KfK 4635
Januar 1990

An Algebraic Characterization of a Class of Petri Nets

W. Korczynski
Institut für Datenverarbeitung in der Technik

Kernforschungszentrum Karlsruhe

KERNFORSCHUNGSZENTRUM KARLSRUHE
Institut für Datenverarbeitung in der Technik

KfK 4635

An Algebraic Characterization of a Class of Petri Nets

W. Korczynski *)

*) Universität Warschau, Fil. Bialystok

KERNFORSCHUNGSZENTRUM KARLSRUHE GMBH, KARLSRUHE

Kernforschungszentrum Karlsruhe GmbH
Postfach 3640, 7500 Karlsruhe 1

ISSN 0303-4003

An Algebraic Characterization of a Class of Petri Nets

Abstract

In this paper an algebraic characterization of a class of Petri nets is given. The nets are characterized by a kind of algebras, which can be considered as a generalization of the concept of the reachability graph of a (marked) Petri net.

Eine algebraische Charakterisation einer Klasse von Petri Netzen

Zusammenfassung

In der Arbeit wird eine algebraische Charakterisierung einer Klasse von Petri Netzen beschrieben. Die Netze werden durch Algebren, die als eine Verallgemeinerung des Begriffs des Erreichbarkeitsgraphs betrachtet werden können, charakterisiert.

Introduction

Petri nets are one of the most convenient models of various concurrent systems. They allow for describing many important dynamic properties of such systems (liveness, safety, deadlocks etc.) by means of relatively simple mathematical tools. There are many various equivalent definitions of Petri nets (see [GeLaTh] for example). Most of them define a Petri net as a relational system (usually as a digraph) with some special properties. This (graphtheoretic) approach seems to be very reasonable for an easy explanation of many important properties of systems but it is not convenient if properties of a class of concurrent systems are considered. The problem here is that in such case one works normally with a kind of "morphisms" i.e. functions which respect or reflect some important properties of objects under consideration (here concurrent systems). The consideration of functions between two relational systems is often relatively difficult and will be always algebraized. The algebraization allows also using well known methods of algebra, which are often more efficient than the methods used normally for studying of relational systems. The "classical" examples of this efficiency can be found in the algebraic topology. There are many various approaches to an "algebraization" of Petri nets. In the approach presented by Meseguer and Montanari [MesMo1] Petri nets are defined as some special art of graphs with an additional structure on the set of vertices. This definition allows to describe a category of Petri nets as a category of graphs *treated as two sorted algebras* in which the second coordinate of a morphism is a monoid homomorphism. In Winskel's theory of events structures, Petri nets determine some special art of two sorted "algebras on multisets". In this approach a Petri net determines an algebra, but the net itself is not determined by any axiomatic defined algebra. The main goal of this algebraization here is a description of some algebraic construction (product and coproduct) on Petri nets.

One of the first "algebraizations" has been proposed by Winkowski (see [Win1], [Win2], [Win3] and [Win4]). In this approach an algebra of processes in a concurrent system had been examined. An axiomatic characterization of this algebra has been given in [Kor1]. In this paper a class of processes in a C/E system has been characterized by means of a relatively simple set of axioms. This characterization has been improved in [Win4]. In all these papers a concurrent system has been characterized as a kind of "monoid with an additional graph - like structure". This graph structure is here understood as a *one sorted* algebra satisfying the well known equations. This approach allows for an "unified" considering places and transitions in a net. The elements of algebras determined by (and determining) Petri nets correspond to the set of elements (i.e. arrows and vertices) of the reachability graph of a Petri net. The same principle will be explored in this paper. Petri nets have been characterized by some algebras, called algebras of of presteps, which are "graphs over free monoids generated by elements (i.e. places and transitions) of a net". In this paper nets without marking any are considered; one can say only a kind of "topological" properties of nets will be characterized in an algebraic way.

It is assumed the reader will be familiar with elementary notions of the theory of Petri nets and abstract algebra. For further study the reader is requested to [Rei1] or [Sta1].

In the paper the standard mathematical notation and terminology is used. The only exeption is that the support set of a relational system (structured set) **A** is denoted sometimes by the symbol /**A**/.

1 Petri Nets

In this section some elementary notions of the theory of Petri nets concerning the sequel will be described. The following is provided as a preamble for subsequent sections.

Definition 1.1.

By a Petri Net it is meant any quadruple

$$N = (P , T , pre , post)$$

with P and T being disjoint sets and pre and post being functions:

$$pre , post: T \times P \longrightarrow Nat$$

For every Petri net N = (P , T , pre , post) its coordinates will be denoted by:

P(N) = P , the set of *places* of the net N

T(N) = T , the set of *transitions* of the net N

pre(N) = pre , the *precondition function* of N

post(N) = post , the *postcondition function* of N.

The following denotation and terminology are used. Let N = (P , T , pre , post) be a Petri net.

Definition 1.2.

a) The function $F : P \times T \cup T \times P \longrightarrow Nat$ defined by the formula

$$F(x,y)=\begin{cases} post(x,y) & \text{if } (x,y)\in T\times P \\ pre(y,x) & \text{if } (x,y)\in P\times T \end{cases}$$

will be called the flow function of the net N and denoted by F(N).

b) The set $P \cup T$ will be called the set of elements of the net N and denoted by X(N).

c) For every element $x \in P \cup T$ the functions

$$x \mapsto {}^{\bullet}x \quad \text{and} \quad x \mapsto x^{\bullet}$$

defined by the formulae:

$${}^{\bullet}x = \{y \in T \cup P \mid F(x,y) \neq 0\} \quad \text{and} \quad x^{\bullet} = \{y \in T \cup P \mid F(y,x) \neq 0\}$$

will be called the precedesor and the successor function of the net N respectively. They will be denoted by the same symbols in every Petri net.

d) For every element $x \in X(N)$ we define

$$d_o(x)=\begin{cases} {}^{\bullet}x & \text{if } x\in T(N) \\ \{x\} & \text{if } x\in P(N) \end{cases}$$

$$d_1(x)=\begin{cases} x^{\bullet} & \text{if } x\in T(N) \\ \{x\} & \text{if } x\in P(N) \end{cases}$$

$$\text{field}(x) = d_o(x) \cup d_1(x)$$

e) For a subset X_o of the set X(N)

$$D_o(X_o) = \bigcup_{x\in X_o} d_o(x) \quad \text{and} \quad D_1(X_o) = \bigcup_{x\in X_o} d_1(x)$$

f) For every $t \in T$ pre_t and $post_t$ are functions defined by the formulae

$$pre_t : P \longrightarrow Nat \ , \ pre_t(x) = pre(t,x)$$

$$post_t : P \longrightarrow Nat \ , \ post_t(x) = post(t,x)$$

In what follows only nets satisfying the following condition are considered:

(ext) $\forall_{x,y \in X(N)}\ {}^\bullet x = {}^\bullet y \ \&\ x^\bullet = y^\bullet \Rightarrow x = y$

2 Presteps in Petri nets

The notion of prestep which is defined in this section plays a fundamental role in consideration about Petri nets. Very loosely speaking one can say that "a prestep is a multiset of mutually independent elements of a Petri net". Here also some simple operations on presteps are defined and some of their properties are shown.

Definition 2.1.

By a prestep in the net **N** it is meant any multiset μ : X(N) --> Nat of the set of elements of the net N such that for every x,y$\in$X(N) it holds:

$$\mu(x) \neq 0 \neq \mu(y) \Rightarrow \text{field}(x) \cap \text{field}(y) = \emptyset.$$

The set of all presteps of a net **N** will be denoted by the symbol PS $_\nabla$(N). For technical purposes it is convenient to introduce a special "impossible prestep" which, generaly speaking, "represents all multisets on the set X(N) not satisfying the condition of the above definition". This prestep will be used to avoid a kind of "partiality" of an algebra of presteps of a Petri net.

Lemma 2.1a.

For every Petri net **N**, every prestep $\alpha \in$ PS(N) and every place p$\in$P(N) such that $\alpha(p) \neq 0$ there exists at most one transition t$\in$T(N) satisfying the condition:

(fpre) $\alpha(t) \neq 0 \ \&\ \text{pre}(t,p) \neq 0$

Proof This is an immediate consequence of the definition of presteps. (If for some transitions t,t'$\in$T(N) we would have pre(t,p) $\neq$ 0 and pre(t',p) $\neq$ 0 then p would be an element of the set field(t) $\cap$ field(t') with $\alpha(t) \neq 0 \neq \alpha(t')$ which contradicts the definition of prestep)

□

Lemma 2.1b.

For every Petri net **N**, every prestep $\alpha \in$ PS(N) and every place p$\in$P(N) such that $\alpha(p) \neq 0$ there exists at most one transition t$\in$T(N) satisfying the condition:

(fpost) $\alpha(t) \neq 0 \ \&\ \text{post}(t,p) \neq 0$

The proof is analogous to the proof of lemma 2.1a.

□

The above lemmas have a very simple interpretation. They guarantee that presteps are conflict-free i.e. no prestep includes two transitions which are in an in- or out-conflict. We define now some operations which make the set of presteps of a Petri net a graph i.e. the operation of the beginning and the end of a prestep.

Let $\alpha \in$ PS(N).

Definition 2.2.

(i) $\delta_o(\alpha) = T \times \{0\} \cup \{(p,n) \in P \times Nat : \exists_{t \in T}\ \alpha(t) \neq 0 \ \&\ n = \alpha(t)\text{pre}(t,p) \vee n = \alpha(p)\}$ for $\alpha \neq \nabla$

(ii) $\delta_1(\alpha) = T \times \{0\} \cup \{(p,n) \in P \times Nat : \exists_{t \in T}\ \alpha(t) \neq 0 \ \&\ n = \alpha(t)\text{post}(t,p) \vee n = \alpha(p)\}$ for $\alpha \neq \nabla$

(iii) $$\delta_o(\nabla) = \nabla = \delta_1(\nabla)$$

The interpretation of the operations defined above is very simple. The prestep $\delta_o(\alpha)$ is a function of the form

$$\delta_i(\alpha) : D_i(\mathrm{dom}\alpha) \longrightarrow \mathrm{Nat} \qquad (i = 0,1)$$

with

$$\delta_o(\alpha)(p) = \begin{cases} \alpha(t)pre(t,p) & \text{if } p \in {}^{\bullet}t \text{ for } a\ transition\ t \in dom\,\alpha \\ \alpha(p) & in\ other\ cases \end{cases}$$

and

$$\delta_1(\alpha)(p) = \begin{cases} \alpha(t)post(t,p) & \text{if } p \in t^{\bullet} \text{ for } a\ transition\ t \in dom\,\alpha \\ \alpha(p) & in\ other\ cases \end{cases}$$

One can say that a place p occurs in the prestep $\delta_o(\alpha)$ with the multiplicity α(t)pre(t,p) iff it is an input-place of a (according to the lemma 2.1a unique) transition t which occurs in this prestep with the multiplicity α(t) or with its "own" multiplicity α(p) if there is no such transition. Analogously, a place p occurs in the prestep $\delta_1(\alpha)$ with the multiplicity α(t)post(t,p) iff it is an output-place of a (according to the lemma 2.1b unique) transition t which occurs in this prestep with the multiplicity α(t) or with its "own" multiplicity α(p) if there is no such transition.

Lemma 2.2.

For every prestep $\alpha \in \mathbf{PS}(N)$ and every transition $t \in T(N)$ we have:

$$(\delta_o(\alpha))(t) = 0 \text{ and } (\delta_1(\alpha))(t) = 0$$

Proof It is an immediate consequence of the definition of the operations δ_o and δ_1.

□

So every prestep being the beginning of a prestep is a multiset in which only places of the net may occur with non-zero multiplicity. One can say it is a multiset over the set of places of the corresponding Petri net i.e. a function of the form

$$\delta_i(\alpha) : P(N) \longrightarrow \mathrm{Nat}$$

Lemma 2.3.

For every prestep $\alpha \in \mathbf{PS}(N)$ it holds

$$(\forall_{t \in T(N)} \alpha(t) = 0) \Rightarrow \delta_o(\alpha) = \alpha \ \&\ \delta_1(\alpha) = \alpha$$

Proof Assume that for every $t \in T(N)$ we have α(t) = 0. Immediately from the definition of the operation δ_o we we obtain:

$$\forall_{p \in P(N)} \forall_{n \in Nat}\ (p,n) \in \delta_o(\alpha) \Leftrightarrow n = \alpha(p)$$

which completes the proof because of the evident equivalence

$$n = \alpha(p) \text{ iff } (p,n) \in \alpha$$

The proof for the operation δ_1 is similar to the above.

□

Corollary 2.1.

The operations δ_o and δ_1 are well defined i.e. for every prestep α the relations $\delta_o(\alpha)$ and $\delta_1(\alpha)$ are presteps.

Proof It is an immediate consequence of the lemmas 2.2. and 2.3. above. (If it were

$$(\delta_o(\alpha))(x) \neq 0 \neq (\delta_o(\alpha))(y) \;\&\; \text{field}(x) \cap \text{field}(y) = \emptyset.$$

for a prestep α and some its elements x and y then (because x and y are places by the abowe lemmas 2.2. and 2.3.) it woul be

$$\alpha(x) = \alpha(y)$$

which contradicts to the assumption that α is a prestep.) For the operation δ_1 is the reasoning similar.

□

Corollary 2.2.

For every prestep $\alpha \in PS(N)$ we have

$$\delta_o(\delta_o(\alpha))=\delta_1(\delta_o(\alpha))=\delta_o(\alpha)$$

and

$$\delta_o(\delta_1(\alpha))=\delta_1(\delta_1(\alpha))=\delta_1(\alpha)$$

i.e. the algebra

$$(PS(N)\,,\,\delta_o\,,\,\delta_1)$$

is a graph.

□

The following lemmas are of rather technical character.

Lemma 2.4a.

For each prestep α and every $x\in P(N)$, $y\in T(N)$ we have:

$$x\in {}^{\bullet}y \;\&\; \alpha(y) \neq 0 \Rightarrow \delta_o(\alpha)(x) \neq 0$$

Proof The condition $x\in {}^{\bullet}y$ is equivalent to $pre(y,x) \neq 0$. So we have

$$\delta_o(\alpha)(x) = \delta_o(\alpha)(y)pre(y,x) \neq 0$$

which completes the proof.

□

Lemma 2.4b.

For each prestep α and every $x\in P(N)$, $y\in T(N)$ we have:

$$x\in y^{\bullet} \;\&\; \alpha(y) \neq 0 \Rightarrow \delta_1(\alpha)(x) \neq 0$$

Proof. The condition $x\in y^{\bullet}$ is equivalent to $post(y,x) \neq 0$. So we have

$$\delta_1(\alpha)(x) = \delta_1(\alpha)(y)post(y,x) \neq 0$$

which completes the proof.

□

Let $\alpha,\beta\in PS(N)$.

Definition 2.3.

$$\alpha + \beta = \begin{cases} \alpha+\beta & \text{if } \forall_{x,y\in X(N)}\alpha(x) \neq 0 \neq \beta(y) \Rightarrow \mathit{field}(x) \cap \mathit{field}(y) = \emptyset \,\&\, \alpha \neq \nabla \neq \beta \\ \nabla & \textit{in other case} \end{cases}$$

The above definition has a very simple interpretation. The "sum" $\alpha + \beta$ of presteps α and β is either equal to their "standard" sum (as multisets) in the case when their domains are disjoint or it is "undefined" in other cases. The following lemmas establish some relations between operations "+", δ_o and δ_1.

Lemma 2.5.

For every presteps $\alpha,\beta \in \mathbf{PS(N)}$ such that $\alpha + \beta \neq \nabla$ we have:

$$\delta_o(\alpha + \beta) = \nabla \text{ iff } \delta_o(\alpha) + \delta_o(\beta) = \nabla$$

and

$$\delta_1(\alpha + \beta) = \nabla \text{ iff } \delta_1(\alpha) + \delta_1(\beta) = \nabla$$

Proof.a $\Rightarrow$) Assume

$$\neg(\delta_o(\alpha + \beta) = \nabla \Rightarrow \delta_o(\alpha) + \delta_o(\beta) = \nabla)$$

i.e.

$$\delta_o(\alpha + \beta) = \nabla \ \& \ \delta_o(\alpha) + \delta_o(\beta) \neq \nabla$$

Immediately from the first part of the above conjunction and definition 2.2. we obtain

$$\alpha + \beta = \nabla$$

which contradicts to the assumption that $\alpha + \beta \neq \nabla$.

b $\Leftarrow$) Assume

(1) $$\delta_o(\alpha) + \delta_o(\beta) = \nabla$$

So for some elements $x,y \in P(N)$ we have

(2) $$(\delta_o(\alpha))(x) \neq 0 \neq (\delta_o(\beta))(y) \ \& \ \text{field}(x) \cap \text{field}(y) \neq \emptyset \ \text{ or } \delta_o(\alpha) = \nabla \ \text{ or } \ \delta_o(\beta) = \nabla$$

If $\delta_o(\alpha) = \nabla$ or $(\delta_o(\beta) = \nabla$ then $\alpha = \nabla$ or $\beta = \nabla$ i.e. $\alpha + \beta = \nabla$ which contradicts to the assumption that $\alpha + \beta \neq \nabla$. So, let us consider the condition

(2) $$(\delta_o(\alpha))(x) \neq 0 \neq (\delta_o(\beta))(y) \ \& \ \text{field}(x) \cap \text{field}(y) \neq \emptyset$$

From the fact that

$$x \in P(N) \ \& \ y \in P(N) \ \& \ \text{field}(x) \cap \text{field}(y) \neq \emptyset$$

we infer (c.f. definition of the operation "field") that x = y. Now we have to consider the following possibilities:

1.$\alpha(x) \neq 0$ & $\beta(x) \neq 0$. In this case we obtain the thesis immediately from the definition of the operations "+" and "δ_o" .

2.$\alpha(x) = 0$. In this case there exists an element $t_\alpha \in T(N)$ such that

$$\alpha(t_\alpha) \neq 0 \ \& \ x \in \text{field}(t_\alpha)$$

If $\beta(x) \neq 0$ then we have

$$\alpha(t_\alpha) \neq 0 \neq \beta(x) \ \& \ \text{field}(t_\alpha) \cap \text{field}(x) \neq \emptyset$$

i.e.

$$\alpha + \beta = \nabla.$$

which contradicts to the assumption that $\alpha + \beta \neq \nabla$.

If $\beta(x) = 0$ then there exists an element $t_\beta \in T(N)$ such that

$$\beta(t_\beta) \neq 0 \ \& \ x \in \text{field}(t_\beta)$$

In this case we have

$$\alpha(t_\alpha) \neq 0 \ \& \ \beta(t_\beta) \neq 0 \ \& \ \text{field}(t_\alpha) \cap \text{field}(t_\beta) \neq \emptyset$$

and we obtain the thesis analogously to the cases above. In the case $\beta(x) = 0$ is the reasoning similar. The proof for the operation δ_1 is analogous.

□

Lemma 2.6.

For every presteps α and β such that $\alpha + \beta \neq \nabla$ it holds

$$\delta_o(\alpha+\beta)=\delta_o(\alpha)+\delta_o(\beta)$$
$$\delta_1(\alpha+\beta)=\delta_1(\alpha)+\delta_1(\beta)$$

Proof. It follows from the above lemma 2.5. that for every presteps α and β satisfying the condition $\alpha + \beta \neq \nabla$ we have:

$$\delta_o(\alpha+\beta)\neq \nabla \text{ iff } \delta_o(\alpha)+\delta_o(\beta)\neq \nabla$$

So we have to prove that for each $x\in X(N)$ it holds:

$$(\delta_o(\alpha+\beta))(x)=(\delta_o(\alpha)+\delta_o(\beta))(x)$$

provided that $\delta_o(\alpha+\beta)\neq \nabla$.

Let $x\in X(N)$ and assume that $\delta_o(\alpha+\beta)\neq \nabla$. It follows from lemma 2.2. that if $x\in T(N)$ then

$$(\delta_o(\alpha+\beta))(x)=(\delta_o(\alpha))(x)=(\delta_o(\beta))(x)=0$$

i.e. both functions (multisets) are equal on the set T(N) of transitions of the net N. Assume $x\in P(N)$. We have to consider the following cases.

1. $(\delta_o(\alpha+\beta))(x)=0$. Now assume

$$(\delta_o(\alpha))(x)+(\delta_o(\beta))(x)\neq 0$$

i.e.

$$(\delta_o(\alpha))(x)\neq 0 \text{ or } (\delta_o(\beta))(x)\neq 0$$

1.1. Assume $(\delta_o(\alpha))(x)\neq 0$. (If $(\delta_o(\beta))(x)\neq 0$ is the reasoning similar.) So we have

$$\alpha(x)\neq 0 \text{ or } \exists_{y\in T(N)}\ \alpha(y)\neq 0\ \&\ x\in {}^\bullet y$$

1.1.1. If $\alpha(x)\neq 0$ then we would have

$$(\alpha+\beta)(x)\neq 0\ \&\ x\in P(N)$$

i.e.

$$\delta_o(\alpha+\beta)(x)\neq 0$$

which contradicts to the assumption 1.

1.1.2. Assume

$$\alpha(y)\neq 0\ \&\ x\in {}^\bullet y$$

for a transition $y\in T(N)$ of the net N. In this case we have

$$(\alpha+\beta)(x)\neq 0\ \&\ x\in {}^\bullet y$$

and as a consequence we obtain that

$$\delta_o(\alpha+\beta)(x)\neq 0$$

which also contradicts to the assumption 1. So it is proven that

(2.6.1) $$(\delta_o(\alpha+\beta))(x)=0\Rightarrow(\delta_o(\alpha+\beta))(x)=(\delta_o(\alpha)+\delta_o(\beta))(x)$$

2. If

$$(\delta_o(\alpha)+\beta))(x)\neq 0$$

then from the definition of the operation δ_o we obtain

$$(\alpha+\beta)(x)\neq 0 \text{ or } \exists_{y\in T(N)}\ (\alpha+\beta)(y)\neq 0\ \&\ x\in {}^\bullet y$$

2.1. Assume $(\alpha+\beta)(x)\neq 0$. In this case we have

$$\text{either } \alpha(x)=0 \text{ or } \beta(x)=0$$

(It is not possible that $\alpha(x)\neq 0\neq \beta(x)$ because $\alpha+\beta\neq \nabla$.) We consider the case $\alpha(x)\neq 0$ only. (The second case is completely analogous.) In this case we obtain from the assumption $x\in P(N)$ that

$$(\delta_o(\alpha))(x)=\alpha(x)\neq 0$$

So in this case both sides of the equality under the consideration are non - zero. We prove that they are equal. First of all we note that it is not possible that there exists a transition $y\in T(N)$ such that

(2.6.2) $$\beta(y)\neq 0\ \&\ x\in {}^\bullet y$$

In fact if y were such a transition then we would have

$$(\alpha + \beta)(x) \neq 0 \neq (\alpha + \beta)(y) \ \& \ x \neq y \ \& \ \text{field}(x) \cap \text{field}(y) = \{x\} \neq \emptyset$$

i.e. $\alpha + \beta$ were not a prestep in contradiction to our assumption. So the condition (2.6.2) must hold. But immediately from this condition and the fact that $\beta(x) \neq 0$ we infer that $(\delta_o(\beta))(x) = 0$ which implies

$$(\delta_o(\alpha + \beta))(x) = \alpha(x) = (\delta_o(\alpha))(x) + 0 = (\delta_o(\alpha))(x) + (\delta_o(\beta))(x)$$

Thus it is proven that

(2.6.3) $$(\alpha + \beta)(x) \neq 0 \ \& \ \alpha(x) \neq 0 \Rightarrow (\delta_o(\alpha + \beta))(x) = (\delta_o(\alpha) + \delta_o(\beta))(x)$$

Assume $\alpha(x) = 0$. Now if $\beta(x) \neq 0$ then the reasoning is analogous. to that in the point 2.1. So assume $\beta(x) = 0$. In this case we have

$$(\alpha + \beta)(x) = \alpha(x) + \beta(x) = 0 + 0 = 0$$

which leads to

$$(\alpha + \beta)(y) \neq 0 \ \& \ x \in {}^{\bullet}y$$

for a transition $y \in T(N)$ of the net **N**. Now we have (c.f. the reasoning in the previous point 2.1.)

$$\text{either } \alpha(y) \neq 0 \text{ or } \beta(y) \neq 0$$

Assume $\alpha(y) \neq 0$. (In the case $\beta(y) \neq 0$ is the reasoning similar) In this case we have

$$(\alpha + \beta)(y) = \alpha(y)$$

and as a consequence we obtain

$$(\delta_o(\alpha + \beta))(x) = \alpha(y)\text{pre}(y,x)$$

If it were $(\delta_o(\beta))(x) \neq 0$ then (because of the fact that $\beta(x) \neq 0$) it would be

$$\beta(y') \neq 0 \ \& \ x \in {}^{\bullet}y$$

for a transition $y' \in T(N)$. But in this case we would have

$$\alpha(x) \neq 0 \neq \beta(y) \ \& \ \text{field}(x) \cap \text{field}(y) \neq \emptyset$$

because of course $x \in \text{field}(x) \cap \text{field}(y)$ i.e. it would be $\alpha + b = \nabla$ in contradiction to the assumption. So it must be

$$(\delta_o(\beta))(x) = 0$$

which implies

$$(\delta_o(\alpha + \beta))(x) = \alpha(y)\text{pre}(y,x) = (\delta_o(\alpha))(x) = (\delta_o(\alpha))(x) + 0 = (\delta_o(\alpha))(x) + (\delta_o(\beta))(x)$$

The implication is thus proven

(2.6.4) $$(\alpha + \beta)(x) \neq 0 \ \& \ \alpha(x) = 0 \Rightarrow (\delta_o(\alpha + \beta))(x) = (\delta_o(\alpha) + \delta_o(\beta))(x)$$

Now "adding" the precedesors of the implication (2.1.) and (2.1') we obtain

(2.6.5) $$(\alpha + \beta)(x) \neq 0 \Rightarrow (\delta_o(\alpha + \beta))(x) = (\delta_o(\alpha) + \delta_o(\beta))(x)$$

which completes proof of the case 2 and we obtain the thesis by (1').

The proof of the equality

$$\delta_1(\alpha + \beta) = \delta_1(\alpha) + \delta_1(\beta)$$

is similar.

□

3 System Algebras

In this section algebras of presteps are characterized in an axiomatic way. This characterization is given by the propositions 3.1. and 3.2.

Definition 3.1.

By a p - algebra we mean any algebra of the form

$$\mathbf{P} = (P, +, 0, \delta_o, \delta_1)$$

satisfying the following conditions:

(p1) the reduct **mon(P)** = (P , + , 0) is a free monoid generated by a set FG(**P**).

(p2) the reduct **graph(P)** = (P , δ_o , δ_1) is a graph.

(p3) for every elements x,y$\in$P the following conditions holds:

(i) $\alpha + \beta \neq \nabla \Rightarrow \delta_o(x + y) = \delta_o(x) + \delta_o(y)$

(ii)$\alpha + \beta \neq \nabla \Rightarrow \delta_1(x + y) = \delta_1(x) + \delta_1(y)$

(p4) $x \in FG(\mathbf{P})$ & $y \in FG(\mathbf{P})$ & $\delta_o(x) = \delta_o(y)$ & $\delta_1(x) = \delta_1(y) \Rightarrow x = y$

Definition 3.2.

An element ∇ of an p - algebra **P** will be called a zero element of **P** iff it is a zero element of the monoid **mon(P)** i.e. for all x$\in$P the condition

$$x + \nabla = \nabla$$

holds and it is an isolated vertex of the graph **graph(P)** i.e. a fixpoint of the operations δ_o and δ_1 with the property that

$$\delta_o(\alpha) = \nabla \Rightarrow \alpha = \nabla \text{ and } \delta_1(\alpha) = \nabla \Rightarrow \alpha = \nabla$$

Proposition 3.1.

For every Petri net **N** the algebra

$$\mathbf{PS(N)} = (PS(N), +, \varnothing, \nabla, \delta_o, \delta_1)$$

is an p - algebra.

Proof. We have to prove that the algebra **PS(N)** satisfies the conditions (p1) - (p3) of the definition 2.1. above.

(p1) It is evident that the algebra **PS(N)** is a monoid. We have to show the set of its free generators. Let us consider the set X(N) of all elements of the net **N**. It is easy to see that every singleton of the form {x} with x$\in$X(N) is a prestep in the net **N**. Let us denote the set of all those presteps by the symbol **sing**(N). Every prestep α can be written in the form:

$$\alpha = k_1 p_1 + k_2 p_2 + \dots + k_n p_n$$

with $n, k_1, k_2, \dots, k_n \in$ Nat and $p_1, p_2, \dots, p_n$ being singletons of the above form. This representation is unique up to the order of the singletons in it.

Now for every monoid **M** and every function f : **sing**(N) ---> /**M**/ the standard extension

$$f^{\#} : \mathbf{PS(N)} \longrightarrow \mathbf{M}$$

$$f^{\#}(k_1 p_1 + k_2 p_2 + \dots + k_n p_n) = k_1 f(p_1) + k_2 f(p_2) + \dots + k_n f(p_n)$$

is the required unique monoid homomorphism.

(p2) We have to show that:

$$\delta_o \bullet \delta_o = \delta_o \bullet \delta_1 = \delta_o$$

and

$$\delta_1 \bullet \delta_o = \delta_1 \bullet \delta_1 = \delta_1$$

The above equalities are immediate consequences of corollary 2.2.

(p3)(i) and (ii) The equalities follow immediately from lemma 2.6.

(p4) This property is an immediate consequence of the extensivity condition (ext).

□

Now we show that the conditions (p1) - (p4) of the definition characterize the algebra of presteps of a net up to isomorphism i.e. we show that every algebra satisfying the conditions (p1) - (p4) is isomorph to the algebra of presteps of a Petri net.

Proposition 3.2.

Every p - algebra is isomorph to the algebra of presteps of a Petri net.

Proof. Let

$$\mathbf{P} = (P, +, 0, \delta_o, \delta_1)$$

be a p - algebra and Atoms(**P**) the set of free generators of the algebra **P**. We put

$$P(\mathbf{P}) = \{x \in Atoms(\mathbf{P}) \mid \delta_o(x) = x = \delta_1(x)\}$$

i.e. P(**P**) is the set of all "atomic" vertices of the graph **graph(P)**. Note that each vertex of the graph **graph(P)** is of the form

$$v = n_1 p_1 +,\ldots,+ n_k p_k$$

with $n_1, \ldots, n_k$ being natural numbers and $p_1, \ldots, p_k$ being elements of the set P(**P**). In fact every element a of the set P (the support of the algebra **P**) is of the form

$$a = n_1 a_1 +,\ldots,+ n_k a_k$$

for some natural numbers $n_1, \ldots, n_k$ and $p_1, \ldots, p_k$ being elements of the set Atoms(**P**). From the uniquess of the representation of this element in the above form and the properties (p3)(i) and (p3)(ii) it follows that for every $1 \le j \le k$ we have

$$\delta_o(a_j) = a_j = \delta_1(a_j)$$

i.e. for every $1 \le j \le k$ we have $a_j \in P(\mathbf{P})$.

Let

$$T(\mathbf{P}) = Atoms(\mathbf{P}) \quad P(\mathbf{P})$$

and let us consider an element $t \in T(\mathbf{P})$. Assume

$$\delta_o(t) = n_1 p_1 +,\ldots,+ n_k p_k$$

and

$$\delta_1(t) = m_1 q_1 +,\ldots,+ m_k q_l$$

Now we define functions

$$pre_t, post_t : P(\mathbf{P}) \longrightarrow Nat$$

in the following way: for $p \in P(\mathbf{P})$

$$pre_t(p) = \begin{cases} n_i & \text{if } p = p_i \text{ for } i \le k \\ 0 & in'other\ case \end{cases}$$

$$post_t(p) = \begin{cases} m_j & \text{if } p = q_j \text{ for } j \le l \\ 0 & in'other\ case \end{cases}$$

Let for every $t \in T(\mathbf{P})$ and $p \in P(\mathbf{P})$ be

$$pre(t,p) = pre_t(p) \text{ and } post(t,p) = post_t(p)$$

It is evident that the quadrupel

$$\mathbf{net(P)} = (P(\mathbf{P}), T(\mathbf{P}), pre, post)$$

is a Petri net.

In order to see that algebras **P** and **PS(net(P))** are isomorph it is sufficient to note that both these algebras are generated by the same set Atoms(**P**).

□

Concluding remarks

The approach described in the paper allows not only for an "algebraization" of the notion of Place/Transition Petri net, but it also seems to be a decent startpoint for an algebraization of many other concepts of nettheory. We mention here two of them.

- The concept of a process in a system (Petri net)

The notion of process in a Place/Transition net can be defined as a generalization of the notion of path in a graph. The "ideology" here is very similar to that used in the theory of traces (see [Maz1] or [AalRoz1]) i.e. a process in a net N can be defined as an equivalence class of a congruence of the free category generated by the graph **graph(PS(N))**. The corresponding congruece is here defined by relatively simple (weak) equation like that described in [Kor1] or [Win4].

- The concept of system (net) morphism.

A net morphism can be defined as a homomorphism of the corresponding algebras of presteps. The so defined morphisms respect the reachability relation of a net which seems to be important in many for the practice relevant cases.

The above problems will be discused in other papers.

Bibliography

[AalRoz1] Aalbersberg I.J. , Rozenberg G. "Theory of Traces" Dept. of Comp. Sc. Rijksuniversiteit Leiden Report No. 86-16

[GeLaTha.] Genrich H.J., Lautenbach K., Thiagarajan P.S. *"Elements of General Net Theory"* in Brauer (ed) "Net Theory and Applications" LNCS 84 Spinger Verlag Berlin, Heidelberg, New York (1980)

[Kor1] Korczynski W. "Aksjomatyczna charakteryzacja algebr procesow systemow wspolbieznych" ICS PAS Report 400 Warsaw (1980) (polish)

[Maz1] Mazurkiewicz A. *"Trace Theory"* in Brauer W., Reisig W., Rozenberg G. (ed) "Petri Nets: Applications and Relationship to Other Models of Concurrency, Advanced in Petri Nets 1986, Part II LNCS 255 Springer Verlag (1987)

[MesMo1] Meseguer J. Montanari U. "Petri Nets are Monoids" Report of SRI-Project No.1243 (1988)

[Rei1 Reisig W. "Petri Netze - eine Einführung" Springer Verlag Berlin, Heidelberg, New York 1986.

[Sta1] Starke P.H. "Petrinetze" Deutscher Verlag der Wissenschaften, Berlin 1980

[Win1] Winkowski J. *"Algebras of Partial Sequences: a Tool to Deal with Concurrency"* in Karpinski (ed) "Fundamentals of Computation Theory" LNCS 56 Springer Verlag Berlin, Heidelberg, New York (1977)

[Win2] Winkowski J. "An Algebraic Characterization of the Behaviour of Non Sequential Systems" Inf. Proc. Letters 6 No. 4 (1977)

[Win3] Winkowski J. "Behaviours of Concurrent Systems" Th. Comp. Sc. 12 (1980)

[Win4] Winkowski J. "An algebraic Description of system Behaviours" Th. Comp. Sc. 21 (1982)