Theoretical Assessment of a Proposal for the Simplified Determination of Critical Loads of Elastic Shells

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Summary

Within the context of the stability analysis of the cryostat of a fusion reactor the question was raised whether or not the rather lengthy conventional stability analysis can be circumvented by applying a simplified strategy based on common linear Finite Element computer programs. This strategy involves the static linear deformation analysis of the structure with and without imperfections. For some simple stability problems this approach has been shown to be successful. The purpose of this study is to derive a general proof of the validity of this approach for thin shells with arbitrary geometry under hydrostatic pressure or dead loading along the boundary.

This general assessment involves two types of analyses:

(1) A general stability analysis for thin shells; this is based on a simple nonlinear shell theory and a stability criterion in form of the neutral (indifferent) equilibrium condition. This result is taken as reference solution.

(2) A general linear deformation analysis for thin imperfect shells and the definition of a suitable scalar parameter (β-parameter) which should represent the reciprocal of the critical load factor.

For both problems approximate solutions are obtained using direct matrix notation. They are based on the associated variational principles and a global Ritz ansatz for the displacement components. The solution of the first problem is restricted to linear prebuckling deformations.

It is shown that the simplified strategy ("β-parameter approach") generally is not capable to predict the actual critical load factor irrespective whether there is a hydrostatic pressure loading or dead loading along the edge of the shell. This general result is in contrast to the observations made for some simple stability problems. Nevertheless, the results of this study do not exclude the possibility that the simplified strategy will give reasonable approximate solutions at least for a restricted class of stability problems. This should be a subject of further analyses.

Theoretische Überprüfung eines Vorschlags zur vereinfachten Bestimmung kritischer Lasten elastischer Schalen

Zusammenfassung

Im Rahmen der Stabilitätsanalyse des Kryostaten eines Fusionsreaktors trat die Frage auf, ob eine in diesem Fall doch recht aufwendige herkömmliche Stabilitätsanalyse nicht vermieden werden könnte, indem eine vereinfachte Vorgehensweise angewandt wird, die auf dem Einsatz üblicher linearer Finite Element Rechenprogramme beruht: Diese Strategie beinhaltet die statische, lineare Deformationsanalyse der Struktur mit und ohne Imperfektionen. Für einige einfache Stabilitätsprobleme war diese Vorgehensweise erfolgreich. Der Zweck dieser Studie ist, einen allgemeinen Beweis der Gültigkeit dieser Vorgehensweise für dünne Schalen beliebiger Geometrie unter hydrostatischer Druckbelastung oder unter verformungsunabhängigen Randlasten herzuleiten.

Die allgemeine Überprüfung beinhaltet zwei Arten von Analysen:

(1) Eine allgemeine Stabilitätsanalyse für dünne Schalen: Diese basiert einerseits auf einer einfachen nichtlinearen Schalentheorie und andererseits auf dem Stabilitätskriterium in Form des neutralen (indifferenten) Gleichgewichts. Das Ergebnis wird als die Referenzlösung angesehen.

(2) Eine allgemeine lineare Deformationsanalyse für dünne, imperfekte Schalen und die Definition eines geeigneten skalaren Parameters (β-Parameter), der den Kehrwert des kritischen Lastfaktors darstellen sollte.

Für beide Probleme werden Näherungslösungen gewonnen, die in direkter Matrizennotation dargestellt werden. Sie basieren auf den zugehörigen Variationsprinzipien in Verbindung mit globalen Ritz-Ansätzen für die Verschiebungskomponenten. Die Lösung des ersten Problems ist dabei auf lineare Vorbeulverformungen beschränkt.

Es wird gezeigt, daß im allgemeinen mit der vereinfachten Vorgehensweise ("β-Parameter Verfahren") der aktuelle kritische Lastfaktor nicht ermittelt wird, gleichgültig, ob hydrostatische Druckbelastung oder feste Randbelastung vorliegt. Dieses allgemeine Ergebnis steht im Gegensatz zu den Feststellungen, die bei einigen einfachen Stabilitätsproblemen gemacht wurden. Dennoch schließt das Ergebnis dieser Studie die Möglichkeit nicht aus, daß die vereinfachte Vorgehensweise brauchbare Näherungslösungen liefert, wenigstens für eine beschränkte Klasse von Problemen. Dies sollte Gegenstand weiterer Untersuchungen sein.

Contents

1. Introduction and Scope of the Study

Within the context of the stability analysis of the cryostat of a fusion reactor the question was raised whether or not a classical stability analysis can be circumvented by applying a suitable strategy based on linear Finite Element computer programs $/\overline{~}1\_/$. Such a strategy can be motivated by observations made within the frame of the elastic stability analysis of straight columns under compressive loading. In the following this simple example is discussed to some extent, such that the basic philosophy of the strategy and its inherent assumptions are understood.

The concept of stability in mechanics is ambiguous. Different concepts based on intuitive arguments are proposed and applied $/\overline{~}2, 3\_/$ which do not necessarily give the same results (critical loads). However, basic research in the past has identified to a large extent for which classes of problems the various approaches are applicable and under what conditions the different stability concepts give the same minimum buckling load.

The standard methods for the analysis of the stability of an equilibrium state are the energy method (or the 2. variation approach) and the equilibrium approach $/\overline{~}2\_/$. For conservative problems both approaches give the same results.

As an introductory example Ziegler $/\overline{~}3\_/$ has analysed the straight column (hinged at both ends) with four different approaches, the two mentioned above, the imperfection method and the vibration method (kinetic stability analysis). These approaches are characterized by the following different questions:

The energy method: What is the value of the load for which the potential energy of the system ceases to be positive definite?

The neutral equilibrium method: What is the value of the load for which the systems admits an adjacent equilibrium configuration under the same loading?

The imperfection method: What is the value of the load for which the static displacements of a system with slightly different geometry - the imperfect system - become excessive or even infinite?

The vibration method: What is the value of the load for which the most general free motion of the system in the vicinity of the equilibrium position ceases to be bounded?

The first three approaches are based on static concepts while the fourth is a kinetic approach. Although there appears to be little connection in these approaches, the result, the minimum buckling load of the hinged column is the same $\underline{/}^-3\underline{}7$.

For the further argumentation the imperfection method is of special interest. Ziegler applied a compressive load with a small excentricity and studied the equilibrium in the deformed configuration of the column. Similarly one may subject the column to an initial deflection $(w_o^* \sin \pi x/\ell)$ of the central line of the column from the line of thrust. The solution of the <u>linearized equilibrium condition formulated in the deformed configuration</u> gives the following relation between the maximum sinusoidal displacement w^* and the applied load P

$$ w^* = \frac{w_o^*}{1 - P/P_c} \quad , \quad P_c = \pi^2 \frac{EJ}{\ell^2} \tag{1.1}$$

where

w_o^* maximum initial deflection

w^* actual maximum deflection measured from the perfectly straight centre line

P_c critical load .

Obviously, for P approaching the value P_c the deflection w^* approaches infinity. Thus, P_c defines the critical load, the buckling load.

For $P/P_c \ll 1$, equ. (1.1) may be developed with respect to P/P_c ; keeping terms linear in P/P_c only, equ. (1) simplifies to

$$ w^* \simeq w_o^* + w_o^* P/P_c $$

or

$$ w^* \simeq w_o^* + w_o^* P/\beta \tag{1.2}$$

where

$$ \beta = \frac{1}{P_c} \, . \tag{1.3}$$

One may easily show that this linearized relation is also obtained if the equilibrium analysis is done in the <u>undeformed</u> configuration of the column subjected to an initial deflection $w_o^* \sin \pi x/\ell$ i.e. an <u>imperfection</u>.

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These results show that a linear elastic analysis - ignoring nonlinear terms in the kinematics and considering the equilibrium conditions only in the undeformed configuration of an <u>imperfect</u> column - allows to determine the buckling load of the column. However, here it is important to note that the "shape of the imperfection" is the same as the eigenfunction of the buckling load.

This observation may motivate a strategy for the calculation of the critical loads for structures whose states of deformation are characterized by a single displacement component:

- Get knowledge about or estimate the buckling mode

- Introduce an imperfection into the structure of magnitude w_0 (e.g. maximum imperfection) which is similar to the buckling mode

- Perform a linear elastic boundary value analysis (analytically or numerically) of the imperfect structures under the same load configuration as the actual structure

- Determine the "maximum deflection w " and calculate the quantity

$$\beta = \frac{w - w_0}{w_0 \, P} \tag{1.4}$$

where w and w_0 are measured with respect to the actual structure and where P is the load factor. Then the critical load factor, i.e. the buckling load[x] is given by

$$P_c = \frac{1}{\beta} \, , \tag{1.5}$$

In cases where the applied load P produces displacements (prebuckling deformations) even without the imperfection then equ. (1.2) should read

$$w = P\alpha + w_0 + w_0 \, P\beta \tag{1.6}$$

($P\alpha$: deflection of the actual structure due to P)

and relation (1.4) should be changed to

$$\beta = \frac{w - P\alpha - w_0}{w_0 \, P} \, , \tag{1.7}$$

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[x] Corresponding to the chosen imperfection

If numerical methods are used two calculations are necessary in this situation:

- numerical analysis of the actual ("perfect") structure under load P to obtain the "maximum deflection (P_α)"
- numerical analysis of the imperfect structure under load P to obtain the "maximum deflection w",

The above strategy is related[x] to another approach for the determination of critical loads of bars as described by Timoshenko and Gere $\underline{/}\ 12,\ p.\ 116\ \underline{/}$:

(α) As a first approximation assume a deflection curve y_1 for the bar. Take this deflection to be an imperfection[xx] of the bar.

(β) Perform a linear analysis of the imperfect bar which gives a deflection y_2 of the bar under the load P.

(γ) A first approximation for the critical load is found by choosing the load P in such a way that the deflection y_1 and y_2 are equal along some sections of the bar[xxx], e.g. the centre section of the bar.

(σ) This procedure may be continued with y_2 as a new imperfection of the bar.

Observing the different notations it may be shown for the simple bar using a sinusoidal imperfection that the result (1.2) is obtained after a single step.

Timoshenko and Gere state that this approach is equivalent to an integration by successive approximations of the differential equation for a buckled bar[xxxx].

The method of sucessive approximations for boundary value as well as eigenvalue problems is a well established mathematical method $\underline{/}\ 13\ \underline{/}$. However, for stability problems which usually correspond to eigenvalue problems, the application of this method requires a complete formulation of this eigenvalue problem $\underline{/}\ 13\ \underline{/}$. Consequently, it is by no means evident that the approach as described by Timoshenko and Gere is equivalent to the method of successive approximations as applied to eigenvalue problems $\underline{/}\ 13\ \underline{/}$, if more complex structures are considered.

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[x] This was brought to my attention by S. Raff.

[xx] Timoshenko and Gere do not use the term "imperfection" but their description can be put in the form as given here.

[xxx] Other rules are possible, e.g. averaging the deflections along the bar before equating.

[xxxx] The method of successive approximations was first applied to the buckling problem by Engesser and Vianello.

The strategy by $/^-1\_/$, loosely described above, has been applied to several stability problems where analytical solutions are available, i.e.

- a flat plate under lateral compressive membrane forces and hinged at all edges and
- a circular ring or long cylindrical shell under hydrostatic pressure.

In these cases exact agreement between the result of the classical stability analysis and this approach was found. This is partially due to the fact that the buckling modes of the structure are known in advance or may be easily guessed. When the buckled shape and thus the imperfection cannot be characterized by a single function or when only a numerical solution for the imperfection structure is available then this approach could possibly be combined with a Fourier analysis and a search for the dominant terms $/^-4\_/$ instead of the determination of the "maximum deflection".

Although it was shown that the above strategy works for some simple stability problems, the basic question remains whether this strategy may be extended to more complex stability problems, e.g. shells and spatial frames; these are the primary structural elements of the cryostat envisaged. The deformation of these structures during buckling are characterized by more than one displacement component, e.g. the normal and the two tangential components of the reference surface of a shell. Consequently the first question is whether the simple relation (1.7) can be extended to this new situation. Secondly, it has to be proved or disproved whether this newly defined scalar quantity "β" is related to the critical load P_c by equ. (1.5). If an affirmative answer is found then the above strategy is also applicable to more complex stability problems.

It is evident that an extension of the list of examples to more complex situations is of limited scientific value. Here a general proof is of interest at least for a certain class of problems of interest. In the following we will consider thin elastic shells of uniform thickness but arbitrary geometry. Two types of loadings are assumed; uniformly distributed hydrostatic pressure and dead loading along the edge such that the stability problem involves only a <u>single load factor</u>.

For this class of problems the general proof or refutation involves two types of analyses:

(1) A general stability analysis for thin shells; a prerequisite for this is a consident, nonlinear shell theory and a stability criterion.

(2) A general linear analysis for thin imperfect shells and a suitable definition of the scalar parameter "β".

It is obvious that neither for the first nor for the second part an exact solution can be obtained. However, it appears to be feasible to derive approximate but general solutions based on variational principles for both types of analyses. Here it is important that all approximations are compatible in both parts of analyses; otherwise their results are not comparable.

Furthermore, for such a general approach it is crucial to apply a compact symbolic notation since otherwise the overview is immediately lost in a whirlpool of equations. Therefore the general theory is formulated using tensor calculus and after introduction of suitable approximations for the displacement fields direct matrix notation is applied.

In the following a more detailed description of the approach is given. The starting point is a rather simple nonlinear shell theory for small strains but moderately large rotations under the Kirchhoff-Love hypothesis. Such a theory had been derived in a different context $\underline{/\ 5\ \underline{/}}$ but is reduced here to purely elastic material response. An associated variational principle is presented using tensorial formulation in the undeformed reference configuration of the shell. Two types of conservative loadings are considered: dead loading along the edge and hydrostatic pressure. It is shown that for the hydrostatic pressure to be conservative the kinematic boundary conditions are restricted. For the rest of the analysis these restrictions are implied.

Following the Ritz method a global approximation for the three tensor components of the displacement vector is made; here it is assumed that a complete set of shape functions satisfying the kinematic boundary conditions is available.[*]

Then a matrix representation of the total potential energy functional for the nonlinear thin shell is derived. Here the introduction of various matrix differential operators and a formal integration of the functional over the reference surface is involved. This reduces the functional to a nonlinear algebraic expression for a column matrix containing the various unknown coefficients of the displacement field approximation. This column matrix is denoted by Z . The fundamental equilibrium state under the given loading is characterized by the vanishing of the first variation of the algebraic energy expression (total potential energy). It is generally obtained as the solution of a nonlinear algebraic equation for Z .

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[*] In practice it is preferable to apply a piecewise approximation (the Finite Element Method), however, for the purpose intended here this would be an unnecessary complication.

For the stability analysis of the fundamental state Z we follow a classical concept known as the "neutral equilibrium approach", in conjunction with the variational principle for the shell. This approach is characterized as follows.

A neutral or equivalently an indifferent equilibrium state is defined by the property that adjacent equilibrium states under the same loading are possible $/\bar{6}\_/$; this implies nonuniqueness of the solution of the nonlinear equations for the fundamental equilibrium state Z . The associated load factor is called critical. This critical state may characterize the transition from stability[x] to instability and frequently this is taken for granted. Therefore we are actually starting off from a nonlinear indifference theory: We have to assess whether there exists an adjacent equilibrium state under the same loading as the fundamental state Z but characterized by the state $Z^* = Z + W$ where W is small compared to Z . For this adjacent state the total potential energy is $\overline{\Pi}^* = \overline{\Pi}(Z + W)$, Since this state is presumed to be in equilibrium, the first variation of $\overline{\Pi}(Z + W)$ with respect to V should vanish. If this condition admits a nontrivial solution for W with a given Z then the state Z is neutral. Since the additional displacement matrix W is small, third and fourth order terms in W may be neglected in the potential energy $\overline{\Pi}^*$. The vanishing of the first variation $\int \overline{\Pi}^*_W$ for all variations δW then leads to a <u>linear homogeneous</u> equation for V . This equation admits a nontrivial solution V if the coefficient matrix, which depends on the fundamental solution Z , is singular. It may be shown that this equation is also obtained if the second variation $\delta^2_Z \overline{\Pi}(Z)$ is required to vanish for all variations δZ .

The evaluation of the above indifference condition implies knowledge of the fundamental state Z . As mentioned above Z is generally governed by a nonlinear equation. However, if the prebuckling deformations are small this equation may be linearized and Z is obtained by inversion of a linear matrix equation. Thus the fundamental state Z is a linear function of the applied loading. This assumption is inherently implied in the further analysis.

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[x] Here, in rather loose terms an equilibrium state is said to be stable whenever in the motion following a sufficiently small initial disturbance (e.g. in the load) the response of the structure (in terms of displacements and velocities) remains as small as desired for all later times.

With this assumption the indifference condition for the case of hydro-
static pressure loading is given by a quadratic eigenvalue problem where
the pressure ρ is the eigenvalue. Further linearization simplifies this to
a linear eigenvalue problem involving only symmetric matrices. From this
the reciprocal of an eigenvalue $1/\rho_i$ may be obtained as the reciprocal of
the Rayleigh quotient in terms of the corresponding eigensolution V_i .
This completes the stability (indifference) analysis of the fundamental
state and this result is used as the reference solution.

The further analysis has to show whether the proposed simplified
strategy - linearized analysis of an imperfect shell and a suitable defini-
tion of the scalar quantity "β" - gives a β-value which is exactly equi-
valent to the inverse Rayleigh quotient of the reference solution. Here
it is implied that the imperfection of the middle surface of the shell,
characterized by the column matrix $\tilde{z}$, is equivalent to the eigen solu-
tion V_i .

The first step is the derivation of a <u>linear</u> theory for slightly <u>im-
perfect</u> shells. This means essentially a general tensorial formulation of
the associated total potential energy where the configuration of the <u>actual</u>
("perfect") shell is used as reference configuration. Naturally, the "per-
fect" configuration is identical to the undeformed configuration of the
shell whose stability is to be analysed. This derivation involves careful
order of magnitude estimations of the kinematical quantities which must be
compatible with corresponding assumptions in the previous derivation of
the nonlinear theory.

Application of the Ritz method with the same shape functions for the
displacements and the imperfections as in the nonlinear case allows to
develop the appropriate matrix formulation of the total potential energy.
The vanishing of the first variation of this potential gives the equilibrium
state of the perfect (i.e. $\tilde{z} = 0$) as well as the imperfect $(\tilde{z} \neq 0)$
shell under the prescribed (e.g. hydrostatic) loading. Because of the
linearity of the problem the solution may be obtained explicitly by matrix
inversion.

Finally a suitable scalar factor "β" is defined by inspection of the
above result and the inverse Rayleigh quotient. Although a partial agree-
ment between "β" and the inverse Rayleigh quotient may be obtained, it is
shown that the simplified strategy generally is not capable to determine
the critical load factor, whether there is a hydrostatic pressure loading
or dead loading along the edges of the shell. This general result is in

contrast to the observations made for the above mentioned simple stability problems. The study closes with a discussion of these results.

2. Fundamentals of a Nonlinear Elastic Shell Theory

The nonlinear theory for an elastic shell to be used is subject to the following restrictions:

(I)   the Kirchhoff-Love hypothesis is assumed to be applicable,

(II)   strains are small but rotations are moderately large,

(III)   the rotation around the normal to the middle surface (reference surface) is assumed to be small compared to the other two rotations,

(IV)   the wall thickness is small compared to the minimum radius of curvature so that in conjunction with the other approximations the metric of the shell space can be approximated by the metric of the middle surface.

2.1 Geometric and Kinematic Preliminaries

The undeformed configuration of the middle surface (reference surface) of the shell is defined by the position vector $\ell \bar{R}$ [*]. A curvilinear coordinate net with coordinates Θ^α, $\alpha = 1,2$ (surface coordinates) is embedded. The base vectors of this coordinate system are defined by

$$\bar{A}_\alpha := \frac{\partial \bar{R}}{\partial \Theta^\alpha} = \bar{R}_{,\alpha} \qquad , \qquad \alpha = 1,2 \qquad (2.1)$$

which are tangential to the coordinate lines. They define a unit vector $\bar{A}_3$ normal to the reference surface

$$\bar{A}_3 := \frac{\bar{A}_1 \times \bar{A}_2}{|\bar{A}_1 \times \bar{A}_2|} \qquad (2.2)$$

such that $\bar{A}_1, \bar{A}_2$ and $\bar{A}_3$ represent a right handed system of base vectors. The contravariant base vectors of the reference surface are then given by

$$\bar{A}^1 = \bar{A}_2 \times \bar{A}_3 \frac{1}{|\bar{A}|} \qquad , \qquad \bar{A}^2 = \bar{A}_3 \times \bar{A}_1 \frac{1}{|\bar{A}|} \qquad \cdot \quad (2.3)$$

$$A = det(\bar{A}_\alpha \cdot \bar{A}_\beta) = [\bar{A}_1 \cdot (\bar{A}_2 \times \bar{A}_3)]^{1/2} \qquad (2.4)$$

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[*] The quantity ℓ is a scale factor of dimension "length"; consequently the length of the vector $\bar{R}$ is dimensionless.

such that

$$\bar{A}_\alpha \cdot \bar{A}^\beta = \delta^\beta_\alpha = \begin{cases} 1 & \alpha = \beta \\ 0 & \alpha \neq \beta \end{cases}, \tag{2.5}$$

The contravariant base vectors $\bar{A}^\alpha$ define a unit normal vector

$$\bar{A}^3 = \frac{\bar{A}^1 \times \bar{A}^2}{|\bar{A}^1 \times \bar{A}^2|} \tag{2.6}$$

which is the same as (2.2)

$$\bar{A}^3 = \bar{A}_3 , \tag{2.7}$$

The line element of the reference surface is given by

$$\ell \, d\bar{R} = \ell \, \bar{R}_{,\alpha} \, d\Theta^\alpha = \ell \, \bar{A}_\alpha \, d\Theta^\alpha$$

$$\ell^2 \, d\bar{R} \cdot d\bar{R} = \ell^2 \, A_{\alpha\beta} \, d\Theta^\alpha \, d\Theta^\beta \tag{2.8}$$

$$A_{\alpha\beta} = \bar{A}_\alpha \cdot \bar{A}_\beta ;$$

here $A_{\alpha\beta}$ are the contravariant components of the metric tensor of the undeformed reference surface. The contravariant components are

$$A^{\alpha\beta} = \bar{A}^\alpha \cdot \bar{A}^\beta \tag{2.9}$$

such that

$$A^{\alpha\beta} A_{\beta\gamma} = \delta^\alpha_\gamma . \tag{2.10}$$

The metric coefficients relate co- and contravariant base vectors:

$$\bar{A}_\alpha = A_{\alpha\gamma} \bar{A}^\gamma \quad , \quad \bar{A}^\alpha = A^{\alpha\gamma} \bar{A}_\gamma . \tag{2.11}$$

The partial derivatives of the covariant base vectors are given in the formulae of Gauss and Weingarten

$$\bar{A}_{\alpha,\beta} = \frac{\partial \bar{A}_\alpha}{\partial \Theta^\beta} = \overset{o}{\Gamma}{}^\gamma_{\alpha\beta} \bar{A}_\gamma + B_{\alpha\beta} \bar{A}_3$$

$$\bar{A}_{3,\alpha} = \frac{\partial \bar{A}_3}{\partial \Theta^\alpha} = - B^\gamma_\alpha \bar{A}_\gamma \tag{2.12}$$

where $\overset{o}{\Gamma}{}^{\gamma}_{\alpha\beta}$ is the Christoffel symbol with respect to the reference surface

$$\overset{o}{\Gamma}{}^{\gamma}_{\alpha\beta} = \frac{1}{2} A^{\gamma\beta}\left(A_{\beta\alpha,\beta} + A_{\beta\beta,\alpha} - A_{\alpha\beta,\beta}\right) \tag{2.13}$$

and $B_{\alpha\beta}$ are the coefficients of the second fundamental form

$$B_{\alpha\beta} = B_{\beta\alpha} = -\bar{A}_{\alpha} \cdot \bar{A}_{3,\beta} = -\bar{A}_{\beta} \cdot \bar{A}_{3,\alpha} = \bar{A}_{3} \cdot \bar{A}_{\alpha,\beta}$$

$$B^{\alpha}_{\beta} = A^{\alpha\gamma} B_{\gamma\beta} \quad , \quad B^{\alpha\beta} = A^{\alpha\gamma} B^{\beta}_{\gamma} \; . \tag{2.14}$$

Let $\bar{v}$ be a surface vector field, i.e. a vector field tangential to the reference surface

$$\bar{v} = v^{\alpha} \bar{A}_{\alpha} = v_{\alpha} \bar{A}^{\alpha}. \tag{2.15}$$

Its partial derivative with respect to the surface coordinate Θ^{β} is given by

$$\bar{v}_{/\beta} = \frac{\partial \bar{v}}{\partial \Theta^{\beta}} = v^{\alpha}{}_{,\beta} \bar{A}_{\alpha} + v^{\alpha} \bar{A}_{\alpha,\beta}$$

$$= v^{\gamma}{}_{:\beta} \bar{A}_{\gamma} + B_{\alpha\beta} v^{\alpha} \bar{A}_{3} \tag{2.16}$$

$$= v_{\gamma:\beta} \bar{A}^{\gamma} + B^{\alpha}_{\beta} v_{\alpha} \bar{A}_{3}$$

where

$$v^{\gamma}{}_{:\beta} := v^{\gamma}{}_{,\beta} + \overset{o}{\Gamma}{}^{\gamma}_{\alpha\beta} v^{\alpha}$$

$$v_{\gamma:\beta} := v_{\gamma,\beta} - \overset{o}{\Gamma}{}^{\alpha}_{\gamma\beta} v_{\alpha} \tag{2.17}$$

is the covariant derivative of the contravariant or covariant components of the surface vector $\bar{v}$; those quantities are the components of a 2nd order surface tensor. Similarly the covariant derivative of a 2nd order surface tensor $\underline{M}$

$$\underline{M} = M^{\alpha\beta} \bar{A}_{\alpha} \otimes \bar{A}_{\beta}$$

may be defined

$$M^{\alpha\beta}{}_{;\delta} = M^{\alpha\beta}{}_{,\delta} + \overset{o}{\Gamma}{}^{\alpha}_{s\delta} M^{s\beta} + \overset{o}{\Gamma}{}^{\beta}_{s\delta} M^{\alpha s} \qquad (2.18)$$

etc.

The position vector $\bar{P}$ of a material point in the shell space for the reference configuration may be represented by

$$\bar{P}(\theta^{\alpha}, \theta) = \ell \bar{R}(\theta^{\alpha}) + \ell \lambda \theta \bar{A}_3(\theta^{\alpha}) \qquad (2.19)$$

where

$\theta \equiv \theta^3$: dimensionless thickness coordinate $-\tfrac{1}{2} \leq \theta \leq \tfrac{1}{2}$

ℓ : scale factor, e.g. minimum radius of curvature

$h(\theta^{\alpha})$: wall thickness

$\lambda = h/\ell$: dimensionless wall thickness .

Parallel to the reference surface other surfaces may be embedded in the shell space; together with the thickness coordinate lines they span a coordinate net in the shell space. The covariant base vectors of this system are given by

$$\bar{G}_{\alpha} = \bar{P}_{,\alpha} = \ell S^{s}_{\alpha} \bar{A}_{s}$$
$$\bar{G}_3 = \bar{P}_{,3} = \ell \lambda \bar{A}_3 \qquad \left.\right\} (2.20)$$

where

$$S^{\alpha}_{\beta} := \delta^{\alpha}_{\beta} - \lambda \theta B^{\alpha}_{\beta} . \qquad (2.21)$$

The covariant components of the metric tensor in the shell space are then given by

$$G_{\alpha\beta} = \bar{G}_{\alpha} \cdot \bar{G}_{\beta} = (\ell)^2 S^{s}_{\alpha} S^{\delta}_{\beta} A_{s\delta}$$
$$= (\ell)^2 \left[A_{\alpha\beta} - 2\lambda \theta B_{\alpha\beta} + (\lambda\theta)^2 B_{\alpha\gamma} B^{\gamma}_{\beta} \right]$$
$$G_{\alpha 3} = \bar{G}_{\alpha} \cdot \bar{G}_3 = 0 \qquad \left.\right\} (2.22)$$

$$G_{33} = \bar{G}_3 \cdot \bar{G}_3 = (\lambda \ell)^2 \; ; \qquad \qquad (2.22)$$

the corresponding contravariant components are

$$G^{\alpha\beta} = \left(\frac{1}{\ell}\right)^2 \left[A^{\alpha\beta} + 2\lambda\Theta B^{\alpha\beta} + 3(\lambda\Theta)^2 B^{\alpha\gamma} B_\gamma^\beta + \ldots \right]$$

$$G^{\alpha 3} = O$$

$$G^{33} = \left(\frac{1}{\lambda\ell}\right)^2 . \qquad \qquad (2.23)$$

A volume element in the shell space is given by

$$dV = \sqrt{G} \; d\Theta^1 d\Theta^2 d\Theta^3$$

$$G = \det (G_{\kappa\iota}) = (\lambda\ell)^2 \det (G_{\alpha\beta}) \qquad \qquad (2.24)$$

and a surface element in the middle surface

$$dA = (\ell)^2 \sqrt{A} \; d\Theta^1 d\Theta^2$$

$$A = \det (A_{\alpha\beta}) . \qquad \qquad (2.25)$$

If $\bar{p}$ is the position vector in the deformed configuration then the displacement of a material point, with coordinate $\Theta^1, \Theta^2, \Theta$ in the reference configuration, is generally given by

$$\bar{u}(\Theta^\alpha, \Theta, t) = \bar{p}(\Theta^\alpha, \Theta, t) - \bar{P}(\Theta^\alpha, \Theta), \qquad (2.26)$$

According to the Kirchhoff-Love hypothesis the displacement vector can be represented by

$$\bar{u} = \ell \bar{v} + \ell \lambda \Theta (\bar{a}_3 - \bar{A}_3), \qquad \qquad (2.27)$$

here

$$\bar{v}(\theta^\alpha, t) = V^\alpha(\theta^\delta, t) \bar{A}_\alpha + W(\theta^\delta, t) \bar{A}_3 \qquad (2.28)$$

is the dimensionless displacement vector of the reference surface and $\bar{a}_3$ the unit normal vector of the deformed surface.

The coordinate lines θ^α = const. are now considered to be convected with the material. Then the covariant base vectors of the deformed middle surface are given by

$$\bar{a}_\alpha = \frac{1}{\ell}\left(\frac{\partial \bar{P}}{\partial \theta^\alpha}\right)_{\theta=0} = \frac{1}{\ell}\left(\bar{P},_\alpha\right)_{\theta=0} \qquad (2.29)$$

With the use of (2.19), (2.26) and (2.27) we obtain

$$\bar{a}_\alpha = \bar{A}_\alpha + \bar{v},_\alpha , \qquad (2.30)$$

The component representation of $\bar{v}$ (equ. (2.28)) and the covariant derivative yields

$$\left.\begin{array}{l} \bar{v}_{/\alpha} = V^\delta_{;\alpha} \bar{A}_\delta - W_\alpha \bar{A}_3 \\[2mm] W_\alpha := -\left(W_{/\alpha} + B^\delta_\alpha V_\delta\right). \end{array}\right\} \qquad (2.31)$$

This allows to give a representation of the unit normal vector $\bar{a}_3$ of the deformed surface in terms of surface displacements:

$$\bar{a}_3 = \frac{\bar{a}_1 \times \bar{a}_2}{|\bar{a}_1 \times \bar{a}_2|} , \qquad (2.32)$$

For arbitrary displacements of the reference surface this represents a rather complex nonlinear expression. From the kinematics of shells under infinitesimal displacement gradients it is well known, that W_α represents the two rotations of the normal of the middle surface with respect to the base vectors $\bar{A}_\alpha$; further, the rigid body rotation of a middle surface element around the normal $\bar{A}_3$ is given by

$$\frac{1}{2}\left(V_{1,2} - V_{2,1}\right)\frac{1}{\sqrt{A}} \quad , \quad A = det\left(A_{\alpha\beta}\right) ;$$

finally the infinitesimal middle surface strains are represented by

$$\frac{1}{2}\left(V_{\alpha:\beta} - V_{\beta:\alpha} - 2W B_{\alpha\beta}\right).$$

Assume that the curvilinear coordinates Θ^{α} are dimensionless[x], then $B_{\alpha\beta}$, $V_{\alpha:\beta}$ and W_{α} are dimensionless.

Now, it is important to note that for most stability problems it is sufficient to introduce certain order of magnitude restrictions on the above kinematic quantities; these restrictions are[xx]:

$$
\left.
\begin{array}{ll}
W_{\alpha} & \sim \lambda \ll 1 \\[3mm]
\frac{1}{2}\left(V_{\alpha:\beta} - V_{\beta:\alpha}\right) & \sim (\lambda)^{2} \\[3mm]
\frac{1}{2}\left(V_{\alpha:\beta} + V_{\beta:\alpha} - 2W B_{\alpha\beta}\right) & \sim (\lambda)^{2}
\end{array}
\right\} \quad (2.33)
$$

i.e. the (linearized) strains as well as the rigid rotation around the normal are small in the order of λ^{2}. The rotations of the normal are only moderately small of order λ .

From the last two conditions it follows that

$$V_{\alpha:\beta} - W B_{\alpha\beta} \quad \sim \quad (\lambda)^{2}, \qquad (2.34)$$

Calculation of the vector product (2.32) and dropping all terms of the order of $(\lambda)^{2}$ or less one gets

$$\bar{a}_{1} \times \bar{a}_{2} = \sqrt{a} \left(\bar{A}_{3} + W^{\alpha} \bar{A}_{\alpha}\right), \qquad (2.35)$$

Thus to a first approximation $\bar{a}_{3}$ will become

$$\bar{a}_{3} \approx \bar{A}_{3} + W^{\alpha} \bar{A}_{\alpha} . \qquad (2.36)$$

---

[x] If the Θ^{α} represent the arc lengths then the scaling factor ℓ should be used to obtain dimensionless coordinates.

[xx] Exluding large prebuckling strains and rotations and the analysis of the post buckling behavior .

It is noted that $\bar{a}_3$ is a unit vector except for terms of the order of $(\lambda)^2$. Combining (2.27) and (2.36) gives

$$\bar{u} = \lambda \left[(V^\alpha + \lambda\Theta W^\alpha)\bar{A}_\alpha + W\bar{A}_3 \right].$$

(2.37)

This approximate displacement field is equal to that of engineering shell theories under infinitesimal displacement gradients $\lfloor \bar{5}, 7 \rfloor$.

If $d\bar{P}$ and $d\bar{p}$ denote the infinitesimal distance vectors of two material points of the referential and the present configuration then the Lagrangian strain tensor $\underline{E}$ is given by

$$d\bar{p} \cdot d\bar{p} - d\bar{P} \cdot d\bar{P} = 2\, d\bar{P}\, \underline{E}\, d\bar{P}$$

$$\underline{E} = E_{MN}\, \bar{G}^M \otimes \bar{G}^N = E^{MN}\, \bar{G}_M \otimes \bar{G}_N \, .$$

(2.38)

With (2.24) one has

$$d\bar{P} = \bar{G}_K\, d\Theta^K$$

and the right hand side of (2.38)$_1$ gives

$$2\, d\bar{P}\, \underline{E}\, d\bar{P} = 2\, d\Theta^M\, E_{MN}\, d\Theta^N$$

$$= 2\, \left\{ d\Theta^\alpha\, E_{\alpha\beta}\, d\Theta^\beta \right.$$

$$+ d\Theta^\alpha\, E_{\alpha 3}\, d\Theta^3 + d\Theta^3\, E_{3\alpha}\, d\Theta^\alpha$$

(2.39)

$$\left. + d\Theta^3\, E_{33}\, d\Theta^3 \right\}.$$

From (2.26) we obtain

$$d\bar{p} = d\bar{u} + d\bar{P}$$

$$= \bar{u}_{,\alpha}\, d\Theta^\alpha + \bar{u}_{,3}\, d\Theta^3 + \bar{G}_\alpha\, d\Theta^\alpha + \bar{G}_3\, d\Theta^3$$

(2.40)

$$= (\bar{G}_\alpha + \bar{u}_{,\alpha})\, d\Theta^\alpha + (\bar{G}_3 + \bar{u}_{,3})\, d\Theta^3$$

Consequently, the left hand side of $(2.38)_1$ may be developed with respect to coordinate differentials

$$d\bar{p} \cdot d\bar{p} - d\bar{P} \cdot d\bar{P}$$

$$= [\bar{G}_\alpha \cdot \bar{u}_{,\beta} + \bar{G}_\beta \cdot \bar{u}_{,\alpha} + \bar{u}_{,\alpha} \cdot \bar{u}_{,\beta}] d\Theta^\alpha d\Theta^\beta$$

$$+ 2[\bar{G}_\alpha \cdot \bar{u}_{,3} + \bar{G}_3 \cdot \bar{u}_{,\alpha} + \bar{u}_{,\alpha} \cdot \bar{u}_{,3}] d\Theta^\alpha d\Theta^3 \qquad (2.41)$$

$$+ [2\bar{G}_3 \cdot \bar{u}_{,3} + \bar{u}_{,3} \cdot \bar{u}_{,3}] d\Theta^3 d\Theta^3.$$

Comparing (2.39) and (2.41) yields

$$\left.\begin{aligned}
E_{\alpha\beta} &= \tfrac{1}{2}[\bar{G}_\alpha \cdot \bar{u}_{,\beta} + \bar{G}_\beta \cdot \bar{u}_{,\alpha} + \bar{u}_{,\alpha} \cdot \bar{u}_{,3}] \\[2mm]
E_{\alpha 3} &= \tfrac{1}{2}[\bar{G}_\alpha \cdot \bar{u}_{,3} + \bar{G}_3 \cdot \bar{u}_{,\alpha} + \bar{u}_{,\alpha} \cdot \bar{u}_{,3}] \\[2mm]
E_{33} &= \bar{G}_3 \cdot \bar{u}_{,3} + \tfrac{1}{2}\bar{u}_{,3} \cdot \bar{u}_{,3} .
\end{aligned}\right\} (2.42)$$

From (2.37) one obtains

$$\left.\begin{aligned}
\bar{u}_{,\alpha} &= \ell[(V^s_{:\alpha} - WB^s_\alpha + \Delta\Theta\, W^s_{:\alpha})\bar{A}_s \\
&\qquad - W_s S^s_\alpha \bar{A}_3] \\[2mm]
\bar{u}_{,3} &= \ell[\Delta W^s \bar{A}_s].
\end{aligned}\right\} (2.43)$$

With (2.24) and (2.43) one obtains the following explicit representation for the covariant Lagrangian strain components in terms of a power series expansion of $(\Delta\Theta)$

$$E_{\alpha\beta} = \frac{1}{2}(\ell)^2 \Big\{ V_{\alpha;\beta} + V_{\beta;\alpha} - 2 W B_{\alpha\beta}$$

$$+ \lambda\theta \big[W_{\alpha;\beta} + W_{\beta;\alpha} - B_\alpha^\varsigma (V_{\varsigma;\beta} - W B_{\varsigma\beta})$$

$$- B_\beta^\varsigma (V_{\varsigma;\alpha} - W B_{\varsigma\alpha}) \big]$$

$$- (\lambda\theta)^2 \big[B_{\varsigma\alpha} W_{;\beta}^\varsigma + B_{\varsigma\beta} W_{;\alpha}^\varsigma \big]$$

$$+ \left(\frac{1}{\ell}\right)^2 \bar{u}_{,\alpha} \cdot \bar{u}_{,\beta} \Big\}$$

$$\bar{u}_{,\alpha} \cdot \bar{u}_{,\beta} = (\ell)^2 \Big\{ W_\alpha W_\beta + (V_{\varsigma;\alpha} - W B_{\varsigma\alpha})(V_{:\beta}^\varsigma - W B_\beta^\varsigma)$$

$$+ (\lambda\theta) \big[W_{\varsigma;\alpha}(V_{;\beta}^\varsigma - W B_\beta^\varsigma) + W_{\varsigma;\beta}(V_{;\alpha}^\varsigma - W B_\alpha^\varsigma)$$

$$- W^\varsigma B_{\varsigma\alpha} W_\beta - W^\varsigma B_{\varsigma\beta} W_\alpha \big]$$

$$+ (\lambda\theta)^2 \big[W_{\varsigma;\alpha} W_{:\beta}^\varsigma + W^\delta B_{\varsigma\alpha} W^\delta B_{\delta\alpha} \big] \Big\}$$

$$E_{\alpha3} = E_{3\alpha} = \frac{1}{2}(\ell)^2 \Big\{ \lambda (V_{:\alpha}^\varsigma - W B_\alpha^\varsigma) W_\varsigma$$

$$+ \lambda(\lambda\theta) \big[W_{:\alpha}^\varsigma W_\varsigma \big] \Big\}$$

$$E_{33} = \frac{1}{2}(\ell)^2 \lambda^2 W^\varsigma W_\varsigma .$$

(2.44)

The order of magnitude assumptions (2.33) and (2.34) and the assumption that $W_{\alpha;\beta}$ is of the order of λ , reduces the Lagrangian strain component $E_{\alpha\beta}$ to

(2.45)

$$E_{\alpha\beta} = (\ell)^2 \left(\alpha_{\alpha\beta} + \lambda\Theta\, \omega_{\alpha\beta} \right)$$

$$\alpha_{\alpha\beta} := \frac{1}{2} \left(V_{\alpha,\beta} + V_{\beta;\alpha} - 2W B_{\alpha\beta} + W_\alpha\, W_\beta \right)$$

$$\omega_{\alpha\beta} := \frac{1}{2} \left(W_{\alpha,\beta} + W_{\beta;\alpha} \right).$$

$$\left. \right\} \quad (2.46)$$

Thus, only terms quadratic in λ are kept in this expression and $\bar{u}_{,\alpha} \cdot \bar{u}_{,\beta}$ is approximated by

$$\bar{u}_{,\alpha} \cdot \bar{u}_{,\beta} = (\ell)^2\, W_\alpha\, W_\beta. \qquad (2.47)$$

This is the only nonlinear term in $E_{\alpha\beta}$. With the above assumptions the shear strain $E_{\alpha3}$ and the thickness strain E_{33} are of the following orders of magnitude

$$E_{\alpha3} \sim (\lambda)^4$$

$$E_{33} \sim (\lambda)^4.$$

$$\left. \right\} \quad (2.48)$$

A rigorous application of the Kirchhoff-Love hypothesis (2.27) yields

$$\bar{u}_{,3} = \lambda\lambda \left(\bar{a}_3 - \bar{A}_3 \right),$$

application of this exact expression to calculate the thickness strain yields

$$E_{33} = \bar{G}_3 \cdot \bar{u}_{,3} + \frac{1}{2} \left(\bar{u}_{,3} \cdot \bar{u}_{,3} \right)$$

$$= (\lambda\lambda)^2 \left(\bar{A}_3 \cdot \bar{a}_3 - 1 \right) + \frac{1}{2} (\lambda\lambda)^2 (\bar{a}_3 - \bar{A}_3)\cdot(\bar{a}_3 - \bar{A}_3)$$

$$= 0 ,$$

a result which is trivial since it is implied in the formulation of (2.27).

Similarly the shear strains $E_{\alpha3}$ should vanish. The result (2.48) is due to the fact that the relation (2.36) for the unit normal $\bar{a}_3$ is only approximate.

2.2 The Energy Functional of the Shell

The further development of the shell equations is based on the Hellinger-Reissner variational principle (mixed variational principle) in the frame of a Lagrangian large displacement formulation $\underline{/ 5 \underline{/}}$. The Hellinger-Reissner variational principle implies a simultaneous variation of the displacement field and the stress field; due to the additional assumption with respect to the stresses further simplifications may be introduced in a mathematical consistent way. This approach has been followed in $\underline{/ 5 \underline{/}}$ within the context of a nonlinear viscoelastic shell and these results may be used here, ignoring the viscous material response.

In this approach the neglection of the stress power of the shear stresses normal to the reference surface and of the thickness stress is an important assumption. The resulting variational principle is of the mixed type involving the membrane force and bending moment tensors as well as the reference surface displacements V^α and W.

Following usual procedures of variational calculus the mixed variational principle may be transformed to a variational principle involving only variations of the displacement field V^α, W. In general terms this variational principle has the following form (virtual work form)

$$\delta \int_{(A)} \mathcal{U}(V^\alpha, W)\, dA - \int_{(A)} \overset{\sim}{\overset{=}{T}} \cdot \delta \bar{u}_{\theta=0}\, dA - \int_{C_F} \overset{\sim}{\overset{=}{F}} \cdot \delta \bar{u}\, dC_F = 0 \qquad (2.49)$$

Here $\mathcal{U}$ is the strain energy function for the elastic shell

$$\mathcal{U} := \frac{1}{2} \frac{Eh}{1+\nu} \left[\alpha^\alpha_\beta \alpha^\beta_\alpha + \frac{\nu}{1-\nu} \alpha^\varrho_\varrho \alpha^\mu_\mu \right.$$
$$\left. + \frac{1}{12}(h)^2 \left(\omega^\alpha_\beta \omega^\beta_\alpha + \frac{\nu}{1-\nu} \omega^\varrho_\varrho \omega^\mu_\mu \right) \right] \qquad (2.50)$$

Further, $\tilde{F}$ is the prescribed stress vector acting on the boundary strip C_F (Fig. 1). This externally applied stress is assumed to be independent of the displacement and rotation of the boundary strip (edge of the shell). Thus, the variational operator $\delta(\cdot)$ can be extracted from the virtual work expression

$$\oint_{C_F} \tilde{\bar{F}} \cdot \delta \bar{u} \, dC_F = \delta \oint_{C_F} \tilde{\bar{F}} \cdot \bar{u} \, dC_F \, . \qquad (2.51)$$

Following the derivation in $\lfloor\bar{\ }8, \text{ p. } 229 \text{ ff}\_\rfloor$ the area element dC_F of the boundary strip may be represented by

$$\left. \begin{array}{l} dC_F = \left(H_{cc}\right)^{1/2} d\ell \, d\Theta \, dC \\[2mm] H_{cc} = 1 - 2 \, \Theta \, \mathcal{B}_{cc} + (\Theta)^2 \left[(\mathcal{B}_{\nu c})^2 + (\mathcal{B}_{cc})^2 \right] ; \end{array} \right\} \qquad (2.52)$$

here $\mathcal{B}_{cc}$, $\mathcal{B}_{\nu c}$ are components of the second fundamental form in the co-ordinate frame spanned by unit vectors $\bar{\nu}$ and $\tilde{\bar{c}}$ (Fig. 1) and dC is an element of the arc length along the boundary curve C .

If the edge loading is defined to be

$$\left. \begin{array}{l} \tilde{N}_\alpha = d\ell \int_{-1/2}^{+1/2} \left(H_{cc}\right)^{1/2} \tilde{\bar{F}} \cdot \bar{A}_\alpha \, d\Theta \\[4mm] \tilde{M}_\alpha = (d)^2 \ell \int_{-1/2}^{+1/2} \left(H_{cc}\right)^{1/2} \tilde{\bar{F}} \cdot \bar{A}_\alpha \, \Theta \, d\Theta \\[4mm] \tilde{Q} = d\ell \int_{-1/2}^{+1/2} \left(H_{cc}\right)^{1/2} \tilde{\bar{F}} \cdot \bar{A}_3 \, d\Theta \end{array} \right\} \qquad (2.53)$$

then

$$\oint_{C_F} \tilde{\bar{F}} \cdot \bar{u} \, dC_F = \ell \oint_C \left\{ \tilde{N}_\alpha V^\alpha + \tilde{M}_\alpha W^\alpha + \tilde{Q} W \right\} dC ;$$

here the displacement field representation (2.37) is implied. The term

$$\tilde{M}_\alpha W^\alpha = \tilde{M}^\alpha W_\alpha = - \tilde{M}^\alpha \left(W_{,\alpha} + \mathcal{B}_{\alpha s} V^s \right)$$

may be partially integrated[x] (see ref. $\lfloor\bar{\ }5\_\rfloor$) to give

---

[x] This implies a continuous distribution of $\tilde{M}^\alpha \underset{R}{\in}_{\alpha s} \nu^s$ along the boundary curve C . The permutation tensor $\underset{R}{\in}_{\alpha\rho}$ in the reference surface is given by

$$\underset{R}{\in}_{11} = 0 = \underset{R}{\in}_{22} \, , \quad \underset{R}{\in}_{12} = - \underset{R}{\in}_{21} = \sqrt{A}$$

$$\oint_C \tilde{M}^\alpha W_\alpha \, dC = -\oint_C \tilde{M}^\alpha \left(B_\alpha^\delta V_\delta + \nu_\alpha \frac{\partial W}{\partial \nu} \ell - \underset{R}{\epsilon}_{\alpha s} V^s \frac{\partial W}{\partial C} \ell \right) dC$$

$$= -\oint_C \left[\tilde{M}^\alpha \left(B_\alpha^\gamma V_\gamma + \nu_\alpha \frac{\partial W}{\partial \nu} \ell \right) + \frac{\partial (\tilde{M}^\alpha \underset{R}{\epsilon}_{\alpha s} V^s \ell)}{\partial C} W \right] dC \tag{2.54}$$

Thus

$$\overline{\overline{\Pi_c}} = -\oint_{C_F} \tilde{\overline{F}} \cdot \bar{u} \, dC_F$$

$$= -\ell \oint_C \left\{ [N^\alpha - \tilde{M}^s B_s^\alpha] V_\alpha \right. \tag{2.55}$$

$$\left. - [\tilde{M}^\alpha \nu_\alpha \ell] \frac{\partial W}{\partial \nu} + \tilde{Q}^* W \right\} dC$$

where $\tilde{Q}^*$ is the ersatz shear force

$$Q^* = \tilde{Q} - \frac{\partial}{\partial C} \left(\tilde{\underset{R}{M}}^\alpha \epsilon_{\alpha\gamma} V^\gamma \ell \right), \tag{2.56}$$

In the virtual work expression (2.49) $\tilde{\overline{T}}$ is the resultant force per unit area acting on the middle surface[*]. If a hydrostatic pressure P is applied to the shell then

$$\tilde{\overline{T}} = -p \, \bar{a}_3 \frac{da}{d\mathcal{A}} ; \tag{2.57}$$

here $da/d\mathcal{A}$ is the ratio of the deformed and undeformed middle surface element. Following the derivation in $[5]$ and ignoring terms less than $(\lambda)^2$ one obtains

$$\tilde{\overline{T}} = -p \left[W^\mu \bar{A}_\mu + (1 + V^\alpha_{;\alpha} - W B_\alpha^\alpha) \bar{A}_3 \right]. \tag{2.58}$$

---

[*] The shell is considered to be thin such that the surface forces may be assumed to be acting on the middle surface.

Obviously this loading depends on the present configuration of the shell. Therefore, the variational operator $\delta(\cdot)$ cannot simply be extracted from the virtual work expression

$$\int_{(A)} \overset{\approx}{T} \cdot \delta \bar{u}_{\Theta=0} \, dA .$$

Nevertheless, under suitable assumptions this is possible. With (2.58) the virtual work expression is

$$\int_{(A)} \overset{\approx}{T} \cdot \delta \bar{u}_{\Theta=0} \, dA = \ell \int_{(A)} p \left\{ W_{,\alpha} \, \delta V^\alpha + \delta \left(\tfrac{1}{2} V^\alpha B_{\alpha\delta} V^\delta \right) \right.$$
$$- \delta \left(W - \tfrac{1}{2} B_\alpha^\alpha (W)^2 \right) \tag{2.59}$$
$$\left. - V^\alpha_{:\alpha} \, \delta W \right\} dA .$$

For the following it will be assumed that the hydrostatic pressure p is uniform; this allows to extract p from the above surface integral. Then the first term on the right hand side may be partially integrated by application of the Green-Gauß theorem for surfaces

$$\int_{(A)} W_{,\alpha} \, \delta V^\alpha \, dA = \oint_C W \, \gamma_\alpha \, \delta V^\alpha \, dC$$
$$- \int_{(A)} W \, \delta V^\alpha_{:\alpha} \, dA . \tag{2.60}$$

The virtual work expression (2.59) may now be given the following form

$$\int_{(A)} \overset{\approx}{T} \cdot \delta \bar{u}_{\Theta=0} \, dA = \ell p \, \delta \int\int_{(A)} \left\{ \tfrac{1}{2} V^\alpha B_{\alpha\delta} V^\delta - \left(W - \tfrac{1}{2} B_\alpha^\alpha (W)^2 \right) \right.$$
$$\left. - V^\alpha_{:\alpha} \, W \right\} dA \tag{2.61}$$
$$+ \ell p \oint_C W \, \gamma_\alpha \, \delta V^\alpha \, dC .$$

Except for the line integral in (2.61) the right hand side has the form of the first variation of a functional. However, under suitable displacement boundary condition

i.e. $W = 0$ on C

or $\quad V_\alpha V^\alpha = 0 \quad$ on C'

$$\left.\rule{0pt}{60pt}\right\}(2.62)$$

this line integral vanishes. The clamped and the hinged edge are examples which satisfy these conditions.

Furthermore, if the shell is a closed one under hydrostatic pressure then no line integral appears on the right hand side of (2.61)[x].

For the following it will be assumed that the one or the other condition applies such that the virtual work of the hydrostatic pressure can be represented by the first variation of the potential $\widetilde{\Pi}_\rho$

$$\int_{(A)} \widetilde{T} \cdot \delta \bar{u}_{\theta=0} \, dA = -\delta \widetilde{\Pi}_\rho \qquad (2.63)$$

$$\widetilde{\Pi}_\rho := + \ell_\rho \int_{(A)} \left\{ W(\Lambda + V^\alpha_{;\alpha}) - \tfrac{1}{2} W B^\alpha_\alpha W - \tfrac{1}{2} V_\alpha B^{\alpha\gamma} V_\gamma \right\} dA$$

$$(2.64)$$

At this place it should be noted that Koiter $\underline{/}9\underline{\,/}$ has calculated the increase in external potential energy of a closed shell under uniform hydrostatic pressure by considering the volume change of the closed shell. Under due consideration of the different notations the result (2.64) is the same as Koiter's for a closed shell.

The virtual work formulation (2.49) can now be transformed to a variational principle: There exists a functional $\widetilde{\Pi}$, the total potential energy,

---

[x] A closed surface may be devided into two parts but continuously connected. For each part a line integral along the fictions edge is obtained; they are equal except for the sign.

$$\overline{\Pi}(V^{\alpha}, W) = \overline{\Pi}_{\alpha} + \overline{\Pi}_{p} + \overline{\Pi}_{c}$$

$$\overline{\Pi}_{\alpha} = \int_{A} \mathcal{O}l \, dA$$

$$\overline{\Pi}_{p} = + l_{p} \int_{A} \left\{ W(1 + V^{\alpha}_{;\alpha}) - \frac{1}{2} W B^{\alpha}_{\alpha} W - \frac{1}{2} V^{\alpha} B_{\alpha\beta} V^{\beta} \right\} dA$$

$$\overline{\Pi}_{c} = - l \oint_{c} \left\{ \left[\tilde{N}^{\alpha} - \tilde{M}^{g} B^{\alpha}_{g} \right] V_{\alpha} - \left[\tilde{M}^{\alpha}_{\gamma_{\alpha}} 1 \right] \frac{\partial W}{\partial \nu} \right. \left. + \tilde{Q}^{*} W \right\} dC$$

$$\left. \vphantom{\int} \right\} \quad (2.65)$$

whose first variation with respect to an admissible displacement field V^{α}, W vanishes:

$$\delta \overline{\Pi} = O .$$

$$(2.66)$$

This principle (2.66) states $[\bar{5}\_]$ that among all admissible displacements V^{α}, W of the middle surface which satisfy the prescribed geometrical boundary conditions, the actual displacements make the total potential energy stationary. This statement is equivalent to the equilibrium and dynamic boundary conditions of the shell $[\bar{5}\_]$. However, these will not and need not be considered here.

3. Derivation of an Approximate Stability (Indifference) Condition

3.1 Displacement Field Approximation and Matrix Representation of Kinematic Quantities

Following the strategy sketched in chapter (1), in a first step we transform the total potential energy functional (2.65) into an algebraic expression with a finite number of degrees of freedom. This is based on an approximate ansatz for the three displacement functions. For this formulation the direct matrix notation is used.

For the present restricted purpose it would be unduly complicated to follow the finite element philosophy. A global approximation for the displacements (Rayleigh-Ritz approach) is sufficient, although in practice it may be difficult to find suitable ansatz functions which satisfy the essential boundary conditions, i.e. the kinematic boundary conditions.

We assume that certain kinematic boundary conditions are prescribed on the edge C or part of it, compatible with one or the other restriction in (2.62). The total potential energy is formulated in terms of the tensor calculus. Therefore and for the purpose of generality and simplicity a transformation of the tensor components of the displacement functions into their physical components is avoided. Thus, the Rayleigh-Ritz ansatz is done directly with respect to the covariant components:

$$V_\alpha = \sum_{i=1}^{M} c_{(\alpha)}^i \, \varphi_{i\,\alpha}(\theta^s) \quad , \quad \alpha = 1, 2$$

$$W = \sum_{i=1}^{N} c^i \, \varphi_i(\theta^s) \quad .$$

(3.1)

We prefer to use the covariant components V_α instead of V^α since V_α is amenable to a simple geometric interpretation[*]. In the above ansatz it is understood that the prescribed functions $\varphi_{i\alpha}(\theta^s)$ and $\varphi_i(\theta^s)$ belong to a complete set of functions $\llbracket 10 \rrbracket$ and satisfy all boundary conditions.

---

[*] V_α is the rectangular projection of $\vec{v}$ on the base vector $\vec{A}_\alpha$:

$$V_\alpha = \vec{v} \cdot \vec{A}_\alpha$$

The three displacement components (3.1) are assembled together in the column matrix

$$U = \begin{bmatrix} V_1 \\ V_2 \\ W \end{bmatrix} , \tag{3.2}$$

Further, the unknown coefficients $\overset{i}{C}_{(\alpha)}$ and $\overset{i}{C}$ are assembled in the column matrix

$$Z = \begin{bmatrix} \overset{1}{C}_{(1)} \\ \vdots \\ \overset{M}{C}_{(1)} \\ \overset{1}{C}_{(2)} \\ \vdots \\ \overset{M}{C}_{(2)} \\ \overset{1}{C} \\ \vdots \\ \overset{N}{C} \end{bmatrix} , \tag{3.3}$$

Then the relations (3.1) have the following matrix representation

$$U_{(\theta^s)} = N_{(\theta^s)} Z , \tag{3.4}$$

where

$$N_{(\theta^s)} = \begin{bmatrix} \overset{1}{\varphi}_{(1)}(\theta^s) \cdots \overset{M}{\varphi}_{(1)}(\theta^s) & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \overset{1}{\varphi}_{(2)}(\theta^s) \cdots \overset{M}{\varphi}_{(2)}(\theta^s) & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \overset{1}{\varphi}_{(\theta^s)} \cdots \overset{N}{\varphi}_{(\theta^s)} \end{bmatrix} \tag{3.5}$$

The covariant components of the middle surface strain tensor $(2.46)_1$ may be split into a linear and nonlinear part such that

$$\left. \begin{aligned} \alpha_{\beta s} &= \Theta_{\beta s} + \eta_{\beta s} \\ \Theta_{\beta s} &:= \tfrac{1}{2}\left(V_{\beta;s} + V_{s;\beta} - 2W B_{\beta s} \right) \\ \eta_{\beta s} &:= \tfrac{1}{2} W_\beta W_s \end{aligned} \right\} \tag{3.6}$$

The two surface tensors $\Theta_{\beta\varsigma}$ and $\eta_{\beta\varsigma}$ are symmetric. Here the question arises in which way the components of each of these tensors may be arranged in a column matrix. Two choices are indicated

$$\Theta = \begin{Bmatrix} \Theta_{11} \\ \frac{1}{2}(\Theta_{12} + \Theta_{21}) \\ \Theta_{22} \end{Bmatrix} \quad OR \quad \begin{Bmatrix} \Theta_{11} \\ \Theta_{12} \\ \Theta_{22} \end{Bmatrix} \quad AND \quad \begin{Bmatrix} \Theta_{11} \\ \Theta_{12} \\ \Theta_{21} \\ \Theta_{22} \end{Bmatrix}$$

$$\eta = \begin{Bmatrix} \eta_{11} \\ \frac{1}{2}(\eta_{12} + \eta_{21}) \\ \eta_{22} \end{Bmatrix} \quad OR \quad \begin{Bmatrix} \eta_{11} \\ \eta_{12} \\ \eta_{22} \end{Bmatrix} \quad AND \quad \begin{Bmatrix} \eta_{11} \\ \eta_{12} \\ \eta_{21} \\ \eta_{22} \end{Bmatrix} \quad (3.7)$$

The first choice[*] has important advantages compared to the other; this will become evident in the following. Thus we introduce the following column matrix

$$\Theta = \begin{Bmatrix} \Theta_{11} \\ \frac{1}{2}(\Theta_{12} + \Theta_{21}) \\ \Theta_{22} \end{Bmatrix} = \begin{Bmatrix} V_{1:1} - WB_{11} \\ \frac{1}{2}(V_{1:2} + V_{2:1} - 2WB_{12}) \\ V_{2:2} - WB_{22} \end{Bmatrix} \quad (3.8)$$

---

[*] With unessential differences this is also the arrangement used in $\underline{/}$ 11, s. 421 $\underline{/}$.

According to the definition of the covariant derivative $(2.17)_2$ we have

$$
\left.\begin{aligned}
V_{1;1} &= V_{1,1} - \overset{o}{\Gamma}{}^{\mu}_{11} V_{\mu} = V_{1,1} - \overset{o}{\Gamma}{}^{1}_{11} V_{1} - \overset{o}{\Gamma}{}^{2}_{11} V_{2} \\
V_{1;2} &= V_{1,2} - \overset{o}{\Gamma}{}^{1}_{12} V_{1} - \overset{o}{\Gamma}{}^{2}_{12} V_{2} \\
V_{2;1} &= V_{2,1} - \overset{o}{\Gamma}{}^{1}_{21} V_{1} - \overset{o}{\Gamma}{}^{2}_{21} V_{2} \\
V_{2;2} &= V_{2,2} - \overset{o}{\Gamma}{}^{1}_{22} V_{1} - \overset{o}{\Gamma}{}^{2}_{22} V_{2} \; .
\end{aligned}\right\} \quad (3.9)
$$

These relations suggest to introduce a differential operator $\mathbb{D}$ such that (3.8) is represented by the following direct matrix notations:

$$
\Theta = \mathbb{D}\, U \, , \tag{3.10}
$$

The operator $\mathbb{D}$ is given by a 3×3 matrix

$$
\mathbb{D}_{(\cdot)} := \frac{1}{2}
\begin{pmatrix}
2\left[(\cdot)_{,1} - \overset{o}{\Gamma}{}^{1}_{11} \right] & -2\,\overset{o}{\Gamma}{}^{2}_{11} & -2 B_{11} \\[4pt]
\left[(\cdot)_{,2} - \overset{o}{\Gamma}{}^{1}_{12} - \overset{o}{\Gamma}{}^{1}_{21}\right] & \left[(\cdot)_{,1} - \overset{o}{\Gamma}{}^{2}_{21} - \overset{o}{\Gamma}{}^{2}_{12}\right] & -\left[B_{12} + B_{21}\right] \\[4pt]
-2\,\overset{o}{\Gamma}{}^{1}_{22} & 2\left[(\cdot)_{,2} - \overset{o}{\Gamma}{}^{2}_{22}\right] & -2 B_{22}
\end{pmatrix}
\tag{3.11}
$$

Now, U is given by the matrix product (3.2) and thus

$$
\Theta = (\mathbb{D}\, N)\, z \; ; \tag{3.12}
$$

this relation shows the linear dependence of the column matrix Θ on the unknown coefficients contained in z .

The two rotations W_{α} of the middle surface normal are assembled in the column matrix

$$ W = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} = - \begin{bmatrix} W_{,1} + B_1^\delta V_\gamma \\ W_{,2} + B_2^\delta V_\gamma \end{bmatrix} , \tag{3.13} $$

The introduction of another differential operator ∇^R allows two write

$$ W = \nabla^R \, u = (\nabla^R \, N) \, z \tag{3.14} $$

where

$$ \nabla^R_{(\cdot)} := \begin{bmatrix} B_1^1 & B_1^2 & (\cdot)_{,1} \\ B_2^1 & B_2^2 & (\cdot)_{,2} \end{bmatrix} . \tag{3.15} $$

The nonlinear parts of the middle surface strain are assembled in the column matrix

$$ \eta = \frac{1}{2} \begin{bmatrix} W_1 \, W_1 \\ \frac{1}{2} [\, W_1 \, W_2 + W_2 \, W_1 \,] \\ W_2 \, W_2 \end{bmatrix} . \tag{3.16} $$

Now it is important to note that (3.16) may be written as the matrix product

$$ \eta = \frac{1}{2} \begin{bmatrix} W_1 & 0 \\ \frac{1}{2} W_2 & \frac{1}{2} W_1 \\ 0 & W_2 \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} . \tag{3.17} $$

The components W_1 and W_2 may be related to W by the following matrix products

$$
\left.
\begin{aligned}
W_1 &= \underset{1}{e}^T W = \underset{1}{e}^T (\nabla N) z \\[2mm]
W_2 &= \underset{2}{e}^T W = \underset{2}{e}^T (\nabla N) z
\end{aligned}
\right\} \quad (3.18)
$$

where $\underset{1}{e}$ and $\underset{2}{e}$ are

$$
\underset{1}{e} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \underset{2}{e} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} . \tag{3.19}
$$

With (3.18) the 2 x 3 matrix in (3.17) is now given by

$$
G_{(z)} := \frac{1}{2} \begin{bmatrix} W_1 & 0 \\[2mm] \frac{1}{2} W_1 & \frac{1}{2} W_2 \\[2mm] 0 & W_2 \end{bmatrix} \tag{3.20}
$$

$$
= \frac{1}{2} \begin{bmatrix} \underset{1}{e}^T (\nabla N) z & 0 \\[2mm] \frac{1}{2} \underset{2}{e}^T (\nabla N) z & \frac{1}{2} \underset{1}{e}^T (\nabla N) z \\[2mm] 0 & \underset{2}{e}^T (\nabla N) z \end{bmatrix} .
$$

Thus, with these notations (3.17) may be written as

$$
\eta = G_{(z)} W = G_{(z)} (\nabla N) z . \tag{3.21}
$$

The change of curvature tensor $\omega_{\alpha\beta}$ (2.46)$_2$ is assembled in the column matrix

$$
\omega = \begin{bmatrix} \omega_{11} \\[2mm] \frac{1}{2}(\omega_{12} + \omega_{21}) \\[2mm] \omega_{22} \end{bmatrix} = \begin{bmatrix} W_{1;1} \\[2mm] \frac{1}{2}(W_{1;2} + W_{2;1}) \\[2mm] W_{2;2} \end{bmatrix} \tag{3.22}
$$

where the covariant derivatives are given by

$$
\left.
\begin{aligned}
W_{1;1} &= W_{1,1} - \overset{o}{\Gamma}_{11}^{\;1} W_1 - \overset{o}{\Gamma}_{11}^{\;2} W_2 \\
W_{1;2} &= W_{1,2} - \overset{o}{\Gamma}_{12}^{\;1} W_1 - \overset{o}{\Gamma}_{12}^{\;2} W_2 \\
W_{2;1} &= W_{2,1} - \overset{o}{\Gamma}_{21}^{\;1} W_1 - \overset{o}{\Gamma}_{21}^{\;2} W_2 \\
W_{2;2} &= W_{2,2} - \overset{o}{\Gamma}_{22}^{\;1} W_1 - \overset{o}{\Gamma}_{22}^{\;2} W_2 \;.
\end{aligned}
\right\}
\tag{3.23}
$$

Analogous to (3.10) we introduce the differential operator $\boxed{\nabla}^{K}$ such that

$$
\omega = \boxed{\nabla}^{K} W
\tag{3.24}
$$

where $\boxed{\nabla}^{K}$ is a 2 x 3 matrix

$$
\boxed{\nabla}^{K}_{(\cdot)} := \frac{1}{2}
\begin{bmatrix}
2[(\cdot)_{,1} - \overset{o}{\Gamma}_{11}^{\;1}] & -2\,\overset{o}{\Gamma}_{11}^{\;2} \\[2mm]
[(\cdot)_{,2} - \overset{o}{\Gamma}_{12}^{\;1} - \overset{o}{\Gamma}_{21}^{\;1}] & [(\cdot)_{,1} - \overset{o}{\Gamma}_{21}^{\;2} - \overset{o}{\Gamma}_{12}^{\;2}] \\[2mm]
-2\,\overset{o}{\Gamma}_{22}^{\;1} & 2[(\cdot)_{,2} - \overset{o}{\Gamma}_{22}^{\;2}]
\end{bmatrix}
\tag{3.25}
$$

This operator consists simply of the first two columns of $\boxed{\nabla}^{D}$. With (3.14) we obtain from (3.24)

$$
\omega = \boxed{\nabla}^{K}(\boxed{\nabla}^{R} v) = (\boxed{\nabla}^{K}\,\boxed{\nabla}^{R}) v = \boxed{\nabla}^{B} v ;
\tag{3.26}
$$

here

$$
\boxed{\nabla}^{B}_{(\cdot)} := \boxed{\nabla}^{K}(\boxed{\nabla}^{R}_{(\cdot)}) \;.
\tag{3.27}
$$

This gives finally

$$
\omega = \boxed{\nabla}^{B} v = (\boxed{\nabla}^{B} N)\, z
$$
$$
\boxed{\nabla}^{B} N = \boxed{\nabla}^{K}(\boxed{\nabla}^{R} N) .
\tag{3.28}
$$

Summarizing the main results we have

$$
u = \begin{pmatrix} V_1 \\ V_2 \\ W \end{pmatrix} = N\, z
$$

$$
\Theta = \begin{pmatrix} \Theta_{11} \\ \frac{1}{2}(\Theta_{12} + \Theta_{21}) \\ \Theta_{22} \end{pmatrix} = \boxed{D}\, u = (\boxed{D}\, N)\, z
$$

$$
W = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} = \boxed{R}\, u = (\boxed{R}\, N)\, z
$$

$$
\eta = \frac{1}{2} \begin{pmatrix} W_1 W_1 \\ \frac{1}{2}(W_1 W_2 + W_2 W_1) \\ W_2 W_2 \end{pmatrix} = G_{(z)}\, W = G_{(z)}\,(\boxed{R}\, N)\, z
$$

$$
\omega = \begin{pmatrix} \omega_{11} \\ \frac{1}{2}(\omega_{12} + \omega_{21}) \\ \omega_{22} \end{pmatrix} = \boxed{K}\, W = \boxed{K}\,(\boxed{R}\, u) = (\boxed{\Theta}\, N)\, z
$$

$$\Big\rangle (3.29)$$

This completes the matrix operator representation of the kinematic quantities.

3.2 Matrix Formulation of the Strain Energy Function for the Thin Shell

The matrix representation of the kinematic quantities is based on the covariant tensor components of the displacements and generalized strains $\alpha_{\alpha\beta}$ and $\omega_{\alpha\beta}$. In terms of these quantities the strain energy function $\mathcal{U}$ may be written in the following form[x]

$$\mathcal{U} = \frac{1}{2}\left\{ \alpha_{s\beta}\, H^{s\alpha\gamma\beta}\, \alpha_{\gamma\alpha} + \frac{1}{12}(\lambda)^2\, \omega_{s\beta}\, H^{s\alpha\gamma\beta}\, \omega_{\gamma\alpha} \right\} \tag{3.30}$$

with the 4. order elasticity tensor

$$H^{s\alpha\gamma\beta} := \frac{E\lambda}{1+\nu}\left(A^{s\alpha} A^{\gamma\beta} + \frac{\nu}{1-\nu} A^{s\beta} A^{\gamma\alpha} \right), \tag{3.31}$$

With the separation $(3.6)_1$ the first term on the right hand side of (3.30) is

$$\begin{aligned}
\alpha_{s\beta}\, H^{s\alpha\gamma\beta}\, \alpha_{\gamma\alpha} &= \Theta_{s\beta}\, H^{s\alpha\gamma\beta}\, \Theta_{\gamma\alpha} \\
&+ \Theta_{s\beta}\, H^{s\alpha\gamma\beta}\, \eta_{\gamma\alpha} + \eta_{s\beta}\, H^{s\alpha\gamma\beta}\, \Theta_{\gamma\alpha} \\
&+ \eta_{s\beta}\, H^{s\alpha\gamma\beta}\, \eta_{\gamma\alpha}.
\end{aligned} \tag{3.32}$$

The explicit evaluation of the first term on the right hand side gives

$$\begin{aligned}
\Theta_{s\beta}\, H^{s\alpha\gamma\beta} &= \Theta_{11}\, H^{1\alpha\gamma1} + \Theta_{12}\, H^{1\alpha\gamma2} + \Theta_{21}\, H^{2\alpha\gamma1} + \Theta_{22}\, H^{2\alpha\gamma2} \\
&= \Theta_{11}\, H^{1\alpha\gamma1} + \frac{1}{2}(\Theta_{12}+\Theta_{21})(H^{1\alpha\gamma2}+H^{2\alpha\gamma1}) + \Theta_{22}\, H^{2\alpha\gamma2}
\end{aligned} \tag{3.33}$$

---

[x] It should be kept in mind that all tensors and operations are defined in the undeformed referential configuration of the shell.

and

$$\Theta_{s\beta} \, H^{s\alpha\gamma\beta} \, \Theta_{\gamma\alpha}$$

$$= \left[\Theta_{11} H^{1111} + \tfrac{1}{2}(\Theta_{12} + \Theta_{21})(H^{1112} + H^{2111}) + \Theta_{22} H^{2112} \right] \Theta_{11}$$

$$+ \left[\Theta_{11} H^{1211} + \tfrac{1}{2}(\Theta_{12} + \Theta_{21})(H^{1212} + H^{2121}) + \Theta_{22} H^{2212} \right] \Theta_{12} \quad (3.34)$$

$$+ \left[\Theta_{11} H^{1121} + \tfrac{1}{2}(\Theta_{12} + \Theta_{21})(H^{1122} + H^{2121}) + \Theta_{22} H^{2122} \right] \Theta_{21}$$

$$+ \left[\Theta_{11} H^{1221} + \tfrac{1}{2}(\Theta_{12} + \Theta_{21})(H^{1222} + H^{2221}) + \Theta_{22} H^{2222} \right] \Theta_{22} .$$

This is combined analogous to (3.33). It is then obvious that

$$\Theta_{s\beta} \, H^{s\alpha\gamma\beta} \, \Theta_{\gamma\alpha} = \Theta^{T} \mathbb{H} \, \Theta \qquad (3.35)$$

where

$$\mathbb{H} := \begin{pmatrix} H^{1111} & (H^{1121} + H^{1211}) & H^{1221} \\[2ex] (H^{1112} + H^{2111}) & \begin{aligned} &(H^{1212} + H^{2121} \\ &+ H^{1122} + H^{2121}) \end{aligned} & (H^{1222} + H^{2221}) \\[2ex] H^{2112} & (H^{2212} + H^{2122}) & H^{2222} \end{pmatrix} \qquad (3.36)$$

With the definition (3.31) and the symmetry of the contravariant metric coefficients $H^{\alpha\beta}$ we see that the 3 x 3 matrix H is symmetric

$$H = H^T. \tag{3.37}$$

Similarly we may write

$$\left.\begin{array}{l} \Theta_{\varsigma\beta} \, H^{\varsigma\alpha\gamma\beta} \, \eta_{\gamma\alpha} \; = \; \eta^T H \Theta \; = \; \Theta^T H \eta \\[2mm] \eta_{\varsigma\beta} \, H^{\varsigma\alpha\gamma\beta} \, \eta_{\gamma\alpha} \; = \; \eta^T H \eta \end{array}\right\} \tag{3.38}$$

and for the deformation energy due to curvature changes we get

$$\omega_{\varsigma\beta} \, H^{\varsigma\alpha\gamma\beta} \, \omega_{\gamma\alpha} \; = \; \omega^T H \omega, \tag{3.39}$$

Consequently, the strain energy function $\mathcal{A}$ is

$$\mathcal{A} = \tfrac{1}{2}\left\{ \Theta^T H \Theta + \Theta^T H \eta + \eta^T H \Theta \right. \\[2mm] \left. + \eta^T H \eta + \tfrac{1}{12}(\lambda)^2 \, \omega^T H \omega \right\}. \tag{3.40}$$

With (3.29) the explicit representations of the right hand terms are

$$\left.\begin{array}{l} \Theta^T H \Theta \; = \; z^T (\nabla N)^T H (\nabla N) \, z \\[2mm] \Theta^T H \eta \; = \; z^T (\nabla N)^T H \, G_{(z)} \, (\nabla N) \, z \\[2mm] \eta^T H \Theta \; = \; z^T (\nabla N)^T G^T_{(z)} \, H \, (\nabla N) \, z \\[2mm] \eta^T H \eta \; = \; z^T (\nabla N)^T G^T_{(z)} \, H \, G_{(z)} \, (\nabla N) \, z \\[2mm] \omega^T H \omega \; = \; z^T (\nabla N)^T H \, (\nabla N) \, z \end{array}\right\} \tag{3.41}$$

The total strain energy potential is according to $(2.65)_2$

$$\widetilde{\Pi}_n = \int_{\mathcal{A}} \alpha \, d\mathcal{A}. \tag{3.42}$$

Here all terms depending on the surface coordinates have to be integrated over the whole middle surface of the shell in the undeformed configuration. This integration leads to the following matrices

$$\mathbb{D}_{(0,0)} := \int_{\mathcal{A}} (\nabla N)^T H (\nabla N) \, d\mathcal{A} \quad = \mathbb{D}_{(0,0)}^T$$

$$\mathbb{D}_{(0,z)} := \int_{\mathcal{A}} (\nabla N)^T H \, G_{(z)} (\nabla N) \, d\mathcal{A}$$

$$\mathbb{D}_{(z,0)} := \int_{\mathcal{A}} (\nabla N)^T G_{(z)}^T H (\nabla N) \, d\mathcal{A} = \mathbb{D}_{(0,z)}^T \quad \Big\}(3.43)$$

$$\mathbb{D}_{(z,z)} := \int_{\mathcal{A}} (\nabla N)^T G_{(z)}^T H \, G_{(z)} (\nabla N) \, d\mathcal{A} = \mathbb{D}_{(z,z)}^T$$

$$\mathbb{B}_{(0,0)} := \frac{1}{12} \int_{\mathcal{A}} (\lambda)^2 (\nabla N)^T H (\nabla N) \, d\mathcal{A} = \mathbb{B}_{(0,0)}^T.$$

With these definitions the strain energy potential is

$$\widetilde{\Pi}_n = \frac{1}{2} z^T \Big\{ \mathbb{D}_{(0,0)} + (\mathbb{D}_{(0,z)} + \mathbb{D}_{(z,0)}) + \mathbb{D}_{(z,z)} + \mathbb{B}_{(0,0)} \Big\} z. \tag{3.44}$$

It should be noted that this is not a quadratic form in z since some of the matrices depend linearly or quadratic on z. The matrix $\{\cdots\}$ is symmetric.

3.3 Matrix Representation of the Hydrostatic Pressure Potential

The potential energy of the hydrostatic pressure p is according to $(2.65)_3$

$$\pi_p = + \ell p \int_{A} \{ W(1 + V^{\alpha}_{;\alpha}) - \frac{1}{2} W B^{\alpha}_{\alpha} W - \frac{1}{2} V^{\alpha} B_{\alpha\beta} V^{\beta} \} dA$$

(3.45)

The trace $V^{\alpha}_{;\alpha}$ is given in explicit terms by

$$V^{\alpha}_{;\alpha} = V_{\varsigma;\alpha} A^{\varsigma\alpha}$$

$$= V_{1,1} A^{11} + V_{1,2} A^{12} + V_{2;1} A^{21} + V_{2;2} A^{22}$$

$$= (V_{1,1} - \overset{o}{\Gamma}{}^{1}_{11} V_{1} - \overset{o}{\Gamma}{}^{2}_{11} V_{2}) A^{11}$$

$$+ (V_{1,2} - \overset{o}{\Gamma}{}^{1}_{12} V_{1} - \overset{o}{\Gamma}{}^{2}_{12} V_{2}) A^{12}$$

$$+ (V_{2,1} - \overset{o}{\Gamma}{}^{1}_{21} V_{1} - \overset{o}{\Gamma}{}^{2}_{21} V_{2}) A^{21}$$

$$+ (V_{2,2} - \overset{o}{\Gamma}{}^{1}_{22} V_{1} - \overset{o}{\Gamma}{}^{2}_{22} V_{2}) A^{22}$$

Introducing a suitable differential operator $\boxed{\nabla}^{p}$ the trace $V^{\alpha}_{;\alpha}$ may be represented as

$$V^{\alpha}_{;\alpha} = E^{T} \boxed{\nabla}^{p} u = E^{T} (\boxed{\nabla}^{p} N) z$$

(3.46)

where

$$
\nabla^{(p)}(\cdot) := \begin{bmatrix}
[(\cdot)_{,1} - \overset{o}{\Gamma}{}^1_{11}]A^{11} & -\overset{o}{\Gamma}{}^2_{11}A^{11} & 0 \\[2mm]
[(\cdot)_{,2} - \overset{o}{\Gamma}{}^1_{12}]A^{12} & -\overset{o}{\Gamma}{}^2_{12}A^{12} & 0 \\[2mm]
-\overset{o}{\Gamma}{}^1_{21} A^{21} & [(\cdot)_{,1} - \overset{o}{\Gamma}{}^2_{21}]A^{21} & 0 \\[2mm]
-\overset{o}{\Gamma}{}^2_{22} A^{22} & [(\cdot)_{,2} - \overset{o}{\Gamma}{}^2_{22}]A^{22} & 0
\end{bmatrix} \qquad (3.47)
$$

and

$$
\mathbb{E} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} . \qquad (3.48)
$$

The normal displacement component W is singled out from the matrix $\mathcal{U}$ by the product

$$
W = \underset{3}{S}^T \cdot \mathcal{U} = \mathcal{U}^T \cdot \underset{3}{S} = \mathbb{Z}^T \mathbb{N}^T \underset{3}{S} \qquad (3.49)
$$

with

$$
\underset{3}{S} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} . \qquad (3.50)
$$

Thus, the first term in the integrand of (3.45) is

$$
W(1 + V^\alpha_{;\alpha}) = \mathbb{Z}^T \mathbb{N}^T \underset{3}{S} + \mathbb{Z}^T \left(\mathbb{N}^T \underset{3}{S} \, \mathbb{E}^T (\nabla \mathbb{N}) \right) \mathbb{Z} \qquad (3.51)
$$

With (3.49) we obtain for the second term

$$\frac{1}{2}\, W\, B_\alpha^\alpha\, W = \frac{1}{2}\, z^T \underbrace{(N^T \underset{3}{S}\, B_\alpha^\alpha\, \underset{?}{S^T}\, N)}_{\text{symmetric}}\, z. \qquad (3.52)$$

The third term is

$$\frac{1}{2}\, V_\alpha\, B^{\alpha\beta}\, V_\beta = \frac{1}{2}\left[V_1\, B^{11}\, V_1 + V_1\, B^{12}\, V_2 + V_2\, B^{21}\, V_1 + V_2\, B^{22}\, V_2\right]$$

and this may be put in the following form

$$\frac{1}{2}\, V_\alpha\, B^{\alpha\beta}\, V_\beta = \frac{1}{2}\, v^T\, K\, v = \frac{1}{2}\, z^T\, N^T\, K\, N\, z \qquad (3.53)$$

with

$$K = \begin{pmatrix} B^{11} & B^{12} & O \\ B^{21} & B^{22} & O \\ O & O & O \end{pmatrix} = K^T, \qquad (3.54)$$

The integration over the middle surface leads to the following matrices, which are independent of the surface coordinates

$$\left. \begin{aligned} p &:= \int_{A} N^T\, \underset{3}{S}\, dA \\[2ex] \overset{*}{K} &:= \int_{A} \left[N^T (K + \underset{3}{S}\, B_\alpha^\alpha\, \underset{3}{S^T})\, N \right]\, dA = \overset{*}{K}{}^T \\[2ex] E &:= \int_{A} N^T (\underset{3}{S}\, \varepsilon^T)\, (\nabla N)\, dA \; . \end{aligned} \right\} \quad (3.55)$$

It should be noted that the matrix $\mathbb{E}$ is nonsymmetric. The potential energy of the hydrostatic pressure is then given by

$$\widetilde{\Pi}_p = + \ell p \left\{ z^T \cdot p - \frac{1}{2} z^T \overset{*}{\mathbb{K}} z + z^T \mathbb{E} z \right\}. \quad (3.56)$$

3.4 Matrix Formulation of the Boundary Loading Potential

The transformation of potential energy of the boundary loading $(2.65)_4$ is
a straight forward matter. With[x]

$$\frac{\partial W}{\partial \nu} = \frac{\partial \Theta^s}{\partial \nu} \frac{\partial W}{\partial \Theta^s} = \frac{\partial \Theta^s}{\partial \nu} \frac{\partial}{\partial \Theta^s}\left(\underset{\sim}{S}^T U\right)$$

$$= \frac{\partial \Theta^s}{\partial \nu} \underset{\sim}{S}^T \left(\frac{\partial}{\partial \Theta^s} N\right) z \qquad\qquad (3.57)$$

we get

$$\tilde{\Pi}_c = -\ell \oint_C \left\{ \left[\tilde{N}^\alpha - \tilde{M}^s B_s^\alpha \right] \underset{\alpha}{S}^T N \right.$$

$$- \left[\ell \tilde{M}^\alpha \nu_\alpha \right] \frac{\partial \Theta^s}{\partial \nu} \underset{\sim}{S}^T\left(\frac{\partial}{\partial \Theta^s} N\right) \qquad (3.58)$$

$$\left. + \tilde{Q}^* \underset{\sim}{S}^T N \right\} z \; dC$$

$$= -\ell \underset{\sim}{t}^T z \; .$$

The definition of the column matrix $\underset{\sim}{t}$ is obvious from the above. Again
it is noted that this integration is to be done in the undeformed reference
state.

---

[x] The coordinate ν is the arc length along the outer normal $\tilde{\nu}$ of the
boundary curve C (Fig. 1) and C is arc length along the curve C

3.5 The Total Potential Energy and the Fundamental Equilibrium State

The total potential energy of the shell in terms of the column matrix Z which contains all unknown parameters characterizing the displacement field is given by

$$\overline{\Pi}_{(z)} = \overline{\Pi}_\alpha + \overline{\Pi}_\rho + \overline{\Pi}_c$$

$$= \frac{1}{2} z^T \left[\overline{D}_{(0,0)} + D_{(0,z)} + D_{(z,0)} + D_{(z,z)} \right. \tag{3.59}$$
$$\left. + B_{(0,0)} \right] z$$
$$+ \ell \rho \left\{ z^T \cdot p - \frac{1}{2} z^T \overset{*}{K} z + z^T E z \right\}$$
$$- \ell \, z^T \cdot t.$$

For various values of Z the total potential $\overline{\Pi}$ takes different values. For the equilibrium state of the shell the potential $\overline{\Pi}$ assumes a stationary value, i.e. the first variation of $\overline{\Pi}$ with respect to Z vanishes:

$$\delta \overline{\Pi}_{(z)} = 0, \tag{3.60}$$

This state is called the fundamental equilibrium state and it clearly depends on the loading p and t. From the stationary condition (3.60) we obtain the fundamental equilibrium state as follows. For most of the terms in (3.59) the derivation of the first variation is a straight forward matter:

$$\delta \left(\overline{\Pi}_\rho + \overline{\Pi}_c \right)$$

$$= \ell \rho \left\{ \delta z^T \cdot p - \frac{1}{2} \delta z^T \overset{*}{K} z - \frac{1}{2} z^T \overset{*}{K} \delta z \right.$$
$$\left. + \delta z^T E z + z^T E \delta z \right\} \tag{3.61}$$
$$- \ell \, \delta z^T \cdot t$$

$$= \delta z^T \left[\ell \rho \{ p - \overset{*}{K} z + (E + E^T) z \} - \ell t \right].$$

The first variation of the strain energy $\widetilde{\Pi}_\alpha$ deserves a more detailed analysis because of the non-quadratic terms. The first variation of these terms gives

$$
\delta \left\{ \frac{1}{2} \, z^T \left[D_{(0,z)} + D_{(z,0)} + D_{(z,z)} \right] z \right\}
$$

$$
= \frac{1}{2} \left\{ \delta z^T D_{(0,z)} \, z + z^T \delta (D_{(0,z)} \, z) \right.
$$

$$
+ \, z^T D_{(z,0)} \, \delta z + \delta(z^T D_{(z,0)}) \, z
$$

$$
\left. + \, \delta(z^T D_{(z,z)} \, z) \right). \tag{3.62}
$$

With the definition $(3.43)_2$ we have

$$
\delta(D_{(0,z)} \, z) = \int_{\mathcal{A}} (\nabla N)^T H \, G_{(\delta z)} (\nabla N) z \, d\mathcal{A}
$$

$$
+ \int_{\mathcal{A}} (\nabla N)^T H \, G_{(z)} (\nabla N) \delta z \, d\mathcal{A} \tag{3.63}
$$

$$
= D_{(0, \delta z)} \, z + D_{(0,z)} \, \delta z \, .
$$

Observing $(3.29)_3$ we see that

$$
\delta W = (\nabla N) \, \delta z \tag{3.64}
$$

such that $G_{(\delta z)}$ may be written as

$$G_{(\delta z)} = \frac{1}{2} \begin{bmatrix} \underset{1}{e}^T \delta w & 0 \\ \frac{1}{2} \underset{2}{e}^T \delta w & \frac{1}{2} \underset{1}{e}^T \delta w \\ 0 & \underset{2}{e}^T \delta w \end{bmatrix} ; \qquad (3.65)$$

thus

$$G_{(\delta z)} (\boxed{\nabla}^R N) z \;=\; G_{(\delta z)} \, w \;=\; G_{(z)} \, \delta w \qquad (3.66)$$

$$=\; G_{(z)} (\boxed{\nabla}^R N) \, \delta z$$

since

$$\frac{1}{2} \begin{bmatrix} \delta W_1 & 0 \\ \frac{1}{2} \delta W_2 & \frac{1}{2} \delta W_1 \\ 0 & W_2 \end{bmatrix} \begin{bmatrix} W_1 \\ W_1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} W_1 & 0 \\ \frac{1}{2} W_2 & \frac{1}{2} W_1 \\ 0 & W_2 \end{bmatrix} \begin{bmatrix} \delta W_1 \\ \delta W_2 \end{bmatrix} \quad (3.67)$$

Consequently, we have the important result

$$D_{(0, \delta z)} z \;=\; D_{(0, z)} \, \delta z \,. \qquad (3.68)$$

This "exchange rule" applies to any two column matrices $\mathbb{y}$ and $\mathbb{v}$ which have the same dimension as $\mathbb{z}$:

$$\mathbb{D}_{(0,\mathbb{y})} \, \mathbb{v} = \mathbb{D}_{(0,\mathbb{v})} \, \mathbb{y} \, . \tag{3.69}$$

This simple rule is closely related to the specific assembly of tensor components $\eta_{\alpha\beta}$ in the column matrix η ; the other alternative shown in (3.7) does not lead to such a simple relation. Thus we have

$$\delta \left(\mathbb{D}_{(0,\mathbb{z})} \, \mathbb{z} \right) = 2 \, \mathbb{D}_{(0,\mathbb{z})} \, \delta\mathbb{z} \, . \tag{3.70}$$

Similarly we find

$$\delta \left(\mathbb{z}^T \mathbb{D}_{(\mathbb{z},0)} \right) = \delta\mathbb{z}^T \mathbb{D}_{(\mathbb{z},0)} + \mathbb{z}^T \mathbb{D}_{(\delta\mathbb{z},0)}$$

$$= 2 \, \delta\mathbb{z}^T \mathbb{D}_{(\mathbb{z},0)} \tag{3.71}$$

and generally

$$\mathbb{v}^T \mathbb{D}_{(\mathbb{y},0)} = \mathbb{y}^T \mathbb{D}_{(\mathbb{v},0)} \, . \tag{3.72}$$

Finally

$$\delta \left(\mathbb{z}^T \mathbb{D}_{(\mathbb{z},\mathbb{z})} \, \mathbb{z} \right) = \delta\mathbb{z}^T \mathbb{D}_{(\mathbb{z},\mathbb{z})} \, \mathbb{z} + \mathbb{z}^T \mathbb{D}_{(\delta\mathbb{z},\mathbb{z})} \mathbb{z}$$

$$+ \mathbb{z}^T \mathbb{D}_{(\mathbb{z},\delta\mathbb{z})} \, \mathbb{z} + \mathbb{z}^T \mathbb{D}_{(\mathbb{z},\mathbb{z})} \, \delta\mathbb{z} \, . \tag{3.73}$$

Observing the definition of $\mathbb{D}_{(\mathbb{z},\mathbb{z})}$ and following the same argument as above one finds

$$\left. \begin{array}{l} \mathbb{z}^T \mathbb{D}_{(\delta\mathbb{z},\mathbb{z})} = \delta\mathbb{z}^T \mathbb{D}_{(\mathbb{z},\mathbb{z})} \\[2em] \mathbb{D}_{(\mathbb{z},\delta\mathbb{z})} \, \mathbb{z} = \mathbb{D}_{(\mathbb{z},\mathbb{z})} \, \delta\mathbb{z} \, . \end{array} \right\} \tag{3.74}$$

In general terms we have

$$\mathbf{y}^T \mathbb{D}_{(v,y)} = v\, \mathbb{D}_{(y,y)}$$

$$\mathbb{D}_{(y,v)}\, \mathbf{y} = \mathbb{D}_{(y,y)}\, v. \Bigg\} \quad (3.75)$$

With this exchange rule (3.73) reduces to

$$\delta\left(\mathbf{z}^T \mathbb{D}_{(z,z)}\, \mathbf{z}\right) = 4\,\delta\mathbf{z}^T \mathbb{D}_{(z,z)}\, \mathbf{z}. \quad (3.76)$$

These results are combined to give an explicit expression of the right hand side of (3.62) in terms of $\delta\mathbf{z}$

$$\delta\left\{\tfrac{1}{2}\,\mathbf{z}^T\left[\mathbb{D}_{(0,z)} + \mathbb{D}_{(z,0)} + \mathbb{D}_{(z,z)}\right]\mathbf{z}\right\}$$

$$= \left\{\delta\mathbf{z}^T\left[\left(\mathbb{D}_{(0,z)} + \mathbb{D}_{(0,z)}^T\right)\mathbf{z} + \mathbb{D}_{(z,0)}\,\mathbf{z}\right.\right. \quad (3.77)$$

$$\left.\left. + 2\,\mathbb{D}_{(z,z)}\,\mathbf{z}\,\right]\right\}.$$

The derivation of the first variation of the quadratic terms in the strain energy is a simple matter which needs not to be elaborated here. For $\delta\Pi$ we get finally

$$\delta\Pi = \delta\mathbf{z}^T\left\{\left[\mathbb{D}_{(0,0)} + \left(\mathbb{D}_{(0,z)} + \mathbb{D}_{(0,z)}^T\right) + \mathbb{D}_{(z,0)}\right.\right.$$

$$\left. + 2\,\mathbb{D}_{(z,z)} + \mathcal{B}_{(0,0)}\right]\mathbf{z} \quad (3.78)$$

$$+ 2\rho\left[p - \overset{*}{K}\mathbf{z} + (E+E')\mathbf{z}\right]$$

$$\left. - \ell t\,\right\}.$$

Since

$$\delta\Pi = 0, \quad \forall\,\delta\mathbf{z} \quad (3.79)$$

we obtain the condition that the bracket $\{\cdots\}$ in (3.78) must vanish. From this we get the following nonlinear algebraic problem for the unknown column matrix $\mathbb{Z}$:

$$\big[\ \mathbb{D}_{(0,0)} + \mathcal{B}_{(0,0)}$$
$$+ \big(\mathbb{D}_{(0,z)} + \mathbb{D}^T_{(0,z)}\big) + \mathbb{D}_{(z,0)} + 2\mathbb{D}_{(z,z)} \qquad (3.80)$$
$$+ \lambda p \big(\mathbb{E} + \mathbb{E}^T - \overset{*}{\mathbb{K}}\big)\big]\mathbb{z} = \ell\big(\mathbb{t} - p\mathbb{p}\big),$$

Two observations should be made. Firstly, the contribution of the hydrostatic pressure to the square matrix $[\cdots]$ on the left side is symmetric since $\overset{*}{\mathbb{K}}$ is symmetric and only the symmetric part of $\mathbb{E}$ is involved. Secondly, the matrix $\mathbb{D}_{(z,0)}$ is nonsymmetric. However, the expression $\mathbb{D}_{(z,0)}\,\mathbb{z}$ can be transformed to read

$$\mathbb{D}_{(z,0)}\,\mathbb{z} = \overset{*}{\mathbb{D}}_{(z)}\,\mathbb{z}$$

where $\overset{*}{\mathbb{D}}_{(z)}$ is symmetric. This derivation will be given later when it is needed in a different context.

If the response of the shell is clearly nonlinear in the pre-buckling state then nonlinear terms have to be included. We will not discuss this case here but assume that in the pre-buckling state the deformations are sufficiently small such that (3.80) reduces to the linear problem

$$\big[\ \mathbb{D}_{(0,0)} + \mathcal{B}_{(0,0)}\ \big]\mathbb{z} = \lambda\big(\mathbb{t} - p\mathbb{p}\big), \qquad (3.81)$$

The solution of this linearized problem is

$$\mathbb{z} = \mathbb{z}_0 = \ell\big[\ \mathbb{D}_{(0,0)} + \mathcal{B}_{(0,0)}\big]^{-1}\big(\mathbb{t} - p\mathbb{p}\big), \qquad (3.82)$$

For a class of problems this approximation may be sufficient. The solution of (3.80) or the approximate solution $\mathbb{z}_0$ (3.82) defines the fundamental state whose stability properties are to be analysed.

3.6 Evaluation of the Stability (Indifference) of the Fundamental State

3.6.1 General Evaluation

The equilibrium state, whose stability is to be analyzed, i.e. the fundamental state, is characterized by the displacement matrix $\mathbb{Z}$, the solution of (3.80). Now it is assumed that the fundamental state is in a state of neutral equilibrium which characterizes the transition from stability to instability. Consequently, one has to assess whether there exists an adjacent equilibrium state under the same pressure loading P and boundary loading $\mathcal{t}$ but characterized by the state

$$\mathbb{Z}^{*} = \mathbb{Z} + \mathbb{V} \tag{3.83}$$

where $\mathbb{V}$ is small[*]compared to $\mathbb{Z}$. Here it should be pointed out that the shape functions $f_{i\alpha}^{\varphi}$ and f_{i}^{φ} in the ansatz (3.1) are sufficient general to include all possible modes of deformation; thus, buckling modes may also be described by the series approximation (3.1).

For the adjacent state the total potential energy is

$$\overline{\mathcal{U}}^{*} = \overline{\mathcal{U}}(\mathbb{Z}^{*}) = \overline{\mathcal{U}}(\mathbb{Z} + \mathbb{V}), \tag{3.84}$$

Since this state is in equilibrium, the first variation of $\overline{\mathcal{U}}(\mathbb{Z} + \mathbb{V})$ with respect to $\mathbb{V}$ must vanish (Fig. 2):

$$\delta_{\mathbb{V}}^{P} \overline{\mathcal{U}}(\mathbb{Z} + \mathbb{V}) = O , \qquad \forall d^{\supset}\mathbb{V}. \tag{3.85}$$

Before this variational condition can be evaluated the potential $\overline{\mathcal{U}}(\mathbb{Z} + \mathbb{V})$ has to be represented in term of powers of $\mathbb{V}$. The general expression of $\overline{\mathcal{U}}(\mathbb{Z})$ for any column matrix $\mathbb{Z}$ is given by (3.59). According to the definition of the various matrices we have

---

[*] Actually we consider _infinitesimal_ additional displacements $\mathbb{V}$.

$$\mathcal{D}_{(0,\,z+v)} = \mathcal{D}_{(0,z)} + \mathcal{D}_{(0,v)}$$

$$\mathcal{D}_{(z+v,\,0)} = \mathcal{D}_{(z,0)} + \mathcal{D}_{(v,0)}$$

$$\mathcal{D}_{(z+v,\,z+v)} = \mathcal{D}_{(z,\,z+v)} + \mathcal{D}_{(v,\,z+v)}$$

$$+ \mathcal{D}_{(z,z)} + \mathcal{D}_{(z,v)} + \mathcal{D}_{(v,z)} + \mathcal{D}_{(v,v)} \qquad (3.86)$$

$$(z+v)^T K^* (z+v) = z^T \overset{*}{K} z + 2 v^T \overset{*}{K} z + v^T \overset{*}{K} v$$

$$(z+v)^T E (z+v) = z^T E z + v^T (E + E^T) z + v^T E v,$$

After some lengthy algebraic calculations the following representation is
obtained

$$\overline{\pi}_{(z+v)} = \overline{\pi}_{(z)} +$$

$$v^T \Big\{ \big[\mathcal{D}_{(0,0)} + \mathcal{B}_{(0,0)} + (\mathcal{D}_{(0,z)} + \mathcal{D}_{(z,0)}) + \mathcal{D}_{(z,0)} + 2 \mathcal{D}_{(z,z)} \qquad \text{1. order in } v$$

$$\qquad + \ell \rho (E + E^T - \overset{*}{K}) z + \ell \rho \rho - \ell t \Big\} +$$

$$(3.87)$$

$$\frac{1}{2} \Big\{ v^T \big[\mathcal{D}_{(0,0)} + \mathcal{B}_{(0,0)} + (\mathcal{D}_{(0,z)} + \mathcal{D}_{(z,0)}) + 4 \mathcal{D}_{(z,z)} \big] v \Big\} + \qquad \text{2. order in } v$$

$$\ell \rho \, v^T \big[E - \tfrac{1}{2} \overset{*}{K} \big] v +$$

$$\big[v^T \mathcal{D}_{(0,z)} v + v^T \mathcal{D}_{(v,0)} z + v^T \mathcal{D}_{(v,z)} z \big] +$$

$$+ \ldots$$

$$+ \frac{1}{2} V^T \left[2 D_{(V, z)} + D_{(0, V)} + D_{(V, 0)} + D_{(z, V)} + D_{(V, z)} \right] V \quad \right] \quad \text{3. order in } V$$

$$+ V^T D_{(V, V)} V \, , \quad \right] \quad \text{4. order in } V$$

Since z characterizes the fundamental equilibrium state it satisfies equ. (3.80) and thus the term linear in V vanishes identically. If we neglect the 3. and 4. order terms the first variation of $\overline{\Pi}(z + V)$ with respect to V gives

$$\delta_V \overline{\Pi}(z + V) =$$

$$\delta V^T \left[D_{(0, 0)} + B_{(0, 0)} + D_{(0, z)} + D_{(z, 0)} + 4 D_{(z, z)} \right] V +$$

$$\ell p \, \delta V^T \left[E + E^T - \overset{*}{K} \right] V +$$

$$\delta V^T \left[D_{(0, z} + D_{(z, 0)} \right] V + \qquad\qquad (3.88)$$

$$\delta V^T \left[D_{(V, 0)} + D_{(V, z)} \right] V +$$

$$V^T \left[D_{(\delta V, 0)} + D_{(0, \delta V)} \right] V \quad = 0$$

and observing (3.72) and (3.75) this may be combined to give

$$\delta_V \overline{\Pi}(z + V) =$$

$$\delta V^T \Big\{ \left[D_{(0, 0)} + B_{(0, 0)} + 2 D_{(0, z)} + 2 D_{(z, 0)} \right. \qquad (3.89)$$

$$\left. + 4 D_{(z, z)} + \ell p \left(E + E^T - \overset{*}{K} \right) \right] V$$

$$+ \left[2 D_{(V, 0)} + 2 D_{(V, z)} \right] z \Big\} \quad = 0$$

for all δW . Thus, the column matrix in the bracket $\{\cdots\}$ has to vanish:

$$\left[\, \mathbb{D}_{(0,0)} + \mathbb{B}_{(0,0)} + 2\left(\mathbb{D}_{(0,z)} + \mathbb{D}_{(z,0)}\right) + 4\,\mathbb{D}_{(z,z)} + \right.$$
$$\left. \lambda\rho\left(\mathbb{E} + \mathbb{E}^T - \overset{*}{\mathbb{K}}\right)\,\right] W + \tag{3.90}$$
$$\left[\, 2\,\mathbb{D}_{(V,0)} + 2\,\mathbb{D}_{(V,z)}\,\right] z .$$

This is a linear homogeneous equation for the additional displacement matrix V . The last term contains V in an implicit form not suitable for the further analysis. Therefore we transform this expression in such a way that V appears explicitly in a matrix product. With

$$\mathbb{D}_{(V,0)} = \mathbb{D}_{(0,V)}^T$$

and the defining expression $(3.43)_2$ for $\mathbb{D}_{(0,V)}$ we have

$$\mathbb{D}_{(V,0)}\, z = \int_{\mathcal{A}} (\nabla N)^T\, \mathbb{G}_{(V)}^T \underbrace{H\,(\nabla N)\,z}_{\text{column matrix}}\, d\mathcal{A}$$

$$\mathbb{G}_{(V)}^T = \frac{1}{2}\begin{bmatrix} \underset{1}{e}^T(\nabla N)V & \frac{1}{2}\underset{2}{e}^T(\nabla N)V & 0 \\[2ex] 0 & \frac{1}{2}\underset{1}{e}^T(\nabla N)V & \underset{2}{e}^T(\nabla N)V \end{bmatrix} \left.\right\} \tag{3.91}$$

The column matrix $H(\nabla N)z$ may be represented by

$$H(\nabla N)z = \begin{Bmatrix} \underset{1}{s}^T H(\nabla N)z \\[2ex] \underset{2}{s}^T H(\nabla N)z \\[2ex] \underset{3}{s}^T H(\nabla N)z \end{Bmatrix} = \begin{Bmatrix} z^T(\nabla N)^T H^T \underset{1}{s} \\[2ex] z^T(\nabla N)^T H^T \underset{2}{s} \\[2ex] z^T(\nabla N)^T H^T \underset{3}{s} \end{Bmatrix} \tag{3.92}$$

$$\underset{1}{S} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad , \quad \underset{2}{S} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad , \quad \underset{3}{S} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad , \tag{3.93}$$

Thus

$$G_{(V)}^T \, H \, (\nabla N) \, z =$$

$$= \frac{1}{2} \begin{bmatrix} z^T (\nabla N)^T H^T \underset{1}{S} \, c^T (\nabla N) v + z^T (\nabla N)^T H^T \underset{2}{S} \frac{1}{2} \underset{2}{c}^T (\nabla N) v \\ z^T (\nabla N)^T H^T \underset{2}{S} \frac{1}{2} \underset{1}{c}^T (\nabla N) v + z^T (\nabla N)^T H^T \underset{3}{S} \underset{2}{c}^T (\nabla N) v \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} z^T (\nabla N)^T H^T \underset{1}{S} & z^T (\nabla N)^T H^T \underset{2}{S} \frac{1}{2} \\ z^T (\nabla N)^T H^T \underset{2}{S} \frac{1}{2} & z^T (\nabla N)^T H^T \underset{3}{S} \end{bmatrix} \begin{bmatrix} \underset{1}{c}^T (\nabla N) v \\ \underset{2}{c}^T (\nabla N) v \end{bmatrix} \tag{3.94}$$

Consequently, the introduction of the symmetric matrix $F_{(z)}$

$$F_{(z)} := \frac{1}{2} \begin{bmatrix} \underset{1}{S}^T H \, (\nabla N) \, z & \frac{1}{2} \underset{2}{S}^T H \, (\nabla N) \, z \\ \frac{1}{2} \underset{2}{S}^T H \, (\nabla N) \, z & \underset{3}{S}^T H \, (\nabla N) \, z \end{bmatrix} \tag{3.95}$$

allows to write (3.94) in the following form

$$G_{(V)}^T \, H \, (\nabla N) \, z = F_{(z)} \, (\nabla N) \, v. \tag{3.96}$$

With this result the right hand side of (3.91)$_1$ may be written as

$$D_{(V,0)} \, z = \overset{*}{D}_{(z)} \, v \tag{3.97}$$

where

$$\overset{*}{\mathbb{D}}_{(z)} := \int_{\mathcal{A}} (\nabla N)^T \, F_{(z)} \, (\nabla N) \, d\mathcal{A}. \tag{3.98}$$

The matrix $\overset{*}{\mathbb{D}}_{(z)}$ is symmetric

$$\overset{*}{\mathbb{D}}_{(z)} = \overset{*}{\mathbb{D}}{}^{T}_{(z)}, \tag{3.99}$$

In an analogous way the matrix product $\mathbb{D}_{(v,z)} z$ is transformed. We get observing $(3.43)_4$

$$\mathbb{D}_{(v,z)} z = \int_{\mathcal{A}} (\nabla N)^T G^T_{(v)} \underbrace{H \, G_{(z)} (\nabla N) z}_{\text{column matrix}} \, d\mathcal{A} \tag{3.100}$$

and

$$G^T_{(v)} H \, G_{(z)} (\nabla N) z =$$

$$= \frac{1}{2} \left[\begin{array}{l} z^T (\nabla N)^T G^T_{(z)} H^T \underset{1}{\underset{\uparrow}{S}} e^T (\nabla N) v + z^T (\nabla N)^T G^T_{(z)} H^T \underset{2}{\underset{2}{S}} \underset{2}{\overset{1}{\,}} e^T (\nabla N) v \\ z^T (\nabla N)^T G_{(z)} H^T \underset{2}{S} \underset{2}{\overset{1}{\,}} e^T (\nabla N) v + z^T (\nabla N)^T G^T_{(z)} H^T \underset{3}{S} e^T (\nabla N) v \end{array} \right]$$

$$= F_{(z,z)} (\nabla N) v. \tag{3.101}$$

Here the symmetric matrix

$$F_{(z,z)} = F^T_{(z,z)} \tag{3.102}$$

is introduced

$$\bar{F}_{(z,z)} := \frac{1}{2} \begin{bmatrix} \underset{1}{S}^T H G_{(z)} (\nabla N) z & \frac{1}{2} \underset{2}{S}^T H G_{(z)} (\nabla N) z \\ \frac{1}{2} \underset{2}{S}^T H G_{(z)} (\nabla N) z & \underset{3}{S}^T H G_{(z)} (\nabla N) z \end{bmatrix} \tag{3.103}$$

and consequently

$$D_{(v,z)} \, z = \overset{*}{D}_{(z,z)} \, v \tag{3.104}$$

where

$$\overset{*}{D}_{(z,z)} := \int_{(A)} (\nabla N)^T F_{(z,z)} (\nabla N) \, dA \tag{3.105}$$

is symmetric. With these transformations the indifference condition (3.90) has the desired mathematical structure

$$\begin{aligned} \Big[\, D_{(0,0)} &+ B_{(0,0)} + 2 \big(D_{(0,z)} + D_{(z,0)} \big) + 4 D_{(z,z)} \\ &+ \big(p \big(E + E^T - K \big) + 2 \big(\overset{*}{D}_{(z)} + \overset{*}{D}_{(z,z)} \big) \big) \Big] v = 0 \end{aligned} \tag{3.106}$$

This is a homogeneous linear equation for the unknown column matrix V. Non-trivial solutions are obtained if the coefficient matrix $[\,\cdots\,]$ is singular for certain loading situations.

It should be noted that the coefficient matrix does not depend explicitly on the prescribed boundary loading but on the hydrostatic pressure P. On the other hand these loadings determine the fundamental state Z anyhow. Finally, the derivation of (3.90) suggests that this result is equivalent to the condition that the 2nd variation of $\widetilde{\Pi}(z)$ should vanish, i.e.

$$\delta^2 \widetilde{\Pi}(z) = 0$$

for all δZ. This may be easily confirmed.

To reduce the complexity of the stability problem the hydrostatic pressure loading and the boundary loading are treated seperately.

3.6.2 Pure Hydrostatic Pressure Loading

Since a pure hydrostatic pressure loading is applied to the shell the boundary loading is set to zero in (3.80):

$$\mathbf{t} = \mathbf{0}.$$

Further, it is assumed that the pre-buckling deformation is rather small such that the solution (3.82) of the linearized problem applies:

$$\mathbf{z} = \mathbf{z}_0 = -\ell p \left[\mathbf{D}_{(0,0)} + \mathbf{B}_{(0,0)} \right]^{-1} \mathbf{p} = +\ell p \, \overset{o}{\mathbf{z}} \qquad (3.107)$$

where

$$\overset{o}{\mathbf{z}} = -\left[\mathbf{D}_{(0,0)} + \mathbf{B}_{(0,0)} \right]^{-1} \mathbf{p} \qquad (3.108)$$

is independent of the pressure p. Inserting this result into (3.106) we obtain

$$\left[\mathbf{D}_{(0,0)} + \mathbf{B}_{(0,0)} + \ell p \left(2\mathbf{D}_{(0,\overset{o}{z})} + 2\mathbf{D}_{(\overset{o}{z},0)} + 2\overset{*}{\mathbf{D}}_{(\overset{o}{z})} \right. \right.$$

$$\left. \left. + \mathbf{E} + \mathbf{E}^T - \overset{*}{\mathbf{K}} \right) + (\ell p)^2 \left(4 \mathbf{D}_{(\overset{o}{z},\overset{o}{z})} + 2 \overset{*}{\mathbf{D}}_{(\overset{o}{z},\overset{o}{z})} \right) \right] \mathbf{v} = \mathbf{0}. \qquad (3.109)$$

Now, the hydrostatic pressure appears explicitly in the coefficient matrix and it is the only parameter which controls the singularity of this matrix. It is obvious that the indifference condition is a quadratic eigenvalue problem.

The pre-buckling deformation was assumed to be small and therefore it seems to be reasonable to ignore the terms quadratic in $\overset{o}{\mathbf{z}}$. Then (3.109) reduces to a general linear eigenvalue problem

$$\left[\mathbf{A} - \lambda \mathbf{C} \right] \mathbf{v} = \mathbf{0}$$

where λ is the eigenvalue

$$\lambda := - \ell p$$

and

$$A := D_{(0,0)} + B_{(0,0)} = A^T$$

$$C := 2 D_{(0,\overset{o}{z})} + 2 D_{(\overset{o}{z},0)} + 2 \overset{*}{D}_{(\overset{o}{z})} - \overset{*}{K} + (E + E^T) = C^T,$$

Assume that this eigenvalue problem has $2M+N$ distinct eigenvalues $\lambda_i = -\ell p_i$ and eigenvectors V_i . Then

$$[A - \lambda_i C] V_i = 0$$

and the reciprocal of the associated Rayleigh quotient is

$$\frac{1}{\lambda_i} = \frac{1}{-\ell p_i} = \frac{V_i^T C V_i}{V_i^T A V_i} \quad ;$$

explicitly we have for the reciprocal of the critical pressure

$$\frac{1}{\ell p_i} = - \frac{V_i^T [2 D_{(0,\overset{o}{z})} + 2 D_{(\overset{o}{z},0)} + 2 \overset{*}{D}_{(\overset{o}{z})} + E + E^T - \overset{*}{K}] V_i}{V_i^T [D_{(0,0)} + B_{(0,0)}] V_i} \quad . \tag{3.110}$$

The lowest eigenvalue λ_{min} defines the critical buckling pressure p_c . If the corresponding eigenvector V_{min} is known then (3.110) allows to calculate the buckling pressure p_c or its reciprocal.

3.6.3 Pure Boundary Loading

If no hydrostatic pressure is present but only loading along the boundary then the equilibrium condition for the fundamental state z reads

$$[D_{(0,0)} + B_{(0,0)} + D_{(0,z)} + D_{(z,0)} + D_{(z,0)} + 2 D_{(z,z)}] z$$

$$\tag{3.111}$$

$$= + \ell t$$

Complete linearization simplifies this to

$$\left[\mathbb{D}_{(0,0)} + \mathcal{B}_{(0,0)}\right]z = \ell t.$$

(3.112)

We assume now that the various contributions to the boundary loading change proportionally such that

$$t = \mathcal{T}\underset{\sim}{t}$$

(3.113)

where $\mathcal{T}$ is a variable load parameter and $\underset{\sim}{t}$ a fixed setting of the boundary loading. Then the solution $\underset{o}{z}$ of (3.112) is linear in $\mathcal{T}$:

$$\underset{o}{z} = \mathcal{T}\ell\left[\mathbb{D}_{(0,0)} + \mathcal{B}_{(0,0)}\right]^{-1}\underset{\sim}{t} = \ell\mathcal{T}\overset{o}{z}$$

(3.114)

where

$$\overset{o}{z} := \left[\mathbb{D}_{(0,0)} + \mathcal{B}_{(0,0)}\right]^{-1}\underset{\sim}{t} ,$$

(3.115)

With these results and definitions the general indifference condition (3.106) deduces to

$$\left[\mathbb{D}_{(0,0)} + \mathcal{B}_{(0,0)} + \ell\mathcal{T}\lambda\left(\mathbb{D}_{(0,\overset{o}{z})} + \mathbb{D}_{(\overset{o}{z},0)} + \overset{*}{\mathbb{D}}_{(\overset{o}{z})}\right)\right]v = 0,$$

(3.116)

here, the terms quadratic in $\overset{o}{z}$ are neglected such that a linear eigenvalue problem is obtained. The reciprocal of the corresponding critical load factor is then given by

$$\frac{1}{\ell\mathcal{T}_i} = -\frac{v_i^T\left[2\mathbb{D}_{(0,\overset{o}{z})} + 2\mathbb{D}_{(\overset{o}{z},0)} + 2\overset{*}{\mathbb{D}}_{(\overset{o}{z})}\right]v_i}{v_i^T\left[\mathbb{D}_{(0,0)} + \mathcal{B}_{(0,0)}\right]v_i}$$

(3.117)

where $\lambda_i = -\ell\tilde{\mathcal{T}}_i$ and v_i are eigenvalues and eigenvectors of (3.116).

4. The Total Potential Energy Functional of an Imperfect Shell for Infinitesimal Strains and Rotations

4.1 Derivation of the Energy Functional with the Initial State of the Perfect Shell as Reference Configuration

The shell whose stability is analysed in chapter (3) is called the "perfect" shell; its initial configuration is denoted by $\mathcal{J}_o$. We may consider now a slightly different "imperfect" shell with the initial configuration free of initial stresses and strains. These imperfections refer only to the geometry of the middle surface. It is assumed that the imperfect shell may be obtained by subjecting the middle surface of the perfect shell to a displacement field - the imperfection - compatible with all kinematic boundary conditions of the perfect shell. According to the strategy described in chapter (1) the imperfect shell is loaded by the same hydrostatic pressure and the same boundary forces and moments.[x]

The energy functional (2.65) is valid for any isotropic elastic shell whose initial configuration is free of stress and strain and whose geometric boundary conditions are compatible with (2.62).[xx] Thus the energy functional of the imperfect shell has the same form as (2.65), however, here the reference configuration is that of the imperfect shell. Following the strategy described in chapter (1) the response of the imperfect shell under the loading has to be calculated with a linearized theory. Consequently, all nonlinear terms in the energy functional of the imperfect shell have to be neglected. Thus, the energy functional of the imperfect shell has the following form

$$\overline{\Pi}_1 = \overline{\Pi}_{1\alpha} + \overline{\Pi}_{1p} + \overline{\Pi}_{1c} \tag{4.1}$$

where

$$\overline{\Pi}_{1\alpha} = \int_{\mathcal{A}_1} \alpha_1 \, d\mathcal{A}_1$$

$$\overline{\overset{\frown}{\mathscr{N}}}_{1P} = + \ell_P \int_{\underset{1}{\mathscr{A}}} \overset{\wedge}{\underset{1}{W}} \, d\overset{\wedge}{\underset{1}{\mathscr{A}}}$$

$$\overline{\overset{\frown}{\mathscr{N}}}_{1c} = -\ell \oint_{\underset{1}{\zeta}} \left\{ \left[\overset{\sim}{\underset{1}{N}}^{\alpha} - \overset{\sim}{\underset{1}{M}}^{\varrho} B_{\varrho}^{\ \alpha} \right] \right.$$

$$\left. - \left[\overset{\sim}{\underset{1}{M}}^{\alpha} \, \nu_{\alpha} \, \ell \right] \frac{\partial \overset{\wedge}{\underset{1}{W}}}{\partial \overset{\wedge}{\nu}} \; + \overset{\sim}{\underset{1}{Q}}^{*} \overset{\wedge}{\underset{1}{W}} \right\} \, d\varsigma \qquad\qquad \Big\} (4.2)$$

and

$$\underset{1}{\mathscr{Q}} = \frac{1}{2} \left\{ \underset{1}{\alpha}_{\varrho\beta} \, \underset{1}{H}^{\varrho\alpha\gamma\beta} \, \underset{1}{\alpha}_{\gamma\alpha} + \frac{1}{12} (\ell)^2 \, \underset{1}{\omega}_{\varrho\beta} \, \underset{1}{H}^{\varrho\alpha\gamma\beta} \, \underset{1}{\omega}_{\gamma\alpha} \right\}$$

$$\underset{1}{H}^{\varrho\alpha\gamma\beta} = \frac{E\ell}{1+\nu} \left(\overset{*}{\underset{1}{A}}^{\varrho\alpha} \overset{*}{\underset{1}{A}}^{\gamma\beta} + \frac{\nu}{1-\nu} \overset{*}{\underset{1}{A}}^{\varrho\beta} \overset{*}{\underset{1}{A}}^{\gamma\alpha} \right). \qquad\qquad \Big\} (4.3)$$

Here $\underset{1}{\alpha}_{\alpha\beta}$ and $\underset{1}{\omega}_{\alpha\beta}$ are the linearized middle surface strain and the change of curvature tensors[x] of the deformed imperfect shell measured with respect to the initial undeformed configuration $\underset{1}{\mathscr{K}}$ of the imperfect shell. Further, the elasticity tensor $\underset{1}{H}^{\varrho\alpha\gamma\beta}$ is defined in the configuration $\underset{1}{\mathscr{K}}$. It is obvious that the linearisation affects only the deformation measures and the potential $\overline{\overset{\frown}{\mathscr{N}}}_{1P}$.

It is noted that a formulation which uses tensorial quantities defined only in configuration $\underset{1}{\mathscr{K}}$ masks the fact that the imperfect shell is only slightly different from the actual "perfect" shell: Such a formulation does not explicitly show a parameter which measures the difference between the two shells. Therefore the energy functional (4.1) is transformed such that all tensorial quantities and operations are defined in the "perfect" con-figuration $\underset{o}{\mathscr{K}}$.[xx]

---

[x] These quantities are normalized.

[xx] This is a prerequisite for the comparison of the results of chapter (3) with the results to be derived for the imperfect shell.

Let $\bar{P}_0$ and $\bar{P}_1$ be the position vectors of a point of the "perfect" and "imperfect" shell (configurations $\mathcal{K}_0$ and $\mathcal{K}_1$)

$$\bar{P}_0 = \ell \left(\bar{R}_0 + \lambda \Theta \bar{A}_3 \right)$$

$$\bar{P}_1 = \ell \left(\bar{R}_1 + \lambda \Theta \bar{A}_3 \right).$$

$$(4.4)$$

$\bar{R}_0$ and $\bar{R}_1$ are the dimensionless position vectors of the two middle surfaces and $\bar{A}_{0\,3}$ and $\bar{A}_{1\,3}$ are the corresponding unit normal vectors (Fig. 3). Note that the thickness parameter $\lambda = h/\ell$ and the scale factor ℓ are taken to be the same; furthermore the surface coordinate of the two middle surfaces and the thickness coordinate are denoted by the same symbols, i.e. Θ^α and Θ , respectively. Naturally, the normalized base vectors of the two systems are different and given by

$$\bar{A}_{0\,\alpha} = \frac{\partial \bar{R}_0}{\partial \Theta^\alpha} \quad , \qquad \bar{A}_{0\,3} = \frac{\bar{A}_{0\,1} \times \bar{A}_{0\,2}}{|\bar{A}_{0\,1} \times \bar{A}_{0\,2}|}$$

and

$$(4.5)$$

$$\bar{A}_{1\,\alpha} = \frac{\partial \bar{R}_1}{\partial \Theta^\alpha} \quad , \qquad \bar{A}_{1\,3} = \frac{\bar{A}_{1\,1} \times \bar{A}_{1\,2}}{|\bar{A}_{1\,1} \times \bar{A}_{1\,2}|} \quad .$$

It is possible to consider a fictious deformation process which maps the perfect configuration $\mathcal{K}_0$ into the configuration $\mathcal{K}_1$ were the coordinate lines (Θ^α, Θ = const) are convected. The appropriate displacement vector of a point having the same coordinates in the two configurations is then defined by

$$\tilde{u} = \bar{P}_1(\Theta^\alpha, \Theta) - \bar{P}_0(\Theta^\alpha, \Theta)$$

$$(4.6)$$

$$= \ell \left[\bar{R}_1(\Theta^\alpha) - \bar{R}_0(\Theta^\alpha) + \lambda \Theta \left(\bar{A}_{1\,3}(\Theta^\alpha) - \bar{A}_{0\,3}(\Theta^\alpha) \right) \right].$$

The dimensionless displacement of the middle surfaces is denoted by

$$\tilde{v} = \bar{R}_1 - \bar{R}_0 \quad .$$

$$(4.7)$$

The component respresentation of this vector with respect to the configuration $\overset{\mathcal{K}}{_0}$ is

$$\overset{\sim}{\underset{0}{v}} = \mathcal{E} \overset{\sim}{\underset{0}{V}}{}^{\alpha} \bar{\underset{0}{A}}_{\alpha} + \mathcal{E} \overset{\sim}{\underset{0}{W}} \bar{\underset{0}{A}}_{3} \quad , \tag{4.8}$$

where $\mathcal{E} \ll 1$ is a measure for the "imperfection" (the difference between the perfect and imperfect shell) and $\overset{\sim}{\underset{0}{V}}{}^{\alpha}$, $\overset{\sim}{\underset{0}{W}}$ are "shape functions". This representation allows to formulate the base vectors of configuration $\overset{\mathcal{K}}{_1}$ in terms of $\bar{\underset{0}{A}}_{\alpha}$, $\bar{\underset{0}{A}}_{3}$ and $\mathcal{E} \overset{\sim}{\underset{0}{V}}{}^{\alpha}$, $\mathcal{E} \overset{\sim}{\underset{0}{W}}$. With (4.5), (4.7) and (4.8) we get

$$\left. \begin{aligned} \bar{\underset{1}{A}}_{\alpha} &= \bar{\underset{1}{R}}_{,\alpha} = \left(\bar{\underset{0}{R}} + \tilde{v} \right)_{,\alpha} \\[2mm] &= \bar{\underset{0}{A}}_{\alpha} + \mathcal{E} \left\{ \left[\overset{\sim}{\underset{0}{V}}{}^{\varsigma}_{;\alpha} - \overset{\sim}{\underset{0}{W}} \underset{0}{B}{}^{\varsigma}_{\alpha} \right] \bar{\underset{0}{A}}_{\varsigma} - \overset{\sim}{\underset{0}{W}}_{\alpha} \bar{\underset{0}{A}}_{3} \right\} \\[2mm] \overset{\sim}{\underset{0}{W}}_{\alpha} &= -\left(\overset{\sim}{\underset{0}{W}}_{/\alpha} + \underset{0}{B}_{\varsigma\alpha} \overset{\sim}{\underset{0}{V}}{}^{\varsigma} \right) \quad ; \end{aligned} \right\} \tag{4.9}$$

here $\overset{\sim}{\underset{0}{V}}{}^{\varsigma}_{;\alpha}$ is the covariant derivative of $\overset{\sim}{\underset{0}{V}}{}^{\varsigma}$ in the coordinate system of configuration $\overset{\mathcal{K}}{_0}$

$$\overset{\sim}{\underset{0}{V}}{}^{\varsigma}_{;\alpha} = \overset{\sim}{\underset{0}{V}}{}^{\varsigma}_{,\alpha} + \overset{0}{\underset{0}{\Gamma}}{}^{\varsigma}_{\delta\alpha} \overset{\sim}{\underset{0}{V}}{}^{\delta} \tag{4.10}$$

and $\underset{0}{B}_{\varsigma\alpha}$ are the covariant coefficients of the second fundamental form for $\overset{\mathcal{K}}{_0}$.

The unit normal vector $\bar{\underset{1}{A}}_{3}$ of the imperfect shell is obtained from (4.5). The vector product $\bar{\underset{1}{A}}_{1} \times \bar{\underset{1}{A}}_{2}$, observing (4.9), is given by

$$\bar{\underset{1}{A}}_{1} \times \bar{\underset{1}{A}}_{2} = \sqrt{\bar{\underset{0}{A}}} \left\{ \bar{\underset{0}{A}}_{3} + \mathcal{E} \left[\overset{\sim}{\underset{0}{W}}_{\alpha} \bar{\underset{0}{A}}{}^{\alpha} + \left(\overset{\sim}{\underset{0}{V}}{}^{\alpha}_{;\alpha} - \overset{\sim}{\underset{0}{W}} \underset{0}{B}{}^{\alpha}_{\alpha} \right) \bar{\underset{0}{A}}_{3} \right] \right.$$

$$+ \mathcal{E}^{2} \left[\left(\overset{\sim}{\underset{0}{W}}_{1} \left(\overset{\sim}{\underset{0}{V}}{}^{2}_{,2} - \overset{\sim}{\underset{0}{W}} \underset{0}{B}{}^{2}_{2} \right) - \overset{\sim}{\underset{0}{W}}_{2} \left(\overset{\sim}{\underset{0}{V}}{}^{2}_{;1} - \overset{\sim}{\underset{0}{W}} \underset{0}{B}{}^{2}_{1} \right) \right) \bar{\underset{0}{A}}{}^{1} \right.$$

$$\left. + \left(\overset{\sim}{\underset{0}{W}}_{2} \left(\overset{\sim}{\underset{0}{V}}{}^{1}_{;1} - \overset{\sim}{\underset{0}{W}} \underset{0}{B}{}^{1}_{1} \right) - \overset{\sim}{\underset{0}{W}}_{1} \left(\overset{\sim}{\underset{0}{V}}{}^{1}_{;2} - \overset{\sim}{\underset{0}{W}} \underset{0}{B}{}^{1}_{2} \right) \right) \bar{\underset{0}{A}}{}^{2} \right]$$

$$+ \cdots \tag{4.11}$$

$$\ldots + \left(\left(\underset{o}{\tilde{V}}{}^1_{;1} - \underset{o}{\tilde{W}} \underset{o}{B}{}^1_1 \right) \left(\underset{o}{\tilde{V}}{}^2_{;2} - \underset{o}{\tilde{W}} \underset{o}{B}{}^2_2 \right) \right.$$

$$\left. - \left(\underset{o}{\tilde{V}}{}^2_{;1} - \underset{o}{\tilde{W}} \underset{o}{B}{}^2_1 \right) \left(\underset{o}{\tilde{V}}{}^1_{;2} - \underset{o}{\tilde{W}} \underset{o}{B}{}^1_2 \right) \right) \underset{o}{\bar{A}}_3 \Big] \Big\}.$$

(4.11)

In analogy to the order of magnitude approximations (2.33) and (2.34) it is assumed that the displacement field $\underset{o}{\tilde{V}}$ characterizing the imperfection obeys the following order of magnitude estimations:

$$\varepsilon \underset{o}{\tilde{W}}_\alpha \qquad\qquad \sim \xi \quad << 1$$

$$\varepsilon \tfrac{1}{2} \left(\underset{o}{\tilde{V}}_{\alpha;\beta} - \underset{o}{\tilde{V}}_{\beta;\alpha} \right) \qquad \sim \xi^2$$

$$\varepsilon \tfrac{1}{2} \left(\underset{o}{\tilde{V}}_{\alpha;\beta} + \underset{o}{\tilde{V}}_{\beta;\alpha} - 2 \underset{o}{\tilde{W}} \underset{o}{B}_{\alpha\beta} \right) \sim \xi^2 \qquad\qquad\text{(4.12)}$$

and

$$\varepsilon \left(\underset{o}{\tilde{V}}_{\alpha;\beta} - \underset{o}{\tilde{W}} \underset{o}{B}_{\alpha\beta} \right) \qquad\qquad \sim \xi^2 .$$

Then all terms quadratic in ε on the right hand side of (4.11) are of the order of ξ^2 and higher[*]. In addition, the second term linear in ε is of order ξ^2. Dropping all terms of order ξ^2 and higher, equation (4.11) simplifies to

$$\underset{1}{\bar{A}}_1 \times \underset{1}{\bar{A}}_2 \simeq \sqrt{\underset{o}{\bar{A}}} \left\{ \underset{o}{\bar{A}}_3 + \varepsilon \underset{o}{\tilde{W}}_\alpha \underset{o}{\bar{A}}{}^\alpha \right\},$$

(4.13)

From this we find $\underset{1}{\bar{A}}_3$ neglecting all terms of order ξ^2 and higher:

$$\underset{1}{\bar{A}}_3 \simeq \underset{o}{\bar{A}}_3 + \varepsilon \underset{o}{\tilde{W}}{}^\alpha \underset{o}{\bar{A}}_\alpha ,$$

(4.14)

---

[*] It should be noted that the surface coordinates Θ^α can be chosen such that the base vectors are unit vectors.

This is a unit vector expect for terms of order ξ^2 and higher. The same result is obtained from (4.11) ignoring the order of magnitude estimations (4.12) but dropping all terms quadratic in ε .

We are now in the position to derive the co- and contravariant metric coefficients $\underset{1}{A}_{\alpha\beta}$ and $\underset{1}{A}^{\alpha\beta}$ in similar terms. The covariant metric coefficients are

$$
\underset{1}{A}_{\alpha\beta} = \underset{1}{\vec{A}}_{\alpha} \cdot \underset{1}{\vec{A}}_{\beta}
$$

$$
= \underset{0}{A}_{\alpha\beta} + \varepsilon \left[\underset{0}{\tilde{V}}_{\alpha;\beta} + \underset{0}{\tilde{V}}_{\beta;\alpha} - 2 \underset{0}{\tilde{W}} \underset{0}{B}_{\alpha\beta} \right] \qquad (4.15)
$$

$$
+ \varepsilon^2 \left[\left(\underset{v}{\tilde{V}}^{s}_{;\alpha} - \underset{0}{\tilde{W}} \underset{0}{B}^{s}_{\alpha} \right) \left(\underset{0}{\tilde{V}}_{s;\beta} - \underset{0}{\tilde{W}} \underset{0}{B}_{s\beta} \right) + \underset{0}{\tilde{W}}_{\alpha} \underset{0}{\tilde{W}}_{\beta} \right] ,
$$

The term linear in ε is of the order ξ^2 and the term quadratic in ε is of the order ξ^2 and higher. Thus, keeping all terms quadratic in ξ the following approximation is obtained

$$
\underset{1}{A}_{\alpha\beta} \simeq \underset{0}{A}_{\alpha\beta} + \varepsilon \left[\underset{0}{\tilde{V}}_{\alpha;\beta} + \underset{0}{\tilde{V}}_{\beta;\alpha} - 2 \underset{0}{\tilde{W}} \underset{0}{B}_{\alpha\beta} \right] + \varepsilon^2 \underset{0}{\tilde{W}}_{\alpha} \underset{0}{\tilde{W}}_{\beta} .
$$

But here we have to follow the assumptions in chapter (1) which demand that terms only linear in ε need to be retained; we get

$$
\underset{1}{A}_{\alpha\beta} \simeq \underset{0}{A}_{\alpha\beta} + \varepsilon \left[\underset{0}{\tilde{V}}_{\alpha;\beta} + \underset{0}{\tilde{V}}_{\beta;\alpha} - 2 \underset{0}{\tilde{W}} \underset{0}{B}_{\alpha\beta} \right] . \qquad (4.16)
$$

This approximation is certainly not consistent with the order of magnitude estimation (4.12); however, it will be seen later that this is of no consequence.

The contravariant components $\underset{1}{A}^{\alpha\beta}$ are defined by

$$
\underset{1}{A}_{s\mu} \underset{1}{A}^{s\alpha} = \delta^{\alpha}_{\beta} .
$$

Starting from (4.16) and neglecting all terms nonlinear in ε we get

$$
\underset{1}{A}^{\mu\nu} \simeq \underset{0}{A}^{\mu\nu} - \varepsilon \underset{0}{A}^{\mu\alpha} \left[\underset{0}{\tilde{V}}_{\alpha;\beta} + \underset{0}{\tilde{V}}_{\beta;\alpha} - 2 \underset{0}{\tilde{W}} \underset{0}{B}_{\alpha\beta} \right] \underset{0}{A}^{\beta\nu} , \qquad (4.17)
$$

For the transformation of the strain energy function $\underset{1}{\mathcal{U}}$ the deformation measures $\underset{1}{\alpha}_{\alpha\beta}$ and $\underset{1}{\omega}_{\alpha\beta}$ have to be given the appropriate form. Under the action of the pressure and the boundary loading the imperfect shell will be deformed; its configuration will change from $\underset{1}{\mathcal{K}}$ to a new configuration $\underset{2}{\mathcal{K}}$. It is assumed that this deformation obeys the Kirchhoff-Love hypotheses. Thus, in full analogy to (2.27) the displacement $\overset{\wedge}{\mathcal{U}}$ of a material point from configuration $\underset{1}{\mathcal{K}}$ to $\underset{2}{\mathcal{K}}$ may be represented as

$$ \overset{\wedge}{\mathcal{U}} = \ell \left(\overset{\wedge}{\mathcal{v}} + \lambda \Theta \left(\bar{a}_3 - \underset{1}{\bar{A}}_3 \right) \right) ; \tag{4.18} $$

here $\overset{\wedge}{\mathcal{v}}$ is the dimensionless displacement vector of the middle surface and $\bar{a}_3 \stackrel{=}{=} \underset{2}{\bar{A}}_3$. the unit normal vector of the deformed imperfect shell. The position of the deformed middle surface may also be measured with respect to the perfect shell surface in configuration $\underset{0}{\mathcal{K}}$ (Fig. 4); this leads to a new displacement vector

$$ \bar{\mathcal{v}} = \overset{\sim}{\bar{\mathcal{v}}} + \overset{\wedge}{\bar{\mathcal{v}}} = \left(\underset{1}{\bar{R}} - \underset{0}{\bar{R}} \right) + \left(\underset{2}{\bar{R}} - \underset{1}{\bar{R}} \right) = \underset{2}{\bar{R}} - \underset{0}{\bar{R}} , \tag{4.19} $$

Since the coordinate frame of configuration $\underset{0}{\mathcal{K}}$ is the reference frame the following coordinate respresentations apply

$$ \overset{\wedge}{\bar{\mathcal{v}}} = \underset{0}{\hat{V}}{}^{\alpha} \underset{0}{\bar{A}}_{\alpha} + \underset{0}{\hat{W}} \underset{0}{\bar{A}}_3 $$

$$ \underset{0}{\bar{\mathcal{v}}} = \underset{0}{V}{}^{\alpha} \underset{0}{\bar{A}}_{\alpha} + \underset{0}{W} \underset{0}{\bar{A}}_3 , \tag{4.20} $$

If the configuration $\underset{1}{\mathcal{K}}$ is used as reference frame we have

$$ \overset{\wedge}{\bar{\mathcal{v}}} = \underset{1}{\hat{V}}{}^{\alpha} \underset{1}{\bar{A}}_{\alpha} + \underset{1}{\hat{W}} \underset{1}{\bar{A}}_3 $$

$$ \underset{1}{\bar{\mathcal{v}}} = \underset{1}{V}{}^{\alpha} \underset{1}{\bar{A}}_{\alpha} + \underset{1}{W} \underset{1}{\bar{A}}_3 . \tag{4.21} $$

Thus with (4.19) and (4.8) we get

$$\overset{\wedge}{\underset{0}{V}}{}^{\alpha} = \underset{0}{V}{}^{\alpha} - \varepsilon \overset{\sim}{\underset{0}{V}}{}^{\alpha}$$

$$\overset{\wedge}{\underset{0}{W}} = \underset{0}{W} - \varepsilon \overset{\sim}{\underset{0}{W}},$$

(4.22)

The deformation of the imperfect shell from configuration $\overset{}{\underset{1}{\mathcal{H}}}$ to $\overset{}{\underset{\iota}{\mathcal{H}}}$ imposes certain strains into the shell. If $d\bar{p}$ denotes the infinitesimal distance vector of two material points in the deformed configuration then analogous to (2.38) we have

$$d\bar{p} \cdot d\bar{p} - d\overset{}{\underset{1}{\bar{P}}} \cdot d\overset{}{\underset{1}{\bar{P}}} = 2\, d\overset{}{\underset{1}{\bar{P}}} \, \overset{}{\underset{1}{\underline{E}}} \, d\overset{}{\underset{1}{\bar{P}}}$$

where $\overset{}{\underset{1}{\underline{E}}}$ is the Lagrangian strain tensor which characterizes the deformation of the imperfect shell. The relevant components of $\overset{}{\underset{1}{\underline{E}}}$ are $\overset{}{\underset{1}{E}}_{\alpha\beta}$; according to (2.42)$_1$ they are given by

$$\overset{}{\underset{1}{E}}_{\alpha\beta} \simeq \frac{1}{2}\left[\overset{}{\underset{1}{\bar{G}}}_{\alpha} \cdot \hat{u}_{,\beta} + \overset{}{\underset{1}{\bar{G}}}_{\beta} \cdot \hat{u}_{,\alpha} \right]$$

(4.23)

if the nonlinear term $\hat{u}_{,\alpha} \cdot \hat{u}_{,\beta}$ is dropped according to the strategy of chapter (1). With (4.4)$_2$ and (4.18) we have

$$\overset{}{\underset{1}{\bar{G}}}_{\alpha} = \overset{}{\underset{1}{\bar{P}}}_{,\alpha} = \ell\left(\overset{}{\underset{1}{\bar{R}}}_{,\alpha} + d\theta\, \overset{}{\underset{1}{\bar{A}}}_{3,\alpha} \right)$$

$$\hat{u}_{,\alpha} = \ell\left(\overset{\wedge}{\bar{v}}_{,\alpha} + d\theta\left(\bar{a}_{3,\alpha} - \overset{}{\underset{1}{\bar{A}}}_{3,\alpha} \right) \right).$$

(4.24)

Substituting this in (4.23) and arranging according to powers of $(d\theta)$ the following expression is obtained

$$\overset{}{\underset{1}{E}}_{\alpha\beta} = \frac{1}{2}\ell^2\Big\{ \overset{}{\underset{1}{\bar{A}}}_{\alpha} \cdot \overset{\wedge}{\bar{v}}_{,\beta} + \overset{}{\underset{1}{\bar{A}}}_{\beta} \cdot \overset{\wedge}{\bar{v}}_{,\alpha}$$

$$+ d\theta\Big[\overset{}{\underset{1}{\bar{A}}}_{\alpha} \cdot \bar{a}_{3,\beta} + \overset{}{\underset{1}{\bar{A}}}_{\beta} \cdot \bar{a}_{3,\alpha} + \overset{}{\underset{1}{\bar{A}}}_{3,\alpha} \cdot \overset{\wedge}{\bar{v}}_{,\beta} + \overset{}{\underset{1}{\bar{A}}}_{3,\beta} \cdot \overset{\wedge}{\bar{v}}_{,\alpha}$$

$$- \Big(\overset{}{\underset{1}{\bar{A}}}_{\alpha} \cdot \overset{}{\underset{1}{\bar{A}}}_{3,\beta} + \overset{}{\underset{1}{\bar{A}}}_{\beta} \cdot \overset{}{\underset{1}{\bar{A}}}_{3,\alpha} \Big) \Big]$$

$$+ (d\theta)^2 \Big[\overset{}{\underset{1}{\bar{A}}}_{3,\alpha} \cdot \Big(\bar{a}_{3,\beta} - \overset{}{\underset{1}{\bar{A}}}_{3,\beta} \Big) + \overset{}{\underset{1}{\bar{A}}}_{3,\beta} \cdot \Big(\bar{a}_{3,\alpha} - \overset{}{\underset{1}{\bar{A}}}_{3,\alpha} \Big) \Big] \Big\}$$

(4.25)

The right hand side has to be formulated in terms of quantities which are entirely defined in the coordinate system of the perfect shell (configuration $\overset{\sim}{\mathcal{H}}_0$). $\underset{1}{\bar{A}}_{\alpha}$ and $\underset{1}{\hat{A}}_3$ are given by (4.9) and (4.14) and thus

$$\underset{1}{\bar{A}}_{3,\alpha} = \left(\varepsilon \underset{0}{\tilde{W}}^{\varrho}_{;\alpha} - \underset{0}{B}^{\varrho}_{\alpha}\right)\underset{0}{\bar{A}}_{\varrho} + \varepsilon \underset{0}{\tilde{W}}^{\varrho} \underset{0}{B}_{\varrho\alpha} \underset{0}{\bar{A}}_3 \; ; \qquad (4.26)$$

further with (4.20)$_1$ we deduce

$$\hat{\underset{1}{v}}_{,\alpha} = \left(\underset{0}{\hat{V}}^{\varrho}_{;\alpha} - \underset{0}{\hat{W}}\underset{0}{B}^{\varrho}_{\alpha}\right)\underset{0}{\bar{A}}_{\varrho} - \underset{0}{\hat{W}}_{\alpha} \underset{0}{\bar{A}}_3 \; . \qquad (4.27)$$

It remains to derive an expression for the unit normal vector $\bar{a}_3$ of the deformed imperfect shell (configuration $\mathcal{H}_2$) compatible with the order of magnitude assumptions. The base vectors of the middle surface in configuration $\overset{}{\mathcal{H}}_2$ are

$$\underset{2}{\bar{A}}_{\alpha} = \underset{2}{\bar{R}}_{,\alpha} = \underset{0}{\bar{R}}_{,\alpha} + \bar{v}_{,\alpha} = \underset{0}{\bar{A}}_{\alpha} + \bar{v}_{,\alpha}$$

where $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (4.28)

$$\bar{v}_{,\alpha} = \left(\underset{0}{V}^{\varrho}_{;\alpha} - \underset{0}{W}\underset{0}{B}^{\varrho}_{\alpha}\right)\underset{0}{\bar{A}}_{\varrho} - \underset{0}{W}_{\alpha} \underset{0}{\bar{A}}_3 \; ;$$

here (4.5)$_1$ and (4.20)$_2$ are implied. Then the unit vector $\bar{a}_3 \cong \underset{2}{\bar{A}}_3$ is given by

$$\bar{a}_3 \cong \frac{\underset{2}{\bar{A}}_1 \times \underset{2}{\bar{A}}_2}{|\underset{2}{\bar{A}}_1 \times \underset{2}{\bar{A}}_2|} \; . \qquad (4.29)$$

The explicit evaluation of this expression follows along the same lines as before (see (2.32) and (4.5)). We obtain

$$\underset{2}{\bar{A}}_1 \times \underset{2}{\bar{A}}_2 = |\underset{0}{A}|^{\frac{1}{2}} \left\{ \underset{0}{\bar{A}}_3 + \underset{0}{W}_{\alpha} \underset{0}{A}^{\alpha} + \left(\underset{0}{V}^{\alpha}_{;\alpha} - \underset{0}{W}\underset{0}{B}^{\alpha}_{\alpha}\right)\underset{0}{\bar{A}}_3 \right.$$

$$\left. + \left[\underset{0}{W}_1\left(\underset{0}{V}^2_{;2} - \underset{0}{W}\underset{0}{B}^2_2\right) - \underset{0}{W}_2\left(\underset{0}{V}^2_{;1} - \underset{0}{W}\underset{0}{B}^2_1\right)\right]\underset{0}{\bar{A}}^1 \right.$$

$$+ \, \cdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (4.30)$$

$$... + \left[\underset{0}{W_2} \left(\underset{0}{V^1_{:1}} - \underset{0}{W} \underset{0}{B^1_1} \right) - \underset{0}{W_1} \left(\underset{0}{V^1_{:2}} - \underset{0}{W} \underset{0}{B^1_2} \right) \right] \underset{0}{\bar{A}^2}$$

$$+ \left[\left(\underset{0}{V^1_{:1}} - \underset{0}{W} \underset{0}{B^1_1} \right) \left(\underset{0}{V^2_{:2}} - \underset{0}{W} \underset{0}{B^2_2} \right) \right.$$ (4.30)

$$\left. - \left(\underset{0}{V^2_{:1}} - \underset{0}{W} \underset{0}{B^2_1} \right) \left(\underset{0}{V^1_{:2}} - \underset{0}{W} \underset{0}{B^1_2} \right) \right] \underset{0}{\bar{A}_3} \Big\} .$$

It is now essential to introduce order of magnitude estimations for the rotations, middle surface strains etc. produced by the displacement field $\bar{v}$.

For these quantities the same assumptions as (4.12) are made

$$\left.\begin{array}{ll}
\underset{0}{W^1_\alpha} & \sim \xi \\[2mm]
\frac{1}{2} \left(\underset{0}{V}_{\alpha:\beta} - \underset{0}{V}_{\beta:\alpha} \right) & \sim \xi^2 \\[2mm]
\frac{1}{2} \left(\underset{0}{V}_{\alpha:\beta} + \underset{0}{V}_{\beta:\alpha} - 2\underset{0}{W} \underset{0}{B}_{\alpha\beta} \right) & \sim \xi^2 \\[2mm]
\left(\underset{0}{V}_{\alpha:\beta} - \underset{0}{W} \underset{0}{B}_{\alpha\beta} \right) & \sim \xi^2 \\[4mm]
\underset{0}{W}_{\alpha:\beta} & \sim \xi \ ,
\end{array}\right\} \quad (4.31)$$

Keeping only terms of order ξ in (4.30) we get the usual approximate expression

$$\bar{a}_3 = \underset{0}{\bar{A}_3} + \underset{0}{W^\alpha} \bar{A}_\alpha .$$ (4.32)

Thus analogous to (4.26)

$$\bar{a}_{3/\alpha} = \left(\underset{0}{W^\rho_{:\alpha}} - \underset{0}{B^\rho_\alpha} \right) \underset{0}{\bar{A}_\rho} + \underset{0}{W^\rho} \underset{0}{B}_{\alpha\rho} \underset{0}{\bar{A}_3} ,$$ (4.33)

We are now in a position to evaluate the expression (4.25). Observing the definition (2.45) we put

$$\underset{1}{E}_{\alpha\beta} = \ell^2 \left(\underset{1}{\alpha}_{\alpha\beta} + \partial\Theta \; \underset{1}{\omega}_{\alpha\beta} + (\partial\Theta)^2 \underset{1}{\zeta}_{\alpha\beta} \right) \tag{4.34}$$

$$\left.
\begin{aligned}
\underset{1}{\alpha}_{\alpha\beta} &:= \frac{1}{2} \left(\bar{A}_{\alpha} \cdot \hat{v}_{,\beta} + \bar{A}_{\beta} \cdot \hat{v}_{,\alpha} \right) \\[2mm]
\underset{1}{\omega}_{\alpha\beta} &:= \frac{1}{2} \left\{ \bar{A}_{\alpha} \cdot \bar{a}_{3,\beta} + \bar{A}_{\beta} \cdot \bar{a}_{3,\alpha} \right. \\[2mm]
&\qquad + \bar{A}_{3,\alpha} \cdot \hat{v}_{,\beta} + \bar{A}_{3,\beta} \cdot \hat{v}_{,\alpha} \\[2mm]
&\qquad \left. - \left(\bar{A}_{\alpha} \cdot \bar{A}_{3,\beta} + \bar{A}_{\beta} \cdot \bar{A}_{3,\alpha} \right) \right\} \\[2mm]
\underset{1}{\zeta}_{\alpha\beta} &:= \frac{1}{2} \left\{ \bar{A}_{3,\alpha} \cdot \left(\bar{a}_{3,\beta} - \bar{A}_{3,\beta} \right) + \bar{A}_{3,\beta} \cdot \left(\bar{a}_{3,\alpha} - \bar{A}_{3,\alpha} \right) \right\}.
\end{aligned}
\right\} \tag{4.35}$$

With the above results the first term in the power series (4.34) is given by

$$\begin{aligned}
\underset{1}{\alpha}_{\alpha\beta} = \frac{1}{2} \Big\{ & \hat{V}_{\alpha;\beta} + \hat{V}_{\beta;\alpha} - 2 \hat{W}_{0} \underset{0}{B}_{\alpha\beta} \\[2mm]
& + \varepsilon \left[\tilde{V}^{\varsigma}_{0;\alpha} - \hat{W}_{0} \underset{0}{B}^{\varsigma}_{\alpha} \right] \left[\hat{V}_{0\varsigma;\beta} - \hat{W}_{0} \underset{0}{B}_{\varsigma\beta} \right] \\[2mm]
& + \varepsilon \left[\tilde{V}^{\varsigma}_{0;\beta} - \hat{W}_{0} \underset{0}{B}^{\varsigma}_{\beta} \right] \left[\hat{V}_{0\varsigma;\alpha} - \hat{W}_{0} \underset{0}{B}_{\varsigma\alpha} \right] \\[2mm]
& + \varepsilon \left[\tilde{W}_{0\alpha} \tilde{W}_{0\beta} + \tilde{W}_{0\beta} \tilde{W}_{0\alpha} \right] \Big\}.
\end{aligned} \tag{4.36}$$

With the order of magnitude assumptions (4.31) and (4.12) and with the relations (4.22) it may be shown that the following estimations apply:

$$\left.
\begin{aligned}
\hat{W}_{0\alpha} &\sim \zeta \\[2mm]
\frac{1}{2} \left(\hat{V}_{0\alpha;\beta} - \hat{V}_{0\beta;\alpha} \right) &\sim \zeta^L \\[2mm]
\frac{1}{2} \left(\hat{V}_{0\alpha;\beta} + \hat{V}_{0\beta;\alpha} - 2 \hat{W}_{0} \underset{0}{B}_{\alpha\beta} \right) &\sim \zeta^L
\end{aligned}
\right\} \tag{4.37}$$

$$\overset{\wedge}{\underset{0}{V}}_{\alpha:\beta} - \overset{\wedge}{\underset{0}{W}} \underset{0}{\beta}_{\alpha\beta} \qquad \sim \xi^2$$

$$\overset{\wedge}{\underset{0}{W}}_{\alpha:\beta} \qquad \qquad \sim \xi \ . \qquad\qquad \left.\begin{array}{c}\\ \\ \\ \end{array}\right\} (4.38)$$

Thus, the expression (4.36) may be simplified if terms only up to the order of ξ^2 are retained; then with the relations (4.22) equation (2.36) reduces to

$$\underset{1}{\alpha}_{\alpha\beta} = \frac{1}{2} \left\{ \left(\underset{0}{V}_{\alpha:\beta} + \underset{0}{V}_{\beta:\alpha} - 2 \underset{0}{W} \underset{0}{\beta}_{\alpha\beta} \right) \right.$$

$$- \varepsilon \left(\underset{0}{\tilde{V}}_{\alpha:\beta} + \underset{0}{\tilde{V}}_{\beta:\alpha} - 2 \underset{0}{\tilde{W}} \underset{0}{\beta}_{\alpha\beta} \right)$$

$$+ \varepsilon \left(\underset{0}{\tilde{W}}_{\alpha} \underset{0}{W}_{\beta} + \underset{0}{\tilde{W}}_{\beta} \underset{0}{W}_{\alpha} \right)$$

$$\left. + \varepsilon^2 \left(\underset{0}{\tilde{W}}_{\alpha} \underset{0}{\tilde{W}}_{\beta} + \underset{0}{\tilde{W}}_{\beta} \underset{0}{\tilde{W}}_{\alpha} \right) \right\} \ . \qquad (4.39)$$

It should be noted that the above order of magnitude estimation and the restriction to terms of order ξ^2 does not allow to drop the term quadratic in the imperfection ε . Nevertheless, the simplified strategy for the buckling analysis requires that the terms quadratic in ε are ignored. Consequently

$$\underset{1}{\alpha}_{\alpha\beta} = \frac{1}{2} \left\{ \left(\underset{0}{V}_{\alpha:\beta} + \underset{0}{V}_{\beta:\alpha} - 2 \underset{0}{W} \underset{0}{\beta}_{\alpha\beta} \right) \right.$$

$$- \varepsilon \left(\underset{0}{\tilde{V}}_{\alpha:\beta} + \underset{0}{\tilde{V}}_{\beta:\alpha} - 2 \underset{0}{\tilde{W}} \underset{0}{\beta}_{\alpha\beta} \right) \qquad (4.40)$$

$$\left. + \varepsilon \left(\underset{0}{\tilde{W}}_{\alpha} \underset{0}{W}_{\beta} + \underset{0}{\tilde{W}}_{\beta} \underset{0}{W}_{\alpha} \right) \right\} \ .$$

One should also be aware of the fact that the expression (4.40) does not contain all terms which are of the order ξ^2 because the nonlinear term $\hat{u}_{,\alpha} \cdot \hat{u}_{,\beta}$ has been dropped at the start.

In a similar way a simplified expression is derived for $\underset{1}{\omega}_{\alpha\beta}$. With the same order of magnitude assumptions it may be seen that the right hand side of $(4.25)_2$ contains terms of the order ξ, ξ^2 and ξ^3. Since $\underset{1}{\omega}_{\alpha\beta}$ is multiplied with the small number λ , and possibly $\lambda \sim \xi$, only the

terms of order $\pmb{\mathit{f}}$ are kept. This assures that only terms of comparable
order of magnitude are retained in $(4.35)_2$. Thus

$$\underset{\scriptstyle 1}{\omega}_{\alpha\beta} = \frac{1}{2} \left\{ \underset{\scriptstyle 0}{W}_{\alpha:\beta} + \underset{\scriptstyle 0}{W}_{\beta:\alpha} - \varepsilon \left(\underset{\scriptstyle 0}{\tilde{W}}_{\alpha:\beta} + \underset{\scriptstyle 0}{\tilde{W}}_{\beta:\alpha} \right) \right\}, \qquad (4.41)$$

It may be shown that the quantity $\underset{\scriptstyle 1}{\mathit{f}}_{\alpha\beta}$ is of order $\pmb{\mathit{f}}$. Thus the
third term in (4.34) can be neglected compared to the others since it
involves the small factor λ^2 . Therefore the power series (4.34) may be
simplified to read

$$\underset{\scriptstyle 1}{E}_{\alpha\beta} \simeq \lambda^2 \left(\underset{\scriptstyle 1}{\alpha}_{\alpha\beta} + \lambda \theta \underset{\scriptstyle 1}{\omega}_{\alpha\beta} \right); \qquad (4.42)$$

this expression contains only terms of order $\pmb{\mathit{f}}^L$ if λ and $\pmb{\mathit{f}}$ are com-
parable. This completes the derivation of the linearized strain measures
$\underset{\scriptstyle 1}{\alpha}_{\alpha\beta}$ and $\underset{\scriptstyle 1}{\omega}_{\alpha\beta}$.

The strain energy function $\underset{\scriptstyle 1}{\mathcal{U}}$ $(4.3)_1$ involves the elasticity
tensor $\underset{\scriptstyle 1}{H}^{\delta\alpha\delta\beta}$

$$\underset{\scriptstyle 1}{H}^{\delta\alpha\delta\beta} = \frac{Eh}{1+\nu} \left[\underset{\scriptstyle 1}{A}^{\delta\alpha} \underset{\scriptstyle 1}{A}^{\delta\beta} + \frac{\nu}{1-\nu} \underset{\scriptstyle 1}{A}^{\delta\beta} \underset{\scriptstyle 1}{A}^{\delta\alpha} \right]. \qquad (4.43)$$

An approximate expression will be derived for this tensor on the basis of
the approximate relation for the contravariant components of the metric
tensor $\underset{\scriptstyle 1}{A}^{\alpha\beta}$ (4.17).
With the definition

$$\tilde{\Theta}_{\alpha\beta} := \frac{1}{2} \left(\underset{\scriptstyle 0}{\tilde{V}}_{\alpha/\beta} + \underset{\scriptstyle 0}{\tilde{V}}_{\beta:\alpha} - 2 \underset{\scriptstyle 0}{\tilde{W}} \underset{\scriptstyle 0}{B}_{\alpha\beta} \right) = \tilde{\Theta}_{\beta\alpha} \qquad (4.44)$$

we may write

$$\underset{\scriptstyle 1}{A}^{\mu\nu} \simeq \underset{\scriptstyle 0}{A}^{\mu\nu} - \varepsilon \lambda \underset{\scriptstyle 0}{A}^{\mu\alpha} \tilde{\Theta}_{\alpha\beta} \underset{\scriptstyle 0}{A}^{\nu\beta} \qquad (4.45)$$

Inserting this into (4.43) we get

$$\underset{\scriptstyle 1}{H}^{\delta\alpha\delta\beta} = \frac{Eh}{1+\nu} \left\{ \underset{\scriptstyle 0}{A}^{\delta\alpha} \underset{\scriptstyle 0}{A}^{\delta\beta} + \frac{\nu}{1-\nu} \underset{\scriptstyle 0}{A}^{\delta\beta} \underset{\scriptstyle 0}{A}^{\delta\alpha} \right.$$

$$\left. - \varepsilon \lambda \left[\tilde{\Theta}_{\xi\eta} \underset{\scriptstyle 0}{A}^{\delta\alpha} \underset{\scriptstyle 0}{A}^{\eta\delta} \underset{\scriptstyle 0}{A}^{\delta\beta} + \tilde{\Theta}_{\xi\eta} \underset{\scriptstyle 0}{A}^{\delta\beta} \underset{\scriptstyle 0}{A}^{\eta\delta} \underset{\scriptstyle 0}{A}^{\delta\alpha} \right. \right.$$

$$+ \cdots \qquad (4.46)$$

$$+ \frac{\nu}{1-\nu} \left(\tilde{\Theta}_{\xi\eta} \underset{0}{A}^{\xi\beta} \underset{0}{A}^{\eta\delta} \underset{0}{A}^{\gamma\alpha} + \tilde{\Theta}_{\xi\eta} \underset{0}{A}^{\xi\alpha} \underset{0}{A}^{\eta\delta} \underset{0}{A}^{\gamma\beta} \right)$$

(4.46)

$$+ \varepsilon^2 \left[\ldots \right] \Big\} .$$

The term quadratic in ε has to be dropped according to the simplified buckling analysis strategy. Putting

$$\underset{0}{H}^{\delta\alpha\gamma\beta} = \frac{E\hbar}{1+\nu} \left\{ \underset{0}{A}^{\delta\alpha} \underset{0}{A}^{\gamma\beta} + \frac{\nu}{1-\nu} \underset{0}{A}^{\delta\beta} \underset{0}{A}^{\gamma\alpha} \right\}$$

(4.47)

we obtain from (4.46)

$$\underset{1}{H}^{\delta\alpha\gamma\beta} = \underset{0}{H}^{\delta\alpha\gamma\beta} - 2\varepsilon \tilde{\Theta}_{\xi\eta} \left[\underset{0}{A}^{\eta\delta} \underset{0}{H}^{\xi\alpha\gamma\beta} + \underset{0}{A}^{\eta\gamma} \underset{0}{H}^{\delta\beta\xi\alpha} \right] .$$

(4.48)

With these results the transformation of the strain energy function $\underset{1}{\mathcal{C}}$ may be performed. We consider first that part of $\underset{1}{\mathcal{C}}$ which represents the strain energy due to stretching of the middle surface, i.e.

$$\frac{1}{2} \underset{1}{\alpha}_{\delta\beta} \underset{1}{H}^{\delta\alpha\gamma\beta} \underset{1}{\alpha}_{\gamma\alpha} ,$$

At this point it is convenient to split $\underset{1}{\alpha}_{\gamma\beta}$ into three terms

(4.49)

$$\underset{1}{\alpha}_{\delta\beta} = \Theta_{\delta\beta} - \varepsilon \tilde{\Theta}_{\delta\beta} + \varepsilon \underset{1}{\gamma}_{\delta\beta}$$

$$\Theta_{\delta\beta} := \frac{1}{2} \left(\underset{0}{V}_{\delta:\beta} + \underset{0}{V}_{\beta:\delta} - 2 \underset{0}{W} \underset{0}{B}_{\delta\beta} \right) \sim \xi^2$$

$$\tilde{\Theta}_{\delta\beta} := \frac{1}{2} \left(\underset{0}{\tilde{V}}_{\delta:\beta} + \underset{0}{\tilde{V}}_{\beta:\delta} - 2 \underset{0}{\tilde{W}} \underset{0}{B}_{\delta\beta} \right) \sim \xi^2/\varepsilon \Bigg\}$$ (4.50)

$$\underset{1}{\gamma}_{\delta\beta} := \frac{1}{2} \left(\underset{0}{\tilde{W}}_{\delta} \underset{0}{W}_{\beta} + \underset{0}{\tilde{W}}_{\beta} \underset{0}{W}_{\delta} \right) \sim \xi^2/\varepsilon$$

On the right the assumed order of magnitudes consistent with (4.31) and (4.12) are indicated.

After some algebra we find

$$
\underset{1}{\alpha}_{s\beta} \underset{1}{H}^{s\alpha\gamma\beta} \underset{1}{\alpha}_{\gamma\alpha} =
$$

$$
= \Theta_{s\beta} \underset{0}{H}^{s\alpha\gamma\beta} \Theta_{\gamma\alpha}
$$

$$
- \varepsilon \left[\tilde{\Theta}_{s\beta} \underset{0}{H}^{s\alpha\gamma\beta} \Theta_{\gamma\alpha} + \Theta_{s\beta} \underset{0}{H}^{s\alpha\gamma\beta} \tilde{\Theta}_{\gamma\alpha} \right]
$$

$$
+ \varepsilon \left[\gamma_{s\beta} \underset{0}{H}^{s\alpha\gamma\beta} \Theta_{\gamma\alpha} + \Theta_{s\beta} \underset{0}{H}^{s\alpha\gamma\beta} \gamma_{\gamma\alpha} \right]
$$

$$
- 2\varepsilon \left[\Theta_{s\beta} \left(\underset{0}{A}^{\gamma\rho} \underset{0}{H}^{s\alpha\gamma\beta} + \underset{0}{A}^{\gamma\gamma} \underset{0}{H}^{\gamma\beta 3\alpha} \right) \tilde{\Theta}_{f\gamma} \Theta_{\gamma\alpha} \right]
$$

$$
+ \varepsilon^2 \left[\cdots \right].
$$

(4.51)

The usual procedure demands to drop the term quadratic in ε. Furthermore, if the order of magnitude assumption indicated in (4.50) is accounted for the fourth row on the right of (4.51) should be neglected. Therefore

$$
\underset{1}{\alpha}_{s\beta} \underset{1}{H}^{s\alpha\gamma\beta} \underset{1}{\alpha}_{\gamma\alpha} = \Theta_{s\beta} \underset{0}{H}^{s\alpha\gamma\beta} \Theta_{\gamma\alpha}
$$

$$
- \varepsilon \left[\tilde{\Theta}_{s\beta} \underset{0}{H}^{s\alpha\gamma\beta} \Theta_{\gamma\alpha} + \Theta_{s\beta} \underset{0}{H}^{s\alpha\gamma\beta} \tilde{\Theta}_{\gamma\alpha} \right]
$$

(4.52)

$$
+ \varepsilon \left[\gamma_{s\beta} \underset{0}{H}^{s\alpha\gamma\beta} \Theta_{\gamma\alpha} + \Theta_{s\beta} \underset{0}{H}^{s\alpha\gamma\beta} \gamma_{\gamma\alpha} \right].
$$

This result clearly indicates that, aside from dropping terms quadratic in ε, the elasticity tensor $\underset{1}{H}^{s\alpha\gamma\beta}$ is approximated by the tensor $\underset{0}{H}^{s\alpha\gamma\beta}$,

The second part of the strain energy

$$
\frac{1}{2} \frac{d^2}{12} \underset{1}{\omega}_{s\beta} \underset{1}{H}^{s\alpha\gamma\beta} \underset{1}{\omega}_{\gamma\alpha}
$$

is due to the change of curvature of the middle surface of the imperfect
shell. With (4.41) the change of curvature tensor is split into two parts

$$\overset{(.)}{\underset{1}{\rho}}_{\vartheta\beta} = \rho_{\vartheta\beta} - \varepsilon \,\tilde{\rho}_{\vartheta\beta} \tag{4.53}$$

$$\left.
\begin{aligned}
\rho_{\vartheta\beta} &:= \tfrac{1}{2}\left(\underset{0}{W}_{\vartheta:\beta} + \underset{0}{W}_{\beta:\vartheta} \right) \qquad \sim \xi \\[2mm]
\tilde{\rho}_{\vartheta\beta} &:= \tfrac{1}{2}\left(\underset{0}{\tilde{W}}_{\vartheta:\beta} + \underset{0}{\tilde{W}}_{\beta:\vartheta} \right) \qquad \sim \xi/\varepsilon
\end{aligned}
\right\} \tag{4.54}$$

In agreement with (4.31) and (4.12) the order of magnitude assumptions
are indicated. Upon substituting this and observing (4.48) we get analogous
to (4.52)

$$\underset{1}{\overset{(.)}{\rho}}_{\vartheta\beta} \; \underset{1}{H}{}^{\vartheta\alpha\delta\beta} \; \underset{1}{\omega}_{\delta\alpha} = \rho_{\vartheta\beta} \; \underset{0}{H}{}^{\vartheta\alpha\delta\beta} \; \rho_{\delta\alpha} \tag{4.55}$$

$$-\varepsilon \left[\tilde{\rho}_{\vartheta\beta} \; \underset{0}{H}{}^{\vartheta\alpha\delta\beta} \; \rho_{\delta\alpha} + \rho_{\vartheta\beta} \; \underset{0}{H}{}^{\vartheta\alpha\delta\beta} \; \tilde{\rho}_{\delta\alpha} \right]$$

$$-2\varepsilon \left[\rho_{\vartheta\beta} \left(\underset{0}{A}{}^{\eta\vartheta} \underset{0}{H}{}^{\vartheta\alpha\delta\beta} + \underset{0}{A}{}^{\eta\delta} \underset{0}{H}{}^{\vartheta\beta\vartheta\alpha} \right) \tilde{\Theta}_{\varepsilon\eta} \, \rho_{\delta\alpha} \right]$$

$$+ \varepsilon^2 \left[\cdots \right].$$

Again terms quadratic in ξ are dropped; keeping terms of order ξ^2
only, (4.55) simplifies to

$$\underset{1}{\omega}_{\vartheta\beta} \; \underset{1}{H}{}^{\vartheta\alpha\delta\beta} \; \underset{1}{\omega}_{\delta\alpha} = \rho_{\vartheta\beta} \; \underset{0}{H}{}^{\vartheta\alpha\delta\beta} \; \rho_{\delta\alpha}$$

$$-\varepsilon \left[\tilde{\rho}_{\vartheta\beta} \; \underset{0}{H}{}^{\vartheta\alpha\delta\beta} \; \rho_{\delta\alpha} + \rho_{\vartheta\beta} \; \underset{0}{H}{}^{\vartheta\alpha\delta\beta} \; \tilde{\rho}_{\delta\alpha} \right]. \tag{4.56}$$

Here again the remark following (4.52) applies. The final representation
of the strain energy $\underset{1}{\mathcal{A}}$ in terms of $\Theta_{\alpha\beta}$, $\tilde{\Theta}_{\alpha\beta}$, $\gamma_{\alpha\beta}$, $\rho_{\alpha\beta}$ and $\tilde{\rho}_{\alpha\beta}$
is

$$\underset{\text{\tiny 1}}{\mathcal{OL}} \approx \frac{1}{2} \Big\{ \Theta_{\varsigma\beta} \, \underset{0}{H}^{\varsigma\alpha\gamma\beta} \, \Theta_{\gamma\alpha}$$

$$- \varepsilon \, \Big[\tilde{\Theta}_{\varsigma\beta} \, \underset{0}{H}^{\varsigma\alpha\gamma\beta} \Theta_{\gamma\alpha} + \Theta_{\varsigma\beta} \, \underset{0}{H}^{\varsigma\alpha\gamma\beta} \tilde{\Theta}_{\gamma\alpha} \Big]$$

$$+ \varepsilon \, \Big[\gamma_{\varsigma\beta} \, \underset{0}{H}^{\varsigma\alpha\gamma\beta} \Theta_{\gamma\alpha} + \Theta_{\varsigma\beta} \, \underset{0}{H}^{\varsigma\alpha\gamma\beta} \gamma_{\gamma\alpha} \Big] \qquad (4.57)$$

$$+ \frac{\lambda^2}{12} \Big(\varsigma_{\varsigma\beta} \, \underset{0}{H}^{\varsigma\alpha\gamma\beta} \varsigma_{\gamma\alpha}$$

$$- \varepsilon \, \Big[\tilde{\varsigma}_{\varsigma\beta} \, \underset{0}{H}^{\varsigma\alpha\gamma\beta} \varsigma_{\gamma\alpha} + \varsigma_{\varsigma\beta} \, \underset{0}{H}^{\varsigma\alpha\gamma\beta} \tilde{\varsigma}_{\gamma\alpha} \Big] \Big) \Big\} .$$

This completes the transformation of $\underset{\text{\tiny 1}}{\mathcal{OL}}$. The strain energy potential $\overset{\frown}{\underset{\text{\tiny 1}}{\Pi}\alpha}$ given by $(4.2)_1$ involves the integration over the surface $\underset{\text{\tiny 1}}{\mathcal{A}}$. This process has to be represented in the configuration $\underset{0}{\mathcal{H}}$. The surface element $d\underset{\text{\tiny 1}}{\mathcal{A}}$ is given by

$$d\underset{\text{\tiny 1}}{\mathcal{A}} = \sqrt{\underset{\text{\tiny 1}}{A}} \; d\Theta^{\alpha} d\Theta^{\beta}$$

$$\underset{\text{\tiny 1}}{A} = \det \big(\underset{\text{\tiny 1}}{\bar{A}}_{\alpha} \cdot \underset{\text{\tiny 1}}{\bar{A}}_{\beta} \big) = \det \big(\underset{\text{\tiny 1}}{A}_{\alpha\beta} \big). \qquad (4.58)$$

Analogous to $\underline{/}^-5$, p. $37\underline{/}$ the ratios $d\underset{\text{\tiny 1}}{\mathcal{A}}/d\underset{0}{\mathcal{A}}$ of the surface elements may be formulated as

$$\frac{d\underset{\text{\tiny 1}}{\mathcal{A}}}{d\underset{0}{\mathcal{A}}} = 1 + \varepsilon \, \Big[\underset{0}{\tilde{V}}^{\alpha}_{;\alpha} - \underset{0}{W} \, \underset{0}{B}^{\alpha}_{\alpha} \Big] \qquad (4.59)$$

where terms quadratic in ε are ignored. The right hand side differs from one by a term linear in ε of order ς^2. Thus

$$d\underset{\text{\tiny 1}}{\mathcal{A}} = \Big(1 + \varepsilon \, \Big[\underset{0}{\hat{V}}^{\alpha}_{;\alpha} - \underset{0}{W} \, \underset{0}{B}^{\kappa}_{\kappa} \Big] \Big) d\underset{0}{\mathcal{A}}. \qquad (4.60)$$

Upon using this in the integral $\overset{\frown}{\underset{\text{\tiny 1}}{\Pi}\alpha}$ $(4.2)_1$ and observing (4.57) we see that extra terms linear in ε will appear in the integrand. However, if only the dominand terms[*] proportional to ς^4 are kept then these extra terms should be dropped; in other words the approximation

---

[*] Here, it is assumed that $\varsigma \sim \lambda$

$$d \underset{1}{c\!\!\!A} \simeq d \underset{0}{c\!\!\!A} \tag{4.61}$$

applies in this case :

$$\overline{\underset{1}{\Pi}_{\mathcal{O}\!L}} \simeq \int_{\underset{0}{c\!\!\!A}} \underset{1}{\mathcal{O}\!L} \, d \underset{0}{c\!\!\!A} \tag{4.62}$$

where $\underset{1}{\mathcal{O}\!L}$ is given by (4.57).

The potential of the hydrostatic pressure given by $(4.2)_2$

$$\overline{\underset{1}{\Pi}_{\rho}} = + \ell \rho \int_{\underset{1}{c\!\!\!A}} \underset{1}{\hat{W}} \, d \underset{1}{c\!\!\!A}$$

is transformed as follows. The quantity $\underset{1}{\hat{W}}$ is the normal component of $\overset{\wedge}{\tilde{v}}$ with respect to the basis in configuration $\underset{1}{\mathcal{K}}$, i.e.

$$\underset{1}{\hat{W}} = \overset{\wedge}{\tilde{v}} \cdot \overline{\underset{1}{A}}_3 \, , \tag{4.63}$$

Substituting from $(4.20)_1$, (4.22) and (4.14) we get

$$\underset{1}{\hat{W}} = \left(\underset{0}{W} - \varepsilon \underset{0}{\tilde{W}} \right) + \varepsilon \left(\underset{0}{V}^{\alpha} - \varepsilon \underset{0}{\tilde{V}}^{\alpha} \right) \underset{0}{\tilde{W}}_{\alpha} \, . \tag{4.64}$$

Together with (4.60) the potential $\overline{\underset{1}{\Pi}_{\rho}}$ may then be written as

$$\overline{\underset{1}{\Pi}_{\rho}} = \ell \rho \int_{\underset{0}{c\!\!\!A}} \left\{ \left(\underset{0}{W} - \varepsilon \underset{0}{\tilde{W}} \right) + \varepsilon \underset{0}{\tilde{W}}_{\alpha} \underset{0}{V}^{\alpha} \right. $$
$$\left. + \varepsilon \left[\underset{0}{\tilde{V}}^{\alpha}_{;\alpha} - \underset{0}{\tilde{W}} \underset{0}{B}^{\alpha}_{\alpha} \right] \underset{0}{W} \right\} d \underset{0}{c\!\!\!A} \, , \tag{4.65}$$

where according to the usual procedure the quadratic ε -terms have been dropped. Analogous to (2.60) the term $\underset{0}{\tilde{W}}_{\alpha} \underset{0}{V}^{\alpha}$ may be partially integrated to give

$$\int_{\underset{0}{c\!\!\!A}} \underset{0}{\tilde{W}}_{\alpha} \underset{0}{V}^{\alpha} \, d \underset{0}{c\!\!\!A} = - \int_{\underset{0}{c\!\!\!A}} \left(\underset{0}{\tilde{W}}_{,\alpha} + \underset{0}{B}_{\alpha\delta} \underset{0}{\tilde{V}}^{\delta} \right) \underset{0}{V}^{\alpha} \, d \underset{0}{c\!\!\!A} \tag{4.66}$$

$$= \ldots ,$$

$$= - \oint_{C_0} \nu_\alpha \tilde{\mathop{W}_0} \mathop{V^\alpha}_0 \, d\mathop{C}_0$$

(4.66)

$$+ \int_{\mathop{A}_0} \tilde{\mathop{W}_0} \mathop{V^\alpha}_{0\ ;\alpha} \, d\mathop{A}_0 - \int_{\mathop{A}_0} \mathop{V^\alpha}_0 \mathop{B}_{\alpha\gamma} \tilde{\mathop{V^\gamma}_0} \, d\mathop{A}_0 ,$$

The original stability problem is subject to the kinematic restrictions (2.62), which assures the existense of a hydrostatic pressure potential in the nonlinear case. These conditions have to be transferred to the linearized problem of the imperfect shell. The conditions are here

$$\mathop{W'}_0 = 0 \qquad \text{and} \qquad \tilde{\mathop{W'}_0} = 0 \quad \text{on} \quad \mathop{C}_0$$

or

$$\nu_\alpha \mathop{V^\alpha}_0 = 0 \qquad \text{and} \qquad \nu_\alpha \tilde{\mathop{V^\alpha}_0} = 0 \quad \text{on} \quad \mathop{C}_0 .$$

$\Big\}$(4.67)

Therefore, the boundary integral on the right vanishes:

$$\oint_{\mathop{C}_0} \nu_\alpha \mathop{V^\alpha}_0 \tilde{\mathop{W}_0} \, d\mathop{C}_0 = 0 .$$

(4.68)

This allows to write the hydrostatic potential in the following form:

$$\overline{\mathop{\Pi}_{1\rho}} = \lambda\rho \int_{\mathop{A}_0} \Big\{ \mathop{W}_0 + \varepsilon (\mathop{W}_0 \tilde{\mathop{V^\alpha}_0}_{;\alpha} + \tilde{\mathop{W}_0} \mathop{V^\alpha}_{0\ ;\alpha})$$

(4.69)

$$- \varepsilon \mathop{W}_0 \mathop{B^\alpha_\alpha}_0 \tilde{\mathop{W}_0} - \varepsilon \mathop{V}_{0\alpha} \mathop{B^{\alpha\beta}}_0 \tilde{\mathop{V}_\beta}_0 \Big\} \, d\mathop{A}_0 .$$

Since the term $\varepsilon \tilde{\mathop{W}_0}$ in the integrand of (4.65) is a constant and any constant may be added to the functional of a variational problem, this term is deleted. This completes the transformation of $\overline{\mathop{\Pi}_{1\rho}}$. At this point the reader is invited to compare this with the hydrostatic potential (2.64) of the actual stability problem.

It remains to transform the potential $(2.2)_3$ of the boundary loading.

For the stability problem to be analyzed it is assumed that the boundary loading is "dead". Actually, the boundary loading consists of surface stress vector $\tilde{F} \equiv \tilde{F}_0$ along the boundary strip $C_F \equiv C_{0F}$ [x]. Generally, under such "dead" surface loading the material particles constituting an infinitesimal portion of a surface will always be subject to the same total vector force. This means that the differential force $\tilde{F} dC_F \equiv \tilde{F}_0 dC_{0F}$ on the boundary strip $C_F \equiv C_{0F}$ is constant throughout the deformation of the shell from the initial to the fundamental state and to the adjacent state. This fact has to be reflected in the formulation of the boundary loading potential $\overline{\Pi}_{1C}$, equ. $(4.2)_3$.

So far, the expression on the right hand side of $(4.2)_3$ is of purely formal nature since the membrane forces, moments etc. have not been defined yet in terms of an appropriate surface stress on the boundary strip C_{1F}. We denote this surface stress by $\tilde{F}_1$. Then the virtual work of these stresses on the strip C_{1F} is given by

$$\oint_{C_{1F}} \tilde{F}_1 \cdot \delta \hat{u} \, dC_{1F} \, , \qquad (4.70)$$

We now recall that the imperfect shell in configuration $\mathcal{K}_1$ may be obtained by a fictitious deformation of the perfect shell in configuration $\mathcal{K}_0$; this deformation will later be related to the actual buckling modes of the perfect shell. In this deformation process the surface elements $dC_F \equiv dC_{0F}$ and dC_{1F} are materially related to each other.

The assumption of dead loading along the boundary strip now implies that the differential force $\tilde{F}_1 dC_{1F}$ is the same as the boundary load in the actual stability problem, i.e.

$$\tilde{F}_1 dC_{1F} = \tilde{F} dC_F \equiv \tilde{F}_0 dC_{0F} \, , \qquad (4.71)$$

Thus, the virtual work (4.70) may be written as

$$\oint_{C_{1F}} \tilde{F}_1 \cdot \delta \hat{u} \, dC_{1F} = \oint_{C_{1F}} \tilde{F}_0 \frac{dC_{0F}}{dC_{1F}} \cdot \delta \hat{u} \, dC_{1F} \, . \qquad (4.72)$$

---

[x] We introduce the subscript *(o)* to distinguish more clearly the configuration *(o)* and *(1)*

The right hand side may be immediately transformed into an integral where all operations are done in configuration $\overset{\sim}{\mathcal{H}}_0$; we get

$$\oint_{\underset{1}{C_F}} \overset{\sim}{\underset{1}{F}} \cdot \delta \hat{u} \, d\underset{1}{C_F} = \oint_{\underset{0}{C_F}} \overset{\approx}{\underset{0}{F}} \cdot \delta \hat{u} \, d\underset{0}{C_F} ,$$

Since $\overset{\approx}{\underset{0}{F}}$ is constant the variational operator may be extracted from this expression such that

$$\oint_{\underset{1}{C_F}} \overset{\sim}{\underset{1}{F}} \cdot \delta \hat{u} \, d\underset{1}{C_F} = \delta \oint_{\underset{0}{C_F}} \overset{\approx}{\underset{0}{F}} \cdot \hat{u} \, d\underset{0}{C_F} = - \delta \overline{\underset{1}{\Pi}_c}$$

and

$$\overline{\underset{1}{\Pi}_c} = - \oint_{\underset{0}{C_F}} \overset{\approx}{\underset{0}{F}} \cdot \hat{u} \, d\underset{0}{C_F} = - \oint_{C_F} \tilde{F} \cdot \hat{u} \, dC_F , \qquad (4.73)$$

Except for the displacement field $\hat{u}$ this expression is identical to (2.55). Therefore we may write[x] observing (4.22)

$$\overline{\underset{1}{\Pi}_c} = - \ell \oint_{\underset{0}{C}} \Big\{ [\tilde{N}^\alpha - \hat{M}^s \mathcal{B}_s^\alpha] \overset{\wedge}{\underset{0}{V}}_\alpha$$
$$- [\tilde{M}^\alpha \nu_\alpha \ell] \frac{\partial \hat{w}}{\partial \nu}$$
$$+ \tilde{Q}^* \overset{\wedge}{\underset{0}{w}} \quad \Big\} \, d\underset{0}{C} \qquad (4.74)$$

$$= - \ell \oint_{\underset{0}{C}} \Big\{ [\tilde{N}^\alpha - \tilde{M}^s \mathcal{B}_s^\alpha] \overset{}{\underset{0}{V}}_\alpha$$
$$- [\hat{M}^\alpha \nu_\alpha \ell] \frac{\partial \overset{}{\underset{0}{w}}}{\partial \nu} + \tilde{Q}^* \overset{}{\underset{0}{w}} \Big\} \, d\underset{0}{C}$$

$$+ \cdots$$

---

[x] Note that the integration is done in configuration $\overset{\sim}{\mathcal{H}}_0$; therefore the displacement vector has to be referred to the base $\overset{\approx}{\underset{0}{A}}_\alpha$ etc.

$$\ldots + \varepsilon \ell \oint\limits_{C_o} \{ [\tilde{N}^\alpha - \tilde{M}^\varsigma \mathcal{B}_\varsigma^\alpha] \overset{\vee}{V_\alpha}$$

$$- [\tilde{M}^\alpha \overset{\vee}{\nu}_\alpha \ell] \frac{\partial \tilde{\overset{\vee}{W}}}{\partial \nu} + \overset{*}{Q}\tilde{\overset{\vee}{W}}] \} \, d\varsigma . \tag{4.74}$$

This completes the evaluation of the boundary loading potential $\overline{\Pi}_{1c}$.

4.2 The Matrix Formulation of the Energy Functional of the Imperfect Shell

In the following we will derive the matrix representation of the three contributions (4.2) to the energy functional of the imperfect shell. Analogous to the argumentation on page (27) global approximations are made for the two displacement fields, i.e. $\overset{\vee}{V}_\alpha$ and $\overset{\vee}{W}$ describing the imperfections and the displacement components $\overset{\vee}{V}_\alpha$ and $\overset{\vee}{W}$ characterizing the deformation of the imperfect shell measured from the perfect shell configuration $\overset{\circ}{\mathcal{H}}$. It is now important to note that the same shape functions $\varphi_{i\alpha}(\Theta^\varsigma)$, $\varphi_i(\Theta^\varsigma)$ must be used as in (3.1), i.e. the approximate ansatz is

$$\overset{\circ}{V}_\alpha = \sum_{i=1}^{M} \overset{i}{\underset{o}{C}}_{(\alpha)} \varphi_{i\alpha}(\Theta^\varsigma)$$

$$\overset{\circ}{W} = \sum_{i=1}^{N} \overset{i}{\underset{o}{C}} \varphi_i(\Theta^\varsigma)$$

$$\overset{\vee}{V}_\alpha = \sum_{i=1}^{M} \overset{\sim i}{\underset{o}{C}}_{(\alpha)} \varphi_{i\alpha}(\Theta^\varsigma) \qquad\qquad (4.75)$$

$$\overset{\vee}{W} = \sum_{i=1}^{N} \overset{\sim i}{\underset{o}{C}} \varphi_i(\Theta^\varsigma) ,$$

Furthermore, it is obvious that the shape functions $\varphi_{i\alpha}$ and φ_i must satisfy all kinematic boundary conditions which are compatible with (4.67). Consequently, we have the following column matrix representations

$$
U = \begin{Bmatrix} \overset{.}{V}_1 \\ \overset{.}{V}_2 \\ \overset{.}{W} \end{Bmatrix} = N_{(\Theta^g)}\, Z
$$

$$
\tilde{U} = \begin{Bmatrix} \overset{.}{\tilde{V}}_1 \\ \overset{.}{\tilde{V}}_2 \\ \overset{.}{W} \end{Bmatrix} = N_{(\Theta^g)}\, \tilde{Z}
$$

$$\left. \right\} \tag{4.76}$$

where the shape function matrix N is given by (4.5) and

$$
Z = \begin{Bmatrix} \overset{1}{\underset{o}{C}}_{(1)} \\ \vdots \\ \overset{M}{\underset{o}{C}}_{(1)} \\ \overset{1}{\underset{o}{C}}_{(2)} \\ \vdots \\ \overset{M}{\underset{o}{C}}_{(2)} \\ \overset{1}{\underset{o}{C}} \\ \vdots \\ \overset{N}{\underset{o}{C}} \end{Bmatrix}
\qquad , \qquad
\tilde{Z} = \begin{Bmatrix} \overset{1}{\underset{o}{\tilde{C}}}_{(1)} \\ \vdots \\ \overset{M}{\underset{o}{\tilde{C}}}_{(1)} \\ \overset{1}{\underset{o}{\tilde{C}}}_{(2)} \\ \vdots \\ \overset{M}{\underset{o}{\tilde{C}}}_{(2)} \\ \overset{1}{\underset{o}{\tilde{C}}} \\ \vdots \\ \overset{N}{\underset{o}{\tilde{C}}} \end{Bmatrix}
\tag{4.77}
$$

Note that for reasons of convenience the subscript (o) which should be attached to Z and $\tilde{Z}$ is deleted. Analogous to (3.29) we define

$$
\Theta = \begin{bmatrix} \Theta_{11} \\ \frac{1}{2}(\Theta_{12} + \Theta_{21}) \\ \Theta_{22} \end{bmatrix} = \mathbb{P}U = (\mathbb{P}'N)Z
$$

$$
\tilde{\Theta} = \begin{bmatrix} \Theta_{11} \\ \frac{1}{2}(\tilde{\Theta}_{12} + \tilde{\Theta}_{21}) \\ \Theta_{22} \end{bmatrix} = \mathbb{P}U = (\mathbb{P}'N)\tilde{Z}
$$

$$
W = \begin{bmatrix} W_{01} \\ W_{02} \end{bmatrix} = \mathbb{R}U = (\mathbb{R}'N)Z
$$

$$
\tilde{W} = \begin{bmatrix} \tilde{W}_{01} \\ \tilde{W}_{02} \end{bmatrix} = \mathbb{R}\tilde{U} = (\mathbb{R}'N)\tilde{Z}
$$

$\left.\right\}$ (4.78)

$$
W_{01} = \mathbb{C}_1^T W \quad , \quad W_{02} = \mathbb{C}_2^T W
$$

$$
\tilde{W}_{01} = \mathbb{C}_1^T \tilde{W} \quad , \quad \tilde{W}_{02} = \mathbb{C}_2^T \tilde{W}
$$

$\left.\right\}$ (4.79)

$$\mathbb{y} = \begin{bmatrix} y_{11} \\ \frac{1}{2}(y_{12} + y_{21}) \\ y_{22} \end{bmatrix} = \begin{bmatrix} \tilde{W}_{01}\, W_{01} \\ \frac{1}{2}\left(\tilde{W}_{01}\, W_{02} + \tilde{W}_{02}\, W_{01} \right) \\ \tilde{W}_{02}\, W_{02} \end{bmatrix}$$

$$= 2 \cdot \frac{1}{2} \begin{bmatrix} \tilde{W}_{01} & 0 \\ \frac{1}{2}\tilde{W}_{02} & \frac{1}{2}\tilde{W}_{01} \\ 0 & \tilde{W}_{02} \end{bmatrix} \begin{bmatrix} W_{01} \\ W_{02} \end{bmatrix} = 2\, G_{(\tilde{z})}\, \mathbb{W}$$

$$= 2\, G_{(\tilde{z})}\, (\mathbb{P}N)\, z \qquad\qquad\qquad (4.80)$$

$$G_{(\tilde{z})} = \frac{1}{2} \begin{bmatrix} \tilde{W}_{01} & 0 \\ \frac{1}{2}\tilde{W}_{02} & \frac{1}{2}\tilde{W}_{01} \\ 0 & \tilde{W}_{02} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} c_1^T(\mathbb{P}N)\tilde{z} & 0 \\ \frac{1}{2}c_2^T(\mathbb{P}N)\tilde{z} & \frac{1}{2}c_1^T(\mathbb{P}N)\tilde{z} \\ 0 & c_2^T(\mathbb{P}N)\tilde{z} \end{bmatrix}$$

$$
\varrho = \begin{Bmatrix} S_{11} \\ \frac{1}{2}(S_{12} + S_{21}) \\ S_{22} \end{Bmatrix} = \begin{Bmatrix} W_{0\,1:1} \\ \frac{1}{2}(W_{0\,1:2} + W_{0\,2:1}) \\ W_{0\,2:2} \end{Bmatrix} \tag{4.81}
$$

$$
= \nabla\!\!\!\!\nabla_K W = \nabla\!\!\!\!\nabla_K (\nabla\!\!\!\!\nabla_K U) = \nabla\!\!\!\!\nabla_B U = (\nabla\!\!\!\!\nabla_B N) Z .
$$

With these definitions and relations the mathematical analogy with the derivations in chapter (3.2) allows to present the strain energy without a detailed derivation in the following form

$$
\mathcal{U} = \frac{1}{2}\Big\{ \Theta^T H \Theta
$$

$$
- \varepsilon \big[\tilde{\Theta}^T H \Theta + \Theta^T H \breve{\Theta} \big]
$$

$$
+ \varepsilon \big[y^T H \Theta + \Theta^T H y \big] \tag{4.82}
$$

$$
+ \frac{\lambda^2}{12}\Big(\varphi^T H \varphi
$$

$$
- \varepsilon \big[\tilde{\varphi}^T H \varphi + \varphi^T H \varphi \big] \Big)\Big\} .
$$

Here the matrix H is the same as (3.36) since the components $H_o^{s\alpha\gamma\beta}$ and $H^{s\alpha\gamma\beta}$ refer to the same configuration $\mathcal{K}_o$.

Substitution of the relations (4.78) to (4.81) in (4.82) gives

$$\Theta^T H \Theta = z^T (\nabla N)^T H (\nabla N) z$$

$$\tilde{\Theta}^T H \Theta = \tilde{z}^T (\nabla N)^T H (\nabla N) z$$

$$\Theta^T H \tilde{\Theta} = z^T (\nabla N)^T H (\nabla N) \tilde{z}$$

$$\psi^T H \Theta = 2 z^T (\nabla N)^T G_{(\tilde{z})} H (\nabla N) z$$

$$\Theta^T H \psi = 2 z^T (\nabla N)^T H G_{(\tilde{z})} (\nabla N) z$$

$$\varphi^T H \varphi = z^T (\nabla N)^T H (\nabla N) z$$

$$\tilde{\varphi}^T H \varphi = \tilde{z}^T (\nabla N)^T H (\nabla N) z$$

$$\varphi^T H \tilde{\varphi} = z^T (\nabla N)^T H (\nabla N) \tilde{z}$$

$$\left.\right\} \quad (4.83)$$

The integration (4.62) is done in the same two-dimensional space as that in (3.42). Consequently we may apply the definition (3.43) for the formal integration of α (4.62) when the relations (4.83) are observed. The final result is with $H = H^T$

$$\tilde{\Pi}_{1\alpha} = \frac{1}{2} \left\{ z^T D_{(0,0)} z + z^T B_{(0,0)} z \right.$$
$$- 2\varepsilon z^T D_{(0,0)} \tilde{z} - 2\varepsilon z^T B_{(0,0)} \tilde{z}$$
$$\left. + 2\varepsilon z^T D_{(\tilde{z},0)} z + 2\varepsilon z^T D_{(0,\tilde{z})} z \right\} .$$

$$(4.84)$$

The potential of the hydrostatic pressure is given by (4.69). The corresponding matrix representation may be derived on the basis of definitions used in chapter (3). We get[*] observing (3.47), (3.48), (3.50) and (3.54)

$$
\begin{aligned}
& \underset{0}{W} = z^T N^T \underset{3}{S} \quad , \quad \underset{0}{\tilde{W}} = \tilde{z}^T N^T \underset{3}{S} \\
& \underset{0}{V^\alpha_{;\alpha}} = \mathcal{E}^T (\mathbb{P} N) z \quad , \quad \underset{0}{\tilde{V}^\alpha_{;\alpha}} = \mathcal{E}^T (\tilde{\mathbb{P}} N) \tilde{z} \\
& \underset{0}{W} \underset{0}{B^\alpha_\alpha} \underset{0}{\tilde{W}} = z^T (N^T_3 \underset{0}{B^\alpha_\alpha} S^T N) \tilde{z} \\
& \underset{0}{V_\alpha} \underset{0}{B^{\alpha\beta}} \underset{0}{\tilde{V}_\beta} = z^T (N^T K N) \tilde{z}
\end{aligned}
\right\} \quad (4.85)
$$

Upon using this and (3.55), we may write the hydrostatic potential (4.69) after integration also as

$$
\underset{\wedge P}{\overline{\Pi}_P} = \ell p \left\{ z^T p + \mathcal{E} z^T (\mathbb{E} + \mathbb{E}^T) \tilde{z} - \mathcal{E} z^T \overset{*}{K} \tilde{z} \right\}, \quad (4.86)
$$

The potential $\overline{\Pi}_C$ of the boundary loading is transformed similarly to the analysis in chapter (3.4). With the same definition as in (3.58) and observing (4.77) we obtain from (4.74) finally

$$
\underset{\wedge C}{\overline{\Pi}_C} = -\ell t^T z + \mathcal{E} \ell t^T \tilde{z}, \quad (4.87)
$$

Note, that the second term is constant and thus may be dropped from the potential energy functional.

The results (4.84), (4.86) and (4.87) imply that the energy functional (4.1) of the imperfect shell has the following matrix representation

---

[*] It should be noted that the coefficients $\underset{0}{B_{\alpha\rho}}$ are the same as $B_{\alpha\beta}$

$$\Pi_1 = \frac{1}{2} \Big\{ z^T D_{(0,0)} z - 2\varepsilon z^T D_{(0,0)} \tilde{z}$$

$$+ 2\varepsilon z^T D_{(\tilde{z},0)} z + 2\varepsilon z^T D_{(0,\bar{z})} z$$

$$+ z^T B_{(0,0)} z - 2\varepsilon z^T B_{(0,0)} \tilde{z} \Big\}$$

$$+ \ell\rho \Big\{ z^T p + \varepsilon z^T (E + E^T)\tilde{z} - \varepsilon z^T \overset{*}{K} \tilde{z} \Big\}$$

$$- \ell H^T (z - \varepsilon \tilde{z}) \qquad\qquad (4.88)$$

$$= \frac{1}{2} z^T \big[D_{(0,0)} + 2\varepsilon D_{(0,\tilde{z})} + 2\varepsilon D_{(\tilde{z},0)} + B_{(0,0)} \big] z$$

$$- \varepsilon z^T \big[D_{(0,0)} + B_{(0,0)} \big] \tilde{z}$$

$$+ \ell\rho \Big\{ z^T p - \varepsilon z^T \overset{*}{K} \tilde{z} + \varepsilon z^T (E + E^T)\tilde{z} \Big\}$$

$$- \ell H^T (z - \varepsilon \tilde{z})$$

5. General Solution of the Linearized Shell Problem with and without Initial
Imperfections on the Basis of the Matrix Variational Formulation

According to the strategy described in chapter (1) two linearized problems
have to be solved:

(a) the perfect shell under a prescribed loading
(b) the imperfect shell under the same loading but acting on the
imperfect structure.

The equilibrium state for these two cases may be obtained from the condi-
tion that the first variation of the energy functional $\overline{\overline{\Pi}}_1$ with respect
to $\mathbb{Z}$ should vanish, i.e.

$$\delta \overline{\overline{\Pi}}_1 = 0 \;, \qquad \forall \delta \overset{\rho}{\mathbb{Z}} \;. \tag{5.1}$$

$\overline{\overline{\Pi}}_1$ was derived for the imperfect shell but naturally the case of the per-
fect shell is included by simply putting $\mathcal{E} = 0$.

In the following we will distinguish between a pure hydrostatic pres-
sure loading and pure boundary loading.

5.1 Solutions for the Pure Hydrostatic Pressure Loading

The first variation of (4.88) gives

$$\delta \overline{\overline{\Pi}}_1 = \delta \mathbb{Z}^T \left\{ \mathbb{D}_{(0,0)} \mathbb{Z} \quad - \mathcal{E} \, \mathbb{D}_{(0,0)} \tilde{\mathbb{Z}} \right.$$
$$+ 2\mathcal{E} \, \mathbb{D}_{(\tilde{\mathbb{Z}},0)} \mathbb{Z} + 2\mathcal{E} \, \mathbb{D}_{(0,\tilde{\mathbb{Z}})} \mathbb{Z}$$
$$\left. + \mathbb{B}_{(0,0)} \mathbb{Z} - \mathcal{E} \, \mathbb{B}_{(0,0)} \tilde{\mathbb{Z}} \right\} \tag{5.2}$$
$$+ \ell p \, \delta \mathbb{Z}^T \left\{ p + \mathcal{E} (\mathbb{E} + \mathbb{E}') \tilde{\mathbb{Z}} - \mathcal{E} \, \overset{*}{\mathbb{K}} \tilde{\mathbb{Z}} \right\}$$

where

$$\mathbb{D}^T_{(\tilde{\mathbb{Z}},0)} = \mathbb{D}_{(0,\tilde{\mathbb{Z}})} \quad , \quad \mathbb{D}'_{(0,\tilde{\mathbb{Z}})} = \mathbb{D}_{(\tilde{\mathbb{Z}},0)}$$

is used. The requirement (5.1) has to be satisfied for all variations $\partial \mathbf{z}$.
The necessary and sufficient condition for this requirement is

$$
\begin{aligned}
\big[\mathbb{D}_{(0,0)} + \mathbb{B}_{(0,0)} &+ 2\varepsilon \big(\mathbb{D}_{(\tilde{z},0)} + \mathbb{D}_{(0,\tilde{z})} \big) \big] \mathbf{z} \\
&= -\ell p \big\{ \mathbf{p} + \varepsilon \big(\mathbb{E} + \mathbb{E}^T \big) \tilde{z} - \varepsilon \overset{*}{\mathbb{K}} \tilde{z} \big\} \\
&+ \varepsilon \big(\mathbb{D}_{(0,0)} + \mathbb{B}_{(0,0)} \big) \tilde{z} \, .
\end{aligned}
\tag{5.3}
$$

This is a system of linear inhomogeneous equations for the components of
the column matrix $\mathbf{z}$ which characterize the equilibrium state of the
shell. For the perfect shell we have $\mathbf{\mathcal{E}} = 0$; thus (5.3) reduces to

$$
\big[\mathbb{D}_{(0,0)} + \mathbb{B}_{(0,0)} \big] \mathbf{z} = -\ell p \, \mathbf{p} \, ;
$$

the corresponding solution is

$$
\underset{p}{\mathbf{z}} = -\ell p \big[\mathbb{D}_{(0,0)} + \mathbb{B}_{(0,0)} \big]^{-1} \mathbf{p} \, .
$$

Introduction of the vector $\overset{\circ}{\mathbf{z}}$ [*]

$$
\overset{\circ}{\mathbf{z}} := -\big[\mathbb{D}_{(0,0)} + \mathbb{B}_{(0,0)} \big] \mathbf{p}
\tag{5.4}
$$

allows to write the solution for the perfect shell in the following form

$$
\underset{p}{\mathbf{z}} = \ell p \, \overset{\circ}{\mathbf{z}} \, .
\tag{5.5}
$$

For the imperfect shell $\big(\mathcal{E} \neq 0 \big)$ the solution of (5.3) is a nonlinear
function of $\mathcal{E}$. Since we are only interested in the linear dependence
on $\mathcal{E}$ we derive an approximate solution for small $\mathcal{E}$. We put

$$
\mathbb{A} := \mathbb{D}_{(0,0)} + \mathbb{B}_{(0,0)}
$$

$$
\mathbb{F} := 2 \big(\mathbb{D}_{(\tilde{z},0)} + \mathbb{D}_{(0,\tilde{z})} \big) \, ;
$$

---

[*] This is the solution for the perfect shell for unit scale factor ℓ
   and unit pressure p

then we may derive

$$(\mathcal{A} + \varepsilon \mathcal{F})^{-1} \simeq \mathcal{A}^{-1} - \varepsilon \mathcal{A}^{-1} \mathcal{F} \mathcal{A}^{-1} ;$$

including terms linear in ε only the solution is

$$\underset{im}{Z} = \varepsilon \tilde{Z} - \ell p \left(\mathcal{D}_{(0,0)} + \mathcal{B}_{(0,0)} \right)^{-1} p$$

$$+ \varepsilon \ell p \left\{ 2 \left[\mathcal{D}_{(0,0)} + \mathcal{B}_{(0,0)} \right]^{-1} \left[\mathcal{D}_{(\tilde{z},0)} + \mathcal{D}_{(0,\tilde{z})} \right] \left[\mathcal{D}_{(0,0)} + \mathcal{B}_{(0,0)} \right]^{-1} p \right.$$

$$\left. + \left[\mathcal{D}_{(0,0)} + \mathcal{B}_{(0,0)} \right]^{-1} \left[\overset{*}{K} - (\mathcal{E} + \mathcal{E}^{T}) \right] \tilde{z} \right\} . \qquad (5.6)$$

Substituting the solution for the perfect shell, equ. (5.5) reduces to

$$\underset{im}{Z} = \varepsilon \tilde{Z} + \ell p \overset{0}{Z}$$

$$- \varepsilon \left\{ 2 \left[\mathcal{D}_{(0,0)} + \mathcal{B}_{(0,0)} \right]^{-1} \left[\mathcal{D}_{(\tilde{z},0)} + \mathcal{D}_{(0,\tilde{z})} \right] \ell p \overset{0}{Z} \right. \qquad (5.7)$$

$$\left. + \ell p \left[\mathcal{D}_{(0,0)} + \mathcal{B}_{(0,0)} \right]^{-1} \left[\mathcal{E} + \mathcal{E}^{T} - \overset{*}{K} \right] \tilde{z} \right\}$$

This result has a similar mathematical structure as (1.6), the conjectured one dimensional (scalar) relation. The first term on the right of (5.7) represents the imperfection measured with respect to the perfect configuration, the second term is the displacement of the perfect structure due to the pressure loading and the third term, linear in the measure ε of the imperfection and linear in the pressure p is the essential term which should contain the information needed to predict the buckling load. This question will be analysed in chapter (6).

5.2 Solution for the Pure Boundary Loading Case

The first variation of (4.88) with respect to z for vanishing hydrostatic pressure p gives

$$[D_{(0,0)} + B_{(0,0)} + 2\varepsilon(D_{(\tilde{z},0)} + D_{(0,\tilde{z})})]z$$

$$= \lambda t + \varepsilon(D_{(0,0)} + B_{(0,0)})\tilde{z} \ . \tag{5.8}$$

We introduce a load factor τ such that

$$t = \tau \underset{\sim}{t} \ . \tag{5.9}$$

Then the solution of (5.8) for the perfect shell $\varepsilon = 0$ is

$$\underset{p}{z} = \lambda\tau[D_{(0,0)} + B_{(0,0)}]^{-1}\underset{\sim}{t} \ , \tag{5.10}$$

With

$$\overset{o}{z}_{o} = [D_{(0,0)} + B_{(0,0)}]^{-1}\underset{\sim}{t} \tag{5.11}$$

we may write for (5.10)

$$\underset{p}{z} = \lambda\tau\overset{o}{z} \ . \tag{5.12}$$

With the same argumentation as in chapter (5.1) the solution for the imperfect shell including only terms linear in ε is

$$\underset{im}{z} = \varepsilon\tilde{z} + \lambda\tau[D_{(0,0)} + B_{(0,0)}]^{-1}\underset{\sim}{t}$$

$$-\varepsilon\lambda\tau\{2[D_{(0,0)} + B_{(0,0)}]^{-1}[D_{(\tilde{z},0)} + D_{(0,\tilde{z})}][D_{(0,0)} + B_{(0,0)}]^{-1}\underset{\sim}{t}$$

$$\tag{5.13}$$

$$= \varepsilon\tilde{z} + \lambda\tau\overset{o}{z}$$

$$-\varepsilon\{2[D_{(0,0)} + B_{(0,0)}]^{-1}[D_{(\tilde{z},0)} + D_{(0,\tilde{z})}]\lambda\tau\overset{o}{z}\}$$

This expression has a completely analogous structure as (5.7) except for
the last term on the right of (5.7).

6. Definition of a Suitable Norm as a Measure for the Critical Load and Comparison with the Rayleigh Quotient of the Actual Stability Problem

The conjecture described in chapter (1) involves scalar relations since we restricted our attention to buckling problems which are characterized by a single load factor. These scalar relations were immediately applicable to define a parameter which could be related to the critical load. However, the solutions for the perfect and imperfect shells are matrix relations. For the purpose of comparison with the scalar Rayleigh quotient a suitable scalar quantity, a norm, has to be defined involving the various column matrixes $\underset{lm}{Z}$, $\tilde{Z}$, $\overset{o}{Z}$ etc. On the basis of the solutions (5.7) or (5.13) this norm may then be brought into a form which allows a comparison with the inverse Rayleigh quotient (3.110) or (3.117). In this course the column matrix $\tilde{Z}$, which contains the weighting factors for the shape functions of the imperfection, must be the same as one of the eigensolutions of the actual buckling problem, i.e.

$$\tilde{Z} = V_i \; , \tag{6.1}$$

preferably the one which corresponds to the lowest critical load.

At first we consider the pure hydrostatic pressure loading. In analogy to (1.7) we calculate the expression $(\underset{lm}{Z} - \varepsilon \tilde{Z} - \ell p \overset{o}{Z})$ which is according to the linearized solution (5.7)

$$\underset{lm}{Z} - \varepsilon \tilde{Z} - \ell p \overset{o}{Z} =$$

$$= - \varepsilon \ell p \left\{ \ell \left[D_{(0,0)} + B_{(0,0)} \right] \left[D_{(\tilde{z},0)} + D_{(0,\tilde{z})} \right]^{-1} \overset{o}{Z} \right. \tag{6.2}$$

$$\left. + \left[D_{(0,0)} + B_{(0,0)} \right]^{-1} \left[E + E^\mathsf{T} - \overset{*}{K} \right] \tilde{z} \right\} ,$$

Applying the theorem (3.69) - the exchange rule - and the relation (3.97) in (6.2), we therefore write

$$\underset{lm}{Z} - \varepsilon \tilde{Z} - \ell p \overset{o}{Z} =$$

$$= - \varepsilon \ell p \left\{ \left[D_{(0,0)} + B_{(0,0)} \right]^{-1} \left[\ell \overset{*}{D}_{(\overset{o}{z})} + \ell D_{(0,\overset{o}{z})} \right. \right.$$

$$\left. \left. + E + E^\mathsf{T} - \overset{*}{K} \right] \tilde{Z} \right\} . \tag{6.3}$$

Guided by the mathematical structure of the inverse Rayleigh quotient (3.110) we introduce a fictitious load matrix

$$\mathbb{P} := [\mathbb{D}_{(0,0)} + \mathbb{B}_{(0,0)}] \tilde{z} \; ; \tag{6.4}$$

thus

$$\tilde{z} = [\mathbb{D}_{(0,0)} + \mathbb{B}_{(0,0)}]^{-1} \mathbb{P} \; . \tag{6.5}$$

Scalar multiplication of (6.3) with $\mathbb{P}$ gives

$$\mathbb{P}^T (\underset{\sim}{z} - \varepsilon \tilde{z} - \ell \rho \overset{o}{z}) =$$

$$= - \varepsilon \ell \rho \left\{ \tilde{z}^T [2 \overset{*}{\mathbb{D}}_{(\overset{o}{z})} + 2 \mathbb{D}_{(0,\tilde{z})} \right.$$

$$+ \mathbb{E} + \mathbb{E}^T - \overset{*}{\mathbb{K}}] \tilde{z} \left. \right\} . \tag{6.6}$$

Since

$$\tilde{z}^T \mathbb{D}_{(0,\overset{o}{z})} \tilde{z} = \left(\tilde{z}^T \mathbb{D}_{(0,\overset{o}{z})} \tilde{z} \right)^T = \tilde{z}^T \mathbb{D}^T_{(0,\overset{o}{z})} \tilde{z}$$

$$= \tilde{z}^T \mathbb{D}_{(\overset{o}{z},0)} \tilde{z} \; , \tag{6.7}$$

relation (6.6) may be written in a form which involves only symmetric matrices

$$\mathbb{P}^T (\underset{\sim}{z} - \varepsilon \tilde{z} - \ell \rho \overset{o}{z}) =$$

$$= - \varepsilon \ell \rho \left\{ \tilde{z}^T [2 \overset{*}{\mathbb{D}}_{(\overset{o}{z})} + (\mathbb{D}_{(0,\overset{o}{z})} + \mathbb{D}_{(\overset{o}{z},0)}) \right. \tag{6.8}$$

$$+ (\mathbb{E} + \mathbb{E}^T) - \overset{*}{\mathbb{K}}] \tilde{z} \left. \right\} .$$

An expression analogous to (1.7) is obtained if we define the parameter as

$$\beta := \frac{\mathbb{P}^T(\underset{in}{\mathbb{Z}} - \varepsilon \tilde{Z} - \ell p \overset{o}{Z})}{+ \ell p \, \varepsilon \, \mathbb{P}^T \tilde{Z}} \, . \tag{6.9}$$

With the definition (6.4) and the result (6.8) we may transform the right side to read

$$\beta = - \frac{\tilde{Z}^T [2 \overset{*}{\mathbb{D}}_{(\overset{o}{Z})} + \mathbb{D}_{(0,\overset{o}{Z})} + \mathbb{D}_{(\overset{o}{Z},0)} + \mathbb{E} + \mathbb{E}^T - \overset{*}{\mathbb{K}}] \tilde{Z}}{\tilde{Z}^T [\mathbb{D}_{(0,0)} + \mathbb{B}_{(0,0)}] \tilde{Z}} \tag{6.10}$$

An analogous expression can be derived for the case of dead loading along the boundary of the shell. Following the same procedure but using solution (5.13) instead of (5.7) we obtain

$$\beta := \frac{\mathbb{P}^T(\underset{in}{\mathbb{Z}} - \varepsilon \tilde{Z} - \ell \tau \overset{o}{Z})}{+ \ell \tau \, \mathbb{P}^T \tilde{Z}}$$

$$\tag{6.11}$$

$$= - \frac{\tilde{Z}^T [2 \overset{*}{\mathbb{D}}_{(\overset{o}{Z})} + \mathbb{D}_{(0,\overset{o}{Z})} + \mathbb{D}_{(\overset{o}{Z},0)}] \tilde{Z}}{\tilde{Z}^T [\mathbb{D}_{(0,0)} + \mathbb{B}_{(0,0)}] \tilde{Z}}$$

where $\overset{o}{Z}$ is defined by (5.11).

We are now in the position to compare the scalar quantities β equ. (6.10) and (6.11), with the associated inverse critical loads (3.110) and (3.117), respectively, if we remember that (6.1) should hold.[*]

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[*] It should be kept in mind that the solutions $\overset{o}{Z}$ as well as the eigen-solutions V_i are different for the case of pure hydrostatic pressure and dead loading along the boundary.

For the case of <u>hydrostatic pressure loading</u> we obtain

$$\beta_i = - \frac{V_i^T [D_{(0,\tilde{z})} + D_{(\tilde{z},0)} + 2 \overset{*}{D}_{(\tilde{z})} + E + E^T - \overset{*}{K}] V_i}{V_i^T [D_{(0,0)} + B_{(0,0)}] V_i} \qquad (6.12)$$

and for <u>dead loading along the boundary</u>[*]

$$\beta_i = - \frac{V_i^T [D_{(0,\tilde{z})} + D_{(\tilde{z},0)} + 2 \overset{*}{D}_{(\tilde{z})}] V_i}{V_i^T [D_{(0,0)} + B_{(0,0)}] V_i} \ , \qquad (6.13)$$

Comparison of (6.12) and (6.13) with the corresponding reciprocals of the critical load factors, equ. (3.110) and (3.117) respectively, shows similarity to a large extent, but nevertheless it is obvious that there is no exact agreement for both loading situations, e.g.

$$\left(\beta_i \right)_{equ. (6.12)} \neq \frac{1}{\ell \, p_i} \ . \qquad (6.14)$$

Thus, even if the imperfection $\tilde{z}$ is chosen identical to one of the eigensolutions of the actual stability problem, the parameter β does not represent the reciprocal of the critical load factor corresponding to the eigensolution. The difference is solely to be seen in the factor (2) of the symmetric matrix $[D_{(0,\tilde{z})} + D_{(\tilde{z},0)}]$.

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[*] Except for the scale factor ℓ .

7. Discussion and Conclusions

At first sight the difference between the reciprocal of the critical load and the β-parameter - solely due to the factor (2) in one of the terms - appears to be trivial error in one of the equations. However, the analysis has been checked several times but no error was found. If we accept the results obtained so far the conclusion is that in general the β-parameter approach will not give results which represent the actual critical load.

On the other hand at least in some applications to simple stability problems (chapter (1)) the β-parameter approach has been shown to be successful; but it should be noted that those analyses were not based on the equations presented in this report. Thus, there exists a restricted class of stability problems where the β-parameter approach gives exact predictions. This suggests to ask for those conditions under which agreement can be obtained. In the following this question will be discussed to some extent.

At first we analyse a few formal aspects. Comparing the reciprocal of the critical load factors (3.110) and (3.117) with the β-parameter (6.12) and (6.13) respectively, it is obvious that agreement is assured if for some column matrix

$$\mathbb{D}_{(0, \overset{o}{z})} = \mathbb{O}$$

which implies

$$\mathbb{D}^T_{(0, \overset{o}{z})} = \mathbb{D}_{(\overset{o}{z}, 0)} = \mathbb{O},$$

The matrix $\mathbb{D}_{(0, \overset{o}{z})}$ depends linearly on $\overset{v}{Z}$ and its elements may be represented as follows

$$\mathbb{D}_{(0, \overset{o}{z})} = \begin{pmatrix} d^T_{11} \overset{o}{z} & d^T_{12} \overset{o}{z} & \cdots \\ d^T_{21} \overset{o}{z} & d^T_{22} \overset{o}{z} & \cdots \\ \vdots & \vdots & \vdots \end{pmatrix}.$$

The vanishing of all elements of $\mathbb{D}_{(0,\dot{z})}$ implies that either

- $d_{mn} \neq 0$ but $d_{mn} \overset{o}{z} = 0$ for all m, n (orthogonality)

or

- $d_{mn} = 0$ for all m and n

or a suitable combination.

For the second case it may be shown that then $\overset{*}{\mathbb{D}}_{(\dot{z})} = 0$. This implies a "degeneration" of the linearized eigenvalue problem (3.109) or (3.116); here the quadratic matrices $\mathbb{D}_{(\dot{z},\dot{z})}$ and $\overset{*}{\mathbb{D}}_{(\dot{z},\dot{z})}$ should be included in the eigenvalue problem. But then the β-parameter approach would be inapplicable anyhow.

However, in the first and third case it cannot be shown that the matrix $\overset{*}{\mathbb{D}}_{(\dot{z})}$ necessarily vanishes in conjunction with $\mathbb{D}_{(0,\dot{z})}$. This appears to be a peculiar situation whose physical implications need to be explored.

Further, agreement between the reciprocal of the critical load factor and the β-parameter may be obtained if an orthogonality condition is satisfied, i.e.

$$V_i^T \mathbb{D}_{(0,\dot{z})} V_i = V_i^T \mathbb{D}_{(\dot{z},0)} V_i = 0$$

such that the mapping $\mathbb{D}_{(0,\dot{z})} V_i$ gives a nonzero column matrix W_i which is orthogonal to V_i

$$V_i^T W = 0 \quad , \quad W_i = \mathbb{D}_{(0,\dot{z})} V_i \; .$$

Again, this appears to be a very special situation and it remains to be analysed whether reasonable physical conditions can be attached to it.

We conclude the discussion of formal aspects by commenting the case that solely the term $V_i^T \overset{*}{\mathbb{D}}_{(\dot{z})} V_i$ vanishes. Then the reciprocal of the critical load factor and the β-parameter agree except for a factor of (2). On the first sight one may suggest to redefine the β-parameter such that complete agreement is obtained. However, such a formal approach is not admissible since the analogy between the new β-parameter and (1.7) is lost.

A further remark is made concerning the discrepancy between the general result derived and the observation made for several specific stability problems. The agreement obtained for the three simple stability problems (chapter (1)) suggests to ask for a common property of the structural

models used. Here it should be pointed out that the axes of the rod or the middle surface of the plate or shell was assumed to be inextensible. Definitely, this assumption is not implied in the two general analyses presented in this study. Therefore, further attention should be put to this property as a possible explanation and as a first step in this direction it is recommended to apply the derived mathematical formalism to one of those simple problems.

The results presented so far do not exclude the possibility that the simplified strategy will give reasonable approximate predictions at least for a limited class of stability problems. This question remains a subject for further analyses.

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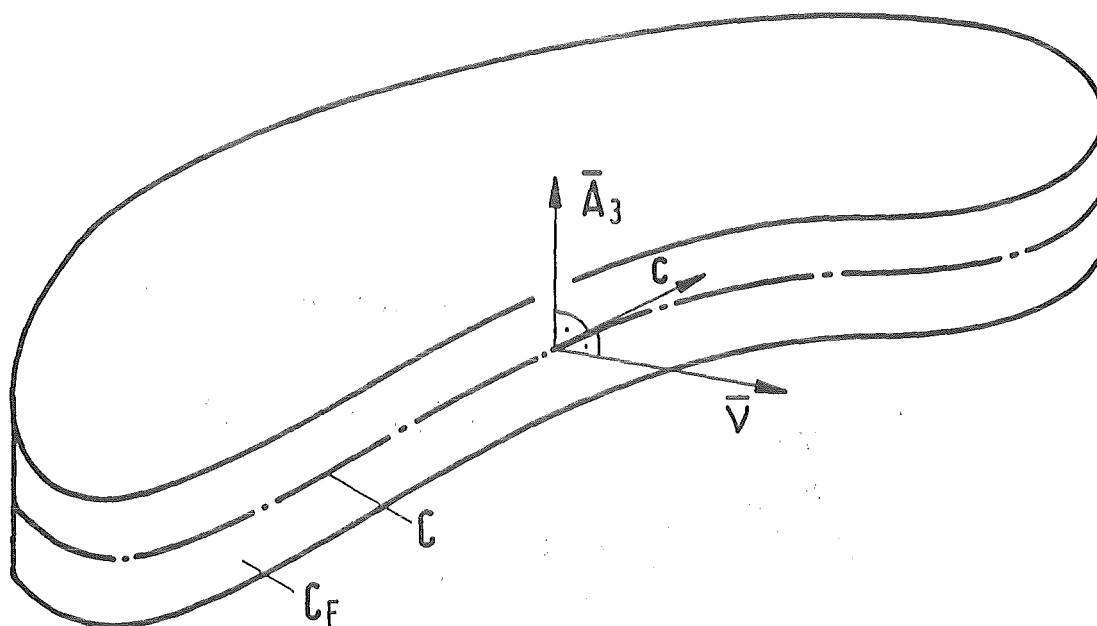
C_F: Boundary strip   c : arc length along C

C : Boundary curve   v : arc length along $\bar{v}$

$\bar{A}_3, \bar{c}, \bar{v}$: Unit vectors

Fig. 1:   Coordiante system along the boundary of the shell
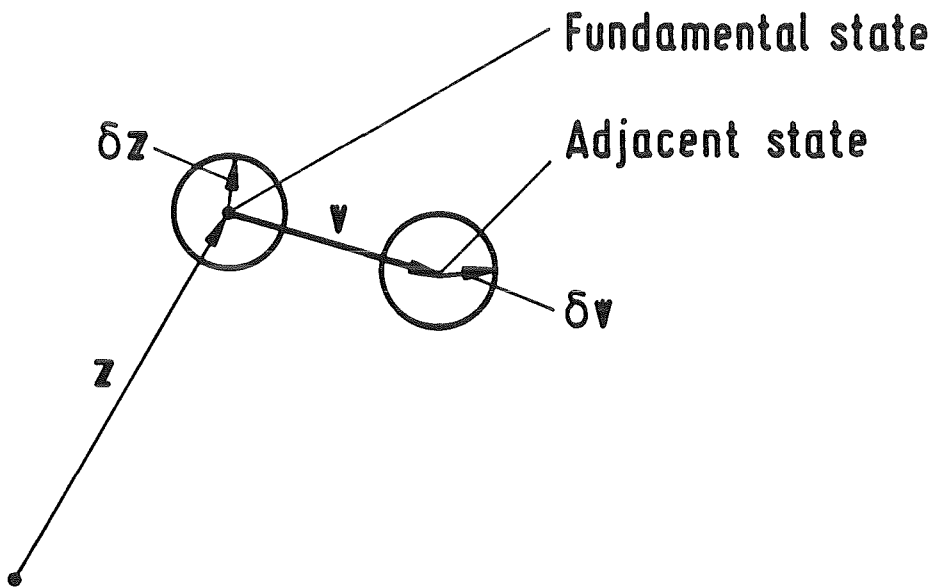
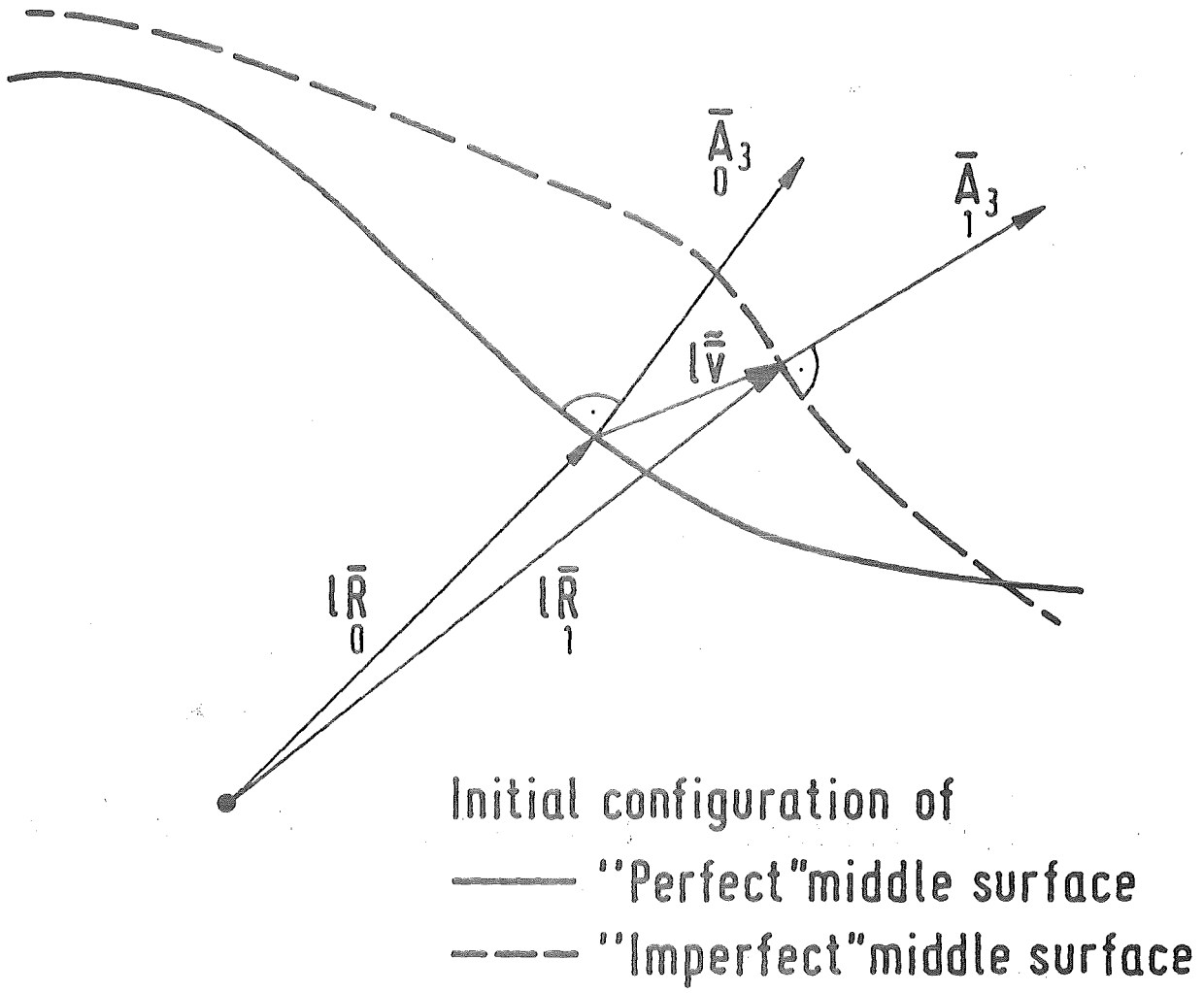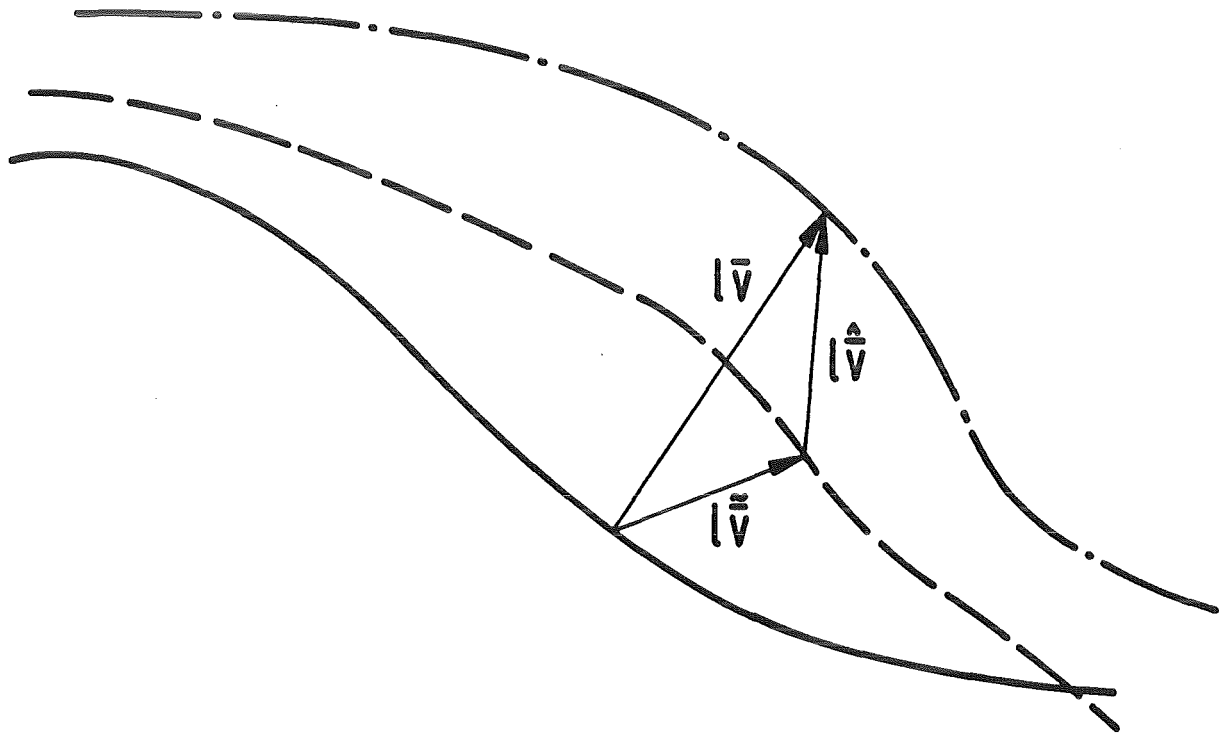Fig. 2: Variations of the fundamental and adjacent equilibrium
in the state space

Fig. 3: "Perfect" and "imperfect" middle surface

Fig. 4: Middle surface configurations and associated displacement vectors