β_k and λ_k = fraction and time constant of the k-th delayed neutron group, this is known to be

$$H_0(\omega) = \left(1 - \sum_{k=1}^{6} \frac{\beta_k i\omega}{\lambda_k + i\omega}\right) \Big/ \left(i\omega l + \sum_{k=1}^{6} \frac{\beta_k i\omega}{\lambda_k + i\omega}\right). \tag{2.3}$$

Let the feedback reactivity be a linear function of $\delta N/N_0$,

$$\varrho_c(t) = \int_{-\infty}^{\infty} g(\vartheta)\, \frac{N(t-\vartheta) - N_0}{N_0}\, d\vartheta. \tag{2.4}$$

The controller response function $g(t)$, which is zero for $t<0$, is assumed to have a Fourier transform $G(\omega)$,

$$G(\omega) = \sum_{j=1}^{J} \frac{a_j}{i\omega - p_j}, \quad (\operatorname{Re} p_j < 0). \tag{2.5}$$

This covers a broad class of functions

$$g(t) = \sum_{j=1}^{J} a_j e^{p_j t} \quad \text{for} \quad t \geqq 0. \tag{2.6}$$

Obviously, the neutron population depends linearly on the input reactivity perturbation,

$$\delta N = N - N_0 = N_0 \int_{-\infty}^{\infty} h(\vartheta)\, \varrho_0(t-\vartheta)\, d\vartheta \tag{2.7}$$

with $h(t) = 0$ for $t<0$. For the Fourier transform $H(\omega)$ of $h(t)$ one confirms easily

$$H(\omega) = \frac{H_0(\omega)}{1 + G(\omega) H_0(\omega)}. \tag{2.8}$$

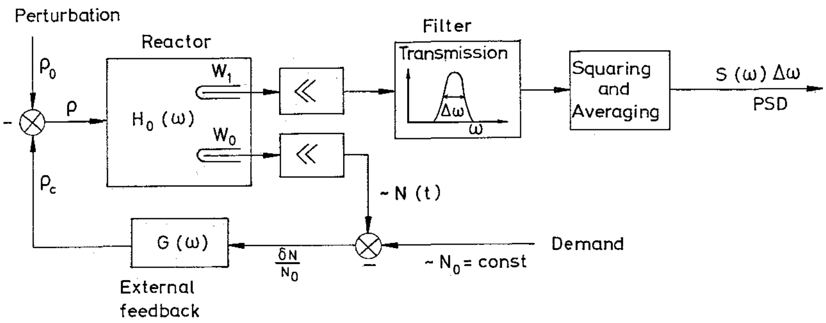

Fig. 1. Scheme of the reactor with external control loop and PSD measuring equipment

With (2.4) and (2.6) $\varrho_c(t)$ can be regarded as a sum of partial feedback reactivities ϱ_j,

$$\begin{aligned} \varrho_c(t) &= \sum_{j=1}^{J} \frac{a_j}{N_0} \int_0^{\infty} e^{p_j \vartheta} \left(N(t-\vartheta) - N_0\right) d\vartheta \\ &= \sum_{j=1}^{J} \varrho_j(t). \end{aligned} \tag{2.9}$$

Each partial term obeys a differential equation

$$\frac{d}{dt} \varrho_j(t) = a_j \frac{N(t) - N_0}{N_0} + p_j \varrho_j(t), \quad j = 1, \ldots, J. \tag{2.10}$$

Part of the control loop is the neutron detector, e. g. a boron-lined ionization chamber. Let $R(t)$ be the neutron detection rate (pulses/second) and $R_0 = W_0 N_0/(\nu l)$ its constant average; W_0 = detector sensitivity (pulses/fission), ν = mean number of fission neutrons per fission. Then

$$\frac{d}{dt} \varrho_j(t) = \frac{a_j}{R_0} R(t) - a_j + p_j \varrho_j(t). \tag{2.10a}$$

a_j/R_0 is seen to be the average increase of this partial feedback reactivity per detected neutron.

3. Heuristic Development

We will now derive non-rigorously and briefly an expression for the PSD (= power spectral density) of the noise in the signal of a detector outside the control loop, i. e. the Fourier transform $S(\omega)$ of the autocorrelation function $R_{11}(\tau)$ of the detector signal,

$$S(\omega) = \sum S_k(\omega) = \int_{-\infty}^{\infty} R_{11}(\tau)\, e^{-i\omega\tau}\, d\tau. \tag{3.1}$$

The terms $S_k(\omega)$ of the sum are related to different sources. This absorption detector has a sensitivity W_1 (pulses/fission) and detects neutrons at an average pulse rate $R_1 = W_1 F$ with $F = N_0/(\nu l)$ = mean fission rate. The autocorrelation function will be defined as

$$R_{11}(\tau) = W_1 F r_1 \delta(\tau) + \frac{W_1^2 N_0^2}{\nu^2 l^2} \lim_{T\to\infty} \cdot \int_{-T}^{T} \frac{(N(t+\tau)-N_0)(N(t)-N_0)\,dt}{2T\cdot N_0^2}. \quad (3.2)$$

If an autocorrelation experiment processes analog signals, the first, singular term of $R_{11}(\tau)$ yields a contribution proportional to the mean square of the detector pulse amplitude. All other contributions are proportional to the square of the mean amplitude. This is taken into account by the factor r_1 [5]. Let the probability density of the pulse amplitudes ξ be $w_1(\xi)$ and let the mean amplitude be $\langle\xi\rangle = 1$. Then

$$r_1 = \langle\xi^2\rangle = \int_0^\infty w_1(\xi)\xi^2 d\xi \geqq 1. \quad (3.3)$$

We require, that the reactivity perturbation $\varrho_0(t)$ has a time average $\bar{\varrho}_0 = 0$, and define its PSD by

$$P_0(\omega) = \int_{-\infty}^{\infty} e^{-i\omega\tau} \lim_{T\to\infty} \int_{-T}^{T} \frac{\varrho_0(t+\tau)\varrho_0(t)\,dt}{2T}\, d\tau. \quad (3.4)$$

In a first approximation the reactor (with its control loop) depends deterministically on $\varrho_0(t)$. We then may insert (2.7) into (3.2), perform the Fourier transformation according to (3.1) and get

$$S_1(\omega) = W_1 F r_1, \quad (3.5)$$

$$S_2(\omega) = W_1^2 F^2 |H(\omega)|^2 P_0(\omega). \quad (3.6)$$

In unperturbed, stationary, subcritical assemblies the dominating source of neutron fluctuations are all branching reactions, especially the fissions [6]. This source of noise is expected also in the reactor considered. On a heuristical basis, we want to treat it as an additional equivalent reactivity perturbation with a white spectrum $P_0'(\omega)$ = const. The appropriate normalization can then be determined by an extrapolation to high frequencies ω, when $G(\omega) \approx 0$, $H(\omega) \approx H_0(\omega)$ and the theory of zeropower correlation experiments [6] is again applicable. Thus, the analog of (3.6) becomes

$$S_3(\omega) = W_1^2 F \frac{\overline{\nu(\nu-1)}}{\nu^2} |H(\omega)|^2. \quad (3.7)$$

Any spurious detector signal $R_s(t)$ in the control loop generates feedback reactivity according to (2.4). One can easily verify that a spurious control detector signal with a PSD $R_{ss}(\omega)$ generates spurious feedback reactivity ϱ_s with a PSD

$$P_s(\omega) = |G(\omega)|^2 R_{ss}(\omega)/(W_0 F)^2. \quad (3.8)$$

According to (3.6) this contributes to $S(\omega)$ a term

$$S_4(\omega) = (W_1/W_0)^2 \cdot |H(\omega)|^2 \cdot |G(\omega)|^2 \cdot R_{ss}(\omega). \quad (3.9)$$

The detector signal in the control loop carries, besides the desired information on the actual neutron population, white noise with a spectral density $W_0 F r_0$, cf. (3.5). The factor $r_0 \geqq 1$ is the analog of r_1, cf. (4.11 a). When all other sources of noise in the control loop (e. g. spurious detector pulses, electronic noise) can be neglected, we have

$$R_{ss}(\omega) = W_0 F r_0 \quad (3.10)$$

in (3.9). With this assumption we can collect the terms S_1 to S_4 and obtain

$$S(\omega) = W_1 F r_1 + W_1^2 F \cdot |H(\omega)|^2 \left[F P_0(\omega) + \frac{\overline{\nu(\nu-1)}}{\nu^2} + \frac{r_0}{W_0}|G(\omega)|^2\right]. \quad (3.11)$$

Before starting to develop the same formula by stochastic methods, we want to point out some of its salient features, which should be observable in reactors with typical control loops and with reactivity perturbations mostly in the lower frequency region. The white noise term S_1, which alone is $\sim W_1$ and which would not be present in a two-detector cross correlation experiment [6], is not considered in this connection.

1. At high power F, the term S_2 dominates for medium frequencies. At high frequencies, when the PSD $P_0(\omega)$ of the reactivity perturbation is attenuated, the term S_3 may become important.

2. At very low power F the term S_3 dominates for medium and high frequencies, when $G(\omega) \approx 0$.

3. At low power and for very low frequencies the term S_4 cannot be neglected, especially when the sensitivity W_0 of the detector in the control loop is low.

4. *Stochastic Formulation*

To get simple equations, we restrict this detailed exposition to a case with one group of delayed neutrons and the simplest controller response function

$$G(\omega) = \frac{a}{i\omega - b}; \quad g(t) = a \cdot e^{bt} \quad \text{for} \quad t \geqq 0. \quad (4.1)$$

The mathematical treatment presented for this case can easily be adapted to the general model assumed in the previous sections.

Let n, c, ϱ_1 be the number of neutrons, number of precursors and the feedback reactivity at time t. As far as nuclear processes are concerned, during an infinitesimal interval $(t, t+dt)$ the following alternatives must be considered:

1. No process occurs.
2. One precursor decays.
3. One neutron is captured or leaks out, but it is not detected by the control detector.
4. One neutron is captured in the control detector, and thus contributes to the feedback reactivity.
5. One neutron produces fission and generates m prompt neutron plus 0 or 1 neutron precursor.

These alternatives have known probabilities. In addition, the feedback reactivity ϱ_1 will shift according to (2.10 a) by an amount

$$\Delta\varrho_1 = -a\,dt + b\varrho_1 dt. \quad (4.2)$$

As we want to neglect in this investigation any additional (electronical or mechanical) noise in the control loop, this shift is a deterministic contribution.

Let $p(n, c, \varrho_1, t)\,d\varrho_1$ be the probability at time t for the numbers n, c and a feedback reactivity in an interval $(\varrho_1, \varrho_1 + d\varrho_1)$, normalized by

$$\sum_{n=0}^{\infty} \sum_{c=0}^{\infty} \int_{-\infty}^{\infty} p(n, c, \varrho_1, t)\, d\varrho_1 \equiv 1. \quad (4.3)$$

The arguments given above lead directly to a balance Eq. (4.5), in which

$$\varrho' = \varrho_1(1 + b\,dt) - a\,dt; \quad d\varrho' = d\varrho_1(1 + b\,dt) \quad (4.4)$$

is used as a substitution on the l.h.s.

$$\begin{aligned} p(n,c,\varrho',t+dt)\,d\varrho' &= p(n,c,\varrho_1,t)\,d\varrho_1 + dt\,d\varrho_1 \\ &\cdot \Big[-\Big(\lambda c + \frac{n}{l}\Big) p(n,c,\ldots) + \lambda(c+1) \\ &\cdot p(n-1,c+1,\ldots) + \frac{n+1}{l}\Big(1 - P_f - \frac{\varrho_0 - \varrho_1}{\nu} - \frac{W_0}{\nu}\Big) \\ &\cdot p(n+1,c,\ldots) + \frac{W_0(n+1)}{\nu l} \int_0^{\infty} w_0(\xi) \\ &\cdot p\Big(n+1, c, \varrho_1 - \frac{a\xi}{R_0}, t\Big) d\xi + \sum_{m=0}^{\infty} \frac{n+1-m}{\nu l} \\ &\cdot (P_f \nu + \varrho_0 - \varrho_1) q_m [(1-\beta\nu)\, p(n+1-m, c, \ldots) \\ &+ \beta\nu\, p(n+1-m, c-1, \ldots)]\Big]. \end{aligned} \quad (4.5)$$

In most r.h.s. terms the arguments ϱ_1, t have been omitted. ϱ_0 is the time-dependent reactivity perturbation, W_0/ν the probability for neutron decay by capture in the control detector. $w_0(\xi)$ gives for this detector the distribution of relative pulse amplitudes.

$P_f = 1/\nu$; at any time the probability for a neutron decay by fission is

$$P_f + (\varrho_0 - \varrho_1)/\nu = (1 + \varrho_0 - \varrho_1)/\nu. \quad (4.6)$$

q_m is the probability for m prompt fission neutrons, $\beta\nu$ the average number of precursors per fission. In (4.5) we assume a) that not more than one precursor is formed in a fission, and b) that prompt fission neutrons and precursors are generated independently. This assumption is reasonable and without significant influence on the final results.

We transform (4.5) by introducing the generating function $f = f(u, v, x, t)$,

$$f(u,v,x,t) = \sum_{n=0}^{\infty} \sum_{c=0}^{\infty} u^n v^c \int_{-\infty}^{\infty} e^{i\varrho_1 x} p(n,c,\varrho_1,t)\, d\varrho_1. \quad (4.7)$$

If we substitute (4.4),

$$e^{i\varrho_1 x} = e^{i\varrho' x}(1 - i x \varrho' b\,dt + i x a\,dt + \cdots) \quad (4.8)$$

on the l.h.s. of (4.5), this part transforms to

$$\begin{aligned} \text{l.h.s.} &= f(u,v,x,t+dt) - b\,dt\,i x \frac{\partial}{i\partial x} f(u,v,x,t) \\ &\quad + a\,dt\,i x f(u,v,x,t) + \cdots. \end{aligned} \quad (4.9)$$

The r.h.s. of (4.5) transforms to

$$\begin{aligned} \text{r.h.s.} &= f + \frac{dt}{l}\Big[-u\frac{\partial f}{\partial u} - \lambda l(v-u)\frac{\partial f}{\partial v} \\ &+ \Big(1 - P_f - \frac{\varrho_0 + W_0}{\nu}\Big)\frac{\partial f}{\partial u} + \frac{1}{\nu i}\frac{\partial^2 f}{\partial x \partial u} \\ &+ \frac{W_0}{\nu}\int_0^{\infty} w_0(\xi) \exp\Big(\frac{i a \xi x}{R_0}\Big) d\xi \frac{\partial f}{\partial u} \\ &+ \Big(P_f + \frac{\varrho_0}{\nu}\Big)\psi(\beta\nu,u,v)\frac{\partial f}{\partial u} - \psi(\beta\nu,u,v)\frac{1}{\nu i}\frac{\partial^2 f}{\partial x \partial u}\Big]. \end{aligned} \quad (4.9\text{a})$$

Herein we use as an abbreviation

$$\begin{aligned} \psi(\beta\nu,u,v) &= \big(1 + \beta\nu(v-1)\big)\sum_{m=0}^{\infty} q_m u^m \approx 1 \\ &+ \nu(1-\beta)(u-1) + \beta\nu(v-1) + \nu^2(D - 2\beta) \\ &\cdot (u-1)^2/2 + \beta\nu^2(u-1)(v-1) + \cdots \end{aligned} \quad (4.10)$$

with the parameter $D = \overline{\nu(\nu-1)}/\bar{\nu}^2 \approx 0.8$ [7]. Terms of order β^2 have been dropped. When we combine (4.9) and (4.9a) we use the expansion (4.10) and a Taylor expansion of the exponential function. This introduces the normalization integrals

$$\int_0^{\infty} w_0(\xi)\, d\xi = \int_0^{\infty} w_0(\xi)\,\xi\, d\xi = 1, \quad (4.11)$$

$$\int_0^{\infty} w_0(\xi)\,\xi^2\, d\xi = r_0 \geqq 1, \quad (4.11\text{a})$$

for the pulse amplitudes of the control detector. We thus obtain

$$\begin{aligned} \frac{\partial}{\partial t} f(u,v,x,t) &= -i a x f + i b x \frac{\partial f}{i\partial x} + \lambda(u-1)\frac{\partial f}{\partial v} \\ &- \lambda(v-1)\frac{\partial f}{\partial v} + \frac{1}{l}\frac{\partial f}{\partial u}\Big[\frac{W_0}{\nu}\Big(\frac{i a x}{R_0} - \frac{r_0 a^2 x^2}{2R_0^2} + \cdots\Big) \\ &+ \big(\varrho_0(1-\beta) - \beta\big)(u-1) + \beta(1+\varrho_0)(v-1) \\ &+ (1+\varrho_0)\nu(D-2\beta)\frac{(u-1)^2}{2} + (1+\varrho_0)\nu\beta(u-1) \\ &\cdot (v-1) + \cdots\Big] - \frac{1}{li}\frac{\partial^2 f}{\partial x \partial u}\Big[(1-\beta)(u-1) + \beta(v-1) \\ &+ \nu(D-2\beta)\frac{(u-1)^2}{2} + \beta\nu(u-1)(v-1) + \cdots\Big]. \end{aligned} \quad (4.12)$$

This partial differential equation is the consistent generalization of the deterministic reactor dynamics equations, including a linear reactivity feedback term.

5. *Necessary Approximations to* (4.12)

From its definition (4.7) we see that the generating function $f(u,v,x,t)$ yields the expected numbers of neutrons and precursors, $N(t)$ and $C(t)$ respectively, the expected feedback reactivity $\varrho_c(t)$ and higher order moments as first or higher order derivatives at $u = v = 1$, $x = 0$.

$$\langle n \rangle(t) = \frac{\partial f}{\partial u}(1,1,0,t) = N(t), \quad (5.1)$$

$$\langle c \rangle(t) = \frac{\partial f}{\partial v}(1,1,0,t) = C(t), \quad (5.1\text{a})$$

$$\langle \varrho_1 \rangle(t) = \frac{\partial f}{i\partial x}(1,1,0,t) = \varrho_c(t), \quad (5.1\text{b})$$

$$\langle \varrho_1 n \rangle(t) = \frac{\partial^2 f}{i\partial x \partial u}(1,1,0,t) \text{ etc.} \quad (5.1\text{c})$$

From the first order derivatives of (4.12) we get an interesting form of the reactor dynamics equations,

$$\frac{dN}{dt} = \frac{\varrho_0(1-\beta) - \beta}{l} N - \frac{1-\beta}{l}\langle \varrho_1 n \rangle + \lambda C, \quad (5.2)$$

$$\frac{dC}{dt} = \frac{\beta(1+\varrho_0)}{l} N - \frac{\beta}{l}\langle \varrho_1 n \rangle - \lambda C, \quad (5.2\text{a})$$

$$\frac{d\varrho_c}{dt} = -a + b\varrho_c + \frac{a W_0 N}{\nu l R_0} = \frac{a(N - N_0)}{N_0} + b\varrho_c. \quad (5.2\text{b})$$

The last of these equations is identical with (2.10). But (5.2), (5.2a) correspond to familiar forms of the reactor kinetics equations only, if we may approximate

$$\langle \varrho_1 n\rangle(t) \approx \langle \varrho_1\rangle(t)\cdot\langle n\rangle(t) = \varrho_c(t)\cdot N(t). \quad (5.3)$$

In general, only with high neutron populations N this is a good approximation. For $\varrho_0 \equiv 0$ one can obtain exact equations for the stationary solutions $\overline{N}$, $\overline{C}$, $\overline{\varrho}_c$ etc. From (5.2) to (5.2b) one easily gets in this case, without external reactivity perturbation,

$$\begin{aligned} &\lambda\overline{C} = \beta\overline{N}/l, \quad \overline{\langle \varrho_1 n\rangle} = 0,\\ &a(\overline{N}-N_0)/N_0 + b\,\overline{\varrho}_c = 0. \end{aligned} \quad (5.4)$$

But the relations $\overline{N} = N_0$ or $\overline{\varrho}_c = 0$ do not follow from (5.2b) and, as higher order equations show, they are only approximately valid for $N_0 \gg 1$. The crucial term in (4.12) is the second order derivative $\partial^2 f/\partial u\,\partial x$, because it puts higher order moments as source terms into differential equations, that should yield lower order moments, e. g. $\langle \varrho_1 n\rangle$ in (5.2), (5.2a). To overcome this difficulty we have to introduce approximations. Our first two assumptions are:

1. The control loop restricts the deviation of the actual numbers n of neutrons from the constant demand N_0 to small amplitudes (this depends, of course, on the right choice of the control loop parameters). Thus, we put

$$p(n,c,\varrho_1,t) = 0 \quad \textit{unless} \quad |n-N_0| \ll N_0. \quad (5.5)$$

2. The demand value N_0 is sufficiently high, so that we may approximate in minor terms of (4.12)

$$p(n,c,\varrho_1,t) \approx p(n-1,c,\varrho_1,t) \quad (5.6)$$

for $|n-N_0| \ll N_0$. Because of (5.5) this holds, then, for all n.

From (5.5) and (5.6) we get approximately

$$\begin{aligned} \frac{\partial^2 f}{\partial u\,\partial x} &= \sum_{n=0}^{\infty}\sum_{c=0}^{\infty} n u^{n-1} v^c \int_{-\infty}^{\infty} i\varrho_1 e^{i\varrho_1 x} p(n,c,\varrho_1,t)\,d\varrho_1\\ &\approx \sum_{n=0}^{\infty}\sum_{c=0}^{\infty} N_0 u^{n-1} v^c \int_{-\infty}^{\infty} i\varrho_1 e^{i\varrho_1 x} p(n-1,c,\varrho_1,t)\,d\varrho_1\\ &= N_0 \frac{\partial f}{\partial x}(u,v,x,t). \end{aligned} \quad (5.7)$$

With this substitution and some reordering of terms (4.12) yields a simplified equation

$$\begin{aligned} \frac{\partial f}{dt} &= \frac{1}{l}\frac{\partial f}{\partial u}\left[(\varrho_0(1-\beta)-\beta)(u-1) + \beta(1+\varrho_0)\right.\\ &\quad\left.\cdot(v-1) + i x a l/N_0\right] + \lambda\frac{\partial f}{\partial v}[(u-1)-(v-1)]\\ &\quad - \frac{N_0}{l}\frac{\partial f}{i\partial x}[(1-\beta)(u-1) + \beta(v-1) - i x b l/N_0]\\ &\quad - i x a f + \psi_1(u-1)^2 + \psi_2(u-1)(v-1) - \psi_3 x^2 + \cdots \end{aligned} \quad (5.8)$$

$\psi_1 = \psi_1(u,v,x,t)$ etc. have the meaning

$$\psi_1 = \frac{\nu(D-2\beta)}{2l}\left[(1+\varrho_0)\frac{\partial f}{\partial u} - N_0\frac{\partial f}{i\partial x}\right], \quad (5.8\text{a})$$

$$\psi_2 = \frac{\beta\nu}{l}\left[(1+\varrho_0)\frac{\partial f}{\partial u} - N_0\frac{\partial f}{i\partial x}\right], \quad (5.8\text{b})$$

$$\psi_3 = \frac{W_0 r_0 a^2}{2\nu l R_0^2}\frac{\partial f}{\partial u} = \frac{r_0 a^2 \nu l}{2 W_0 N_0^2}\frac{\partial f}{\partial u}. \quad (5.8\text{c})$$

From the first order derivatives of (5.8) at $u=v=1$, $x=0$, we now get instead of (5.2), (5.2a)

$$\frac{dN}{dt} = \frac{\varrho_0(1-\beta)-\beta}{l}N - \frac{\varrho_c(1-\beta)}{l}N_0 + \lambda C, \quad (5.9)$$

$$\frac{dC}{dt} = \frac{\beta(1+\varrho_0)}{l}N - \frac{\varrho_c\beta}{l}N_0 - \lambda C. \quad (5.9\text{a})$$

(5.2b) is left unchanged. Thus, the approximation (5.7) leads to neutron kinetics Eqs. (5.9), (5.9a), which are linearized in respect to the treatment of the feedback reactivity.

Equations which determine the second order moments $\langle n(n-1)\rangle$, $\langle nc\rangle$, $\langle n\varrho_1\rangle$, $\langle c(c-1)\rangle$, $\langle c\varrho_1\rangle$, $\langle \varrho_1^2\rangle$ may be similarly generated as the appropriate second order derivatives of (5.8) at $u=v=1$, $x=0$. Such equations now cannot contain higher order moments as source terms. Thus, in principle, moments of any order can be successively computed from (5.8).

Practical considerations favour the inclusion of some more approximations:

3. To make the linearization of the reactor kinetics equations possible, we demand

$$|\varrho_0|, |\varrho_1|, |\varrho_c| \ll 1. \quad (5.10)$$

In analogy to (5.7) we then get

$$\frac{\varrho_0}{l}\frac{\partial f}{\partial u} \approx \frac{\varrho_0 N_0}{l} f. \quad (5.11)$$

4. We are content with approximate solutions that yield first and second order moments, which are correct within the previously stated approximations. Then we may drop all higher order terms of (5.8) and approximate the minor terms ψ_1, ψ_2, ψ_3 by

$$(1+\varrho_0)\frac{\partial f}{\partial u} - N_0\frac{\partial f}{i\partial x} \approx N_0 f, \quad (5.12)$$

$$\psi_1(u-1)^2 \approx \frac{\nu(D-2\beta)}{2}\frac{N_0}{l}(u-1)^2 f, \quad (5.13)$$

$$\psi_2(u-1)(v-1) \approx \beta\nu\frac{N_0}{l}(u-1)(v-1) f, \quad (5.13\text{a})$$

$$\psi_3 x^2 \approx \frac{r_0 a^2 \nu l}{2 W_0 N_0} x^2 f. \quad (5.13\text{b})$$

Including all these approximations into (5.8), we get its final form,

$$\begin{aligned} \frac{\partial f}{\partial t} &= \frac{1}{l}\frac{\partial f}{\partial u}\left[-\beta(u-1) + \beta(v-1) + \frac{i x a l}{N_0}\right]\\ &\quad + \lambda\frac{\partial f}{\partial v}[(u-1)-(v-1)] - \frac{N_0}{l}\frac{\partial f}{i\partial x}\\ &\quad \cdot\left[(1-\beta)(u-1) + \beta(v-1) - \frac{i x b l}{N_0}\right]\\ &\quad + \frac{N_0}{l} f\left[\varrho_0(1-\beta)(u-1) + \varrho_0\beta(v-1) - \frac{i x a l}{N_0}\right.\\ &\quad \left. + \frac{(D-2\beta)\nu}{2}(u-1)^2 + \beta\nu(u-1)(v-1) - \frac{r_0 a^2 \nu l^2 x^2}{2 W_0 N_0^2}\right]. \end{aligned} \quad (5.14)$$

In this final stage we get a set of approximate, linearized reactor dynamics equations, the last of which is again identical with (5.2b) and (2.10).

$$\begin{aligned} \frac{dN}{dt} &= -\frac{\beta}{l}N + \lambda C - \frac{N_0(1-\beta)}{l}(\varrho_c - \varrho_0),\\ \frac{dC}{dt} &= \frac{\beta}{l}N - \lambda C - \frac{N_0\beta}{l}(\varrho_c - \varrho_0),\\ \frac{d\varrho_c}{dt} &= \frac{a}{N_0}N + b\varrho_c - a. \end{aligned} \quad (5.15)$$

Note. The step from (4.12) to (5.8), involving the approximation (5.7), was *necessary* to make the mathematical treatment feasible. The next step from (5.8) to (5.14) is *convenient*, as it leads to simpler expressions. In any special case straight-forward procedures can check the importance of the neglections made after (5.8).

6. Mathematical Background

We will now formulate a few simple lemmas, which apply to equations of type (5.14), (5.15) in general.

Assume a M-row square matrix (a_{mn}) with real, constant elements. In our special case, $M=3$ and the a_{mn} are the matrix elements of (5.15)

$$\begin{aligned} &a_{11}=-\beta/l, && a_{12}=\lambda, && a_{13}=-(1-\beta)N_0/l,\\ &a_{21}=\beta/l, && a_{22}=-\lambda, && a_{23}=-\beta N_0/l,\\ &a_{31}=a/N_0, && a_{32}=0, && a_{33}=b. \end{aligned} \tag{6.1}$$

All eigenvalues α_m of the matrix (a_{mn}) shall have negative real parts

$$\text{Real part } \alpha_m<0 \quad m=1,\ldots,M. \tag{6.1a}$$

Otherwise the reactor with its control loop would not be stable. We will define a set of Green's functions $h_{mn}(t)$, for $m, n=1,\ldots,M$, by putting

$$\begin{aligned} &\frac{d}{dt}h_{mn}(t)=\sum_{i=1}^{M} a_{mi}h_{in}(t) \quad \text{for} \quad t\geqq 0,\\ &h_{mn}(t)=0 \quad \text{for} \quad t<0,\ h_{mn}(0)=\delta_{mn}, \end{aligned} \tag{6.2}$$

δ_{mn} = Kronecker's symbol. These Green's functions are sums of exponentials

$$h_{mn}(t)=\sum_{i=1}^{M} C_{mni}e^{\alpha_i t} \quad \text{for} \quad t\geqq 0, \tag{6.2a}$$

which decay to zero for $t\to\infty$ in view of (6.1a).

Lemma 1. When the source functions $b_m(t)$ are bounded and otherwise wellbehaved, functions

$$d_m(t)=\sum_{i=1}^{M}\int_{-\infty}^{\infty} h_{mi}(t-\vartheta)\,b_i(\vartheta)\,d\vartheta \tag{6.3}$$

exist for $m=1,\ldots,M$. The integrals converge for Green's functions of type (6.2a). These functions are solutions of

$$\frac{d}{dt}d_m(t)=\sum_{i=1}^{M} a_{mi}d_i(t)+b_m(t). \tag{6.4}$$

For a proof, one has simply to insert (6.3), (6.2) into (6.4).

(There are other solutions to (6.4) but, as one can easily show, only the solutions (6.3) are bounded for all times t.)

Lemma 2. If for $t'\geqq t$ and $m=1,\ldots,M$

$$\frac{d}{dt'}x_m(t')=\sum_{i=1}^{M} a_{mi}x_i(t')+b_m(t') \tag{6.5}$$

are valid, then

$$x_m(t+\tau)=\sum_{i=1}^{M}\left[h_{mi}(\tau)\,x_i(t)+\int_{t}^{\infty} h_{mi}(t+\tau-\vartheta)\,b_i(\vartheta)\,d\vartheta\right] \tag{6.6}$$

is true for all $\tau\geqq 0$ and $m=1,\ldots,M$.

For a proof, substitute $t'=t+\tau$ and insert (6.6) and (6.2) into (6.5). For $\tau=0$ one gets the correct initial values $x_m(t)$.

As a special case, with $b_m\equiv 0$, and as a useful corollary we obtain

Lemma 3. For all $t,\tau\geqq 0$ the Green's functions defined by (6.2) obey the relations

$$h_{mn}(t+\tau)=\sum_{i=1}^{M} h_{mi}(\tau)\,h_{in}(t). \tag{6.7}$$

Lemma 4. Let $c_{mn}(t)$ be elements of a symmetric M-row matrix (c_{mn}),

$$c_{mn}=c_{nm}, \quad m,n=1,\ldots,M. \tag{6.8}$$

When the $c_{mn}(t)$ are bounded and otherwise wellbehaved, we can define functions

$$g_{mn}(t)=\sum_{i,k=1}^{M}\int_{-\infty}^{\infty} h_{mi}(t-\vartheta)h_{nk}(t-\vartheta)\,c_{ik}(\vartheta)\,d\vartheta. \tag{6.9}$$

These functions are elements of a M-row symmetric matrix (g_{mn})

$$g_{mn}=g_{nm}, \quad m,n=1,\ldots,M \tag{6.10}$$

and they are solutions of

$$\frac{d}{dt}g_{mn}(t)=\sum_{j=1}^{M}(a_{mj}g_{jn}+a_{nj}g_{jm})+c_{mn}(t). \tag{6.11}$$

Proof. a) The symmetry (6.10) follows from an interchange of the summation indices i, k in (6.9) and the symmetry (6.8).

b) Exploiting (6.2) one obtains from (6.9)

$$\sum_{j=1}^{M} a_{mj}g_{jn}=\sum_{i,k=1}^{M}\int_{-\infty}^{t} h_{nk}(t-\vartheta)\frac{d}{dt}h_{mi}(t-\vartheta)\,c_{ik}(\vartheta)\,d\vartheta, \tag{6.12}$$

$$\sum_{j=1}^{M} a_{nj}g_{jm}=\sum_{k,i=1}^{M}\int_{-\infty}^{t} h_{mi}(t-\vartheta)\frac{d}{dt}h_{nk}(t-\vartheta)\,c_{ki}(\vartheta)\,d\vartheta. \tag{6.12a}$$

But, as $c_{ki}=c_{ik}$, we get further from (6.2)

$$\begin{aligned} \frac{d}{dt}g_{mn}&=\frac{d}{dt}\sum_{i,k=1}^{M}\int_{-\infty}^{t} h_{mi}(t-\vartheta)h_{nk}(t-\vartheta)\,c_{ik}(\vartheta)\,d\vartheta\\ &=\sum_{i,k=1}^{M}\delta_{mi}\delta_{nk}c_{ik}(t)+\sum_{j=1}^{M}(a_{mj}g_{jn}+a_{nj}g_{jm}). \end{aligned} \tag{6.13}$$

This yields directly (6.11).

Lemma 5. Let $\boldsymbol{y}$ be a M-component vector and $f(\boldsymbol{y},t)$ a function, which obeys

$$\frac{\partial f}{\partial t}=\sum_{m,n=1}^{M} a_{mn}y_m\frac{\partial f}{\partial y_n}+\left[\sum_{m=1}^{M} b_m y_m+\sum_{m,n=1}^{M} c_{mn}y_m y_n\right]f \tag{6.14}$$

with parameters a_{mn}, $b_m(t)$, $c_{mn}(t)$ as stated previously. A constant boundary value is

$$f(0,t)\equiv 1. \tag{6.14a}$$

One solution of this problem is

$$f(\boldsymbol{y},t)=\exp\left[\sum_{m=1}^{M} d_m(t)\,y_m+\sum_{m,n=1}^{M} g_{mn}(t)\,y_m y_n\right] \quad (6.15)$$

with the functions $d_m(t)$ and $g_{mn}(t)$ as given by (6.3) and (6.9) respectively.

Proof. a) The boundary value (6.14a) is obvious.

b) The ansatz (6.15) reduces the problem (6.14) to the verification of

$$\frac{d}{dt}\left[\sum_{m=1}^{M} d_m y_m+\sum_{m,n=1}^{M} g_{mn} y_m y_n\right]=\sum_{m,n=1}^{M} a_{mn} y_m d_n + \sum_{m=1}^{M} b_m y_m+\sum_{m,n=1}^{M}\left(a_{mn} y_m \sum_{j=1}^{M} 2 g_{nj} y_j + c_{mn} y_m y_n\right). \quad (6.16)$$

Use of the symmetry $g_{nj}=g_{jn}$ has been made. From (6.16) all linear terms in y_m cancel on account of (6.4). All second order terms in $y_m y_n$ cancel as well, as can be seen after reordering the term

$$\sum_{m,n=1}^{M} 2a_{mn}\sum_{j=1}^{M} g_{nj} y_m y_j = \sum_{m,j=1}^{M} a_{mj}\sum_{n=1}^{M} g_{jn} y_m y_n+\sum_{n,j=1}^{M} a_{nj}\sum_{m=1}^{M} g_{jm} y_n y_m \quad (6.16\text{a})$$
$$=\sum_{m,n=1}^{M} y_m y_n \sum_{j=1}^{M}(a_{mj} g_{jn}+a_{nj} g_{jm}).$$

Thus on account of (6.4) and (6.11), the relation (6.16) becomes an identity.

The significance of this result becomes obvious, when one compares (6.14) and (5.14). With $M=3$ these relations are equivalent, as we may in (5.14) substitute

$$y_1=u-1, \qquad y_2=v-1, \qquad y_3=ix. \quad (6.17)$$

The boundary condition (6.14a) applies as the usual normalization of a probability generating function, cf. (4.3), (4.7).

The matrix elements a_{mn} are given in (6.1). For the source functions $b_m(t)$ we obtain from (5.14)

$$b_1=\frac{\varrho_0(t)N_0(1-\beta)}{l}, \quad b_2=\frac{\varrho_0(t)N_0\beta}{l}, \quad b_3=-a. \quad (6.18)$$

Comparing (5.15) and (6.4), we see that under these conditions the functions $d_m(t)$ are components,

$$d_1(t)=N(t), \quad d_2(t)=C(t), \quad d_3(t)=\varrho_c(t), \quad (6.19)$$

of one solution of the reactor dynamics Eqs. (5.15). Actually, it is the only solution without transients and that is bounded for all times t.

In our special case, the matrix elements c_{mn} are constants,

$$c_{11}=\frac{(D-2\beta)\nu N_0}{2l}, \quad c_{12}=c_{21}=\frac{\beta\nu N_0}{2l},$$
$$c_{13}=c_{22}=c_{23}=c_{31}=c_{32}=0, \quad c_{33}=\frac{r_0 a^2 \nu l}{2W_0 N_0}, \quad (6.20)$$

As a consequence, also the matrix elements g_{mn}, which are defined by (6.9), must be constant. To perform the integration of (6.9), one has first to obtain by some numerical method the Green's functions $h_{mn}(t)$ from (6.2).

Thus, we can solve (5.14) and obtain a generating function of type (6.15). This function applies to a controlled reactor, which responds to a reactivity perturbation $\varrho_0(t)$ and is stationary otherwise. All initial transients must have decayed, as they have been intentionally excluded during the derivation of this solution.

From this generating function we can, then, compute first and second order moments of such quantities as the numbers of neutrons and precursors, n and c, or the feedback reactivity ϱ_1. We should, of course, not completely forget the approximations implicit in (5.14).

7. *Correlation Function for the Detector Signal*

For the signal of the independent detector, which is outside the control loop, we want to use now a more concise definition of the autocorrelation function, viz.

$$R_{11}(\tau)=W_1 F r_1 \delta(\tau) + \frac{W_1^2 N_0^2}{\nu^2 l^2}\lim_{T\to\infty}\int_{-T}^{T}\frac{N(t)N^*(t+\tau)\,dt}{2T\cdot N_0^2}-R_1^2. \quad (7.1)$$

Herein $N(t)$ is the expected number of neutrons at time t, i. e. the usual solution of (5.15) or (2.7). $N^*(t+\tau)$ is the expected number of neutrons at time $(t+\tau)$ *provided* that one neutron was detected (this applies for $\tau>0$) at time t. When we compare (7.1) with the Eq. (3.2), we see that the necessary refinement is now the distinction between $N^*(t+\tau)$ and $N(t+\tau)$.

To compute $N^*(t+\tau)$ for $\tau>0$ we define $p^*(n, c, \varrho_1, t+\tau)$ = probability density for the set of values (n, c, ϱ_1) at time $(t+\tau)$ *after* one detector pulse at time t. With this probability density we may set up a corresponding generating function $f^*(u, v, x, t+\tau)$ similar to (4.7). It can be developed as follows:

1. At time t the probability for detecting a neutron is proportional to the actual neutron population n.

2. An absorbing detector reduces for each pulse the number of neutrons by one unit.

3. An independent detector leaves other parameters unchanged.

Thus in the limit $\tau\to 0$

$$p^*(n-1,c,\varrho_1,t)\sim n\cdot p(n,c,\varrho_1,t). \quad (7.2)$$

For the corresponding generating function this means

$$f^*(u,v,x,t)\sim\frac{\partial}{\partial u}f(u,v,x,t). \quad (7.2\text{a})$$

The necessary normalization $f^*(1,1,0,t+\tau)\equiv 1$ then yields the initial value for $\tau\approx 0$

$$f^*(u,v,x,t)=\frac{\partial f}{\partial u}(u,v,x,t)\Big/\frac{\partial f}{\partial u}(1,1,0,t). \quad (7.3)$$

With the substituted variables (6.17) and with (5.1) this reads also

$$f^*(\boldsymbol{y},t)=\frac{\partial f}{\partial y_1}(\boldsymbol{y},t)\Big/\frac{\partial f}{\partial y_1}(0,t)=\frac{\partial f}{\partial y_1}(\boldsymbol{y},t)/N(t). \quad (7.3\text{a})$$

Eq. (5.14) or the equivalent (6.14) is valid also for this generating function $(\tau\geqq 0)$:

$$\frac{\partial f^*}{\partial\tau}(\boldsymbol{y},t+\tau)=\sum_{i,k=1}^{M} a_{ik} y_i \frac{\partial f^*}{\partial y_k}(\boldsymbol{y},t+\tau) + \left[\sum_{i=1}^{M} b_i(t+\tau)\,y_i+\sum_{i,k=1}^{M} c_{ik} y_i y_k\right] f^*(\boldsymbol{y},t+\tau). \quad (7.4)$$

f^* is distinguished from f, as given by (6.15), only by the restriction $\tau \geqq 0$ in (7.4) and the specified initial value (7.3a).

From now on, all derivatives shall have the argument $\boldsymbol{y}=0$. Then (7.4) gives for $i=1, \ldots, M$

$$\frac{\partial^2 f^*}{\partial y_i \partial \tau}(t+\tau)=\sum_{k=1}^{M} a_{ik}\frac{\partial f^*}{\partial y_k}(t+\tau)+b_i(t+\tau). \quad (7.5)$$

Therefore Lemma 2 is applicable. Especially, for $i=1$

$$\frac{\partial f^*}{\partial y_1}(t+\tau) = \sum_{k=1}^{M}\left[h_{1k}(\tau)\frac{\partial f^*}{\partial y_k}(t)+\int_t^{\infty} h_{1k}(t+\tau-\vartheta)\,b_k(\vartheta)\,d\vartheta\right]. \quad (7.6)$$

On account of (7.3a) this leads to

$$\frac{\partial f}{\partial y_1}(t)\cdot\frac{\partial f^*}{\partial y_1}(t+\tau)=\sum_{k=1}^{M} h_{1k}(\tau)\frac{\partial^2 f}{\partial y_1\partial y_k}(t) + \sum_{k=1}^{M}\frac{\partial f}{\partial y_1}(t)\int_t^{\infty} h_{1k}(t+\tau-\vartheta)\,b_k(\vartheta)\,d\vartheta. \quad (7.7)$$

From (6.15) we obtain (at $\boldsymbol{y}=0$!)

$$\frac{\partial^2 f}{\partial y_1\partial y_k}(t)=\frac{\partial f}{\partial y_1}(t)\,\frac{\partial f}{\partial y_k}(t)+2g_{1k}(t). \quad (7.8)$$

An equation of type (7.6) holds also for $\partial f/\partial y_1$, so that (7.7) and (7.8) combine to give

$$\frac{\partial f}{\partial y_1}(t)\cdot\frac{\partial f^*}{\partial y_1}(t+\tau)=\frac{\partial f}{\partial y_1}(t)\cdot\frac{\partial f}{\partial y_1}(t+\tau) + 2\sum_{k=1}^{M} g_{1k}(t)\,h_{1k}(\tau). \quad (7.9)$$

At $\boldsymbol{y}=0$, the derivatives define the expected numbers of neutrons $N(t)$ and $N^*(t+\tau)$, so that

$$N(t)\cdot N^*(t+\tau) = N(t)\cdot N(t+\tau)+2\sum_{m=1}^{M} g_{1m}(t)\,h_{1m}(\tau). \quad (7.9\text{a})$$

For $g_{1m}(t)$ we may substitute (6.9) and apply Lemma 3,

$$\sum_{m=1}^{M} h_{1m}(\tau)\,g_{1m}(t) = \sum_{i,k,m=1}^{M} h_{1m}(\tau)\int_{-\infty}^{\infty} h_{1i}(t-\vartheta)\,h_{mk}(t-\vartheta)\,c_{ik}(\vartheta)\,d\vartheta \quad (7.10)$$
$$= \sum_{i,k=1}^{M}\int_{-\infty}^{\infty} h_{1i}(t-\vartheta)\,h_{1k}(t+\tau-\vartheta)\,c_{ik}(\vartheta)\,d\vartheta.$$

In our case the c_{ik}, as given by (6.20), are constant, so that finally for $\tau>0$

$$N(t)\cdot N^*(t+\tau)=N(t)\cdot N(t+\tau) + 2\sum_{i,k=1}^{M} c_{ik}\int_{-\infty}^{\infty} h_{1i}(\vartheta)\,h_{1k}(\vartheta+\tau)\,d\vartheta. \quad (7.11)$$

For negative τ, the initial pulse of a delayed coincidence is at time $(t+\tau)$, the final pulse at time t. This means, that for $\tau<0$ the integrand of (7.1) is

$$N(t+\tau)\cdot N^*(t)=N(t+\tau)\cdot N(t) + 2\sum_{i,k=1}^{M} c_{ik}\int_{-\infty}^{\infty} h_{1i}(\vartheta+\tau)\,h_{1k}(\vartheta)\,d\vartheta. \quad (7.11\text{a})$$

But, as the c_{ik} are symmetric, (7.11) is identical with (7.11a) and may be inserted into (7.1) for any τ.

$N(t)$ and $N(t+\tau)$ are expectation values, given by (2.7), and $R_1=W_1N_0/(\nu l)$. From (7.1) we then obtain a final expression

$$R_{11}(\tau)=W_1Fr_1\delta(\tau)+W_1^2F^2\int_{-\infty}^{\infty} d\vartheta\int_{-\infty}^{\infty} d\vartheta' \lim_{T\to\infty}\cdot\int_{-T}^{T}\frac{dt}{2T}\,h(\vartheta)\,h(\vartheta')\,\varrho_0(t+\tau-\vartheta)\,\varrho_0(t-\vartheta') \quad (7.12)$$
$$+W_1^2F\sum_{i,k=1}^{M}\frac{2c_{ik}}{N_0\nu l}\int_{-\infty}^{\infty} h_{1i}(\vartheta)\,h_{1k}(\vartheta+\tau)\,d\vartheta.$$

8. Evaluation of the PSD

From the definition (3.1) of the PSD, as the Fourier transform of (7.12), we see that we have to deal with 4 contributions:

1. White detector noise from the first term of (7.12). This is obviously $S_1(\omega)$ as given by (3.5).

2. Noise caused by the reactivity perturbation ϱ_0. Fourier transforming the second term of (7.12) yields $S_2(\omega)$ as given by (3.6) and (3.4).

3. Noise from branching processes $S_3(\omega)$, connected with c_{11} and $c_{12}=c_{21}$ of the last term.

4. Noise originating in the reactivity feedback loop $S_4(\omega)$, connected with c_{33}. — All other coefficients c_{ik} are zero, cf. (6.20).

From the last term of (7.12) we get

$$S_3(\omega)+S_4(\omega)=W_1^2F\sum_{m,n=1}^{M}\frac{2c_{mn}}{N_0\nu l}H_{1m}(-\omega)\,H_{1n}(\omega), \quad (8.1)$$

with $H_{mn}(\omega)=$ Fourier transform of $h_{mn}(t)$. The $H_{mn}(\omega)$ may be expressed in terms of $H(\omega)$. From the kinetics Eqs. (5.15) follows that $H(\omega)$, defined by (2.7), obeys in our case (2.8) with $G(\omega)$ from (4.1) and

$$H_0(\omega)=\left(1-\frac{\beta i\omega}{\lambda+i\omega}\right)\Big/\left(i\omega l+\frac{\beta i\omega}{\lambda+i\omega}\right). \quad (8.2)$$

From (6.2) we can derive that the functions $H_{mn}(\omega)$ obey for $m, n=1, \ldots, M$ equations

$$i\omega H_{mn}(\omega)=\sum_{j=1}^{M} a_{mj}H_{jn}(\omega)+\delta_{mn}, \quad (8.3)$$

with the a_{mj} from (6.1) and $\delta_{mn}=$ Kronecker's symbol. By elementary algebra one gets

$$H_{11}(\omega)=\frac{l}{1-\dfrac{\beta i\omega}{\lambda+i\omega}}H(\omega),\quad H_{13}(\omega)=\frac{N_0H(\omega)}{b-i\omega}, \quad (8.4)$$

$$H_{12}(\omega)=\frac{\lambda}{\lambda+i\omega}H_{11}(\omega)\approx\frac{\lambda l H(\omega)}{\lambda+i\omega}. \quad (8.4\text{a})$$

Thus the contribution from branching processes, i. e. fissions, according to (8.1) and (6.20) becomes

$$S_3(\omega)=W_1^2F|H_{11}(\omega)|^2\left[\frac{2c_{11}}{N_0\nu l}+\frac{4c_{12}\lambda^2}{N_0\nu l(\lambda^2+\omega^2)}\right] = W_1^2FD|H(\omega)|^2A(\omega) \quad (8.5)$$

with $D=\overline{\nu(\nu-1)}/\nu^2$ and

$$A(\omega)=\frac{\lambda^2+\omega^2(1-2\beta/D)}{\lambda^2+\omega^2(1-\beta)^2}\approx 1. \quad (8.5\text{a})$$

But with $D \approx 0.8$ [7] and $\beta < 10^{-2}$ $A(\omega)$ is almost constant and equal to 1. In this approximation $S_3(\omega)$ as given by (3.7) is confirmed.

The remaining term of (8.1) with c_{33} yields directly $S_4(\omega)$ as given by (3.9) and (3.10).

9. Concluding Remarks

So far the expression (3.11) has been confirmed here for one special case. The general development of the stochastic treatment with 6 groups of delayed neutrons and a controller response function (2.5), (2.6) with J terms has also been carried out. The general procedure is to introduce vectors

$$\boldsymbol{c} = [c_1, \ldots, c_6], \quad \boldsymbol{\rho} = [\varrho_1, \ldots, \varrho_J], \tag{9.1}$$

specifying the numbers of precursors and the actual values of all partial feedback reactivities ϱ_j defined by (2.9), (2.10).

The corresponding probability density $p(n, \boldsymbol{c}, \boldsymbol{\rho}, t)$ is then used to define a generating function $f(u, \boldsymbol{v}, \boldsymbol{x}, t)$ similar to (4.7) with the vectors $\boldsymbol{v}$ and $\boldsymbol{x}$ related to $\boldsymbol{c}$ and $\boldsymbol{\rho}$. The subsequent development becomes very similar to that presented above and leads, with the approximations of section 5, to a confirmation of (3.11) for the general case.

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