KfK 5297
April 1994

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ISSN 0303-4003

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Abstract

Game theoretical models and analysis are provided for the sequential material accountancy problem. We model the n-period problem as a general sequential game played between the Operator and the Inspector. The game is analyzed through the solution concept of (Nash) equilibrium. We study several versions of the game corresponding to various assumption on the payoffs and the strategy sets. The first model solved is what we refer to as *the static game*. This is a game in which detection time is unimportant and the operator has to decide about his diversion plan at the beginning of the game (and he cannot deviate from it in a later stage). The solution of this game is obtained by its decomposition into two simpler games: a zero-sum game which determines the diversion plan and the statistical test (which turned out to be the *CUMUF* test) and a second, non zero-sum game which determines the diversion probability and the false alarm probability.

Next we return to the sequential game and prove that under the assumptions underlying the statistical analysis, the *CUMUF* test emerges as part of the solution of the game i.e., as the inspector's strategy in equilibrium. Then we consider a 'really sequential' game in which early detection is important and in which the operator can *retreat* (in view of high observed intermediate *MUF*) from completing a diversion plan that he has started. We find the structure of the equilibrium and the equilibrium equations of this game . These equations turn out to be too complex to be solved analytically, hence we provide numerical solutions which give interesting insight into the problem.

Spieltheoretische Untersuchungen zur Materialbilanzierung

Zusammenfassung

Spieltheoretische Modelle und Untersuchungen für das sequentielle Materialbilanzierungsproblem werden vorgestellt. Das n-Perioden Problem wird als allgemeines sequentielles Spiel modelliert, das zwischen einem "Betreiber" und einem "Inspektor" gespielt wird. Anhand des Lösungskonzepts des Nash-Equilibriums wird das Spiel analysiert. Verschiedene Versionen des Spiels werden untersucht, die mit verschiedenen Annahmen über die Auszahlungsfunktionen und den Strategiemengen der Spieler korrespondieren. Zuerst wird das sogenannte "statische Spiel" betrachtet. In diesem Spiel ist die "Entdeckungszeit" ohne Einfluß und der Betreiber muß seine Entwendungsstrategie zu Beginn des Spieles unwiderruflich festlegen. Die Lösung des Spiels erhält man durch seine Aufteilung in zwei einfachere Spiele: eines Null-Summen-Spiels, das die Entwendungsstrategie und den statistischen Test (der sich als der CUMUF-Test herausstellt) bestimmt und eines weiteren, Nicht-Null-Summen-Spiels, das die Entwendungswahrscheinlichkeit und die Fehlalarmrate ermittelt.

Dann wird das sequentielle Spiel untersucht und bewiesen, daß sich der CUMUF-Test unter gewissen Annahmen als Teil der Lösung des Spiels herausstellt, d.h. er ist die Strategie des Inspektors im Equilibrium. Als nächstes wird das "echt sequentielle" Spiel untersucht, in dem eine frühzeitige Aufdeckung einer Abzweigung von Bedeutung ist und in dem der Betreiber seinen (a priori) Entwendungsplan nicht vollenden muß, falls zu große Bilanzergebnisse auftreten. Die Struktur des Equilibriums sowie die Equilibriumsgleichungen werden angegeben. Da diese Gleichungen zu komplex sind für eine analytische Behandlung, werden numerische Lösungen vorgestellt, die einen interessanten Einblick in das Problem ermöglichen.

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1 Introduction

Material accountancy is a methodology designed to control materials with particular properties - rare, unpleasant, dangerous, precious - that are used by man in the course of his economic and social activities but their use requires the exercise of special care. In particular, material accountancy is practiced to a loss or diversion of materials for purposes unknown, but illegal according to some agreement, law or treaty.

Examples for this use of material accountancy are diverse environmental problems; here it is to be guaranteed that certain pollutants are released to the environment only within permitted standards. Other examples are arms control and disarmament agreements where the appropriate use of troops, equipment or special materials has to be controlled. In fact, it was the *Treaty for Non-Proliferation of Nuclear Weapons* and its verification provisions, performed by the *International Atomic Energy Agency* (IAEA) in Vienna through its Nuclear Material Safeguards Systems, which stimulated the most detailed work in this area.

How can this use of the material accountancy principle be formulated in a simple and applicable way? A straightforward formulation might state that any material entering a well defined area - called *material balance area* - cannot simply disappear i.e., either it is still there, or it has left it. A quantitative description of this principle for one material balance area and one inventory period $[t_0, t_1]$ goes as follows.

At time t_0 there is an amount I_0 of the material (in the material balance area) which we call the *beginning real inventory.* The algebraic sum of the amounts of material that enter and leave the material balance are in the time interval $[t_0, t_1]$, is called the *net flow* and denoted by D. The real inventory at time t_0 plus the net flow in $[t_0, t_1]$ is the *book inventory* $B = I_0 + D$ at time t_1 i.e., the amount of material that *should be* in the material balance area at time t_1. The amount of material actually contained in the material balance area at time t_1 is called the *ending real inventory* and denoted by I_1. If all material contained in and passing through the material balance area in the time interval $[t_0, t_1]$ is carefully accounted for, if no material has disappeared or has been diverted, and if there are no measurement errors, then the difference between the book inventory B and the real inventory I_1 should be zero. This is simply a consequence of the law of the conservation of matter. However, since not all of these conditions are always satisfied, the difference between these two quantities at the end of the inventory period, which for historical reasons has been called *Material Unaccounted For* (*MUF*),

$$MUF = B - I_1 = I_0 + D - I_1,$$

is not always zero. Thus arises the problem of finding out the various causes for this difference being nonzero and, of trying to separate them.

Statistically, this problem is treated as follows: We assume that *MUF* is a normally distributed random variable (r.v.) with variance σ^2 given by

$$\sigma^2 = \mathrm{var}(MUF) = \mathrm{var}(I_0) + \mathrm{var}(D) + \mathrm{var}(I_1).$$

There are two hypothesis H_0 and H_1 regarding the expected value $E(MUF)$, namely

$$\begin{array}{llll} H_0 & : & E(MUF) = 0 & \text{(legal behavior)} \\ H_1 & : & E(MUF) = \mu > 0 & \text{(illegal behavior).} \end{array}$$

The standard hypothesis testing procedure to decide which of the two hypothesis is true, is based on the observed value of MUF - denoted by $\widehat{MUF}$ - and is of the form:

Reject H_0 (and accept H_1) if and only if $\widehat{MUF} > s$.

The threshold s is determined by the *false alarm* (type I error) *probability* α, through the relation

$$1 - \alpha = P_0\{MUF \leq s\} = \phi\left(\frac{s}{\sigma}\right),$$

where P_0 denotes the probability given that H_0 is true, and $\phi(\cdot)$ is the cumulative standard normal distribution. The efficiency of this test is measured, as usual, by its power (one minus type II error probability, β). This is the *detection probability* given by

$$\begin{aligned} 1 - \beta &= P_1\{MUF > s\} \geq 1 - \phi\left(\frac{s-\mu}{\sigma}\right) \\ &= \phi\left(\frac{\mu}{\sigma} - U(1-\alpha)\right), \end{aligned}$$

where P_1 denotes the probability given that H_1 is true, and $U(\cdot)$ is the inverse of $\phi(\cdot)$. Note that $1-\beta(\alpha)$ is a concave function of α satisfying $1-\beta(0) = 0$ and $1-\beta(1) = 1$.

In general, a sequence of n inventory periods $[t_0, t_1]$, ..., $[t_{n-1}, t_n]$ is considered for each of which, a single material balance statistics is observed namely

$$MUF_i = I_{i-1} + D_i - I_i, \quad i = 1, \ldots, n.$$

The test for deciding whether or not an illegal action has taken place, involves several new problems: On the technical level, the statistical analysis is more complicated since MUF_i, $i = 1, \ldots, n$ are correlated, due to the intermediate inventory which appears in two adjacent MUF_i. On a more fundamental level, it is not clear what is now an appropriate measure of efficiency, in particular if one has some kind of sequential decision procedure in mind. It is also not clear what is now the alternative hypothesis H_1 corresponding to illegal behavior, since there are many different possibilities for loss or diversion of material within the n inventory periods. This adds to the model a second decision maker, the *operator.* This is the decision maker who decides whether to behave legally or illegally and what form of illegal behavior to choose.

One is thus driven from the standard statistical analysis to a *game theoretical analysis* involving two decision makers: the *operator* who decides whether to divert and how to divert material, and the *inspector* who controls the operator by observing data, establishing the material balances, and decides whether or not to declare that an illegal action has taken place. In a slightly different variant of the situation, the operator has to report a certain set of data which he may try to falsify. The inspector then may verify some of the data by independent measurements and declare whether or not an illegal action (falsification) has taken place.

2 Game theoretical framework

In a general multistage inspection game, there are two players: the *operator*, denoted as player **O** , and the *inspector*, denoted as player **I** . The game proceeds in n stages. In stage i, $i = 1, \ldots, n$, player **O** has an action set M_i consisting of all (pure - nonrandomized) actions available to him at that stage. The action set of the inspector **I** is the same in all stages, and consists of two elements A (alarm) and C (clear or continue).

$X_1, \ldots, X_n$ are the signals (or observations) at stages $1, \ldots, n$ respectively. Typically, X_i is a random variable (r.v.) with values in a finite dimensional Euclidean space E_i. The distribution of these r.v. are given by transition probabilities $f_1, \ldots, f_n$:

$$f_i \; : \; M_1 \otimes E_1 \otimes, \ldots, \otimes M_{i-1} \otimes E_{i-1} \otimes M_i \longrightarrow \Delta(E_i),$$

where by convention, $M_0 = E_0 = \emptyset$ and for any set E, we denote by $\Delta(E)$ the space of probability distributions on E. That is, for any *history* of i actions $m_1, \ldots, m_i$, $(m_i \in M_i)$ and $i-1$ observations $x_1, \ldots, x_{i-1}$, $(x_i \in E_i)$, $f(\cdot \mid m_1, x_1, \ldots, m_{i-1}, x_{i-1}, m_i)$ is the probability density of the r.v. X_i observed at stage i.

For $i = 1, \ldots, n$, I_i and O_i are real functions on the product space $M_1 \otimes, \ldots, \otimes M_i$. These are the *payoff functions* to **I** and **O** respectively, if the game stops by an alarm at stage $i < n$. Finally, I and O are two real functions on $M_1 \otimes, \ldots, \otimes M_n$, the payoff functions when the game ends after n stages with no alarm called.[2]

The game is played as follows:

- At stage 1, the operator **O** chooses an action $m_1 \in M_1$ (not observable by the inspector **I**). An observation x_1 is drawn from a r.v. X_1 with density $f_1(\cdot \mid m_1)$. This observation becomes common knowledge to both players.

- The inspector chooses either C, in which case the game continues to stage 2, or A, in which case the game terminates (by an alarm) with payoffs $(I_1(m_1), O_1(m_1))$.

- Inductively: At stage $i = 2, \ldots, n$, if the game has not been terminated before, player **O** chooses $m_i \in M_i$, and an observation x_i is made from the r.v. with density $f_i(\cdot \mid m_1, x_1, \ldots, m_{i-1}, x_{i-1}, m_i)$.

  Player **I** chooses either C, in which case the game continues to stage $i+1$, if $i < n$, and terminates with payoffs $(\tilde{I}(m_1, \ldots, m_n), \tilde{O}(m_1, \ldots, m_n))$ if $i = n$, or he chooses A, in which case the game terminates (by an alarm) with payoffs $(I_i(m_1, \ldots, m_i), O_i(m_1, \ldots, m_i))$.

Note that by this notation we assume that the payoffs are determined only by the actions of the operator and do not depend on the observations. This is quite reasonable if no alarm is called, in case of alarm this implies the implicit assumption that the actual diversions are checked and found out after the alarm.

[2]In order to incorporate the standard notation of both game theory and statistics, used extensively in this paper, we denote the operator's actions by m since we reserve the letter μ for the conventional use as the expected value of the *MUF* random variable.

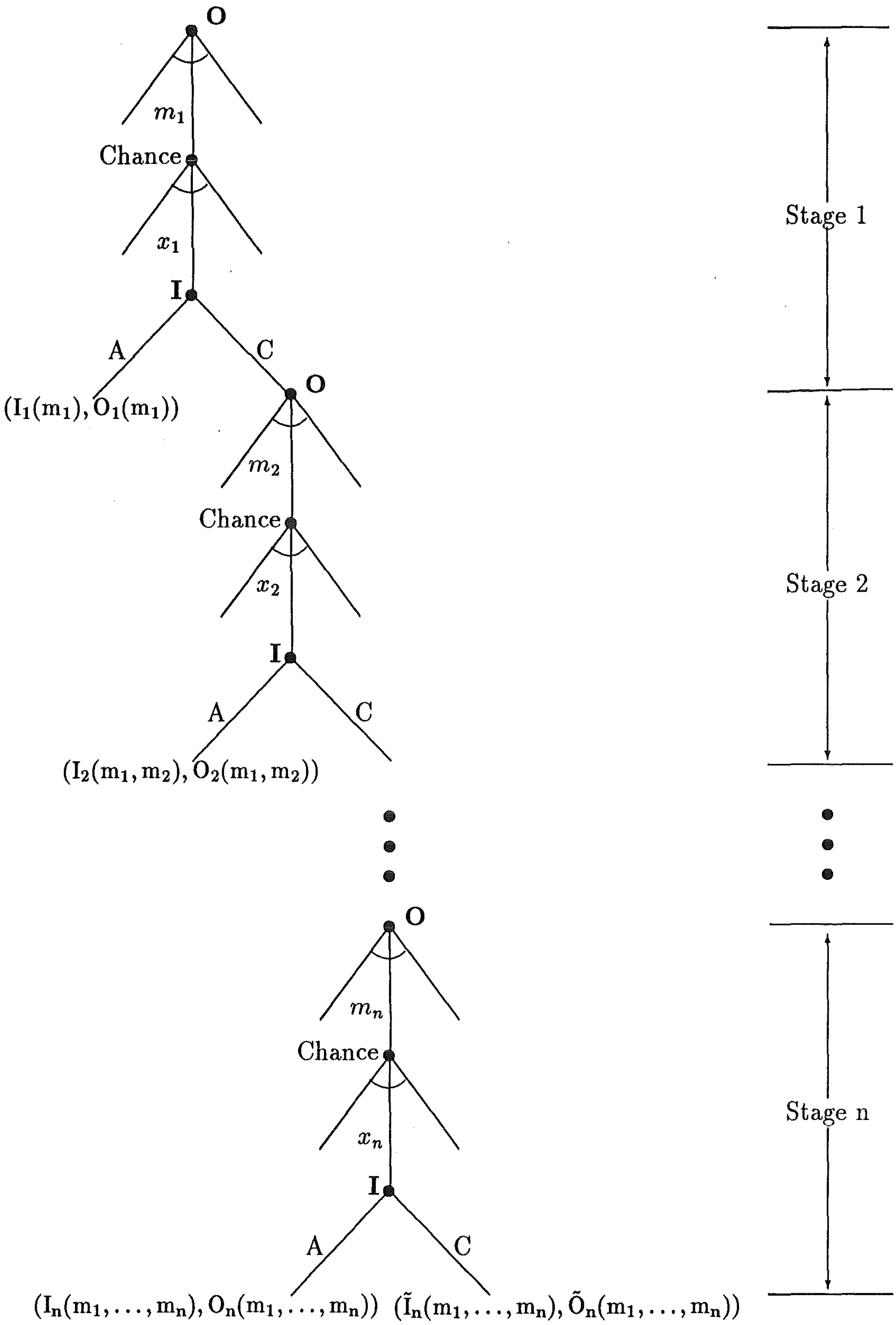

Figure 1: The game in extensive form.

2.1 Restrictions on players' strategies

Our first step in analyzing the above described general model, is to consider a special case corresponding to the situation in which *time is not important* - in spite of the sequential nature of the problem, and of the model, the only issues of interest are whether or not an illegal action took place, and whether or not an alarm was called.

In addition to being a natural simple case to start with, this is an attempt to relate our model to the existing, mainly statistical, analysis which is basically *static*, as far as the decision making process is concerned. To do this, we impose restrictions on the players' strategies capturing the underlying assumptions of most existing analysis.

Assumption 2.1
- *The operator's diversion strategy is completely determined at the beginning of the game, as he chooses either not to divert -* ($m_i = 0,\ i = 1,\ldots,n$)*, or to divert according to a plan of the form* $m = (m_1,\ldots,m_n)$ *with* $m_i \geq 0$ *and* $\sum_i m_i = CM$*, for a fixed constant* $CM > 0$.
- *The inspector may call an alarm only at the end of the* n*-th stage, after having observed the whole vector* $x = (x_1,\ldots,x_n)$.

The first part of the assumption says that even if **I** decides on calling an alarm already at time $i < n$ he may as well wait till the end of the game since time is not important. In the second part of the assumption, CM is to be thought of as a "Critical Mass" in the sense that any diversion less than CM is worthless to **O** and harmless to **I** . Similarly, any diversion greater than CM has the same effect as a diversion of CM. We shall return to this point later. By changing units, we shall assume without loss of generality that $CM = 1$. So by this assumption, **O** decides at the beginning whether to divert or not. If he chooses to divert, he selects a distribution of the critical amount along the n stages and he has no opportunity to change this any more.

In view of Assumption 2.1, the game is basically a "one stage game" in which **O** moves first to choose ℓ (legal behavior) or $\bar{\ell}$ (illegal behavior) together with a diversion vector $m = (m_1,\ldots,m_n)$. Then a random vector x is observed and the inspector decides whether or not to call an alarm. This extensive form game, which we denote by Γ_0, is described in Figure 2. The encircled area in the figure denotes the *information set* of the inspector (i.e., he does not know whether the operator chose ℓ or $\bar{\ell}$.)

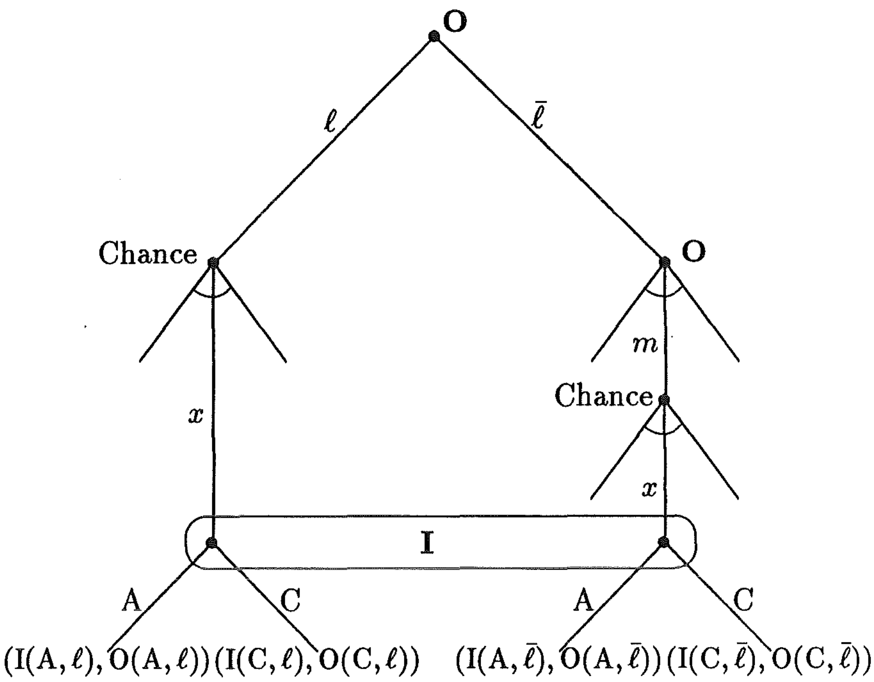

Figure 2: The restricted game Γ_0 in extensive form.

To analyze the game Γ_0 we first choose convenient payoff scales. Table 1 gives the payoffs for **I** and **O** at the end points of the game i.e., for all combinations of ℓ or $\bar{\ell}$ with A or C.

		O*perator*	
		ℓ	$\bar{\ell}$
I*nspector*	C	(0,0)	(−1,1)
	A	$(-e,-h)$	$(-a,-b)$

Table 1: The payoffs for Γ_0.

The entries of Table 1 are ordered pairs (x, y) in which x is the payoff for **I** and y is the payoff for **O** . As can be seen from the table, we chose the status-quo payoff to be 0 for both players $(I(\ell, C) = O(\ell, C) = 0)$, and the payoffs for undetected illegal behavior, $(\bar{\ell}, C)$, are 1 for **O** and -1 for **I** . Since this is the worst event for **I** , it is reasonable to assume $0 < e < 1$, $0 < a < 1$ and $0 < h < b$. Sometimes it is also assumed that $e < a$ which means that for the inspector, a detection of diversion, which means a "failure of safeguards", is worse than the inconvenience of a false alarm.

A pure strategy of **I** is an alarm set $\mathbf{A} \subset E_1 \otimes \ldots \otimes E_n$ with the interpretation that **I** calls an alarm, at the end of the n-th period, if and only if $(x_1, \ldots, x_n) \in \mathbf{A}$. A mixed strategy s is a probability distribution on pure strategies.

A pure strategy of **O** is a choice between ℓ and $\bar{\ell}$ and a diversion plan $m = (m_1, \ldots, m_n)$ satisfying $\sum_i m_i = 1$. A mixed strategy is a probability distribution on such pairs. Since the game under consideration is of perfect recall (see Kuhn (1953)), any mixed strategy is equivalent to a *behavioral* strategy $t = (q, p)$ where q is the probability of $\bar{\ell}$ and p is the probability distribution on diversion plans m if $\bar{\ell}$ is chosen.

A pair of strategies (s, t), determines two conditional probabilities:

$$\alpha(s,t) = P_{s,t}\{A \mid \ell, s\} \quad \textit{and} \quad \beta(s,t) = P_{s,t}\{C \mid \bar{\ell}, s, t\},$$

where $P_{s,t}$ is the probability distribution on outcomes induced by the strategies (s, t) (and the probabilities f_i ; $i = 1, \ldots, n$.) That is, $\alpha(s, t)$ is the *false alarm probability* (type I error probability), and $\beta(s, t)$ is the *non detection probability* (type II error probability). Note that $\alpha(s, t)$ does not depend on t while $\beta(s, t)$ is linear in p. Hence we write $\alpha(s)$ and $\beta(s, p) = \int \beta(s, m)dp(m)$.

The game Γ_0 can now be described as a game in normal form: $\Gamma_0 = (S, T, I, O)$ where S is the set of mixed strategies for **I** , T is the set of behavioral strategies for **O** , I and O are the payoff functions for **I** and **O** respectively given by

$$\begin{aligned} I(s,t) &= -q\,(a + (1-a)\beta(s,p)) - (1-q)e\alpha(s) && (1)\\ O(s,t) &= -q\,(b - (1+b)\beta(s,p)) - (1-q)h\alpha(s) && (2) \end{aligned}$$

2.2 Solution of the static game

The basic solution concept in non zero-sum games is the Nash Equilibrium (which we shall simply refer to as equilibrium), defined as follows:

Definition 2.2 *A pair of strategies (s^*, t^*) is called an equilibrium point if*

$$\begin{aligned} I(s^*,t^*) &\geq I(s,t^*) \quad \forall s \in S\\ O(s^*,t^*) &\geq I(s^*,t) \quad \forall t \in T \end{aligned}$$

The corresponding $I(s^*, t^*)$ and $O(s^*, t^*)$ are then called equilibrium payoffs of the game.

First, we observe that the game has no extreme equilibria i.e., no equilibrium in which **O** plays $\bar{\ell}$ with probability 0 or 1, and no equilibrium in which **I** calls an alarm with probability 0 or 1. This is easily seen from the payoffs structure in Table 1. In particular, this means that $0 < q < 1$ must hold in any equilibrium. Consequently, it follows from (1) and the definition of equilibrium that (in any equilibrium,) p must be a maximizer of $\beta(s, p)$ that is, the support of p must contain only diversion vectors m maximizing $\beta(s, m)$. For any strategy s of **I** we denote therefore

$$\beta(s) := \sup_p \beta(s,p) = \sup_m \beta(s,m). \qquad (3)$$

For an equilibrium point (s, t), the payoff functions (1) and (2) can thus be written as

$$\begin{aligned} I(s,t) &= -q\,(a + (1-a)\beta(s)) - (1-q)e\alpha(s) && (4)\\ O(s,t) &= -q\,(b - (1+b)\beta(s)) - (1-q)h\alpha(s) && (5) \end{aligned}$$

Note that we actually eliminated the diversion vector m from the strategy of **O** since in equilibrium, it has to be determined by (3). It will however reappear in an auxiliary game to be discussed later.

Proceeding to determine q and s in equilibrium, observe that player **O** 's payoff is $-h\alpha(s)$ if he chooses ℓ and $(1+b)\beta(s) - b$ if he chooses $\bar{\ell}$. It follows that in the (α, β) space player **O** 's best reply to s is (see Figure 3):

- Choose ℓ if $\beta(s) < \frac{b}{1+b} - \frac{h}{1+b}$.
- Choose $\bar{\ell}$ if $\beta(s) > \frac{b}{1+b} - \frac{h}{1+b}$.
- Choose any of the two if $\beta(s) = \frac{b}{1+b} - \frac{h}{1+b}$.

In Figure 3 we give the inspector's payoffs in the (α, β) square as he uses a strategy s and the operator uses his best reply.

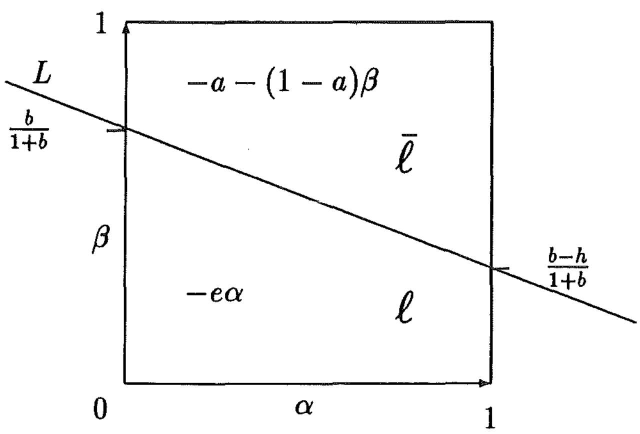

Figure 3: Payoff for **I** in Γ_0, when **I** uses (α, β) test and **O** plays a best reply.

Note that the inspector's payoff is discontinuous along the line L where it is also not defined since the operator, being indifferent between ℓ and $\bar{\ell}$, can use any mixture of them as a best reply to s.

To maximize his payoff, the inspector is not free "to pick a point" in the (α, β) square but rather, he chooses a strategy s which determines the point (α, β). In other words, the domain in Figure 3 is restricted by the line $\beta = \beta(\alpha)$ of the relation between α and the β attainable by the most powerful test, that is

$$\beta(\alpha) = \inf_s \{\beta(s) \mid \alpha(s) = \alpha\},$$

or, substituting $\beta(s)$ from (3),

$$\beta(\alpha) = \inf_{\{s \mid \alpha(s) = \alpha\}} \sup_m \beta(s, m). \tag{6}$$

The function $\beta(\alpha)$ is determined by distribution of the r.v. $X = (X_1, \ldots, X_n)$, and very mild assumptions on this (satisfied for instance by the multivariate normal distribution which we will assume in this paper) yields:

Assumption 2.3 *The line $\beta = \beta(\alpha)$ is well defined for all $\alpha \in [0,1]$. It is differentiable, convex, strictly decreasing and satisfies $\beta(0) = 1, \quad \beta(1) = 0$.*

Looking back to equation (6), it suggests that $\beta(\alpha)$ may be viewed as *minimax value* of some zero sum game. In fact define an auxiliary game G_α, parameterized by α, as follows:

Definition 2.4 *For any $\alpha \in (0,1)$, the game G_α is the zero-sum game with the players* **I** *and* **O** *in which*

- *the pure strategy space of player* **I** *(the maximizer) is the set of all strategies (i.e., Statistical tests) with type I error probability not exceeding α,*

$$\Sigma_\alpha = \{s \mid \alpha(s) \leq \alpha\},$$

- *the pure strategy space of player* **O** *(the minimizer) is the set $\mathcal{M}$ of all diversion vectors*

$$\mathcal{M} = \{m = (m_1, \ldots, m_n) \mid m_i \geq 0, \ i = 1, \ldots, n, \ \sum_i m_i = 1\},$$

- *the payoff (from* **O** *to* **I** *) when (s, m) is played is the detection probability $1 - \beta(s, m)$.*

In other words, the game G_α captures one part of our original game namely, the situation resulting after the operator has decided to behave illegally and the inspector has decided to use tests with false alarm probability of at most α. Thus, G_α does not depend directly on the parameters of the original game, but only through the fact that these parameters determine the value of α in equilibrium.

Proposition (2.3) can now be rewritten in terms of G_α as follows:

Proposition 2.5 *For any $\alpha \in (0,1)$, the game G_α has a value $v(\alpha) = 1 - \beta(\alpha)$ and both players have optimal strategies (i.e.,* sup *and* inf *may be replaced respectively by* max *and* min.*) Furthermore, $v(\alpha)$ is differentiable, concave, strictly increasing with $v(0) = 0$ and $v(1) = 1$.*

For multivariate normally distributed X, the game G_α was solved by Avenhaus (1986) using the Neyman Pearson Lemma (see e.g. Rohatgi (1976)). To state this result, let $X = (X_1, \ldots, X_n)$ be multivariate normally distributed with mean $m = (m_1, \ldots, m_n)$ and covariance matrix Σ, and denote by $\mathbf{1}$ the column $n-$dimensional vector with all coordinates equal 1; $\mathbf{1}' = (1, \ldots, 1)$. Avenhaus' result can now be stated as:

Theorem 2.6 *If $X \sim N(m, \Sigma)$, then the value of the game G_α is*

$$v = 1 - \beta = \phi\left(\frac{1}{\sqrt{\mathbf{1}'\Sigma\mathbf{1}}} - U(1-\alpha)\right),$$

where ϕ is the cumulative standard normal distribution, and U is its inverse. The optimal strategy s_α of the inspector is given by the alarm set A_α:

$$A_\alpha = \{x \mid x' \cdot \mathbf{1} > \sqrt{e'\Sigma\mathbf{1}}U(1-\alpha)\},$$

and the optimal strategy of the operator is the (pure) diversion vector

$$m = \frac{1}{\mathbf{1}'\Sigma\mathbf{1}}\Sigma\mathbf{1}.$$

The interesting points to be noticed here are:

- The operator has a *pure* optimal strategy and thus he need not use a mixed strategy (i.e., randomizing over diversion plans.)
- The operator's optimal test is based on the statistics $x'\mathbf{1}$ that is, it depends *only on the sum of observations x_i*. This is the theoretical basis for the commonly used *CUMUF* test which will reappear in our analysis later on.
- The statistics $x' \cdot \mathbf{1}$ is, as may be easily verified, the material balance statistic for the whole reference period which means that *intermediate real inventories are ignored.*

Returning now to our game, we add the line $\beta = \beta(\alpha)$ to Figure 3 to obtain Figure 4 in which we maximize the inspector's payoff.

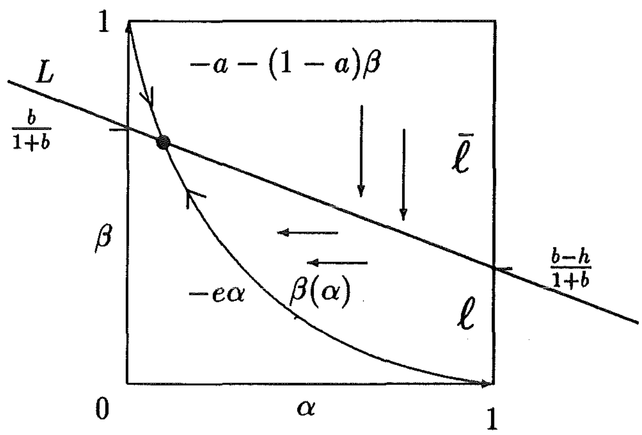

Figure 4: The equilibrium values (α^*, β^*)

As we see in Figure 4, in the region (above the line $\beta = \beta(\alpha)$) in which **O** uses ℓ, the payoff $(-e\alpha)$ decreases in α, while in the region in which **O** uses $\bar{\ell}$, the payoff

$(-a-(1-a)\beta)$ decreases in β. It follows that the maximum is attained at the intersection of the line $\beta = \beta(\alpha)$ and the straight line L, we thus have:

Corollary 2.7 *If (s^*, t^*) is an equilibrium of the game Γ_0, then the corresponding (α^*, β^*) is the unique solution of*

$$\beta(\alpha^*) = \frac{b}{b+1} - \frac{h}{b+1}\alpha^*. \tag{7}$$

Since (α^*, β^*) is on the line L, by the definition of this line, s^* makes the operator indifferent between ℓ and $\bar{\ell}$ and so, adopting an illegal behavior with *any probability* q is a best reply to s^*. However, the payoff of the operator does depend on q since it is given by

$$I(s^*, q) = -q[a + (1-a)\beta(\alpha^*)] - (1-q)e\alpha^*. \tag{8}$$

The value q^* of q at equilibrium is determined by the condition that α^* is a best reply to q^* that is, it is the value of α maximizing

$$I(s, q^*) = -q^*[a + (1-a)\beta(\alpha)] - (1-q^*)e\alpha. \tag{9}$$

Due to the properties of $\beta(\alpha)$ and due to $0 < e < a < 1$, this function has a one local maximum at the point α^* satisfying the first order condition:

$$\frac{e}{q^*} = e - (1-a)\beta'(\alpha^*), \tag{10}$$

where $\beta'(\alpha^*)$ is the derivative with respect to α of $\beta(\alpha)$, evaluated at α^*. Note that $0 < q^* < 1$ as expected from our assumptions. Also, it can be easily verified that α^* satisfying (10) is also the global maximum of (9) i.e., the *unique best reply* to q^*. It follows that the equilibrium values (α^*, q^*) are the solution to equations (7) and (10). Given these values, the rest of the equilibrium strategies (i.e., s^* and the diversion plan m^*) are determined as optimal strategies in the auxiliary game G_{α^*}.

Our discussion of the equilibrium point can now be summarized by the following:

Theorem 2.8 *The game Γ_0 given in Figure 2 has a unique equilibrium $(s^*, (q^*, m^*))$ in which*

- *the false alarm probability $\alpha^* = \alpha(s^*)$ is the solution of (7),*
- *the probability q^* of illegal behavior is given by the solution of (10), and*
- *the strategy s^* and the diversion plan m^* are optimal strategies in the game G_{α^*}.*

Note that the uniqueness claimed in Theorem 2.8 is only for (α^*, β^*, q^*). There may well be more than one s^* or m^* although those may also be unique in special cases (such as the above mentioned multivariate normal distribution case.)

A special case worth noticing is when $\beta(\alpha) = 1 - \alpha$. In this case equations (7) and (10) yield

$$\alpha^* = \frac{1}{1+b-h} \quad \text{and} \quad q^* = \frac{e}{1+e-a},$$

which is the (unique) mixed equilibrium of the matrix game in Table 1. This corresponds to a situation in which the inspector has no specific test procedure for detecting illegal behavior of the operator; he just decides to call an alarm or not without basing his decision on any information. $(\alpha^*, 1-\alpha^*)$ is then just a mixed strategy, α^* being the probability of calling an alarm.

3 A sequential model

The above assumptions that led to the solution of the game using the Neyman Pearson Lemma in the auxiliary game, are admittedly quite restrictive in that they actually suppress the dynamic nature of the problem: time is explicitly assumed to be unimportant, and each player makes *one time* decision. Our aim in the rest of this paper is to relax these assumptions and solve a game theoretical model in which time is important, and the decision process is really sequential. Given the difficulty of the problem, our analysis will be for a *two period* model, although some of the results and ideas are valid for a longer horizon model which we plan to develop in the next stage of this research. More specifically, the features captured in our model are:

- The significant amount of diverted material is 1 (by appropriate choice of units.)
- The operator has no utility for less than 1 unit of mass diverted, and has no increment of utility for any additional mass beyond 1. This implies that (under any reasonable inspection policy) any second period diversion which is neither 0 nor completion to 1 is dominated.
- The operator has the option to *retreat* in second the period from his diversion plan (i.e., not completing the diversion he started with) if it is "too risky".
- Alarm as such has a negative utility for both players. However, the more material "caught" to be diverted the worse it is for the operator and (relatively) better it is for the inspector.
- Time is valuable: Any amount of material diverted at first period, carries an *additional penalty* for the inspector to the second period.
- The negative effect for the inspector of an undetected diversion, increases with the amount diverted (and may have discontinuous jump at the critical mass 1.)

The two person non zero-sum game with players **O** (operator) and **I** (inspector) is played in two periods as follows:

Period 1

- **O** chooses $m \in [0,1]$ (mass diverted in first period.)
- A random variable $X_1 \sim N(m, \sigma_1^2)$ is observed by both players (denote this observation by x_1, this is MUF_1).

- I chooses one of the two actions: C_1 (continue to second period) or A_1 (declare an alarm) which stops the game and assigns payoffs $(-a(m), -b(m))$ to **I** and **O** respectively.

Period 2

- **O** chooses one of the two actions: D (complete diversion to 1) or R (retreat.)
- A random variable X_2 is observed by both players (denote this observation by x_2, this is MUF_2.) The conditional distribution of X_2 given m and x_1 is

$$
\begin{aligned}
X_2(D, m, x_1) &\sim N(1 - m + \rho\frac{\sigma_2}{\sigma_1}(x_1 - m)\ ,\ (1-\rho^2)\sigma_2^2) \\
X_2(R, m, x_1) &\sim N(\rho\frac{\sigma_2}{\sigma_1}(x_1 - m)\ ,\ (1-\rho^2)\sigma_2^2)
\end{aligned}
$$

  (σ_1, σ_2 and ρ are given parameters of the joint distribution of (X_1, X_2).)

- **I** chooses one of the two actions: C_2 (no alarm) or A_2 (declare an alarm.) In both cases the game ends and payoffs are made as indicated in Figure 5.

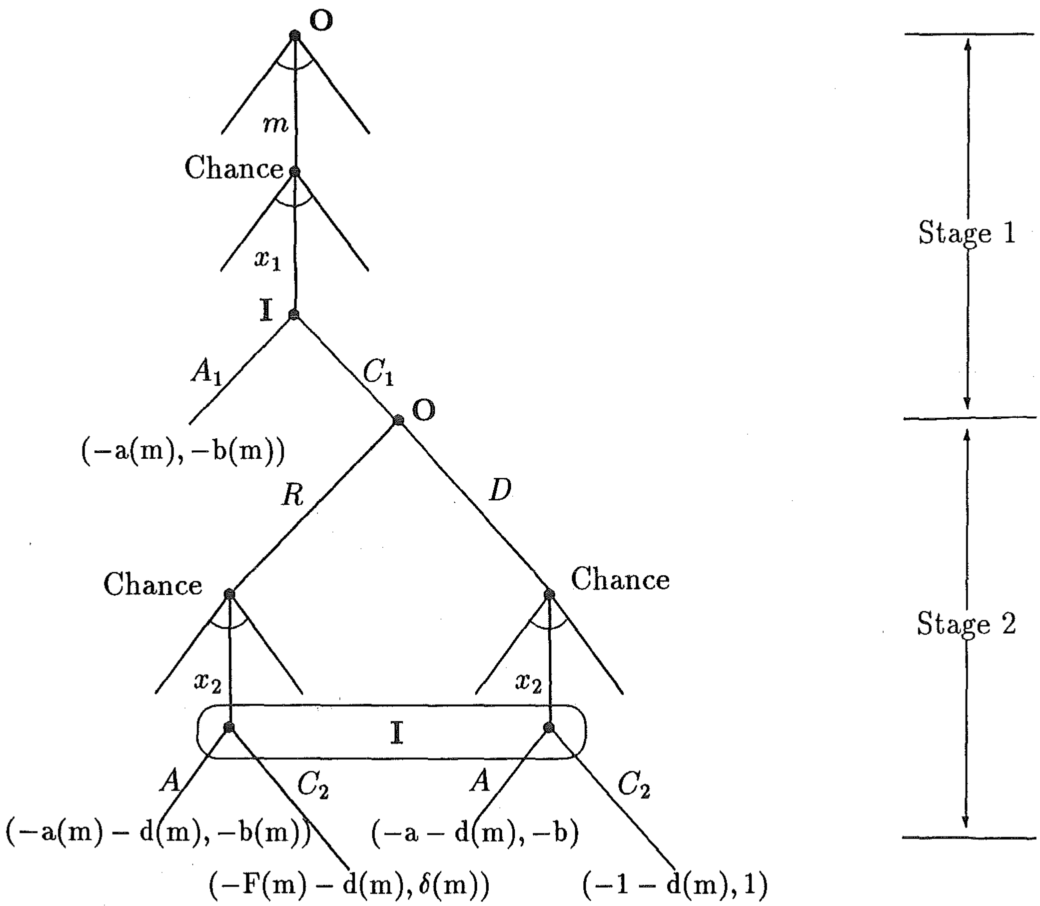

Figure 5: The extensive form game.

Payoff functions

In the payoffs indicated in Figure 5 $\delta(m)$ is a step function which is 1 if $m = 1$ and 0 if $m < 1$. All functions and constants are non negative. We make the following assumptions:

- Undetected diversion of critical amount 1, which is the worst event for **I** and the best for **O** , corresponds to payoffs $(-1, 1)$ (by appropriate choice of utility units.)
- The disutility of **I** from undetected diversion of amount m is $F(m)$ which we assume increasing in m, $F(0) = 0$ and $F(1) = 1$.
- Any alarm results in a disutilities $-a(m)$ and $-b(m)$, for **I** and **O** respectively, where m is the amount which is actually diverted. We assume that $a(m)$ decreases in m, and $b(m)$ increases in m, with $a(1) = a < 1$ and $b(1) = b > 0$ corresponding to the disutilities for detection of a full diversion. The motivation for this assumption is that, since as long as $m < 1$ the inspector has no direct disutility from a diversion of amount m, his disutility $-a(m)$ from an alarm is the damage for his reputation and reliability: As m is higher, the alarm is 'more justified' and hence there is less damage for the inspector's reputation. As for the operator; a larger m is more evident for 'intended illegal action' which may provoke a more severe punishment and more damage for his reputation.
- The function $d(m)$ is the (additional) disutility to the inspector from early diversion. We assume that it is an increasing function of m and $d(0) = 0$.

We do not assume continuity of the payoff functions, which may typically be discontinuous at the critical mass $m = 1$.

Strategies

Noting that the game under consideration is a game of *perfect recall*, there is no loss of generality, and in fact it is more natural in our context, to consider only *behavioral strategies* (Aumann (1964)), which are:

- A strategy of the inspector is an ordered pair $s = (a_1, a_2)$, where a_1 is a transition probability from $\mathcal{R}$ to $\{A_1, C_1\}$ and a_2 is a transition probability from $\mathcal{R}^2$ to $\{A_2, C_2\}$.

  Interpretation: if $X_1 = x_1$ then **I** chooses A_1 (i.e., calls an alarm) with probability $a_1(x_1)$. If the game reaches the second period with $X_1 = x_1$ and $X_2 = x_2$, then **I** calls an alarm according to the cumulative probability distribution $a_2(x_1, x_2)$. Given a strategy $s = (a_1, a_2)$, for any $x_1 \in \mathcal{R}$ let

  $$\mathbf{A}_2(x_1) = \{x_2 \in \mathcal{R} \mid a_2(x_1, x_2) = A_2\} \text{ and } \mathbf{C}_2(x_1) = \{x_2 \in \mathcal{R} \mid a_2(x_1, x_2) = C_2\}.$$

  So given x_1 and given that no alarm was called at first period, then $\mathbf{A}_2(x_1)$ is the set of MUF_2 observations that will surely provoke an alarm while $\mathbf{C}_2(x_1)$ is the set of MUF_2 observations that will surely not provoke an alarm.

- A strategy of the operator is an ordered pair $t = (q, p)$ where q is a measure on $[0, 1]$, the probability distribution of the first period diversion m, and p is a transition probability from $[0, 1] \times \mathcal{R}$ to $\{D, R\}$ that is, $p(m, x_1) = P(D \mid m, x_1)$ is the probability of completing the diversion at second period given that m was already diverted at first period, the observed X_1 (i.e., MUF_1) was x_1 and the **I** did not call an alarm at first period.

Σ and T denote the strategy sets of **I** and **O** respectively. Observe that both sets are convex. Any pair of strategies (s, t) determines a probability distribution on outcomes and hence expected payoff functions for the players which we denote by $I(s, t)$ (for the inspector) and $O(s, t)$ (for the operator.)

4 The structure of equilibrium

Our main objective is to study the equilibrium, as given by Definition 2.2, of the above described game. The major problem here is the size and complexity of the strategy sets which makes it practically impossible even to write the equilibrium conditions in a workable form. We prove first that for finding equilibrium points, only much smaller sets of strategies may be considered. Only *threshold* strategies can be equilibrium strategies. More precisely we have the following theorems the proofs of which are given in the Appendix.

Theorem 4.1 *(Second period test for the inspector.) If s is an equilibrium strategy of* **I** *then:*

(i) *To any x_1 not followed by an alarm at first period, there is a critical value $c_2(x_1) \in (-\infty, \infty)$ such that:*

$$x_2 > c_2(x_1) \implies a_2(x_1, x_2) = 1 \quad \text{(i.e., call alarm surely.)}$$
$$x_2 < c_2(x_1) \implies a_2(x_1, x_2) = 0 \quad \text{(i.e., surely, alarm is not called.)}$$

(ii) *The function $c_2(x_1)$ is decreasing in x_1*

Theorem 4.2 *(Second period test for the operator.) If t is an equilibrium strategy of* **O** *then:*

(i) *for (almost) each x_1 there is $m^*(x_1)$ such that*

$$m > m^*(x_1) \implies p(m, x_1) = 1$$
$$m < m^*(x_1) \implies p(m, x_1) = 0$$

(ii) *The function $m^*(x_1)$ is increasing in x_1. Thus, the inverse function $x^*(m)$ is increasing in m.*

The function x^* is a threshold function: If **O** diverted m at the first period, then he completes his diversion if $x_1 \leq x^*(m)$ and retreats if $x_1 > x^*(m)$.

The last theorem in this section establishes the first period threshold in the equilibrium inspection strategy.

Theorem 4.3 *(First period test for the inspector.) If s is an equilibrium strategy of* **I** *then the alarm set at the first period is determined by a critical value c_1, that is:*

$$\begin{array}{lll} x_1 > c_1 & \Longrightarrow & a_1(x_1) = 1 \text{ (i.e., call alarm surely.)} \\ x_1 < c_1 & \Longrightarrow & a_1(x_1) = 0 \text{ (i.e., surely, alarm is not called.)} \end{array}$$

When $x_1 = c_1$, the inspector is indifferent between calling an alarm or not. However, in our model this event has zero probability.

5 The *CUMUF* test as an equilibrium strategy

To 'calibrate' our game theoretical model we shall first relate it to the existing statistical treatment of the problem. One of the commonly used statistical test is the *CUMUF* test according to which no alarm is called before the end of the last period, and the statistics for the test is *the sum* of the *MUF* 's in all periods. Can this test emerge from our game model ? In this section we prove that in fact, the *CUMUF* test is part of the solution of our game model if we make some appropriate assumptions.

5.1 Assumptions underlying the statistical treatment

Evidently, to obtain the *CUMUF* as a game theoretical solution, we have to impose some restrictions on the game (i.e., on the payoffs and on the strategies) which reflect the underlying assumptions of this statistical test. These are basically that the payoffs are proportional to the false alarm probability and the probability of undetected diversion. This is captured by the following assumption (which incorporates our previous assumption regarding the critical amount of 1 as a significant diversion.)

Assumption 5.1

$$\begin{array}{rrcl} (i) & a(m) & = & a \ \ \forall m \in \mathcal{R} \\ (ii) & b(m) & = & b \ \ \forall m \in \mathcal{R} \\ (iii) & d(m) & = & 0 \ \ \forall m \in [0,1] \\ (iv) & F(m) & = & \begin{cases} 0 & m < 1 \\ 1 & m \geq 1 \end{cases} \end{array} \tag{11}$$

for some constants $0 < a < 1$ *and* $b > 0$.

In words this assumption means that the damage of an alarm, which need not be the same for both agents, is independent of the amount of material actually diverted ((i) and (ii)). Undetected diversion is relevant if and only if it exceeds the critical value 1 ((iv), recall that incorporated in our model is also the similar assumption regarding the operator.) And finally, part (iii) states that detection time is unimportant.

Additional restriction is needed on the **O** 's strategy set due to the fact that the statistical framework is basically static: The operator has to decide *at the beginning of the game* whether or not to divert. If he diverts a positive amount m at the first period, he also (surely) diverts $1-m$ at the second period.

Assumption 5.2 *The operator is restricted to strategies $t = (q, m^*)$ in which $m^*(x_1)$ is constant in x_1.*

With these assumptions, a strategy of the operator is a pair $t = (q, m)$ in which q is the probability of diversion and m is the amount to be diverted at the first period. The inspector's strategy is $s = (c_1, c_2(\cdot))$, and the payoff functions become

$$\begin{aligned} I(s,t) &= (1-q)I(s,0) + qI(s,m) \\ O(s,t) &= (1-q)O(s,0) + qO(s,m) \end{aligned}$$

where

$$\begin{aligned} I(s,0) &= -a + \frac{a}{\sigma_1\sqrt{2\pi}} \int_{-\infty}^{c_1} \phi\left(\frac{c_2(x_1) - \rho\frac{\sigma_2}{\sigma_1}x_1}{\sigma_2\sqrt{1-\rho^2}}\right) \exp\left(-\frac{x_1^2}{2\sigma_1^2}\right) dx_1 \\ O(s,0) &= -b + \frac{b}{\sigma_1\sqrt{2\pi}} \int_{-\infty}^{c_1} \phi\left(\frac{c_2(x_1) - \rho\frac{\sigma_2}{\sigma_1}x_1}{\sigma_2\sqrt{1-\rho^2}}\right) \exp\left(-\frac{x_1^2}{2\sigma_1^2}\right) dx_1 \\ I(s,m) &= -a - \frac{1-a}{\sigma_1\sqrt{2\pi}} \int_{-\infty}^{c_1} \phi\left(\frac{c_2(x_1) - 1 + m - \rho\frac{\sigma_2}{\sigma_1}(x_1 - m)}{\sigma_2\sqrt{1-\rho^2}}\right) \exp\left(-\frac{(x_1-m)^2}{2\sigma_1^2}\right) dx_1 \\ O(s,m) &= -b + \frac{1+b}{\sigma_1\sqrt{2\pi}} \int_{-\infty}^{c_1} \phi\left(\frac{c_2(x_1) - 1 + m - \rho\frac{\sigma_2}{\sigma_1}(x_1 - m)}{\sigma_2\sqrt{1-\rho^2}}\right) \exp\left(-\frac{(x_1-m)^2}{2\sigma_1^2}\right) dx_1 \end{aligned}$$

where ϕ is the cumulative normal distribution function,

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp(-\frac{u^2}{2}) du.$$

Remarks

- $I(s,0)$ and $O(s,0)$ increase both in c_1 and c_2 reflecting the fact that if no diversion is committed it is better for both agents to avoid a false alarm.
- $I(s,m)$ is a decreasing function both of c_1 and c_2 and goes to $-a$ as $c_1 \to -\infty$ reflecting the fact that if a diversion of 1 was committed, the best thing that can happen to **I** is to call an alarm and get $-a$, his highest possible payoff in this event.

5.2 Equilibrium

Recalling that, according to Definition 2.2, an equilibrium point is a pair of strategies (s^*, t^*) satisfying

$$\begin{aligned} I(s^*,t^*) &\geq I(s,t^*) \ \forall s \in S \\ O(s^*,t^*) &\geq O(s^*,t) \ \forall t \in T, \end{aligned}$$

we have first the following observations,

- There exists no equilibrium with $q = 0$ (i.e., **O** surely does not divert). This is because the best reply of **I** to that is $c_1 = c_2 = \infty$ namely, never call an alarm. But this cannot be an equilibrium strategy for the operator since then **O** can profit by diverting safely.
- $c_1 = -\infty$ (or $c_2 \equiv -\infty$), together with $q = 1$ consists of an equilibrium in which **O** surely diverts and **I** always calls an alarm independently of the *MUF* observations. This is of course a non interesting equilibrium of an *open conflict* between **I** and **O** .
- The 'open conflict' equilibrium is the only equilibrium in which $q = 1$. This is obvious since the only best reply to a sure diversion is a sure alarm. We conclude that except for this trivial equilibrium in any equilibrium we have $0 < q < 1$.

Next we have

Proposition 5.3 *In equilibrium, for almost all x_1 (with respect to the normal density), $c_2(x_1) < \infty$.*

Proof If H_0 is the null hypothesis that no diversion took place and H_1 is the alternative hypothesis that there was a diversion, then the likelihood ratio of (x_1, x_2) for these two hypothesis is

$$L(x_1, x_2) = \frac{e^{-\frac{1}{2\sigma_1^2}x_1^2} e^{-\frac{1}{2\sigma^2}(x_2 - \rho\frac{\sigma_2}{\sigma_1}x_1)^2}}{e^{-\frac{1}{2\sigma_1^2}(x_1-m)^2} e^{-\frac{1}{2\sigma^2}(x_2-(1-m)-\rho\frac{\sigma_2}{\sigma_1}(x_1-m))^2}},$$

where $\sigma^2 = (1-\rho^2)\sigma_2^2$. For any x_1, this likelihood ratio tends to zero as $x_2 \to \infty$ (recall that $q > 0$). It follows that for some finite $c_2(x_1)$,

$$x_2 > c_2(x_1) \Rightarrow \quad -a > 0 \cdot Pr(H_0|x_1, x_2) + (-1) \cdot Pr(H_1|x_1, x_2),$$

and therefore **I** calls alarm whenever $x_2 > c_2(x_1)$.

∎

Assume that $(s^*, t^*) = (c_1, c_2(\cdot); q, m)$ is an equilibrium with $0 < q < 1$. Since **O** is choosing both diversion and non diversion with positive probability, it must be that both alternatives yields for him the same expected payoff (otherwise t^* could be improved by choosing the better alternative with probability one) i.e., $O(s^*, 0) = O(s*, m)$ which is,

$$\frac{\int_{-\infty}^{c_1} \phi\left(\frac{c_2(x_1)-(1-m)-\rho\frac{\sigma_2}{\sigma_1}(x_1-m)}{\sigma_2\sqrt{1-\rho^2}}\right) \exp\left(-\frac{(x_1-m)^2}{2\sigma_1^2}\right) dx_1}{\int_{-\infty}^{c_1} \phi\left(\frac{c_2(x_1)-\rho\frac{\sigma_2}{\sigma_1}x_1}{\sigma_2\sqrt{1-\rho^2}}\right) \exp\left(-\frac{x_1^2}{2\sigma_1^2}\right) dx_1} = \frac{b}{1+b}. \tag{12}$$

The expected payoff of **O** must attain its maximum at the equilibrium value m that is

$$m = \arg\max O(s^*, \tilde{m}),$$

which is (omitting the constants),

$$m = \arg\max \left\{ \int_{-\infty}^{c_1} \phi\left(\frac{c_2(x_1) - (1-\tilde{m}) - \rho\frac{\sigma_2}{\sigma_1}(x_1 - \tilde{m})}{\sigma_2\sqrt{1-\rho^2}}\right) \exp\left(-\frac{(x_1-\tilde{m})^2}{2\sigma_1^2}\right) dx_1 \right\}. \tag{13}$$

Equations (12) and (13) express the condition that t^* is a *best reply* to s^*. The fact that s^* is a best reply to t^* is expressed by the following two condition:

Following x_1 and $c_2(x_1)$, the inspector must be indifferent whether or not to call an alarm in the second stage. This follows from the continuity of the payoff function $I(s,t)$ in x_2, since **I** prefers alarm when $x_2 > c_2(x_1)$ and not alarm when $x_2 < c_2(x_1)$. The payoff for alarm is $-a$ and for non alarm is $(-1) \cdot Pr(H_1|x_1, c_2(x_1))$ hence we must have

$$-a = (-1) \cdot Pr(H_1|x_1, c_2(x_1)),$$

which is

$$a = \frac{q e^{-\frac{1}{2\sigma_1^2}(x_1-m)^2} e^{-\frac{1}{2\sigma^2}(x_2-(1-m)-\rho\frac{\sigma_2}{\sigma_1}(x_1-m))^2}}{q e^{-\frac{1}{2\sigma_1^2}(x_1-m)^2} e^{-\frac{1}{2\sigma^2}(x_2-(1-m)-\rho\frac{\sigma_2}{\sigma_1}(x_1-m))^2} + (1-q) e^{-\frac{x_1^2}{2\sigma_1^2}} e^{-\frac{1}{2\sigma^2}(x_2-\rho\frac{\sigma_2}{\sigma_1}x_1)^2}}. \tag{14}$$

Finally, the first stage threshold c_1 must be the maximizer of the inspector's payoff function, that is (omitting constants):

$$\begin{aligned} c_1 \;=\; & \arg\max \Big\{ (1-q)a \int_{-\infty}^{\tilde{c}} \phi\left(\frac{c_2(x_1)-\rho\frac{\sigma_2}{\sigma_1}x_1}{\sigma_2\sqrt{1-\rho^2}}\right) \exp\left(-\frac{x_1^2}{2\sigma_1^2}\right) dx_1 \\ & -q(1-a) \int_{-\infty}^{\tilde{c}} \phi\left(\frac{c_2(x_1)-1+m-\rho\frac{\sigma_2}{\sigma_1}(x_1-m)}{\sigma_2\sqrt{1-\rho^2}}\right) \exp\left(-\frac{(x_1-m)^2}{2\sigma_1^2}\right) dx_1 \Big\}. \end{aligned} \tag{15}$$

The four equations (12) - (15) are the equilibrium conditions. It turns out that they uniquely determine the equilibrium strategies (s^*, t^*).

Theorem 5.4 (i) *The game has an equilibrium (s^*, t^*) in which the strategies $s^* = (c_1, c_2(\cdot))$ and $t^* = (q, m)$ are determined as follows:*

- $c_1 = \infty$.

- $c_2 = C - x_1$, *where C is the unique solution of the equation:*

$$\frac{\phi\left(\dfrac{C-1}{\sqrt{\sigma_1^2+2\rho\sigma_1\sigma_2+\sigma_2^2}}\right)}{\phi\left(\dfrac{C}{\sqrt{\sigma_1^2+2\rho\sigma_1\sigma_2+\sigma_2^2}}\right)} = \frac{b}{1+b}. \tag{16}$$

- $m = \dfrac{1+\rho\frac{\sigma_2}{\sigma_1}}{1+2\rho\frac{\sigma_2}{\sigma_1}+\frac{\sigma_2^2}{\sigma_1^2}}$.
- $\dfrac{1}{q} = 1 + \left(\dfrac{1}{a}-1\right)\exp\left\{-\dfrac{m^2}{2\sigma_1^2} - \dfrac{\left(1-m\left(1+\rho\frac{\sigma_2}{\sigma_1}\right)-C\right)^2 - C^2}{2\sigma_2^2(1-\rho^2)}\right\}$.

(ii) *Under Assumption 8.1 (to be stated later), this equilibrium is unique.*

Proof Appendix 8.4

In the equilibrium described in Theorem 5.4, no test is made at the first period and the statistics for the second period test is $x_1+x_2 = MUF_1 + MUF_2$, and an alarm is called if $MUF_1 + MUF_2 > C$. This is precisely the *CUMUF* test.

It is interesting to see the relationship between Theorem 5.4 and Theorem 2.8 (for the special case of two stages). Both Theorems provide the solution (equilibrium) to the same problem but with two different representations. In Theorem 2.8 the strategy of the inspector is solved by determined the false alarm probability α which is then used to solve the auxiliary game G_α to obtain the threshold C of the Neyman Pearson test. In Theorem 5.4 there is no auxiliary game and C is determined directly (not through α) by equation (16) which is closely related to the likelihood ratio and the Neyman-Pearson test. The false alarm probability α, as well as the diversion probability q and the non detection probability β are then computed explicitly from C.

Note also that the value of the diversion vector, $(m, 1-m)$, is the same in both theorems and it is $(1/2, 1/2)$ if $\sigma_1 = \sigma_2$, independently of ρ.

Part (ii) of Theorem 5.4 asserts that except for extreme configurations of distribution structure which do not satisfy Assumptions 8.1, this *CUMUF* equilibrium is the only equilibrium of the game. In particular this is the case if $\sigma_1 = \sigma_2$. When Assumptions 8.1 is not satisfied (but still satisfy $\rho < 0$), an additional equilibrium exists which is given in the following two Theorems, the proofs of which are given in Appendix 8.4 as well.

Theorem 5.5 *If $\rho < -\sigma_1/\sigma_2$ then there is (only) one more equilibrium (s^*, t^*) in which the strategies $s^* = (c_1, c_2(\cdot))$ and $t^* = (q, m)$ are determined as follows:*

- $c_1 = \infty$.

- $c_2 = C + \rho\frac{\sigma_2}{\sigma_1}x_1$, *where C is the unique solution of the equation:*

$$\frac{\phi\left(\dfrac{C-1}{\sigma_2\sqrt{1-\rho^2}}\right)}{\phi\left(\dfrac{C}{\sigma_2\sqrt{1-\rho^2}}\right)} = \frac{b}{1+b}. \tag{17}$$

- $m = 0$.

- *q is the same function of C and m as in Theorem 5.4.*

Theorem 5.6 *If $\rho < -\sigma_2/\sigma_1$ then there is (only) one more equilibrium (s^*, t^*) in which the strategies $s^* = (c_1, c_2(\cdot))$ and $t^* = (q, m)$ are determined as follows:*

- $c_1 = \infty$.

- $c_2 = C + \frac{\sigma_2}{\rho\sigma_1}x_1$, *where C is the unique solution of the equation:*

$$\frac{\phi\left(\dfrac{-\rho C - \frac{\sigma_2}{\sigma_1}}{\sigma_2\sqrt{1-\rho^2}}\right)}{\phi\left(\dfrac{-\rho C}{\sigma_2\sqrt{1-\rho^2}}\right)} = \frac{b}{1+b}. \tag{18}$$

- $m = 1$.

- *q is the same function of C and m as in Theorem 5.4.*

To get the intuition of the last two theorems, fix $\rho < 0$ and consider the ratio σ_1/σ_2. If σ_1 is much smaller than σ_2 (so as to have $\rho < -\sigma_1/\sigma_2$) then **O** puts all diversion in second stage since MUF_1 is measured much more precisely than MUF_2. In the reverse situation, when $\rho < -\sigma_2/\sigma_1$), then $m = 1$ i.e., all the diversion is made at the first stage.

6 Timely detection and the retreat option

After 'calibrating' our game theoretical model, by deriving the *CUMUF* test as part of the solution of a special case, we turn now back to the general model presented in section 3. Our object is to take now into account the two features which make the model 'truly sequential' namely,

- The importance of timely detection, which is captured by the extra-penalty function $d(m)$. This is a negative payoff $-d(m)$ incurred by the inspector for a diversion of m at the first stage.

- The option of the operator to make a decision after the first stage of the game whether to go on with his diversion plan or to retreat since it looks too risky to do so in view of the first stage inspection (MUF_1).

These two elements of the model are linked naturally together; The option of the inspector to make decisions in the process of the game i.e., to call an alarm at any stage, existed already in the static model treated in section 2 and the special case of the sequential model treated in section 5. The fact that it turns out that the inspector *does not* call an alarm before the end of the game, is a feature of the solution which is due to the assumption that time is not important which is, $d(m) \equiv 0$. If we depart from this assumption, as we shall see, the inspector *will use* his option to call an alarm in the first stage (that is $c_1 < \infty$) and it is most natural to provide the operator with a similar option to reconsider his situation and decide about his moves after each stage of the game.

We shall first write down the equilibrium equations for the general model which turn out to be rather complex and hardly manageable analytically. In view of that we take a special simple case of payoff functions which is 'the simplest change' of the payoff structure in section 5 incorporating the above mentioned new elements. This will also allow an easy comparison between the two models and hence an easy measure of the effect of these two 'sequential' elements.

6.1 Equilibrium equations

By virtue of Theorems 4.1 - 4.3, as long as we are interested only in the study of equilibria, the a priori huge sets of strategies can be reduced to strategies of rather simple forms: The inspector strategies are, as before, a pair $s = (c_1, c_2(\cdot))$ of first and second stage threshold for alarm. A strategy of the operator is a pair $t = (q, m^*(\cdot))$ where $q \in \Delta[0,1]$ is a probability distribution of m - the quantity which he diverts at the first stage, and $m^*(\cdot)$ is a threshold function $m^* : \mathcal{R}^{\to}[0,1]$ to be used as follows: If m was diverted at the first stage and the observed $MUF\,1$ was x_1:

- Retreat if $m^*(x_1) < m$.
- Complete diversion (i.e., divert $1 - m$) if $m^*(x_1) \geq m$.

Even after this restriction of the strategies, the corresponding sets S and T are still too large and it is impractical to find an equilibrium directly from the definition:

$$I(s,t) \geq I(s',t) \quad \forall s' \in S$$

$$O(s,t) \geq O(s,t') \quad \forall t' \in T.$$

We therefore derive some simpler equations to be satisfied by the equilibrium strategies. These equations are derived not directly from the definition but rather from the following simple property of the equilibrium strategies: If in equilibrium a player is indifferent between two actions, it must be that the two actions yield him the same expected payoff. In particular if he randomizes between two actions, he must be indifferent between them (since otherwise he should use only the preferred action) and hence they both yield the same (maximal) expected payoff. Using this simple principle we have the following:

- Given that **O** has already diverted the amount m, the value x_1 was observed and **I** did not call an alarm (since $x_1 < c_1$), then **O** has two alternatives: Either to retreat (R), in which case his expected payoff is (denoting $\sigma = \sigma_2\sqrt{1-\rho^2}$)

$$O(R \mid m, x_1) = \delta(m)\phi\left(\frac{c_2(x_1)-\mu_0}{\sigma}\right) - b(m)\left[1-\phi\left(\frac{c_2(x_1)-\mu_0}{\sigma}\right)\right]$$

which is

$$O(R \mid m, x_1) = -b(m) + (\delta(m)+b(m))\phi\left(\frac{c_2(x_1)-\mu_0}{\sigma}\right),$$

or to complete the diversion (D) in which case his expected payoff is

$$O(D \mid m, x_1) = -b + (1+b)\phi\left(\frac{c_2(x_1)-\mu_1}{\sigma}\right).$$

Now, at the threshold $m = m^*(x_1)$ the operator is indifferent between these two alternatives hence $O(R \mid m^*(x_1), x_1) = O(D \mid m^*(x_1), x_1)$ from which we have that the equation

$$(1+b)\phi\left(\frac{c_2(x_1)-\mu_1}{\sigma}\right) = (b-b(m)) + (\delta(m)+b(m))\phi\left(\frac{c_2(x_1)-\mu_0}{\sigma}\right) \tag{19}$$

has to be satisfied at $m = m^*(x_1)$.

- When **O** chooses a certain diversion m he will later, depending on the value of x_1 choose R or D and receive the higher of $O(R \mid m, x_1)$ and $O(D \mid m, x_1)$. Therefore his expected payoff (before observing x_1) is

$$\int_{-\infty}^{s_1} M(m, x_1)\varphi_m(x_1)\, dx_1 \; - \int_{s_1}^{\infty} b(m)\varphi_m(x_1)\, dx_1\ , \tag{20}$$

where $M(m, x_1)$ stands for:

$$\max\left\{-b(m) + (\delta(m)+(m))\phi\left(\frac{c_2(x_1)-\mu_0}{\sigma}\right),\ -b+(1+b)\phi\left(\frac{c_2(x_1)-\mu_1}{\sigma}\right)\right\}.$$

It follows that support of the distribution $q(m)$ must be contained in the set of maximizers of the function in (20). In particular if this function attains its maximum at finitely many points m, then only these values of m can have a positive q probability.

- Consider the second stage decision of the inspector: If $x_1 < c_1$ was observed at the first stage and $x_2 = c_2(x_1)$ at the second stage, then by the definition of $c_2(\cdot)$ (and the continuity of the payoff functions in c_2), the inspector should be indifferent between calling or not calling an alarm. Hence the two actions yield in that case the same expected payoff i.e.,

$$
\begin{aligned}
&-\int_0^{m^*(x_1)-} a(m)dq(m \mid x_1, c_2(x_1)) - \int_{m^*(x_1)}^1 a dq(m \mid x_1, c_2(x_1)) = \\
&-\int_0^{m^*(x_1)-} F(m)dq(m \mid x_1, c_2(x_1)) - \int_{m^*(x_1)}^1 F dq(m \mid x_1, c_2(x_1)),
\end{aligned}
$$

which is

$$
\begin{aligned}
&\int_0^{m^*(x_1)-}(a(m)-F(m))\varphi_m(x_1)e^{-\frac{1}{2}\left(\frac{c_2(x_1)-\mu_0}{\sigma}\right)^2} dq(m) = \\
&\int_{m^*(x_1)}^1 (F-a)\varphi_m(x_1)e^{-\frac{1}{2}\left(\frac{c_2(x_1)-\mu_1}{\sigma}\right)^2} dq(m)\ .
\end{aligned}
\tag{21}
$$

- Finally consider I 's decision at the first stage. If $x_1 = c_1$ is observed then, by the definition of c_1, the inspector is indifferent between calling or not calling an alarm at the first stage since both alternatives yield the same expected payoff that is:

$$
\begin{aligned}
&-\int_0^1 a(m)dq(m \mid x_1 = c_1) = \\
&-\int_0^1 d(m)dq(m \mid x_1 = c_1) \\
&-\int_0^{m^*(c_1)-}\left[a(m)\left(1-\phi\left(\frac{c_2(c_1)-\mu_0}{\sigma}\right)\right) + F(m)\phi\left(\frac{c_2(c_1)-\mu_0}{\sigma}\right)\right] dq(m \mid x_1 = c_1) \\
&-\int_{m^*(c_1)}^1 \left[a\left(1-\phi\left(\frac{c_2(c_1)-\mu_1}{\sigma}\right)\right) + F\phi\left(\frac{c_2(c_1)-\mu_1}{\sigma}\right)\right] dq(m \mid x_1 = c_1),
\end{aligned}
$$

which can be also written as:

$$
\begin{aligned}
&\int_0^1 d(m)\varphi_m(s_1)\ dq(m) = \\
&\int_0^{m^*(s_1)-}(a(m)-a)\varphi_m(c_1)\ dq(m) \\
&-\int_0^{m^*(c_1)-}[F(m)-a(m)]\phi\left(\frac{c_2(c_1)-\mu_0}{\sigma}\right)\varphi_m(c_1)\ dq(m) \\
&-\int_{m^*(c_1)}^1 [F-a]\phi\left(\frac{c_2(c_1)-\mu_1}{\sigma}\right)\varphi_m(c_1)\ dq(m)\ .
\end{aligned}
\tag{22}
$$

The four conditions (19) to (22) are to be satisfied at any equilibrium point (s,t). However these are not straightforward equations to solve and there is no hope for closed analytic solutions.

As a first step we start with a special and relatively simple set of parameters we introduced in Assumption 5.1 except that (iii) is replaced by

$(iii')\quad d(m) = d \cdot m \ \ \forall m \in [0,1]$,

where $d > 0$ that is, we are assuming the extra damage due to late detection to be linear in the amount m diverted in the first period.

Except for providing a simple case to start with, the adoption of Assumptions 5.1 will enable us to compare the equilibria in the two models and see the effect of the two new features namely, the importance of time (expressed by the linear function dm) and the possibility of the operator to retreat from a diversion move he started with.

Under the above assumptions the four equilibrium conditions take somewhat simpler forms, namely,

$$\frac{\phi\left(\dfrac{c_2(x_1)-\mu_1(m,x_1)}{\sigma}\right)}{\phi\left(\dfrac{c_2(x_1)-\mu_0(m,x_1)}{\sigma}\right)}=\frac{b}{1+b} \quad \text{at } m=m^*(x_1)\ \forall x_1\in\mathcal{R}\,. \tag{23}$$

The second condition is that the support of the distribution $q(m)$ must be contained in the set of maximizers of the function $G(m)$ given by:

$$G(m)=-b+\int_{-\infty}^{s_1}\max\left\{b\phi\left(\frac{c_2(x_1)-\mu_0}{\sigma}\right),\ (1+b)\phi\left(\frac{c_2(x_1)-\mu_1}{\sigma}\right)\right\}\varphi_m(x_1)\,dx_1\,. \tag{24}$$

The third equation becomes:

$$a\int_0^{m^*(x_1)-}\varphi_m(x_1)e^{-\frac{1}{2}\left(\frac{c_2(x_1)-\mu_0}{\sigma}\right)^2}\,dq(m)=(1-a)\int_{m^*(x_1)}^{1}\varphi_m(x_1)e^{-\frac{1}{2}\left(\frac{c_2(x_1)-\mu_1}{\sigma}\right)^2}\,dq(m)\,. \tag{25}$$

Finally the last equation is:

$$\begin{aligned}&\alpha\textstyle\int_0^1 m\varphi_m(c_1)\,dq(m)=\\&a\textstyle\int_0^{m^*(c_1)-}\phi(\frac{c_2(c_1)-\mu_0}{\sigma})\varphi_m(c_1)\,dq(m)-(1-a)\int_{m^*(c_1)}^{1}\phi(\frac{c_2(c_1)-\mu_1}{\sigma})\varphi_m(c_1)\,dq(m)\,.\end{aligned} \tag{26}$$

6.2 Partial results

The major difficulty in attempting to solve equations (23) to (26) is the fact that one of the "unknowns" is a probability distribution q on $[0,1]$ which determines the first period diversion. In spite of richness of the set from which q can be chosen, we strongly conjecture that in equilibrium, q belongs to a much smaller (and hence more manageable) set of distributions namely, the support of q consists of only two values of m, one of which is $m=0$. First we can prove that this distribution is the simplest possible, that is:

Theorem 6.1 *There is no equilibrium in which q is singular distribution which assigns probability 1 to some value $m\in[0,1]$.*

The proof which is rather lengthy and technical is given in Appendix 8.5.

Assume that in the strategy of **O** the probability distribution q is of finite support. That is consider **O** 's strategies in which the first period diversion can take one of the values $(m_1,m_2,\ldots,m_n)$ with probabilities $(q_1,q_2,\ldots,q_n)$ respectively. Assume without loss of generality that $0\le m_1<m_2<\ldots<m_n\le 1$. For this to be part of an equilibrium strategy, the second period behavior is given by a set of critical values $x_1^*,x_2^*,\ldots,x_n^*$ with the interpretation that given that m_i was diverted at the

first period, then if $x_1 \leq x_i^*$, divert $1-m_i$ at the second period, otherwise retreat from the diversion plan. Furthermore, since $x^*(m)$ is increasing in m (by Theorem 4.2) we have: $x_1^* < x_2^* < \ldots < x_n^*$. We draw some conclusions about an equilibrium involving this type of strategy.

To begin with, is it possible that $x_1^* > -\infty$? This would mean that in the event $-\infty < x_1 \leq x_1^*$ there is a sure completion of diversion at the second period. The best reply to this is a sure alarm (at first or second period). This of course cannot be the situation in equilibrium (since **O** would increase his payoff by retreating in this event.)

On the other hand, $x_1^* = -\infty$ and $m > 0$ means that with positive probability q_1, a positive amount is diverted in the first period, followed by a sure retreat at the second period. This gain cannot be the situation in equilibrium since replacing m_1 by 0 would decrease the false alarm probability and hence increase **O** 's payoff (recall that **O** has no positive utility from total diversion less than 1.) We conclude that:

- In equilibrium it must be the case that $m_1 = 0$, $q_1 > 0$ and $x_1^* = \infty$. That is, with positive probability **O** does not divert at the first period and then surely does not divert at the second period. In other words: *With positive probability he behaves legally.*

Considering now the other end, $x_n^* < \infty$ means that whenever $x_1 > x_n^*$ it is certain that **O** will not complete the diversion at the second period. It is impossible that when $x_1 > x_n^*$ is observed, **I** does not call an alarm at the first period and does call an alarm at the second period. (Since he knows with certainty that no additional diversion was made at the second period, if he decides to call an alarm at the second period, he is better off doing it at the first period and saving the element dm.) So there are two possibilities:

- **I** does not call an alarm (in either period) whenever $x_1 > x_n*$, or
- **I** calls an alarm at the first period whenever $x_1 > x_n*$.

The first alternative means $c_2(x_1) = \infty$ for all $x_1 > x_n*$. But since c_2 is decreasing in x_1 (by Theorem 4.1), it follows that $c_2 \equiv \infty$ i.e., **I** never calls an alarm at the second period which of course cannot be the case in equilibrium.

The second alternative means that $c_1 \leq x_n^*$. In such a case, there is no loss of generality in assuming $c_1 = x_n*$. In fact if $c_1 < x_n^*$, nothing is changed if we replace x_n^* by c_1 (since the threshold x^* is relevant only when there is no alarm at the first period.)

This discussion is summarized as follows:

Proposition 6.2 *In equilibrium in which q has a finite support, there is a positive probability for legal action ($m_1 = 0$ and $x_1^* = -\infty$) and the first period* **I** *threshold, c_1 is also the threshold for* **O** *'s completion if he has diverted the largest amount m_n.*

6.3 The one-diversion plan equilibrium

In view of our result that no equilibrium exists with a singular q, the simplest case to consider is that in which the support of q consists of two points. By the conclusion of the previous section this support must be $\{0, m\}$ with the corresponding completion thresholds being $-\infty$ and c_1. Thus the equilibrium has the following structure:

- The operator behaves legally (in both periods) with probability q and with probability $1-q$ he diverts m at the first period and completes the diversion at the second period if no alarm was called before.
- The inspector uses a two period threshold strategy $s = (c_1, c_2)$ which is to call an alarm at the first period if $x_1 > c_1$ and otherwise to call an alarm at the second period if $x_2 > c_2(x_1)$.

The interesting thing about this conclusion is that *in no event does the operator retreat* after starting a diversion plan. Although he has this option, he does not use it in equilibrium in which q has a two points support. But now this equilibrium becomes very similar, in structure, to the equilibrium found in Section 5 where we assumed that there is no possibility for **O** to retreat.

The conditions to determine the equilibrium strategies $\tau = (q, m)$ and $s = (c_1, c_2)$ are similar except that now we will not be able to conclude that $c_1 = \infty$: Because of the "time value term" dm we must find $c_1 < \infty$ that is, *the inspector may well call an alarm at the first period.*

The equilibrium conditions are equations (12), (13) and (14) as in the previous section and a fourth equation is the analog of equation (15) which incorporate the factor dm. It is more convenient to write this equation differently, expressing the fact that at $x_1 = c_1$, the inspector is indifferent between calling an alarm (at the first period) or not:

$$a(1-q(m \mid c_1))\phi\left(\frac{s_2(c_1) - \rho\frac{\sigma_2}{\sigma_1}c_1}{\sigma}\right) = q(m \mid c_1)\left[dm + (1-a)\phi\left(\frac{s_2(c_1) - \mu_1(m, c_1)}{\sigma}\right)\right], \tag{27}$$

where

$$q(m \mid c_1) = \frac{qe^{-\frac{1}{2\sigma_1^2}(c_1-m)^2}}{qe^{-\frac{1}{2\sigma_1^2}(c_1-m)^2} + (1-q)e^{-\frac{1}{2\sigma_1^2}c_1^2}}$$

is the conditional probability of diversion given $x_1 = c_1$.

Equation (14) provided the function c_2 (given by Equations (36), (37), (38) and (39) in Section 8.4), we are left actually with three unknowns c_1, q, and m to be found by solving the three equations (12) (13) and (27). These equations were solved numerically and the results are given in the next section.

6.4 Numerical results

The one-diversion plan equilibrium developed in the previous section was computed by solving numerically equations (12), (13) and (27) (with c_2 given by Equations (36), (37), (38) and (39).

Input

The input of the numerical procedure consists of the statistical distribution parameters: σ_1, σ_2 and ρ, and the payoff parameters: b (alarm penalty) for the operator, a (alarm penalty) and d time element for the inspector.

Output

The output is an equilibrium point, which consists of a pair of strategies: q and m for the operator and c_1 and $c_2(\cdot)$ for the inspector. From these we computed several characteristics of the equilibrium, the most interesting of which are:

- *FAP* - The false alarm probability.
- β The probability of no detection, given that a diversion took place (type II error).
- $Q = q\beta$ - The probability that a diversion will take place and will not be detected.
- I - The expected payoff of the inspector.
- O - The expected payoff of the operator.

We were interested in the dependence of the equilibrium on the payoff parameters (penalties) b - of the operator and a - of the inspector. All computations were made for several values of the 'time factor' parameter d. The results are given by the tables in the following tables which are self explanatory: In all cases we took $\sigma_1 = \sigma_2$. Each line in the table corresponds to a game in which a, b, d and ρ are given at the heading of the table and the left column. The other numbers on the line correspond to the resulting equilibrium. Recall that θ is the coefficient of x_1 in c_2. In other words, the statistics of the second period test is $x_2 + \theta x_1$. According to the results of Section 5, $\theta = 1$ when $d = 0$ (i.e., no importance for timely detection.)

In the first table the solution is given for varying values of b - the operator's 'penalty' resulting from an alarm.

Table 2: The Equilibrium solution for varying values of b.

$a = 0.8 \quad d = 0.05 \quad \rho = -0.6$									
b	c_1	q	m	θ	FAP	β	Q	I	O
0.4	1.0388	0.8655	0.36	0.8692	0.4962	0.1439	0.12455	-0.7824	-0.1985
0.8	1.9448	0.7725	0.46	0.9608	0.2503	0.3332	0.25740	-0.7316	-0.2002
1.2	2.4989	0.7036	0.49	0.9900	0.1550	0.4609	0.32429	-0.6813	-0.1860
1.6	2.9191	0.6503	0.50	1.0000	0.1075	0.5492	0.35714	-0.6379	-0.1720
2.0	3.2836	0.6078	0.5	1.0000	0.0799	0.6134	0.37282	-0.6011	-0.1599
2.4	3.5720	0.5730	0.5	1.0000	0.0624	0.6619	0.37927	-0.5699	-0.1496
2.8	3.8089	0.5438	0.5	1.0000	0.0503	0.6998	0.38055	-0.5431	-0.1409
3.2	4.0089	0.5189	0.5	1.0000	0.0417	0.7301	0.37885	-0.5199	-0.1335
3.6	4.1812	0.4974	0.5	1.0000	0.0353	0.7550	0.37554	-0.4996	-0.1270
4.0	4.3322	0.4785	0.5	1.0000	0.0303	0.7757	0.37117	-0.4817	-0.1213
4.4	4.4663	0.4618	0.5	1.0000	0.0264	0.7933	0.36635	-0.4656	-0.1164
4.8	4.5866	0.4469	0.5	1.0000	0.0233	0.8083	0.36123	-0.4513	-0.1119
5.2	4.6955	0.4335	0.5	1.0000	0.0208	0.8213	0.35603	-0.4382	-0.1079
5.6	4.7949	0.4213	0.5	1.0000	0.0186	0.8327	0.35082	-0.4264	-0.1044
6.0	4.8862	0.4102	0.5	1.0000	0.0168	0.8427	0.34568	-0.4155	-0.1011

$a = 0.8 \quad d = 0.2 \quad \rho = -0.6$									
b	c_1	q	m	θ	FAP	β	Q	I	O
0.4	0.5128	0.8587	0.18	0.7241	0.5675	0.1233	0.10588	-0.7918	-0.2273
1.0	1.5022	0.7273	0.36	0.8692	0.2195	0.3902	0.28379	-0.7323	-0.2195
1.6	2.0175	0.6445	0.42	0.9231	0.1170	0.5434	0.35022	-0.6701	-0.1872
2.0	2.2688	0.6035	0.44	0.9417	0.0851	0.6099	0.36807	-0.6347	-0.1704
2.4	2.4316	0.5694	0.46	0.9608	0.0658	0.6594	0.37546	-0.6044	-0.1579
2.8	2.6029	0.5411	0.47	0.9704	0.0525	0.6982	0.37780	-0.5777	-0.1469
3.3	2.7779	0.5113	0.48	0.9802	0.0412	0.7358	0.37621	-0.5489	-0.1360
3.6	2.9064	0.4958	0.48	0.9802	0.0361	0.7543	0.37398	-0.5333	-0.1300
4.1	3.0155	0.4729	0.49	0.9900	0.0299	0.7799	0.36881	-0.5108	-0.1225
4.4	3.1146	0.4608	0.49	0.9900	0.0269	0.7929	0.36537	-0.4983	-0.1182
4.8	3.2354	0.4462	0.49	0.9900	0.0236	0.8081	0.36057	-0.4831	-0.1132
5.6	3.3598	0.4208	0.5	1.0000	0.0188	0.8325	0.35032	-0.4574	-0.1054
6.0	3.4501	0.4098	0.5	1.0000	0.0170	0.8426	0.34530	-0.4458	-0.1019

$a = 0.8$ $d = 0.6$ $\rho = -0.6$									
b	c_1	q	m	θ	FAP	β	Q	I	O
0.4	0.2657	0.8539	0.07	0.6461	0.6186	0.1091	0.09316	-0.7948	-0.2473
0.6	0.6181	0.7981	0.14	0.6949	0.4548	0.2050	0.16361	-0.7905	-0.2722
0.9	0.9804	0.7331	0.20	0.7391	0.3041	0.3296	0.24163	-0.7685	-0.2737
1.3	1.3273	0.6684	0.26	0.7857	0.1930	0.4562	0.30492	-0.7362	-0.2507
2.0	1.7171	0.5897	0.32	0.8349	0.1060	0.5961	0.35152	-0.6809	-0.2118
2.4	1.8982	0.5570	0.34	0.8519	0.0800	0.6494	0.36172	-0.6532	-0.1919
2.8	2.0098	0.5292	0.36	0.8692	0.0642	0.6896	0.36494	-0.6291	-0.1797
3.1	2.1043	0.5114	0.37	0.8779	0.0546	0.7148	0.36555	-0.6124	-0.1694
3.5	2.2196	0.4906	0.38	0.8868	0.0450	0.7427	0.36437	-0.5919	-0.1575
3.8	2.2762	0.4765	0.39	0.8957	0.0398	0.7601	0.36219	-0.5785	-0.1513
4.3	2.3882	0.4560	0.40	0.9048	0.0327	0.7848	0.35787	-0.5575	-0.1404
4.8	2.4723	0.4382	0.41	0.9139	0.0276	0.8048	0.35266	-0.5391	-0.1323
5.2	2.5277	0.4254	0.42	0.9231	0.0244	0.8183	0.34810	-0.5264	-0.1266
5.5	2.5887	0.4168	0.42	0.9231	0.0222	0.8274	0.34486	-0.5163	-0.1220
6.0	2.6376	0.4031	0.43	0.9324	0.0195	0.8406	0.33885	-0.5022	-0.1163

The following is a sample of some graphs of interest drawn from this table.

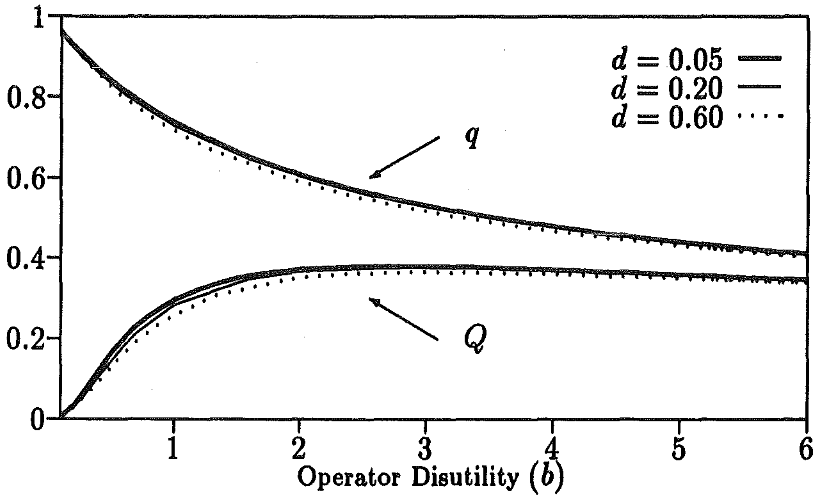

Figure 6: Diversion probability q and the probability Q of undetected diversion as functions of the penalty b.

Here we see that although the diversion probability q decreases as the penalty b

gets larger, the total probability of overlooked diversion increases in b for low values of b (but then decreases as expected).

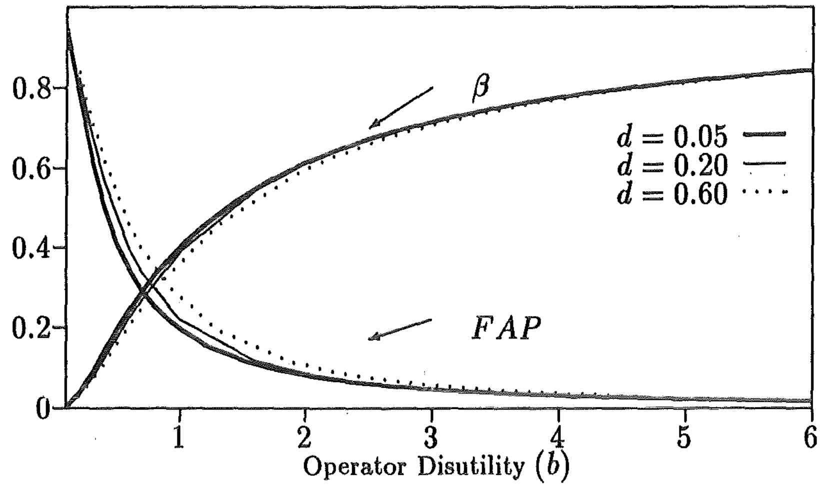

Figure 7: The false alarm probability α (FAP) and type II error probability β as functions of the penalty b.

The interesting thing in this figure is that β increases as the penalty b increases. This is because q decreases i.e., the operator commits much less violations and consequently larger proportion of the alarms ar false.

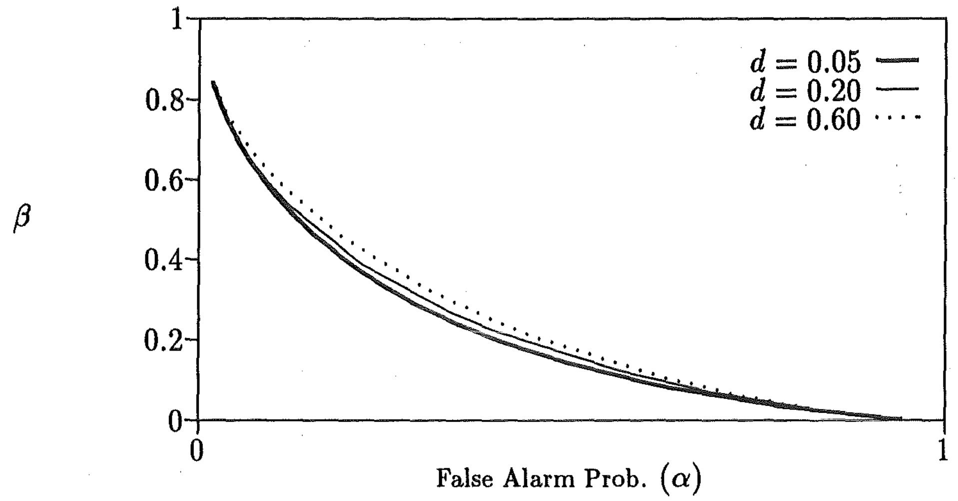

Figure 8: The relation between β and α resultiong from varying the penalty value b.

Here we see the classical dependence of the probabilities of first and second type errors. Note that each game corresponds to *a single* point on this line. To obtain the line we varied the game by varying the value of the penalty b.

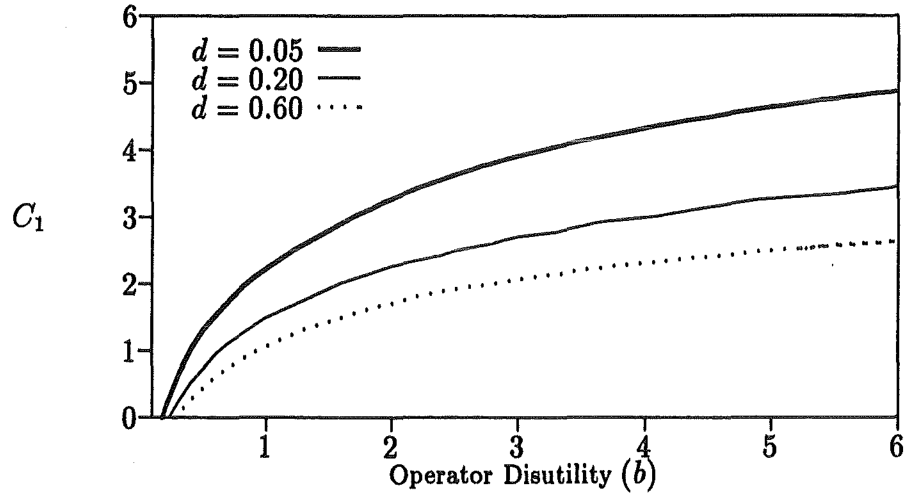

Figure 9: The first period threshold for alarm, c_1, as a function of the penalty b.

As expected, the higher is the penalty b the less likely it is for the inspector to call an alarm in the first period.

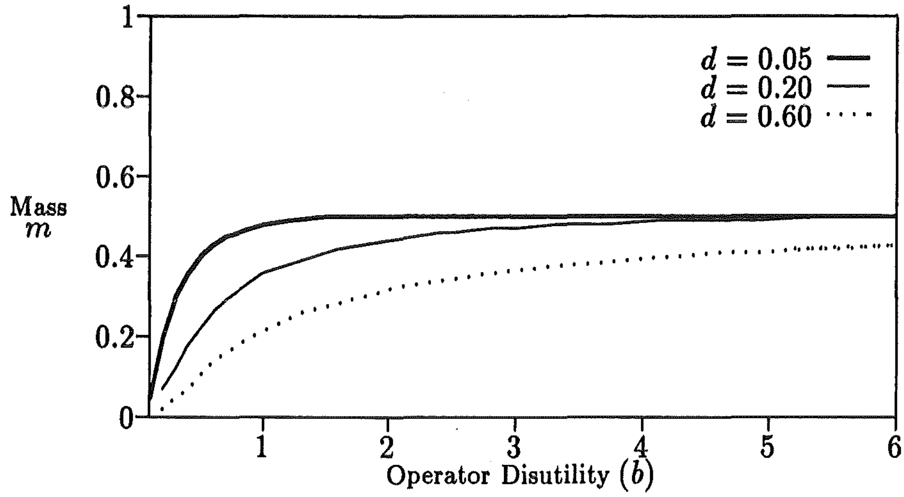

Figure 10: The amount m diverted at the first periodas a function of the penalty b.

The influence of the penalty b on the amount m diverted at the first period is non significant in a large range of b. Recall that with no time element ($d = 0$), the equilibrium value is constant in b ($1/2$ if $\sigma_1 = \sigma_2$).

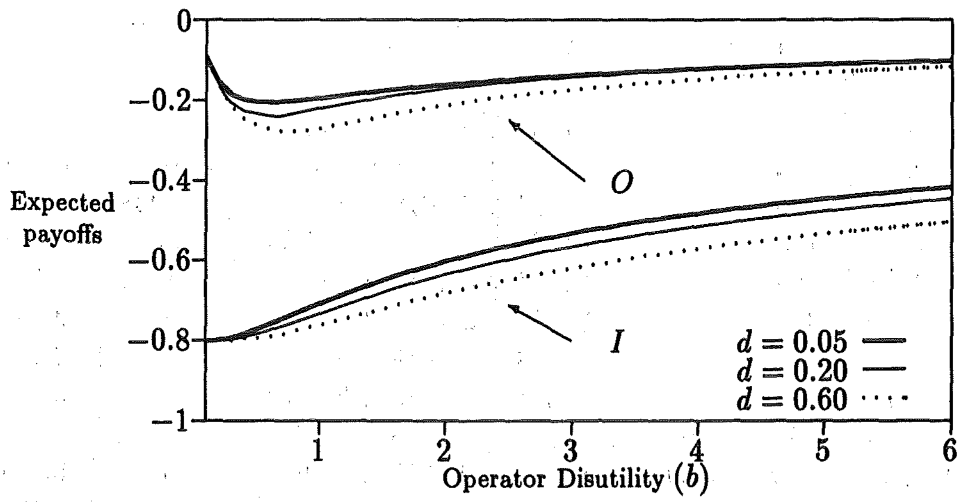

Figure 11: The inspector's expected payoff I, and the operator's expected payoff O as functions of the penalty b.

Interestingly, the expected payoffs of ***both players*** increase as b increases. This is true for the inspector on the whole region, while for the operator this is the case except for very low values of b. Note the ***all payoffs are negative*** although if the operator would behave legally and the inspector would never call an alarm, the payoff would be higher namely zero for both. However, this is not a stable situation.

As for the effect of the time factor d on the solution, we see that it has non significant effect on the 'performance' of the equilibrium as measured by FAP, β, q and Q. It does affect however both players' strategies namely c_1, and m.

The following table contains the solutions for varying values of the inspector's disutility, a, from an alarm.

Table 3: The Equilibrium solution for varying values of a.

$b = 1.2 \quad d = 0.05 \quad \rho = -0.6$									
a	c_1	q	m	θ	FAP	β	Q	I	O
0.100	3.7005	0.0621	0.5	1	0.1534	0.4618	0.02868	-0.0480	-0.1840
0.150	3.6562	0.0951	0.5	1	0.1534	0.4618	0.04392	-0.0748	-0.1841
0.200	3.6086	0.1296	0.5	1	0.1534	0.4618	0.05985	-0.1038	-0.1840
0.250	3.5578	0.1657	0.5	1	0.1534	0.4618	0.07652	-0.1349	-0.1841
0.300	3.5030	0.2034	0.5	1	0.1534	0.4618	0.09393	-0.1685	-0.1841
0.350	3.4437	0.2429	0.5	1	0.1534	0.4618	0.11217	-0.2046	-0.1841
0.400	3.3791	0.2843	0.5	1	0.1534	0.4618	0.13129	-0.2435	-0.1841
0.450	3.3082	0.3277	0.500	1.0000	0.1534	0.4618	0.15133	-0.2853	-0.1841
0.500	3.2298	0.3733	0.500	1.0000	0.1535	0.4618	0.17239	-0.3302	-0.1841
0.550	3.1421	0.4213	0.500	1.0000	0.1535	0.4617	0.19451	-0.3786	-0.1842
0.600	3.0428	0.4718	0.500	1.0000	0.1536	0.4617	0.21783	-0.4306	-0.1843
0.650	2.9287	0.5250	0.500	1.0000	0.1537	0.4616	0.24234	-0.4866	-0.1844
0.700	2.8316	0.5813	0.495	0.9950	0.1539	0.4615	0.26827	-0.5467	-0.1846
0.800	2.5333	0.7036	0.485	0.9851	0.1548	0.4610	0.32436	-0.6812	-0.1858
0.850	2.3243	0.7702	0.475	0.9753	0.1564	0.4602	0.35445	-0.7561	-0.1877
0.900	2.0737	0.8408	0.450	0.9512	0.1601	0.4581	0.38517	-0.8361	-0.1921

$b = 1.2 \quad d = 0.2 \quad \rho = -0.6$									
a	c_1	q	m	θ	FAP	β	Q	I	O
0.050	2.6576	0.0303	0.490	0.9900	0.1543	0.4613	0.01398	-0.0252	-0.1852
0.100	2.6079	0.0619	0.490	0.9900	0.1545	0.4612	0.02855	-0.0523	-0.1854
0.150	2.5551	0.0948	0.490	0.9900	0.1547	0.4611	0.04371	-0.0815	-0.1857
0.200	2.5333	0.1292	0.485	0.9851	0.1548	0.4610	0.05956	-0.1127	-0.1858
0.250	2.4723	0.1651	0.485	0.9851	0.1552	0.4608	0.07608	-0.1463	-0.1862
0.300	2.4407	0.2026	0.480	0.9802	0.1554	0.4607	0.09334	-0.1822	-0.1865
0.350	2.3691	0.2417	0.480	0.9802	0.1560	0.4604	0.11128	-0.2209	-0.1872
0.400	2.3243	0.2828	0.475	0.9753	0.1564	0.4602	0.13014	-0.2621	-0.1877
0.450	2.2711	0.3258	0.470	0.9704	0.1570	0.4598	0.14980	-0.3062	-0.1884
0.500	2.2079	0.3709	0.465	0.9656	0.1578	0.4594	0.17039	-0.3534	-0.1893
0.600	2.0737	0.4681	0.450	0.9512	0.1601	0.4581	0.21444	-0.4577	-0.1921
0.650	1.9947	0.5206	0.440	0.9417	0.1620	0.4571	0.23797	-0.5152	-0.1943
0.700	1.9216	0.5759	0.425	0.9277	0.1640	0.4560	0.26261	-0.5763	-0.1969
0.800	1.7085	0.6961	0.385	0.8913	0.1729	0.4511	0.31401	-0.7104	-0.2075
0.850	1.5860	0.7619	0.350	0.8605	0.1805	0.4470	0.34057	-0.7828	-0.2167
0.875	1.5023	0.7961	0.330	0.8433	0.1865	0.4444	0.35379	-0.8203	-0.2227
0.925	1.3080	0.8694	0.265	0.7897	0.2060	0.4344	0.37767	-0.8966	-0.2447

$b = 1.2$ $d = 0.6$ $\rho = -0.6$									
a	c_1	q	m	θ	FAP	β	Q	I	O
0.100	1.9216	0.0607	0.425	0.9277	0.1640	0.4560	0.02768	-0.0609	-0.1969
0.150	1.8893	0.0930	0.420	0.9231	0.1651	0.4554	0.04235	-0.0942	-0.1981
0.200	1.8522	0.1265	0.415	0.9185	0.1664	0.4547	0.05752	-0.1295	-0.1997
0.300	1.7914	0.1983	0.400	0.9048	0.1691	0.4530	0.08983	-0.2067	-0.2030
0.350	1.7474	0.2362	0.395	0.9002	0.1708	0.4525	0.10688	-0.2489	-0.2049
0.400	1.7085	0.2763	0.385	0.8913	0.1729	0.4511	0.12464	-0.2933	-0.2075
0.450	1.6684	0.3182	0.375	0.8824	0.1752	0.4499	0.14316	-0.3402	-0.2102
0.550	1.5864	0.4084	0.350	0.8605	0.1805	0.4470	0.18255	-0.4420	-0.2166
0.600	1.5339	0.4572	0.335	0.8476	0.1846	0.4446	0.20327	-0.4971	-0.2216
0.700	1.4081	0.5626	0.300	0.8182	0.1956	0.4389	0.24693	-0.6155	-0.2345
0.750	1.3517	0.6194	0.280	0.8018	0.2006	0.4377	0.27111	-0.6789	-0.2384
0.825	1.2515	0.7133	0.240	0.7699	0.2117	0.4336	0.30929	-0.7794	-0.2484
0.875	1.1192	0.7849	0.190	0.7316	0.2337	0.4191	0.32895	-0.8456	-0.2785
0.900	1.0352	0.8237	0.155	0.7058	0.2505	0.4075	0.33566	-0.8767	-0.3031

The effect of a on the resulting equilibrium can be seen in the following figures.

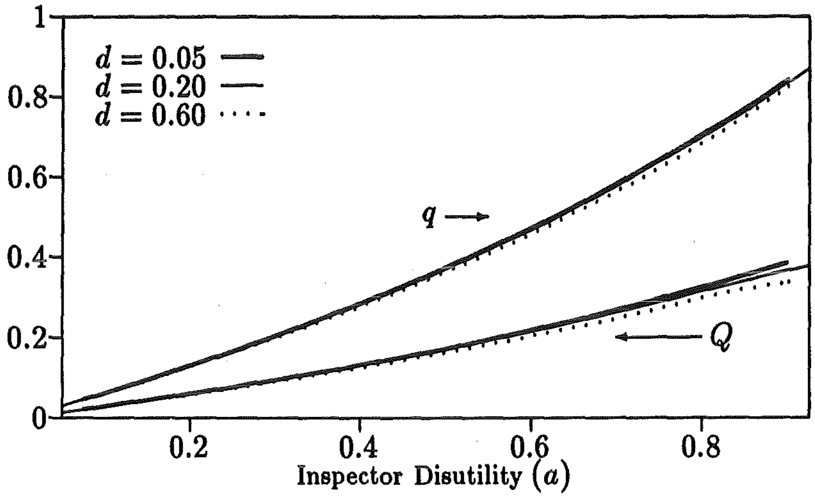

Figure 12: Diversion probability q and the probability Q of undetected diversion as functions of a.

As expected, the higher is a the higher is the probability of violation (both detected and undetected), since the inspector is reluctant to call an alarm and sure enough, the operator takes advantage of this situation. Note, however, that the quality of the inspector's test, as measured by α (FAP) and β, is not affected by a. This is seen in the following figure.

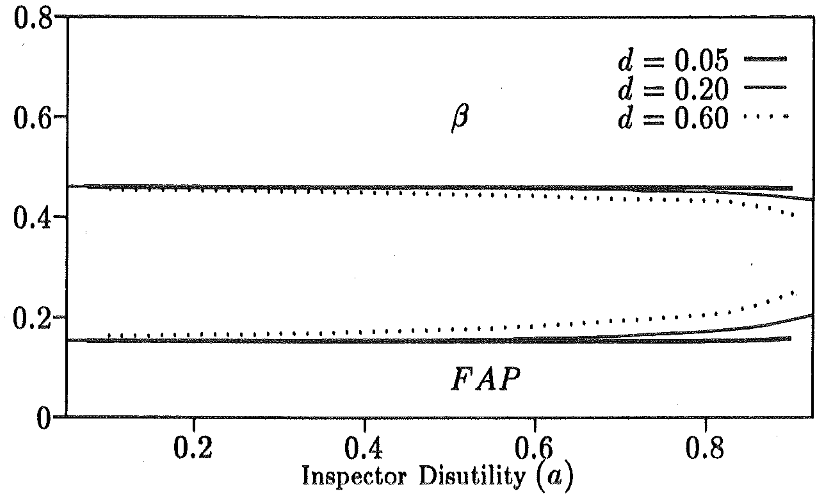

Figure 13: The false alarm probability α (FAP) and type II error probability β as functions of a.

The next figure shows that threshold for calling alarm at the first period, (c_1), decreases mildly in a. So although alarm is more painful for the inspector, he applies it more, in anticipation of higher tendency of the operator for illegal behavior.

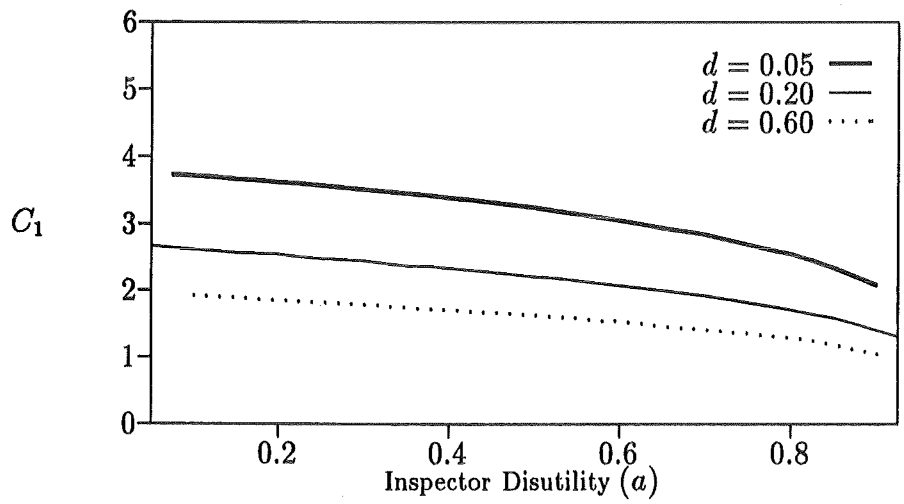

Figure 14: The first period threshold for alarm, c_1 as a function of a.

The effect of a and d on the mass m diverted at first period and on the players' payoffs are given by the last two figures.

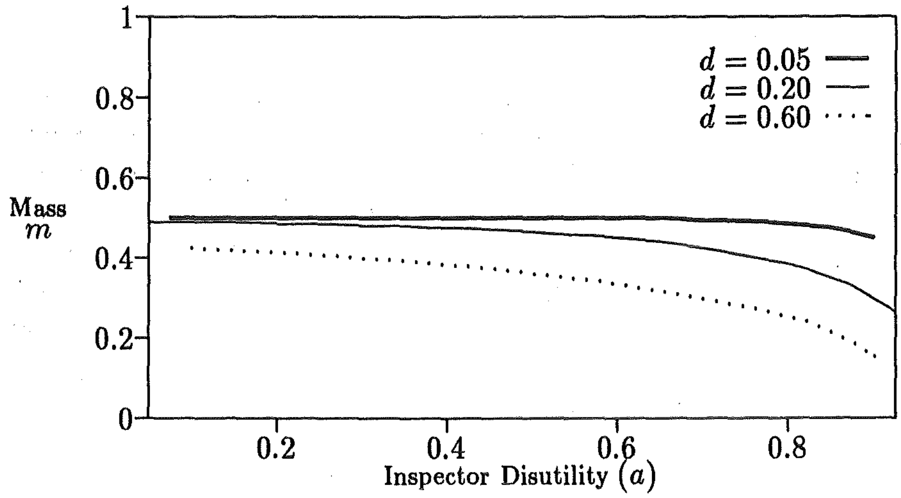

Figure 15: The amount m diverted at the first period as a function of a.

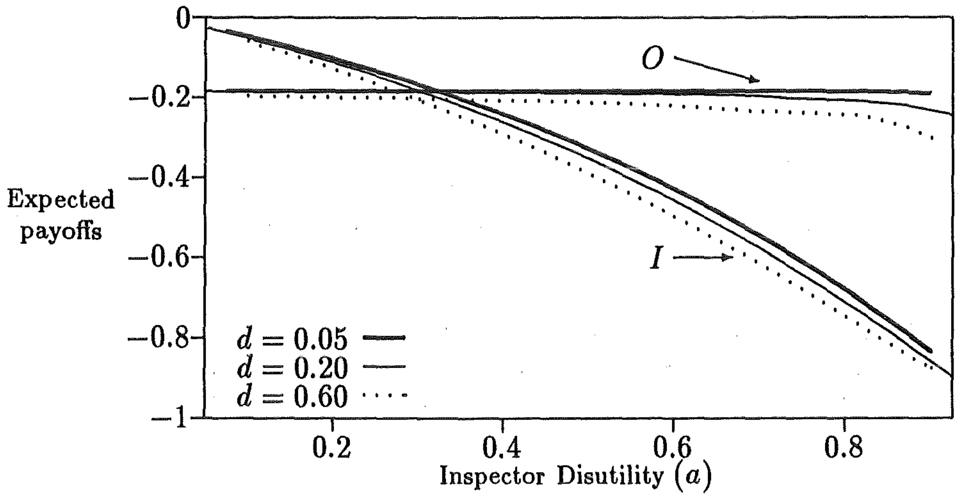

Figure 16: The inspector's expected payoff I, and the operator's expected payoff O as functions of a.

The inspector's expected utility decreases in a while that of the operator is practically independent of a. This is another expression of the non zero-sum nature of the game.

Here again we see that although d has a significant effect on the equilibrium strategies, it has negligible effect on the performance characteristics of the outcome.

7 Concluding remarks

On deterrence strategies

Our result that in any equilibrium the operator must be choosing legal behavior with positive probability, has a very interesting and practically important consequence to our problem. Consider the variant of the game, in which the operator announces his strategy in advance and commits himself to use it. Such a game is called an **inspector leadership game** and it was treated in Avenhaus, Okada and Zamir (1990). We proved there that the only equilibrium of this game is one in which the operator plays legally (even though he is indifferent between playing legally or not). In other words, although the operator's equilibrium payoff is the same in both games, the diversion probability q is zero in the leadership game and consequently the equilibrium payoff of the inspector is higher.

Although this result is appealing, the operator is still not deterred from illegal action since he is indifferent. However, if the inspector uses inspection strategies which are slightly more severe than the equilibrium strategy (say $s_\epsilon = (c_1 - \epsilon, c_2 - \epsilon)$), then the operator **will strictly prefer legal behavior** on illegal one, in other words, these are *deterrence inspection strategies.*

More than two inventory periods

Given the complexity of the solution, it was natural to restrict our solutions of the sequential model for only for two inventory periopds. However, we expect that most of the results can be genaralized for more than two inventory periods. More precisely we conjecture that:

- We expect that for any number n of inventory periods, the equilibrium strategies of both players will be *threshold strategies.* In other word, we expect a genalization of Theorems 4.1, Theorems 4.2, and Theorems 4.3 according to which, in equilibrium, at each period i,
  - The inspector calls an alarm if and only if the current MUF_i is greater than a critical value c_i which depends on all previous MUF values.
  - The operator's diversion depends on the previous MUF values and he retreats from his diversion plans if MUF_i exceeds a certain critical value.
- With the analogue of Assumption 5.1, a theorem similar to Theorem 5.4 should be proveable that is, the theshold values of the inspector's strategy are:

$$c_1 = \ldots = c_{n-1} = \infty \quad \text{and} \quad c_n = C - \sum_1^{n-1} MUF_i,$$

  for some constant C.

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8 Appendix

In the proofs we need the following assumption on the joint distribution of (X_1, X_2):

Assumption 8.1 $\max\left(-\frac{\sigma_1}{\sigma_2}, -\frac{\sigma_2}{\sigma_1}\right) \leq \rho < 0.$

This condition is satisfied in our case since

$$X_1 = MF_1 = I_0 + D_1 - I_1 \quad \textit{and} \quad X_2 = MF_2 = I_1 + D_2 - I_2,$$

and assuming that I_i and D_i are pairwise independent it follows that $Cov(MF_1, MF_2) = -Var(I_1) < 0$, implying $\rho < 0$. Also

$$\begin{aligned} \sigma_1^2 = Var(MF_1) &= Var(I_0) + Var(D_1) + Var(I_1) \\ &\geq Var(I_1) = -Cov(MF_1, MF_2) = -\rho\sigma_1\sigma_2 \end{aligned}$$

which implies $\rho \geq -\sigma_1/\sigma_2$. Similarly

$$\sigma_2^2 = Var(MF_2) \geq Var(I_1) = -\rho\sigma_1\sigma_2,$$

which implies $\rho \geq -\sigma_2/\sigma_1$.

Next we need the following general property of the cumulative standard normal distribution

$$\phi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{x^2}{2}} dx.$$

Lemma 8.2 *For any $\delta > 0$, the ratio $\phi(y-\delta)/\phi(y)$ is strictly increasing in y for all $y \in (-\infty, \infty)$ with*

$$\lim_{y \to -\infty} \frac{\phi(y-\delta)}{\phi(y)} = 0 \quad \text{and} \quad \lim_{y \to \infty} \frac{\phi(y-\delta)}{\phi(y)} = 1.$$

Proof The limit of this ratio as $y \to \infty$ is 1 since this is the limit of both numerator and denominator. The second limit is verified using L'Hospital's rule:

$$\lim_{y \to -\infty} \frac{\phi(y-\delta)}{\phi(y)} = \lim_{y \to -\infty} \frac{e^{-\frac{1}{2}(y-\delta)^2}}{e^{-\frac{1}{2}y^2}} = \lim_{y \to -\infty} e^{2y\delta - \delta^2} = 0.$$

The condition

$$\frac{d}{dy}\left(\frac{\phi(y-\delta)}{\phi(y)}\right) > 0$$

is equivalent to

$$\phi(y)\, e^{-\frac{1}{2}(y-\delta)^2} - \phi(y-\delta)\, e^{-\frac{1}{2}y^2} > 0,$$

which is

$$G(y) \stackrel{\text{def}}{=} \frac{\phi(y)}{\phi(y-\delta)}\, e^{(2y\delta - \frac{\delta^2}{2})} > 1 \quad \forall y. \tag{28}$$

To prove (28) we show:

$$\lim_{y\to-\infty} G(y) = 1 \tag{29}$$

$$G'(y) > 0 \quad \forall y. \tag{30}$$

The first is proved by L'Hospital's rule:

$$\begin{aligned}\lim_{y\to-\infty} G(y) &= \lim_{y\to-\infty} \frac{e^{-\frac{1}{2}y^2} e^{(2y\delta-\frac{\delta^2}{2})} - \delta\phi(y)\, e^{(y\delta-\frac{\delta^2}{2})}}{e^{-\frac{1}{2}(y-\delta)^2}} \\ &= \lim_{y\to-\infty} \left(1 + \delta\,\frac{\phi(y)}{e^{-\frac{1}{2}y^2}}\right) = 1\end{aligned}$$

(In the last step we again used L'Hospital's rule:

$$\lim_{z\to-\infty} \frac{\phi(z)}{e^{-\frac{1}{2}z^2}} = \lim_{z\to-\infty} \frac{e^{-\frac{1}{2}z^2}}{-ze^{-\frac{1}{2}z^2}} = \lim_{z\to-\infty} \frac{-1}{z} = 0.)$$

To prove (30) we have

$$\begin{aligned} G'(y) > 0 &\iff (\ln G(y))' > 0 \\ &\iff \frac{\phi'(y)}{\phi(y)} - \frac{\phi'(y-\delta)}{\phi(y-\delta)} + \delta > 0 \\ &\iff \frac{1}{R(-y)} - \frac{1}{R(-y+\delta)} + \delta > 0 \\ &\iff \int_{-y}^{-y+\delta} \frac{d}{du}\left(\frac{1}{R(u)}\right) du < \delta \end{aligned}$$

Here $R(z)$ is the Mill's ratio defined by,

$$R(z) = \frac{1-\phi(z)}{\phi'(-z)} = \frac{\phi(-z)}{\phi'(z)} = \frac{\phi(-z)}{\phi'(-z)}$$

which implies

$$\frac{\phi'(z)}{\phi(z)} = \frac{1}{R(-z)}.$$

Recall now the following property of the Mill's ratio (see Owen (1980)):

$$0 < \frac{d}{dz}\left(\frac{1}{R(z)}\right) < 1 \qquad \forall z,$$

This implies the last inequality concerning the integral of this derivative, completing the proof of the lemma.

■

Corollary 8.3 *For any $0 < \eta < 1$ and any $\delta > 0$, there is a unique $Z = Z(\delta)$ satisfying*

$$\frac{\phi(Z(\delta)-\delta)}{\phi(Z(\delta))} = \eta. \tag{31}$$

Lemma 8.4 *For any $\eta \in (0,1)$, the function $Z(\delta)$ defined by Corollary 8.3 is strictly increasing in δ at all $\delta \in (-\infty,\infty)$.*

Proof Differentiating both sides of equation (31), with respect to δ, we get:

$$\frac{\phi(Z(\delta))\phi'(Z(\delta)-\delta)(Z'(\delta)-1)-\phi(Z(\delta)-\delta)\phi'(Z(\delta))Z'(\delta)}{\phi^2(Z(\delta))}=0,$$

which is equivalent to:

$$\phi'(Z(\delta)-\delta)(Z'(\delta)-1)=\eta\phi'(Z(\delta))Z'(\delta)$$

or

$$(Z'(\delta)-1)e^{-\frac{1}{2}(Z(\delta)-\delta)^2}=\eta Z'(\delta)e^{-\frac{1}{2}(Z(\delta))^2}.$$

or

$$Z'(\delta)\left(e^{\delta Z(\delta)-\frac{1}{2}\delta^2}-\eta\right)=e^{\delta Z(\delta)-\frac{1}{2}\delta^2}$$

Since the right hand side of the last equation is always positive, $Z'(\delta)$ cannot vanish and hence cannot change sign, therefore it is enough to show that it is positive for some δ, or equivalently, it remains to show the existence of $\delta > 0$ for which $\exp\{\delta Z(\delta)-\frac{1}{2}\delta^2\} > \eta$. In fact choose any $\delta > 0$ large enough so as to satisfy $\phi(-\delta/2)/\phi(\delta/2) < \eta$ and let us show that such δ satisfies the desired inequality:

$$\frac{\phi(-\frac{\delta}{2})}{\phi(\frac{\delta}{2})}=\frac{\phi(\frac{\delta}{2}-\delta)}{\phi(\frac{\delta}{2})}<\eta.$$

Since $\phi(Z+\delta)/\phi(Z)$ is increasing and reaching η at $Z = Z(\delta)$ it follows that $Z(\delta) > \delta/2$. Consequently, the interval $[Z(\delta)-\delta, Z(\delta)]$ lies to the right of the interval $[-\delta/2, \delta/2]$ (which is of the same length) and therefore the center of $[Z(\delta)-\delta, Z(\delta)]$ lies to the right of 0 (see Figure 17).

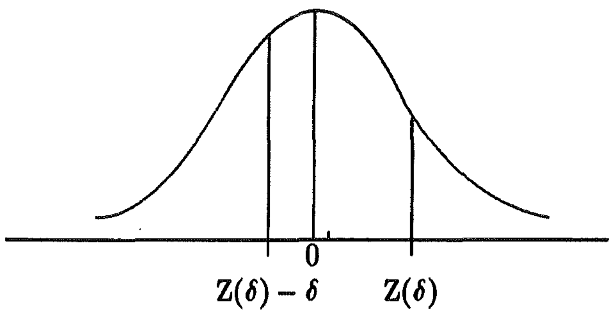

Figure 17: The density of the Standard Normal Distribution.

By the symmetry of the normal density φ, this implies $\varphi(Z(\delta)-\delta) > \varphi(Z(\delta))$ and finally we have:

$$e^{\delta Z(\delta)-\frac{1}{2}\delta^2}=\frac{\varphi(Z(\delta)-\delta)}{\varphi(Z(\delta))}>1>\eta,$$

completing the proof of the Lemma.

■

8.1 Proof of Theorem 4.1

Denote by $\mathbf{Z}$ the random variable which is the total diversion of material in the two periods and denote its cumulative distribution (which depends of course on the strategies) by Q. We first prove:

Proposition 8.5 *For any pair of strategies (s,t) and for any given x_1, the conditional distribution $Q(\mathbf{Z} \mid x_1, x_2)$ is stochastically increasing in x_2 for fixed x_1 and stochastically increasing in x_1 for fixed x_2.*

Proof We shall prove that $\forall\theta \in [0,1]$ the probability $P(\mathbf{Z} < z \mid x_1, x_2)$ decreases in x_2 and strictly decreases whenever it is not 0 or 1.

Let $\sigma^2 = (1-\rho^2)\sigma_2^2$ and define the following functions:

$$\begin{aligned}
\mu_0(m,x_1) &= \rho\frac{\sigma_2}{\sigma_1}(x_1 - m)\\
\mu_1(m,x_1) &= 1 - m + \rho\frac{\sigma_2}{\sigma_1}(x_1 - m) = 1 - m + \mu_0(m,x_1)\\
\varphi_{x_1}(m) &= \frac{1}{\sqrt{2\pi\sigma_1^2}}e^{-\frac{1}{2\sigma_1^2}(x_1-m)^2}\\
\varphi_{x_2}(\mu) &= \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{1}{2\sigma^2}(x_2-\mu)^2}.
\end{aligned}$$

The probability we are interested in is

$$P(\mathbf{Z} < z \mid x_1, x_2) = \frac{\int_{[0,z)} \varphi_{x_1}(m)\,(1-p(m,x_1))\,\varphi_{x_2}(\mu_0(m,x_1))\,dq(m)}{\int_{[0,1]} \varphi_{x_1}(m)\,[(1-p(m,x_1))\,\varphi_{x_2}(\mu_0(m,x_1)) + p(m,x_1)\,\varphi_{x_2}(\mu_1(m,x_1))]\,dq(m)}$$

Since $\varphi_{x_1}(m)$ and $\varphi_{x_2}(\mu)$ are always strictly positive it follows that if $P(\mathbf{Z} < z \mid x_1, x_2)$ equals 0 or 1 for one value of x_2 then it is constant in x_2 and hence (weakly) decreasing. Let us therefore assume that it is neither 0 nor 1 and prove that it strictly decreasing, or equivalently that its reciprocal is strictly increasing. We write it as

$$\frac{1}{P(\mathbf{Z} < z \mid x_1, x_2)} = 1 + f_1(x_1,x_2) + f_2(x_1,x_2)$$

where

$$\begin{aligned}
f_1(x_1,x_2) &= \frac{\int_{[z,1]} \varphi_{x_1}(m)\,(1-p(m,x_1))\,\varphi_{x_2}(\mu_0(m,x_1))\,dq(m)}{\int_{[0,z)} \varphi_{x_1}(m)\,(1-p(m,x_1))\,\varphi_{x_2}(\mu_0(m,x_1))\,dq(m)}\\
f_2(x_1,x_2) &= \frac{\int_{[0,1]} \varphi_{x_1}(m)\,p(m,x_1)\,\varphi_{x_2}(\mu_1(m,x_1))\,dq(m)}{\int_{[0,z)} \varphi_{x_1}(m)\,(1-p(m,x_1))\,\varphi_{x_2}(\mu_0(m,x_1))\,dq(m)}
\end{aligned}$$

Since $P(\mathbf{Z} < z \mid x_1, x_2) < 1$ not both $f_1(x_1,x_2) = 0$ and $f_2(x_1,x_2) = 0$. It is therefore sufficient to prove that for $i = 1,2$,

$$f_i(x_1,x_2) > 0 \Longrightarrow f_i(x_1,x_2) \text{ strictly increases in } x_2.$$

Let $\xi \in [0, z)$ be any point in the support of the denominator of $f_i(x_1, x_2)$. If $f_1(x_1, x_2) > 0$ then

$$\begin{aligned} F(\xi, x_1, x_2) &\stackrel{\text{def}}{=} \frac{\varphi_{x_2}(\mu_0(\xi, x_1))}{\int_{[z,1]} \varphi_{x_1}(m)\,(1 - p(m, x_1))\,\varphi_{x_2}(\mu_0(m, x_1))\,dq(m)} \\ &= \frac{1}{\int_{[z,1]} \varphi_{x_1}(m)\,(1 - p(m, x_1))\frac{\varphi_{x_2}(\mu_0(m,x_1))}{\varphi_{x_2}(\mu_0(\xi,x_1))}\,dq(m)}. \end{aligned}$$

Now for $m \in [z, 1]$

$$\begin{aligned} \frac{\varphi_{x_2}(\mu_0(m, x_1))}{\varphi_{x_2}(\mu_0(\xi, x_1))} &= \frac{e^{-\frac{1}{2\sigma^2}(x_2 - \rho\frac{\sigma_2}{\sigma_1}(x_1 - m))^2}}{e^{-\frac{1}{2\sigma^2}(x_2 - \rho\frac{\sigma_2}{\sigma_1}(x_1 - \xi))^2}} \\ &= e^{\frac{1}{2\sigma^2}[\rho\frac{\sigma_2}{\sigma_1}(\xi - m)(2x_2 - \rho\frac{\sigma_2}{\sigma_1}(2x_1 - \xi - m))]}. \end{aligned}$$

Since $\rho < 0$ and $\xi < z \leq m$, this function is increasing in each of the variables x_1 and x_2. Therefore so is the integral of such functions i.e., $1/F(\xi, x_1, x_2)$ is increasing in each of the variables x_i and hence $F(\xi, x_1, x_2)$ is decreasing in x_2. Since

$$\frac{1}{f_1(x_1, x_2)} = \int_{[0,z)} \varphi_{x_1}(\xi)\,(1 - p(\xi, x_1))\,F(\xi, x_1, x_2)\,dq(\xi)$$

and the integrand is increasing in x_i it follows that $f_1(x_1, x_2)$ is (strictly) increasing in x_i.

The case $f_2(x_1, x_2) > 0$ is treated similarly: This time we have to show that $\varphi_{x_2}(\mu_1(m, x_1))/\varphi_{x_2}(\mu_0(\xi, x_1))$ strictly increases in x_i for all $m \in [0, 1]$. In fact

$$\begin{aligned} \frac{\varphi_{x_2}(\mu_1(m, x_1))}{\varphi_{x_2}(\mu_0(\xi, x_1))} &= \frac{e^{-\frac{1}{2\sigma^2}(x_2 - (1-m) - \rho\frac{\sigma_2}{\sigma_1}(x_1 - m))^2}}{e^{-\frac{1}{2\sigma^2}(x_2 - \rho\frac{\sigma_2}{\sigma_1}(x_1 - \xi))^2}} \\ &= e^{\frac{1}{2\sigma^2}[((1+\rho\frac{\sigma_2}{\sigma_1}\xi) - m(1+\rho\frac{\sigma_2}{\sigma_1}))(2x_2 - 2\rho\frac{\sigma_2}{\sigma_1}x_1)]}. \end{aligned}$$

Since by Assumption 8.1

$$(1 + \rho\frac{\sigma_2}{\sigma_1}\xi) - m(1 + \rho\frac{\sigma_2}{\sigma_1}) > (1 + \rho\frac{\sigma_2}{\sigma_1})(1 - m) \geq 0,$$

the coefficient of x_2 is strictly positive and hence $\varphi_{x_2}(\mu_1(m, x_1))/\varphi_{x_2}(\mu_0(\xi, x_1))$ strictly increases in x_2 and also in x_1 (since $\rho < 0$), completing the proof of the Proposition. ∎

In the next Proposition we prove that although $\mathbf{A}_2(x_1) = \mathcal{R}$ may also be considered as a threshold test in which the critical value is $-\infty$, it will not occur in equilibrium. Intuitively, the reason is that since time is valuable, if the inspector is determined to call an alarm at second period independently of MF_2, he should not wait but rather call the alarm at first period.

Proposition 8.6 *If (s, t) is an equilibrium, then whenever x_1 is not followed by a first period alarm, $\mathbf{A}_2(x_1) \neq \mathcal{R}$.*

Proof If $\mathrm{A}_2(x_1) = \mathcal{R}$ (and $m < 1$) then the operator's best reply is not to complete diversion at x_1 (since $b(m) < b$.) Therefore $Q(\mathbf{Z} \mid x_1) = q(m \mid x_1)$ and the net difference in the $\mathbf{I}$'s payoff between calling alarm at first period or in second period is

$$-\int a(z)dQ(z \mid x_1) + \int (A(m) + d(m))dq(m \mid x_1) = \int d(m)dq(m \mid x_1).$$

This is always strictly positive unless $q(m = 0 \mid x_1) = 1$ which can not be the case since if it was, the inspector's best reply would be not to call alarm at all. We conclude that the inspector is strictly better off calling the alarm at first period in contradiction to the fact that no alarm was called after x_1.

■

In the next Proposition we prove that sure diversion of critical mass can not be part of equilibrium.

Proposition 8.7 *If (s,t) is an equilibrium, then for almost all x_1*

$$q(\{m : p(m, x_1) = 1\} \mid x_1) < 1.$$

Proof $q(\{m : p(m, x_1) = 1\} \mid x_1) = 1$ means that given x_1, the operator surely completes the critical diversion. Since $-1 < -a$, the inspector's best reply is to call an alarm at the second period if he did not call at the first one. Since $-b < -b(m)$ for any $m < 1$, the operator's best reply to that is never to complete the diversion at second period. So the only way to satisfy $q(\{m : p(m, x_1) = 1\} \mid x_1) = 1$ is by $q(\{m = 1\}) = 1$ but this can certainly not be part of an equilibrium. Since the inspector's best reply would be to call an alarm at first period to which the best reply is not to divert at all.

■

Next we prove that for any $\mathcal{MF}_1$, a certainty of not calling alarm at second period can not be part of an equilibrium.

Proposition 8.8 *If (s,t) is an equilibrium then for almost all x_1 which were not followed by an alarm, $\mathrm{C}_2(x_1) \neq \mathcal{R}$.*

Proof Since X_2 has a continuous distribution, $\mathrm{C}_2(x_1) = \mathcal{R}$ means that given x_1 the inspector will surely not call an alarm at second period. The best reply of the operator is $p(m, x_1) = 1$, $\forall m$, hence $q(\{m : p(m, x_1) = 1\} \mid x_1) = 1$ in contradiction with Proposition 8.7.

■

Corollary 8.9 *If (s,t) is an equilibrium, then the conclusion in Proposition 8.5 can be strengthened as follows: $\forall z \in [0,1]$, the probability $P(\mathbf{Z} < z \mid x_1, x_2)$ decreases (weakly) in x_i and $P(\mathbf{Z} < 1 \mid x_1, x_2)$* strictly *decreases in x_i.*

Proof In the proof of Proposition 8.5 we showed that $\forall z \in [0,1]$, $P(\mathbf{Z} < z \mid x_1, x_2)$ strictly decreases in x_i provided it is strictly between 0 and 1. So it is therefore sufficient to show that $0 < P(\mathbf{Z} < 1 \mid x_1, x_2) < 1$ for almost all (x_1, x_2). In fact $P(\mathbf{Z} < 1 \mid x_1, x_2) = 0$ means $P(\mathbf{Z} = 1 \mid x_1, x_2) = 1$ hence $q(\{m : p(m, x_1) = 1\} \mid x_1) = 1$ which cannot be the case in equilibrium by Proposition 8.7.

Similarly $P(\mathbf{Z} < 1 \mid x_1, x_2) = 1$ implies $P(\mathbf{Z} < 1 \mid x_1) = 1$. Therefore either $\mathbf{C}_2(x_1) = \mathcal{R}$ which cannot be the case in equilibrium. By Proposition 8.8 or $\mathbf{A}_2(x_1) = \mathcal{R}$, which cannot be by Proposition 8.6. (If I is indifferent between the two possibilities, the operator can ϵ—change his q to make it $\mathbf{C}_2(x_1) = \mathcal{R}$ and then complete diversion safely.)

∎

We are now in the position to conclude the proof of Theorem 4.1.

Proof Given $t = (q, p)$ and given x_1 and x_2, the conditional expected payoff for **I** if he calls an alarm is

$$U(A_2) = -\int_{[0,1]} a(z)\, dQ(z \mid x_1, x_2) - \int_{[0,1]} d(m)\, dq(m \mid x_1, x_2)$$

and if he does not call an alarm (i.e., chooses C_2) it is

$$U(C_2) = -\int_{[0,1]} F(z)\, dQ(z \mid x_1, x_2) - \int_{[0,1]} d(m)\, dq(m \mid x_1, x_2).$$

Therefore the net (conditional) utility from calling an alarm is

$$DI(x_1, x_2) =: U(A_2) - U(C_2) = \int_{[0,1]} (F(z) - a(z))\, dQ(z \mid x_1, x_2).$$

By Corollary 8.9 this is a strictly increasing function of x_2 (noting that the integrand is strictly increasing in m since $F(m)$ is increasing and $a(m)$ is decreasing.) It follows that for any x_1 after which no alarm was called at the first period, there is *at most* one critical value $c_2(x_1)$ for which $DI(x_1, x_2) = 0$, that is:

$$\begin{aligned} x_2 < c_2(x_1) &\implies U(A_2) < U(C_2) \\ x_2 > c_2(x_1) &\implies U(A_2) > U(C_2). \end{aligned}$$

Such a value in fact exists since otherwise either $\mathbf{C}_2(x_1) = \mathcal{R}$ or $\mathbf{A}_2(x_1) = \mathcal{R}$ which cannot happen in equilibrium (Proposition 8.8 and Proposition 8.6.)

To prove that $c_2(x_1)$ is decreasing in x_1 recall that $c_2(x_1)$ is the value of x_2 for which the $DI(x_1, x_2) = 0$, and the difference $DI(x_1, x_2)$ is strictly increasing in x_2. It is therefore sufficient to prove that for fixed x_2,

$$x_1 > x_1' \implies DI(x_1, x_2) > DI(x_1', x_2).$$

To see this we have:

$$\begin{aligned} DI(x_1, x_2) &= \int_{[0,1]} (F(z) - a(z))\, dQ(z \mid x_1, x_2) \\ &= \int_{[0,1)} (F(z) - a(z))\, dQ(z \mid x_1, x_2) + Q(\{\mathbf{Z} = 1\} \mid x_1, x_2)(1 - a). \end{aligned}$$

By Proposition 8.5, for all θ we have $Q(\mathbf{Z} < \theta \mid x_1', x_2) \geq Q(\mathbf{Z} < \theta \mid x_1, x_2)$ and the inequality is strict for $\theta = 1$ (Corollary 8.9.) Since $F(m) - a(m)$ is increasing in m, it follows that for $x_1 > x_1'$,

$$\int_{[0,1)} (F(z) - a(z))\, dQ(z \mid x_1, x_2) > \int_{[0,1)} (F(z) - a(z))\, dQ(z \mid x_1', x_2).$$

As for the second part, since $1 - a > 0$, and $Q(\{\mathbf{Z} = 1\} \mid x_1, x_2) = 1 - Q(\{\mathbf{Z} < 1\} \mid x_1, x_2)$ is strictly increasing in x_1 (by 8.5), we have:

$$Q(\{\mathbf{Z} = 1\} \mid x_1, x_2)(1 - a) > Q(\{\mathbf{Z} = 1\} \mid x_1', x_2)(1 - a),$$

concluding the proof of the theorem.

■

Note that at $x_2 = c_2(x_1)$ the inspector is indifferent between calling alarm or not and hence any randomization between the two is a best reply. But in our continuous model this is an event of probability 0.

8.2 Proof of Theorem 4.2

Given that $\mathbf{O}$ is using an equilibrium strategy t and given that m was diverted in the first period, x_1 was observed and no alarm was called in first period, assume that the critical value for the inspector's second period test (Theorem 4.1) is $c_2 = c_2(x_1)$.

If the operator completes the diversion his (conditional) expected payoff is

$$D(m) = -b(1 - \phi(\frac{c_2 - \mu_1}{\sigma})) + \phi(\frac{c_2 - \mu_1}{\sigma})$$

and if he retreats his expected payoff is

$$R(m) = -b(m)(1 - \phi(\frac{c_2 - \mu_0}{\sigma})).$$

We show now that the difference between the two is strictly increasing in m. In fact

$$\Delta(m, c_2) =: D(m) - R(m) = -b + (b + 1)\phi(\frac{c_2 - \mu_1}{\sigma}) + b(m)(1 - \phi(\frac{c_2 - \mu_0}{\sigma}))$$

also

$$\begin{aligned} \frac{\partial}{\partial m}(\frac{c_2 - \mu_0}{\sigma}) &= \rho \frac{1}{\sigma} \frac{\sigma_2}{\sigma_1} < 0 \\ \frac{\partial}{\partial m}(\frac{c_2 - \mu_1}{\sigma}) &= \frac{1}{\sigma}(1 + \rho \frac{\sigma_2}{\sigma_1}) > 0. \end{aligned}$$

Therefore $\phi(\frac{c_2 - \mu_0}{\sigma})$ decreases and $\phi(\frac{c_2 - \mu_1}{\sigma})$ increases in m. Since $b(m)$ increases in m and $b > 0$ it follows that the difference strictly increases in m.

■

Proposition 8.10 *For almost all x_1 after which no alarm was called, and for any $m < 1$ diverted at first period, the conditional expected payoff difference $\Delta(m, c_2)$ is strictly increasing in c_2.*

Proof As we computed in the proof of the previous Proposition the expected net payoff for completion is

$$\Delta(m) = -b + b(m) + \phi(\frac{c_2 - \mu_1}{\sigma}) + \phi(\frac{c_2 - \mu_1}{\sigma}) - b(m)\phi(\frac{c_2 - \mu_0}{\sigma}).$$

It is clearly enough to prove that the difference of the last two terms is strictly increasing in c_2. In fact

$$b\phi(\frac{c_2 - \mu_1}{\sigma}) - b(m)\phi(\frac{c_2 - \mu_0}{\sigma}) = \phi(\frac{c_2 - \mu_1}{\sigma}) \left(b - b(m)\frac{\phi(y + \delta)}{\phi(y - \delta)}\right)$$

where

$$\delta = \frac{\mu_1 - \mu_0}{2\sigma} \text{ and } y = \frac{c_2 - \mu_1}{\sigma} + \delta.$$

Noticing that $\delta > 0$ since $m < 1$, the result now follows since the ratio $\phi(y + \delta)/\phi(y - \delta)$ is strictly decreasing in y by Lemma 8.2.

■

To prove the monotonicity of $m^*(x_1)$, note first that if both players use equilibrium strategies then $m^*(x_1) < 1$ for almost all x_1 after which no alarm was called. Since if $m^*(x_1) = 1$ has a positive probability then at that event the inspector is sure that the diversion will not be completed at second period. Therefore he will either call alarm or not independently of the value of x_2, which we know cannot happen in equilibrium. Therefore by Proposition 8.10 and the monotonicity of $c_2(x_1)$, Theorem 4.1 (ii),

$$x_1 > x_1' \Longrightarrow \Delta(m^*(x_1), c_2(x_1')) > \Delta(m^*(x_1), c_2(x_1)) = 0.$$

So when $m^*(x_1)$ was already diverted and the first signal was x_1', the operator should complete diversion with certainty that is $m^*(x_1) > m^*(x_1')$, completing the proof of Theorem 4.2.

■

8.3 Proof of Theorem 4.3

Recall that for **O** the first stage strategy is a probability distribution q on $[0, 1]$ (the probability distribution of the random amount m of material he diverts at the first stage,) while for the inspector the first stage strategy is any subset A_1 of the real line $\mathcal{R}$(the alarm set for the first stage: Call an alarm if $x_1 \in A_1$).

Given $q(m)$ and an observation of x_1, denote by $q(m \mid x_1)$ the conditional probability distribution of m given x_1. Since the density of x_1 given m is

$$\varphi_m(x_1) = \frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{1}{2\sigma_1^2}(x_1 - m)^2},$$

the conditional density $q(m \mid x_1)$ is given by

$$dq(m \mid x_1) = \frac{\varphi_m(x_1)}{\int_0^1 \varphi_m(x_1) dq(m)} dq(m).$$

We first prove the following result saying roughly that the higher is x_1, the more likely it seems to **I** that m was high.

Proposition 8.11 *For any given $q(m)$, the distribution $q(m \mid x_1)$ is stochastically increasing in x_1.*

Proof We want to show that for any $\theta \in [0,1]$ the conditional probability $P(M \leq \theta \mid x_1)$ decreases in x_1. In fact

$$P(M \leq \theta \mid x_1) = \frac{\int_0^\theta \varphi_m(x_1) dq(m)}{\int_0^1 \varphi_m(x_1) dq(m)} \quad .$$

Since $\varphi_m(x_1) > 0$ for all m and all x_1 it follows that if $P(M \leq \theta \mid x_1) = 0$ then $\int_0^\theta dq(m) = 0$ and hence $P(M \leq \theta \mid x_1) = 0$ for all x_1 in accordance to the claimed monotonicity. So we may assume that $P(M \leq \theta \mid x_1) > 0$ and let us prove that its reciprocal is monotonically increasing in x_1:

$$\frac{1}{P(M \leq \theta \mid x_1)} = 1 + \frac{\int_\theta^1 \varphi_m(x_1) dq(m)}{\int_0^\theta \varphi_m(x_1) dq(m)} = 1 + f(x_1)$$

where

$$f(x_1) = \frac{\int_\theta^1 \varphi_m(x_1) dq(m)}{\int_0^\theta \varphi_m(x_1) dq(m)} \quad .$$

Again if $f(x_1) = 0$ for a certain value of x_1, it remains so for all values of x_1 in accordance to the claimed monotonicity. If $f(x_1) > 0$, let ξ $[0, \theta)$ be any point in the support of its denominator then,

$$\begin{aligned} 0 &< \frac{\varphi_m(\xi)}{\int_\theta^1 \varphi_m(x_1) dq(m)} \\ &= \frac{1}{\int_\theta^1 e^{-\frac{1}{2\sigma_1^2}[(x_1-m)^2-(x_1-\xi)^2]} dq(m)} \\ &= \frac{1}{\int_\theta^1 e^{-\frac{1}{2\sigma_1^2}(\xi-m)(2x_1-m-\xi)} dq(m)} \quad . \end{aligned}$$

Since $\xi < \theta \leq m$ the integrand in the last expression is increasing in x_1 and hence the whole expression is decreasing in x_1 and it remains so when it is integrated $\int_0^\theta \cdots dq(m)$ to obtain $1/f(x_1)$ which is therefore decreasing in x_1. We conclude finally that $f(x_1)$ is increasing in x_1.

■

Let (s,t) be an equilibrium and consider the decision problem of **I** after the first stage, given x_1 and given the strategy of **O** : If he calls an alarm his (conditional expected payoff is

$$I(A_1 \mid x_1) = -\int_0^1 a(m)dq(m \mid x_1). \tag{32}$$

If he chooses C_1 (not alarm) then, by Theorem 4.1 he will use a threshold test $c_2(x_1)$ to decide whether or not to call an alarm at the second stage. His expected payoff (based on x_1 only, before observing x_2) is

$$\begin{aligned} I(C_1 \mid x_1) \quad = \quad & -\int_0^{m^*(x_1)} F(m)\phi(\frac{c_2(x_1)-\mu_0(m,x_1)}{\sigma})dq(m \mid x_1) \\ & -\int_{m^*(x_1)}^1 F\phi(\frac{c_2(x_1)-\mu_1(m,x_1)}{\sigma})dq(m \mid x_1) \\ & -\int_0^{m^*(x_1)} a(m)\left(1-\phi(\frac{c_2(x_1)-\mu_0(m,x_1)}{\sigma})\right)dq(m \mid x_1) \\ & -\int_{m^*(x_1)}^1 a\left(1-\phi(\frac{c_2(x_1)-\mu_1(m,x_1)}{\sigma})\right)dq(m \mid x_1) \\ & -\int_0^1 d(m)dq(m \mid x_1). \end{aligned}$$

This can be written as:

$$\begin{aligned} I(C_1 \mid x_1) \quad = \quad & -\int_0^{m^*(x_1)}(F(m)-a(m))\phi(\tfrac{c_2(x_1)-\mu_0(m,x_1)}{\sigma})dq(m \mid x_1) \\ & -\int_0^{m^*(x_1)} a(m)dq(m \mid x_1) - \int_{m^*(x_1)}^1 (F-a)\phi(\tfrac{c_2(x_1)-\mu_1(m,x_1)}{\sigma})dq(m \mid x_1) \\ & -\int_{m^*(x_1)}^1 a\,dq(m \mid x_1) - \int_0^1 d(m)dq(m \mid x_1). \end{aligned} \tag{33}$$

By Equations (32) and (33) the net expected difference between the two decisions is

$$\begin{aligned} I(A_1 \mid x_1) - I(C_1 \mid x_1) \quad = \quad & \int_0^{m^*(x_1)}(F(m)-a(m))\phi(\tfrac{c_2(x_1)-\mu_0(m,x_1)}{\sigma})dq(m \mid x_1) \\ & +\int_{m^*(x_1)}^1 (F-a)\phi(\tfrac{c_2(x_1)-\mu_1(m,x_1)}{\sigma})dq(m \mid x_1) \\ & -\int_{m^*(x_1)}^1 (a(m)-a)dq(m \mid x_1) \\ & \int_0^1 d(m)dq(m \mid x_1). \end{aligned} \tag{34}$$

We shall show that the right hand side of Equation (34) is increasing in x_1. In fact by our assumptions, both $d(m)$ and $-(a(m)-a)$ are increasing functions of m, and by Proposition 8.11 the conditional distribution $q(m \mid x_1)$ is stochastically increasing in x_1, hence each of the last two terms in (34) is increasing in x_1. Therefore it suffices to show that the first two terms are also increasing in x_1. To see this let us write ϕ as the integral of the appropriate normal density $\varphi_{x_2}(\mu)$, interchange the order of integration, and rewrite these terms as $D(x_1)$,

$$\begin{aligned} D(x_1) \quad = \quad & \int_{-\infty}^{c_2(x_1)} \left\{\int_0^{m^*(x_1)}(F(m)-a(m))\varphi_{x_2}(\mu_0(m,x_1))dq(m \mid x_1,x_2)\right. \\ & \left. +\int_{m^*(x_1)}^1 (F-a)\varphi_{x_2}(\mu_1(m,x_1))dq(m \mid x_1,x_2)\right\} dx_2\,. \end{aligned} \tag{35}$$

Now, the expression in $\{\ldots\}$ is precisely $I(A_2 \mid x_1, x_2) - I(C_2 \mid x_1, x_2)$, the conditional expected difference of payoffs between calling an alarm or not, given x_1 and x_2. This is the expression $DI(x_1, x_2)$ in the proof of Theorem 4.1. As we proved there, this is an increasing function in each of the two variables x_1 and x_2. Since in the region of integration $x_2 \in (-\infty, c_2(x_1))$ the inspector chooses C_2, it follows that it is also negative in that region. We conclude that

$$D(x_1) = \int_{-\infty}^{c_2(x_1)} DI(x_1, x_2) dx_2 ,$$

is a (negative) increasing function of x_1 and hence the existence of a threshold c_1 (the zero point of $D(x_1)$) such that the inspector calls an alarm at the first stage if and only if $x_1 > c_1$.

∎

Note that the critical c_1 can in principle be $\pm\infty$. The following proposition shows that in equilibrium $-\infty < c_1$.

Proposition 8.12 *In equilibrium there exists a sufficiently small (but finite) x_1 such that $I(C_1 \mid x_1) > I(A_1 \mid x_1)$ holds.*

Proof If this was not true there would be an equilibrium in which **I** calls an alarm at the first stage independently of the value of x_1. The only best reply for that, from the point of view of **O** is not to divert at all at first stage (since $-b(m) < 0$ for all $m > 0$). We claim that these two strategies are not in equilibrium. In fact a strategy of **O** which does not divert any material at first stage will make the full diversion at second stage if x_1 is smaller than some critical value x_1^* and the best reply of **I** to that is not to call alarm at first stage and call one at second stage if $x_1 < x_1^*$.

∎

8.4 Proof of Theorems 5.4, 5.5 and 5.6

In this section we prove Theorems 5.4 which states that, under Assumption 8.1 on the joint distribution of *MUF*$_1$ and *MUF*$_1$, the *CUMUF* test is part of the (only) equilibrium of the two stage game. Theorems 5.5 and 5.6 provide the equilibrium (which does not involve *CUMUF*) in the extreme distributions which do not satisfy Assumption 8.1.

We first solve $c_2(x_1)$ from equation (14) by straightforward manipulations to obtain:

$$c_2(x_1) = C - \theta x_1, \tag{36}$$

where, denoting $d = 1 - m(1 + \rho\frac{\sigma_2}{\sigma_1})$ and noting that $d > 0$ (since $\rho < 0$, by Assumption 8.1),

$$\theta = \frac{\sigma_2^2}{\sigma_1^2}\frac{(1-\rho^2)m}{d} - \rho\frac{\sigma_2}{\sigma_1} . \tag{37}$$

$$C = \frac{1}{2}\left[d - \frac{\sigma_2^2(1-\rho^2)}{d}\left(2\ln H - \frac{m^2}{\sigma_1^2}\right)\right] \tag{38}$$

$$H = \frac{q(1-a)}{(1-q)a}\,. \tag{39}$$

Next we prove:

Proposition 8.13 *In equilibrium $c_1 = \infty$ (i.e., there is no alarm at first stage).*

Proof This will follow from equation (15) if we prove that the inspector's payoff is strictly increasing in c_1 for any finite c_1,

$$\frac{\partial}{\partial c_1} I(c_1, c_2(\cdot); q, m) > 0.$$

By substitution of c_2 from (36) into $I(c_1, c_2(\cdot); q, m)$ and taking the partial derivative with respect to c_1 we obtain:

$$\begin{aligned}\frac{\partial I(c_1, c_2(\cdot); q, m)}{\partial c_1} &= \frac{(1-q)a}{\sigma_1\sqrt{2\pi}}\phi\left(\frac{C - \theta c_1 - \rho\frac{\sigma_2}{\sigma_1}c_1}{\sigma_2\sqrt{1-\rho^2}}\right)\exp\left(-\frac{c_1^2}{2\sigma_1^2}\right)\\ &\quad -\frac{q(1-a)}{\sigma_1\sqrt{2\pi}}\phi\left(\frac{C - \theta c_1 - (1-m) - \rho\frac{\sigma_2}{\sigma_1}(c_1 - m)}{\sigma_2\sqrt{1-\rho^2}}\right)\exp\left(-\frac{(c_1-m)^2}{2\sigma_1^2}\right),\end{aligned}$$

and the condition $\partial I/\partial c_1 > 0$ is therefore equivalent to

$$\frac{\phi\left(\frac{C-\theta c_1 - \rho\frac{\sigma_2}{\sigma_1}c_1}{\sigma_2\sqrt{1-\rho^2}}\right)}{\phi\left(\frac{C-\theta c_1-(1-m)-\rho\frac{\sigma_2}{\sigma_1}(c_1-m)}{\sigma_2\sqrt{1-\rho^2}}\right)}\exp\left(\frac{(c_1-m)^2}{2\sigma_1^2} - \frac{c_1^2}{2\sigma_1^2}\right) > H. \tag{40}$$

Let us define

$$\delta = \frac{d}{\sigma_2\sqrt{1-\rho^2}} > 0\,, \quad \tilde{C} = \frac{C}{\sigma_2\sqrt{1-\rho^2}}\,,$$
$$y = \tilde{C} - \frac{m}{\delta\sigma_1^2}\,, \qquad \mu = \exp\left(\delta\tilde{C} + \frac{\ln H}{2} - \frac{m^2}{\sigma_1^2}c_1\right).$$

With this notation, it follows from equation (16) that

$$e^{\frac{\delta^2}{2}} = \mu, \tag{41}$$

and inequality (40) becomes

$$\frac{\phi(y)}{\phi(y-\delta)}e^{\delta y} > \mu, \tag{42}$$

and we want to prove that it holds for all for all c_1 i.e., for all $y \in (-\infty, \infty)$ (since y is linear in c_1 with a negative coefficient). To see this we first show that

$$\lim_{y\to-\infty}\frac{\phi(y)}{\phi(y-\delta)}e^{\delta y} = \mu. \tag{43}$$

In fact, using L'Hospital's rule,

$$\begin{aligned}
\lim_{y\to-\infty}\frac{\phi(y)}{\phi(y-\delta)}e^{\delta y} &= \lim_{y\to-\infty}\frac{\delta e^{\delta y}\phi(y)+\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}e^{\delta y}}{\frac{1}{\sqrt{2\pi}}e^{-\frac{(y-\delta)^2}{2}}}\\
&= \lim_{y\to-\infty}\frac{\delta\sqrt{2\pi}\phi(y)+e^{-\frac{y^2}{2}}}{e^{-\frac{y^2+\delta^2}{2}}}\\
&= e^{\frac{\delta^2}{2}}=\mu.
\end{aligned}$$

In the last step we used the fact that $\lim_{y\to-\infty}\phi(y)e^{y^2/2}=0$ and then we applied equation (41).

The proof of inequality (42) will be completed by proving that the derivative of the left hand side (with respect to y) is positive for all y. To do that we recall the *Mill's Ratio* defined as:

$$R(y)=\frac{1-\phi(y)}{\phi'(y)}=\frac{\phi(-y)}{\phi'(y)},$$

and make use of its property (see Jonston and Kotz (1985) and Owen (1980)):

$$0<\frac{d}{dy}\left(\frac{1}{R(y)}\right)<1, \quad \forall y\in(-\infty,\infty). \tag{44}$$

The left hand side of equation (42) has a positive derivative if and only if its derivative has a positive derivative which is the inequality

$$(\ln\phi(y))'-(\ln\phi(y-\delta))'+\delta>0.$$

Noting that $(\ln\phi(y))'=1/R(-y)$, this is equivalent to:

$$\frac{1}{R(-y)}-\frac{1}{R(-y+\delta)}+\delta>0, \quad \text{or} \quad \int_{-y}^{-y+\delta}\frac{d}{dx}\left(\frac{1}{R(x)}\right)dx<\delta,$$

which is true by inequality (44). This concludes the proof of Proposition 8.13 establishing that $c_1=\infty$.

∎

We now make use of the following property of the normal distribution (see e.g. Owen (1980) page 403): For any real constants α and β,

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(\alpha+\beta x)\exp\left(-\frac{x^2}{2}\right)dx=\phi\left(\frac{\alpha}{\sqrt{1+\beta^2}}\right). \tag{45}$$

Substitute $c_2(x_1)=C-\theta x_1$ and $c_1=\infty$ in the equilibrium conditions:

From (12), by changing integration variables we have:

$$\begin{aligned}
\frac{b}{1+b} &= \frac{\int_{-\infty}^{\infty}\phi\left(\frac{C-\theta x_1-(1-m)-\rho\frac{\sigma_2}{\sigma_1}(x_1-m)}{\sigma_2\sqrt{1-\rho^2}}\right)\exp\left(-\frac{(x_1-m)^2}{2\sigma_1^2}\right)dx_1}{\int_{-\infty}^{\infty}\phi\left(\frac{C-\theta x_1-\rho\frac{\sigma_2}{\sigma_1}x_1}{\sigma_2\sqrt{1-\rho^2}}\right)\exp\left(-\frac{x_1^2}{2\sigma_1^2}\right)dx_1}\\
&= \frac{\int_{-\infty}^{\infty}\phi\left(\frac{C-1+m(1-\theta)-(\rho\sigma_2+\theta\sigma_1)x}{\sigma_2\sqrt{1-\rho^2}}\right)\exp\left(-\frac{x^2}{2}\right)dx}{\int_{-\infty}^{\infty}\phi\left(\frac{C-(\rho\sigma_2+\theta\sigma_1)x}{\sigma_2\sqrt{1-\rho^2}}\right)\exp\left(-\frac{x^2}{2}\right)dx}.
\end{aligned}$$

Using (45) we get

$$\frac{\phi\left(\dfrac{C-1+m(1-\theta)}{\sqrt{\sigma_2^2(1-\rho^2)+(\theta\sigma_1+\rho\sigma_2)^2}}\right)}{\phi\left(\dfrac{C}{\sqrt{\sigma_2^2(1-\rho^2)+(\theta\sigma_1+\rho\sigma_2)^2}}\right)}=\frac{b}{1+b}\,. \tag{46}$$

From (15) we get:

$$\begin{aligned} m &= \arg\max\left\{\int_{-\infty}^{\infty}\phi\left(\frac{C-\theta x_1)-(1-\tilde{m})-\rho\frac{\sigma_2}{\sigma_1}(x_1-\tilde{m})}{\sigma_2\sqrt{1-\rho^2}}\right)\exp\left(-\frac{(x_1-\tilde{m})^2}{2\sigma_1^2}\right)dx_1\right\} \\ &= \arg\max\left\{\int_{-\infty}^{\infty}\phi\left(\frac{C-\theta(m+\sigma_1 x)-(1-\tilde{m})-\rho\sigma_2 x}{\sigma_2\sqrt{1-\rho^2}}\right)e^{-\frac{x^2}{2}}dx\right\}. \end{aligned}$$

Apply equation (45) with

$$\alpha=\frac{C-1+\tilde{m}(1-\theta)}{\sigma_2\sqrt{1-\rho^2}} \quad \text{and} \quad \beta=-\frac{\theta\sigma_1+\rho\sigma_2}{\sigma_2\sqrt{1-\rho^2}}$$

to obtain:

$$m=\arg\max\phi\left(\frac{C-1+\tilde{m}(1-\theta)}{\sigma_2^2+\theta^2\sigma_1^2+2\rho\theta\sigma_1\sigma_2}\right)=\arg\max\left(\frac{C-1+\tilde{m}(1-\theta)}{\sigma_2^2+\theta^2\sigma_1^2+2\rho\theta\sigma_1\sigma_2}\right). \tag{47}$$

This equation is clearly satisfied for any m if $\theta=1$. Therefore choose $\theta=1$ and m which satisfies equation (37) i.e.,

- $m=\dfrac{1+\rho\frac{\sigma_2}{\sigma_1}}{1+2\rho\frac{\sigma_2}{\sigma_1}+\frac{\sigma_2^2}{\sigma_1^2}}$.

- Equation (46) for $\theta=1$ yields

$$\frac{\phi\left(\dfrac{C-1}{\sqrt{\sigma_1^2+2\rho\sigma_1\sigma_2+\sigma_2^2}}\right)}{\phi\left(\dfrac{C}{\sqrt{\sigma_1^2+2\rho\sigma_1\sigma_2+\sigma_2^2}}\right)}=\frac{b}{1+b}\,,$$

  which is equation (16).

- Finally, the expression for q is a straightforward solution of equations (38) and (39).

This completes the proof of part (i) of Theorem 5.4. To see under which conditions this equilibrium is unique consider again Equation (47); since the right hand side is linear in $\tilde{m}$ with coefficient $(1-\theta)$, then either the coefficient is zero - the case which we considered - or it is one of the following two cases:

Case 1 If $\theta > 1$, the maximum is at $m = 0$ which then implies, by equation (37) that

$$1 < \theta = -\rho\frac{\sigma_2}{\sigma_1}$$

i.e., $\rho < -\sigma_1/\sigma_2$ in contradiction with Assumption 8.1.

Case 2 If $\theta < 1$, the maximum is at $m = 1$ which then implies, by equation (37) that

$$1 < \theta = -\frac{\sigma_2}{\rho\sigma_1}$$

i.e., $\rho < -\sigma_2/\sigma_1$, again in contradiction with Assumption 8.1.

We conclude that under Assumption 8.1 the only possibility is $\theta = 1$, completing the proof of (ii).

■

To prove Theorems 5.5 and 5.6 note that $c_1 = \infty$, and equations (37), (38),(39), (46) were proved using only the part $\rho < 0$ from Assumption 8.1. Therefore the only other equilibria can occur through cases 1 and 2 mentioned above.
Case 1 can occur when $\rho < -\sigma_1/\sigma_2$ and then $m = 0$ and $\theta = -\rho\sigma_2/\sigma_1$. Substituting this in Equation (46) yields Equation (17) for C and completes the proof of Theorem 5.5. Similarly,
Case 2 can occur when $\rho < -\sigma_2/\sigma_1$ and then $m = 1$ and $\theta = -\sigma_2/(\rho\sigma_1)$. Substituting this in Equation (46) yields Equation (18) for C and completes the proof of Theorem 5.6.

■

8.5 Proof of Theorem 6.1

In this section we prove Theorem 6.1 stating that there is no equilibrium in which **O** diverts a certain quantity q at the first stage. We start by making the following observations:

1. A sure diversion of $m = 1$ at the first stage is clearly not possible in equilibrium. Hence the support of q must be some $m < 1$.

2. There cannot be a finite x_1^* such that player **O** completes the diversion if and only if $x_1 \leq x_1^*$ since then **I** will call an alarm (in first or second period) whenever $x_1 \leq x_1^*$ (since he knows a full diversion has taken place.) This means that with positive probability (namely $P\{x_1 \leq x_1^*\}$ the operator will complete the diversion *knowing that he will certainly be detected.* Obviously **O** would be better off not diverting.

3. There cannot be an equilibrium in which **O** diverts m in the first period and then surely completes diversion (or surely retreat) for an interval of x_1 of positive measure, since the best reply of **I** would be either a sure alarm or a sure non-alarm (on that interval), which cannot be an equilibrium strategy.

4. It follows that if q is concentrated at a single value m then **O** 's strategy is of the form:

   - Divert m at first period and then, as x_1 is observed and no alarm was called, complete diversion with probability $0 < \theta(x_1) < 1$ (and retreat with probability $1 - \theta(x_1)$).

We also need the following, rather intuitive, result:

Proposition 8.14 *There is no equilibrium with $c_1 = -\infty$ except for that with $q = 1$ (the 'open conflict equilibrium' mentioned in section 5.2).*

Proof The idea of the proof is rather simple: $c_1 = -\infty$ means that the inspector always calls an alarm at the first stage, independently of x_1. His payoff in this case is $-a$. This is certainly not the best thing to do if there is a positive probability $1 - q$ that the operator behaves legally. More specifically consider the strategy $\sigma_c = (\infty, C - x_1)$ that is: never call an alarm at first stage and call alarm at the second stage if $x_2 > C - x_1$. This is the *CUMUF* test with threshold C. We claim that, for small enough constant C, the strategy σ_c yields for **I** a payoff strictly larger than $-a$. To prove this we have to show that the conditional payoff to **I** given that there was no alarm is larger than $-a$ (since this event has a positive probability for any finite C). To do this let

$$\begin{aligned} P_0 &= Pr\{\text{no diversion}|\text{no alarm}\} \\ &= \tfrac{1-q}{\sigma_1\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi\left(\tfrac{C - x_1 - \rho\frac{\sigma_2}{\sigma_1}x_1)}{\sigma_2\sqrt{1-\rho^2}}\right) \exp\left(-\tfrac{x_1^2}{2\sigma_1^2}\right) dx_1 \end{aligned} \tag{48}$$

and

$$\begin{aligned} P_1 &= Pr\{\text{diversion}|\text{no alarm}\} \\ &= \tfrac{q}{\sigma_1\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi\left(\tfrac{C-1-(x_1-m)-\rho\frac{\sigma_2}{\sigma_1}(x_1-m)}{\sigma_2\sqrt{1-\rho^2}}\right) \exp\left(-\tfrac{(x_1-m)^2}{2\sigma_1^2}\right) dx_1. \end{aligned} \tag{49}$$

The probability of no alarm is $P_0 + P_1$ and the conditional payoff of **I** in this event is $0 \cdot P_0 - 1 \cdot P_1$ thus we have to show that for sufficiently small C we have $-P_1 > -a(P_0 + P_1)$ or, equivalently,

$$\frac{P_0}{P_1} > \frac{1-a}{a}. \tag{50}$$

In the expression for P_0 change variable, $x = x_1/\sigma_1$ and use (45) for $\alpha = C/\sigma_2\sqrt{1-\rho^2}$ and $\beta = -(\sigma_1 + \rho\sigma_2)/\sigma_2\sqrt{1-\rho^2}$. Do similarly for P_1 to obtain

$$P_0 = (1-q)\phi\left(\frac{C}{\sqrt{1+(\sigma_1+\rho\sigma_2)^2}}\right) \quad \text{and} \quad P_1 = q\phi\left(\frac{C-1}{\sqrt{1+(\sigma_1+\rho\sigma_2)^2}}\right).$$

Thus we have

$$\frac{P_0}{P_1} = \frac{1-q}{q}\frac{\phi(y+\delta)}{\phi(y-\delta)},$$

where

$$y = \frac{C - \frac{1}{2}}{\sqrt{1 + (\sigma_1 + \rho\sigma_2)^2}} \quad \text{and} \quad \delta = \frac{\frac{1}{2}}{\sqrt{1 + (\sigma_1 + \rho\sigma_2)^2}} .$$

Since, by Lemma 8.2, the ratio at the right hand side tends to ∞ as $y \to -\infty$ (i.e., as $C \to -\infty$), it follows that for C sufficiently small, the right hand side will be larger than $(1 - a)/a$ and inequality (50) will be satisfied. This completes the proof of Proposition 8.14.

■

So, let $s = (c_1, c_2(\cdot))$ and $\tau = (m, \theta(\cdot))$ be a pair of equilibrium strategies where c_2 (second period threshold) and θ (completion probability) are functions of x_1.

The fact that $0 < \theta(x_1) < 1$ implies that **O** is indifferent between completion and retreating. For $x_1 < c_1$ this yields precisely equation (23):

$$\frac{\phi\left(\dfrac{c_2(x_1) - \mu_1(m, x_1)}{\sigma}\right)}{\phi\left(\dfrac{c_2(x_1) - \mu_0(m, x_1)}{\sigma}\right)} = \frac{b}{1+b},$$

which is to hold now for the fixed m and for all $x_1 < c_1$.

Letting $\sigma Z = c_2(x_1) - \mu_0(m, x_1)$ and $\delta = (1 - m)/\sigma$ this is

$$\frac{\phi(Z - \delta)}{\phi(Z)} = \frac{b}{1+b} \tag{51}$$

with $\delta > 0$ (since $m < 1$). By Corollary 8.3, there is a unique solution which we denote by $Z(m)$ and hence, for all $x_1 < c_1$ we have:

$$c_2(x_1) = Z(m) + \mu_0 = Z(m) - \rho\frac{\sigma_2}{\sigma_1}m + \rho\frac{\sigma_2}{\sigma_1}x_1. \tag{52}$$

For $x_1 < c_1$ and $x_2 = c_2(x_1)$, player **I** is indifferent between calling or not calling an alarm at the second period which implies:

$$-a - dm = -dm - \theta(x_1 \mid c_2(x_1)),$$

or,

$$\theta(x_1 \mid c_2(x_1)) = a, \tag{53}$$

where $\theta(x_1 \mid c_2(x_1))$ is the conditional probability of diversion given the observations x_1 and $x_2 = c_2(x_1)$. That is,

$$\theta(x_1 \mid c_2(x_1)) = \frac{\varphi(c_2(x_1) \mid m, D)}{\theta(x_1)\varphi(c_2(x_1) \mid m, D) + (1 - \theta(x_1))\varphi(c_2(x_1) \mid m, R)},$$

where $\varphi(c_2(x_1) \mid m, D)$ and $\varphi(c_2(x_1) \mid m, R)$ are densities of x_2 at $x_2 = c_2(x_1)$, given x_1 and given that **O** diverted or retreated respectively i.e.,

$$\varphi(c_2(x_1) \mid m, D) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(c_2(x_1) - \mu_1)^2},$$

and

$$\varphi(c_2(x_1) \mid m, R) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(c_2(x_1)-\mu_0)^2}.$$

Substituting $c_2(x_1)$ from (52) we see that $\varphi(c_2(x_1) \mid m, D)$ and $\varphi(c_2(x_1) \mid m, R)$ do not depend on x_1 (only on m.) It follows from equation (53) that:

- The completion probability $\theta(x_1)$ is constant for $x_1 \leq c_1$.

Denoting this constant probability by θ, the equilibrium strategy of **O** can then be described by the pair of numbers (m, θ). Given such a strategy, the decision of the inspector after the observation of x_1, is either to call an alarm and get a payoff $-a$ or not call an alarm and get an expected payoff:

$$-dm+\theta\left[\phi\left(\frac{c_2(x_1)-\mu_1}{\sigma}\right)-a\left(1-\phi\left(\frac{c_2(x_1)-\mu_1}{\sigma}\right)\right)\right]+(1-\theta)\left[-a\left(1-\phi\left(\frac{c_2(x_1)-\mu_0}{\sigma}\right)\right)\right].$$

Again by substituting $c_2(x_1)$ from equation (52), this expression is independent of x_1. If it is smaller than $-a$, then **I** calls an alarm for all x_1 i.e., $c_1 = -\infty$. If on the other hand the above expression is greater than $-a$, then $c_1 = \infty$. By Proposition 8.14, it cannot be that in equilibrium $c_1 = -\infty$. Therefore we conclude that $c_1 = \infty$ i.e., the inspector never calls an alarm at the first period. (The case in which the expected payoff for no alarm is equal to $-a$ is also excluded in equilibrium since this would make **I** indifferent between $c_1 = -\infty$ and $c_1 = \infty$, but then **O** could decrease slightly m or θ to induce **I** to choose $c_1 = \infty$ yielding a higher payoff to **O**.)

With no alarm at the first period, the expected payoff for the operator is (using his indifference between D and R and substituting c_2 from (52)):

$$\begin{aligned} O(s,\tau) &= -b\left(1-\phi\left(\frac{c_2(x_1)-\mu_0}{\sigma}\right)\right) \\ &= -b+b\phi\left(\frac{Z(m)}{\sigma}\right). \end{aligned} \tag{54}$$

Since $Z(m)$ is determined by m (and the parameters of the game of course), the payoff of **O** at the equilibrium is a function of m which therefore, has to attain its maximum at the equilibrium value of m. We claim that this value is $m = 0$.

Proposition 8.15 *The equilibrium payoff for the operator as given in (54), attains its maximum at $m = 0$.*

Proof Since ϕ is an increasing function, it is enough to see that $Z(m)$ decreases in m. In fact, recalling that $Z(m)$ is the unique solution of equation (51), we have to show that this solution is decreasing in m or equivalently, increasing in δ (since $\delta = 1 - m$), and this is proved Lemma 8.4.

∎

The intuition behind this result is the following: The inspector does not call an alarm at the first period and sets a critical value of x_2 for the second period. Since it is known that **O** is diverting m, the observation x_1 is not informative and the threshold for second period alarm depends therefore only on m (and on x_1 only through the dependence of x_1 and x_2). Denoting this threshold by $C(m)$ it is clear that it decreases in m (since the higher m the easier is to complete the diversion and the more alert has **I** to be.) Now, since **O** is indifferent between completing or not completing the diversion, his payoff equals that due to false alarm when he does not complete diversion which is $-b(1-\phi(C(m)))$. Clearly **O** is interested therefore in increasing the threshold $c(m)$ i.e., decreasing m.

So far we proved that if there is an equilibrium with singular distribution q, it must be of the form:

- **O** commits no diversion at the first period.
- **I** does not call an alarm at the first period.
- At the second period, **O** diverts the full 1 with probability θ (and with probability $1-\theta$ makes no diversion).
- **I** calls an alarm at the second period if and only if x_2 exceeds a critical value $c_2(x_1)$.

To find such an equilibrium we know already by (52) that

$$c_2(x_1) = K + \rho \frac{\sigma_2}{\sigma_1} x_1,$$

for some constant K (which is $Z(0)$ in equation (52).)

Equation (53) now becomes:

$$\begin{aligned} a &= \theta(x_1 \mid c_2(x_1)) \\ &= \frac{e^{-\frac{1}{2\sigma^2}(c_2(x_1)-\rho\frac{\sigma_2}{\sigma_1}x_1-1)^2}}{\theta e^{-\frac{1}{2\sigma^2}(c_2(x_1)-\rho\frac{\sigma_2}{\sigma_1}x_1-1)^2} + (1-\theta)e^{-\frac{1}{2\sigma^2}(c_2(x_1)-\rho\frac{\sigma_2}{\sigma_1}x_1)^2}} \\ &= \frac{e^{-\frac{1}{2\sigma^2}(K-1)^2}}{\theta e^{-\frac{1}{2\sigma^2}(K-1)^2} + (1-\theta)e^{-\frac{1}{2\sigma^2}K^2}}. \end{aligned}$$

This last equation determines θ.

Can this be an equilibrium pair of strategies ? Clearly, the inspector's strategy is a best reply to the operator's strategy since with no diversion at the first period, the threshold strategy is the best test (by the Neyman Pearson Lemma). Is **O** 's strategy a best reply to **I** 's strategy ? His payoff at this point is (by (54)) his payoff for no diversion since he is indifferent between the two alternatives:

$$O(s,\tau) = -b\left(1-\phi\left(\frac{c_2(x_1)-\rho\frac{\sigma_2}{\sigma_1}x_1}{\sigma}\right)\right) = -b + b\phi\left(\frac{K}{\sigma}\right).$$

Suppose now that he uses a different strategy $\tilde{\tau}$ according to which he diverts the full amount 1 at the first period. With this reply, his expected payoff is:

$$\begin{aligned} O(s,\tilde{\tau}) &= -b\left(1-\phi\left(\frac{c_2(x_1)-\rho\frac{\sigma_2}{\sigma_1}x_1}{\sigma}\right)\right)+\phi\left(\frac{c_2(x_1)-\rho\frac{\sigma_2}{\sigma_1}x_1}{\sigma}\right) \\ &= -b+b\phi\left(\frac{K}{\sigma}\right)+\phi\left(\frac{K}{\sigma}\right) > O(s,\tau). \end{aligned}$$

This means that τ is not a best reply to s. We conclude that there is no equilibrium with a sure diversion of some quantity at the first period, completing the proof of the theorem.

∎

Acknowledgment

We highly acknowledge the contribution of D. Rothenstein and E. Matanya, Shmuel Zamir's students, for performing the numerical work at the end of this report.