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DYNAMIC DENSITY DEPENDENCE OF THE ALPHA-NUCLEON FORCE IN FOLDING MODELS OF INELASTIC SCATTERING OF ALPHA-PARTICLES

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Abstract

The importance of a dynamic density dependence of the α-particle-bound nucleon force is demonstrated by deformed folding model analyses of elastic and inelastic scattering of 104 MeV α-particles from ^{50}Ti and ^{52}Cr. Approximations are discussed and technical details are given.

DYNAMISCHE DICHTEABHÄNGIGKEIT DER ALPHA-NUKLEON-KRAFT IN FALTUNGSMODELLEN DER UNELASTISCHEN STREUUNG VON ALPHA-TEILCHEN

Die Bedeutung einer dynamischen Dichteabhängigkeit der α-Teilchen-Nukleon-Wechselwirkung wird in Faltungsmodell-Analysen der elastischen und unelastischen Streuung von 104 MeV α-Teilchen an ^{50}Ti und ^{52}Cr demonstriert. Formalismus und Approximationen werden diskutiert.

A most sucessful description and semi-microscopic interpretation of α-particle scattering from nuclei is provided by single folding models generating the real part of the α-particle-nucleus optical potential by a convolution of an effective α-particle-bound nucleon interaction $V_{eff}^{\alpha-N}(\vec{r}_\alpha,\vec{r})$ with the nucleon-density distribution or the transition density of a nuclear transition, respectively (see e.g. Rebel et al. 1974). It has been shown that interesting information about the size and the shape of nuclei can be extracted from high-quality scattering data on this basis (see e.g., Gils et al. 1980). The extension of the folding model to inelastic scattering often invokes a collective model description of the excitation of nuclear states, introducing a permanently or dynamically deformed density distribution (thus providing the transition densities in terms of the"deformation" of the ground state density). In the particular case of a vibrational model the transition densities are derived from a nucleon distribution $\rho(\vec{r})$ with a dynamically deformed surface usually introduced by a vibrating half-way radius (of a Fermi-type distribution)

$$c\ (\hat{r}) = c_0\ \left[1 + \sum_{\lambda\mu} \alpha^*_{\lambda\mu}\ Y_{\lambda\mu}\ (\hat{r})\right] \tag{1}$$

Here the $\alpha^*_{\lambda\mu}$ are combinations of phonon-creation and annihilation operators of the multipolarity (λ,μ) with $\lambda \geqq 2$. In the framework of this model the transition density is essentially obtained from a Taylor series expansion of the "deformed" density around $c = c_0$. The vibrational model form factor proves to be a rather general parametrization of the radial shape of the transition density, even for transitions where a collective excitation may not be taken too literally. This enables a relatively model independent extraction of isoscalar transition rates from (α,α') data. (Rebel et al. 1981, Corcalciuc et al. 1983).

An important question in explicit folding model analyses is the problem of an adequate and reliable α-particle-bound nucleon effective interaction. Many previous calculations have used density-independent interactions calibrated e.g. for the case of elastic scattering from ^{40}Ca and describing rather well the diffraction region of the differential scattering cross sections. However, density-independent interactions fail to reproduce the cross section at larger scattering angles for higher energy projectiles when the particles probe deeper into the nucleus and would provide information about the densities at smaller radii. Such a feature has been shown in elastic (Friedman et al. 1978) as well as in inelastic scattering of α-particles (Pesl et al. 1983). This lacuna of the effective interaction has been remedied in elastic scattering analyses (Gils et al. 1980) by introducing a density-dependent factor $g(\rho)$ by writing

$$V_{eff}^{\alpha-N}\,(\vec{r}_\alpha,\vec{r}) = V_{DI}\,(\vec{r}_\alpha,\vec{r}) \cdot g(\rho) \qquad (2)$$

accounting for saturation with increasing density.

A particular choice of $V_{eff}^{\alpha-N}$ as sucessfully used by Friedman et al. (1978) for analyses of elastic scattering of 104 MeV α-particles is the Gaussian form

$$V_{DI}(\vec{r}_\alpha,\vec{r}) = V_o \exp\left(-|\vec{r}_\alpha-\vec{r}|^2/a^2\right) \qquad (3)$$

$$g(\rho) = 1 - \gamma\, \rho^{2/3}(r)$$

with V_o = 64.6 MeV a = 1.798 fm and γ = 1.9 fm^2

Effects of density-dependent interactions on inelastic scattering are well known in proton scattering and influence the nuclear shape information extracted by deformed folding studies of (p,p') data for permanently deformed nuclei (Hamilton and Mackintosh 1977, 1978, Mackintosh 1978 e.g.). The sensitivity of inelastic α-particle scattering cross sections to the saturation properties of the effective interaction is somewhat surprising and contrary to the general belief that inelastic α-particle scattering is mainly determined by the low-density surface region.

In the present paper we would like to draw attention on the fact that precise (α,α') scattering data in the 100 MeV region, extending to large angles are even sensitive to the details of the manner in which the density dependence is introduced in the explicit folding model calculations.
Our starting point is an observation of Pesl et al (1983) who analysed, in a coupled channel procedure, the differential cross sections of elastic and inelastic scattering of 104 MeV α-particles from ^{50}Ti and ^{52}Cr using the force given by eq. (3). Instead of the density-independent (real) potential of the $0 \rightarrow I_f = L$ transition

$$<I_f = L||U||I_i = 0>_{DI} = -i^L(2L+1)^{-1/2}\ \beta_L^m\ c_o \int \frac{\partial\rho}{\partial r} \cdot V_L(r_\alpha,r)r^2\ dr$$

$$+ \text{ second order terms} \qquad (4)$$

(with β_L^m the "deformation" parameter of the density and V_L the L-th multipole component of an density-independent effective interaction $V_{eff}^{\alpha-N}$) they calculated the real part of the transition potentials by

$$<I_f = L||U||I_i = 0>_{SDD} =$$

$$= -i^L(2L+1)^{-1/2}\ \beta_L^m\ c_o \int \frac{\partial\rho}{\partial r}\ g(\rho) \cdot V_L^{DI}(r_\alpha,r)r^2\ dr$$

$$+ \text{ second order terms} \qquad (5a)$$

assuming a static density dependence by

$$g(\rho) = 1 - \gamma\ \rho_{L=0}^{2/3}\ (r) \qquad (5b)$$

with $\rho_{L=0}$ being the spherical part of the dynamically deformed distribution $\rho(\vec{r})$. Indeed, as compared to the results based on e.q. (4) considerably improved fits for large angle scattering could be obtained, but at the expense of quite unreasonable values of the transition rates and of the deformation parameters of the density distribution. The values of the "deformation" parameters were found to be rather small, in fact similar to the values of potential deformation resulting from an usual extended optical potential analysis (see Tab. 1). This can hardly be assumed to be reasonable.

In the present work we do not restrict the density dependence to the monopole part of the deformed density $\rho(\vec{r})$ and allow the saturation factor to follow dynamic changes of the density $\rho(\vec{r})$ without introducing any additional parameters. Since the inclusion of the density dependence can be considered as a replacement of the nuclear density $\rho(\vec{r})$ in the folding integral by an effective density (Srivastava and Rebel 1984)

$$\rho_{eff}(\vec{r}) = \rho(\vec{r})\ [1 - \gamma\ \rho^{2/3}\ (\vec{r})] \qquad (6a)$$

the dynamic density dependence factor $g'(\rho)$ replacing $g(\rho)$ in e.q. (5a) follows from an expansion of the effective density $\rho_{eff}(\vec{r})$,(in first order, say)

$$g'(\rho) = (1 - \frac{5}{3}\gamma\, \rho_0^{\,2/3}(r)) \qquad (6b)$$

with $\rho_0(r)$ being the monopole part of $\rho(\vec{r})$. While by eq. (5b) the same factor $g(\rho)$ appears for elastic and inelastic scattering, the dynamical density dependence leads to different saturation factors, additionally dependent on the order of deformation.

The influence of the nuclear shape on the saturation factor is quite obvious in cases of permanent deformation, here, infact, a static effect taken into account by including all multipoles of $\rho(\vec{r})$ in a generalized expression like eq. (5b).

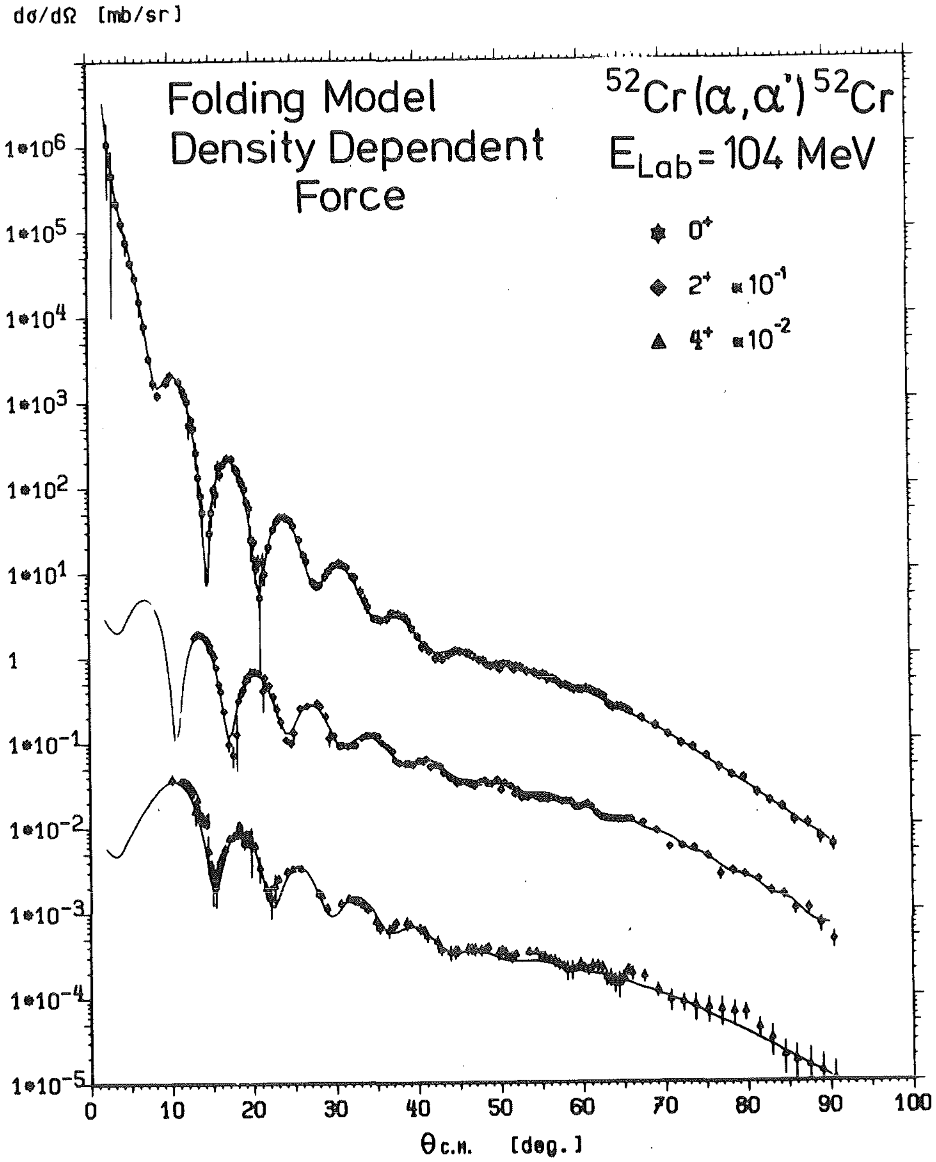

Fig. 1: 104 MeV α-particle scattering from ^{52}Cr described by a folding model with dynamic density dependence of the α-nucleon interaction

The following results of coupled channel analyses of 104 MeV α-particle scattering from the 0^+, 2_1^+ and 4_1^+ states of ^{50}Ti and ^{52}Cr (Pesl et al. 1983) will show that the effect of a dynamical density dependence is substantial and cannot be ignored.

Tab. 1 presents the results of ordinary extended optical model analysis (EOM) with a Wood-Saxon squared real part of the potential, of the procedure with static density dependence (SDD) and with dynamic density dependence (DDD). The latter procedure gives the natural result that the potential distribution is less deformed than the density distribution and gives isoscalar rates which are consistent with values found by implicit procedures or electromagnetic results (see Tab. 2). We mention that for these N = Z nuclei the electromagnetic values (Endt 1979) need not be in perfect agreement with isoscalar values. The MIFP values result from a modified implicit folding procedure (Srivastava and Rebel 1984). The excellent agreement of the theoretical cross sections calculated with dynamical density dependence with the experimental data is displayed for the case $^{52}Cr(\alpha,\alpha')$ in Fig. 1.

We conclude that the dynamic density-dependence of the effective α-particle-nucleon interaction plays a vital role in providing a consistent semi-microscopic description of inelastic scattering of higher-energy α-particles from vibrational nuclei.

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Nucleus	Proc.	V_o [MeV]	r_v/c_o [fm]	a_v/a_m [fm]	W_o [MeV]	r_w [fm]	a_w [fm]	β_{02}	β_{04}	β_{24}	$J_V/4A$ [MeV fm³]	χ^2/F 0^+	2^+	4^+
^{50}Ti	EOM	143.5	1.43	1.17	18.6	1.61	0.57	0.12	0.06	0.075	311	1.9	5.7	2.1
	SDD	-	1.04	0.52	18.1	1.58	0.63	0.12	0.09	0.075	297	2.5	4.1	3.5
	DDD	-	1.07	0.49	18.1	1.58	0.64	0.18	0.125	0.010	299	3.7	5.4	3.2
^{52}Cr	EOM	151.8	1.39	1.20	22.3	1.51	0.72	0.13	0.05	0.075	303	2.8	6.3	1.7
	SDD	-	1.08	0.46	20.7	1.54	0.68	0.12	0.08	0.075	295	2.1	2.7	2.2
	DDD	-	1.08	0.46	21.4	1.51	0.72	0.175	0.11	0.05	293	2.4	3.1	2.9

Tab. 1: Results of $0^+ - 2_1^+ - 4_1^+$ -coupled channel analyses of $^{50}Ti(\alpha,\alpha')$ and $^{52}Cr(\alpha,\alpha')$ differential cross sections (E_α = 104 MeV), described on the basis of an anharmonic vibrational model. The parameters W_o, r_w, a_w describe the strength and geometry of the imaginary part (WS form).
For sake of simplicity the deformation of the imaginary potential is taken to be identical to that of the real part or of the underlying density distribution (Fermi shape) in the folding procedure.

G_2 [s.p.u.]	^{50}Ti	^{52}Cr
EM [a]	5.8 ± 0.4	11 ± 1
MIFP [b]	6.1 ± 0.3	7.5 ± 0.1
SDD	3.5	4.4
DDD	8.1	8.7
G_4 s.p.u.		
EM [a]	-	3.4 ± 0.6
MIFP [c]	4.2	3.0
SDD	1.8	1.8
DDD	3.7	2.9

a EM = Electromagnetic values (Endt, 1979)
b MIFP Applied to values obtained by Rebel et al. 1981
c MIFP Applied to values obtained by Pesl et al. 1983

Tab. 2: Isoscalar transition rates G_L (in single-particle units) for $0^+ \rightarrow 2_1^+$ and $0^+ \rightarrow 4_1^+$ transitions in ^{50}Ti and ^{52}Cr

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Appendix A

The Deformed Folding Model with Density-Dependent Forces.

With a density-dependent effective α-bound-nucleon interaction of the type

$$V_{eff}^{\alpha-N}(\vec{r}_\alpha, \vec{r}) = V_{DI}(|\vec{r} - \vec{r}|)\ [1 - \gamma\rho^{2/3}(\vec{r})] \qquad (A1)$$

the real-part $U_R(\vec{r}_\alpha)$ of the deformed optical potential is given by

$$U_R(\vec{r}_\alpha) = \int V_{DI}(|\vec{r}_\alpha - \vec{r}|)\ \rho(\vec{r})\ [1 - \gamma\rho^{2/3}(\vec{r})]\ d\vec{r} \qquad (A2)$$

Obviously this is quite similar to the density-independent formulation (Rebel et al. 1974) if we replace the deformed density distribution $\rho(\vec{r})$ by a sort of effective density (Srivastava and Rebel 1984)

$$\rho_{eff}(\vec{r}) = \rho(\vec{r}) - \gamma\rho^{5/3}(\vec{r}) \qquad (A3)$$

Introducing the deformation of the matter distribution by an angular dependence of the half-way radius,

$$c(\vec{r}) = c_o\ [1 + \sum_{\lambda\mu} \alpha_{\lambda\mu} Y_{\lambda\mu}\ (\hat{r})] \qquad (A4)$$

in the vibrational case e.g., we expand the effective density in powers t of the operators $\alpha_{\lambda\mu}$

$$\rho_{eff}(\vec{r}) = \rho_{eff}^{(o)} + \sum_{\lambda\mu t}\ [\rho_{eff}^{(t)}(r) \cdot \alpha_{\lambda\mu}^{(t)}] Y_{\lambda\mu}(\hat{r}) \qquad (A5)$$

with

$$\alpha_{\lambda\mu}^{(1)} = \alpha_{\lambda\mu}$$

$$\alpha_{\lambda\mu}^{(t)} = [\alpha_\lambda \otimes \alpha_\lambda \ldots]_{\lambda\mu} \qquad \text{etc.}$$

and

$$\rho_{eff}^{(t)} = \frac{c_o^{\ t}}{t!}\ \frac{\partial \rho_{eff}}{\partial c_o^{\ t}} \qquad (A6)$$

We consider the "Fermi"-distributions

$$\rho_q = \rho_{oo}\ [1 + S]^{-q} \qquad (A7)$$

with

$$S = \exp\left(\frac{r - c_o}{a}\right) \qquad (A8\ a)$$

and ρ_{oo} resulting from the normalization of distribution with q = 1.

Thus, we get with

$$B = 1 \;/\; [a \cdot (1+S)] \tag{A8 b}$$

$$\frac{d\rho_q}{dc_o} = \rho_q(r)\; q\; S \cdot B \tag{A9 a}$$

$$\frac{d^2\rho_q}{dc_o{}^2} = \frac{d\rho_q}{dc_o}\;(q\;S - 1)\;B \tag{A9 b}$$

or in general by use of the relation

$$\frac{d\rho_q}{dc_o} = \frac{q}{a}\;[\rho_q - \rho_{q+1}] \tag{A9c}$$

With these expressions eq. (A6) is written

$$\rho_{eff}^{(t)} = \frac{c_o{}^t}{t!} \quad \frac{\partial^t \rho_1}{\partial_{c_o}{}^t} - \gamma \rho_{oo}{}^{2/3}\; \frac{\partial^t \rho_{5/3}}{\partial_{c_o}{}^t} \tag{A10}$$

or more explicitely

$$\rho_{eff}^{(o)}(r) = \rho_1(r)\;\; [1 - \;\gamma\; \rho_1^{2/3}(r)] \tag{A11}$$

$$\rho_{eff}^{(1)}(r) = c_o{}^2\; \frac{\partial\rho_1}{\partial c_o}\;\; [1 - \tfrac{5}{3}\;\gamma\rho_1^{2/3}(r)] \tag{A11 b}$$

$$\rho_{eff}^{(2)}(r) = c_o{}^2\; \{\; \frac{\partial^2\rho_1}{\partial c_o^2}\quad [1 - (\tfrac{5}{3})^2\;\gamma\rho_1^{2/3}(r)]$$

$$- \frac{\partial\rho_1}{\partial c_o} \cdot \rho_1^{2/3}(r) \cdot \frac{5}{3} \cdot (\frac{5}{3} - 1)\;\gamma B\;\} \tag{A11 c}$$

Appendix B: The Approximation $\rho((\vec{r}_\alpha + \vec{r})/2) \cong \rho(\vec{r})$

Our explicit folding model calculations are based on the approximation

$$\rho^{2/3}\left(\frac{\vec{r}_\alpha+\vec{r}}{2}\right) \cong \rho^{2/3}(\vec{r}) \tag{B1}$$

in the density-dependent factor of the effective interaction.
This approximation facilitates the formalism considerably and is also the basis of the results for geometrical properties of folding potentials (Srivastava 1982) and for recent modifications of implicit folding procedures (Srivastava and Rebel 1984).

With $\vec{s} = \vec{r}_\alpha - \vec{r}$, we see in

$$U_R(\vec{r}_\alpha) = \int \rho(\vec{r}_\alpha - \vec{s})\,[1-\gamma\,\rho^{2/3}(\vec{r}_\alpha - \vec{s}/2)]\; V_{DI}(\vec{s})\; d\vec{s} \tag{B2}$$

whereas ρ varies with $\vec{s}$ the $\rho^{2/3}$-term varies with $\vec{s}/2$.
As $\rho^{2/3}$-variation is also more smooth, in particular in the tail region, for a sufficiently small range of $V_{DI}(s)$ we can replace $\rho^{2/3}(\vec{r}_\alpha - \vec{s}/2)$ by the average value and define

$$\rho_{eff} = \rho(\vec{r})\;[1-\gamma\frac{1}{J_v}\int \rho^{2/3}(\vec{r} + \vec{s}/2)\; V_{DI}(s)\; d\vec{s}\,] \tag{B3}$$

with

$$J_v = \int V_{DI}(s)d\vec{s} = \pi^{3/2}\, b^3\, V_o$$

when assuming a Gaussian shape $V_{DI} = V_o \exp(-\,s^2/b^2)$.
Expanding $\rho^{2/3}(\vec{r} + \vec{s}/2)$ in a Taylor series

$$\rho_{eff}(\vec{r}) = \rho(\vec{r})\,[\,1 - \gamma\,\frac{1}{J_v}\int e^{(\vec{s}/2).\vec{\nabla}}\;\rho^{2/3}(\vec{r})\; V_{DI}(s)d\vec{s}\,] \tag{B4}$$

(where $\vec{\nabla}$ operates only on $\rho^{2/3}(\vec{r})$) and integrating over $d\vec{s}$, we get

$$\rho_{eff}(\vec{r}) = \rho(\vec{r})\,[\,1 - \gamma\; e^{\frac{b^2}{16}\nabla^2}\;\rho^{2/3}(\vec{r})\,] \tag{B5}$$

With a Fourier transformation

$$\rho^{2/3}(\vec{r}) = \frac{1}{(2\pi)^3} \int d\vec{k}\, e^{i\vec{k}\cdot\vec{r}} \tilde{\rho}^{2/3}(\vec{k})$$

ρ_{eff} is written

$$\rho_{eff}(\vec{r}) = \rho(\vec{r}) \, [1 - \gamma \frac{1}{(2\pi)^3} \int d\vec{k}\, e^{-\frac{b^2k^2}{16}} \rho^{2/3}(k)\, e^{i\vec{k}\cdot\vec{r}}]$$

$$\approx \rho(\vec{r}) \, [1 - \gamma\, \rho^{2/3}(\vec{r}) + O(\nabla^2)] \qquad \text{(B6)}$$

showing the approximation (B1) as leading term provided $O(\nabla^2)$ is sufficiently small.

We estimate the term $O(\nabla^2)$ for a Gaussian shape $\rho(r) = \rho_o\, e^{-r^2/a^2}$ and $\rho^{2/3}(r) = \rho_o{}^{2/3}\, e^{-2/3\; r^2/a^2}$

Introducing the Fourier transform

$$\tilde{\rho}^{2/3}(\vec{k}) = \text{const}\cdot e^{-\frac{3}{8} k^2 a^2}$$

the effective density (eq. B6) is written

$$\rho_{eff}(r) = \rho(r) \;\; [1 - \text{const}\; e^{-r^2/\sigma^2}]$$

$$\sigma^2 = 3a^2/2 + b^2/4 = \langle r^2\rangle_\rho \;\; [1 + \frac{b^2}{4\langle r^2\rangle_\rho}] \; ,$$

$\langle r^2\rangle_\rho$ being the ms radius of the density distribution.
Thus the neglect of $O(\nabla^2)$ appears to be reasonable,
if

$$1 \gg \frac{b^2}{4\langle r^2\rangle_\rho} \approx 0.6\; A^{-2/3} \qquad \text{(B7)}$$

$$\approx 0.05$$

for b = 2 fm and A = 40. $(\langle r^2\rangle_\rho \cong (3/5)\cdot(1.2\cdot A^{1/3})^2)$

Considering eqs. (B2-6) identical arguments could replace $\rho^{2/3}(\frac{\vec{r}_\alpha + \vec{r}}{2})$ by $\rho^{2/3}\,(\vec{r}_\alpha)$ (considerably simplifying the folding integral).
However, the extraction of a particular multipole component of the potential

would involve all multipoles of ρ since

$$U_R(\vec{r}_\alpha) = [1 - \gamma \sum_{lm} T_{lm}(r_\alpha)\, Y_{lm}(\hat{r}_\alpha)]\, x \qquad \text{(B8a)}$$

$$\{\sum_{LM} [\int \rho_{LM}(r)\, V_{LM}(r_\alpha, r)\, r^2 dr]\, Y_{LM}(\hat{r}_\alpha)\}$$

or

$$U_{LM}(r_\alpha) = X_{LM}(r_\alpha) - \gamma\, Z_{LM}(r_\alpha) \qquad \text{(B8b)}$$

with

$$X_{LM}(r_\alpha) = \int \rho_{LM}(r)\, V_{LM}(r_\alpha, r) r^2 dr$$

$$Z_{LM}(r_\alpha) = \sum_{\substack{l_1 l_2 \\ m_1 m_2}} T_{l_1 m_1}(r_\alpha) \cdot X_{l_2 m_2}(r_\alpha)$$

$$x \left[\frac{(2l_1+1)(2l_2+1)(2L+1)}{4\pi}\right]^{1/2} (-1)^M$$

$$x \begin{pmatrix} l_1 & l_2 & L \\ m_1 & m_2 & -M \end{pmatrix} \begin{pmatrix} l_1 & l_2 & L \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{(B8c)}$$

We many recognize that any dependence from the position r_α of the α-projectile would complicate the formulation. In fact, this cannot be excluded a priori. But we know from detailed studies of elastic α-particle scattering (Gils 1983) that the influence of the density $\rho(r_\alpha)$ at the site of the α-particle is small. Most likely it simulates some kind of a polarization of the α-particle diving into ρ.. Just, such effects seem to be neglected by the approximation (B1).

Appendix C: Multipoles of the folded potential for a density dependent force

In this appendix we consider the formulation of a deformed folding model using various forms of the saturation factor $g(\rho)$ and of the density independent part $V_{DI}(|\vec{r}_\alpha-\vec{r}\ )$ of the effective interaction.

$$g(\rho) = [1 - \alpha\ \rho^{2/3}(\vec{r})]\ [1 - \beta\ \rho^{2/3}(\vec{r}_\alpha)] \tag{C1}$$

and

$$g(\rho) = [1 - \gamma\ \rho^{2/3}\ (\frac{\vec{r} + \vec{r}_\alpha}{2}\ )] \tag{C2}$$

For V_{DI} we consider three different shapes

$$V_{DI} = V_o\ \delta(\vec{r}_\alpha - \vec{r}) \tag{C3}$$

$$V_{DI} = V_o\ \exp(-\ |\vec{r}_\alpha - \vec{r}|^2/\mu^2) \tag{C4}$$

$$V_{DI} = V_o\ \frac{\exp(-\ |\vec{r}_\alpha - \vec{r}|/\mu)}{(|\vec{r}_\alpha - \vec{r}|/\mu)} \tag{C5}$$

The multipoles

$$V_{lm}(r_\alpha,r) = \int V_{DI}(\vec{r}_\alpha,\vec{r}) \cdot Y^*_{lm}(\hat{r}_\alpha)\ Y_{lm}(\hat{r})\ d\Omega_\alpha\ d\Omega$$

are analytically given by

$$V_{lm} = \frac{1}{r_\alpha^2}\ \delta(r_\alpha - r) \qquad \text{for the } \delta\text{-force} \tag{C3}$$

$$V_{lm} = V_o\ \exp(-\ \frac{r_\alpha^2 + r^2}{\mu^2}\ ) \cdot 4\pi\ i^l\ j_l(-\ \frac{2i\ r_\alpha\ r}{\mu^2}\ )$$

for the Gaussion force (C4)

$$V_{lm} = V_o\ \ 4\pi\ j_l\ \ (i\ \frac{r_<}{\mu}\ )\ h_l^{(1)}\ (\frac{i\ r_>}{\mu}\ ) \qquad \text{for the Yukawa force (C5)}$$

where $r_<(r_>)$ is the smaller (larger) value of r_α and r and $h_l^{(1)}$ is the spherical Hankelfunction of the first kind and j_l is spherical Bessel function.

In the case of a density-independent force $(g(\rho) = 1)$ the multipoles of the interaction potential are just given by

$$U_{lm}(r_\alpha) = \int \rho_{lm}(r) \cdot V_{lm}(r_\alpha,r)\, r^2\, dr = X_{lm}(r_\alpha) \tag{C6}$$

where $\rho_{lm}(r)$ are the multipoles of the deformed density distribution

$$\rho(\vec{r}) = \sum_{lm} \rho_{lm}(r)\, Y_{lm}(\hat{r}) \tag{C7}$$

For the form (C1) the real part of the interaction potential is given by

$$\begin{aligned} U_R(\vec{r}_\alpha) &= [1 - \beta\, \rho^{2/3}(\vec{r}_\alpha)] \int \rho(\vec{r})\, [1 - \alpha\, \rho^{2/3}(\vec{r})]\, V_{DI}\, d\vec{r} \\ &= W(\vec{r}_\alpha) - \beta\, \rho^{2/3}(\vec{r}_\alpha)\, W(\vec{r}_\alpha) \end{aligned} \tag{C8}$$

The multipoles of $W(r_\alpha)$ are

$$W_{lm}(r_\alpha) = X_{lm}(r_\alpha) - \alpha\, Z_{lm}(r_\alpha) \tag{C9}$$

with

$$Z_{lm}(r_\alpha) = \int (\rho^{5/3}(\vec{r}))_{lm}\, V_{lm}(r,r_\alpha)\, r^2\, dr \tag{C10}$$

Introducing the multipoles T_{lm} by

$$\rho^{2/3}(\vec{r}_\alpha) = \sum_{lm} T_{lm}(r_\alpha) \cdot Y_{lm}(\hat{r}_\alpha) \tag{C11}$$

we write

$$U_{lm}(r_\alpha) = X_{lm}(r_\alpha) - \alpha\, Z_{lm}(r_\alpha) - \beta\, \tilde{Z}_{lm}(r_\alpha) \tag{C12}$$

with

$$\tilde{Z}_{lm}(r_\alpha) = \sum_{\substack{l_1m_1\\l_2m_2}} (-1)^m \left[\frac{(2l_1+1)(2l_2+1)(2l+1)}{4\pi}\right]^{1/2} \begin{pmatrix} l_1 & l_2 & l \\ m_1 & m_2 & -m \end{pmatrix}$$

$$\times \begin{pmatrix} l_1 & l_2 & l \\ 0 & 0 & 0 \end{pmatrix} W_{l_1m_1}(r_\alpha) \cdot T_{l_2m_2}(r_\alpha) \qquad \text{(C13)}$$

Through the term $\beta\,\tilde{Z}_{lm}$ all multipoles ρ_{lm} do contribute to a particular multipole U_{lm}, which makes practical calculations some-what unpleasant. The form (C2), which appears physically less justified than (C1), is even a more complicated case.

Denoting $(\vec{r}_\alpha + \vec{r})/2 = \vec{R}$ we expand

$$\rho^{2/3}(\vec{R}) = \sum_{l_1m_1} \frac{T_{l_1m_1}(R)}{R^{l_1}} \cdot R^{l_1} \cdot Y_{l_1m_1}(\hat{R}) \qquad \text{(C14)}$$

It is reasonable to assume that for small $|\vec{R}| < \varepsilon$, $\rho(\vec{r})$ can be considered to be spherical so that $T_{lm} = 0$ for $l \neq 0$ at $|\vec{R}| < \varepsilon$.

Now taking a Slater-expansion of $T_{l_1m_1}(R) / R^{l_1}$ we get

$$\frac{T_{l_1 m_1}(R)}{R^{l_1}} = \sum_{l_2m_2} (-1)^{l_2} P_{l_1 m_1 l_2}(r_\alpha, r)\; Y_{l_2 m_2}(\hat{r}_\alpha)\; Y^*_{l_2 m_2}(\hat{r}) \qquad \text{(C15)}$$

In the above expression, we note that the Slater coefficients P are independent of m_2, as $T_{l_1 m_1}(R)/R^{l_1}$ is spherically symmetric. We also get, decomposing the solid harmonics, $R^{l_1} Y_{l_1 m_1}$ (see Appendix D)

$$R^{l_1} Y_{l_1 m_1}(\hat{R}) = \sum_{l_3 m_3} Q_{l_1 l_3}(r_\alpha, r)\,(-1)^{m_1} \cdot \begin{pmatrix} l_1 - l_3 & l_3 & l_1 \\ m_1 - m_3 & m_3 & -m_1 \end{pmatrix} \cdot Y_{l_1 - l_3,\, m_1 - m_3}(\hat{r}_\alpha)\; Y_{l_3 m_3}(\hat{r}) \qquad \text{(C.16)}$$

where

$$Q_{l_1 l_3}(r_\alpha, r) = \left[\frac{4\pi\,(2l_1+1)!}{(2l_3+1)!\,(2l_1-l_3+1)!} \right]^{1/2} (-1)^{l_1} \sqrt{(2l_3+1)} \cdot \left(\frac{r_\alpha}{2}\right)^{l_1-l_3} \left(\frac{r}{2}\right)^{l_3} \qquad \text{(C.17)}$$

we also note that, in the Slater expansion,

$$V_{DI}(\vec{r}_\alpha, \vec{r}) = \sum_{l_4, m_4} V_{l_4}(r_\alpha, r)\; Y_{l_4 m_4}(\hat{r}_\alpha)\; Y^*_{l_4 m_4}(\hat{r}) \qquad \text{(C.18)}$$

V_{l_4} is independent of m_4 as V_{DI} is central.

Thus we write

$$U(\vec{r}_\alpha) = X(\vec{r}_\alpha) - \gamma \; S(\vec{r}_\alpha) \qquad (C.19)$$

where

$$S(\vec{r}_\alpha) = \int \rho(\vec{r}) \; \rho^{2/3}\left(\frac{\vec{r}_\alpha + \vec{r}}{2}\right) V_{DI}(\vec{r}_\alpha,\vec{r}) \; d\vec{r} \qquad (C.20)$$

$$= \sum_{\substack{l \; l_1 \; l_2 \; l_3 \; l_4 \\ m \; m_1 \; m_2 \; m_3 \; m_4}} I_{lm\,l_1\,l_2\,l_3\,l_4}(r_\alpha).$$

$$Y_{l_2 m_2}(\hat{r}_\alpha) \; Y_{l_1-l_3,\, m_1-m_3}(\hat{r}_\alpha) \; Y_{l_4 m_4}(\hat{r}_\alpha).$$

$$\int Y_{lm}(\hat{r}) \; Y^*_{l_2 m_2}(\hat{r}) \; Y_{l_3 m_3}(\hat{r}) \; Y^*_{l_4 m_4}(\hat{r}) \; d\,\Omega_r .$$

$$\begin{pmatrix} l_1 - l_3 & l_3 & l_1 \\ m_1 - m_3 & m_3 & -m_1 \end{pmatrix} \qquad (C.21)$$

where

$$I_{lml_1m_1l_2l_3l_4}(r_\alpha) = (-1)^{l_2+m_1} \int \rho_{lm}(r) \; P_{l_1m_1l_2}(r_\alpha,r) \; Q_{l_1l_3}(r_\alpha,r).$$

$$V_{l_4}(r_\alpha,r) \; r^2 \, dr \qquad (C.\ 22)$$

Using the results (D.2) and (D.3) from Appendix D, we get

$$Y_{l_2 m_2}(\hat{r}_\alpha)\; Y_{l_4 m_4}(\hat{r}_\alpha)\; Y_{l_1-l_3, m_1-m_3}(\hat{r}_\alpha)$$

$$= \sum_{\substack{L'M'\\ L''M''}} \left(\frac{2L'+1}{4\pi}\right) \left[(2l_2+1)(2l_4+1)(2\overline{l_1-l_3}+1).\;(2L''+1)\right]^{1/2}$$

$$\begin{pmatrix} l_2 & l_4 & L' \\ m_2 & m_4 & -M' \end{pmatrix} \begin{pmatrix} l_1-l_3 & L' & L'' \\ m_1-m_3 & M' & -M'' \end{pmatrix}$$

$$\begin{pmatrix} l_2 & l_4 & L' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_1-l_3 & L' & L'' \\ 0 & 0 & 0 \end{pmatrix}$$

$$(-1)^{M'+M''}\; Y_{L''M''}(\hat{r}_\alpha) \tag{C.23}$$

and

$$\int Y^*_{l_2 m_2}(\hat{r})\; Y^*_{l_4 m_4}(\hat{r}) Y_{lm}(\hat{r})\; Y_{l_3 m_3}(\hat{r})\; d\Omega_r$$

$$= (-1)^{m_2+m_3+m_4} \int Y_{l_2 -m_2}(\hat{r})\; Y_{l_4 -m_4}(\hat{r})\; Y_{lm}(\hat{r})\; Y^*_{l_3 -m_3}(\hat{r})\; d\Omega_r$$

$$= (-1)^{m_2+m_3+m_4} \sum_{LM} \frac{2L+1}{4\pi} \left[(2l_2+1)(2l_4+1)(2l+1)(2l_3+1)\right]^{1/2}$$

$$(-1)^{M-m_3} \begin{pmatrix} l_2 & l_4 & L \\ -m_2 & -m_4 & -M \end{pmatrix} \begin{pmatrix} L & l & l_3 \\ M & m & m_3 \end{pmatrix}$$

$$\begin{pmatrix} l_2 & l_4 & L \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} L & l & l_3 \\ 0 & 0 & 0 \end{pmatrix} \tag{C.24}$$

Thus, the full expression for $S(\vec{r}_\alpha)$ becomes

$$S(\vec{r}_\alpha) = \sum_{\substack{l\; l_1\; l_2\; l_3\; l_4\; L\; L'\; L'' \\ m\; m_1\; m_2\; m_3\; m_4\; M\; M'\; M''}} I^{(r_\alpha)}_{lm\, l_1 l_2 l_3} \cdot Y^{(\hat{r}_\alpha)}_{L''\, M''} \left(\frac{2L+1}{4\pi}\right) \left(\frac{2L'+1}{4\pi}\right) (-1)^{M+M'+M''+m_2+m_4+l_2+l_4+L}$$
$$\left[(2l_2+1)^2\,(2l_4+1)^2\,(2l+1)\,(2l_3+1)\,(2\overline{l_1-l_3}+1)\,(2L''+1)\right]^{1/2}$$
$$\begin{pmatrix} l_2 & l_4 & L' \\ m_2 & m_4 & -M' \end{pmatrix} \begin{pmatrix} l_1-l_3 & L' & L'' \\ m_1-m_3 & M' & M'' \end{pmatrix}$$
$$\begin{pmatrix} l_2 & l_4 & L \\ m_2 & m_4 & M \end{pmatrix} \begin{pmatrix} L & l & l_3 \\ M & m & m_3 \end{pmatrix} \begin{pmatrix} l_1-l_3 & l_3 & l_1 \\ m_1-m_3 & m_3 & -m_1 \end{pmatrix}$$
$$\begin{pmatrix} l_2 & l_4 & L' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_1-l_3 & L' & L'' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_2 & l_4 & L \\ 0 & 0 & 0 \end{pmatrix}$$
$$\begin{pmatrix} L & l & l_3 \\ 0 & 0 & 0 \end{pmatrix} \tag{C.25}$$

Here we note that unless $m_2+m_4 = M'$, the contribution to the above sum would be zero, and thus $(-1)^{M'+m_2+m_4} = (-1)^{2M'} = 1$.

Performing the summation over m_2 and m_4 we get using

$$\sum_{m_2 m_4} (2L+1) \begin{pmatrix} l_2 & l_4 & L' \\ m_2 & m_4 & -M' \end{pmatrix} \begin{pmatrix} l_2 & l_4 & L \\ m_2 & m_4 & M \end{pmatrix} = \delta_{LL'}\ \delta_{M-M'} \tag{C.26}$$

$$S_{L''M''}(r_\alpha) = \sum_{\substack{l \; l_1 \; L \; l_2 l_3, l_4 \\ m \; m_1 \; M}} I_{lm\, l_1\, m_1\, l_2, l_3, l_4}(r_\alpha)\; (-1)^{M+M''+l_2+l_4+L}\; \frac{(2L+1)(2l_2+1)(2l_4+1)}{(4\pi)^2} .$$

$$\left[(2l+1)\;(2l_3+1)\;(2\overline{l_1-l_3}+1)\;(2L''+1)\right]^{1/2}$$

$$\begin{pmatrix} l_1 - l_3 & L & L'' \\ m_1 - m_3 & -M & M'' \end{pmatrix} \begin{pmatrix} l_1 - l_3 & l_3 & l_1 \\ m_1 - m_3 & m_3 & -m_1 \end{pmatrix}$$

$$\begin{pmatrix} L & l & l_3 \\ M & m & m_3 \end{pmatrix} \begin{pmatrix} l_2 & l_4 & L \\ 0 & 0 & 0 \end{pmatrix}^2 \begin{pmatrix} l_1-l_3 & L & L'' \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} L & l & l_3 \\ 0 & 0 & 0 \end{pmatrix} \qquad (C.27)$$

Unfortunately, this expression can not be simplified any further, and thus makes the computation of the multipoles of the deformed folded potential unmanageable in any meaningful study.

Appendix D: Useful expressions

Here we give the some of the relations used for deriving results in the appendix C, some of which are not available in text-books, and have been derived by us (see b and c below)

a) Decomposition of solid harmonics

If $\vec{r} = \vec{a} + \vec{b}$, we have

$$r^{l} \; Y_{lm}(\vec{r}) = \sum_{\lambda\mu} \left[\frac{4\pi \; (2l+1)!}{(2\lambda+1)! \; (2(l-\lambda)+1)!} \right]^{1/3} a^{l-\lambda} . \; b^{\lambda} .$$

$$(-1)^{l+m} . \; (2\lambda+1)^{1/2} . \begin{pmatrix} l-\lambda & \lambda & l \\ m-\mu & \mu & -m \end{pmatrix} .$$

$$Y_{l-\lambda,m-\mu}(\hat{a}) \qquad Y_{\lambda\mu}(\hat{b})$$

(D 1)

b) Contraction of three-spherical harmonics.

$$\overline{Y_{l_1m_1}(\Omega) \; Y_{l_2m_2}(\Omega)} \; Y_{l_3m_3}(\Omega)$$

$$= \left[\sum_{L\,M} \left[\frac{(2l_1+1)(2l_2+1)(2L+1)}{4\pi} \right]^{1/2} Y_{LM}(\Omega) \; (-1)^{M} . \right.$$

$$\left. \begin{pmatrix} l_1 & l_2 & L \\ M_1 & m_2 & -M \end{pmatrix} \begin{pmatrix} l_1 & l_2 & L \\ 0 & 0 & 0 \end{pmatrix} \right] Y_{l_3m_3}(\Omega)$$

$$= \sum_{\substack{L\,M \\ L'\,M'}} \left[\frac{(2l_1+1)(2l_2+1)(2L+1)(2L+1)(2l_3+1)(2L'+1)}{(4\pi)^2} \right]^{1/2} .$$

$$(-1)^{M+M'} \qquad Y_{L'M'}(\Omega).$$

$$\begin{pmatrix} l_1 & l_2 & L \\ m_1 & m_2 & -M \end{pmatrix} \begin{pmatrix} L & l_3 & L' \\ M & m_3 & -M' \end{pmatrix}$$

$$\begin{pmatrix} l_1 & l_2 & L \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} L & l_3 & L' \\ 0 & 0 & 0 \end{pmatrix}$$

(D.2)

c) Angular integration of four spherical harmonics

$$\int Y_{l_1 m_1}(\Omega) \quad Y_{l_2 m_2}(\Omega) \quad Y_{l_3 m_3}(\Omega) \quad Y^*_{l_4 m_4}(\Omega) \quad d\Omega$$

$$= \sum_{LM} \left[\frac{(2l_1+1)(2l_2+1)(2L+1)(2L+1)(2l_3+1)(2l_4+1)}{(4\pi)^2} \right]^{1/2}$$

$$(-1)^{M+m_4} \begin{pmatrix} l_1 & l_2 & L \\ m_1 & m_2 & -M \end{pmatrix} \begin{pmatrix} L & l_3 & l_4 \\ M & m_3 & -m_4 \end{pmatrix}$$

$$\begin{pmatrix} l_1 & l_2 & L \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} L & l_3 & l_4 \\ 0 & 0 & 0 \end{pmatrix}$$

(D.3)