KfK 2942
EUR 6406e
August 1980

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by

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ISSN 0303-4003

Abstract

According to general agreement, the nuclear material safeguards procedure is organized in such a way that the plant operator generates all data necessary for the establishment of a material balance, that the inspectors verify the operator's data with the help of independent measurements and that if there are no significant differences between the operator's data and the inspector's findings the material balance is established with the help of the operator's data. This procedure implies two tests of significance: one difference test (D-test) for the comparison of the operator's and the inspector's data, and one for the material balance establishment (MUF-test).

In the theoretical part it is shown that under the assumption, that in case of diversion the operator falsifies *all* data by a class specific amount, it is optimal in the sense of the probability of detection to use the difference MUF-D as the test statistics. However, as there are arguments for keeping the two tests separately, and furthermore, as it is not clear that the combined test statistics is optimal for any diversion strategy, the overall guaranteed probability of detection for the bivariate test is determined.

A numerical example is given applying the theoretical tools presented in the theoretical part. Using the material balance data of a Highly Enriched Uranium (HEU) fabrication plant the variances of MUF, D (no diversion) and MUF-D are calculated with the help of the standard deviations of operator and inspector measurements. The two inventories of the material balance are stratified. The samples sizes of the strata and the total inspection effort for data verification are determined by game theoretical methods (attribute sampling).

On the basis of these results the overall detection probability of the combined system (data verification and material accountancy) is determined both for the MUF-D test and the bivariate (D, MUF) test as a function of the goal quantity. The results of both tests are evaluated for different diversion strategies.

Verifizierung von Nuklearen Materialbilanzen: Allgemeine Theorie und Anwendung auf eine Fabrikationsanlage für hochangereichertes Uran

Zusammenfassung

Entsprechend einer allgemeinen Übereinkunft ist das Safeguard-Verfahren für nukleares Material so organisiert, daß der Inspektor die Betreiberdaten anhand unabhängiger Messungen überprüft und dann, falls keine signifikanten Unterschiede zwischen Inspektor- und Betreiberdaten auftreten, die Materialbilanz mit den Betreiberdaten schließt. Diese Vorgehensweise impliziert zwei Signifikanztests: zum einen den Differenzentest (D-Test) für den Vergleich der Inspektor- und Betreiberdaten und zum andern den Materialbilanztest (MUF-Test).

Im theoretischen Teil der Arbeit wird gezeigt, daß, falls im Falle einer Entwendung der Betreiber alle Daten um einen klassenspezifischen Betrag verfälscht, der (MUF-D)-Test optimal in Bezug auf die Entdeckungswahrscheinlichkeit ist. Es gibt jedoch Argumente dafür, die beiden Tests getrennt zu behandeln. Da ferner nicht klar ist, ob der kombinierte Test für beliebige Entwendungsstrategien optimal ist, wird die Gesamtentdeckungswahrscheinlichkeit für den bivariaten Test bestimmt.

An einem numerischen Beispiel wird das in der Arbeit dargestellte theoretische Handwerkzeug angewendet. Dazu werden die Daten einer Fabrikationsanlage für hochangereichertes Uran benutzt und die Varianzen von MUF, D (im Falle einer Nichtentwendung) und MUF-D mit Hilfe der Standardabweichungen der Inspektor- und Betreibermessungen berechnet. Die zwei Inventare der Materialbilanz werden stratifiziert. Ferner werden die Stichprobenumfänge für die einzelnen Strata mit Hilfe spieltheoretischer Methoden bestimmt und der Gesamtinspektionsaufwand für die Datenverifizierung berechnet (attribute sampling).

Auf der Grundlage dieser Ergebnisse wird die Gesamtentdeckungswahrscheinlichkeit des kombinierten Systems (Datenverifizierung und Materialbilanz) ermittelt. Dazu werden sowohl der MUF-D, als auch der bivariate (D, MUF)-Test verwendet und die Güte als Funktion der Zielmenge bestimmt. Die Ergebnisse beider Tests werden für verschiedene Entwendungsstrategien ausgewertet.

Contents

1. Introduction

Like in any other material processing industrial plant, material balances are established in nuclear plants, which serve the purpose of process surveillance, evidencing the whereabouts of the material processed, and pursue additional objectives. Besides these objectives, the material balance in nuclear plants takes still another and very significant role resulting from the Treaty on the Non-Proliferation of Nuclear Weapons (NPT). The Treaty on the Non-Proliferation of Nuclear Weapons provides international safeguards allowing to inspect the fissionable material used for peaceful application by the nuclear industries of signatory states. The responsible organization for these international safeguards is the International Atomic Energy Agency (IAEA) in Vienna and for the member countries of the European Communities the Safeguards Directorate in Luxembourg.

After lengthy preparation and negotiations the details, i.e. the principles and organization of these international safeguards, were fixed in 1971 and stipulated in an IAEA model agreement /1/. According to this model agreement the most important tool of safeguarding is the principle of material accountancy supplemented by containment and surveillance. Safeguarding has been so organized that the operators of nuclear plants record the source data allowing to establish the material balance, transmit them in a summarized form to the competent national and regional authority, respectively, which, in this turn, submits them to the international authority, the IAEA. The IAEA verifies the data received by its own independent measurements; in case that the IAEA measurements agree with that of the plant operator within the accuracy of measurement, the international authority will accept these data and establish the material balance, exclusively relying on the data supplied by the operators.

The 1971 model agreement did not stipulate all details of safeguards. Although the target was described verbally by "timely detection of the diversion of significant amounts", no quantitative statements were made as to the meaning of "timely" and "significant". Neither was the probability defined with which a particular diversion was to be detected. For this reason, a number of analyses for existing plants were performed after 1971 which to give an idea of the numerical values of the variables specified. Such analyses were also made on the German side /2/, /3/.

In this work a similar analysis is performed for the NUKEM plant in Wolfgang near Hanau which fabricates primarily fuel elements containing highly enriched Uranium. The objective of this work in a more restricted sense is to analyze the application of the safeguards system of the safeguards authorities to a nuclear plant processing high enriched uranium, with

- the efficiency of the material balance accountancy system,
- the optimum distribution of inspection efforts required for data verification,
- the efficiency of data verification, and
- the efficiency of total safeguarding

being the major subjects of the study. The efficiency is considered as the optimum and guaranteed probability of detection as a function of the inspection effort and the alleged amount of fissionable material diverted. The objective of this work in a broader sense was to analyze requirements feasible technically for the numerical values of the significant amounts indicated above. In the last years, in particular, this discussion regained a world-wide interest; the figures assumed under the aspect of safeguarding were not always in conformance with the technical possibilities available.

In this evident from the definition of the efficiency concept that strategies of diversion have to be considered when analyzing this variable. Therefore, such a procedure does not at all imply the imputation that an individual operator actually intends to divert fissionable material. Still more so, among the conceivable strategies of diversion the most favorable strategy of diversion must be determined for the operator (theoretically acting in an unlawful manner) so that the guaranteed probability of detection is really determined by optimizing the inspection effort. According to the organization of the safeguards system two categories of diversion strategies have to be considered, namely

- the strategies by which material is diverted without data falsification, taking advantage of the inaccuracy of measurements and relying on the hope that the (inaccurate) material balance does not allow detection of this diversion; and
- the strategies by which the material balance data are falsified so that the material balance "evens up" and the difference between the reported and the true value can be diverted.

Finally, to determine the efficiency of the total system, combinations of both categories of strategies have to be considered.

The statements above automatically lead to the layout of this work: In the first part the theoretical considerations are presented which lead to the formalism needed for the optimization of safeguards measures and, furthermore, for the determination of the efficiency of the system. As a large part of these considerations has already been published (see, e.g., /4/, /5/), only those results are derived in full length, which are new, and the numerical application of which is a genuine purpose of this study.

In the second part, a description of the nuclear plant under consideration and an indication of the major technical parameters, the NUKEM material accountancy system, as practical today, and the verification system, as conceived, will be presented.

It should be underlined here that although the used measurement accuracies have been given by the competent NUKEM members, an experimental confirmation of these data is missing in some cases. This means that the calculated variances and probabilities of detection, respectively, are incontestable theoretically, but that they still wait for experimental verification by "integral experiments" prior to their practical use (see e.g. /6/, /7/).

In the last part the theory outlined in the first part is applied to the NUKEM data: The accuracy of the accountancy system, the optimal sample sizes of the data verification system, and the efficiencies of the total system is determined for alternative test procedures. It should be noted that it was this latter question which stimulated this work, as there was the problem, whether or not a test procedure, which is the best one in a clearly defined *statistical sense*, should be replaced by another one, which has *practical advantages*. This problem could only be discussed at the hand of a concrete example such as the NUKEM plant.

PART I

THEORETICAL CONSIDERATIONS

In this part, the theory of a safeguards system based on the material accountability principle and on the procedure where the inspector verifies the data of the operator which are used for the material balance test is developed.

Basically, it is a game theoretical approach. More precisely, the inherent conflict situation between the operator who - perhaps - might divert material and the inspector who has to detect any diversion, is described by a zero-sum-game with the probability of detection as payoff to the inspector (a first principle's justification for this approach has been given in Ref. /5/).

In the following, the theory will be developed in all its details relevant for practical applications, only some formal proofs will be deleted as they have already been published in easily accessible Journals or Conference Proceedings.

2. Material Accountability

In this chapter the basic formulae for the establishment of the material balance at the end of one inventory period are put together. This formalism has been documented many times (see, e.g., /4/, /5/); we repeat it here as it will be used throughout this work.

2.1 Material Balance Principle

Let us consider a well defined material balance area of a nuclear facility that contains at a given time to some nuclear material into which enter the receipts R and from which leave the shipments S during a given interval of time $[t_o,t_1]$.

The material contained in the material balance area at time t_o is called the *physical inventory* I_o. The physical inventory at t_o plus the throughput in $[t_o,t_1]$ gives the *book inventory* B at t_1, i.e., the amount of material that should be contained in the material balance area at time t_1:

$$B := I_o+R-S \ . \qquad (2\text{-}1)$$

The amount of material actually contained in the material balance area at time t_1 is called the physical inventory I_1.

If all material contained in and passing through, the material balance area in the interval of time $[t_o,t_1]$ is carefully accounted for, and if no material has disappeared or has been diverted then the difference between the book inventory B at t_1 and the physical inventory I_1 should be zero. This is simply a consequence of the law of conservation of matter. However, as not all of these conditions must be satisfied, the difference between these two quantities at the end of one inventory period, which for historical reasons has been called material unaccounted for (MUF)[1)]

$$MUF := B-I_1 = I_o+R-S-I_1 \ , \qquad (2\text{-}2)$$

is not always zero. Thus arises the problem of finding out the various causes of this difference being nonzero and, furthermore, of trying to separate them.

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1) It would be better to call this quantity 'book-physical inventory difference', as in most cases the material is accounted for, but with measurement errors. Infact this term has been used for some time (see, e.g., Stewart /8/).

Before going on, we will formulate the 'MUF-equation' (2.2) in a somewhat different way: Let us assume that all the data necessary for the establishment of the material balance may be classified into K classes of material where the i-th class contains N_i batches, i=1...K. Then, if x_{ij} is the operator's measurement result of the material content of the j-th batch of the i-th class, i=1...K, j=1...N_i, the difference MUF between the book and the ending physical inventory may be written as the algebraic sum of these measurement data:

$$\mathrm{MUF} = \sum_{i=1}^{K} \sum_{j=1}^{N_i} x_{ij} \cdot \delta_{ij} \quad , \tag{2-3}$$

with $\delta_{ij}=1$, if x_{ij} belongs to beginning inventory or receipts, and with $\delta_{ij}=-1$, if x_{ij} belongs to shipments or ending inventory.

2.2 Measurement Errors

We consider first the case that the material content of the j-th item of the i-th class is determined with the help of a *unique measurement* (e.g., active interrogation). Let T_{ij} be the true value of this material content, and let $e_{O_{ij}}$ be the random error of the measurement, d_{O_i} be the calibration error of the measurement common to all measurements of this class. Then the result x_{ij} of this measurement can be written as

$$x_{ij} = T_{ij}+e_{O_{ij}}+d_{O_i}, \quad i=1...K, \quad j=1...N_i \quad . \tag{2-4}$$

We assume that the errors are normally distributed random variables with zero expectation values and known variances, and that errors from different classes are independent:

$$E(e_{O_{ij}}) = E(d_{O_i}) = 0 \ ;$$

$$\mathrm{var}(e_{O_{ij}}) =: \sigma^2_{O_{ri}} \ ;$$

$$\mathrm{cov}(e_{O_{ij}}, e_{O_{i'j'}}) = 0 \quad \text{for} \quad (i,j)\neq(i',j') \ ;$$

$$\mathrm{var}(d_{O_i}) =: \sigma^2_{O_{si}} \ ;$$

$$\mathrm{cov}(d_{O_i}, d_{O_{i'}}) = 0 \quad \text{for} \quad i\neq i' \ ;$$

$$\mathrm{cov}(e_{O_{ij}}, d_{O_{i'}}) = 0 \ .$$

The variance of one single measurement is then given by

$$\mathrm{var}(X_{ij}) = \sigma^2_{O_{ri}} + \sigma_{O_{si}} \quad , \quad i=1...K, \quad j=1...N_i \quad , \tag{2-6}$$

and the variance of the sum of all measurements of the i-th class is given by

$$\mathrm{var}(\sum_j X_{ij}) = N_i \cdot \sigma^2_{O_{ri}} + N_i^2 \cdot \sigma^2_{O_{si}} \quad , \quad i=1...K \; . \tag{2-7}$$

We consider second the case that the material content of the j-th item of the i-th class is determined with the help of *three different steps* each causing a measurement error (e.g., sampling, volume determination, concentration determination). Then the result x_{ij} of this measurement can be written as

$$x_{ij} = (V_{ij} + e^V_{O_{ij}} + d^V_{O_i}) \cdot (C_{ij} + e^S_{O_{ij}} + e^C_{O_{ij}} + d^C_{O_i}) \tag{2-8}$$

where $V_{ij}[\ell]$, and $C_{ij}[\mathrm{kgU}/\ell]$ are the true values of volume and concentration, where $e^V_{O_{ij}}$, $e^S_{O_{ij}}$ and $e^C_{O_{ij}}$ are the random errors of volume determination, sampling and concentration determination, and where $d^V_{O_i}$ and $d^C_{O_i}$ are the calibration errors of volume and concentration determination. (We assume that there is no 'persistent' sampling error, as we assume that the solution is homogenized in such a way that the concentration in one sample deviates only randomly from the average concentration in the solution).

We assume again that the errors are normally distributed random variables with zero expectation values and known variances, and that errors from different classes are independent:

$$E(e^V_{O_{ij}}) = E(e^S_{O_{ij}}) = E(e^C_{O_{ij}}) = E(d^V_{O_{ij}}) = E(d^C_{O_{ij}}) = 0 \; ;$$

$$\mathrm{var}(e^V_{O_{ij}}) =: \sigma^2_{OV_{ri}} \; ;$$

$$\mathrm{var}(e^S_{O_{ij}}) =: \sigma^2_{OS_{ri}} \; ;$$

$$\mathrm{var}(e^C_{O_{ij}}) =: \sigma^2_{OC_{ri}} \; ;$$

$$\mathrm{cov}(e^\ell_{O_{ij}}, e^\ell_{O_{i'j'}}) = 0 \quad \text{for} \quad (i,j) \neq (i',j') \quad , \quad \ell=V,S,C \; ;$$

$$\mathrm{var}(d^V_{O_{ij}}) =: \sigma^2_{OV_{si}} \; ;$$

$$\mathrm{var}(d^{C}_{O_{ij}}) =: \sigma^2_{OC_{si}} ;$$

$$\mathrm{cov}(d^{\ell}_{O_i}, d^{\ell}_{O_{i'}}) = 0 \quad \text{for} \quad i \neq i' , \quad \ell = V,S,C ;$$

$$\mathrm{cov}(e^{\ell}_{O_{ij}}, d^{\ell}_{O_{i'}}) = 0 \quad \text{for} \quad \ell = V,S,C . \qquad (2\text{-}9)$$

The variance of the measurement of the material content of one item is then given by the following formula, if one neglects second order terms:

$$\mathrm{var}(X_{ij}) = V^2_{ij} \cdot (\sigma^2_{OS_{ri}} + \sigma^2_{OC_{ri}} + \sigma^2_{OC_{si}}) + C^2_{ij} \cdot (\sigma^2_{OV_{ri}} + \sigma^2_{OV_{si}}) ,$$

$$i=1\ldots K, \; j=1\ldots N_i , \qquad (2\text{-}10)$$

and the variance of the sume of all measurements of the i-th class is given by

$$\mathrm{var}(\sum_j X_{ij}) = \sum_j V^2_{ij} \cdot (\sigma^2_{OS_{ri}} + \sigma^2_{OC_{ri}}) + (\sum_j V_{ij})^2 \cdot \sigma^2_{OC_{si}} +$$

$$+ \sum_j C^2_{ij} \cdot \sigma^2_{OV_{ri}} + (\sum_j C_{ij})^2 \cdot \sigma^2_{OV_{si}} . \qquad (2\text{-}11)$$

If we assume that the true values of volume and concentration of the different items of this class are the same,

$$V_{ij} = V_i , \quad C_{ij} = C_i , \quad i=1\ldots K ,$$

then formula (2-11) simplifies to

$$\mathrm{var}(\sum_j X_{ij}) = N_i \cdot \left[V^2_i \cdot (\sigma^2_{OS_{ri}} + \sigma^2_{OC_{ri}}) + C^2_i \cdot \sigma^2_{OV_{ri}} \right] +$$

$$+ N^2_i \cdot \left[V^2_i \cdot \sigma^2_{OC_{si}} + C^2_i \cdot \sigma^2_{OV_{si}} \right] . \qquad (2\text{-}11')$$

Finally, if we introduce the relative variances (squares of the coefficients of variation)

$$\delta^2_{OS_{ri}} = \frac{\sigma^2_{OS_{ri}}}{C^2_i} , \quad \delta^2_{OC_{ri}} = \frac{\sigma^2_{OC_{ri}}}{C^2_i} , \quad \delta^2_{OV_{ri}} = \frac{\sigma^2_{OV_{ri}}}{V^2_i} , \quad \delta^2_{OC_{si}} = \frac{\sigma^2_{OC_{si}}}{C^2_i} , \quad \delta^2_{OV_{si}} = \frac{\sigma^2_{OV_{si}}}{V^2_i} ,$$

then we can write formula (2-11') in the following form

$$\mathrm{var}(\sum_j X_{ij}) = X^2_i \cdot \left[N_i \cdot (\delta^2_{OS_{ri}} + \delta^2_{OC_{ri}} + \delta^2_{OV_{ri}}) + N^2_i \cdot (\delta^2_{OC_{si}} + \delta^2_{OV_{si}}) \right] , \qquad (2\text{-}11'')$$

where $X_i = V_i \cdot C_i$ [kgU] is the average U-content of one item of the i-th class.

According to formula (2-3) the variance of the Material Unaccounted For is then given by

$$\mathrm{var(MUF)} = \sum_i \mathrm{var}(\sum_j X_{ij}) \quad , \qquad (2\text{-}13)$$

where $\mathrm{var}(\sum_j X_{ij})$ is either given by formula (2-7) or by formula (2-11) resp. its simplified versions (2-11') and (2-11"). It should be noted that there may be random losses which may contribute to the variance of the Material Unaccounted For. If this is true, then formula (2-13) has to be generalized appropriately.

2.3 Material Balance Test

In the following, we aggregate all our measurements to initial physical inventory I_O, receipts R, shipments S and ending physical inventory I_1. We write

$$\begin{aligned} I_O &= E(I_O)+e_O \\ R &= E(R)+e_R \\ S &= E(S)+e_S \\ I_1 &= E(I_1)+e_1 \quad , \end{aligned} \qquad (2\text{-}14a)$$

where $E(I_O)$, $E(R)$, $E(S)$ and $E(I_1)$ are the true values of I_O, R, S and I_1, where the expectation values of the errors are zero:

$$E(e_O) = E(e_R) = E(e_S) = E(e_1) = 0 \quad , \qquad (2\text{-}14b)$$

and where the variances of these errors are known and written as follows:

$$\begin{aligned} \mathrm{var}(e_O) &= \sigma^2_{I_O} \\ \mathrm{var}(e_R) &= \sigma^2_R \\ \mathrm{var}(e_S) &= \sigma^2_S \\ \mathrm{var}(e_1) &= \sigma^2_{I_1} \quad . \end{aligned} \qquad (2\text{-}14c)$$

The measurement errors may cause a nonzero book-physical inventory difference, as already explained. In order to understand this, we write eg. (2-1) with the help of (2-14a) in the following form:

$$\mathrm{MUF} = E(I_O)+e_O+E(R)+e_R-E(S)-e_S-E(I_1)-e_1 \quad . \qquad (2\text{-}15)$$

If no material is missing, we have because of the conservation of matter

$$E(I_O)+E(R)-E(S)-E(I_1) = 0 \qquad (2\text{-}16)$$

and therefore

$$MUF = e_O+e_R-e_S-e_1 \quad . \qquad (2\text{-}17)$$

This however, leads immediately to

$$E(MUF) = 0 \quad . \qquad (2\text{-}18)$$

We call this relation the *null hypothesis* H_O.

We now can formulate our problem which is to find out whether the non-vanishing book-physical-inventory difference is caused only by measurement errors. In statistical terms: we have to test the null hypothesis H_O. We achieve this by choosing a significance threshold s for the sample value (realized value) of the book-physical-inventory difference $\hat{MUF}$ and by deciding

$$H_O \text{ correct if } \hat{MUF} \leq s \quad . \qquad (2\text{-}19)$$

The value of the significance threshold s is fixed with the help of the *probability of error of the first kind* α, which is defined by

$$\alpha := \text{prob}\{MUF>s \,|\, H_O\} \quad . \qquad (2\text{-}20)$$

In words, α is the probability that 'H_O not correct' will be stated if, in fact, H_O is true. The problem of the appropriate choice of the value of α will be discussed in Chapter 4.

If the result of the measurement is

$$\hat{MUF} > s \quad ,$$

we conclude that 'the null hypothesis H_O is not correct' or 'the alternative hypothesis H_1 is correct'. The nature of the problem determines whether we want to formulate the alternative hypothesis H_1 explicitely. Let us assume that it is reasonable to formulate H_1 in the following way:

$$H_1: \; E(MUF) = M \quad , \qquad (2\text{-}21)$$

where M is a quantity greater than zero. The choice of the appropriate value of M will also be discussed in Chapter 4. In this case, we can characterize the test by the *probability of error of the second kind* β, which is defined by

$$\beta := \text{prob}\{MUF \leq s \,|\, H_1\} \quad . \qquad (2\text{-}22)$$

In words, β is the probability that 'H_1 not correct' will be stated if, in fact, H_1 is true (or, in line with standard statistical terminology, it is the probability that the statement 'H_0 not correct' will *not* be made).

The probabilities of errors first and second kind for normally and independently distributed measurement errors and random losses are given by (see, e.g., Ref. /5/)

$$1-\alpha = \phi(\frac{S}{\sigma}) \tag{2-23}$$

and furthermore

$$1-\beta = \phi(\frac{M}{\sigma} - U_{1-\alpha}) \quad , \tag{2-24}$$

where

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp(-\frac{t^2}{2})\,dt \tag{2-25}$$

is the normal or Gaussian distribution function, U its inverse, and, with the definitions (2-14c),

$$\sigma^2 = \sigma_{I_0}^2 + \sigma_R^2 + \sigma_S^2 + \sigma_{I_0}^2 \quad . \tag{2-26}$$

Since the purpose of the test procedure described so far is to detect unusual losses or diversion, for obvious reasons we call the probability of error of the first kind, α, the *false alarm probability*, and we call one minus the probability of the error of the second kind, $1-\beta$, the *probability of detection*. Because of the central importance of eg. (2-24), which establishes a relation between false alarm probability α, variance of measurements σ^2, amount M assumed to be diverted, and probability of detection we discuss it here in some detail. We see immediately:

- The probability of detection increases with increasing amount M assumed to be missing (or diverted). This property is a natural requirement in any detection system.
- The probability of detection increases with decreasing standard deviation σ. This is reasonable, too. If one remembers that the standard deviation ordinarily decreases with increasing effort (money or man-hours), this property means that the probability of detection increases with increasing effort.
- The probability of detection increases with increasing false alarm probability. This is a well-known property of any detection system (e.g., fire

alarm system): the more sensitive the system is, the higher is its false alarm rate.

3. Data Verification

It has been outlined in the introduction that the material data, which are reported by the plant operator, are verified by the safeguards authority with the help of independent measurements.

In this chapter we will develop the theory for data verification procedures the main purpose is the determination of optimum sample sizes for different classes of material and furthermore, the determination of the efficiency of such procedures for a given total verification effort.

As the problem in its most general form is by far too complicated for any analytical treatment, we will consider special cases with respect to the diversion strategy chosen by the plant operator (models A and B) and with respect to the sampling procedure chosen by the inspector (attribute and variable sampling).

3.1 Verification of Inventory Data

Inventory data verification is by its very nature a time independent problem: We assume that the plant operations have been stopped, that the operator has reported all inventory data. Furthermore, let us assume that the inspector verifies n_i of the N_i batch data in the i-th class with the help of independent measurements on a random sampling basis, and let the measurement result be y_{ij}, $i=1...K$, $j=1...n_i$. (For simplicity we have assumed that the data are rearranged in such a way, the the first n_i of the N_i batch data of the i-th class are verified.

Already in the foregoing chapter we have pointed out that frequently one material content determination consists of several independent measurements, e.g., volume-, concentration- and isotopic composition, and that consequently independent sampling plans could in principle be established for these measurements. For simplicity we assume here that always - if at all - the whole material content measurement of one batch is verified. The following considerations are based on the case that the verification of the material data of one item is performed with the help of a unique measurement; the more complicated case can be treated as well, but leads to more complicated formulae.

As the inspector must not use the same instruments as the operator, we have under the assumption, that no data are falsified by the operator (null hypothesis H_O)

$$y_{ij} = T_{ij}+e_{I_{ij}}+d_{I_i} , \quad i=1...K, \; j=1...n_i , \tag{3-1}$$

where the random errors $e_{I_{ij}}$ and the calibration errors $d_{I_{ij}}$, common to all measurements of the i-th class, are again assumed to be random variables with zero expectation values and known variances:

$$E(e_{I_{ij}}) = E(d_{I_i}) = 0 ;$$

$$var(e_{I_{ij}}) =: \sigma^2_{I_{ri}} ;$$

$$cov(e_{I_{ij}},e_{I_{i'j'}}) = 0 \quad for \quad (i,j)\neq(i',j') ;$$

$$var(d_{I_i}) =: \sigma^2_{I_{si}} ;$$

$$cov(d_{I_i},d_{I_{i'}}) = 0 \quad for \quad i\neq i' ;$$

$$cov(e_{I_{ij}},d_{I_{i'}}) = 0 \quad . \tag{3-2}$$

We now define two different falsification scenarios, which however do not comprise all possible falsification strategies.

We call *model* A the case that *all* data of the i-th class are falsified by the same class specific amount (alternative hypothesis H_1); we therefore have

$$y_{ij} = T_{ij}+\mu_i+e_{I_{ij}}+d_{I_i} , \quad i=1...K, \; j=1...n_i . \tag{3-3}$$

We call *model* B the case that only $r_i (\leq N_i)$ data of the i-th class are falsified by the same class specific amount (alternative hypothesis H_1); we therefore have for those data which are falsified, and which are selected by the inspector,

$$y_{ij} = T_{ij}+\mu_i+e_{I_{ij}}+d_{I_i} , \quad i=1...K, \; j=1...\ell , \tag{3-4}$$

where ℓ is a hypergeometrically distributed random variable, whereas we have for those data, which are not falsified and which are selected by the inspector,

$$y_{ij} = T_{ij}+e_{I_{ij}}+d_{I_i} \ , \quad i=1...K, \ j=1...n_i-\ell \ . \qquad (3-5)$$

It has been outlined already in the introduction that data falsification represents a second class of diversion strategies: The data are falsified in such a way that the book-physical-inventory difference MUF is not significant thus, the amount M_1 of material which is equivalent to the difference between the sum of the true and the reported data can be diverted, if the data verification procedure did not lead to significant differences between the reported data and the inspector's findings.

In the last years several formalisms have been developed for the determination of optimum sample sizes. In the following we first will present in some detail the game theoretical formalism for *model A* and for a special case of *model B* because according to our conviction this formalism is the appropriate one, as it explicitely takes into account the interent conflict situation. Thereafter, we discuss further approaches and compare these approaches with the help of some numerical calculations.

3.1.1 Game Theoretical Treatment of *Model A*

As the inspector is not interested in estimating the true values T_{ij}, but only in knowing whether or not data have been falsified, he will form a test on the basis of the differences $y_{ij}-x_{ij}$. For *model A* we can determine the optimum test statistics in the sense of the Neyman-Pearson-Lemma (i.e., that test statistics, which leads among all tests with false alarm probability α to the highest probability of detection) for a given total falsification. We will present here only the result of the analysis, the formal proof can be found in the literature /9/:

Let us consider the differences

$$z_{ij} := x_{ij}-y_{ij} \ , \quad i=1...K, \ j=1...n_i \ , \qquad (3-6)$$

where x_{ij} and y_{ij} are material data of the j-th item of the i-th class, reported by the operator and found by the inspector. Let the null hypothesis H_0 be given by

$$E(z_{ij}) = 0 \quad \text{for} \quad i=1...K, \ j=1...n_i \ , \qquad (3-7)$$

and the alternative hypthesis H_1

$$E(Z_{ij}) = \mu_i : \sum_i \mu_i \cdot N_i = M>O \quad \text{for} \quad i=1...K, \quad j=1...n_i \ . \tag{3-8}$$

Then the critical region of the Neyman-Pearson test for H_O and *any* H_1 with fixed value of M is given by the set

$$\left\{ D := \sum_{i=1}^{K} \frac{N_i}{n_i} \cdot \sum_{j=1}^{n_i} Z_{ij} : D>s \right\} \ . \tag{3-9}$$

The test characteristic or probability of detection 1-β defined by

$$1-\beta := \text{prob}\{D>s | H_1\} \tag{3-10}$$

as a function of the error first kind or false alarm probability α defined by

$$\alpha := \text{prob}\{D>s | H_O\} \tag{3-11}$$

is given by the following expression

$$1-\beta = \phi\left(\frac{M}{\sqrt{\sum_i \frac{N_i^2}{n_i} \cdot (\sigma_{ri}^2 + n_i \cdot \sigma_{si}^2)}} - U_{1-\alpha} \right) , \tag{3-12}$$

where σ_{ri}^2 and σ_{si}^2 are defined by

$$\sigma_{ri}^2 := \sigma_{Ori}^2 + \sigma_{Iri}^2 \ , \quad \sigma_{si}^2 := \sigma_{Osi}^2 + \sigma_{Isi}^2 \ , \quad i=1...N \ .$$

The optimal diversion strategy

$$(\mu_1^*, ..., \mu_K^*) : \sum_i \mu_i^* \cdot N_i = M \tag{3-13}$$

is given by the following expression

$$\mu_i^* = \frac{M}{\sigma^2(C)} \cdot \left[\left(\sum_k N_k \cdot \sigma_{rk} \cdot \sqrt{\varepsilon_k} \right) \cdot \sigma_{ri} \cdot \sqrt{\varepsilon_i} + C \cdot N_i \cdot \sigma_{si}^2 \right] , \quad i=1...K \ , \tag{3-14}$$

where

$$\sigma^2(C) := \left(\sum_i N_i \cdot \sigma_{ri} \cdot \sqrt{\varepsilon_i} \right)^2 + C \cdot \sum_i N_i^2 \cdot \sigma_{si}^2 \ .$$

We now assume that the verification of the data of one item of the i-th class requires the effort ε_i, i=1...K (man hours or money), and that the inspector has the total effort C for the inventory at his disposal. The problem of determining the optimal inspection sample sizes therefore consists in optimizing the probability of detection (3-12) with respect to the sample sizes n_i under the boundary condition

$$\sum_i \varepsilon_i \cdot n_i = C \ . \tag{3-15}$$

With the help of the Lagrange formalism we get the following optimal sample sizes n_i^*

$$n_i^* = \frac{C}{\sum_k N_k \cdot \sigma_{rk} \cdot \sqrt{\varepsilon_k}} \cdot \frac{N_i \cdot \sigma_{ri}}{\sqrt{\varepsilon_i}} \ , \quad i=1\ldots K \ , \tag{3-16}$$

and the *guaranteed* probability of detection

$$1-\beta^* = \phi\left(\frac{M \cdot \sqrt{C}}{\sigma(C)} - U_{1-\alpha}\right) \ . \tag{3-17}$$

As we would expect, this guaranteed probability of detection is a monotonuously increasing function of the total amount M assumed to be diverted, and the total verification effort C.

3.1.2 Game Theoretical Treatment of *Model B*

Contrary to the situation in the case of *model A* it is not possible to determine the optimum test statistic in the sense of the Neyman-Pearson-Lemma for *model B*. It is, however, reasonable, to use also in this case the D-statistics as test statistics. But still it is not possible to determine analytically optimum sample sizes without making assumptions about the parameter values. An approximate solution to this problem, where the measurement errors are taken into account, is presented in Ref. /9/; here we will consider a different case which thereafter will be compared with other approaches.

Attribute sampling (as opposed to *variable sampling*) is a procedure the purpose of which is to test whether or not the data of one item are falsified by an amount which is large compared to the measurement uncertainties. In the following we will consider this case which means that we completely ignore the measurement errors.

Let us first present the relation between total number N of items, inspection sample size n, number of falsified batch data r and probability of detection 1-β for one class of material, as it serves as a basis for all forthcoming considerations:

In case of *drawing without replacement* the probability of no detection is given by the following expression

$$\beta = \frac{\binom{r}{0}\cdot\binom{N-r}{n}}{\binom{N}{n}} = \frac{\binom{N-r}{n}}{\binom{N}{n}} \qquad (3\text{-}18)$$

which is a special case of the well known hypergeometric formula. Because of the identity

$$\frac{\binom{N-n}{r}}{\binom{N}{n}} = \frac{\binom{N-r}{n}}{\binom{N}{r}}$$

we can write this formula in the two following ways:

$$\beta = \prod_{j=0}^{n-1} \left(1-\frac{r}{N-j}\right) = \prod_{j=0}^{r-1} \left(1-\frac{n}{N-j}\right) \quad . \qquad (3\text{-}19)$$

For small inspection sample sizes, i.e., for n<<N, we can write it therefore as

$$\beta \approx \left(1-\frac{r}{N}\right)^{n} \qquad (3\text{-}20)$$

which ist the exact formula for the *drawing with replacement* case. For small number of falsifications we can write this as

$$\beta \approx \left(1-\frac{n}{N}\right)^{r} \quad . \qquad (3\text{-}21)$$

If the product $r\cdot n$ is much smaller than N, then both formulae (3-20) and (3-21) coincide to

$$\beta \approx 1-\left(\frac{r\cdot n}{N}\right)^{r} \quad . \qquad (3\text{-}22)$$

Game Theoretical Formalism /9/

Let us assume that we have K classes of material, and that every class consists of k_i subclasses with N_{k_i} batches, $1\le i\le K$, $1\le j\le k_i$. Every batch contains the amount μ_{ij} of material; the inspection effort for the verification of one element of the i-th class is ε_i, $1\le i\le K$.

The problem is to optimize the sample sizes n_{ij} in the various subclasses under the boundary condition of a fixed total verification effort C,

$$C \ge \sum_{i=1}^{K} \varepsilon_i \cdot \sum_{j=1}^{k_i} n_{ij} \quad , \qquad (3\text{-}23)$$

assuming that the operator wants to divert the total amount M of material by falsifying r_{ij} data of the j-th subclass of the i-th class by the amount μ_{ij},

$$M \leq \sum_{i=1}^{K} \sum_{j=1}^{k_i} \mu_{ij} \cdot r_{ij} \quad . \tag{3-24}$$

It should be noted that in the sense of the attribute sampling we have assumed that the operator falsifies - if at all - the data of r_{ij} batches by their full amount.

Let us start our approach with the formulae for the probability of detection $1-\beta$, which are generalizations of formula (3-19) to K classes,

- in case of drawing with replacement

$$1-\beta = 1- \prod_{i=1}^{K} \prod_{j=1}^{k_i} \left(1- \frac{r_{ij}}{N_{ij}}\right)^{n_{ij}} \tag{3-25}$$

- in case of drawing without replacement

$$1-\beta = 1- \prod_{i=1}^{K} \prod_{j=1}^{k_i} \frac{\binom{r_{ij}}{0}\binom{N_{ij}-r_{ij}}{n_{ij}}}{\binom{N_{ij}}{n_{ij}}} = 1- \prod_{i=1}^{K} \prod_{j=1}^{k_i} \prod_{m=0}^{n_{ij}-1} \left(1- \frac{r_{ij}}{N_{ij}-m}\right) \quad . \tag{3-26}$$

Again, for $n_{ij} \ll N_{ij}$ formula (3-26) is passing into formula (3-25).

It can be derived from very general game theoretic considerations that the optimum sample sizes of the inspector are solutions of a two-person zero-sum game with $-\beta$ as the payoff to the inspector and the following sets of strategies:

$$\left\{(n_{11} \ldots n_{k_K K}) \times (r_{11} \ldots r_{k_K K}) : \sum_{i=1}^{K} \varepsilon_i \sum_{j=1}^{k_i} n_{ij} = C \ , \quad \sum_{i=1}^{K} \sum_{j=1}^{k_i} \mu_{ij} \cdot r_{ij} = M \ , \right.$$

$$\left. n_{ij} \geq 0, \ r_{ij} \geq 0, \ j=1 \ldots k_i, \ i=1 \ldots K \right\} \ .$$

If the sample sizes n_{ij} and r_{ij} are treated as continous variables, then the optimal strategies and the probability of detection at the saddle point are

$$n_{ij}^* = \frac{C \cdot \exp(-\kappa\cdot\varepsilon_i)}{\sum\limits_{\ell=1}^{K} \varepsilon_\ell \cdot \exp(-\kappa\cdot\varepsilon_\ell) \sum\limits_{h=1}^{k_\ell} \mu_{\ell h} N_{\ell h}} \cdot \mu_{ij} \cdot N_{ij} \,, \qquad (3\text{-}27)$$

$$1 \le i \le K \,, \quad 1 \le j \le k_i$$

$$r_{ij}^* = N_{ij} \cdot (1-\exp(-\kappa\cdot\varepsilon_i)) \,, \qquad (3\text{-}28)$$

$$1-\beta(\underline{n}^*, \underline{r}^*) = 1-\exp(-\kappa\cdot C) \;; \qquad (3\text{-}29)$$

here, the parameter κ is uniquely determined by the following relation

$$\sum_{i=1}^{K} \sum_{j=1}^{k_i} N_{ij} \cdot \mu_{ij} \cdot \exp(-\kappa\cdot\varepsilon_i) = \sum_{i=1}^{K} \sum_{j=1}^{k_i} \mu_{ij} \cdot N_{ij} - M \;. \qquad (3\text{-}30)$$

Two-step-procedure

From the practical point of view it is desirable to develop formulae which can be used as easy as possible by the inspectors at the plant site. The formalism developed so far lends itself to such a procedure as will be shown now.

From (3-27) we get the following distribution of the total given effort on the K classes, i.e., we get the following optimal class sample sizes n_i^*, $1 \le i \le K$:

$$n_i^* := \sum_{j=1}^{k_i} n_{ij}^* = \frac{C \cdot \exp(-\kappa\cdot\varepsilon_i)}{\sum\limits_{\ell=1}^{K} \varepsilon_\ell \cdot \exp(-\kappa\cdot\varepsilon) \cdot \sum\limits_{h=1}^{k_\ell} \mu_{\ell h} N_{\ell h}} \cdot \sum_{j=1}^{k_i} \mu_{ij} \cdot N_{ij} \qquad (3\text{-}31)$$

$$= \frac{C \cdot \exp(-\kappa\cdot\varepsilon_i)}{\sum\limits_{\ell=1}^{K} \varepsilon_\ell \cdot \exp(-\kappa\cdot\varepsilon_\ell) \cdot \bar{\mu}_\ell \cdot \sum\limits_{h=1}^{k_\ell} N_{\ell h}} \cdot \bar{\mu}_i \cdot \sum_{j=1}^{k_i} N_{ij} \quad ;$$

here, the quantity $\bar{\mu}_i$ defined by

$$\bar{\mu}_i := \frac{\sum\limits_{j=1}^{k_i} \mu_{ij} \cdot N_{ij}}{\sum\limits_{j=1}^{k_i} N_{ij}} \tag{3-32}$$

represents the expectation value of a random variable. The parameter κ in equation (3-31) is now the solution of the following equation

$$\sum_{i=1}^{K} \exp(-\kappa \cdot \varepsilon_i) \cdot \bar{\mu}_i \cdot \sum_{j=1}^{k_i} N_{ij} = \sum_{i=1}^{K} \bar{\mu}_i \sum_{j=1}^{k_i} N_{ij} - M \quad . \tag{3-33}$$

with $C_i^* := n_i^* \cdot \varepsilon_i$ we have

$$C_i^* = \frac{C \cdot \exp(-\kappa \cdot \varepsilon_i)}{\sum\limits_{\ell=1}^{K} \varepsilon_\ell \cdot \exp(-\kappa \cdot \varepsilon_\ell) \bar{\mu}_\ell \cdot \sum\limits_{h=1}^{k_\ell} N_{\ell h}} \cdot \bar{\mu}_i \cdot \varepsilon_i \sum_{j=1}^{k_i} N_{ij} \quad . \tag{3-34}$$

The advantage of these formulae is that they can be calculated before the actual inventory verification procedure at the safeguards authority's headquarters, if there exists some idea about the values of μ_i, $1 \le i \le K$, e.g. from earlier investigations. At the plant site the inspector then can determine the sample sizes n_{ij}^* for the subclasses, after having obtained the values of μ_{ij} and N_{ij}, $1 \le i \le K$, $1 \le j \le k_i$, according to the following formula

$$n_{ij}^* = \frac{C_i^*}{\bar{\mu}_i \cdot \varepsilon_i \cdot \sum\limits_{h=1}^{k_i} N_{ih}} \cdot \mu_{ij} \cdot N_{ij} \tag{3-35}$$

where $\bar{\mu}_i$ and C_i^* are given by (3-32) and (3-34).

Let us still consider the following problem. The values of the $\bar{\mu}_i$, $1 \le i \le K$ came from data of foregoing inspections thus, it is possible but there are some differences between these and actual data which are not to be neglected. In such a case the following procedure is possible: during inspection the inspector calculates the actual $\bar{g}_i$, $1 \le i \le K$, with the help of formula (3-32) and the actual data and we assume

$$\bar{\mu}_i - \bar{g}_i = \lambda \ , \qquad 1 \le i \le K \ , \qquad \lambda \in \mathbb{R} \quad . \tag{3-36}$$

With this λ we can generalize our above formula:

$$n_i^*(\lambda) := \frac{C(\lambda)\cdot\exp(-\kappa(\lambda)\cdot\varepsilon_i)}{\sum_{\ell=1}^{K}\varepsilon_\ell\cdot\exp(-\kappa(\lambda)\cdot\varepsilon_\ell)(\bar{\mu}_\ell+\lambda)\sum_{h=1}^{k_\ell}N_{\ell h}}\cdot(\bar{\mu}_i+\lambda)\sum_{j=1}^{k_i}N_{ij} \qquad (3\text{-}37)$$

where $\kappa(\lambda)$ is the solution of

$$\sum_{i=1}^{K}\exp(-\kappa(\lambda))\ (\bar{\mu}_i+\lambda)\sum_{j=1}^{k_i}N_{ij} = \sum_{i=1}^{K}(\bar{\mu}_i+\lambda)\sum_{j=1}^{k_i}N_{ij} - M \qquad (3\text{-}38)$$

and the inspection effort is

$$C(\lambda) = -\frac{\ell n\ \beta}{\kappa(\lambda)}\ , \qquad (3\text{-}39)$$

where $1-\beta$ is the given probability of detection. So we have the inspection effort in one class

$$C_i^*(\lambda) := \varepsilon_i\cdot n_i^*(\lambda)\ ,\quad 1\le i\le K$$

$$= \frac{C(\lambda)\cdot\exp(-\kappa(\lambda)\cdot\varepsilon_i)}{\sum_{\ell=1}^{K}\varepsilon_\ell\cdot\exp(-\kappa(\lambda)\varepsilon_\ell)(\bar{\mu}_\ell+\lambda)\sum_{h=1}^{k_\ell}N_{\ell h}}\cdot\varepsilon_i\cdot(\bar{\mu}_i+\lambda)\cdot\sum_{j=1}^{k_i}N_{ij}\ . \qquad (3\text{-}40)$$

The formulae (3-37) - (3-40) can be computed before inspection for various λ. At the beginning of the inspection the inspector has to determine the λ and with the results in his list he is able to calculate the sample sizes

$$n_{ij}^* = \frac{C_i^*(\lambda)}{(\mu_i+\bar{\lambda})\cdot\varepsilon_i\sum_{h=1}^{k_i}N_{ih}}\cdot\mu_{ij}\cdot N_{ij} \qquad \begin{matrix}1\le j\epsilon k_i\\ 1\le i\le K\end{matrix} \qquad (3\text{-}41)$$

If we now assume that $\varepsilon_1=\ldots=\varepsilon_K=\varepsilon$ then we know that $\kappa(\lambda)$ is solution of

$$\exp(-\kappa(\lambda)\cdot\varepsilon)\cdot\sum_{i=1}^{K}\bar{\mu}_i\sum_{j=1}^{k_i}N_{ij} + \lambda\cdot\exp(-\kappa(\lambda)\cdot\varepsilon)\cdot N = \sum_{\ell=1}^{K}\mu_i\sum_{j=1}^{k_i}N_{ij} - M + \lambda\sum_{i=1}^{K}\ \sum_{j=1}^{k_i}N_{ij}$$

$$(3\text{-}42)$$

which leads to

$$\exp(-\kappa(\lambda)\cdot\varepsilon) = 1-\frac{M}{\sum_{i=1}^{K}\bar{\mu}_i \sum_{j=1}^{k_i} N_{ij}+\lambda} \quad .$$

So we have the result that $\kappa(\lambda)$ is monotonically decreasing in λ and $C(\lambda)$ monotonically increasing in λ.

Measurement Errors

Although in this section attribute sampling is considered, it is interesting to determine the influence of random errors on the probability of detection. (It should be kept in mind, however, that in the case that measurement errors cannot be ignored, the sample sizes (3-27) and (3-28) are *not* optimal.

Let us assume that the classes are homogeneous (or equivalently, let us restrict on the consideration of class sample sizes). If the D-statistics (3-9) is used as test statistics, then the probability of detecting an falsification of total size M is approximately given by (see /5/)

$$1-\beta = \phi\left(\frac{M-\sigma_{D|H_0}\cdot U_{1-\alpha}}{\sigma_{D|H_1}}\right) , \tag{3-43}$$

where the variances $\sigma^2_{D|H_0}$ and $\sigma^2_{D|H_1}$ are given by

$$\sigma^2_{D|H_0} = \sum_{i=1}^{K} N_i^2\cdot\left(\frac{\sigma^2_{ri}}{n_i^*}+\sigma^2_{si}\right)$$

$$\sigma^2_{D|H_1} = \sigma^2_{D|H_0} + \sum_i \mu_i^2\cdot r_i^*\cdot(N_i-r_i^*)\cdot\left(\frac{1}{n_i^*}-\frac{1}{N_i}\right) \quad , \tag{3-44}$$

where σ^2_{ri} and σ^2_{si} are given by

$$\sigma^2_{ri} = \sigma^2_{Ori}+\sigma^2_{Iri} \ , \quad \sigma^2_{si} = \sigma^2_{Osi}+\sigma^2_{Isi} \ ,$$

and where n_i^* and r_i^* are given by (3-27) and (3-28) with

$$n_i^* = \sum_j n_{ij}^* \ , \quad r_i^* = \sum_j r_{ij}^* \quad .$$

3.1.3 Comments about Attribute Sampling According to the IAEA Technical Manual, Part F /10/

Short description

Let M be the goal quantity and $1-\beta$ the total probability of detection to be guaranteed by the sampling scheme. If there are K classes of material, and if every batch of the i-th class, i=1...K, contains the amount μ_i of material, then the operator has to falsify r_i batch data of the i-th class, where

$$r_i = \frac{M}{\mu_i} \ , \quad i=1...K \ , \tag{3-45}$$

if he wants to divert the total amount M of material, which corresponds to the goal quantity, from the i-th class. According to formula (3-21) in the case

$$r_i << N_i \ , \quad i=1...K \ ,$$

the class probability of not detection β_i is then given by

$$\beta_i = \left(1 - \frac{n_i}{N_i}\right)^{r_i} \ , \quad i=1...K \ . \tag{3-46}$$

If however the operator wants to divert the amount M of material by distributing this diversion over the K classes according to

$$M = \sum_{i=1}^{K} M_i \ , \tag{3-47}$$

where M_i is the amount of material to be diverted from the i-th class, i=1...K, then he has to falsify $\tilde{r}_i$ batch data of the i-th class, where

$$\tilde{r}_i = \frac{M_i}{\mu_i} \ , \quad i=1...K \ . \tag{3-48}$$

In this case the probability of no detection $\tilde{\beta}_i$ for the i-th class is given by

$$\tilde{\beta}_i = \left(1-\frac{n_i}{N_i}\right)^{\tilde{r}_i}, \quad i=1\ldots K . \tag{3-49}$$

From (3-45) and (3-48) we get

$$\tilde{r}_i = \frac{M_i}{\mu_i} = \frac{M_i}{M}\cdot r_i , \quad i=1\ldots K ,$$

therefore we get with (3-49)

$$\tilde{\beta}_i = \left(1-\frac{n_i}{N_i}\right)^{\frac{M_i}{M}\cdot r_i} = \beta_i^{\frac{M_i}{M}} .$$

If the inspector now determines his sample sizes n_i, $i=1\ldots K$, according to (3-46) such that the class probabilities of no detection are all equal to β, then we get for the overall probability of no detection in case that the diversion is distributed over the K classes

$$\prod_{i=1}^{K} \tilde{\beta}_i = \prod_{i=1}^{K} \beta_i^{\frac{M_i}{M}} = \prod_{i=1}^{K} \beta^{\frac{M_i}{M}} = \beta . \tag{3-50}$$

In other words: If the inspector determines his sample sizes n_i, $i=1\ldots K$, such that the class probability β of no detection is guaranteed under the assumption that the total amount M is diverted from one class, then the total probability of no detection is again β under the assumption, that the diversion is distributed over the K classes.

Comments

a) As already mentioned, this procedure is only approximately valid for $r_i \ll N_i$, $i=1\ldots K$.

For N=5, r=3 and n=2, e.g., we get

$$(1-\frac{n}{N})^n = .0216 ; \quad \prod_{j=0}^{r-1}\left(1-\frac{n}{N-j}\right) = .1 .$$

b) In order to be able to use this procedure it is necessary that the total amount M of material can be diverted from one class.

c) Different inspection efforts ε_i, i=1...K, are not taken into account in this formalism, this is achieved only in the game theoretic formalism.

For comparison purposes, we derive from the game theoretic formula (3-27) the following optimum sample sizes for $\varepsilon_1=\varepsilon_2=\ldots=\varepsilon_k=:\varepsilon$

$$n_i^* = \frac{\ln \beta}{\ln\left(1-\frac{M}{\sum_{j=1}^{K}\mu_j\cdot N_j}\right)\sum_{j=1}^{K}\mu_j\cdot N_j}\cdot\mu_i\cdot N_i\ ; \quad 1\leq i\leq K \quad . \qquad (3\text{-}27a)$$

In Table 3.1 two numerical examples are given for illustrative purposes. Under the condition $\varepsilon_i=\varepsilon$ for i=1,2,...,K it is possible to show under certain assumptions an analytical equivalence [1] of

$$n_i = N_i(1-\beta_i^{1/r_i}) = N_i(1-\beta_i^{\mu_i/M}) \qquad i=1,2,\ldots,K \qquad (3\text{-}46a)$$

which corresponds to the formula in Manual F and formula (3-27a). If we assume

$$M << \sum_{j=1}^{K}\mu_j\cdot N_j$$

then we have

$$\ln\left(1-\frac{M}{\sum_{j=1}^{K}\mu_j\cdot N_j}\right) = -\frac{M}{\sum_{j=1}^{K}\mu_j\cdot N_j}$$

and

$$n_i^* = -N_i\cdot\frac{\mu_i\cdot\ln\beta}{M} \quad .$$

Now we expand $\beta^{\mu_i/M}$ in MacLaurins' series, retaining only the first order term with μ_i/M as variable. Thus

$$\beta^{\mu_i/M} = 1 + \ln\beta^{\mu_i/M}$$

and

---

[1]) This equivalence has been pointed out by J. Jaech, Exxon, Richland, Wash. State.

	n_1		n_2	
	Manual F	Game theory	Manual F	Game theory
<u>Example 1</u> $N_1 = 80$ $N_2 = 10$ $\mu_1 = 100$ g $\mu_2 = 1500$ g	<4.65> = 5	<4.25> = 5	<5.27> = 6	<7.97> = 8
<u>Example 2</u> $N_1 = 2000$ $N_2 = 30$ $\mu_1 = 5$ g $\mu_2 = 1000$ g	<5.89> = 6	<5.61> = 6	<13.52> = 14	<16.82> = 17

<u>Table 3-1:</u> Comparison of 'optimum' sample sizes according to Manual F and according to the game theoretic formalism for 2 classes of material with the same inspection effort for one batch of each class, $M = 5000$ g, $\beta = 0.05$

$$1 - \beta^{\mu_i/M} = \frac{\mu_i}{M} \cdot \ln \beta$$

and we have that the n_i of formula (3-46a) is equal to the n_i^* of formula (3-27a).

3.1.4 Comments about Mass Value Sampling (/11/)

Short description

Let us assume that there is a set of N batches which together contain the amount x of material; the single batches may contain different parts of material. Then the mass-value sampling recommends the following procedure:

- Choose a mass unit z, e.g., the minimum discrepancy which can be detected by the inspector's quantitative technique; the population size in mass units then is P=x/z.
- Calculate the sample size v(z), which is required for the detection of a diversion of the amount M of material, by postulating the probability 1-β for the detection of at least one falsification in mass units; with r=M/z one gets from (3-21)

$$v(z) = P \cdot \left(1-\beta^{\frac{1}{r}}\right) = \frac{x}{z} \cdot \left(1-\beta^{\frac{1}{r}}\right) = \frac{x}{z} \cdot \left(1-\beta^{\frac{z}{M}}\right) \quad . \qquad (3\text{-}51)$$

- Draw the sample sizes as follows: Make a random order of the batches and from a cumulative sum of the mass units for the total amount x of material, select v(z) different random numbers up to P or select the units for examination at intervals of P/v(z) starting from a random point which is less than P/v(z).

Comments

From an exact analytical point of view there are some questions and points in mass value sampling which cannot be resolved satisfactorily:

a) How shall the mass value z be determined?

b) What is the probability to detect at least one falsified batch?

c) Different inspection efforts ε_i, i=1...K, for different classes of material are not taken into account in this formalism.

Whereas not very much can be said to points a) and c), we can give an upper limit for the probability of no detection, depending on the values of x and β:

From (3-51) we conclude, that for z>0 v(z) is a monotone decreasing function in z with

$$\lim_{z \to 0+} v(z) = -\frac{x}{M} \cdot \ln \beta =: v \ . \qquad (3\text{-}52)$$

The probability to detect one falsified batch datum by drawing one batch is greater or equal to

$$\frac{M}{z} \Big/ \frac{x}{z} = \frac{M}{x} \ , \qquad (3\text{-}53)$$

thus, the probability of detecting no falsified batch datum is smaller or equal to

$$1 - \frac{M}{x} \ . \qquad (3\text{-}54)$$

If we draw with replacement v(z) mass units, then we get with (3-51) for the probability w of detecting no falsified batch datum the following upper limit:

$$w \le \left(1 - \frac{M}{x}\right)^{v(z)} = \left(1 - \frac{M}{x}\right)^{\frac{x}{z} \cdot (1-\beta^{\frac{z}{M}})} =: \delta(z) \ ; \qquad (3\text{-}55)$$

therefore, solving this equation for β, we get with

$$\beta = \left(1 - \frac{z}{x} \cdot \frac{\ln \delta(z)}{\ln(1 - \frac{M}{x})}\right)^{\frac{M}{z}} \qquad (3\text{-}56)$$

the following results: As $\delta(z)$ is a strictly increasing function in z, the inspector should choose his mass unit z as small as possible. With the help of (3-56), he can determine the value of β for a given upper limit $\delta(z)$ of detecting no falsification and thereafter, he can determine his sample size v(z) with the help of (3-51).

Let us still have a look at the limiting case given by (3-52): Because of

$$-\frac{x}{M} \cdot \ln \beta \ge v(z) \quad \text{for all} \quad z \ge 0 \qquad (3\text{-}57)$$

we have

$$\delta := (1-\frac{M}{x})^{-\frac{x}{M}\cdot\ln\beta} < (1-\frac{M}{x})^{v(z)} \quad , \tag{3-58}$$

or with (3-55),

$$\delta < \delta(z) \quad \text{for all} \quad z>0 \; . \tag{3-59}$$

If we calculate the limiting mass unit sample size v according to (3-52), then we get with (3-55)

$$w \leq (1-\frac{M}{x})^{-\frac{x}{M}\cdot\ln\beta} \tag{3-60}$$

the following result: In the limiting case $z\to 0$ the inspector can with the help of (3-58) determine the value of β for a given limit δ of the probability of detecting no falsification. Thereafter, he can determine his mass value sample size v with the help of (3-52).

It should be noted that in both cases, $z>0$ and $z\to 0$, the sample sizes $v(z)$ and v are determined by postulating an upper limit, $\delta(z)$ and δ, for the probability of detecting no *batch datum* falsification, and not by postulating a value for the - only technically interesting - probability β of detecting no *mass unit datum* falsification.

3.2 Remark on the Verification of Flow Data

In analyzing the problem of verifying inventory data one assumes that all data of the operator are available at the same time and furthermore, that the batches are also available at the same time such that the inspector can select according to a sampling plan some of the batches in order to verify their data with the help of independent measurements.

Flow data, i.e., data of input or output batches, are generated in a sequential manner and the batches will not be available at the same time as they have either disappeared in the production process or have been shipped before later batches are ready for shipment. Thus, the inspector has to decide from batch to batch whether or not he will verify its data.

In analyzing this problem one has to make a distinction (which is irrelevant in the inventory verification case): One either has to assume that the operator is decided a priori to falsify a certain part of the data, or one has to assume that he will decide from batch to batch whether or not he will falsify its data. In the first case the situation is not so different from that of the inventory verification, in fact one arrives at similar results. In the second case the situation becomes difficult; so far no satisfying theory has been developed for several classes of material.

In the practical application presented in the second and third part of this paper these questions play no role because of the plant and operational conditions. Therefore, we will not present the theory here but refer the interested reader to the literature /5/.

4. Systems Effectiveness

It has been mentioned already in the introduction that according to the safeguards procedure the operator of a nuclear plant may consider two principally different sets of diversion strategies, namely

- the strategies by which material is diverted without data falsification, relying on the hope that the inaccurate material balance does not allow detection of this diversion and
- the strategies by which the material balance data are falsified so that the material balance 'evens up' and the difference between the reported and the true value can be diverted.

Generally speaking, the two safeguards measures, which counter the two strategies just mentioned, culminate in the execution of two tests, namely the D-test and the MUF-test, with null and alternative hypotheses

$$H_0: E(D) = E(MUF) = 0$$

$$H_1: E(D) = M_1 \ , \ E(MUF) = M_2 \ , \qquad (4\text{-}1)$$

and with the boundary condition of a fixed overall false alarm probability. So far, we have suboptimized the two test procedures. Now, the question arises whether or not there exists an 'optimal' combined test procedure, and furthermore, what the efficiency of that procedure is. In the following, we will analyze two different procedures of the inspector, thereafter, we will discuss their qualitative aspects.

A technical problem is given by the fact that the two test statistics MUF and D are not independent because the generator's data are used in both these statistics.

4.1 Separate Tests for the Material Balance Establishment and Data Verification

The considerations of this section are valid both for *models* A and B if we assume that the D-statistics is normally distributed (this is exactly true for *model* A and approximately true for *model* B). We only have to take into account that the variances of the D-statistics under the null and under the alternative hypotheses are the same for *model* A, and different for *model* B (see (3-44)). For the sake of generality, we always will write $\sigma_{D|H_0}$ and $\sigma_{D|H_1}$ and keep in mind that for *model* A these two variances are equal.

Bivariate Test

Let us assume that first the data verification test is performed and thereafter the material balance test. Let s_1 and s_2 be the significance thresholds for these two tests. Then the single error first kind probabilities α_1 and α_2 are given by

$$1-\alpha_1 := \text{prob}\{-D\leq s_1 | H_0\} \;;\quad 1-\alpha_2 := \text{prob}\{MUF\leq s_2 | H_0\} \qquad (4\text{-}2)$$

and the total error first kind probability (total false alarm probability) α is given by

$$1-\alpha := \text{prob}\{-D\leq s_1 \wedge MUF\leq s_2 | H_0\} \qquad (4\text{-}3)$$

which leads to the following expression (/5/)

$$1-\alpha = \frac{1}{2\pi\sqrt{1-\rho_{H_0}^2}} \cdot \int_{-\infty}^{U_{1-\alpha_1}} dt_1 \int_{-\infty}^{U_{1-\alpha_2}} dt_2 \exp\left(-\frac{t_1^2-2\rho_{H_0} t_1 t_2+t_2^2}{2(1-\rho_{H_0}^2)}\right), \qquad (4\text{-}4)$$

where the correlation ρ_{H_0} is given by calculating $\text{cov}(D,MUF)=\sigma_{MUF}^2$

$$\rho_{H_0} := \frac{\sigma_{MUF}^2}{\sigma_{MUF}\cdot\sigma_{D|H_0}} = \frac{\sigma_{MUF}}{\sigma_{D|H_0}} \qquad (4\text{-}5)$$

and where U is the inverse of the Gaussian distribution function. In Figure 4-1 this relation between α_1 and α_2 is represented graphically for fixed value of α with ρ as parameter.

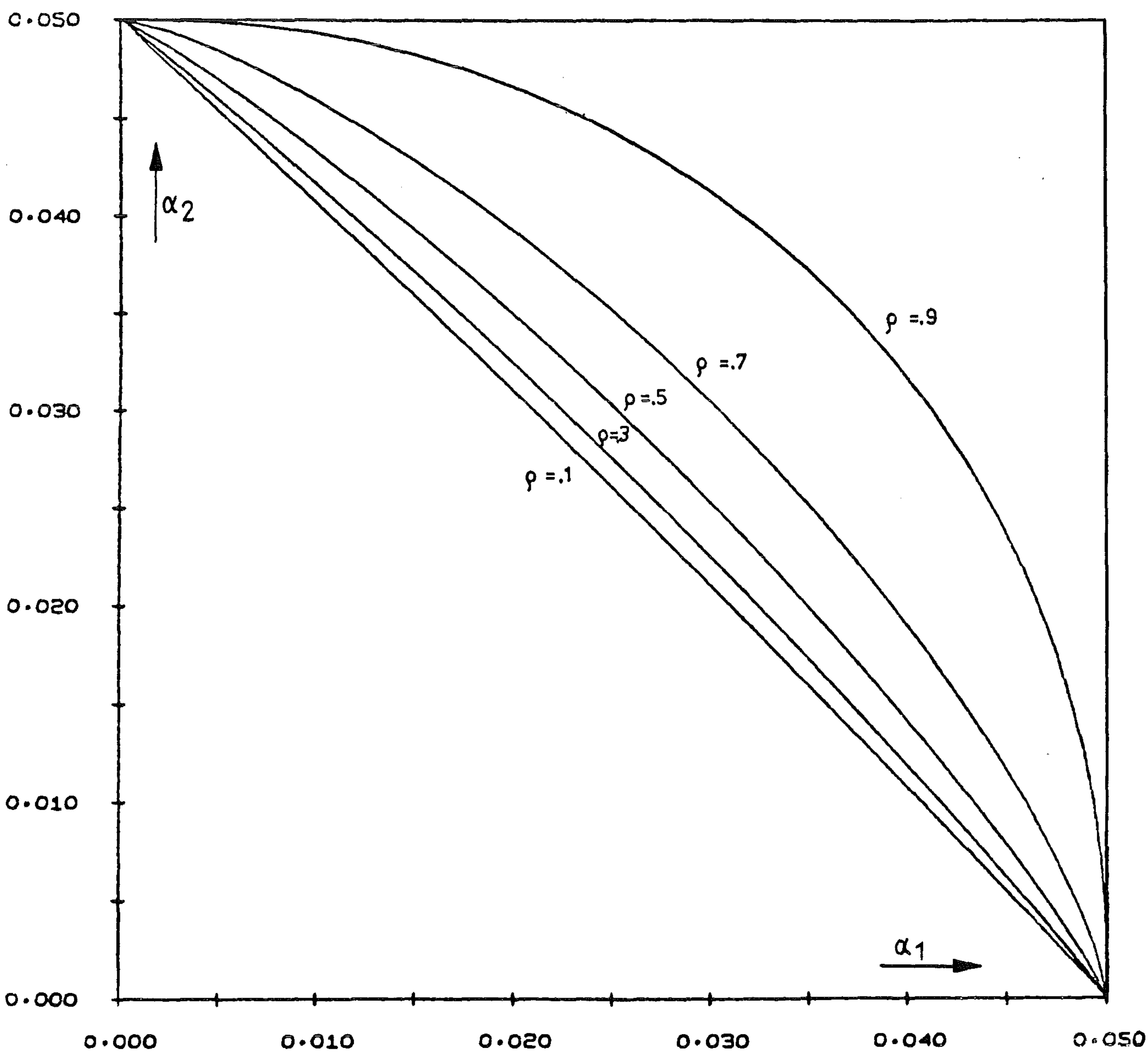

Figure 4-1: Graphical representation of the relation

$$1-\alpha = \frac{1}{2\pi\cdot\sqrt{1-\rho^2}} \int_{-\infty}^{U_{1-\alpha_1}} dt_1 \int_{-\infty}^{U_{1-\alpha_2}} dt_2 \exp\left(-\frac{t_1^2-2\rho t_1 t_2+t_2^2}{2(1-\rho^2)}\right)$$

between α_1 and α_2 for $\alpha=0.05$ and ρ as parameter.

The total error second kind probability (one minus the total probability of detection) is given by

$$\beta := \text{prob}\{-D \le s_1 \wedge MUF \le s_2 | H_1\} \tag{4-6}$$

which leads to the following expression

$$\beta = \frac{1}{2\pi\sqrt{1-\rho_{H_1}^2}} \cdot \int_{-\infty}^{\frac{\sigma_{D|H_0}}{\sigma_{D|H_1}} \cdot U_{1-\alpha_1} - \frac{M_1}{\sigma_{D|H_1}}} dt_1 \int_{-\infty}^{U_{1-\alpha_2} - \frac{M_2}{\sigma_{MUF}}} dt_2 \exp\left(-\frac{t_1^2 - 2\rho_{H_1} t_1 t_2 + t_2^2}{2(1-\rho_{H_1}^2)}\right), \tag{4-7}$$

where the correlation ρ_{H1} is given by

$$\rho_{H_1} := \frac{\sigma_{MUF}^2}{\sigma_{MUF} \cdot \sigma_{D|H_1}} = \frac{\sigma_{MUF}}{\sigma_{D|H_1}} \quad . \tag{4-8}$$

The optimal significance thresholds s_1 and s_2, or in other words, the optimum single false alarm probabilities α_1 and α_2 for a given total false alarm probability are defined as those single false alarm probabilities which minimize the probability of no detection under the assumption that the goal quantity M of material is distributed in an optimal way from the point of view of the operator. Thus, we have to solve the following optimization problem

$$\min_{\alpha_1, \alpha_2} \max_{M_1, M_2} \beta(M_1, M_2; \alpha_1, \alpha_2) \tag{4-9}$$

where $\beta(M_1, M_2; \alpha_1, \alpha_2)$ is given above, where furthermore α_1, α_2 is subject to the boundary condition (4-4), and where finally M_1 and M_2 are subject to the boundary condition $M_1+M_2=M$.

Solution of the Optimization Problem

Let us carry through first the maximization of β with respect to M_1 and M_2. If we eliminate M_2 by $M-M_1$ and use for the derivation of a function of the type

$$F(x) = \int_{-\infty}^{g(x)} dt\, f(t,x) \tag{4-10}$$

the well-known formula

$$\frac{d}{dx} F(x) = f(g(x),x) \cdot \frac{d}{dx} g(x) + \int_{-\infty}^{g(x)} dt \frac{d}{dx} f(t,x) \quad , \qquad (4\text{-}11)$$

we get with

$$\sigma_{D|H_i} =: \sigma_i \ , \quad \rho_{D|H_i} =: \rho_i \ , \qquad i = 0, 1, \qquad (4\text{-}12)$$

the following expression for the derivation of β with respect to M_1:

$$\sqrt{2\pi} \cdot \frac{\partial}{\partial M_1} \beta(M_1,\alpha_1,\alpha_2) = -\frac{1}{\sigma_1} \cdot \exp\left(-\frac{1}{2}\left[\frac{\sigma_0}{\sigma_1} \cdot U_{1-\alpha_1} - \frac{M_1}{\sigma_1}\right]^2\right) \cdot \phi\left(\frac{U_{1-\alpha_2} - \frac{M-M_1}{\sigma_{MUF}} - \rho_1 \cdot \left[\frac{\sigma_0}{\sigma_1} U_{1-\alpha_1} - \frac{M_1}{\sigma_1}\right]}{\sqrt{1-\rho_1^2}}\right)$$

$$+ \frac{1}{\sigma_{MUF}} \cdot \exp\left(-\frac{1}{2}\left[U_{1-\alpha_2} - \frac{M-M_1}{\sigma_{MUF}}\right]^2\right) \cdot \phi\left(\frac{\frac{\sigma_0}{\sigma_1} \cdot U_{1-\alpha_1} - \frac{M_1}{\sigma_1} - \rho_1 \cdot \left[U_{1-\alpha_2} - \frac{M-M_1}{\sigma_{MUF}}\right]}{\sqrt{1-\rho_1^2}}\right) , \qquad (4\text{-}13)$$

and the optimal value M_1^* of M_1 for arbitrary α_1 and α_2 is given by

$$\left.\frac{\partial}{\partial M_1} \beta(M_1,\alpha_1,\alpha_2)\right|_{M_1^*} = 0 \ . \qquad (4\text{-}14)$$

Second, we carry through the minimization of β with respect to α_1 and α_2. We assume that α_2 is eliminated by using the relation between $\alpha, \alpha_1, \alpha_2$ and ρ; in fact we get by implicite differentiation

$$0 = \exp\left(-\frac{U_{\alpha_2}^2}{2}\right) \cdot \phi\left(\frac{U_{1-\alpha_1} - \rho_0 \cdot U_{1-\alpha_2}}{\sqrt{1-\rho_0^2}}\right) \cdot \frac{dU_{\alpha_2}}{dU_{\alpha_1}} + \exp\left(-\frac{U_{\alpha_1}^2}{2}\right) \cdot \phi\left(\frac{U_{1-\alpha_2} - \rho_0 \cdot U_{1-\alpha_1}}{\sqrt{1-\rho_0^2}}\right) . \qquad (4\text{-}15)$$

By using the relation

$$\frac{dU_{1-\alpha}}{d\alpha} = -\frac{dU_\alpha}{d\alpha} = -\sqrt{2\pi} \cdot \exp\left(\frac{U_\alpha^2}{2}\right) \qquad (4\text{-}16)$$

we obtain from (4-15)

$$0 = \phi\left(\frac{U_{1-\alpha_2} - \rho_0 \cdot U_{1-\alpha_1}}{\sqrt{1-\rho_0^2}}\right) + \phi\left(\frac{U_{1-\alpha_1} - \rho_0 \cdot U_{1-\alpha_2}}{\sqrt{1-\rho_0^2}}\right) \cdot \frac{d\alpha_2}{d\alpha_1} \quad . \qquad (4\text{-}17)$$

Therefore, we get the following expression for the derivation of β with respect to α_1:

$$\frac{\partial}{\partial \alpha_1} \beta(M_1, M_2, \alpha) = -\exp\left(-\frac{1}{2}\cdot\left[\frac{\sigma_0}{\sigma_1}\cdot U_{1-\alpha_1} - \frac{M_1}{\sigma_1}\right]^2\right)\cdot\phi\left(\frac{U_{1-\alpha_2} - \frac{M_2}{\sigma_{MUF}} - \rho_1\cdot\left[\frac{\sigma_0}{\sigma_1}\cdot U_{1-\alpha_1} - \frac{M_1}{\sigma_1}\right]}{\sqrt{1-\rho_1^2}}\right)\cdot$$

$$\frac{\sigma_0}{\sigma_1}\cdot\exp\left(\frac{U_{\alpha_1}}{2}\right) + \exp\left(-\frac{1}{2}\left[U_{1-\alpha_2} - \frac{M_2}{\sigma_{MUF}}\right]^2\right)\cdot\phi\left(\frac{\frac{\sigma_0}{\sigma_1}\cdot U_{1-\alpha_1} - \frac{M_1}{\sigma_1} - \rho_1\cdot\left[U_{1-\alpha_2} - \frac{M_2}{\sigma_{MUF}}\right]}{\sqrt{1-\rho_1^2}}\right)\cdot$$

$$\cdot \exp\left(\frac{U^2_{\alpha_2}}{2}\right)\cdot\frac{d\alpha_2}{d\alpha_1} \quad . \qquad (4\text{-}18)$$

Using the determinant (4-14) for the optimal value M_1^* of M_1, we get the following expression for the derivation of β with respect to α_1 at the point (M_1^*, M_2^*):

$$\frac{\partial}{\partial \alpha_1} \beta(M_1^*, M_2^*, \alpha_1) = \frac{\sigma_0}{\sigma_1}\cdot\exp\left(-\frac{1}{2}\left[\frac{\sigma_0}{\sigma_1}\cdot U_{1-\alpha_1} - \frac{M_1^*}{\sigma_1}\right]^2\right)\cdot\phi\left(\frac{U_{1-\alpha_2} - \frac{M_2^*}{\sigma_{MUF}} - \rho_1\cdot\left[\frac{\sigma_0}{\sigma_1}\cdot U_{1-\alpha_1} - \frac{M_1}{\sigma_1}\right]}{\sqrt{1-\rho_1^2}}\right)$$

$$\cdot\left[\exp\left(\frac{U^2_{\alpha_1}}{2}\right) + \frac{\sigma_{MUF}}{\sigma_0}\cdot\exp\left(\frac{U^2_{\alpha_2}}{2}\right)\cdot\frac{d\alpha_2}{d\alpha_1}\right] \quad . \qquad (4\text{-}19)$$

As the first three factors of this relation are always greater zero, we obtain with (4-17) the following determinant for the optimal values of α_1 and α_2:

$$\exp\left(\frac{U^2_{\alpha_1}}{2}\right)\cdot\phi\left(\frac{U_{1-\alpha_1} - \rho_0\cdot U_{1-\alpha_2}}{\sqrt{1-\rho_0^2}}\right) - \rho_0 \cdot\exp\left(\frac{U^2_{\alpha_2}}{2}\right)\cdot\phi\left(\frac{U_{1-\alpha_2} - \rho_0\cdot U_{1-\alpha_1}}{\sqrt{1-\rho_0^2}}\right) = 0 \qquad (4\text{-}20)$$

which shows, together with the false alarm relation (4-4) that the optimal values of α_1 and α_2 are *independent upon the goal quantity* M.

In Figure 4-2 relation (4-20) has been represented graphically, and in Figure 4-3 both the relations (4-4) and (4-20) together with their intersections (dashed line) have been represented; the set of these intersections is the set of optimal solutions (α_1^*, α_2^*).

Properties of the Solution

In order to see whether or not the solutions found above are really solutions of the optimization problem (4-9), we have to determine the second derivatives of β with respect to M_1 and α_1. As these derivates are somewhat lengthy, and as no clear analytical conclusions can be drawn, we only report that so far all numerical calculations have shown, that the solutions (4-18) and (4-20) infact are solutions to the optimization problem (4-9).

There is a geometrical interpretation of eq. (4-20), which also allows a graphical determination of the optimal significance thresholds:

The necessary condition for the optimal values of α_1 and α_2 is according to (4-19) given by

$$\exp\left(\frac{U_{\alpha_1}^2}{2}\right) + \frac{\sigma_{MUF}}{\sigma_O} \cdot \exp\left(\frac{U_{\alpha_2}^2}{2}\right) \cdot \frac{d\alpha_2}{d\alpha_1} = 0 \quad . \tag{4-21}$$

This represents a differential equation the general solution of which is with $\sigma_{MUF}|\sigma_O = \rho_O$:

$$U_{\alpha_1} + \rho_O \cdot U_{\alpha_2} = \text{const} \ . \tag{4-22}$$

This represents a set of straight lines with slope $\frac{1}{\rho_O}$ in the $(U_{\alpha_1}, U_{\alpha_2})$-plane (see Figure 4-4). In order that there exists a unique solution (α_2^*, α_1^*), the constant must be determined in such a way that the straight line touches the line which represents eq. (4-4) for fixed values of α and ρ_O. As the straight line has the slope $-\frac{1}{\rho_O}$,

$$\frac{dU_{\alpha_2}}{dU_{\alpha_1}} = -\frac{1}{\rho_O} \quad ,$$

we get from (4-15) a condition for the tangential point (α_2^*, α_1^*), which is just again (4-20).

It should be noted, however, that Figure 4-3 is better suited for the

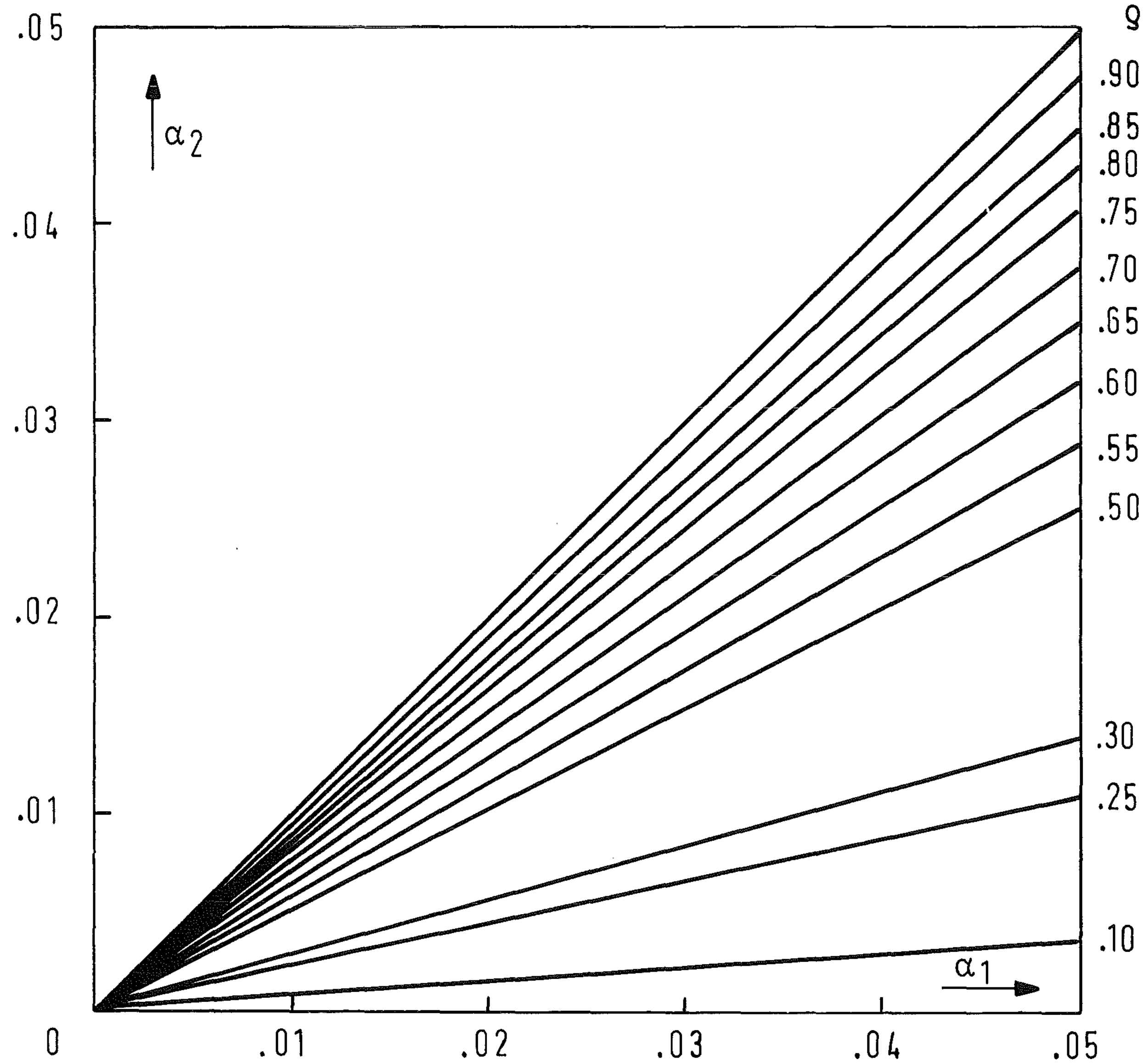

Figure 4-2: Graphical representation of the relation

$$\exp\left(\frac{U_{\alpha_1}^2}{2}\right) \cdot \phi\left(\frac{U_{1-\alpha_1} - \rho \cdot U_{1-\alpha_2}}{\sqrt{1-\rho^2}}\right) - \rho \cdot \exp\left(\frac{U_{\alpha_2}^2}{2}\right) \cdot \phi\left(\frac{U_{1-\alpha_2} - \rho \cdot U_{1-\alpha_1}}{\sqrt{1-\rho^2}}\right) = 0$$

between α_1 and α_2 for α=0.05 and ρ as parameter.

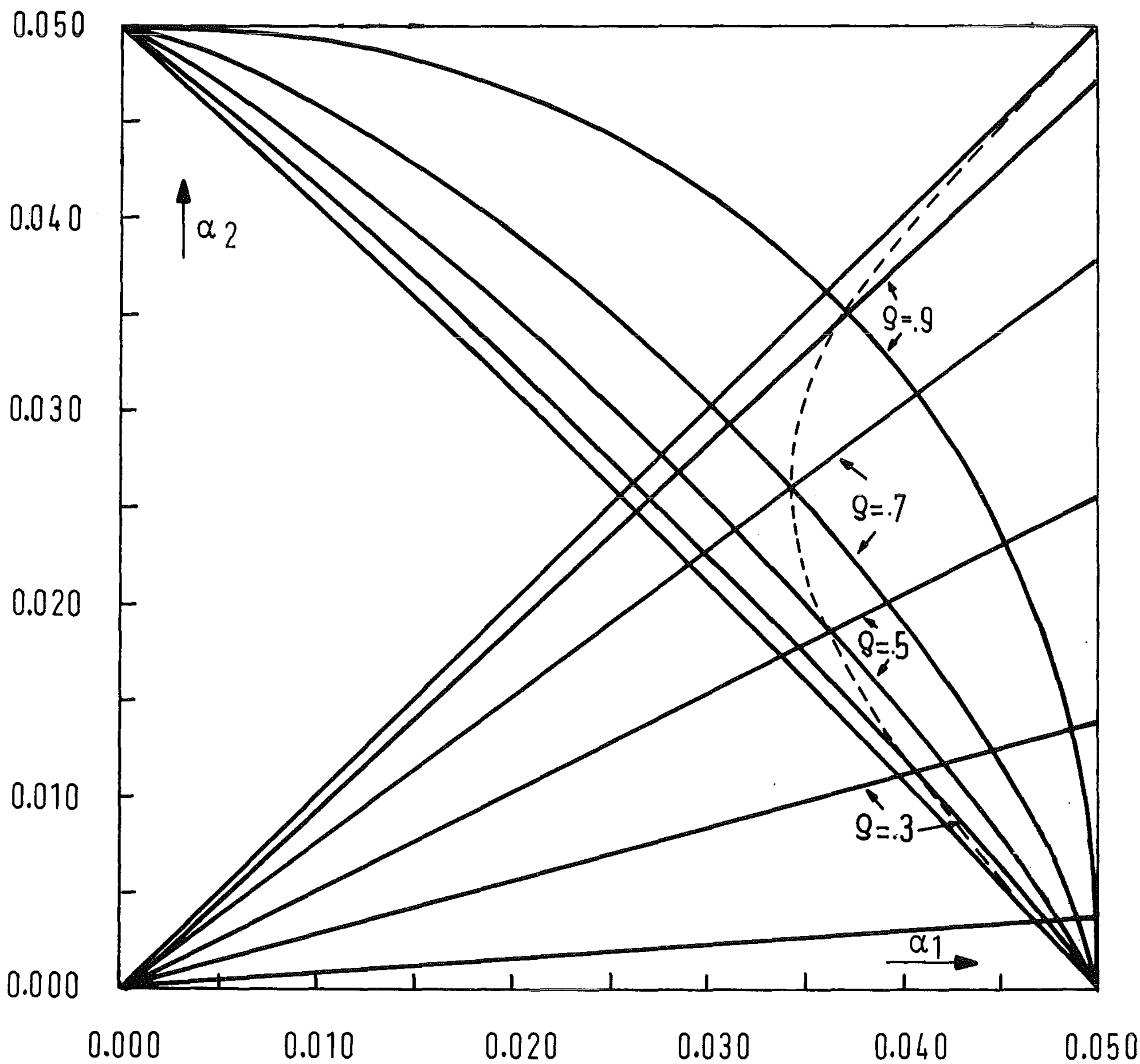

Figure 4-3: Graphical representation of the intersection (dashed line) of the two relations given in Figures 4-1 and 4-2.

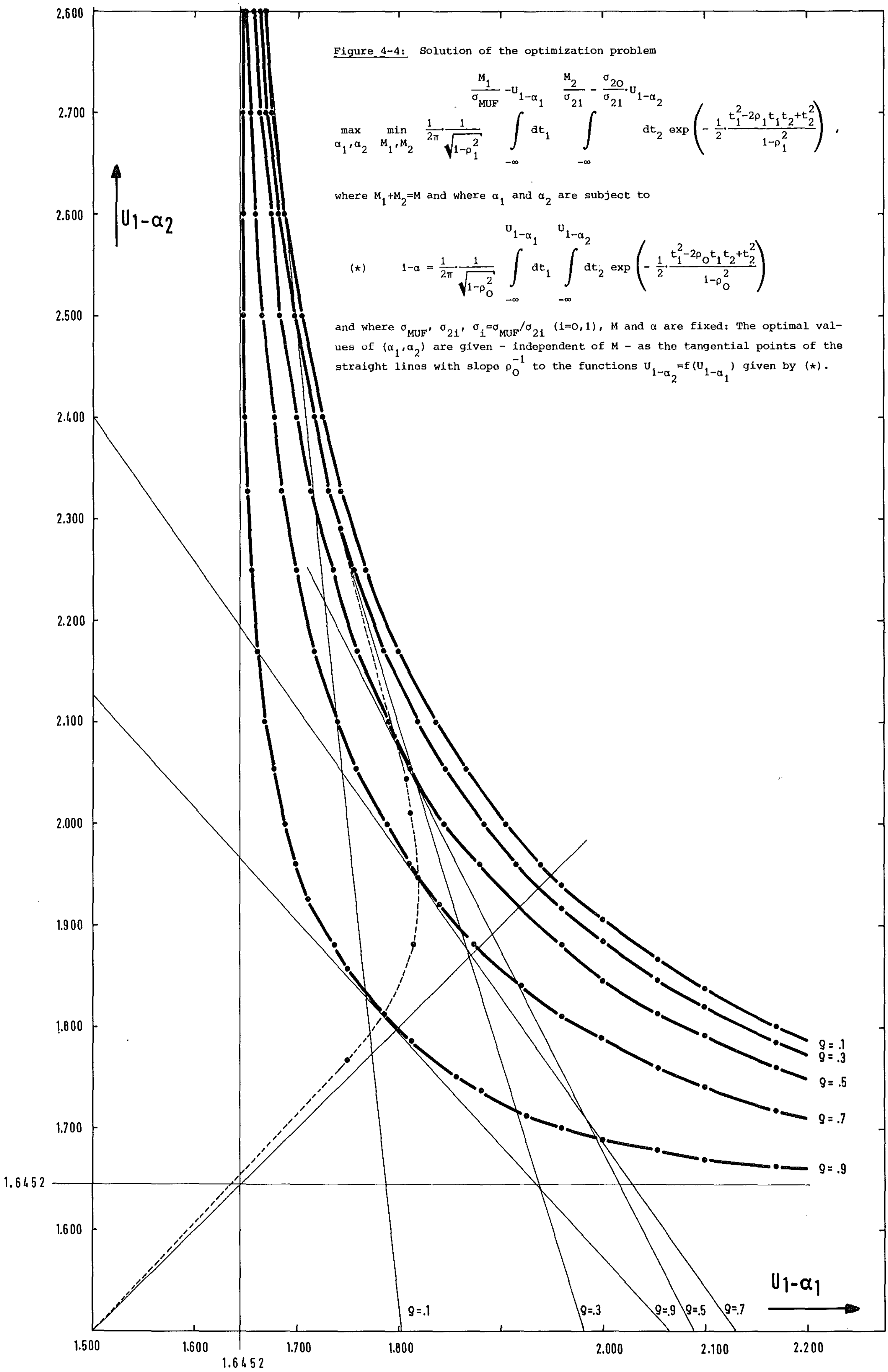

Figure 4-4: Solution of the optimization problem

$$\max_{\alpha_1,\alpha_2}\ \min_{M_1,M_2}\ \frac{1}{2\pi}\cdot\frac{1}{\sqrt{1-\rho_1^2}}\int_{-\infty}^{\frac{M_1}{\sigma_{MUF}}-U_{1-\alpha_1}} dt_1 \int_{-\infty}^{\frac{M_2}{\sigma_{21}}-\frac{\sigma_{20}}{\sigma_{21}}\cdot U_{1-\alpha_2}} dt_2\ \exp\left(-\frac{1}{2}\cdot\frac{t_1^2-2\rho_1 t_1 t_2+t_2^2}{1-\rho_1^2}\right),$$

where $M_1+M_2=M$ and where α_1 and α_2 are subject to

$$(*)\qquad 1-\alpha=\frac{1}{2\pi}\cdot\frac{1}{\sqrt{1-\rho_0^2}}\int_{-\infty}^{U_{1-\alpha_1}} dt_1 \int_{-\infty}^{U_{1-\alpha_2}} dt_2\ \exp\left(-\frac{1}{2}\cdot\frac{t_1^2-2\rho_0 t_1 t_2+t_2^2}{1-\rho_0^2}\right)$$

and where σ_{MUF}, σ_{2i}, $\sigma_i=\sigma_{MUF}/\sigma_{2i}$ (i=0,1), M and α are fixed: The optimal values of (α_1,α_2) are given - independent of M - as the tangential points of the straight lines with slope ρ_0^{-1} to the functions $U_{1-\alpha_2}=f(U_{1-\alpha_1})$ given by (*).

numerical determination of (α_2^*, α_1^*).

Even though we cannot give an analytical expression for the guaranteed probability of detection $1-\beta(M_1^*, M_2^*, \alpha_1^*, \alpha_2^*)$, it can easily be shown that it is a monotonously increasing function of $M=M_1^*+M_2^*$:

As the optimal values (α_1^*, α_2^*) are independent of M, we get from (4-7) with $M_2=M-M_1$ the following expression for the partial derivation of β with respect to M

$$\sqrt{2\pi}\cdot\frac{\partial}{\partial_M}\beta = -\exp\left(-\frac{1}{2}\cdot\left[\frac{\sigma_O}{\sigma_1}\cdot U_{1-\alpha_1}-\frac{M_1}{\sigma_1}\right]^2\right)\cdot\phi\left(\frac{U_{1-\alpha_2}-\frac{M_2}{\sigma_M}-\rho_1\cdot\left[\frac{\sigma_O}{\sigma_1}\cdot U_{1-\alpha_1}-\frac{M_1}{\sigma_1}\right]}{\sqrt{1-\rho_1^2}}\right)\cdot\frac{1}{\sigma_1}\cdot\frac{dM_1}{dM}$$

$$-\exp\left(-\frac{1}{2}\cdot\left[U_{1-\alpha_2}-\frac{M_2}{\sigma_M}\right]\right)\cdot\phi\left(\frac{\frac{\sigma_O}{\sigma_1}\cdot U_{1-\alpha_1}-\frac{M_1}{\sigma_M}-\rho_1\cdot\left[U_{1-\alpha_2}-\frac{M_2}{\sigma_M}\right]}{\sqrt{1-\rho_1^2}}\right)\cdot\frac{1}{\sigma_M}\cdot\left[1-\frac{dM_1}{dM}\right].$$

If we use the determinant (4-14) for the optimal values (M_1^*, M_2^*), then we get

$$\sqrt{2\pi}\cdot\frac{\partial}{\partial_M}\beta(M_1^*, M_2^*;\alpha_1^*, \alpha_2^*) = -\frac{1}{\sigma_M}\cdot\exp\left(-\frac{1}{2}\cdot\left[U_{1-\alpha_2}-\frac{M_2}{\sigma_M}\right]^2\right)\cdot\phi\left(\frac{\frac{\sigma_O}{\sigma_1}\cdot U_{1-\alpha_1}-\frac{M_1}{\sigma_1}-\rho_1\cdot\left[U_{1-\alpha_2}-\frac{M_2}{\sigma_M}\right]}{\sqrt{1-\rho_1^2}}\right)$$

which is always smaller zero, therefore we get with (4-4) the result

$$\frac{\partial}{\partial_M}(1-\beta(M))>0\ , \quad 1-\beta(M=0) = \alpha \quad . \qquad (4\text{-}23)$$

4.2 Combined Material Balance Establishment and Data Verification Test

Let us come back to the question for the 'optimum' test for the two hypotheses given by (4-1). The best test with respect the probability of detection, for a given value of the false alarm probability is given by the Neyman-Pearson Lemma and can be derived as follows /12/:

Neyman-Pearson-Test

The critical region of the Neyman-Pearson test is given by the following set of realizations $\underline{t}':=(t_1, t_2)$ of (D,MUF) values

$$\left\{(t_1,t_2): \frac{f_1(t_1,t_2)}{f_0(t_1,t_2)} \geq k\right\} \; ; \tag{4-24}$$

here, the constant k is determined by the error of the first kind probability, f_0 is the (exact) joint density of MUF and D under H_0,

$$f_0(t_1,t_2) = \frac{1}{2\pi}\cdot\frac{1}{|\underline{\underline{\Sigma}}_0|}\cdot\exp\left(-\frac{1}{2}\cdot\underline{t}'\cdot\underline{\underline{\Sigma}}_0^{-1}\cdot\underline{t}\right) \; , \tag{4-25a}$$

f_1 is the (approximate) joint density of MUF and D under H_1,

$$f_1(t_1,t_2) = \frac{1}{2\pi}\cdot\frac{1}{|\underline{\underline{\Sigma}}_1|}\cdot\exp\left(-\frac{1}{2}(t-M)'\cdot\underline{\underline{\Sigma}}_1^{-1}\cdot(\underline{t}-\underline{M})\right), \tag{4-25b}$$

where $\underline{M}'=(-M_1,M_2)$, and where $\underline{\underline{\Sigma}}_0$ and $\underline{\underline{\Sigma}}_1$ are the covariance matrices under H_0 and H_1:

$$\underline{\underline{\Sigma}}_i = \begin{pmatrix} \sigma^2_{D|H_i} & \sigma^2_{MUF} \\ \sigma^2_{MUF} & \sigma^2_{MUF} \end{pmatrix}, \quad i=0,1 \; . \tag{4-26}$$

The inverses of the matrices are given by

$$\underline{\underline{\Sigma}}_i^{-1} = \frac{1}{\sigma^2_{D|H_i}-\sigma^2_{MUF}}\cdot\begin{pmatrix} 1 & -1 \\ -1 & \rho^{-2}_{H_i} \end{pmatrix}, \quad i=0,1 \; . \tag{4-27}$$

The critical region of the test therefore is given by the following set of realizations $\underline{t}$ of (D,MUF) values:

$$\left\{(t_1,t_2): \frac{1}{2}\cdot t'\cdot\underline{\underline{\Sigma}}_0^{-1}\cdot\underline{t}-\frac{1}{2}\cdot t'\cdot\underline{\underline{\Sigma}}_1^{-1}\cdot\underline{t}+\underline{t}'\cdot\underline{\underline{\Sigma}}_1^{-1}\cdot\underline{M} \geq k'\right\} \tag{4-28}$$

which means that the optimum test statistics is given by the following expression:

$$\frac{1}{2}\cdot\underline{t}'\cdot\underline{\underline{\Sigma}}_0^{-1}\cdot t-\frac{1}{2}\cdot\underline{t}'\cdot\underline{\underline{\Sigma}}_1^{-1}\cdot\underline{t}+\underline{t}'\cdot\underline{\underline{\Sigma}}_1^{-1}\cdot\underline{M} \; . \tag{4-29}$$

This test statistics clearly depends upon the diversion strategy ($\mu_1,\ldots,\mu_K$, $r_1,\ldots,r_K$, M_2). As it would not be reasonable, to make assumptions on all these parameters, one should proceed as follows: One determines the distribution of this statistics and thereafter the probability of detection for a given false alarm probability as a function of these parameters. Thereafter one minimizes the probability of detection with respect to all diversion strategies resulting in a total diversion:

$$\left\{(\mu_1,\ldots,\mu_K, r_1,\ldots,r_K; M_2): \sum_i |\mu_i|\cdot r_i+M_2 = M > 0\right\} \quad .$$

This way one gets the guaranteed probability of detection and, more important, the appropriate test statistics.

As this program is too complicated for any analytical treatment, we now restrict our analysis to *model A*, i.e. the case that all item data are falsified by a class specific amount μ_i, i=1...K. If we assume that for a given total falsification M_1 the values of μ_i, i=1...K, are already optimally chosen, the following set of distribution strategies remain to be considered:

$$\left\{(M_1,M_2): M_1+M_2 = M > 0\right\} \quad . \tag{4-30}$$

Under this assumption we have

$$\sigma^2_{D|H_1} = \sigma^2_{D|H_1} =: \sigma^2_D \; , \qquad \rho_{H_0} = \rho_{H_1} =: \rho \quad ; \tag{4-31}$$

therefore, with

$$\underline{\underline{\Sigma}}_0=\underline{\underline{\Sigma}}_1 =: \underline{\underline{\Sigma}} \tag{4-32}$$

the test statistics is

$$\underline{t}'\cdot\underline{\underline{\Sigma}}^{-1}\cdot\underline{M} \; , \tag{4-33}$$

which is normally distributed with expectation values

$$E(\underline{t}'\cdot\underline{\underline{\Sigma}}^{-1}\cdot\underline{M}) = \begin{cases} 0 & \\ & \text{under} \\ \underline{M}'\cdot\underline{\underline{\Sigma}}^{-1}\cdot\underline{M} & \end{cases} \begin{matrix} H_0 \\ \\ H_1 \end{matrix} \tag{4-34a}$$

and variance

$$\mathrm{var}(\underline{t}'\cdot\underline{\underline{\Sigma}}^{-1}\cdot\underline{M}) = \underline{M}'\cdot\underline{\underline{\Sigma}}^{-1}\cdot\underline{M} \quad , \tag{4-34b}$$

therefore the probability of detection is given by

$$1-\beta = \phi\left(\sqrt{\underline{M}'\cdot\underline{\underline{\Sigma}}^{-1}\cdot\underline{M}} - U_{1-\alpha}\right). \tag{4-35}$$

As $1-\beta$ is a monotone function of

$$\underline{M}'\cdot\underline{\underline{\Sigma}}^{-1}\cdot\underline{M} = \frac{1}{\sigma_D^2-\sigma_{MUF}^2}\cdot(M_1^2+2M_1M_2+\rho^{-2}\cdot M_2^2)$$

the optimum distribution strategy (M_1^*,M_2^*) for a goal quantity M is given by

$$M_1^* = M; \quad M_2^* = 0 \,. \tag{4-36}$$

In this case the test statistics is given by

$$\underline{t}'\cdot\underline{\underline{\Sigma}}^{-1}\cdot\underline{M} = \frac{M}{\sigma_D^2-\sigma_{MUF}^2}\cdot(MUF-D) \;, \tag{4-37}$$

which is up to a constant factor the well known MUF-D statistics. The guaranteed probability of detection is

$$1-\beta_{opt} = \phi\left(\frac{M}{\sqrt{\sigma_D^2-\sigma_{MUF}^2}} - U_{1-\alpha}\right). \tag{4-38}$$

From the fact that the optimal diversion strategy is to exclusively falsify data one might draw the wrong conclusion that the MUF-test would not be necessary. Infact, if there would be no MUF test, then the operator would divert the whole amount M by diversion into MUF, and the probability of detection based on the D-test alone would be equal to the false alarm probability.

4.3 Comparison of the Different Test Procedures

In the following we compare the different test procedures *quantitatively* with the help of their detection probabilities, thereafter we present some *qualitative* arguments.

Let us compare firstly the Neyman-Pearson-test (NP-test) with the bivariate (D,MUF)-test. By definition the bivariate test cannot lead to a higher probability of detection then the NP-test. Moreover, it can be shown that there are diversion strategies where the NP-test really leads to a higher detection probability than the bivariate test.

Secondly, let us look at the (MUF-D)-test and the bivariate (D,MUF)-test. In the case M_1=M we know that the (MUF-D)-test is an NP-test. It is an interesting question whether or not the bivariate test leads for other distribution strategies to a higher detection probability. To answer this question we have to consider two cases.

In the first case

$$var(D)>2 \cdot var(MUF)$$

it can be shown theoretically that there exist strategies and combinations of false alarm probabilities α_1 and α_2 where the (D,MUF)-test is better than the (MUF-D)-test. A numerical example is given in Figure 4-5.

In the second case

$$var(D)<2 \cdot var(MUF)$$

we only have examples where the (MUF-D)-test is always better than the bivariate test (see, e.g. Figure 4-6). Furthermore, we can show theoretically, that in this case there exist combinations of α_1 and α_2 where the (MUF-D)-test is always better than the bivariate test. It is an open question, if also in this case it is possible, to come with a suitable combination of α_1 and α_2 for some inspection strategies to a higher detection probability with the bivariate test than with the (MUF-D)-test.

There are, however, criteria other than the overall probability of detection, and arguments, which have to be taken into account. One important qualitative argument in favor of the (MUF-D) statistic is that it does not depend on the operator's systematic error or, in other words it is essentially a *MUF statistic adjusted for operator's bias, as estimated by the inspector.* This is an important point because (a) such information may be difficult to obtain; (b) even if given by the operator, he may purposely give a high value for his systematic error, a value that would be difficult to verify of refute. On the other hand, there are arguments in favor of the procedure where separate tests for MUF and D are performed. For illustrative purposes only one major argument shall be given here: The operator who collects all the data necessary for the establishment of the material balance sometimes will perform the MUF-test for plant internal purposes. Now, if the safeguards authority establishes the material balance for the same period by using her own data in addition to those of the operator, there exist two statements, which in an extreme case might be contradictory, and which would require complicated second action levels for a clarification.

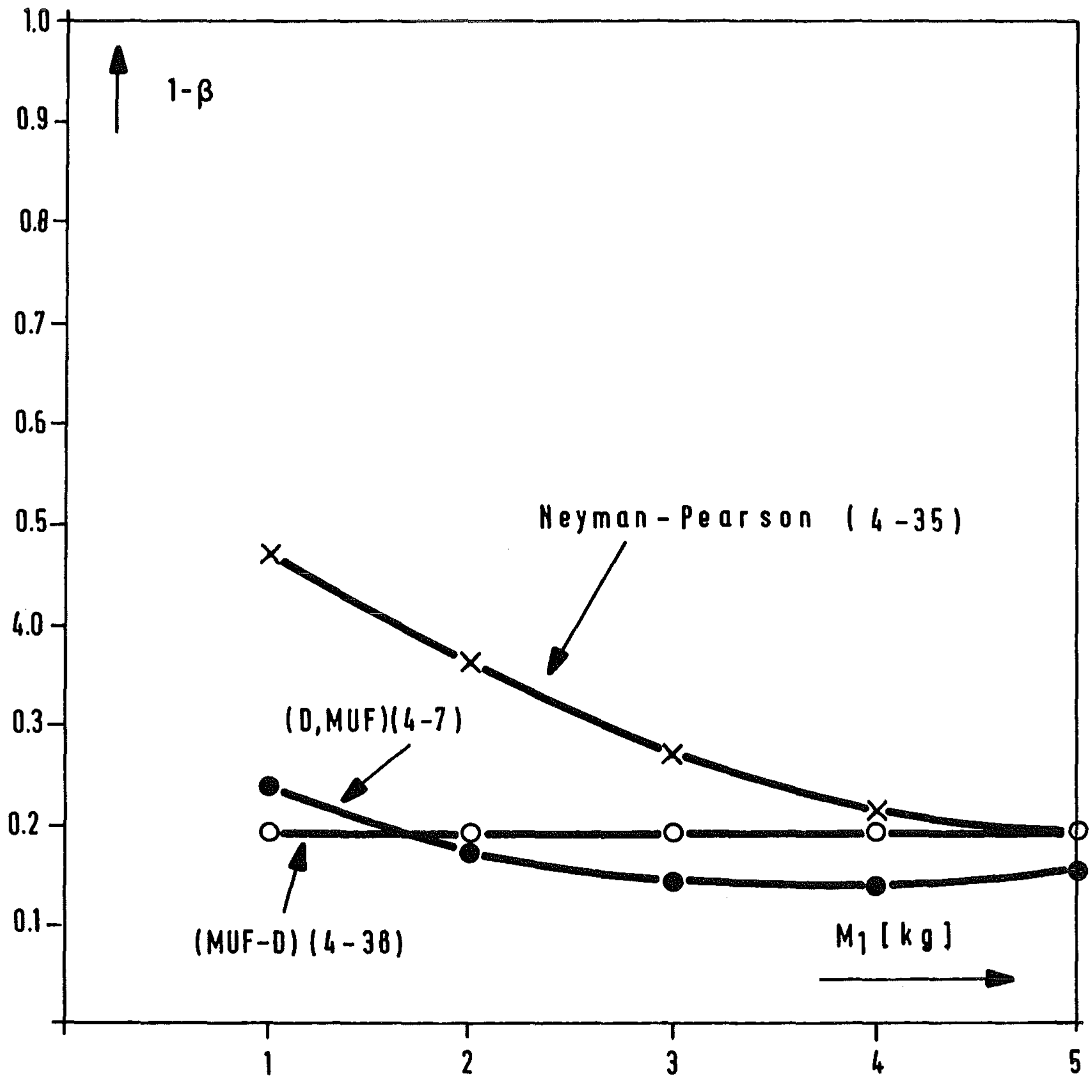

Figure 4-5: Probabilities of detection 1-β for *Model A* and different test procedures as functions of M_1 resp. M_2 with $M_1+M_2=M$ for M=5[kg], $\alpha=0.05$, $\sigma_D^2=52.6[kg^2]$, $\sigma_{MUF}^2=8.33[kg^2]$.

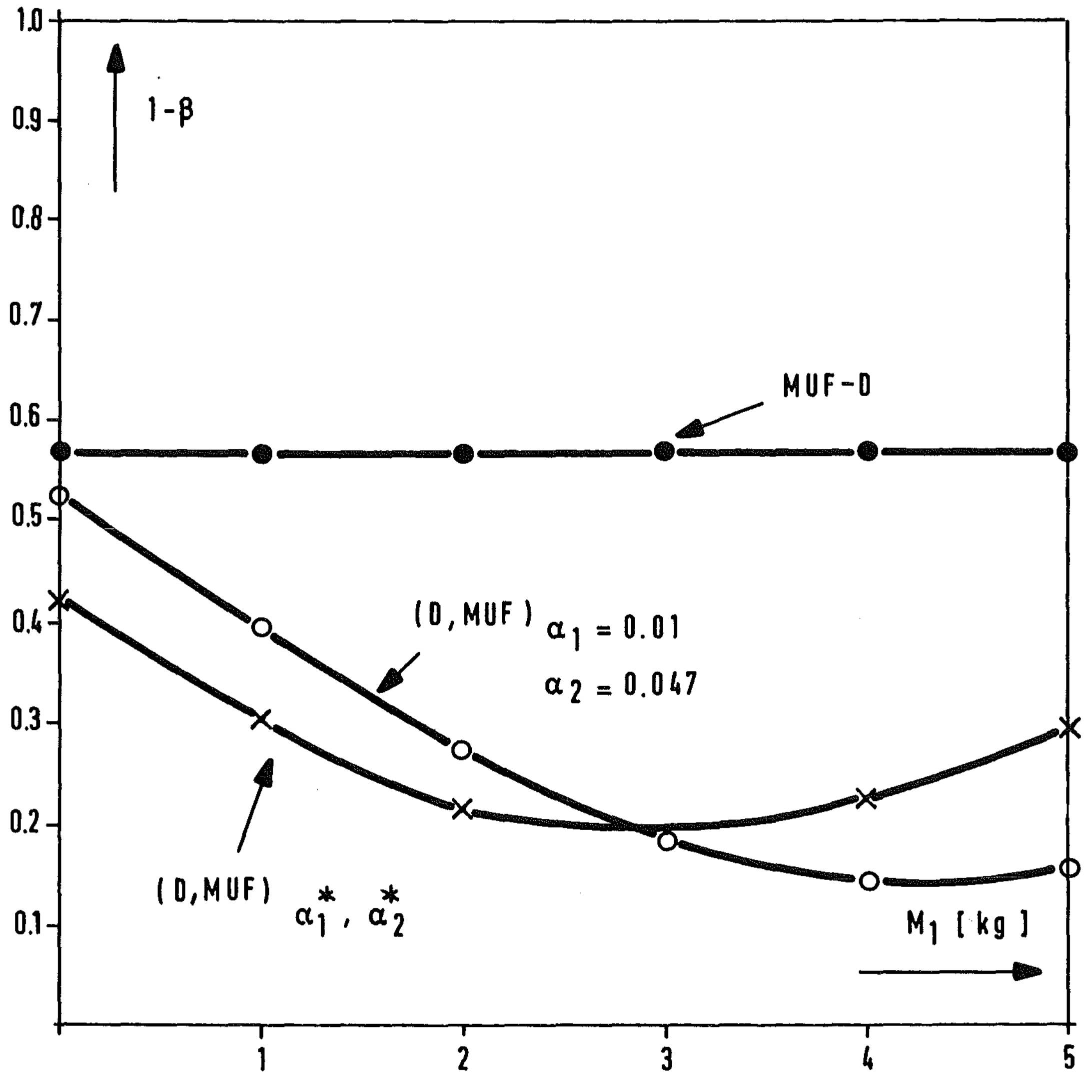

Figure 4-6: Detection probability 1-β for *Model A* and different test procedures as function of M_1 resp. M_2 with $M_1+M_2=M$ for M=5kg, $\alpha=0.05$, $\sigma_D^2=16kg^2$, $\sigma_{MUF}^2=8.33kg^2$.

PART II

PLANT DATA

The theory which has been outlined in the first part shall now be applied to a concrete case of a nuclear material fabrication plant. It is the purpose of this part, to collect all plant data which are necessary for the optimization of safeguards measures and for the evaluation of the safeguards system. In the following, we first describe the plant operations in general terms, thereafter, we present the relevant data for one representative inventory period.

5. Description of Plant Operations [1])

It is difficult to characterize the NUKEM fabrication plant in Hanau, Federal Republic of Germany, from the safeguards point of view just by one sentence saying that the plant was laid out to produce fuel elements for material testing reactors and for pebble bed high temperature reactors.

Although this is the main production activity in terms of flow of highly enriched uranium the safeguards dimensions of NUKEM cannot be understood, if some important other features are not mentioned.

These are briefly as follows:

- The plant works with high and low enriched uranium where the boundary is 20 % enrichment.
- The starting product for the main fabrication is highly enriched uranium hexafluoride.
- There is a large scrap recovery unit in which scrap recovery campaigns are also run for clients.
- Chemical, metallurgical and mechanical treatment of various types, of uranium are possible and to a large extent the technical means and skills are available.
- Large stocks of uranium of all enrichments are at hand in a variety of forms and dimensions. Part of it is stored for clients.
- The stock of uranium is split into several thousands of accountancy units spread over a large number of locations.
- Part of the uranium stock is mixed with Thorium, the latter being the major component.
- The production units are not linked so that the production activity in the plant considered as a whole never comes to a complete standstill.

In the following only the highly enriched part is considered.

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[1]) This chapter follows closely an earlier publication (/13/).

For highly enriched uranium the production area can be subdivided into for main fuel preparation and processing steps as follows:

- the chemical processing areas
- the MTR lines (alloy and cermet lines)
- the HTR line (kernel, particle and pebble lines)
- the uranium oxide line (pellet and rod line)
  this line will be operated completely i.e. with pellet pressing, rod loading and fuel element assembling in the future.

A schematical representation of the material flow in the NUKEM plant is given in Figure 5-1.

As in many companies the general policy is to allow for a strict separation of responsibility between production activities and control and safety functions. This means in practice that both the control/safety departments on the one side and the production departments on the other side are directly subordinated to the management.

At the various levels in the organisation the responsibility of the individuals are stated in a letter of appointment, which must be signed for agreement by both the management and the appointed person.

The main safeguards-relevant responsibilities from the plant side are:

The supervisor for NM in storage and for accountancy: He is in charge of the continuous recording and monthly reporting of all incoming and outgoing NM. He has to ensure that all batches in the store are correctly labelled with external or internal shipper's data. He has to record the movement of NM-batches from production account to production account each time the NM-batches are not going back immediately to the storage. Finally he has to check the information provided on the tag for the material batches which are brought back to the store.

Responsibility for NM in the process: The individual who has to handle NM is responsible for this material after registration of the movement in the general ledger. He is relieved of this responsibility after handing over the material to the storage supervisor or to the next production account in which case he must inform the supervisor of the movement.

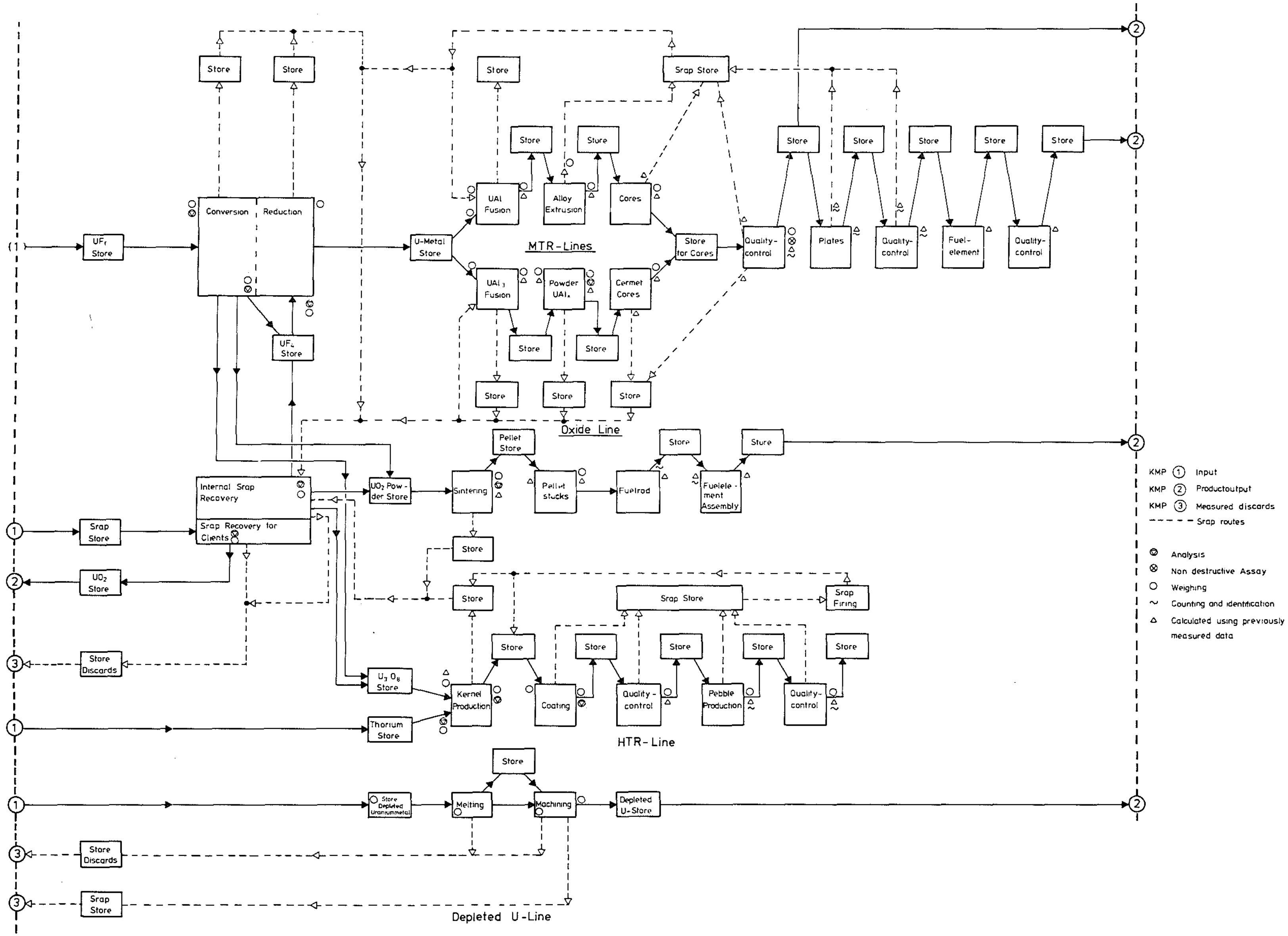

Figure 5-1: Nuclear material flow chart of the NUKEM with nuclear material control points (after Ref. /13/).

Material accounting: The nuclear material accountancy is divided into two parts. One part is used for the monthly reports to EURATOM and consists of a card register in which the input and output of the plant is recorded on a monthly time basis. The second is a records-system which follows all movements between the storage and the plant and, if necessary, within the plant on a daily basis. For this purpose the plant is divided into several responsibility areas and for each area (Fig. 5-1) an account is kept at the central storage book-keeping section. A continuous supervision of all nuclear materials is possible here, because generally the uranium is put into the store between two processing or control stages. In addition, the persons responsible for the plant areas report their uranium stocks once a day to permit cross-checking. Each batch is identified by an accompanying card, showing, inter alia, the amount of uranium and U-235.

The accounting system has run since 1975 by means of an electronic data banking system which makes it possible to draw at any time physical inventory listings for all material on storage.

The inspection effort can be quantified only when, the information provided about the plant activity includes per typical input or output: the amount of material per material balance period and per shipment, the number of shipments, the typical item and the accuracy of the data.

The main safeguards activities *of the inspectors* at the NUKEM plant are the following:

- typical HEU feed material

(1) UF 6	- check delivery notes and seals - take a sample out of each bottle (16 kg of HEU per bottle) - check gross weight - seal bottle
(2) U metal	- check delivery notes and seals - observe weighing - take sample in each container - seal container.
(3) Scrap cores and plates and other scrap	- check delivery notes and seals - count and identify cores and plates - weigh

- seal container
- sample homogeneous batches at the recovery.

For material (2) and (3) in addition, immediate verifications can be performed with the Sb-Be interrogation device and the U-235 amount determined. On feed material statistical sampling is rarely justified and offers little advantage.

- typical product material

(1) MTR production: The number of elements of the same type produced during one material balance period is small so that all MTR elements are measured with the γ scanner, all for consistency checks, some, for which a standard is available on an absolute basis. MTR elements for reactors in the european community are provided with a rivet tamper resistant identification seal. All element containers are sealed for shipment.

(2) RHF cores : Common standards for the cores are available. The cores are verified with the γ scanner in fixed geometry. Although a large number of cores are produced the measurement time is so short (10 seconds) that all cores are measured.

(3) U Metal UO_2 : Population numbers being small not statistical sampling is performed:
- compound weight is verified
- samples are taken from each homogeneous batch
- items are measured (100% basis) with the Sb-Be interrogation device
- seals are put on all containers.

(4) HTR pebbles : In this case statistical sampling under tamper resistant conditions has been applied to the total production.

- Items on inventory (highly enriched uranium)

In addition to the typical materials listed above the full intermediate "good" product spectrum is at hand; typical items are UA1 billets, boxes with cores, sandwiches, plates, cans with $UA1_K$ buttons, $UA1_K$ powder, UF4 powder, containers with kernels, uncoated particles, coated particles, UO2 powder, U3O8 powder.

These materials are mostly in such containers that their U-235 content determination in an attribute in variable mode is possible with the Sb-Be activitation device.

Furthermore as can be seen from the flow chart, scraps and waste are generated in small but relevant amounts, part of it being of the directly recyclable type. Except for a few items containing minor amounts of U-235 in large and ill defined matrices the same measurement device (Sb-Be) is used.

At this point an important question arises about the sampling effort to be made on such an inventory. Obviously the most reasonable way seems to divide the whole set of batches to be verified into a number of subsets or strata of like objects. This stratification is performed on the basis of the information available under the "heading material description" in the PIL or in the items lists. The existing codes have proved to be satisfactory and a better definition is certainly recommended. Since the practical application of this concept left a large number of items uncovered or resulted in unwanted mixing of different items in a class, other ways have been explored. In a nutshell, the total field is divided into number of weight classes and a number of item type classes, the latter only for those types of materials which are unmistakably constituting distinct classes of material. In the case of NUKEM the latter are the fuel pebbles, the UF6 cylinders, the Uranium metal and the plates for the MTR fuel elements. Weight classes are chosen so that the distribution of the elements in the classes is such that to speak of standard deviations of the elements still make some sense.

The idea underlying this subdivision is that the field is in principle made up by measured or measurable units. It is clear then that one of the governing criteria is to be found in the ability a measuring device has to confirm that a certain, defined attribute is present or not. In the ideal situation one instrument should have this ability for *all* the elements in the field and they would thus belong to a single class. In the case of NUKEM this instrument is the Sb-Be device. The NUKEM inventory is accordingly divided into ten classes of objects.

6. Material Accountability and Data Verification

For reasons mentioned in the foregoing Chapter there does not exist a stationary production state in the NUKEM plant therefore, it is not possible to give general representative figures for throughput and inventory. Instead, a concrete inventory period from October 1977 to April 1978 has been selected, which will provide the numerical data for the calculations in the third part. In the following, only the U-235 data are considered.

The authors took the item list containing beginning physical inventory (BI), ending physical inventory (EI) and inventory change (IC) which NUKEM handed to EURATOM. Now we made a stratification according chemical, physical and geometrical viewpoints and nondestructive analysis measurement methods.

In Tables 6-1, 6-2 and 6-3 the (sligthly adjusted) data for initial physical inventory, throughput and ending physical inventory for the period are presented. In Tables 6-4 and 6-5 the average relative standard deviations of the errors of the operator's and the inspector's (destructive) measurements are given, and in Table 6-6 the average relative standard deviations of the errors of the inspector's nondestructive measurements are given. Finally, in Table 6-7 the times needed for one measurement taken by the inspektor at the plant site are presented, as they are necessary for the establishment of sampling plans for the physical inventory.

Verification of flow data is performed either by independent measurements of the inspector or simply by observation of the operator's measurements.

Table 6-1: Physical Inventory Data from Beginning Inventory.

Class	Material	Total Isotopic Weight [kg U-235]	Number of Items	Average Isotopic Weight per Item [kg U-235]
1	UF_6	384.8	53	7.26
2	MTR, RHF Elements	28.9	53	0.55
3	HTR Elements	379.9	*) $3.8*10^5$	.001
4	Fuel Plates	79.5	4998	.016
5	Fuel Rods	10.3	147	.07
6	Pure Metals	108.8	30	3.63
7	Intermediate Products	334.1	2539	.132
8	Waste, Heterogeneous Scrap, Liquids	8.2	76	.132
	Total	1334.5		

*) 380 batches with 1000 items per batch.

Table 6-2: Inventory Changes, October 1977 to April 1978.

	Material	Isotopic Weight [kg]	Number of Items	Isotopic Weight per Item [kg]
Input	UF_6	174.0	12	14.5
	UO_2 Powder and Metal Alloys	56.0	39	1.44
	Scrap	1.2	1	1.20
	Total	231.2		
Output	MTR Elements	58.3	232	.25
	HTR Elements	31.0	$3.1*10^4$	.001
	Fuel Rods	10.3	147	.07
	UO_2 Powder	13.5	15	.9
	Pure Metals	5.0	4	1.25
	Liquid Waste	.2	10	.02
	Total	118.2		

Table 6-3: Physical Inventory Data from Ending Inventory.

Class	Material	Total Isotopic Weight [kg]	Number of Items	Average Isotopic Weight per Item [kg]
1	UF_6	399.8	58	6.89
2	MTR, RHF Elements	38.4	131	0.29
3	HTR Elements	403.5	4.04*) $*10^5$	.001
4	Fuel Plates	46.0	2452	.019
5	Fuel Rods	-	-	-
6	Pure Metals	118.5	38	3.12
8	Intermediate Products	434.1	6050	.072
9	Waste, Heterogeneous Scraps Liquids	7.2	64	.113
	Total	1447.5		

*) 404 batches with 1000 items per batch.

Table 6-4: Relative Standard Deviations of Errors of *Operator's* Destructive Measurements.

Material	Weighing syst.	Weighing random	U-Analysis syst.	U-Analysis random	Isotopic Analysis syst.	Isotopic Analysis random
UF_6 1)	- 3)	$2*10^{-3}$	$3*10^{-4}$	$2.5*10^{-4}$	$1*10^{-3}$	$2*10^{-3}$
MTR and RHF Elements 2)	- 3)	-	-	-	$1*10^{-3}$	$2*10^{-3}$
HTR Elements	$5*10^{-4}$	$5*10^{-4}$	$1*10^{-3}$	$2*10^{-3}$	$5*10^{-4}$	$1*10^{-3}$
Fuel Plates (MTR) 2)	- 3)	-	-	-	$1*10^{-3}$	$2*10^{-3}$
Fuel Rods (KNK)	$1*10^{-5}$	$5*10^{-4}$	$1*10^{-3}$	$5*10^{-4}$	$5*10^{-4}$	$1*10^{-3}$
Pure Metals	$1*10^{-5}$	$3*10^{-4}$	$1*10^{-3}$	$5*10^{-4}$	$5*10^{-4}$	$1*10^{-3}$
Intermediate Products 1)	- 3)	$5*10^{-4}$	$2.5*10^{-3}$	$1*10^{-3}$	$1*10^{-3}$	$2*10^{-3}$
Waste, Heterogeneous Scrap, Liquids	$1*10^{-2}$	$1*10^{-2}$	.2	.2	$1*10^{-2}$	$1*10^{-2}$
UO_2 Powder	- 3)	$2*10^{-3}$	$3*10^{-4}$	$2.5*10^{-4}$	$5*10^{-4}$	$1*10^{-3}$
Metal Alloys	- 3)	$1*10^{-4}$	$1*10^{-3}$	$5*10^{-4}$	$5*10^{-4}$	$1*10^{-3}$

1) 'gross-tare' weighing.

2) Measurement of characteristic radiation.

3) As the net weight is determined by the difference of gross and tare weights, the systematic error of the weighing cancels.

Table 6-5: Relative Standard Deviations of Errors of *Inspector's* Destructive Measurements.

Material	Weighing 3) syst.	Weighing 3) random	U-Analysis syst.	U-Analysis random	Isotopic Analysis syst.	Isotopic Analysis random
UF_6	- 3)	$2*10^{-3}$	$5*10^{-4}$ 2)	$5*10^{-4}$	$1*10^{-4}$ 2)	$1*10^{-4}$
MTR, RHF	- 3)	-	-	-	$5*10^{-4}$ 2)	$5*10^{-4}$
HTR	- 3)	-	-	-	$1*10^{-2}$	$5*10^{-3}$
Fuel Plates	- 3)	-	-	-	$5*10^{-4}$ 2)	$5*10^{-4}$
Fuel Rods	- 3)	-	-	-	$5*10^{-4}$ 2)	$5*10^{-4}$
Pure Metals	$1*10^{-5}$	$3*10^{-4}$	$1*10^{-4}$ 2)	$1*10^{-4}$	$1*10^{-4}$ 2)	$1*10^{-4}$
UO_2 Powder	- 3)	-	$5*10^{-4}$	$5*10^{-4}$	$5*10^{-4}$	$5*10^{-4}$
Metal Alloys	- 3)	-	$5*10^{-4}$	$5*10^{-4}$	$5*10^{-4}$	$5*10^{-4}$
Intermediate Products	- 3)	$5*10^{-4}$	$5*10^{-4}$ 2)	$5*10^{-4}$	$5*10^{-4}$ 2)	$5*10^{-4}$
Waste etc. 1)	- 3)	-	-	-	-	-

1) No independent measurement taken by the inspektor.

2) No data available, therefore the same value has been taken as for the relative standard deviation of the random error.

3) See footnote 3) of Table 6-4.

Table 6-6: Relative Standard Deviations of Errors of *Inspector's* Measurements for Attribute and for Variable Sampling.

Material	Attribute (nondestructive)	
	syst.	random
Fuel Plates	$1*10^{-3}$	$2*10^{-3}$
Fuel Rods	$2*10^{-3}$ 1)	$3*10^{-3}$ 1)
Pure Metals	$2*10^{-2}$	$5*10^{-2}$
MTR-Elements	$5*10^{-2}$ 2)	$5*10^{-2}$ 2)
Intermediate Products	$2*10^{-2}$	$5*10^{-2}$
Waste, Heterogeneous Scrap, Liquids	$2*10^{-1}$ 3)	$2*10^{-1}$ 3)

1) γ-scanning for attribute sampling.

2) Active interrogation with Cf-source.

3) Unreliable data.

Table 6-7: Time needed for Inspector's Inventory Verification Measures at the Plant Site [1].

Material	Measure	Time needed for one Item [min]
UF_6	Seal Check	3
MTR, RHF Elements	Seal Check	.5 [2]
HTR Elements	Seal Check	.5
Fuel Plates	γ-scanner	4
Fuel Rods	γ-scanner	4
Pure Metals	Sb-Be	6
Intermediate Products	Sb-Be	6
Waste, Heterogeneous Scrap, Liquids	Sb-Be	6

1) i.e., not that time needed for destructive analyses in authorized laboratories.

2) For one batch.

PART III

NUMERICAL CALCULATIONS

In this part we apply the theoretical results which have been laid down in Part I to the data of a concrete plant which have been presented in Part II. As a practical result, we get inspection sample sizes for the various material classes and furthermore, the efficiency of the material accountability and data verification procedures as well as that of the combined safeguards system.

The analysis of the efficiency of safeguards measures in a concrete plant is a prerequisite for an implementation of those measures in that plant. Moreover, however, we believe that such a numerical exercise helps to demonstrate the ability of statistical and game theoretical methods in evaluating nuclear material safeguards systems.

7. Material Accountability

According to Tables 6.1 to 6.3 the actual values of beginning and ending inventories, receipts and shipments are

$$\hat{I}_0 = 1334.5 \text{ [kg U-235]}$$

$$\hat{R} = 231.2 \text{ [kg U-235]}$$

$$\hat{S} = 118.2 \text{ [kg U-235]}$$

$$\hat{I}_1 = 1447.5 \text{ [kg U-235]} ,$$

therefore the actual value of MUF is with (2-2)

$$\hat{MUF} = 1334.5+231.2-118.2-1447.5 = 0 \text{ [kg U-235]} .$$

Next, we determine the variance of MUF (σ^2_{MUF}). We calculated σ^2_{MUF} with average values because it was not our aim to get σ^2_{MUF} as precise as possible. But at this place we have to say that there exists a computer program called NUMSAS (see /14/) which is possible to take into account various degrees of enrichment, content of U-235 and different standard deviations for determing σ^2_{MUF}. In case the U-235-content of one item of one class is determined with the help of only one measurement, the variance of the measurement error for the whole class is given by formula (2-7). In many cases, the U-235-content is determined by weighing, U-analysis and U-235-analysis (see also /10/). The variance of the measurement errors in the i-th class is in these cases given by

$$X_i^2 \cdot (N_i^2 \cdot \delta^2_{irO} + N_i^2 \cdot \delta^2_{isO}) , \qquad (7-1)$$

where

$$\delta^2_{irO} = \delta^2_{iWrO} + \delta^2_{iUArO} + \delta^2_{iIArO}$$

$$\delta^2_{isO} = \delta^2_{iWsO} + \delta^2_{iUAsO} + \delta^2_{iIAsO} ,$$

where δ^2 denotes the relative variances and X_i the average true U-235-content in the i-th class, and where the single indices have the following meaning:

O : operator's measurement
i : number of the class
r : random error

s : systematic error

W : weighing

UA: Uranium analysis

IA: Isotopic analysis .

This formula corresponds to formula (2-11") except for the facts that we have three different measurement steps instead of only two and that sampling errors are ignored. The variance of MUF, σ^2_{MUF} is then given by the sum over all class variances. With the help of the data given in Tables 6-1 to 6-4 we get

$$\sigma^2_{MUF} = 8.33\ [(\text{kg U-235})^2]\ .$$

Table 7-1 gives the contribution of the single strata to σ^2_{MUF}.

Table 7-1: Contribution of the Single Strata to σ^2_{MUF}.

Stratum	Beginning Inventory BI[kg^2]	Ending Inventory EI[kg^2]	BI+EI	$\frac{\sigma_i}{x_i}$
1	0.1839	0.19627	0.38017	$7.9*10^{-4}$
2	0.000885	0.0015	0.00239	$7.3*10^{-4}$
3	0.218594	0.24695	0.46554	$8.7*10^{-4}$
4	0.0064	0.00217	0.00857	$7.4*10^{-4}$
5	0.000133	0.0	0.00013	$1.1*10^{-3}$
6	0.015355	0.0187	0.03406	$8.1*10^{-4}$
7	0.814573	1.37583	2.19040	$1.9*10^{-3}$
8	2.74396	2.13538	4.87934	$1.4*10^{-1}$
Inventory Change			0.36853	$1.7*10^{-3}$

σ^2_{MUF} = 8.33[kg^2]

This leads to the standard deviation

$$\sigma_{MUF} = 2.87 \text{ [kg U-235]} .$$

According to formula (2-23) the significance threshold s of the MUF-test is given by

$$s = \sigma_{MUF} \cdot U_{1-\alpha} ;$$

for the values 0.01, 0.05 and 0.1 of the false alarm probability we therefore get the following values of the significance threshold s:

α	$U_{1-\alpha}$	s[kg U-235]
0.01	2.33	6.68
0.05	1.65	4.72
0.1	1.28	3.68

In Figure 7-1 the dependence of the detection probability of the goal quantity M with the false alarm probability as parameter according to formula (2-24) is represented graphically. From Figure 7-1 one can draw the conclusion that a goal quantity of about 9[kg] U-235 can be detected with a probability of about 95 % if the false alarm probability is 5 % and one makes the amount of the whole measured material is about 3 tons U-235.

8. Data Verification

In this chapter we separately have to discuss models A and B which have been introduced in Section 3.1.

8.1 *Model A*

We assume in addition that the inspector

- does not make a seal check for the strata 2 and 3 of the beginning and ending inventories as stated in Table 6-7 (instead, he checks MTR-elements with γ-scanning with an inspection effort of 4 minutes and MTR-elements with the Sigma-machine and an inspection effort of 1 minute per item),
- only makes a seal check for the UF_6-containers (so we get zero for the variances of the random and systematic errors of the inspector which means that the assumption of section 3.2.1 of all variances being greater zero is not valid),

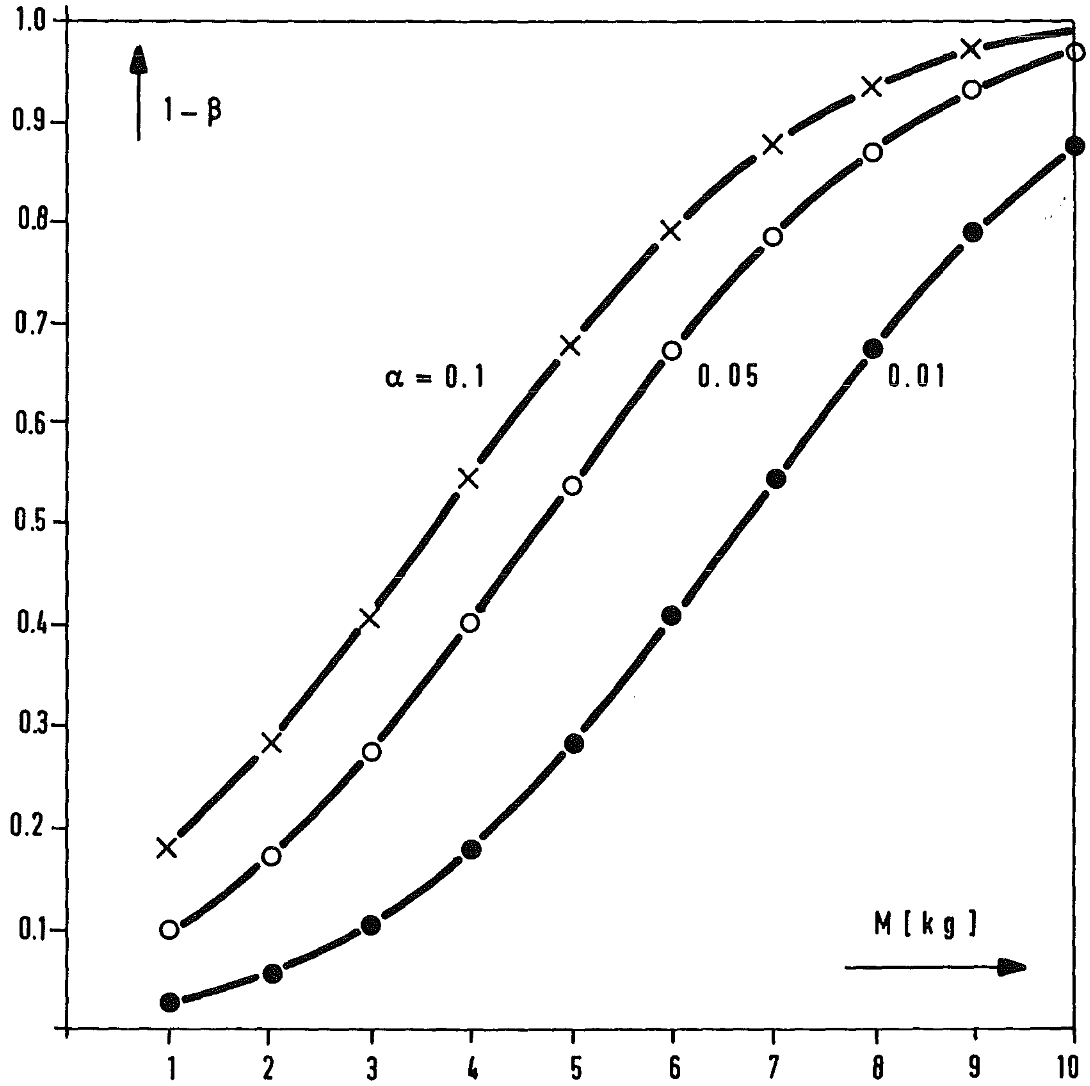

Figure 7-1: Probability of detection 1-β as a function of the goal quantity M for the MUF-test according to formula (7-1) with σ_{MUF}=2.87[kg] and false alarm probability α as parameter.

- does not make independent measurements of the inventory changes, but only supervises the operator's measurements, see chapter 6.

Starting from the data presented in Tables 6-1 and 6-2 to 7, we get for $\alpha=\beta=0.05$, with the help of formulae (3-14) and (3-16) for various values of the goal quantity M_1 the inspection and falsification strategies (sample sizes) which have been tabulated in Table 8-1.

It should be noted that in principle the total inspection effort varies with the goal quantity M_1 for fixed values of α and β. In our example, however, we do not get numerical differences because we round the sample size up to integer values.

With these results we are able to calculate the variance of the D-statistics. Formula (3-14) gives

$$\mathrm{var}(D) := \mathrm{var}(D|H_0) = \mathrm{var}(D|H_1) = \frac{1}{C}\cdot\sigma^2(C)$$

which leads to

$$\mathrm{var}(D) = 52.6[\mathrm{kg}^2] \quad .$$

Thus, the standard deviation is

$$\sqrt{\mathrm{var}(D)} = 7.25[\mathrm{kg}] \quad .$$

It should be noted that the falsification strategy according to *Model* A seems highly artificial. On one hand, one could imagine, that a constant falsification of all data of one class will be built in the production (or measurement) process. On the other hand, this would represent a very inflexible strategy with all its potential hazards to the operator.

8.2 *Model B*

We assume in addition that the inspector

- verifies the inventory data with the methods reported in Table 6-7,
- does not make independent measurements of the inventory changes, but only supervises the operator's measurements.

Furthermore, we assume that the operator

- falsifies those material data which he intends to falsify by the total amount of material of the unit.

Table 8-1: Optimal Inspection and Falsification Strategies for the Beginning and Ending Inventory for *Model A* According to Formulae (3-14, 3-16 and 3-17) with $\alpha=\beta=0.05$.

Goal Quantity for Total Falsification M_1[kg]	Stratum	Total Number of Items N_i	Inspection Strategy n_i^*	Falsification Strategy μ_i^*[g]
	BI 1)			
	1	53	1	$8.54*10^{-1}$
	2	988	1	$2.84*10^{-3}$
	3	380000	2	$6.24*10^{-4}$
	4	4998	1	$1.53*10^{-3}$
	5	147	1	$4.24*10^{-3}$
	6	30	1	$2.51*10^{-1}$
	7	2539	1	$2.25*10^{-2}$
1	8	76	1	1.67
	EI 2)			
	1	58	1	$8.12*10^{-1}$
	2	338	1	$1.08*10^{-2}$
	3	404000	2	$6.55*10^{-4}$
	4	2452	1	$1.80*10^{-3}$
	5	0	0	0
	6	38	1	$2.16*10^{-1}$
	7	6050	1	$1.30*10^{-2}$
	8	64	1	1.69
	BI			
	1	53	1	1.70
	2	988	1	$5.67*10^{-3}$
	3	380000	2	$1.25*10^{-3}$
	4	4998	1	$3.05*10^{-3}$
	5	147	1	$8.47*10^{-3}$
	6	30	1	$5.01*10^{-1}$
	7	2539	1	$4.50*10^{-2}$
2	8	76	1	3.34
	EI			
	1	58	1	1.62
	2	338	1	$2.16*10^{-2}$
	3	404000	2	$1.31*10^{-3}$
	4	2452	1	$3.60*10^{-3}$
	5	0	0	0
	6	38	1	$4.32*10^{-1}$
	7	6050	1	$2.60*10^{-2}$
	8	64	1	3.38
	BI			
	1	53	1	2.56
	2	988	1	$8.50*10^{-3}$
	3	380000	2	$1.87*10^{-3}$
3	4	4998	1	$4.57*10^{-3}$
	5	147	1	$1.27*10^{-2}$
	6	30	1	$7.51*10^{-1}$
	7	2539	1	$6.74*10^{-2}$
	8	76	1	5.01

(Continue →)

Goal Quantity for Total Falsification M_1[kg]	Stratum	Total Number of Items N_i	Inspection Strategy n_i^*	Falsification Strategy μ_i^*[g]
	EI			
	1	58	1	2.43
	2	338	1	$3.23*10^{-2}$
	3	404000	2	$1.97*10^{-3}$
3	4	2452	1	$5.39*10^{-3}$
	5	0	0	0
	6	38	1	$6.47*10^{-1}$
	7	6050	1	$3.90*10^{-2}$
	8	64	1	5.07
	BI			
	1	53	1	3.41
	2	988	1	$1.13*10^{-2}$
	3	380000	2	$2.50*10^{-2}$
	4	4998	1	$6.09*10^{-3}$
	5	147	1	$1.69*10^{-2}$
	6	30	1	1.00
	7	2539	1	$8.98*10^{-2}$
4	8	76	1	6.67
	EI			
	1	58	1	3.24
	2	338	1	$4.31*10^{-2}$
	3	404000	2	$2.63*10^{-3}$
	4	2452	1	$7.19*10^{-3}$
	5	0	0	0
	6	38	1	$8.62*10^{-1}$
	7	6050	1	$5.19*10^{-2}$
	8	64	1	6.75
	BI			
	1	53	1	4.25
	2	988	1	$1.41*10^{-2}$
	3	380000	2	$3.13*10^{-3}$
	4	4998	1	$7.60*10^{-3}$
	5	147	1	$2.11*10^{-2}$
	6	30	1	1.25
	7	2539	1	1.12
5	8	76	1	8.34
	EI			
	1	58	1	4.05
	2	338	1	$5.38*10^{-2}$
	3	404000	2	$3.28*10^{-3}$
	4	2452	1	$8.97*10^{-3}$
	5	0	0	0
	6	38	1	1.08
	7	6050	1	$6.49*10^{-2}$
	8	64	1	8.44
	BI			
	1	53	1	5.10
	2	988	1	$1.69*10^{-2}$

(Continue →)

Goal Quantity for Total Falsification M_1[kg]	Stratum	Total Number of Items N_i	Inspection Strategy n_i^*	Falsification Strategy μ_i^*[g]
	3	380000	2	$3.75*10^{-3}$
	4	4998	1	$9.11*10^{-3}$
	5	147	1	$2.53*10^{-2}$
	6	30	1	1.50
	7	2539	1	$1.35*10^{-1}$
6	8	76	1	10.0
	EI			
	1	58	1	4.85
	2	338	1	$6.44*10^{-2}$
	3	404000	2	$3.94*10^{-3}$
	4	2452	1	$1.08*10^{-2}$
	5	0	0	0
	6	38	1	1.29
	7	6050	1	$7.78*10^{-2}$
	8	64	1	10.1
	BI			
	1	53	1	8.47
	2	988	1	$2.81*10^{-2}$
	3	380000	2	$6.27*10^{-3}$
	4	4998	1	$1.51*10^{-2}$
	5	147	1	$4.20*10^{-2}$
	6	30	1	2.49
	7	2539	1	$2.24*10^{-1}$
10	8	76	1	16.6
	EI			
	1	58	1	8.05
	2	338	1	$1.07*10^{-1}$
	3	404000	2	$6.59*10^{-3}$
	4	2452	1	$1.78*10^{-2}$
	5	0	0	0
	6	38	1	2.14
	7	6050	1	$1.29*10^{-1}$
	8	64	1	16.8
	BI			
	1	53	1	11.8
	2	988	1	$3.92*10^{-2}$
	3	380000	2	$8.81*10^{-3}$
	4	4998	1	$2.11*10^{-2}$
	5	147	1	$5.85*10^{-2}$
	6	30	1	3.46
	7	2539	1	$3.12*10^{-1}$
14	8	76	1	23.2
	EI			
	1	58	1	11.2
	2	338	1	$1.49*10^{-1}$
	3	404000	2	$9.25*10^{-3}$
	4	2452	1	$2.49*10^{-2}$
	5	0	0	0
	6	38	1	2.98

(Continue →)

Goal Quantity for Total Falsification M_1[kg]	Stratum	Total Number of Items N_i	Inspection Strategy n_i^*	Falsification Strategy μ_i^*[g]
14	7	6050	1	$1.81*10^{-1}$
	8	64	1	23.5
	BI			
	1	53	1	15.1
	2	988	1	$5.01*10^{-2}$
	3	380000	2	$1.14*10^{-2}$
	4	4998	1	$2.70*10^{-2}$
	5	147	1	$7.49*10^{-2}$
	6	30	1	4.44
	7	2539	1	$4.00*10^{-1}$
18	8	76	1	29.8
	EI			
	1	58	1	14.3
	2	338	1	$1.91*10^{-1}$
	3	404000	2	$1.19*10^{-2}$
	4	2452	1	$3.18*10^{-2}$
	5	0	0	0
	6	38	1	3.82
	7	6050	1	$2.32*10^{-1}$
	8	64	1	30.2
	BI			
	1	53	1	18.4
	2	988	1	$6.10*10^{-2}$
	3	380000	2	$1.39*10^{-2}$
	4	4998	1	$3.28*10^{-2}$
	5	147	1	$9.11*10^{-2}$
	6	30	1	5.40
	7	2539	1	$4.88*10^{-1}$
22	8	76	1	36.4
	EI			
	1	58	1	17.5
	2	988	1	$2.32*10^{-1}$
	3	404000	2	$1.46*10^{-2}$
	4	2452	1	$3.87*10^{-2}$
	5	0	0	0
	6	38	1	4.65
	7	6050	1	$2.82*10^{-1}$
	8	64	1	36.8
	BI			
	1	53	1	20.8
	2	988	1	$6.91*10^{-2}$
	3	380000	2	$1.58*10^{-2}$
	4	4998	1	$3.72*10^{-2}$
	5	147	1	$1.03*10^{-1}$
	6	30	1	6.11
	7	2539	1	$5.54*10^{-1}$
25	8	76	1	41.3
	EI			
	1	58	1	19.8

(Continue →)

Goal Quantity for Total Falsification M_1[kg]	Stratum	Total Number of Items N_i	Inspection Strategy n_i^*	Falsification Strategy μ_i^*[g]
25	2	338	1	$2.62*10^{-1}$
	3	404000	2	$1.66*10^{-2}$
	4	2452	1	$4.39*10^{-2}$
	5	0	0	0
	6	38	1	5.27
	7	6050	1	$3.21*10^{-1}$
	8	64	1	41.7

---

1) BI = Beginning Inventory

2) EI = Ending Inventory

The data which are needed for the calculation of the optimal inspection and falsification strategies according to formulae (3-28) and (3-31) are taken from Tables 6-1, 6-3 and 6-7. If the sample sizes of the inspector exceeded the total number of items in the stratum, we used a two-step-procedure: The inspector was assumed to verify *all* material data of this stratum and redistributed his remaining effort among the other strata again according to formula (3-28). As we did not get necessarily integers, we always rounded the figures up in the inspector's case, and we rounded up in the operator's case only if $r_i^*>0.05$. In addition we put $r_i^*=0$ if $n_i^*=N_i$ (the fact that we can get $r_i^*>0$ if $n_i^*=N_i$ is a consequence of the drawing with replacement scheme). A consequence of this procedure is that the diverted amount of material does not correspond exactly to the goal quantity.

As easily can be seen, the optimal strategies depend again on the goal quantity M_1. The results of the calculations are put together in Tables 8-2 and 8-3. Contrary to the situation in *Model* A, where the operator had to determine a-priori the optimal falsification amounts μ_i^*, we postulated here a probability of detection of $1-\beta=0.95$ for each the beginning and the ending inventory, as the operator a-priori had to decide only about the distribution of the falsification of the total amounts for both inventories.

Contrary to the situation in case of *Model* A the total verification effort varies for a fixed value of β for both inventories with varying goal quantity M_1. In Figures 8-1 and 8-2 the dependance between these two quantities is represented graphically.

For the determination of the systems effectiveness, based on the MUF- and on the D-statistics, we calculate the variance of the D-statistics according to formula (3-44) with the help of the sample sizes given in Tables 8-2 and 8-3. The result is given in Table 8-4. In Table 8-5 the results for C=66[min] are given.

Table 8-2: Optimal Inspection and Falsification Strategies for the Beginning Inventory for *Model B* for β=0.05.

Goal Quantity M_1^1[kg]	Stratum	Number of Items	Optimal Inspection Strategy n_i^*	Optimal Diversion Strategy r_i^*	Inspection Effort [min]
0.5	1	53	53	0	15061.5
	2	53*)	53	0	
	3	380*)	380	0	
	4	4998	479	3	
	5	147	62	1	
	6	30	30	0	
	7	2539	2007	2	
	8	76	50	1	
1	1	53	53	0	7807.5
	2	53*)	53	0	
	3	380*)	380	0	
	4	4998	240	5	
	5	147	31	1	
	6	30	30	0	
	7	2539	1003	4	
	8	76	50	1	
1.5	1	53	53	0	5389.5
	2	53*)	53	0	
	3	380*)	380	0	
	4	4998	160	7	
	5	147	21	1	
	6	30	30	1	
	7	2539	668	6	
	8	76	17	1	
2	1	53	53	0	4179
	2	53*)	44	0	
	3	380*)	380	0	
	4	4998	120	10	
	5	147	16	1	
	6	30	30	0	
	7	2539	501	7	
	8	76	13	1	
2.5	1	53	53	0	3449
	2	53*)	36	0	
	3	380*)	380	0	
	4	4998	96	12	
	5	147	13	1	
	6	30	30	0	
	7	2539	401	9	
	8	76	10	1	
3	1	53	53	0	2966
	2	53*)	30	0	
	3	380*)	380	0	
	4	4998	80	14	
	5	147	11	1	
	6	30	30	0	
	7	2539	334	11	
	8	76	9	1	

(Continue →)

Goal Quantity M_1^1[kg]	Stratum	Number of Items	Optimal Inspection Strategy n_i^*	Optimal Diversion Strategy r_i^*	Inspection Effort [min]
5	1	53	53	0	1912.5
	2	53*)	18	0	
	3	380*)	229	1	
	4	4998	48	23	
	5	147	7	1	
	6	30	30	0	
	7	2539	200	18	
	8	76	5	1	
7	1	53	53	0	1496.5
	2	53*)	13	0	
	3	380*)	164	1	
	4	4998	35	32	
	5	147	5	1	
	6	30	30	0	
	7	2539	143	25	
	8	76	4	1	
9	1	53	53	0	1216
	2	53*)	10	1	
	3	380*)	128	1	
	4	4998	27	41	
	5	147	4	2	
	6	30	30	0	
	7	2539	111	31	
	8	76	3	1	
11	1	53	53	0	1095.5
	2	53*)	8	1	
	3	380*)	105	1	
	4	4998	22	50	
	5	147	3	2	
	6	30	30	0	
	7	2539	91	38	
	8	76	3	2	
12.5	1	53	53	0	949
	2	53*)	8	1	
	3	380*)	92	1	
	4	4998	20	57	
	5	147	3	2	
	6	30	26	1	
	7	2539	80	43	
	8	76	2	2	

---

*) Number of batches.

Table 8-3: Optimal Inspection and Falsification Strategies for the Ending Inventory for *Model B* for β=0.05.

Goal Quantity M_1^2[kg]	Stratum	Number of Items	Optimal Inspection Strategy n_i^*	Optimal Diversion Strategy r_i^*	Inspection Effort [min]
0.5	1	58	58	0	17713.5
	2	131*)	131	0	
	3	404*)	404	0	
	4	2452	280	1	
	5	0	0	0	
	6	38	38	0	
	7	6050	2610	4	
	8	64	44	0	
1	1	58	58	0	9177
	2	131*)	114	0	
	3	404*)	404	0	
	4	2452	140	2	
	5	0	0	0	
	6	38	38	0	
	7	6050	1304	8	
	8	64	22	1	
1.5	1	58	58	0	6318
	2	131*)	76	0	
	3	404*)	404	0	
	4	2452	93	3	
	5	0	0	0	
	6	38	38	0	
	7	6050	869	11	
	8	64	15	1	
2	1	58	58	0	4891
	2	131*)	58	0	
	3	404*)	404	0	
	4	2452	70	4	
	5	0	0	0	
	6	38	38	0	
	7	6050	652	15	
	8	64	11	1	
2.5	1	58	58	0	4031
	2	131*)	46	0	
	3	404*)	404	0	
	4	2452	56	5	
	5	0	0	0	
	6	38	38	0	
	7	6050	521	19	
	8	64	9	1	
3	1	58	58	0	3463.5
	2	131*)	39	0	
	3	404*)	404	0	
	4	2452	47	6	
	5	0	0	0	
	6	38	38	0	
	7	6050	434	22	
	8	64	8	1	

(Continue →)

Goal Quantity M_1^2 [kg]	Stratum	Number of Items	Optimal Inspection Strategy n_i^*	Optimal Diversion Strategy r_i^*	Inspection Effort [min]
5	1	58	58	0	2237
	2	131*)	23	1	
	3	404*)	243	1	
	4	2452	28	10	
	5	0	0	0	
	6	38	38	0	
	7	6050	260	37	
	8	64	5	1	
7	1	58	58	0	1717.5
	2	131*)	17	1	
	3	404*)	174	1	
	4	2452	20	14	
	5	0	0	0	
	6	38	38	0	
	7	6050	186	52	
	8	64	4	1	
9	1	58	58	0	1422.5
	2	131*)	13	1	
	3	404*)	136	1	
	4	2452	16	18	
	5	0	0	0	
	6	38	38	0	
	7	6050	144	66	
	8	64	3	1	
11	1	58	58	0	1199
	2	131*)	11	1	
	3	404*)	111	1	
	4	2452	13	22	
	5	0	0	0	
	6	38	32	1	
	7	6050	118	81	
	8	64	2	1	
12.5	1	58	58	0	1086
	2	131*)	10	1	
	3	404*)	98	1	
	4	2452	12	25	
	5	0	0	0	
	6	38	29	1	
	7	6050	104	92	
	8	64	2	1	

---

*) Number of batches.

Table 8-4: Variance and Standard Deviation of the D-Statistic as Functions of the Goal Quantity M_1 for *Model B* with β=0.05.

Goal Quantity M_1[kg]	$var(D\|H_O)$ [kg^2]	$\sqrt{var(D\|H_O)}$ [kg]	$var(D\|H_1)$ [kg^2]	$\sqrt{var(D\|H_1)}$ [kg]
1	39.03	6.25	39.09	6.25
2	39.14	6.26	39.45	6.28
3	39.23	6.26	39.97	6.32
4	39.33	6.27	40.66	6.38
5	39.44	6.28	41.60	6.45
6	39.50	6.28	42.62	6.53
10	39.90	6.32	48.80	6.99
14	40.16	6.34	57.70	7.60
18	40.55	6.37	69.34	8.33
22	40.90	6.40	85.89	9.27
25	41.36	6.43	101.62	10.08

It has been assumed that the inspector measures independently all items of UF_6, MTR and HTR.

Table 8-5: Variance and Standard Deviation of the D-Statistic as Functions of the Goal Quantity M_1 for *Model B* and Total Inspection Effort C=66[min].

Goal Quantity M_1[kg]	$var(D\|H_0)$ [kg^2]	$\sqrt{var(D\|H_0)}$ [kg]	$var(D\|H_1)$ [kg^2]	$\sqrt{var(D\|H_1)}$ [kg]
1	44.11	6.64	86.14	9.28
2	44.11	6.64	126.77	11.26
3	44.11	6.64	511.07	22.61
4	44.11	6.64	543.45	23.31
5	44.11	6.64	583.92	24.16
6	44.11	6.64	619.16	24.88
10	44.11	6.64	767.06	27.70
14	44.11	6.64	914.20	30.24
18	44.11	6.64	1047.56	32.37
22	44.11	6.64	1193.25	34.54
25	44.11	6.64	1298.78	36.04

It has been assumed that the inspector measures independently all items of UF_6, MTR and HTR.

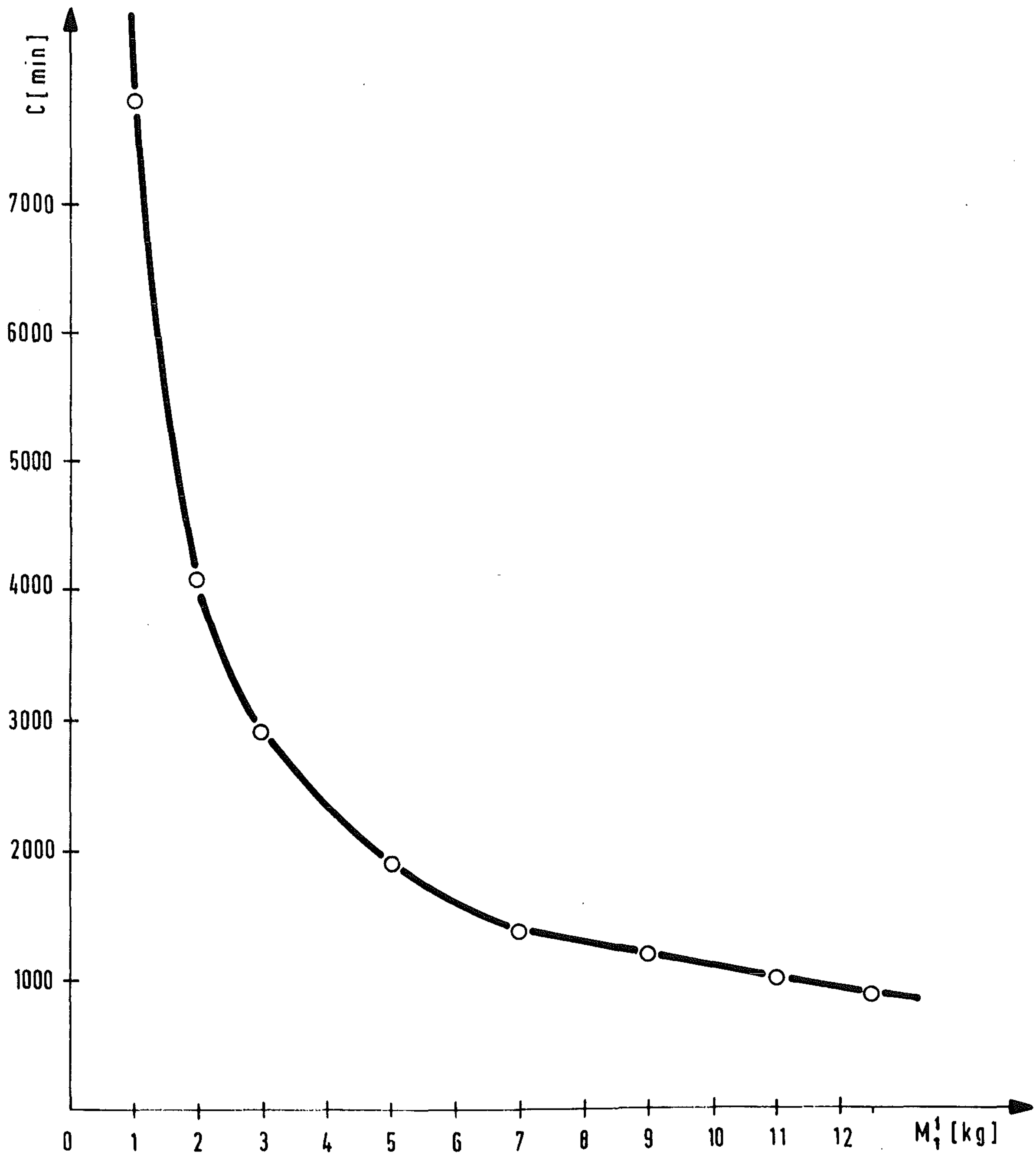

Figure 8-1: Inspection effort C as a function of goal quantity M_1^1 for beginning inventory in *model B* with β=0.05.

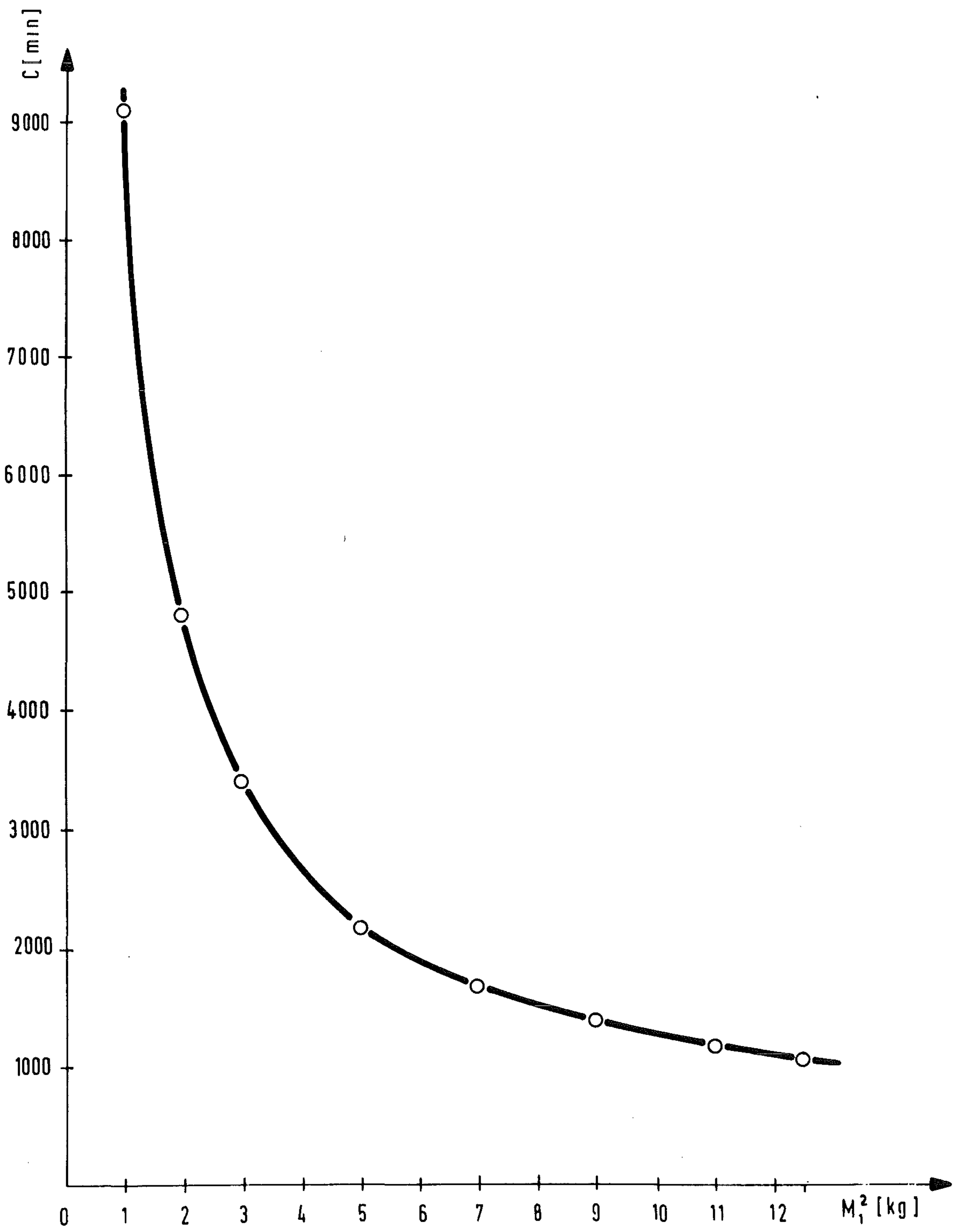

Figure 8-2: Inspection effort C as a function of goal quantity M_1^2 for ending inventory in *model B* with β=0.05.

9. Detection Probability of the Combined System

In this chapter we determine the efficiency, i.e., the overall probability of detection for the combined safeguards system based on the MUF and on the D-statistics. Again we have to discuss models A and B separately.

We proceed as follows: We start with a fixed value of the verification effort C which has been determined in such a way that the probability for detecting a data falsification is presumed equal to $1-\beta=0.95$ (which in case of *Model A* is numerically independent of the value of the goal quantity M_1) for a given value of $\alpha=0.05$. It is clear that in the course of the optimation procedure the probability of detecting a data falsification alone may shift to a value different from 0.95.

The total probability of detection for the MUF-D-test is according to formula (3-12) given by

$$1-\beta = \phi\left(\frac{M-U_{1-\alpha}\cdot \sigma^2_{D|H_0}-\sigma^2_{MUF}}{\sqrt{\sigma^2_{D|H_1}-\sigma^2_{MUF}}}\right) \tag{9-1}$$

whereas the total probability of detection for the (D,MUF)-test is according to formula (4-7) given by

$$1-\beta = \frac{1}{2\pi}\frac{1}{\sqrt{1-\rho_1^2}}\int_{-\infty}^{\frac{M_1}{\sigma_{D|H_1}}-\frac{\sigma_{D|H_0}}{\sigma_{D|H_1}}\cdot U_{1-\alpha_1}} dt_1 \int_{-\infty}^{\frac{M_2}{\sigma_{MUF}}-U_{1-\alpha_2}} dt_2 \, \exp\left(-\frac{1}{2}\cdot\frac{t_1^2-2\rho_1 t_1 t_2+t_2^2}{1-\rho_1^2}\right) \tag{9-2}$$

where the single false alarm probabilities α_1 and α_2 are determined by the optimization procedure (4-9), and where in both cases the variances $\sigma^2_{D|H_0}$ and $\sigma^2_{D|H_1}$ depend on the specific model under consideration.

9.1 *Model A*

From section 4.2 we know that the (MUF-D)-test is a best test if the operator diverts all material via the data falsification strategy. In Figures 9-1 and 9-2 we have represented graphically the probability of the de-

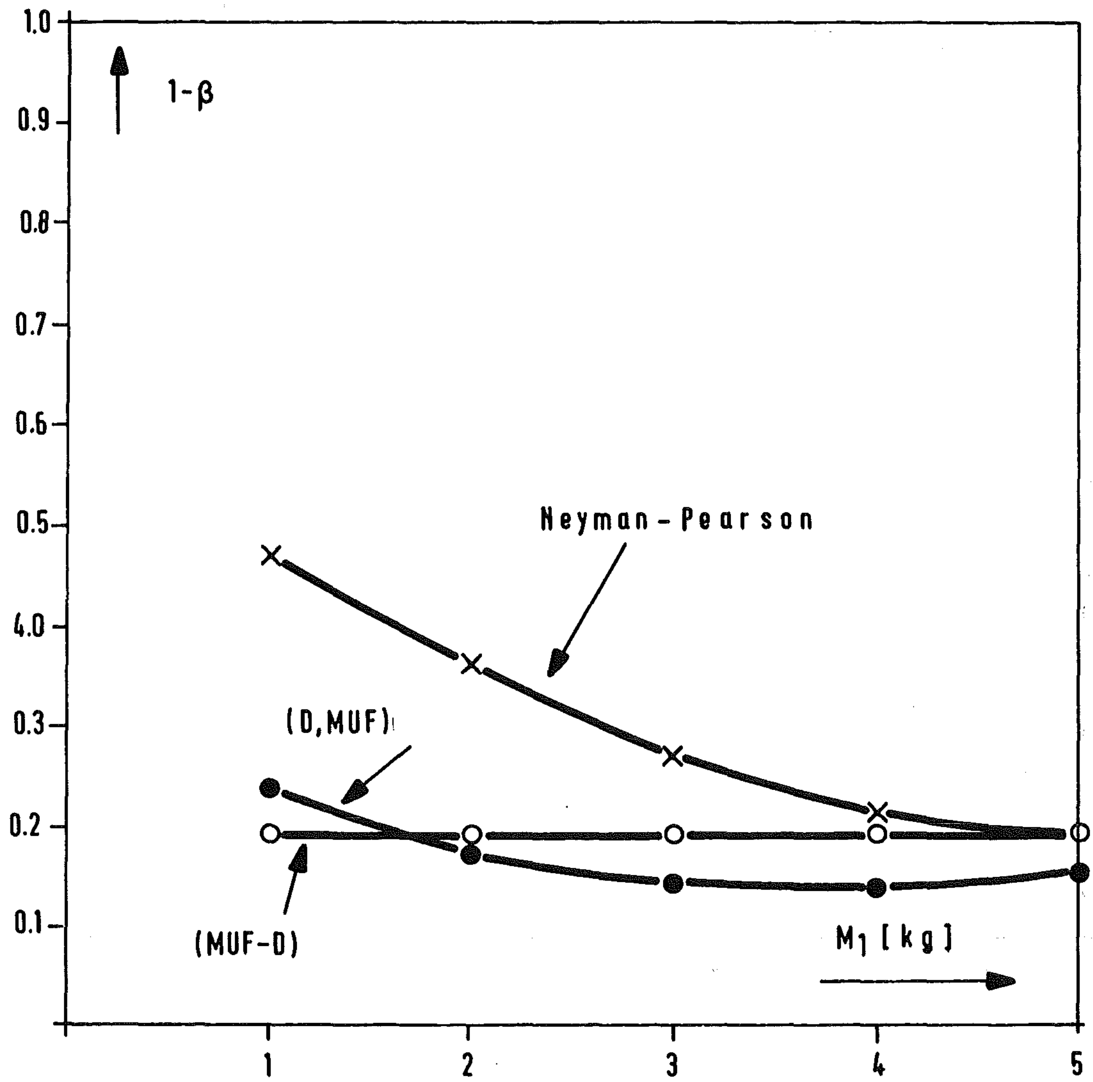

Figure 9-1: Probability of detection of the (MUF-D)-, Neymann-Pearson- and (D,MUF)-test for different diversion strategies in *Model A*; goal quantity $M=M_1+M_2=5kg$, $\alpha=0.05$.
The figure shows for each M_1 the detection probability for the diverted amount $M-M_1+M_1/2$.

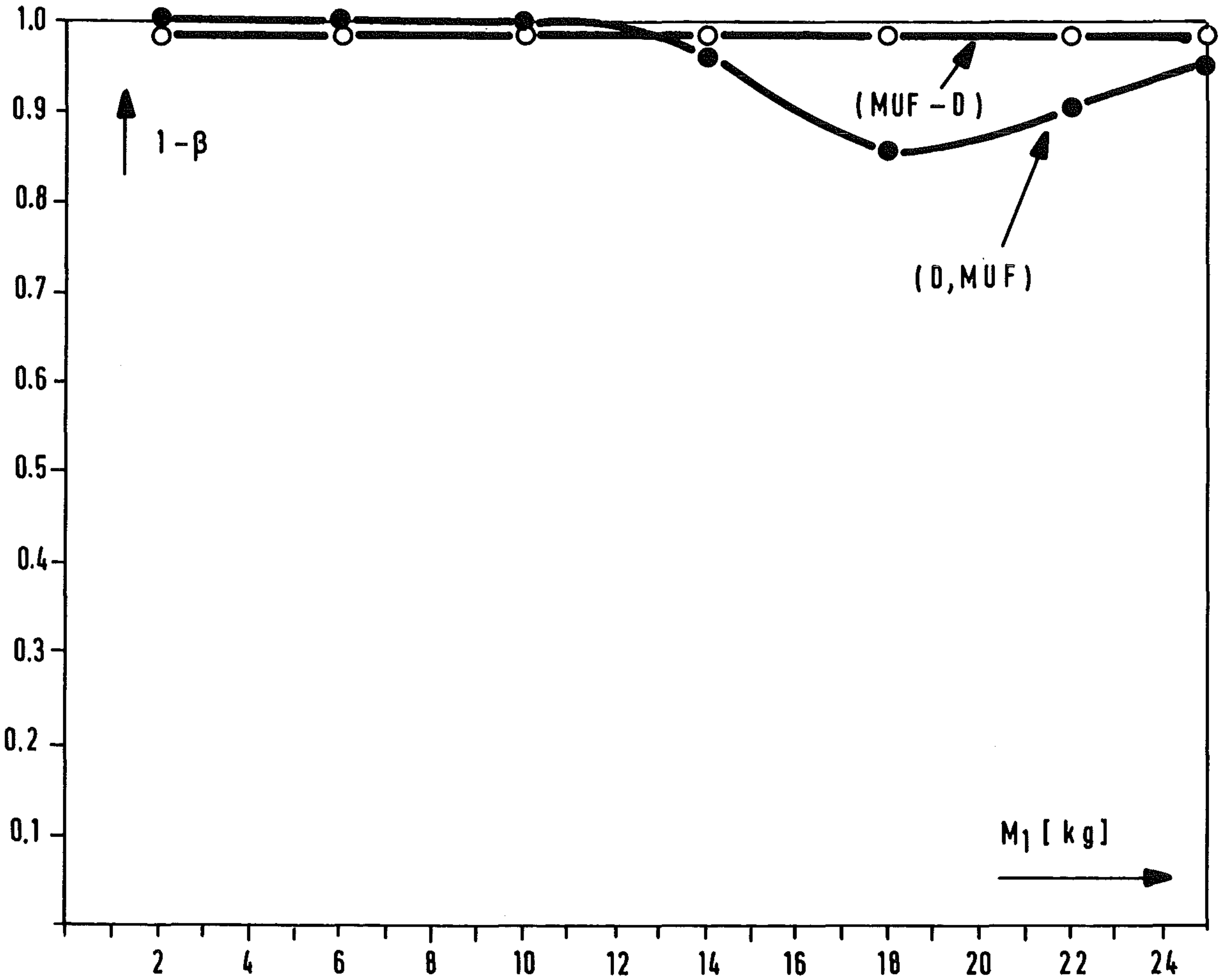

Figure 9-2: Probability of detection of the (MUF-D)- and the (D,MUF)-test for different diversion strategies in *Model A*; goal quantity $M=M_1+M_2=25kg$, $\alpha=0.05$.

The figure shows for each M_1 the detection probability for the diverted amount $M-M_1+M_1/2$.

tection for the MUF-D-test as a function of the total data falsification (M_1) for fixed values of the goal quantity M according to formula (3-12) resp. (9-1) which is independent of M_1 because in the variances

$$\sigma^2_{D|H_0} = \sigma^2_{D|H_1} = \sum_i \frac{N_i^2}{n_i} (\sigma^2_{ri} + n_i \cdot \sigma^2_{si}) \qquad (9\text{-}3)$$

the sample sizes n_i are determined with the help of formula (3-16) for one fixed value of M_1 according to the procedure outlined above.

It should be noted here that the operator has to falsify data consistently: If he reports, e.g., less material as initial inventory than really in the plant, then he also has to report less material as ending inventory than really in the plant (or less inputs, or more outputs). This means that, if he falsifies all data by an amount M_1, he actually can divert only half of this amount, i.e. he can divert $M_1/2$.

In Figures 9-1 and 9-2, also the probability of detection for the bivariate (D,MUF)-test is represented graphically as a function of the total data falsification (M_1) for fixed values of the goal quantity M according to (4-7) resp. (9-2), with optimized false alarm probabilities α_1^* and α_2^*. We see that in a range of small values of M_1 the bivariate (D,MUF)-test leads to a higher probability of detection, however, in the larger range the MUF-D-test is better.

9.2 *Model B*

Contrary to the case of *model* A, section 4.2 does not tell us anything about the optimality of the MUF-D-test for the two hypotheses $H_0: M_1 = 0$ and $H_1: M_1 = M' > 0$, $M_2 = M - M'$ for model B. In Figures 9-3 and 9-4 we have represented graphically the probability of detection for the MUF-D-test as a function of the total data falsification (M_1) for fixed values of the goal quantity M according to the (here only approximatively valid) formula (3-43) resp. (9-1), where in the variances

$$\sigma^2_{D|H_0} = \sum_i \frac{N_i^2}{n_i} (\sigma^2_{ri} + n_i \cdot \sigma^2_{si})$$

$$\sigma^2_{D|H_1} = \sigma^2_{D|H_0} + \sum_i \mu_i^2 \cdot r_i \cdot (N_i - r_i) \cdot \left(\frac{1}{n_i} - \frac{1}{N_i}\right) \qquad (9\text{-}4)$$

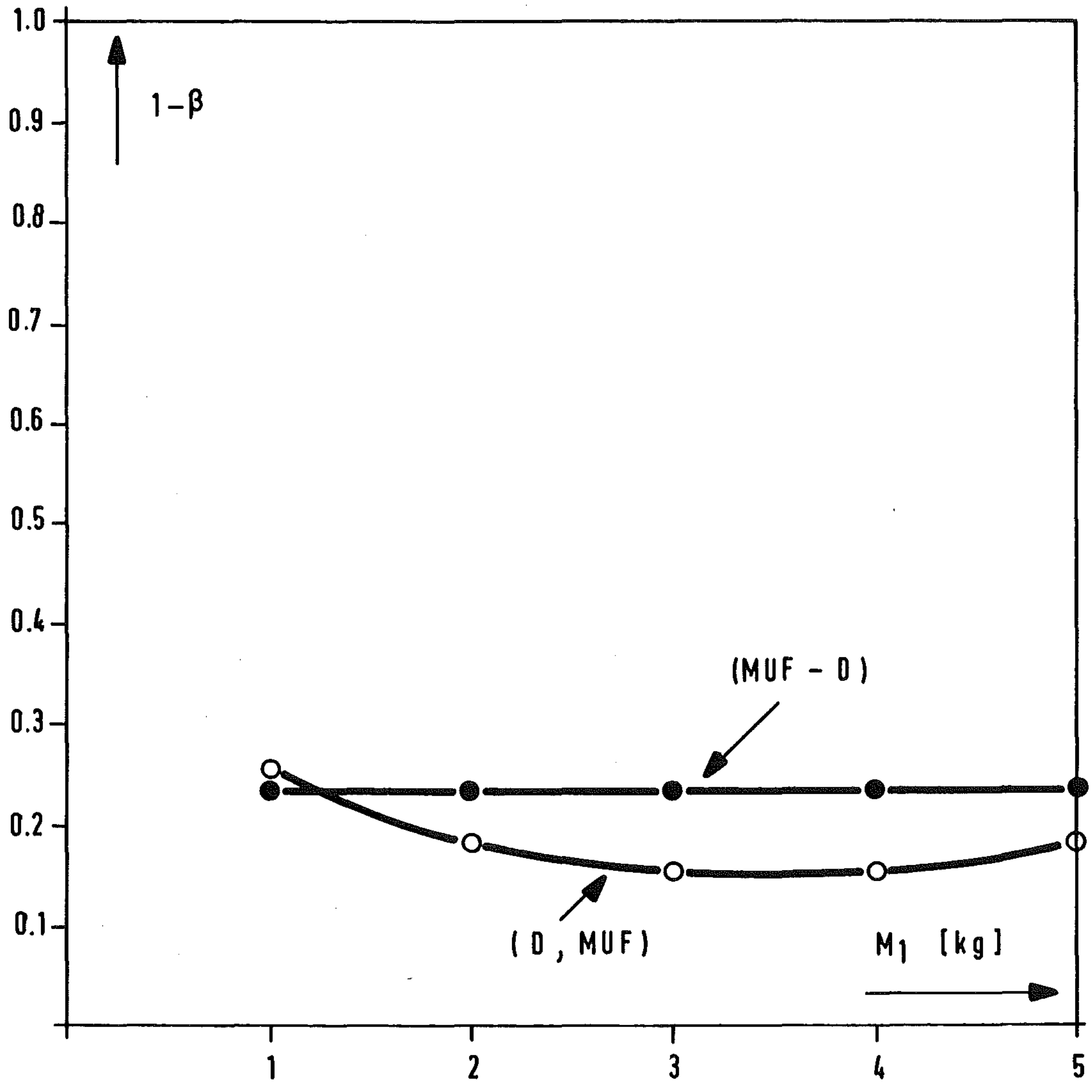

Figure 9-3: Probability of detection of the (MUF-D)- and the bivariate (D,MUF)-test for different diversion and inspection strategies in *Model B*; goal quantity $M=M_1+M_2=5$kg, variances as in Table 8-4, $\alpha=0.05$.

The figure shows for each M_1 the detection probability for the diverted amount $M-M_1+M_1/2$.

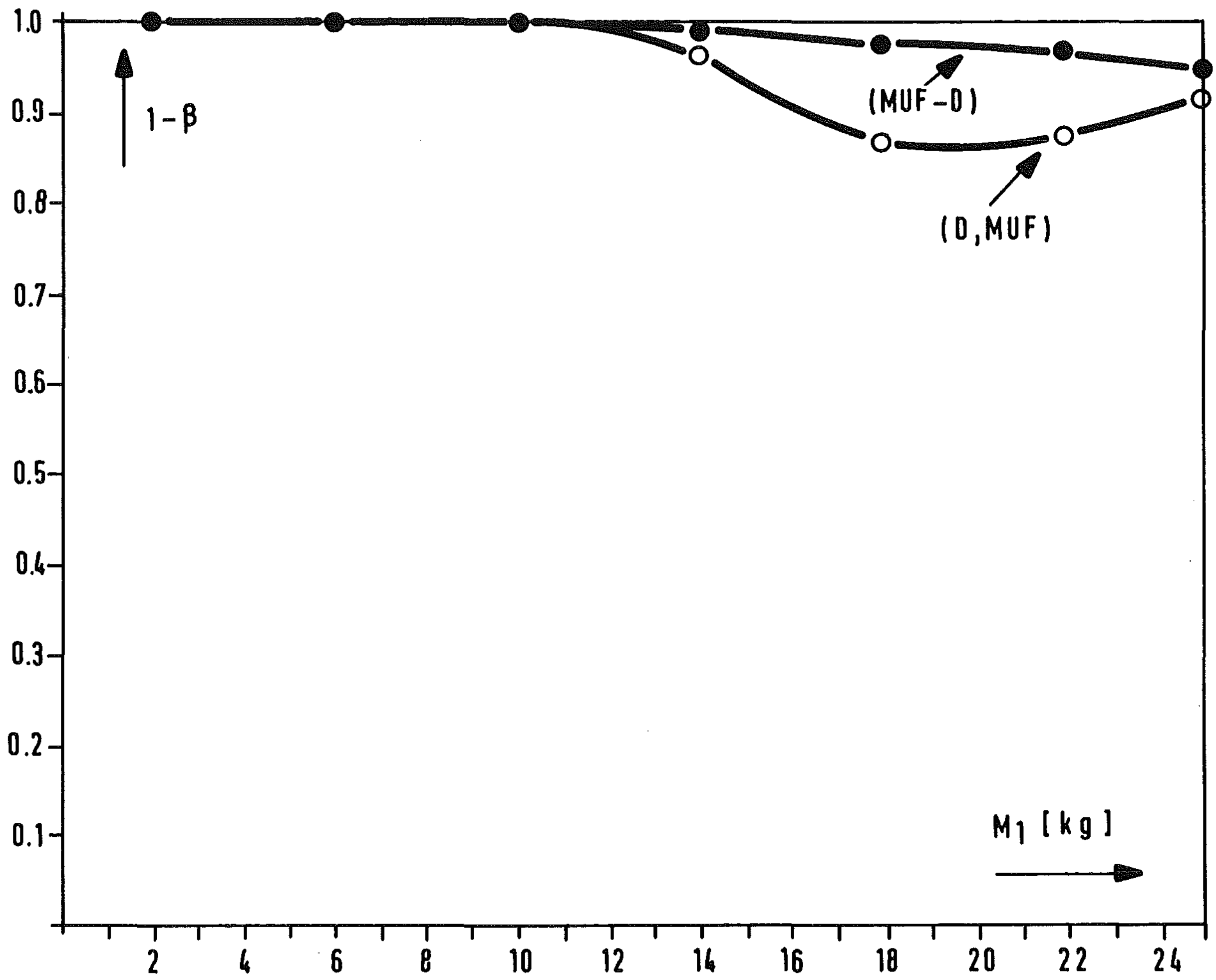

Figure 9-4: Probability of detection of the (MUF-D)- and the (D,MUF)-test for different diversion strategies in *Model B*; goal quantity $M=M_1+M_2=25$kg, variances as in Table 8-4, $\alpha=0.05$.
The figure shows for each M_1 the detection probability for the diverted amount $M-M_1+M_1/2$.

the sample sizes n_i are determined with the help of formula (3-27) for one fixed value of M_1 according to the procedure outlined in the introduction of this chapter.

In Figures 9-3 and 9-4, also the probability of detection for the bivariate (D,MUF)-test is represented graphically as a function of the total data falsification (M_1) for fixed values of the goal quantity M according to (4-7) resp. (9-2), with optimized false alarm probabilities α_1^* and α_2^*. Again, we see that in a range of small values of M_1, the bivariate (D,MUF)-test leads to a higher probability of detection, however, in the larger range the MUF-D-test is better.

In Figure 9-5, the probability of detection both for the MUF-D and the (D,MUF)-test is represented graphically as a function of the total data falsification (M_1) for fixed values of the goal quantity M with variances $\sigma^2_{D|H_0}$ and $\sigma^2_{D|H_1}$ taken from table 8-5, i.e. for a fixed verification effort C. If we compare the data in Figure 9-5 with the corresponding data in Figure 9-3, we notice that a smaller verification effort leads to a higher probability of detection! This contradiction can be explained as follows: Formula (3-16) for the optimal sample sizes of the inspector can in certain limiting cases also be obtained by an optimization procedure where the variance $\sigma^2_{D|H_1}$ is minimized. The probability of detection (9-1), however, is *maximized* for minimized variance $\sigma^2_{D|H_1}$ only if

$$M-U_{1-\alpha}\cdot\sqrt{\sigma^2_{D|H_O}-\sigma^2_{MUF}} > 0 \ , \qquad (9-5)$$

otherwise the probability of detection is *minimized.* In case of the probability of detection (9-2) we observe a similar phenomenon. The conclusion to be drawn is that our procedure - determination of the sample sizes with the help of formula (3-27) and performance of the D-test leads only to reasonable results with respect to the (approximated) probabilities of detection (9-1) and (9-2), if the goal quantity M is not too small compared to the standard deviations of the measurement and sampling errors (see, e.g., (9-3)).

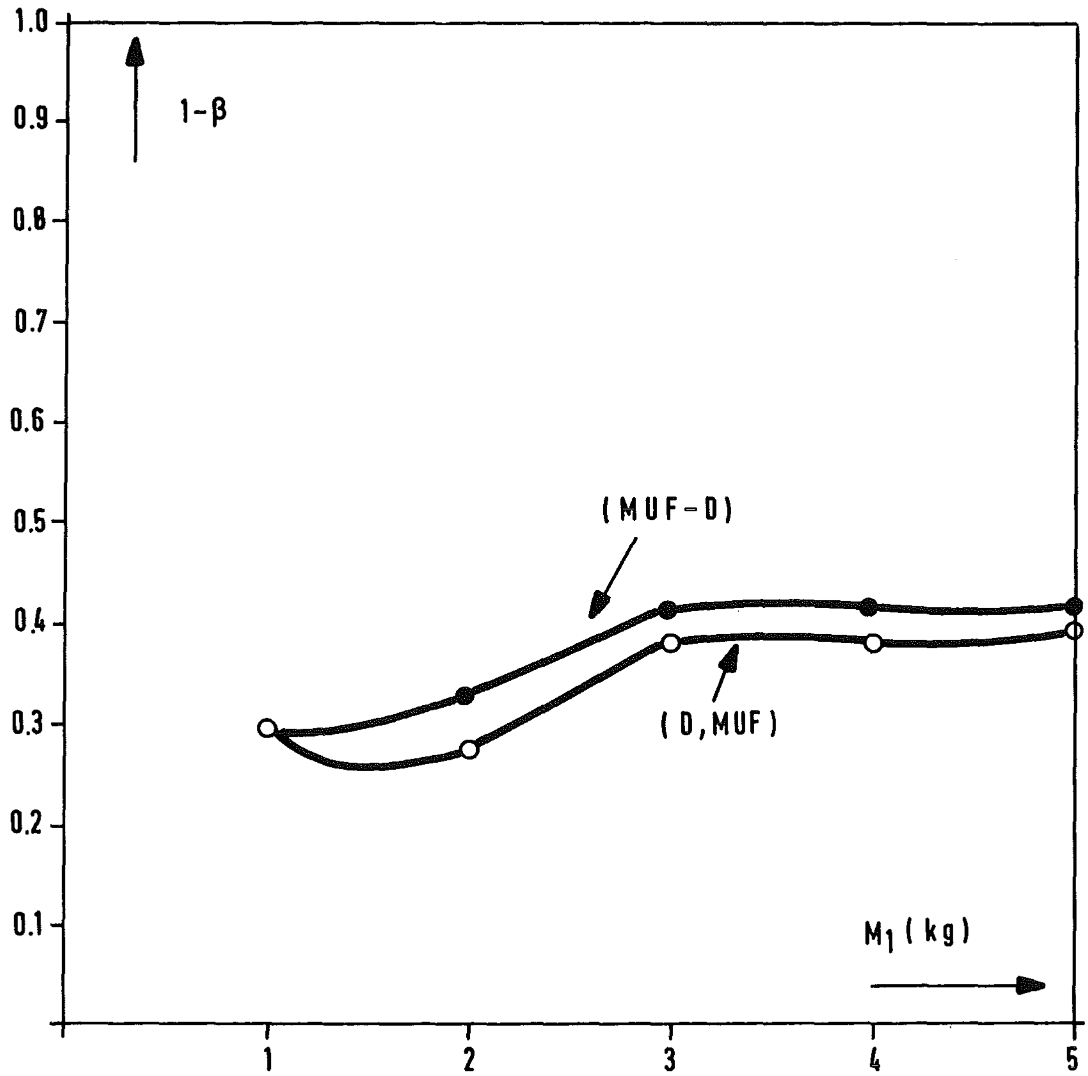

Figure 9-5: Probability of detection of the (MUF-D)- and the (D,MUF)-test for different diversion strategies in *Model B*; goal quantity $M=M_1+M_2=5kg$, variances as in Table 8-7, $\alpha=0.05$.
The figure shows for each M_1 the detection probability for the diverted amount $M-M_1+M_1/2$.

10. Concluding Remarks

In this paper the detection probability of alternative statistical evaluation schemes for the nuclear material safeguards measures have been analyzed at the hand of concrete fabrication plant data. It has been shown that it is a prerequisite for such an analysis to take into account all possible diversion and control strategies which in turn depend on the evaluation schemes.

Because of the complexity of the system a priori it was clear that an evaluation scheme could not be found which was superior to all other schemes. This is true also for the Neyman-Pearson-test, which by definition is the best test if the diversion strategy is completely specified, because the inspector cannot know this strategy and therefore has to make assumptions on global strategies. The minimax-procedure which is a consequence of this lack of one of the analyzed procedures (MUF-D, or (D,MUF), or others). It was the intent of the *analytical investigations* to clarify this situation as far as possible, and it was the intent of the numerical calculations to show the numerical differences of the detection probabilities of the different procedures. It has also been mentioned that in view of these results, *practical arguments* will have to determine the final decisions.

There are many aspects of nuclear material safeguards based on the material balance principle which have not been tackled in this paper. We only mention the question of the subdivision of plants into several material balance areas, or the question of the number of inventory periods during a reference time which has to do with the question of the appropriate detection time. There are studies on all of these questions however, it would be too early at this step of development to try to put together all these ideas: the present paper might give an idea how many 'models', 'alternatives' and 'schemes' would have to be taken into account. Instead, on first has to try to rule out some of these by common sense arguments; thereafter only one can proceed with theoretical investigations.

One major aspect of this study was to find out, under which assumptions simplified formulae or procedures can be justified, infact, inspectors at the plant site or in the headquarters cannot be expected to handle too complicated a formalism. For this reason the data verification problems we considered in greater detail, and it was shown, e.g., under which conditions presently used formulae for the determination of inspection sample sizes are approximately valid. Concrete proposals, however, were not made in this study because this should be left to the practitioners, as already stated.

This is true even more for the question whether or not the safeguards system in its present form is considered to fulfill the requirements of the Non-Proliferation-Treaty, at least in the case of the plant considered here. The effifiency, i.e. the relation between probability of detection, goal quantity, false alarm probability, and inspection effort has been calculated numerically for an inventory period of 6 months: It remains to be decided by the safeguards authorities whether one can live with these results or not and then, consequently, has to search for further means.

Acknowledgement

The authors would like to thank Mr. F. Schinzer, NUKEM Hanau, for fruitful discussions and comments about plant data and measurement systems, and Dr. G. Spannagel, Kernforschungszentrum Karlsruhe, for valuable advice in connection with the computations underlying Figures 4-1 to 4-4.

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