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Tritiumbilanzierung und nichtmeßbare Inventare in Fusionsreaktoren

ZUSAMMENFASSUNG

Zur Entwicklung der Technologie der Kernfusion werden derzeitig nur relativ kleine Tritiummengen eingesetzt. Somit ist es ausreichend, sogenannte „konventionelle“ Bilanzierungstechniken einzusetzen. Es ist jedoch vorhersehbar, daß die Handhabung von Tritium ausgeweitet werden wird – und somit auch die Tritiummenge. In diesem Fall können nur fortgeschrittene Bilanzierungsmethoden die neuen Erfordernisse erfüllen. In dieser Arbeit wird eine solche fortgeschrittene Bilanzierungstechnik entwickelt und auf idealisierte Experimente im Tritiumlabor Karlsruhe (TLK) sowie einen ebenfalls idealisierten Brennstoffkreislauf vom ITER-Typ angewandt. Diese Aufgabe umfaßt zuerst die Modellierung von Brennstoffkreislauf-Operationen und liefert somit die sogenannten „echten“ Daten der Prozeßinventare. Weil sowohl die Untersysteme des Brennstoffkreislaufs wie auch deren Kopplung untereinander nennenswerten Veränderungen unterliegen, müssen zuerst flexible Simulationsmodelle eingesetzt werden. Zweitens muß ein Meßmodell die echten Daten bearbeiten, es muß Datenreduktion durchgeführt werden, und es müssen mathematisch-statistische Methoden eingesetzt werden, um die Inventare zu verifizieren. Ein dritter Schritt der statistischen Behandlung zielt darauf ab, festzustellen, ob eine Tritiumanomalie, z. B. ein Tritiumverlust, vorliegt.

Da die statistische Analyse auf Probleme führt, deren Lösungen nicht in der Literatur zu finden waren, werden im Anhang die Ergebnisse der entsprechenden Untersuchungen in mathematisch-abstrakter Form dargestellt.

ABSTRACT

For the time being fusion technology development involves relatively small quantities of tritium. Consequently, it is sufficient to apply so-called "conventional" accountancy tools. However, it is foreseeable that tritium operations - and thus the amount of tritium - will increase substantially. An advanced accountancy methodology will satisfy the resulting new requirements. In this study such an advanced accountancy methodolody is developed and applied to the situation envisaged with idealized experiments of the Karlsruhe Tritium Laboratory (TLK) as well as an idealized ITER-type fuel cycle. Firstly, this task comprises modeling of fuel cycle operations, providing the "true" data of the in-process inventories. As both the fuel cycle subsystems

and networking themselves are susceptible to changes, a measurement model takes care of the true data, handles data reduction, and applies mathematical methods to confirm the final inventories on a statistical basis. Then, in a third step, the test statistics might verify whether or not a tritium anomaly, e.g. a tritium loss has occurred.

Since the statistical analysis generates problems the solutions of which are not part of the standard statistical literature in the Annex the results of the related original work is presented in mathematical-abstract form.

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TRITIUM ACCOUNTANCY AND UNMEASURABLE INVENTORIES IN FUSION REACTORS

1. INTRODUCTION

Experience meanwhile gathered in tritium operation of several Tokamaks enables us to investigate in more detail also the pending questions concerning tritium inventory taking and accountancy. Careful tritium inventory taking should be provided above all for ITER (International Thermonuclear Experimental Reactor) - this will certainly become a requirement irrespective of the ITER site ultimately selected.

However, these still open questions have been discussed more thoroughly in recent time only. The problems typical of inventory taking and accountancy named in these discussions are: Which aspects should be considered in setting up Material Balance Areas (MBA)? How does the frequency of inventory taking influence the reliability of accountancy? In which way can a so-called "hidden inventory" exert an influence on the balance? Which effects are caused by the waste streams? It is generally known that these questions can be answered by means of computer simulations [1, 2]. Even if validated measuring values are not yet available, the areas can be defined by parameter variation and sensitivity analyses where difficulties might occur and where further efforts would be rewarding.

It should be recalled that numerous results are already in hand, especially as regards the methodology [3]. However, we should also underline that a proposed solution should be demonstrated for a concrete case in order to be able to make a dependable statement.

It will be investigated here which will be the consequences on inventory taking and hence on accountancy in such cases where among othe aspects, timely detection and localization of anomalies have to be taken into account, and where a hidden inventory cannot be ruled out.

First statistical principles tell us that it is best to consider only one overall balance in time and space and that any subdivision of a reference time into several inventory periods or subdivision of a plant into several MBAs with the purpose of timely detecting or localizing anomalies goes at the expense of the overall accountancy effectiveness. However, this is only the general picture; it may be different in special cases.

Since the reasons for such a counterintuitive behavior are not so obvious, in the Annex the statistical analysis of these problems is presented in abstract form. It may be considered as the essence of all statistical procedures discussed here.

Startup of tritium operation in a fusion reactor is linked to the question of the amounts of tritium involved which can be "bound" in the plasma vessel and in the adjacent tritium carrying components. Established knowledge has been that especially carbon containing wall linings absorb hydrogen but, depending on the operating condition, release again at least some of it. According to knowledge presently available, tritium inventories bound in this way are not accessible to inventory taking by measurement.

Most of this inventory will probably accumulate in the lining of the First Wall; this process will go on gradually after commencement of tritium operation, maybe until a saturation value will be attained. Depending on the strategy of operation, e.g. by prolongated conditioning, this inventory will undergo changes; a substantial fraction might again become accessible to measurement. These processes are presently supported solely by credibility guesses. Consequently, as will be described in more detail below, parameterized simulation can be reasonably applied.

It should be mentioned in this context that "hidden" inventory, inventory taking and accountancy are not basically novel phenomena. A typical example would be the inventory in a chemical process column which cannot be measured (in situ); above all during inventory taking in reprocessing plants the same questions are encountered albeit under the strict requirements of nuclear safeguards covered by international agreements [4].

For the numerical calculations which illustrate quantitatively the properties of the accountancy effectiveness and the effects of timely detecting and localizing anomalies we have used data of idealized experiments in the Karlsruhe Tritium Laboratory (TLK). Only for the illustration of the effects of hidden inventories we have used estimated data of an idealized ITER fuel cycle.

2. PROCESS MODEL

In this investigation the stochastic version of the Karlsruhe Tritium Model (KATRIM) was used [5]. This model relies on a set of linear differential equations which, in their general form, can be written as

$$\dot{y}_k = \sum_i a_{ik} y_i - y_k \sum_i a_{ki}, \qquad i = 1,2,\ldots, i \neq k$$

where y_k and y_i are the tritium inventories of the subsystems k and i of the fuel cycle and a_{ik} and a_{ki} are so-called "transfer coefficients." Thus, we obtain with y_k the variation with the time of the inventory in the subsystem k as a function of those inventories of subsystems which contribute to this special case.

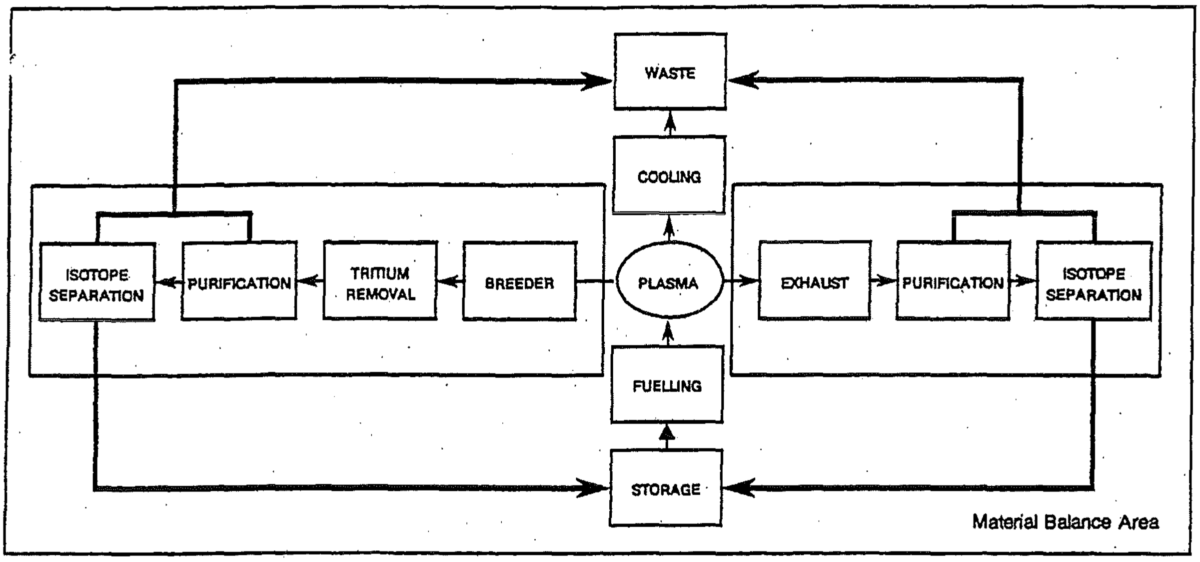

Figure 2.1: Idealized ITER fuel cycle.

As the operating conditions of the subsystems undergo random variations, this distribution applies to the respective inventories too - however within reasonable area boundaries. This means, however, that also the differential equations must be solved anew taking into account new starting conditions after each change of an operating condition. It should be mentioned here that the general framework, e.g. the availability of subsystems, can be specified. Application of this flexible tool invariably leads to numerical approaches to solutions; the computer capacity available today produces in this case results of sufficient accuracy, even if comprehensive Monte Carlo simulations are made.

The process model described allows any specifications to be made as regards the number of subsystems and their networking. However, not the study of cycle variants will be treated with priority here. Therefore, the flowsheet traced in Figure 2.1 will be used as an example.

We assume that the random errors of all measurements are normally distributed. The systematic error contribution is not known. It follows from the analysis described below that any systematic error contributions during inventory taking exert little influence on the results of this study. Nevertheless, it should be mentioned here once more that this error component and hence its consequences will remain unknown until suitably organized interlaboratory programs (so-called Round-Robin tests) will furnish the necessary information [6].

3. MEASUREMENT MODEL

Once the process model has been established, the plant operator has to conceive a system of measurement devices with the help of which for all material balance areas and inventory periods all inventories, receipts and shipments can be measured. This may pose technical problems since frequently material flows are measured by inventory differences thus, independent measurements of flows and inventories are difficult to obtain. In addition, since in applying all these measurement methods errors cannot be avoided, one has to know their variances in order that the statistical analyses to be discussed subsequently can be performed.

The estimation of the variances of the random measurement errors does not pose a major problem in general. There are, however, also systematic errors (biases) of various kinds some of which can only be estimated with the help of the afore mentioned interlaboratory or Round Robin tests and which may require considerable technical, organisatorial and analytical effort. Nevertheless, this has to be done since systematic errors are crucial for accountancy effectiveness as a whole. As a result of all of these technical and analytical efforts, the so-called covariance matrix of the whole measurement system, defined in space and time, has to be established.

4. ACCOUNTANCY PRINCIPLE AND TEST PROCEDURE

Let us consider one material balance area and one inventory period $[t_0, t_1]$. Let the real inventories at t_0 and t_1 be I_0 and I_1, and let the receipts and shipments during this period be R_1 and S_1, which means that the book inventory at t_1, i.e. the inventory which should be there, is $B_1 = I_1 + R_1 - S_1 = I_0 + A_1$. To establish the material balance for this material balance area and this inventory period means to compare the book inventory with the real inventory at t_1.

For this purpose, we define the balance statistics

$$L_1 := B_1 - I_1 = I_0 + A_1 - I_1 . \qquad (4\text{-}1)$$

If there were no measurement errors and losses we should find $L_1 = 0$; in case there are losses running up to the amount μ_1 we should find $L_1 = \mu_1$. Since, however, measurement errors can never be avoided, a statistical test has to be performed in order to decide whether or not a non-zero value of L_1 can be explained by measurement errors alone.

Standard statistical procedures lead to the error first and second kind probabilities α_1 and $ß_1$ as follows:

Let H_0 be the null hypothesis which means that no material is missing, and H_1 the alternative hypothesis which means that the amount μ_1 of material is missing, in formulae

$$H_0 : E_0(L_1) = 0, \quad H_1 : E_1(L_1) = \mu_1 > 0. \tag{4-2}$$

Now, with the help of the observed test statistics L_1 it is decided

$$\begin{aligned} L_1 \leq s_1 &: H_0 \text{ is true} \\ L_1 > s_1 &: H_1 \text{ is true,} \end{aligned} \tag{4-3}$$

where s_1 is the so-called significance threshold. The value of s_1 is determined with the help of the error first kind probability α, also called false alarm probability, which is defined as the probability to decide „H_1 is true" if in fact H_0 is true,

$$\alpha_1 = \text{prob}\,(L_i > s_1 \mid H_0). \tag{4-4}$$

If we assume that the measurement errors are normally distributed and define the variance of L_1 as

$$\text{var}\,(L_1) = \sigma_1^2, \tag{4-5}$$

then we get

$$\alpha_1 = \Phi\left(\frac{s_1}{\sigma_1}\right), \tag{4-6}$$

where Φ (x) is the standard normal or Gaussian distribution defined by

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{t^2}{2}\right) dt. \tag{4-7}$$

The efficiency of this procedure is measured by the error second kind probability β_1, namely the probability to decide „H_0 is true" if in fact H_1 is true,

$$\beta_1 = \text{prob}\,(L_1 \leq s_1 \mid H_1) . \tag{4-8}$$

Explicitely, it is given by

$$\beta_1 = \Phi\left(\frac{s_1}{\sigma_1} - \frac{\mu_1}{\sigma_1}\right) \tag{4-9}$$

or, if we eliminate s_1 with the help of (4-6),

$$\beta_1 = \Phi\left(\Phi^{-1}(1-\alpha_1) - \frac{\mu_1}{\sigma_1}\right), \tag{4-10}$$

where Φ^{-1} is the inverse of Φ. In the following we will call $1-\beta$ *accountancy effectiveness.*

5. SEQUENCE OF INVENTORY PERIODS

Now, let us consider a sequence of inventory periods $[t_0, t_1]$, ... $[t_{n-1}, t_n]$, and i = 1, ... n. In order to define an appropriate test procedure, one has to consider an anomaly scenario $(\mu_1, ..., \mu_n)$, where μ_i, i = 1, ..., n, is the anomaly occurring in the i^{th} inventory period. Since it would be highly arbitrary just to select one specific scenario, a *game theoretical* treatment is mandatory.

Any game theoretical analysis requires at least two players. The first player is the plant operator. Since we do not know the pattern of anomalies if occurring, we have to assume that the technical system as the 'second player' decides first whether or not to 'introduce' an anomaly; and if so to choose an anomaly scenario that is most adverse to the operator.

A two person game is defined by the sets of strategies of the two players - in our case the set of possible test procedures on one hand and the set of anomalies on the other hand - and the payoffs to the players. If these payoffs are independent of the value and of the time of occurrence of the anomalies, then it turns out that it suffices that a two person zero sum game has to be considered with the accountancy effectiveness (equals the probability of detecting an anomaly) as payoff to the operator, with the false alarm probability as parameter. Just in passing it should be noted that this procedure is standard statistical practice.

This, in turn enables us to apply the well known Neyman-Pearson lemma [7] which gives advice how to construct the best test in the sense of accountancy effectiveness. The result is that for a fixed total anomaly $\mu = \mu_1 + ... + \mu_n$ the test statistic is just the overall balance for the reference time $[t_0, t_n]$ which then is used as described in Section 4.

This means, however, that the aspect of detecting any anomaly in time is ignored. Thus, if one wants to take into account this aspect, after each inventory taking indeed a test has to be performed. Such a procedure is the subject of the following sections.

5.1 Two Inventory Periods

Next, we consider two consecutive inventory periods, $[t_0, t_1]$ and $[t_1, t_2]$. For the reasons just mentioned we apply a test procedure to both periods separately. In that case, however, we face a problem related to the total error probabilities: the test statistics use a common inventory, namely I_1; therefore, they cannot be handled as independent statistics. This leads us to the bivariate normal distribution. The corresponding analysis has been performed [8] and is presented in abstract form in the Annex. This study follows another method which, in a broader context, is presented in [3].

We assume that the test performed after the first inventory period did not indicate any significant difference between $I_0 + A_1$ and I_1. It is customary, in that case, to use for the next inventory period a starting inventory S_1, derived as weighted mean using $I_0 - A_1$ and I_1, which has a variance as small as possible [9]; the result is

$$S_1 = \frac{\sigma_0^2 + \sigma_{A1}^2}{\sigma_0^2 + \sigma_{A1}^2 + \sigma_I^2} \cdot I_1 + \frac{\sigma_I^2}{\sigma_0^2 + \sigma_{A1}^2 + \sigma_I^2} \cdot (I_0 - A_1). \tag{5-1}$$

The variance var (S_1) of S_1, is given by

$$\frac{1}{\text{var}\,(S_1)} = \frac{1}{\sigma_1^2} + \frac{1}{\sigma_0^2 + \sigma_{A1}^2}. \tag{5-2}$$

From (5-2) we derive that the variance of S_1 is smaller than the variance of I_1 and smaller than the variance of $I_0 + A_1$; (5-1) shows that the mean S_1 is more influenced by that component, which has a smaller variance.

For the second inventory period we select the following accountancy statistics:

$$LR_2 = S_1 + A_2 - I_2\,, \tag{5-3}$$

where A_2 and I_2 denote the sum of inputs and outputs occurring during the second period and the inventory measured after the second period, respectively:

$$t_2: \; I_2, \mathrm{var}\,(I_2) = \sigma_{I2}^2, A_2, \mathrm{var}\,(A_2) = \sigma_{A2}^2 \, . \tag{5-4}$$

Now, it is interesting that L_1 and LR_2 display the property that

$$\mathrm{cov}\,(L_1, LR_2) = 0, \tag{5-5}$$

which means L_1 and LR_2 are not correlated and, because we assumed them to be normally distributed, are independent. Thus, all joint probabilities for the two inventory periods will be products of the single inventory period probabilities. For the total probabilities of error, α and ß, we obtain:

$$1 - \alpha = (1 - \alpha_1) \cdot (1 - \alpha_2), \; \beta = \beta_1 \cdot \beta_2 \, , \tag{5-6}$$

where α_1 and $ß_1$ are given by (4-6) and by (4-10), where $ß_2$ is calculated by

$$\beta_2 = \Phi\left(\Phi^{-1}(1 - \alpha_2) - \frac{\mu_{R2}}{\sigma_{R2}} \right)$$

$$\mu_{R2} = \frac{\sigma_1^2}{\sigma_0^2 + \sigma_{A1}^2 + \sigma_1^2} \cdot \mu_1 + \mu_2 \tag{5-7}$$

$$\sigma_{R2}^2 = \mathrm{var}\,(LR_2) = \mathrm{var}\,(S_1) + \sigma_{A2}^2 + \sigma_{I1}^2$$

and where μ_1, and μ_2 are the real losses during the two inventory periods.

Here we should note that the procedure described above, which represents a special application of the so-called Kalman filter theory does not necessarily lead to a higher accountancy effectiveness, compared to other procedures. However, it displays substantial technical advantages, especially if a long-term set of inventory periods has to be considered.

5.2 The General Case

For the i^{th} inventory period $[t_{i-1}, t_i]$, i = 1, 2, ..., n, we define the i^{th} material balance statistics LR_i, as

$$L_{Ri} = S_{i-1} + A_i - I_i \, , \tag{5-8}$$

where the starting inventory S_{i-1}, is given by the weighted mean of the previous, appropriately defined, book inventory $S_{i-2} - A_{i-1}$, and the previous final inventory I_{i-1}, and where S_{i-1} has a minimum variance. For S_{i-1}, we get the following recursive relation

$$S_{i-1} = \frac{\text{var}(S_{i-2}) + \sigma^2_{Ai-1}}{\text{var}(S_{i-2}) + \sigma^2_{Ai-1} + \sigma^2_{Ii-1}} \cdot L_{i-1} + \frac{\sigma^2_{Ai-1}}{\text{var}(S_{i-2}) + \sigma^2_{Ai-1} + \sigma^2_{Ii-1}} \cdot (S_{i-2} + A_{i-1}). \tag{5-9}$$

The variance of this starting inventory S_{i-1}, can be determined by a simple recursive relation,

$$\frac{1}{\text{var}(S_{i-1})} = \frac{1}{\sigma^2_{Ii-1}} + \frac{1}{\text{var}(S_{i-2}) + \sigma^2_{Ai-1}}. \tag{5-10}$$

from which the following recursive relation of the variance of the i[th] material balance statistics L_i, is obtained:

$$\frac{1}{\text{var}(LR_i) - \sigma^2_{Ii} - \sigma^2_{Ai}} = \frac{1}{\sigma^2_{Ii-1}} + \frac{1}{\text{var}(LR_{-1i}) - \sigma^2_{Ii-1}}, i = 2, 3, \ldots, n$$

$$\text{var}(LR_1) = \text{var}(L_I) = \sigma^2_{I0} + \sigma^2_{A1} + \sigma^2_{I1}. \tag{5-11}$$

5.3 Total Accountancy Effectiveness for n Periods

We assume that, during the i[th] period, the real loss μ_i occurs; i.e., the expected value of the i[th] balance statistic L_i is

$$E_1(L_i) = \mu_i, i = 1, 2, \ldots, n. \tag{5-12a}$$

Then the expected value μ_{Ri} of LR_i,

$$E_1(LR_i) = \mu_{Ri}, i = 1, 2, \ldots, n. \tag{5-12b}$$

satisfied the recursive relation

$$\mu_{Ri} = \frac{\sigma^2_{Ii-1}}{\text{var}(LR_{i-1})} \cdot \mu_{Ri-1} + \mu_i, i = 2, 3, \ldots, n.$$

$$\mu_{Ri} = \mu_1. \tag{5-13}$$

We should mention that, under steady state process conditions,

$$\sigma^2_{Ii} =: \sigma^2_I,\ \sigma^2_{Ai} =: \sigma^2_A,\ i = 1, 2, \ldots, n \tag{5-14}$$

the recursive relations (5-10), (5-11), and (5-13) can be solved explicitely and have interesting asymptotic properties. However, these solutions are complicated; for numerical calculations we prefer the recursive equations given.

We obtain the total accountancy effectiveness for n periods as

$$1-\beta = 1-\prod_{i=1}^{n} \beta_i\,; \tag{5-15}$$

where the single probabilities of the error of the second kind are given by

$$\beta_i = \Phi\left(\Phi^{-1}(1-\alpha) - \frac{\mu_{Ri}}{\sigma_{Ri}}\right); \tag{5-16}$$

and the total error first kind probability α is given by:

$$1-\alpha = \prod_{i=1}^{n} (1-\alpha_i)\,, \tag{5-17}$$

and μ_{Ri} as well as σ_{Ri}^2 = var (LR_i), i = 1, 2, ... n, are calculated by (5-13) and (5-11), respectively.

Let us recall at this point that the scaling of the interval, $[t_0, t_n]$ into n inventory periods was considered as a given scenario. Now, we might ask for the maximum accountancy effectiveness 1-ß, if the probability of error α, as well as the total loss μ, are given. It can be proven with the Lemma of Neyman and Pearson [7] that the optimum test procedure makes use of the balance derived over the total accountancy period, namely:

$$L = I_0 + \sum_{i=1}^{n} A_i - I_n\,. \tag{5-18}$$

In other words, we discard any credit from the physical inventory takings I_1, I_2, ... I_{n-1}. Therefore, the physical inventory takings carried out at the points in time, t_1, t_2, ... t_{n-1}, derive their importance only from the objective to detect any loss in due time. But timeliness is not covered by the "accountancy effectiveness" measure.

In order to grasp this aspect we might define the run length RL, of the test procedure as the number of inventory periods covered until the test procedure indicates that losses may have occurred. The timely detection of a loss can then be measured by given quantiles of the run length distribution or by the average run length E(RL). From what has been said above, however, it follows that there is a trade-off between a high overall probability of detection and timeliness. Since this cannot be reconciled within the framework developed here, but has to be treated with the help of practical arguments, we will not go into the further details of these operations at the present stage of development, but discuss it in the next chapter.

5.4 Numerical Examples

A typical experimental situation in the Karlsruhe Tritium Laboratory (TLK) is demonstrated in Figure 5.1: We assume tritium transfer within a cycle.

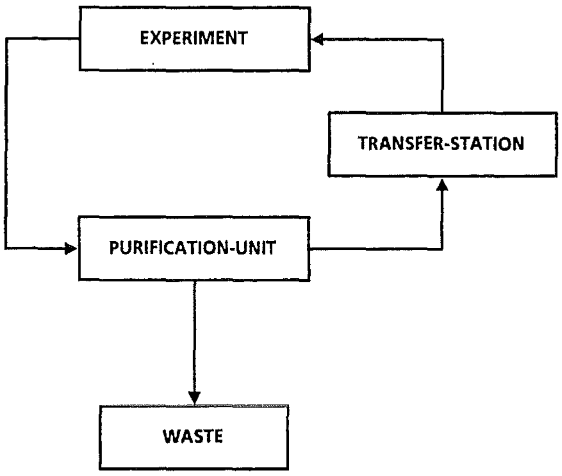

Fig. 5.1: Idealized process cycle of TLK activities.

The cycle consists of three compartments, namely the Experiment, the Purification Unit, and the Transfer Station. The Experiment needs purified tritium to meet its research objectives; it will discard tritium together with impurities. The purification of this tritium will cause a tritium loss. In addition, further tritium losses are to be expected during the transfer and use of tritium batches; they are included in the Waste compartment shown in Figure 5.1. We consider this cycle as a typical, though idealized, part of TLK.

The special data used in this example are compiled in Table 5.1. Using these data we calculate

$$\sqrt{\text{var (I)}} = 0.2 \cdot 0.005 = 0.001 \quad [\text{g}^2] \tag{5-19a}$$

and, for the waste accumulated during one year, we obtain

$$\sqrt{\text{var (A)}} = 0.2 \cdot 0.01 \cdot 0.2 = 0.0004 \quad [\text{g}] \tag{5-19b}$$

which means that the variance of the waste measurement can be neglected when compared to the inventory measurement. Therefore, we might interpret our numerical example in the following way: Accounting after (4-1) consists in a comparison of inventories.

Inventory of the Transfer Station	0.2 g
Coefficient of variation of inventory determination	0.5%
Total period	1 year
Maximum number of inventory periods	12 $year^{-1}$
Total waste per year	1% of the inventory
Coefficient of variation of waste measurement	20%

Table 5.1: Data of the first numerical example.

From (5-19) we derive, for any period length, the variance of the balance statistics:

$$\mathrm{var}\,(L_I) = 2 \cdot \mathrm{var}\,(I) = 0.002^2 \quad [g^2]\,. \tag{5-20}$$

If we consider a loss μ, and assume a probability of error first kind α = 0.05, we derive from (4-9) the accountancy effectiveness,

$$1 - \beta = \Phi\left(\frac{\mu}{0.002} - 1.645\right). \tag{5-21}$$

Generally, (5-15) gives the total accountancy effectiveness related to this special example and n inventory periods. Figure 5.2 shows the total accountancy effectiveness as a function of the total annual loss μ; the number of inventory periods serves as a parameter.

In the next numerical example we assume that the annual total waste amounts to 2.5 % of the inventory. Then we calculate

$$2 \cdot \mathrm{var}\,(I) + \mathrm{var}\,(A) = 2 \cdot 0.001^2 + 0.001^2 = 0.00173^2 \quad [g^2]\,. \tag{5-22}$$

For one inventory period per year, the assumed loss μ, and for a probability of error first kind α = 0.05, we obtain the total accountancy effectiveness from (4-9):

$$1 - \beta = \Phi\left(\frac{\mu}{0.00173} - 1.645\right). \tag{5-23}$$

If we consider n inventory periods, the variance of the waste measurement per inventory period is given by

$$\sigma_{An}^2 = \frac{1}{n} \cdot \mathrm{var}\,(A) =: \frac{\sigma_{A1}^2}{n};$$

the variance of the transformed test statistics LR_i, i = 1, ..., n, is defined by (5-8) with σ_{1i}^2 = var (I) and $\sigma_{Ai}^2 = \sigma_{An}^2$, i = 1, ..., n. The expected value μ_{Ri}, is given by (5-12b) where we introduce $\mu_i = \mu/n$. The total accountancy effectiveness for n periods is calculated with (5-15). Figure 5.3 shows the total accountancy effectiveness as a function of the total annual loss μ, and for various numbers of inventory periods per year.

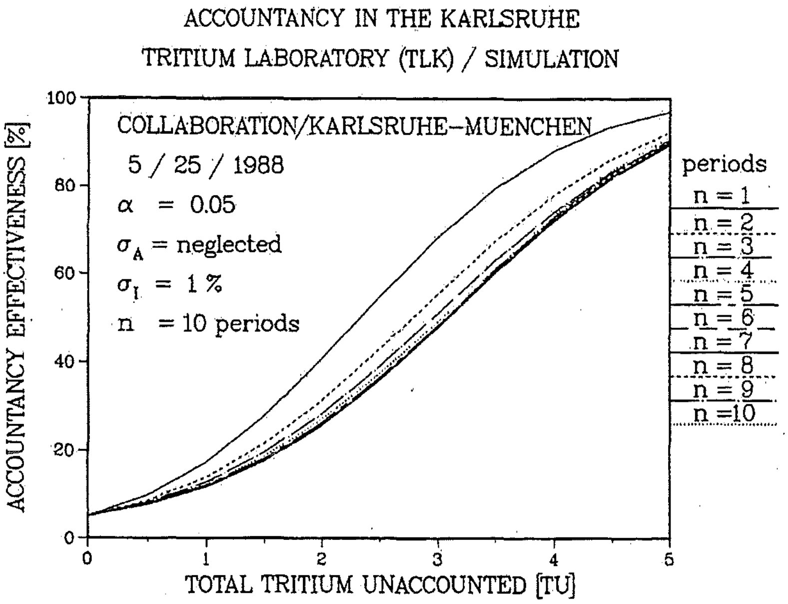

Fig. 5.2: Total accountancy effectiveness according to (5-15) for the numerical example given in Table 5.1.

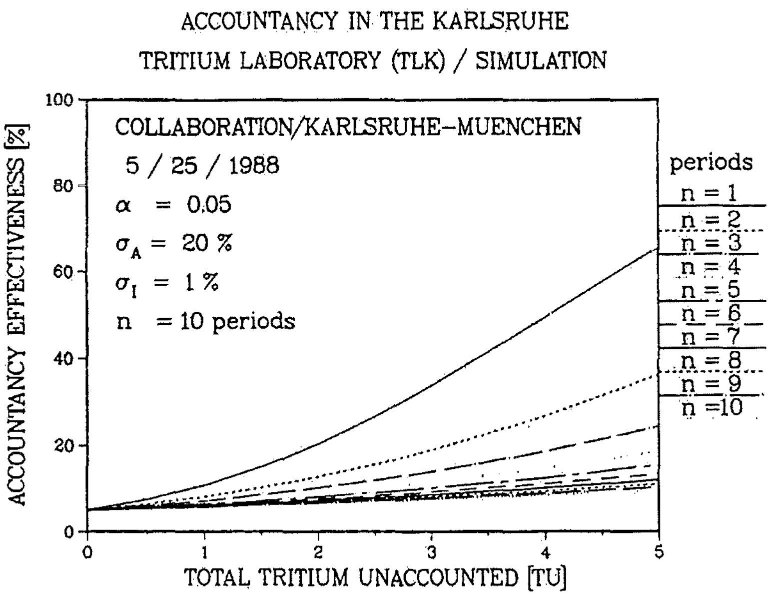

Fig. 5.3: Total accountancy effectiveness according to (5-15) for the data given in Table 5.1, but a total annual waste amounting to 2.5 % of the inventory.

6. TIMELINESS

The result just described implies that a statement on an anomaly can be given only at the end of the reference time $[t_0, t_n]$. There is however also the aspect of timely detecting an anomaly, which, as we just saw, can be met only at the expense of accountancy effectiveness. If so, it has to be valued in terms of payoffs to the players. This means that the solution of the game, in particular, the test procedure of the operator depends on these payoffs.

It should be mentioned that in case one considers - at least theoretically - an infinite sequence of inventory periods and furthermore, exponentially discounted payoffs, then it suffices to use as payoff to the operator the average run length if an anomaly occurs, i.e., the expected time until detecting an anomaly, with the expected time until the first false alarm as a parameter [10]. Even, if this is justified the average run length need not exist and furthermore, there is no equivalent to the Neyman-Pearson lemma, which advises us how to construct the best sequential test.

In practice it will not be possible to estimate payoff functions which do take into account the timeliness aspect. Therefore, only a pragmatical approach is possible: One defines inventory periods according to practical, i.e. plant operations specific criteria and uses sequentical test procedures which have qualified for other control purposes and performs sensitivity studies with respect to anomaly patterns. Since this problem is not genuine to tritium accountancy, one can draw on well documented long-term experience from other areas [3].

For the sake of illustration, just three of them are presented here. First, one can simply perform single tests of the kind described in the fourth section, with the same single false alarm probability for each single test. Since the test statistics are correlated at least via the intermediate inventories, it is an analytical resp. numerical problem to determine the accountancy effectiveness and the detection time distribution (or some quantile) and furthermore, control the propagation of false alarm probabilities.

Second, one can use test statistics ('CUMUF') which are just the material balances from the very beginning until the time in question. This may be justified again by the Neyman-Pearson lemma, but as before, the propagation of the false alarm probabilities represents a problem.

Third there is the so-called CUSUM- or Page's Test [11] which is similar to the CUMUF Test. It is widely used in industrial quality control and therefore, has been investigated intensely.

In addition, it should be mentioned that a Kalman filter approach [12] has been studied. This means basically, as outlined in the fifth chapter, that the single material balance statistics are transformed to uncorrelated, and therefore, since all measurement errors are assumed to be normally or Gaussian distributed, independent statistics. The latter ones then, again can be used in the different ways described above.

6.1 Inventory Taking and Balance

Inventory taking makes part of routine work of each plant operator. Frequently, the circumstances of plant operation even prescribe the dates for inventory taking. Many substances - especially all radioactive materials - are e.g. monitored by the authority as well which prepares material balances and in that case also fixes the dates of inventory taking. For the sake of simplicity, the dates of changes in operation are used also for inventory taking in this study. So, a tritium balance for the interval (t_i, t_{i+1}) reads:

$$I(i) + \text{breeding gain} - \text{burnup} - I(i+1) \quad \text{for } i = 0, 1, 2, \ldots \tag{6-1}$$

where I(i) are the in-process inventories measured at the dates indicated above and the breeding gain and burnup are calculated for the intervals above and taken into account as variables free of errors. In practice, they can likewise be measured variables.

If no measurement errors or anomalies occur, the difference above should equal zero; in case of anomalies $\mu > 0$, the difference should be μ. Since measurement errors cannot be avoided, a statistical test must be performed which helps to decide whether a non-vanishing difference can be attributed solely to measurement errors.

6.2 Overall Balances

It has been mentioned before, and it has already been described in detail in earlier publications [13] that an overall balance covering space and time gives a particularly good quality of balance accountancy does not produce indications regarding the location or time span during which a supposed anomaly occurs.

Let the total inventory I(i) at time t_i be:

$$I(i) = \sum_{j=1}^{n} I(i,j) \quad \text{for } i = 0, 1, 2, \ldots, n, \tag{6-2}$$

where I(i, j) is the inventory of the j^{th} subsystem at time i and where n is the total number of reactor subsystems to be considered. In the simple case systematic errors are neglected so that the variance σ_i^2 of the measurement errors of the total inventory reads:

$$\sigma_i^2 = \sum_{j=1}^{n} \sigma_{i,j}^2 , \tag{6-3}$$

where $\sigma_{i,j}^2$ is the variance of the measurement errors of the j^{th} subsystem at time t_i.

For the interval (t_0, t_n) the total balance in terms of space and time reads, provided that the breeding gain and burnup are supposed to be fixed variables,

$$I(0) - I(n). \tag{6-4}$$

To be able to decide whether an anomaly has occurred or not within that interval, the following onesided test is performed in the same way as outlined in chapter 4:

$$\begin{aligned} I(0) - I(n) \le s: &\quad \text{no anomaly} \\ I(0) - I(n) > s: &\quad \text{anomaly}. \end{aligned} \tag{6-5}$$

The significance threshold s is fixed by means of the error first kind or false alarm probability α as

$$1 - \alpha = \text{prob}\left(I(0) - I(n) \le s \,\middle|\, H_0 \right) \tag{6-6}$$

where H_0 denotes the zero hypothesis which means the absence of an anomaly. The quality of balance accountancy 1 - ß is then expressed by the formula

$$1 - \beta = \text{prob}\left(I(0) - I(n) > s \,\middle|\, H_1 \right) \tag{6-7}$$

where H_1 means the alternative hypothesis, e.g. the anomaly $\mu > 0$.

For measurement errors with a normal distribution the relationship between 1 - ß, α, μ and σ_0^2 and σ_n^2 is given by:

$$1-\beta=\Phi\left(\frac{\mu}{\sqrt{\sigma_0^2+\sigma_n^2}}-\Phi^{-1}(1-\alpha)\right), \qquad (6\text{-}8)$$

where $\Phi(.)$ is the standard normal distribution given by (4-7) and $\Phi^{-1}(.)$ its inverse.

6.3 Partial Balances for a Time Span

To be able to fix the duration of anomalies appearing, partial balances must be considered, e.g. balances of the form

$$I(i)-I(i+1),\ i=0,1,2,\dots. \qquad (6\text{-}9)$$

To be able to prepare these partial balances, tests can be performed which are similar to the tests described before. However, a problem is encountered when the significance thresholds are fixed. Although they could be fixed as above by means of predetermined single false alarm probabilities, this is of little use: As the balance statistics are not independent, the total false alarm probability cannot be determined from the single false alarm probabilities, at least not by analysis.

Besides, it is not automatically the suitable criterion of specifying the test because the timely detection of anomalies initiates the study of partial balances. By contrast, it would be appropriate to consider mean run lengths, i.e. the times expected to elapse until detection of an anomaly. However, they can be determined solely by means of computer simulation.

Accordingly, the significance threshold s for the sequential test method

$$\begin{aligned} &I(i)-I(i+1)\le s,\ i=0,1,2,\dots,: && \text{continue}\\ &I(i)-I(i+1)>s: && \text{stop}\end{aligned} \qquad (6\text{-}10)$$

is fixed in such a way that under H_0 the mean run length L_0 is specified. In practice, this is achieved by an iterative method in which s is previously defined, the corresponding mean run length L_0 is estimated from a sufficient number of simulation runs and, subsequently, s is varied until the desired mean run length is established. A meaningful starting value is obtained on the assumption that all the test statistics are independent; in this case we obtain, assuming $\sigma_i^2 = \sigma^2$ for i = 0, 1, 2, ...

$$L_0 = \sum_{i=1}^{\infty} i \cdot (1-\alpha)^{i-1} \cdot \alpha = \frac{1}{\alpha} = \frac{1}{\Phi\left(s/\sqrt{2} \cdot \sigma\right)}. \quad (6\text{-}11)$$

where α is the single false alarm probability.

The mean run length L_1 until occurrence of an anomaly corresponds here to the quality of balance accountancy in the overall balance test. It obviously depends on the chosen scenario, i.e. on the way in which the total anomaly is distributed among the time intervals (t_0, t_1), (t_1, t_2) …, (t_{n-1}, t_n).

Evidently, other sequential test methods are conceivable as well and have been examined in connection with nuclear material balance accountancy. A familiar test method consists in summing up all single test statistics for a given time t_i which means that the overall balances are tested from the start until the given point in time

$$\begin{aligned} & I(0) - I(i) \leq s, \ i = 1, 2, \ldots : \text{ continue} \\ & I(0) - I(i) > s: \qquad \text{stop} \end{aligned} \quad (6\text{-}12)$$

where the new significance threshold s is fixed as described above.

The advantages and drawbacks of this method over the method above are obvious:

- A protracted anomaly will be detected more effectively with the second method applied because at each date of balance accountancy the total anomaly previously appearing is recorded.
- An abrupt anomaly can be identified to occur within a period at the end of which it is detected using the first test method.

The third sequential test method which should be mentioned here is a test method in which instead of the original balance statistics those are used which have undergone an independence transformation, see chapter 5 and [14]. This gives the following test statistics:

$$\begin{aligned} & Z(1) = I(0) - I(1) \\ & Z(i) = \frac{1}{i} \cdot \sum_{j=0}^{i-1} I_j - I_i, i = 2, 3, \ldots, \end{aligned} \quad (6\text{-}13)$$

which means that at the end of the i^{th} period the mean value of the previous inventories is compared with the inventory measured last. The variances of these new test statistics are:

$$\text{var}\,(Z(i)) = \frac{i+1}{i} \cdot \sigma^2, i = 1, 2, \ldots . \tag{6-14}$$

They are dependent on the period but they quickly arrive at an asymptotic value.

If the significance points for the individual tests are chosen such that the single false alarm probabilities adopt the same value α, the mean run length under H_0 is according to (6-11):

$$L_0 = \frac{1}{\alpha}. \tag{6-15}$$

Consequently, the significance threshold s_i of the individual tests can be conveniently fixed as follows by specification of L_0:

$$s_i = \sqrt{\text{var}\,(Z(i))} \cdot \Phi^{-1}(1-\alpha), i = 1, 2, \ldots . \tag{6-16}$$

As already said, scenarios must be defined for determination of the mean run length L_1 under H_1, i.e. upon occurrence of anomalies.

If, for a given anomaly, $ß_i$ is the probability for not detecting this anomaly after the i^{th} intenvory period, then the average run length L_1 under H_1 is

$$L_1 = \sum_{i=1}^{\infty} i \cdot \prod_{j=1}^{i-1} ß_j \cdot (1-\beta_i) \tag{6-17}$$

Two alternatives will be considered below as examples.

(a) It is assumed that during each inventory period the same amount μ of material gets lost. Then the expectations of the transformed test statistics read:

$$E_1\,(Z(i)) = \frac{i+1}{2} \cdot \mu \quad \text{for } i = 1, 2, \ldots . \tag{6-18}$$

and, accordingly, the single detection probabilities read:

$$1 - \beta_i = \Phi\left(\frac{\mu}{\sigma} \cdot \frac{\sqrt{i \cdot (i+1)}}{2} - \Phi^{-1}(1-\alpha)\right) \quad \text{for } i = 1, 2, \ldots . \tag{6-19}$$

(b) It is assumed that only during the first period the amount μ gets lost. Then the expectations of the transformed test statistics read:

$$E_1(Z(i)) = \frac{\mu}{i} \quad \text{for } i = 1, 2, \ldots, \tag{6-20}$$

and, accordingly, the single detection probabilities read:

$$1 - \beta_i = \Phi\left(\frac{\mu}{\sigma} \cdot \frac{1}{\sqrt{i \cdot (i+1)}} - \Phi^{-1}(1-\alpha)\right) \quad \text{for } i = 1, 2, \ldots. \tag{6-21}$$

In any case, these average run lengths satisfy the following conditions:

$$L_1 < L_0 \quad \text{and} \quad L_1(\mu_i = 0) = L_0 \quad \text{for } i = 1, 2, \ldots,$$

which is reasonable; the first property corresponds to that of an unbiased test if non-sequential test procedures are considered.

6.4 Numerical Examples

All random measurement errors are specified as variables with normal distributions. In concentration measurements also those methods can be applied which make use of the radioactivity of tritium. For the sake of simplicity, normal distribution is assumed in that case as well. The random measurement uncertainties regarding volume, pressure, temperature and concentration have been compiled in Table 6.1.

Measured Variable	Range of Coefficient of Variation (%)
Volume V	$1 \leq \delta V \leq 3$
Pressure p	$0.1 \leq \delta p \leq 1$
Temperature T	$0.5 \leq \delta T \leq 2$
Concentration C	$1 \leq \delta C \leq 5$

Table 6.1: Ranges of typical measurement uncertainties.

The values in the table will be referred to in the following calculations. All measured variables are supposed to be 2% in the determination of the coefficient of variation used in inventory taking. This is a simple and obvious numerical value, which is justified considering the uncertainties of the starting data compiled in Table 6.1 and the remarks below.

It should be stressed once more that the selected parameters do not impose a restriction on simulation. In particular, they do not impair the goal pursued, namely a comparison of balance accountancy methods because actually all methods are affected by the same uncertainties of starting data. This finding can obviously be elucidated in more detail, e.g. by sensitivity analyses.

Figure 6.1 presents an example of the inventory dynamics of the total material balance area (MBA) shown in Figure 2.1. Likewise, Figure 6.2 presents a simulation result for a false alarm rate α = 5% and under the assumption that no anomaly occurs. The line indicates the significance threshold s.

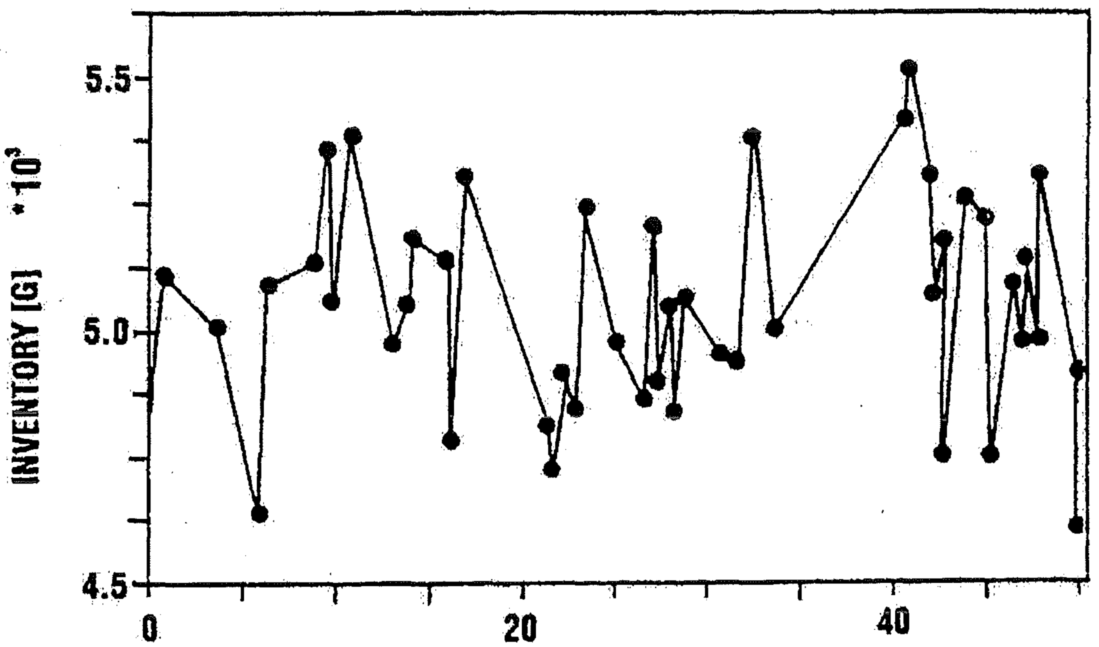

Fig. 6.1: Tritium inventory dynamics within the material balance area indicated in Figure 2.1.

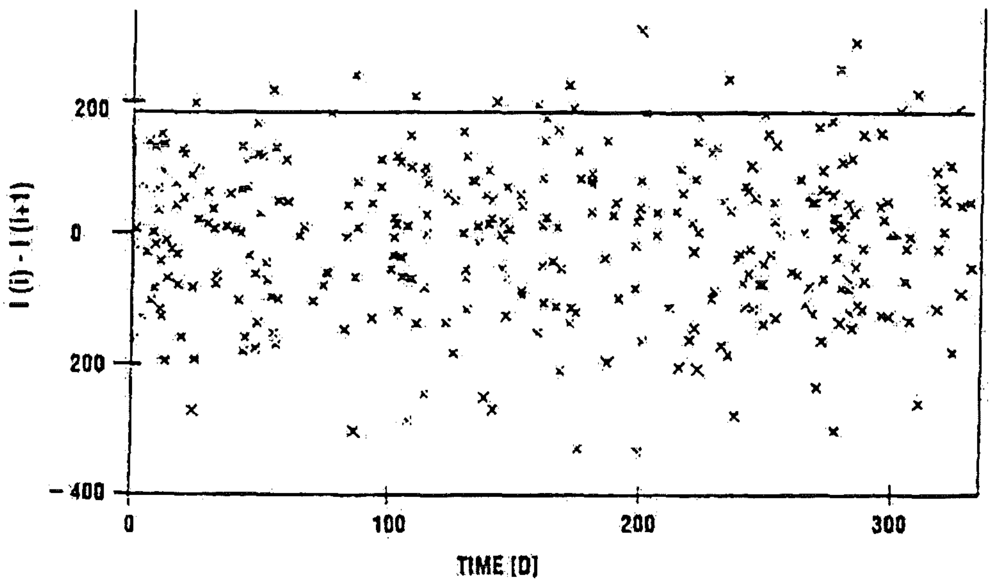

Fig. 6.2: Simulated inventory differences under the assumption that no anomaly occurs; the line indicates the significance threshold s for α = 5%.

Figure 6.3 shows a result using the first test procedure. The anomaly follows a uniform distribution; the false alarm rates are 5% and 10%. Obviously, the mean run length L_1 decreases as a function of the size of the anomaly.

The data shown in Figure 6.4 were obtained using the approximation of the third test procedure. The anomaly occurred during the first inventory period.

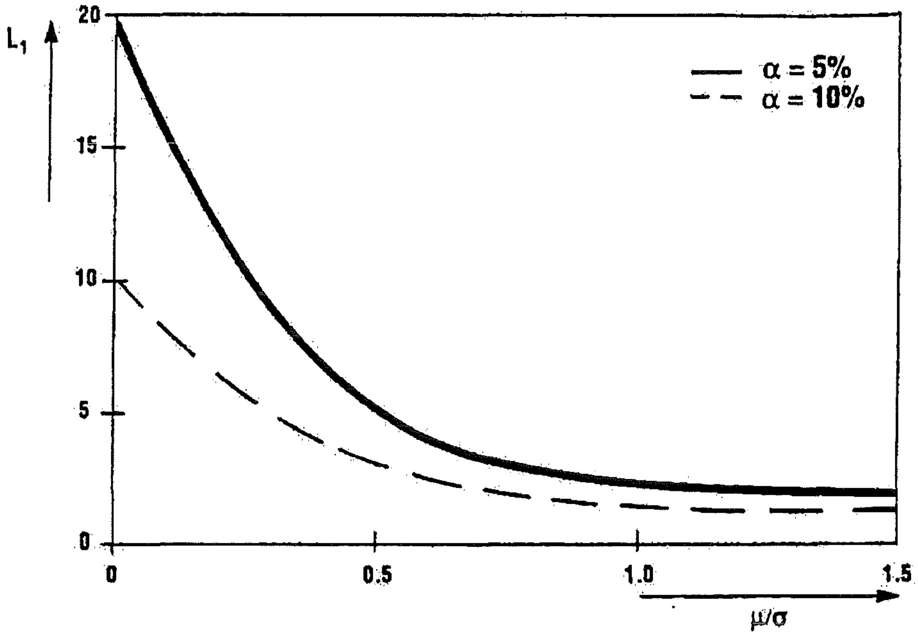

Fig. 6.3: Mean rung length L_1 for the first test procedure with α = 5% and 10% and an anomaly with uniform distribution.

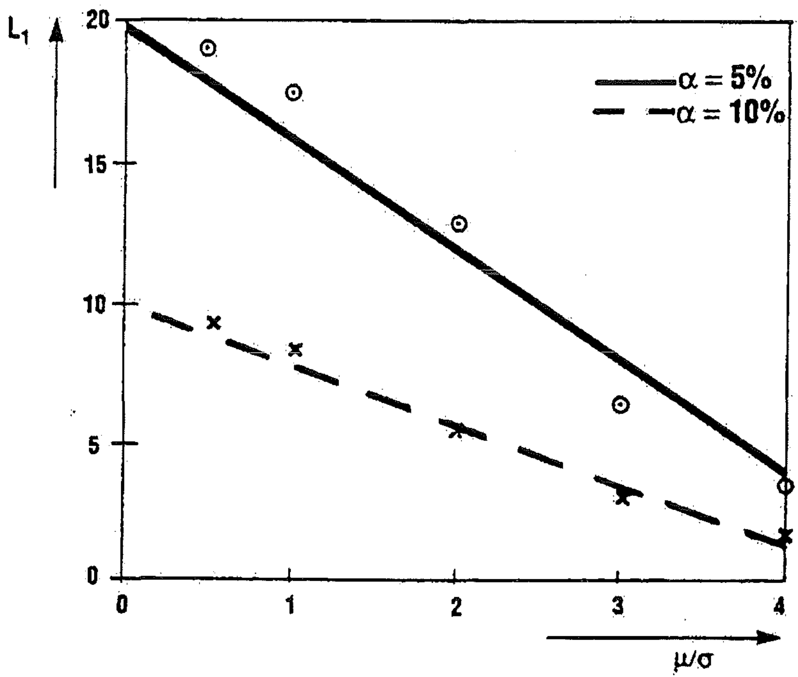

Fig. 6.4: Mean run length for the third test procedure with α = 5% and 10%; the anomaly occurred during the first inventory period.

7. LOCALIZATION

So far, we have only considered one material balance area. Now, one may have reasons, as mentioned initially to subdivide the facility into a series, or any configuration of material balance areas and apply the formalism as used for a sequence of inventory periods. The result is again, that in the sense of accountancy effectiveness it is optimal just to have one grand Material Balance Area, comprising the whole facility under consideration. However, there is also, among others, the criterion of localizing an anomaly which may lead to a subdivision of the facility into several material balance areas. It should be emphasized a last time that this criterion can only be met at the expense of accountancy effectiveness and timeliness.

Thus, in the practical case, one may arrive at a complicated time space network of inventory periods and material balance areas. A small scale example has been presented recently [10]. Statistical procedures are available, but have to be adjusted, both to the structure of the accountancy system as well as the measurement model. If handled appropriately (e.g. if computer based systems are used [11]), these procedures permit the plant operator to safely and timely detect any major anomaly and even, to localize it.

7.1 Example

Consider the system represented by Figure 7.1 which represents an idealized part of the Karlsruhe Tritium Laboratory (TLK). It is evident from this figure that besides to the store and the transfer station, the studies relate to an experiment, the mobile transfer station and the cleanup system. It is assumed that a certain amount of tritium needed in the experiment is withdrawn from the store and supplied to the experiment via the transfer station. After some time of experimentation this probably contaminated tritium is transferred to the cleanup system via the mobile transfer station from which the cleaned tritium is returned to the store via the transfer station.

The underlying sequence of the process has been entered in Table 7.1. In our model the assumption is made that so-called "accountancy tanks" are available at the following places:

- in the store,
- in the transfer station,
- in the experiment (only input),
- in the mobile transfer station,
- in the cleanup system (only output).

Time and interval, resp.	LAG	TTS	EXP	TTSm	REI
			Initial Phase		
t_0	I_0^{LAG}	$I_0^{TTS}(=0)$	$I_0^{EXP}(=0)$	$I_0^{TTSm}(=0)$	$I_0^{REI}(=0)$
$[t_0, t_1]$[1)]	$A_0^{LAG}=I_0^{LAG}-I_1^{LAG}$	$E^{TTS}=I_1^{TTS}-I_0^{TTS}$	-[2)]	-	-
t_1	$I_1^{LAG}(=0)$	I_1^{TTS}	$I_1^{EXP}(=0)$	$I_1^{TTSm}(=0)$	$I_1^{REI}(=0)$
$[t_1, t_2]$[3)]	-	$A^{TTS}=I_1^{TTS}-I_2^{TTS}$	$E^{EXP}=I_2^{EXP}-I_1^{EXP}$	-	-
t_2	$I_2^{LAG}(=0)$	$I_2^{TTS}(=0)$	I_2^{EXP}	$I_2^{TTSm}(=0)$	$I_2^{REI}(=0)$
$[t_2, t_3]$			Phase of Experiment		
t_3	$I_3^{LAG}(=0)$	$I_3^{TTS}(=0)$	X[4)]	$I_3^{TTSm}(=0)$	$I_3^{REI}(=0)$
$[t_3, t_4]$[5)]	-	-	X	$E^{TTSm}=I_4^{TTSm}-I_3^{TTSm}$	X
t_4	$I_4^{LAG}(=0)$	$I_4^{TTS}(=0)$	X	$I_4^{TTSm}(=0)$	$I_4^{REI}(=0)$
$[t_4, t_5]$[6)]	-	-	-	$A^{TTSm}=I_4^{TTSm}-I_5^{TTSm}$	X
t_5	$I_5^{LAG}(=0)$	$I_5^{TTS}(=0)$	X	$I_5^{TTSm}(=0)$	X
$[t_5, t_6]$			Cleanup Phase		
$[t_6]$	$I_6^{LAG}(=0)$	$I_6^{TTS}(=0)$	X	$I_6^{TTSm}(=0)$	I_6^{REI}
$[t_6, t_7]$[7)]	-	$E^{TTS}=I_7^{TTS}-I_6^{TTS}$	-	-	$A^{REI}=I_6^{REI}-I_7^{REI}$
$[t_7]$	$I_7^{LAG}(=0)$	I_7^{TTS}	X	$I_7^{TTSm}(=0)$	$I_7^{REI}(=0)$
$[t_7, t_8]$[8)]	$E^{LAG}=I_8^{LAG}-I_7^{LAG}$	$A^{TTS}=I_7^{TTS}-I_8^{TTS}$	X	-	X
$[t_8]$	I_8^{LAG}	I_8^{TTS}	X	$I_8^{TTSm}(=0)$	$I_8^{REI}(=0)$

1) Transfer from LAG to TTS
2) No change of inventory
3) Transfer from TTS to EXP
4) No measurement
5) Transfer from EXP to TTSm
6) Transfer from TTSm to REI
7) Transfer from REI to TTS
8) Transfer from TTS to LAG

Table 7.1: Sequence of the process of tritium transfer from the store (LAG) via the transfer station (TTS) to the experiment (EXP) and back to the store via the mobile transfer station (TTSm) and the cleanup system (REI); inventories, receipts and shipments are indicated by I, E and A.

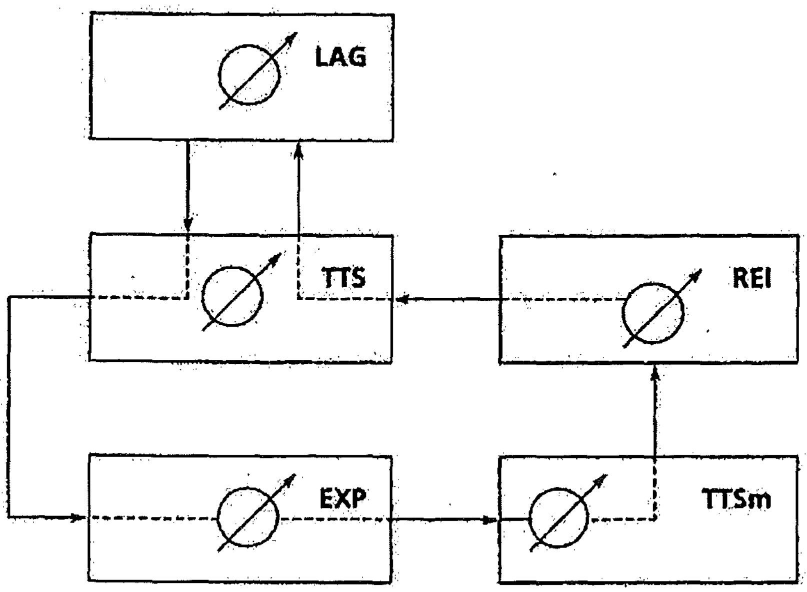

Fig. 7.1: The store (LAG), transfer station (TTS), experiment (EXP), mobile transfer station (TTSm) and cleanup system (REI) process units of the Tritium Laboratory. Most of the process units are interconnected by pipework. The positions of the measuring points have been indicated by Ø.

Analyses are feasible in principle at all systems mentioned above. Supplementing the process steps in Table 7.1, material balance areas must be included in these model assumptions: Consequently, we postulate in the model that anomalies shall be amenable to location in these zones.

We studied in the previous chapter how the aspect of the "timeliness" of detection of a supposed anomaly affects the quality of accountancy. We have shown in particular in chapter 5 which way the statistical dependencies arising in connection with this problem can be treated by means of an independence transformation, see [16].

By the example of a more realistic tritium involving process than that examined before the influence of a desired location of a supposed anomaly on the quality of accountancy will be investigated in this paper. Here the statistical dependencies will not be treated by means of an independence transformation but they will be rather taken into account explicitely in the determination of the false alarm probability and accountancy effectiveness because they occur only once.

7.2 Balances

Let $I_i = 1, 2, \ldots$ be the physical inventories of a material balance area at time t_i, and let $R_{i,i+1}$ and $S_{i,i+1}$, respectively, be the receipts and shipments, respectively, of this area during the interval $[t_i, t_{i+1}]$. Then, the difference $X_{i,i+1}$ at time t_{i+1} between the book inventory $I_i + R_{i,i+1} - S_{i,i+1}$ and physical ending inventory is given for interval by the expression

$$X_{i,i+1} = I_i + R_{i,i+1} - S_{i,i+1} - I_{i+1}, \quad i = 0, 1, 2, \ldots. \qquad (7\text{-}1)$$

If during this interval no material gets lost, the true value (expected value) of $X_{i,i+1}$ equals zero; otherwise it is equivalent to the loss. However, since all inventories and transfers considered are associated with measuring errors, it must be decided on the basis of a suitable statistical method whether a non-vanishing value of $X_{i,i+1}$ can be explained or not by measuring errors.

We are interested in balances covering the total period $[t_0, t_8]$ and applying these to the overall system and the subsystems LAG+TTS, EXP and REI. These balances are composed of "elementary" balances which are valid for shorter intervals because it results from (7-1) and

$$X_{i+1,i+2} = I_{i+1} + R_{i+1,i+2} - S_{i+1,i+2} - I_{i+2} \qquad (7\text{-}2)$$

by addition of the balance equation with $R_{i,i+1} + R_{i+1,i+2} = R_{i,i+2}$ and the same for S

$$X_{i,i+1} + X_{i+1,i+2} = X_{i,i+2} = I_i + R_{i,i+2} - S_{i,i+2} - I_{i,i+2} \qquad (7\text{-}3)$$

which applies to the longer interval. Moreover, the „elementary" balances are of interest in cases where we require the timeliness of detection of anomalies. This problem was treated already in chapter 6.

7.3 Interim Balances for the Subsystems

Reasonable points in time for the establishment of interim balances are determined by the times of inventory taking in the subsystems, i.e. by the times t_0 (LAG), t_1 (TTS), t_2 (EXP), t_4 (TTSm), t_6 (REI) and t_8 (LAG).

Reasonable material balance areas for the process sequence under consideration are LAG + TTS, EXP and REI. Depending on the situation, the mobile transfer station TTSm is assigned to the experiment and the cleanup system, respectively.

Thus, we have theoretically for the three material balance areas the five periods of inventory taking $[t_0, t_1]$ ..., $[t_6, t_8]$, i.e. a total of 15 "elementary" balance equations. As a matter of fact, some of them are unimportant because changes do not occur in all material balance areas during certain periods or no inventory is present there. Not trivial balances can be written

$$X_{0,2}^{LAG+TTS}, X_{6,8}^{LAG+TTS}, X_{0,2}^{EXP}$$

$$X_{2,4}^{EXP+TTSm}, X_{4,6}^{TTSm+REI}, X_{6,8}^{REI} \ .$$

The equations describing these balances will be constructed below with reference to Table 7.1.

At time t_2 only the balance equations for the areas LAG + TTS and EXP are of interest. It holds

$$X_{0,2}^{LAG+TTS} = I_0^{LAG} - I_1^{TTS} \tag{7-4}$$

$$X_{0,2}^{EXP} = I_0^{EXP} + I_2^{EXP} - I_1^{EXP} - 0 - I_2^{EXP} \equiv 0 \ . \tag{7-5}$$

As rigid pipework connection is assumed between the transfer station and the experiment, the expectation values of S^{TTS} and R^{EXP}during the interval $[t_1, t_2]$ can be written $E(S^{TTS}) = E(R^{EXP})$, as has already been mentioned, and hence,

$$X_{0,2}^{EXP} = I_1^{TTS} - I_2^{EXP} \ , \tag{7-6}$$

which is, since different measurements are considered, not identically zero, contrary to (7-5).

The experiment is performed during the interval $[t_2, t_3]$. We, therefore, suppose that tritium is not accessible for balancing purposes. It is assumed that at time t_4 tritium is removed from the experiment by means of the mobile transfer station which is equipped with an accountancy tank so that we can write for the interval $[t_2, t_4]$.

$$X_{2,4}^{EXP+TTSm} = I_2^{EXP} - I_4^{TTSm} \tag{7-7}$$

where the superscript 'TTSm' stands for mobile transfer station. At time t_4 the mobile transfer station is coupled to the experiment in order to take over tritium.

It is assumed that at time t_5 the tritium taken over from the experiment is transferred to the cleanup system.

As the mobile transfer station is coupled to the cleanup system both make up a common material balance area. Consequently, the balance equation for this area and the interval $[t_4, t_6]$ at time t_6 are expressed by

$$X_{4,6}^{TTSm+REI} = I_4^{TTSm} - I_6^{REI} \ . \tag{7-8}$$

Cleaned up tritium is transferred into the store by means of the transfer station so that two material balance areas can be formed, namely "cleanup" and "transfer station plus store."

We suppose that at time t_8 transfer into the store is completed. Thus, at time t_8 the following expression holds for cleanup during the interval $[t_6, t_8]$

$$X = X_6^{REI} - I_7^{TTS} \ . \tag{7-9}$$

Again, the output of the cleanup system $S_{7,8}^{REI}$, has been replaced with the input $R_{7,8}^{TTS}$ according to

$$R_{7,8}^{TTS} = I_7^{TTS} - I_8^{TTS} \ , \tag{7-10}$$

because rigid pipework is provided between the cleanup system and the transfer station and only one measurement is performed so that tritium losses can be ruled out through so-called "containment surveillance" measures. Obviously, we could have written as well

$$S_{6,8}^{REI} = I_6^{REI} - I_7^{REI} \ ; \tag{7-11}$$

In that case $X_{6,8}^{REI}$ would be identically equal to zero.

At time t_8 the following relation holds for the transfer station and the store during the interval $[t_6, t_8]$:

$$X_{6,8}^{TTS+LAG} = I_7^{TTS} - I_8^{LAG} \ . \tag{7-12}$$

7.4 Overall Balances for the Subsystems

We now consider the total interval $[t_0, t_8]$ and compose the balance equations for this interval and the subsystem previously considered.

For the material balance area LAG+TTS the following expression holds

$$X_{0,8}^{LAG+TTS} = I_0^{LAG} - I_7^{TTS} - I_1^{TTS} - I_8^{LAG} \ , \tag{7-13}$$

which is exactly the sum of the balances $X_{0,2}^{LAG+TTS}$ and $X_{6,8}^{LAG+TTS}$, which has been expected in accordance with the remark made at the beginning of this chapter.

We assume that the mobile transfer station serves solely as a measuring point for tritium leaving the experiment. In other words, we suppose that at the outlet of the experiment an accountancy tank is provided.

To the „experiment" area the following expression applies:

$$X_{0,8}^{EXP} = X_{2,4}^{EXP+TTSm} . \tag{7-14}$$

For the „cleanup" area and with the assumption made above for the mobile transfer station it holds

$$X_{0,8}^{REI} = X_{4,6}^{TTSm+REI} . \tag{7-15}$$

7.5 Overall Balance for the Whole System

We ultimately consider the balance for the whole system (GES) applicable to the total interval $[t_0, t_8]$. It holds

$$X_{0,8}^{GES} = I_0^{LAG} - I_8^{LAG} . \tag{7-16}$$

At the beginning of this chapter, see eqs. (7-1) to (7-3), we said that this balance would result as the sum of the three balances indicated in the preceding section for the subsystems. That this is not true is attributable to the option already mentioned in Section 7.3 for determination of inputs and outputs of the subsystems.

7.6 Total Accountancy Effectiveness

The following statistical analysis requires that the inventory takings are based on measurements which are independent of each other, with normal distribution of the associated measuring errors. Moreover, the variances used here must be known. As a rule, these prerequisites are fulfilled.

The total accountancy effectiveness, $1 - \beta_{0,8}^{GES}(\mu)$, i.e. the probability of detecting an anomaly μ, where μ equals the missing tritium amount, at the end of the interval $[t_0, t_8]$, is defined by

$$1-\beta_{08}^{GES}(\mu)=\text{prob}\left(X_{08}^{GES}>s^{GES}\,\middle|\,E(X_{08}^{GES})=\mu\right), \tag{7-17}$$

where s^{GES} is the significance threshold of the related statistical test (for statistical nomenclature and details see Ref. [3].

In conformity with the prerequisites formulated, $X_{0,8}^{GES}$ is normally distributed with the variance

$$\text{var}\left(X_{0,8}^{GES}\right)=\text{var}\left(I_0^{LAG}\right)+\text{var}\left(I_8^{LAG}\right), \tag{7-18}$$

so that the following expression holds

$$1-\beta_{08}^{GES}(\mu)=\Phi\left(\frac{\mu-s^{GES}}{\sqrt{\text{var}\,(X_{0,8}^{GES})}}\right), \tag{7-19}$$

where Φ is the Gaussian or normal distribution, as given by (4-7).

The significance threshold is fixed by specifying the probability of false alarm α. As indicated in chapter 4, the latter is defined as the probability of detecting an anomaly if in reality no such anomaly exists, and it is expressed by

$$1-\alpha=\Phi\left(\frac{s^{GES}}{\sqrt{\text{var}\,(X_{0,8}^{GES})}}\right). \tag{7-20}$$

If we eliminate s^{GES} using the relation above, we finally obtain

$$1-\beta_{08}^{GES}(\mu)=\Phi\left(\frac{\mu}{\sqrt{\text{var}\,(X_{0,8}^{GES})}}-\Phi^{-1}(1-\alpha)\right), \tag{7-21}$$

where Φ^{-1} is the inverse of $\Phi(.)$.

It follows from this relation in quantitative terms that the accountancy effectiveness increases with increasing μ and α, respectively, but decreases with increasing variance.

Figures 7.2 and 7.3 show by way of example how the accountancy effectiveness undergoes variations as a funtion of μ and α. The variances have been derived from the presently discussed measurement uncertainties which are on the order of percent.

7.7 Total Accountancy Effectiveness for Local Balances With Identical Single False Alarm Probabilities

We consider again the interval $[t_0, t_8]$. The material balance area is divided into three partial areas, namely „store plus transfer station,“ „experiment“ and „cleanup system“. We obtain in this way the total accountancy effectiveness for the respective local balances according to

$$1-\beta_{08}^{GES}(\mu) = 1-\text{prob}\left(X_{08}^{LAG+TTS} \le s^1 \,\middle|\, E\left(X_{08}^{LAG+TTS}\right)=\mu_1\right)\cdot \text{prob}\left(X_{08}^{EXP} \le s^2, X_{08}^{REI} \le s^3 \,\middle|\, E\left(X_{08}^{EXP}, X_{08}^{REI}\right) = \mu_2, \mu_3; \mu_2+\mu_3 = \mu-\mu_1\right), \qquad (7\text{-}22)$$

where

s_ℓ = significance threshold for the statistical test applicable to the partial area ℓ, and

μ_ℓ = loss in the partial area ℓ, with ℓ=1≡LAG+TTS, 2≡EXP and 3≡REI.

The total false alarm probability α^{GES} is obtained from $1-\beta_{0,8}^{GES}(\mu)$, if we put $\mu_1 = \mu_2 = \mu_3 = 0$.

If the same individual false alarm probabilities α_E specified in advance for the individual tests, we obtain the following relation between α^{GES} and α_E

$$1-\alpha_{GES} = (1-\alpha_E)\cdot B\left(\Phi^{-1}(1-\alpha_E), \Phi^{-1}(1-\alpha_E); \rho\right), \qquad (7\text{-}23)$$

where the function B (h, k; ρ), defined by

$$B(h,k;\rho) = \frac{1}{2\pi}\frac{1}{\sqrt{1-\rho^2}}\int_{-\infty}^{h}dx\int_{-\infty}^{k}dy\left(\exp\left(-\frac{1}{2}\cdot\frac{1}{1-\rho^2}(x^2-2\rho x y+y^2)\right)\right) \qquad (7\text{-}24)$$

is the bivariate normal distribution with the correlation ρ at the point (h, k).

In our case this correlation ρ reads

$$\rho = \frac{\text{cov}\left(X_{08}^{EXP}, X_{08}^{REI}\right)}{\sqrt{\text{var}\left(X_{08}^{EXP}\right)\cdot \text{var}\left(X_{08}^{REI}\right)}} = \frac{-\text{var}\left(I_4^{TTSm}\right)}{\sqrt{\left(\text{var}\left(I_2^{EXP}\right)+\text{var}\left(I_4^{TTSm}\right)\right)\cdot\left(\left(\text{var}\left(I_4^{TTSm}\right)+\text{var}\left(I_6^{REI}\right)\right)\right)}}, \qquad (7\text{-}25)$$

where $\text{cov}\left(X_{0,8}^{EXP}, X_{08}^{REI}\right)$ is the covariance between X_{08}^{EXP} and X_{08}^{REI}.

If we divide by $\mathrm{var}\,(I_4^{TTSm})$, we obtain

$$\rho = \frac{1}{\sqrt{\left(1+\frac{\mathrm{var}\,(I_2^{EXP})}{\mathrm{var}\,(I_4^{TTSm})}\right)\cdot\left(1+\frac{\mathrm{var}\,(I_6^{REI})}{\mathrm{var}\,(I_4^{TTSm})}\right)}}. \tag{7-26}$$

Thus, the total accountancy effectiveness for local balances with equal individual false alarm probabilities as a function of the overall anomaly μ is given by the expression

$$1-\beta_{08}^{GES}(\mu) = 1-\Phi\left(\Phi^{-1}(1-\alpha_E)-\frac{\mu_1}{\sqrt{\mathrm{var}\,(I_8^{TTS})+\mathrm{var}\,(I_8^{LAG})}}\right)\cdot$$

(7-27)

$$B\left(\Phi^{-1}(1-\alpha_E)-\frac{\mu_2}{\sqrt{\mathrm{var}\,(I_2^{EXP})+\mathrm{var}\,(I_4^{TTSm})}},\Phi^{-1}(1-\alpha_E)-\frac{\mu_3}{\sqrt{\mathrm{var}\,(I_4^{TTSm})+\mathrm{var}\,(I_6^{REI})}};\rho\right)$$

where the total anomaly μ is composed by addition of the individual anomalies μ_1, μ_2 and μ_3.

7.8 Consideration of Waste Streams

The results presented before relate to a process sequence (see Table 7.1) without waste streams. Should there be a waste stream, the algorithm given above must be extended accordingly.

In our extended model we assume two waste streams, namely one in the experiment and one in the cleanup system.

I_3^{EXA} and I_6^{REA} denote the waste inventories of the experiment and cleanup system, respectively.

Again, we first consider the meanwhile extended global balance

$$X_{0,8}^{GES} = I_0^{LAG} - I_8^{LAG} - I_3^{EXA} - I_6^{REA}. \tag{7-28}$$

We had envisaged three areas for the local balances; following the extension, the local balance equations read as follows:

„Experiment“ area:

$$X_{0,8}^{EXP} = I_2^{EXP} - I_4^{TTSm} - I_3^{EXA} \qquad (7\text{-}29)$$

„Cleanup system“ area:

$$X_{0,8}^{REI} = I_4^{TTSm} - I_6^{REI} - I_6^{REA} . \qquad (7\text{-}30)$$

The „store plus transfer station“ area considered above is free from weaste streams in this model and therefore it does not occur here. The associated probabilities are then obtained by an analogous approach.

7.9 Numerical Calculations

For the following calculations we will refer to the data for determination of the variation coefficients in inventory taking which were communicated to us by experts and start from a value $\delta = \sqrt{\mathrm{var}\,(x)}\,E\,(x) = 0.01$. This is a plain and obvious numerical value and its use seems to be justified considering the uncertainty of the source data.

Considering the "cleanup" subsystem, we assume that 0.5 and 1.0 Ci/g, respectively, of getter material are bound. As the variation coefficients in the determination of the bound tritium we take the two values 0.2 and 0.3, respectively [17].

We have selected here two cases from the great number of cases treated by us. In both cases we assume that no wastes arise in the experiment. The accountancy effectiveness has been represented in the following two figures versus the total anomaly.

In Figure 7.2 the case is considered that no noticeable waste volumes arise during cleanup and that all measuring points are equal in rank. The upper plot describes the case where localization of a supposed anomaly is no significant aspect whereas it is taken into account in the bottom plot.

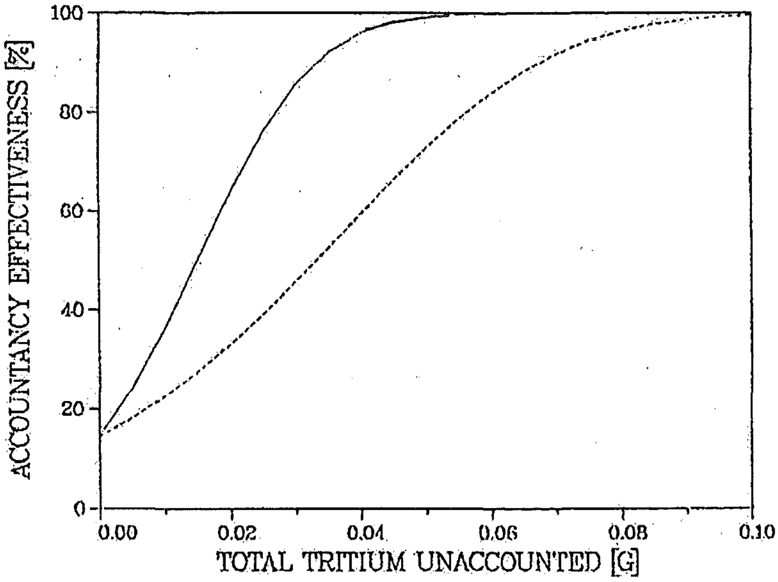

Fig. 7.2: Accountancy effectiveness as a function of total tritium unaccounted: No waste, equal quality of key measurement points. Upper curve no localization of supposed anomaly; lower curve with localization, $\mu_1=\mu_2=\mu_3=\mu/3$.

In Figure 7.3 the case is considered that noticeable waste volumes arise during cleanup which can be measured only with relatively little accuracy as mentioned above. All the other measuring points are again deemed equal with respect to accuracy. The two plots are defined as in Figure 7.2.

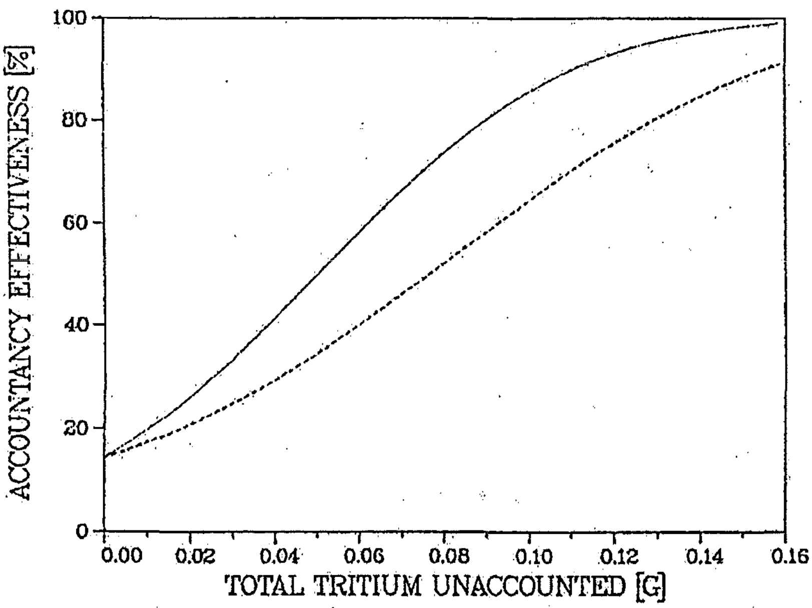

Fig. 7.3: Similar to upper figure – but occurrence of poorly measured waste streams.

At the beginning of this chapter we mentioned that it is optimal in the sense of the overall accountancy effectiveness if only one overall balance is formed and evaluated. Figures 7.2 and 7.3 indicate „the penalty of localization." Nevertheless, it may happen that for some range of anomalies the accountancy effectiveness of the overall balance may be smaller than that of the local balances. Naturally, the same may be observed if we subdivide a reference time into several inventory periods. An explanation for this – at first sight – counterintuitive behavior is given in the Annex.

8. HIDDEN INVENTORIES

With the current ITER design it must be anticipated that noticeable fractions of the tritium inventory are, in principle, not amenable to measurement. The meanwhile discussed materials for the plasma facing reactor components give rise to particularly great tritium retention, i.e. a particularly great "hidden inventory." Although at least some of this hidden inventory can be recovered during conditioning and thus becomes again accessible to measurement, it is quite obvious that this will exert a crucial influence on the effectivity of accountancy.

It should be added that in such cases each authority must consider also the possibility of unlawful withdrawal of tritium. So, for this very reason, it will also be of importance to the operator whether elucidation of such anomalies will cause extended interruptions in operation. It should be mentioned that quite similar problems arise in fissile material safeguards by the IAEA.

The general theory elaborated to be able to answer the questions addressed as well as the simulation models required for the numerical computations were explained in detail in a former publication [3]. Therefore, they will be represented here once more in a summarized form only.

8.1 Accountancy Principle

We consider the general case of a Material Balance Area and a reference time interval $[t_0, t_n]$ which we divide into n inventory periods $[t_{i-1}, t_i]$, i=1, ..., n. At the time t_{i-1} the inventory I_{i-1} in the MBA is measured.During the interval $[t_{i-1}, t_i]$ the inputs R_i and the outputs S_i are measured which, together with the initial inventory at the time t_i, add up to become the so-called "book inventory" B_i.

At the time t_i the real inventory I_i is measured again: If no anomalies (e. g. material losses) occur, both inventories should agree within the measurement inaccuracy.

We will consider now the reactor together with the nuclear fuel cycle as one MBA. It should be stressed that a subdivision of the MBA might be required if specific problems have to be solved. We suppose here that within the reference period under consideration tritium neither enters nor leaves the plant so that $R_i = S_i = 0$, $i = 1, ..., n$. We further assume that not all of the inventory can be measured but only a fraction of it which we term P_i, $i = 1, ..., n$. Let us except from the assumption above the time t_0 at which the complete inventory I_0 is taken.

All variables introduced so far are associated with independent and normally distributed measuring errors, i. e., expressed in the currently employed terminology, the variances would be

$$\text{var}(I_0) = \sigma_0^2, \text{var}(P_i) = \sigma_P^2 . \qquad (8\text{-}1)$$

We will now deal with material balance statistics

$$Y_i = I_0 - P_i, \quad i = 1, ..., n. \qquad (8\text{-}2)$$

Under the null hypothesis H_0 no anomalies appear, i. e. the expectations of Y_i would be

$$E_0(Y_i) = E(I_0) - E(P_i) = M_i, \quad i = 1, ..., n, \qquad (8\text{-}3)$$

where M_i is the non-measured process inventory at the time t_i. We now write (with F = fluctuation)

$$M_i = \Theta + F_i, E_0^F(F_i) = 0, \text{ var}_0^F(F_i) = \sigma_F^2 \qquad (8\text{-}4)$$

i. e. we consider M_i to be a random variable with the expectation Θ and the variance σ_F^2. So it results from (8-2) with the null hypothesis H_0

$$E_0(Y_i) = \Theta \qquad (8\text{-}5)$$

$$\text{var}(Y_i) = \sigma_0^2 + \sigma_P^2 + \sigma_F^2 =: \sigma_i^2, i = 1, ..., n. \qquad (8\text{-}6)$$

Under the alternative hypothesis H_1 we assume that the anomaly μ_i occurs in the i^{th} period of inventory taking $[t_{i-1}, t_i]$. Then it follows from (8-2)

$$E_1(Y_i) = \Theta + \sum_{j=1}^{i} \mu_j \,, \tag{8-7}$$

while the variance (8-6) does not undergo changes.

8.2 Statistical Analysis

It is an obvious goal of tritium accountancy to detect with the highest possible certainty an anomaly appearing within a reference period. If we assume that all moments previously indicated are known, the respective test method can be written

$$Y_n - \Theta \leq s: \quad \text{accept } H_0 \,, \tag{8-8}$$

and otherwise suppose H_1, where the significance threshold s can be fixed using the previously defined false alarm probability α according to

$$1 - \alpha = \text{prob}\,(Y_n - \Theta \leq s \,|\, H_0) = \Phi\left(\frac{s}{\sigma_n}\right) \tag{8-9}$$

with Φ (.) being the standard normal distribution as given by (4-7). With (8-9) the accountancy effectiveness, i.e., the probability for detecting an anomaly, as a function of α is given by

$$1 - \beta = \Phi\left(1/\sigma_n \cdot \sum_{j=1}^{n} \mu_j - \Phi^{-1}(1-\alpha)\right) \tag{8-10}$$

where Φ^{-1} (.) is the inverse of Φ (.).

We now assume that an anomaly should be detected in time. For this, a scalar criterion must be defined. Let t_0, t_1, t_2, ... again be the times at which at least some of the inventory is measured. Supposing that the anomaly occurs at time t_0, then $T = t_i$ is the actual detection time provided that the null hypothesis is rejected in a sequential test at $T = t_i$. As this will not happen with certainty, T is a (discrete) random variable implemented at t_1, t_2, ..., and its expectation, termed average run length,

$$E(T) = \sum_i t_i \cdot \text{prob}\,(T = t_i)\,, \tag{8-11}$$

is a suitable measure of timely detection as discussed in chapter 6. By analogy with the non-sequential test method described before we specify the average run length

L_0 on the null hypothesis H_0 and, thus, have now to solve the task of finding a sequential test method minimizing L_1 with the value of L_0 unchanged.

Unfortunately, in contrast to the case of the non-sequential problem, no general solution exists to this problem of optimization. Therefore, "reasonable" test methods are considered and their average run lengths are examined as a function of the parameters determining these methods, especially the various conceivable anomalies (abrupt, protracted).

Another problem results from the fact that the detection probabilities prob ($T = t_i$) are generally highly complex expressions and, therefore, analytical comparisons cannot be made; only simulation procedures are helpful here.

We focus on the most simple and intuitive test method, namely

$$(Y_i - \Theta) / \text{var}(Y_i)^{1/2} \leq s : \text{continue} \qquad (8\text{-}12)$$

and otherwise suppose H_1, with var (Y_i) as given by (8-6), and the significance threshold s fixed in such a manner that under the null hypothesis H_0 the previously defined value L_0 of the average run length is not exceeded.

8.3 Numerical Results

It is evident which elements are needed for simulation: The lowermost level is always given by the so-called true data from a reasonable process simulation, in our case KATRIM [5]. Then the data are evaluated with respect to accountancy. Finally, a statement can be made on the accountancy efficiency.

The process data and the numerical simulation parameters used in our calculations are summarized in Table 8.1.

Our process simulation considers periods during which the machine has to be conditioned: We assume that conditioning recovers the trapped tritium completely. During operation, the trapped tritium inventory is of the order of several 100 g up to 1000 g; standard deviation is assumed to be 100 g.

As a loss pattern, we use an abrupt anomaly, occurring shortly after machine start-up.

Finally, it should be mentioned that not the average run length L_0 on the null hypothesis is fixed a priori, but the single inventory period false alarm probability α.

Parameter	Value
Reference time	1 [yr]
Inventory period (on the average)	3 [d]
Time horizon	5000 [periods]
Anomaly	100 - 500 [grams]
Inventory	5000 [grams]
Coefficients of variation - pressure p - temperature T - concentration C	 $0.1 < \delta p < 1$ [%] $0.5 < \delta T < 2$ [%] $1 < \delta C < 5$ [%]

Table 8.1: Basic data used in the process simulation.

In Figures 8.1, 8.2, and 8.3 results of numerical calculations are presented: the mean run length L_1 versus the anomaly μ, the accountancy effectiveness versus the anomaly μ, and the accountancy effectiveness versus the mean run length (anomaly μ eliminated).

These and other simulation runs suggest the following conclusions: The accountancy effectiveness increases, the mean run length decreases with increasing anomaly which is reasonable. Both, the accountancy effectiveness and the mean run length depend strongly on the size of the mean trapped inventory. If, however, the anomaly μ is eliminated, then the resulting relation between accountancy effectiveness and mean run length is practically independent of the size of the mean trapped inventory. Therefore, we present this relation as the *characteristics of the ITER tritium accountancy system*.

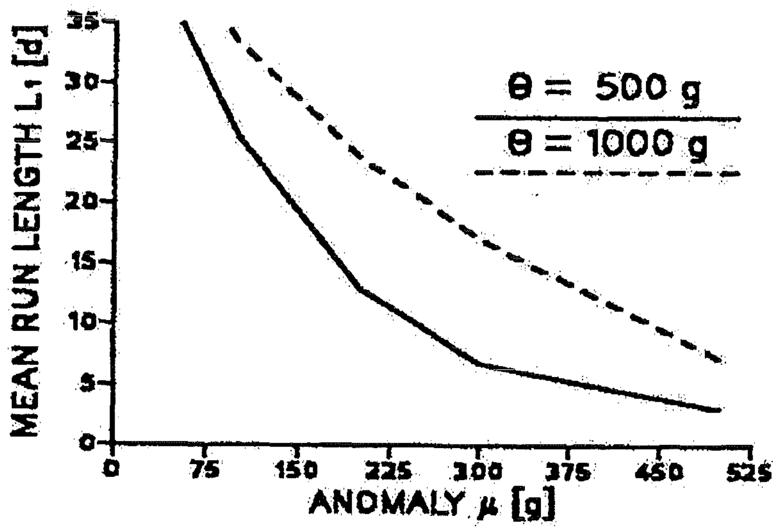

Fig. 8.1: Mean run length L_1 versus anomaly μ (Θ = 500, 1000 g; α = 2%)

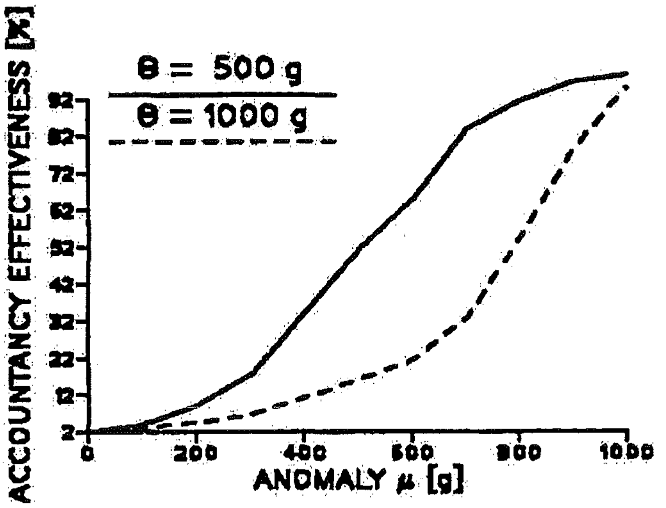

Fig. 8.2: Accountancy effectiveness versus anomaly (Θ = 500, 1000 g; α= 2%)

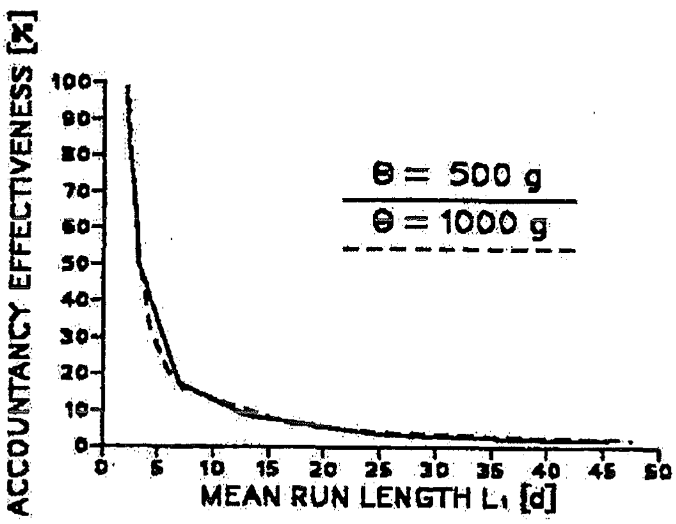

Fig. 8.3: Accountancy effectiveness versus mean run length (Θ = 500, 1000 g; α = 2%)

It should be mentioned that the relatively large amount of trapped tritium might be considered to be an accountancy problem. However, we hope that conditioning which is mandatory in any case will help to meet the required accountancy objective.

9. PERSPECTIVES

According to present knowledge the first power stations equipped with nuclear fusion reactors will operate on a deuterium-tritium mix which will call for the nuclear fuel cycle to be complete. Its components are being investigated already now in the laboratories of large research establishments. Especially the Karlsruhe Research Center operates a so-called "tritium laboratory".

Tritium is radioactive and rather expensive, two severe reasons which support the need of careful tritium accounting. In addition, it is worthwhile to mention that management of the fuel cycle will benefit from the implementation of an optimised accountancy.

Special attention should be paid to the discussion going on in the Federal Republic of Germany. It has been initiated recently by reports about some "occurrences" in handling tritium. The discussion has, e.g., caused the highest court in the Federal republic of Germany to decide to the effect that tritium - even if present in any small amount - shall be classified in principle as a weapon grade material.

At an international meeting of experts working on tritium R&D, an expert from one of the best known laboratories recently expressed his feelings as follows: "I should re-mind you again, tritium accountancy is a very tough business" [18].

For all these reasons, the problem must be solved with the best tools available, which include modern mathematical-statistical procedures. As problems of this type had been studied for many years in greater detail at KfK within the Fissile Materials Accountancy Project, an obvious approach was to examine also the balancing activities required for the fuel cycle of fusion reactors. Moreover, at the beginning of those studies at KFK, the Tritium Laboratory mentioned above was in its planning stage.

This report is a compilation of the activities performed in this field over the past ten years. In addition to the basic balancing issue, process models and measurement models adapted to the problems at hand had to be developed.

Three main areas were studied for tritium balancing, namely timeliness, the localization of an anomaly, and the influence of the unmeasurable tritium inventory. Typical results of these computer simulations can be represented in terms of the balancing quality or the mean time to discovery of an anomaly.

As there had been no binding design of ITER in the past, no comprehensive accountancy system could be developed. It is still unknown what problems will have to be solved within a comprehensive framework of balancing tritium in an ITER fuel cycle.

It should also be emphasized that, so far, only anomalies have been considered, such as unforeseeable losses. Deliberate diversions, e.g. for military purposes, raise entirely new questions (data verification) which, however, have already been covered within the framework of the above mentioned Fissile Materials Accountancy Project.

As a consequence, the problem cannot be considered to have been solved as far as the final goal is concerned, namely balancing a tritium fuel cycle (e.g. for ITER). Yet, the main aspects of a tritium balancing system have been dealt with, and the framework thus has been established which then needs to be filled in by experts in the light of the special conditions to be expected.

This will require, on the basis of a concrete ITER design, harmonization of the requirements to be met by a balancing system (necessary quality, discovery time, localization, etc.) with the aspect of feasibility, which should also include reasonable costs. Only in the light of these criteria can a workable and durable balancing system be designed.

ACKNOWLEDGMENT

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ANNEX

Intercomparison of Two Methods of Testing Hypotheses on Bivariate Normally Distributed Random Variables with known Covariance Matrix

In this annex statistical problems are dealt with which arise in analyzing tritium accountancy problems in general, but in particular in connection with special questions occurring in a tritium laboratory [1]. The analysis of these problems is presented in a mathematical abstract setting in order to emphasize their general nature.

Two test procedures are described here which are applied to testing hypotheses on the expectation of two bivariate normaly distributed random variables with known covariance matrix. The first procedure is the Neyman-Pearson test for a single alternative hypothesis. The second is a procedure where the two hypotheses concerning the expectations of the two random variables are tested separately. Furthermore, two assumptions are made: First, that the expectations under the hypotheses are known individually, and, second, that only their sum is known.

Both test procedures are analyzed under both assumptions. Whereas, by definition, the Neyman-Pearson test is better than the second test under the first assumption, it depends on the values of the parameters which of the two tests is better under the second assumption.

A.1 The Problem

Let two bivariate normally distributed random variables X_1 and X_2 with known co-variance matrix be given. Under the null hypothesis H_0 let the expectations of the two random variables be zero, whereas under the alternative hypothesis H_1 they are assumed to adopt the initially known values μ_1 and μ_2. It is proposed that through observation of the two random variables a choice is made in favor of one of the two hypotheses, with the probability of error of the first kind α given in advance.

Let us further assume that not μ_1 and μ_2, but only the sum $\mu = \mu_1 + \mu_2$ be known and that also for solving this problem an adequate decision making procedure has to be found.

Let us finally assume that not a single choice has to be made in favor of one the two hypotheses H_0 and H_1, but that this choice has to be made separately for the two random variables.

The best suited decision making procedure for two simple hypotheses, i.e. hypotheses unambiguously fixing the corresponding distribution functions, is described by the lemma detected by Neyman and Pearson [2]. So, if the values of μ_1 and μ_2 are given, the decision making procedure can be indicated immediately; this is done in Section A.2.1. If only the sum $\mu = \mu_1 + \mu_2$ is given, a minimax approach is adequate, i.e. μ_1 and μ_2 are pessimistically determined in Section A.2.2 such that the probability of error of the second kind ß is maximized.

If a separate decision is to be made regarding the two random variables under the two hypotheses, the test procedure is characterized by two significance points which are determined by the two single probabilities of errors of the first kind. The latter are then determined in Section A.3.1 for the given overall probability of error of the first kind in such a way that the overall probability of error of the second kind is minimized. If, again, only the sum $\mu = \mu_1 + \mu_2$ is given, the procedure is the same as in Section A.3.1; this will be performed in Section A.3.2.

By definition, the Neyman-Pearson test is the best suited test for given values of μ_1 and μ_2. If, however, only the sum $\mu = \mu_1 + \mu_2$ is supposed to be known, application of the two test procedures described here can obviously produce the result that for given values of μ_1 and μ_2 the probability of error of the second kind is smaller in the second minimax test than in the first. This will be demonstrated in Section A.4.

A.2 Neyman-Pearson Tests

On the prerequisites made by us, the two random variables X_1 and X_2 are bivariate normally distributed with known covariance matrix and the expectations

$$E_0(X_1, X_2) = (0,0) \quad \text{under } H_0, \quad \text{and} \tag{2-1a}$$

$$E_1(X_1, X_2) = (\mu_1, \mu_2) \quad \text{under } H_1. \tag{2-1b}$$

So, the common density under hypothesis H_0 is expressed by

$$f_0(x_1, x_2) = \frac{1}{2\pi} \cdot \frac{1}{\sqrt{1-\rho^2}} \cdot \frac{1}{\sigma_1} \cdot \frac{1}{\sigma_2} \cdot \exp\left(-\frac{1}{2} \cdot \frac{1}{1-\rho^2}\right) \cdot \left(\frac{x_1^2}{\sigma_1^2} - 2\rho \frac{x_1}{\sigma_1^2} \cdot \frac{x_2}{\sigma_2} + \frac{x_2^2}{\sigma_2^2}\right) \tag{2-2}$$

and under H_1 by

$$f_1(x_1, x_2) = \frac{1}{2\pi} \cdot \frac{1}{\sqrt{1-\rho^2}} \cdot \frac{1}{\sigma_1} \cdot \frac{1}{\sigma_2} \cdot$$

$$\cdot \exp\left(-\frac{1}{2} \cdot \frac{1}{1-\rho^2} \cdot \frac{(x_1 - \mu_1)^2}{\sigma_1^2} - 2\rho \frac{x_1 - \mu_1}{\sigma_1} \cdot \frac{x_2 - \mu_2}{\sigma_2} + \frac{(x_2 - \mu_2)^2}{\sigma_2^2} \right), \quad (2\text{-}3)$$

where the second moments are assumed to be known and given by

$$\mathrm{var}\,(X_i) = \sigma_i^2, i = 1, 2, \quad \mathrm{cov}\,(X_1, X_2) = \rho \cdot \sigma_1 \cdot \sigma_2 . \quad (2\text{-}4)$$

A.2.1 Values μ_1 and μ_2 given

Let us define the critical range of observations which leads to the rejection of H_0.

The critical range Cr of the best test for the given probability of error of the first kind α is, according to the Neyman-Pearson lemma [2], expressed by

$$Cr = \left\{ (x_1, x_2) : \frac{f_1\,(x_1, x_2)}{f_0\,(x_1, x_2)} > \lambda \right\}, \quad (2\text{-}5)$$

where λ is determined by α. Using (2-2) and (2-3), this yields the explicit form

$$\left\{ (x_1, x_2) : \frac{x_1}{\sigma_1} \cdot \left(-\frac{\mu_1}{\sigma_1} + \rho \cdot \frac{\mu_2}{\sigma_2} \right) + \frac{x_2}{\sigma_2} \cdot \left(-\frac{\mu_2}{\sigma_2} + \rho \cdot \frac{\mu_1}{\sigma_1} \right) < \lambda' \right\}. \quad (2\text{-}6)$$

Now the probabilities of errors of the first and second kinds, α and ß, are defined by

$$\alpha := \mathrm{prob}\,((X_1, X_2) \in Cr \mid H_0) \cdot \quad (2\text{-}7)$$

$$\beta := \mathrm{prob}\,((X_1, X_2) \notin Cr \mid H_1). \quad (2\text{-}8)$$

Since linear combination of bivariate normally distributed random variables are again normally distributed, see for example Ref. [3], with appropriate moments, these probabilities can immediately be expressed by quantiles of the standard normal distribution. If we eliminate λ' by means of α, this leads to the probability of error of the second kind according to

$$\beta_{NP}\,(\mu_1, \mu_2) = \Phi\left(\Phi^{-1}\,(1-\alpha) - \sqrt{\frac{1}{1-\rho^2} \cdot \left(\frac{\mu_1^2}{\sigma_1^2} - 2\,\rho \frac{\mu_1}{\sigma_1} \cdot \frac{\mu_1}{\sigma_1} + \frac{\mu_2^2}{\sigma_2^2} \right)} \right) \quad (2\text{-}9)$$

where Φ (.) is the standard normal distribution

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{x} \exp\left(-\frac{t^2}{2}\right) dt \tag{2-10}$$

and Φ^{-1} (.) its inverse.

A.2.2 Values $\mu_1 + \mu_2$ given

If the values of μ_1 and μ_2 are not known individually, but only their sum $\mu = \mu_1 + \mu_2$ is known, it is reasonable to define that test which is based on the least favorable values of μ_1 and μ_2 in terms of the probability of error of the second kind. So, definition of that test leads to the problem of optimization

$$\max_{0 \le \mu_1 \le \mu} \beta_{NP} (\mu_1, \mu - \mu_1). \tag{2-11}$$

As $\beta_{NP} (\mu_1, \mu - \mu_1) \equiv \beta_{NP} (\mu_1)$ is a monotone function of the argument, it will be sufficient to consider instead the problem of optimization

$$\max_{0 \le \mu_1 \le \mu} \left(\frac{\mu_1^2}{\sigma_1^2} - 2\rho \frac{\mu_1}{\sigma_1} \cdot \frac{\mu - \mu_1}{\sigma_2} + \frac{(\mu - \mu_1)^2}{\sigma_2^2} \right) \tag{2-12}$$

the solution of which is given by

$$\mu_1^* = \mu \frac{\sigma_1^2 + \rho\, \sigma_1 \sigma_2}{\sigma_1^2 + 2\rho\, \sigma_1 \sigma + \sigma_2^2}, \tag{2-13}$$

which can be shown as follows: We write as can be verified easily

$$\frac{\mu_1^2}{\sigma_1^2} - 2\rho \frac{\mu_1}{\sigma_1} \cdot \frac{\mu - \mu_1}{\sigma_2} + \frac{(\mu - \mu_1)^2}{\sigma_2^2} = \left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} + \frac{2\rho}{\sigma_1 \sigma_2} \right) \cdot (\mu_1 - \mu_1^*)^2 + \frac{1 - \rho^2}{\sigma_1^2 + \sigma_2^2 + 2\rho\, \sigma_1 \sigma_2} \cdot \mu^2,$$

where μ_1^* is given by (2-13). From this the assertion follows immediately.

The related probability $\beta_{NP}^* = \beta_{NP} (\mu_1^*)$ is expressed by

$$\beta^*_{NP} = \Phi\left(\Phi^{-1}(1-\alpha) - \frac{\mu}{\sqrt{\sigma_1^2 + 2\rho\ \sigma_1\ \sigma_2 + \sigma_2^2}}\right). \qquad (2\text{-}14)$$

It can be easily understood that the critical range of this test is

$$\{(X_1, X_2): X_1 + X_2 > \sqrt{\sigma_1^2 + 2\rho\,\sigma_1\,\sigma_2 + \sigma_2^2} \cdot \Phi^{-1}(1-\alpha)\}. \qquad (2\text{-}15)$$

Thus, it appears that the probability of error of the second kind in this minimax test is independent of the single values of μ_1 and μ_2 as long as one retains the sum $\mu = \mu_1 + \mu_2$

$$\beta^*_{NP}(\mu_1, \mu - \mu_1) = \beta^*_{NP} = \Phi\left(\Phi^{-1}(1-\alpha) - \frac{\mu}{\sqrt{\sigma_1^2 + 2\rho\ \sigma_1\ \sigma_2 + \sigma_2^2}}\right). \qquad (2\text{-}16)$$

We will return to this point in Section A.4.

It should be added here that the solution found is a saddle point solution: when we call δ any test with the given error first kind probability, α, and call ß (δ, μ_1) the error second kind probability as a function of the test and of the hypothesis H_1 given by μ_1, the Neyman-Pearson test δ^* and μ_1^*, according to (2-13), satisfy the so-called saddle point criterion

$$\beta(\delta^*, \mu_1) \leq \beta(\delta^*, \mu_1^*) \leq \beta(\delta, \mu_1^*) \quad \text{for all } \delta \text{ and } \mu_1 . \qquad (2\text{-}17)$$

This means that the order of optimization does not matter as regards the test δ and μ_1 which is the more remarkable since the double optimization problem cannot be solved in an order different from that explicitely followed here.

A.3 Separate Tests

We start again from the test problem which is characterized by the formulae (2-1) to (2-4). But now we do not try to find the best test for the given error first kind probability, α, since we wish to make a separate choice between the two hypotheses H_0 and H_1 for the two random variables X_1 and X_2. This leads to the critical range

$$\{(X_1, X_2): x_1 > \lambda_1 \ \text{ or } \ X_2 > \lambda_2\}, \qquad (3\text{-}1)$$

where the two significance thresholds are fixed by the single error first kind probabilities which, in turn, have to be determined in such a way that the resulting error first kind probability adopts the given value α.

A.3.1 Values μ_1 and μ_2 given

In conformity with (3-1), the overall error first kind probability, α, is defined by

$$1-\alpha = \text{prob}\left(X_1 \leq \lambda_1 \text{ and } X_2 \leq \lambda_2 \mid H_0\right). \tag{3-2}$$

With (2-2) this leads to

$$1-\alpha = \int_{-\infty}^{\lambda_1} dx_1 \int_{-\infty}^{\lambda_2} dx_2 \, f_0\left(x_1, x_2\right) \tag{3-3}$$

or, by suitable transformation, to

$$1-\alpha = \int_{-\infty}^{\lambda_1/\sigma_1} dt_1 \int_{-\infty}^{\lambda_2/\sigma_2} dt_2 \, \frac{1}{2\pi} \cdot \frac{1}{\sqrt{1-\rho^2}} \cdot \exp\left(-\frac{1}{2} \cdot \frac{1}{1-\rho^2} \cdot \left(t_1^2 - 2\sigma\, t_1\, t_2 + t_2^2\right)\right). \tag{3-4}$$

This can be written as

$$1-\alpha = B\left(\frac{\lambda_1}{\sigma_1}, \frac{\lambda_2}{\sigma_2}; \rho\right), \tag{3-5}$$

where B (h, k, ρ) is the distribution of two bivariate standard normally distributed random variables,

$$B(h,k,\rho) = \frac{1}{2\pi} \cdot \frac{1}{\sqrt{1-\rho^2}} \cdot \int_{-\infty}^{h} dt_1 \int_{-\infty}^{k} dt_2 \, \exp\left(-\frac{1}{2} \cdot \frac{1}{1-\rho^2} \cdot \left(t_1^2 - 2\rho\, t_1\, t_2 + t_2^2\right)\right). \tag{3-6}$$

Now the single error first kind probabilities, α_1 and α_2, are given by

$$1-\alpha_1 = \text{prob}\left(X_i \leq \lambda_i \mid H_0\right) \quad \text{for } i = 1, 2, \ldots. \tag{3-7}$$

i.e., explicitly by

$$1-\alpha_i = \Phi\left(\frac{\lambda_i}{\sigma_i}\right) \quad \text{for } i = 1, 2. \tag{3-8}$$

Accordingly, (3-5) can also be written as

$$1-\alpha = B\left(\Phi^{-1}(1-\alpha_1), \Phi^{-1}(1-\alpha_2); \rho\right) \tag{3-9}$$

In order to fix λ_1 and λ_2 and α_1 and α_2, respectively, for a given value of α, we consider the overall probability of error of the second kind which is defined as

$$\beta_G(\mu_1, \mu_2) = \text{prob}\,(X_1 \leq \lambda_1 \text{ and } X_2 \leq \lambda_2 \mid H_1). \tag{3-10}$$

and which, according to definition (3-6) of B (h, k; ρ) is given by

$$\beta_G(\mu_1, \mu_2) = B\left(\Phi^{-1}(1-\alpha_1) - \frac{\mu_1}{\sigma_1}, \Phi^{-1}(1-\alpha_2) - \frac{\mu_2}{\sigma_2}; \rho\right). \tag{3-11}$$

Evidently, we will define α_1 and α_2 such that β_G is minimized which means that we have to solve the following optimization problem:

$$\min_{\alpha_1, \alpha_2} B\left(\Phi^{-1}(1-\alpha) - \frac{\mu_1}{\sigma_1}, \Phi^{-1}(1-\alpha_2) - \frac{\mu_2}{\sigma_2}; \rho\right) \tag{3-12}$$

where the boundary condition (3-5) has to be taken into account.

Unfortunately, the optimization produces values of α_1 and α_2 which are highly complex and, above all, dependent on μ_1 and μ_2. As this test will not be further considered in this note, the respective formulae will not be indicated here.

A.3.2 Values $\mu_1 + \mu_2$ given

If the values of μ_1 and μ_2 are not given individually, but only their sum $\mu = \mu_1 + \mu_2$ is known, we proceed as in Section A.2.2 which means that we now solve the optimization problem

$$\min_{\alpha_1, \alpha_2} \max_{0 \leq \mu_1 \leq \mu} B\left(\Phi^{-1}(1-\alpha_1) - \frac{\mu_1}{\sigma_1}, \Phi^{-1}(1-\alpha_2) - \frac{\mu - \mu_1}{\sigma_2}; \sigma\right) \tag{3-13}$$

where α_1 and α_2 have to satisfy the boundary condition (3-5).

The solution of this minimax problem was already found earlier for a special case [13] and will be cited here: The optimum values of α_1, α_2, μ_1 and μ_2 are given as solutions of the following system of equations:

$$\sigma_1 \cdot \exp\left(\frac{1}{2}\left(\Phi^{-1}(1-\alpha_1)\right)^2 \cdot \Phi\left(\frac{1}{\sqrt{1-\rho^2}} \cdot \left(\Phi^{-1}(1-\alpha_1) - \rho \cdot \Phi^{-1}(1-\alpha_2)\right)\right)\right) + \tag{3-14}$$

$$+\sigma_2 \cdot \exp\left(\frac{1}{2}\left(\Phi^{-1}(1-\alpha)\right)^2 \cdot \Phi\left(\frac{1}{\sqrt{1-\rho^2}} \cdot \left(\Phi^{-1}(1-\alpha_2) - \rho \cdot \Phi^{-1}(1-\alpha_1)\right)\right)\right) = 0$$

$$\sigma_1 \cdot \exp\left(\frac{1}{2}\left(\Phi^{-1}(1-\alpha_1) - \frac{\mu_1}{\sigma_1}\right)^2 \cdot \Phi\left(\frac{1}{\sqrt{1-\rho^2}} \cdot \left(\Phi^{-1}(1-\alpha_1) - \frac{\mu_1}{\sigma_1}\right)\right)\right) =$$

$$= \sigma_2 \cdot \exp\left(\frac{1}{2}\left(\Phi^{-1}(1-\alpha) - \frac{\mu_2}{\sigma_2}\right)\right)^2 \tag{3-15}$$

$$\cdot \Phi\left(\frac{1}{\sqrt{1-\rho^2}} \cdot \left(\Phi^{-1}(1-\alpha_2) - \frac{\mu_2}{\sigma_2} - \rho\left(\Phi^{-1}(1-\alpha_1) - \frac{\mu_1}{\sigma_1}\right)\right)\right),$$

$$\mu_1 + \mu_2 = \mu, \tag{3-16}$$

where $\Phi(.)$ is the density of the standard normal distribution and $\Phi^{-1}(.)$ its inverse.

A graphical solution of (3-14) and (3-9) for $\sigma_1 = \sigma_2$ and given value of is presented in [4], Figure 5.1, p. 215, resp. Figure 5.2, p. 216.

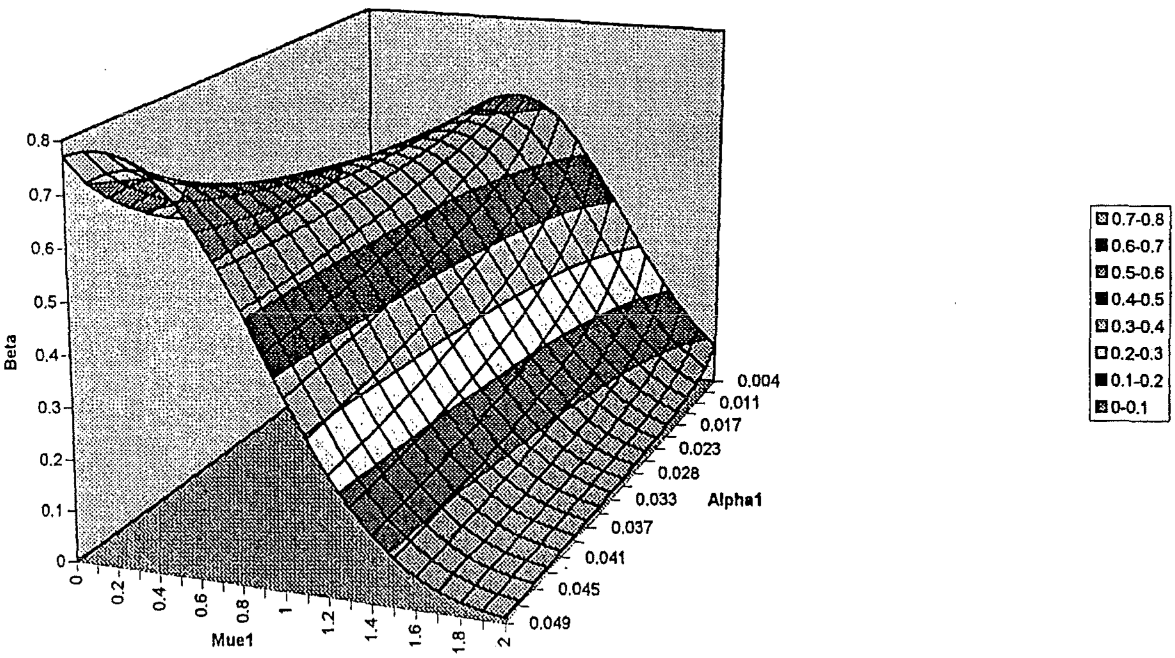

Fig. A.3.1: Graphical representation of Table A.3.1.

It should be mentioned that equations (3-9) and (3-14) to (3-16) represent only necessary conditions for the solution of the optimization problem (3-13). Since it seems to be impossible to show analytically that these equations in fact solve the optimization problem (3-12), numerical calculations have been performed which confirm our conjecture. An example, taken from [5], is given by Table A.3.1 and Figure A.3.1.

If one looks at the equations (3-14) and (3-9) which determine α_1 and α_2, one finds that they do not depend on the value of μ, but only on the ratio of σ_1/σ_2, ρ and α which is of great advantage in practical application.

The optimum values of α_1 and α_2 given by (3-14) and (3-9) can be interpreted in geometric terms. Then, with

$$\Phi^{-1}(1-\alpha_1)=x \quad \text{and} \quad \Phi^{-1}(1-\alpha_2)=y \tag{3-17}$$

we write the condition (3-9) in the form

$$1-\alpha=\frac{1}{2\pi}\cdot\frac{1}{\sqrt{1-\rho^2}}\int_{-\infty}^{x}dt_1\int_{-\infty}^{y}dt_2\exp\left(-\frac{1}{2}\frac{1}{1-\rho^2}\cdot(t_1^2-2\rho\,t_1\,t_2+t_2^2)\right). \tag{3-18}$$

By implicit differentiation with respect to x we obtain, using the Leibniz formula,

$$0=\frac{1}{2\pi}\frac{1}{\sqrt{1-\rho^2}}\int_{-\infty}^{y}dt_2\left(-\frac{1}{2}\frac{1}{1-\rho^2}\cdot(x^2-2\rho\,t_2\,x+t_2^2)\right)$$
$$+\frac{1}{2\pi}\frac{1}{\sqrt{1-\rho^2}}\int_{-\infty}^{y}dt_1\exp\left(-\frac{1}{2}\frac{1}{1-\rho^2}\cdot(t_1^2-2\rho\,t_1\,y+y^2)\right)\frac{dy}{dx}$$

and following completion of the square and integration

$$0=\exp\left(-\frac{x^2}{2}\right)\cdot\Phi\left(\frac{y-\rho x}{1-\rho^2}\right)+\exp\left(-\frac{y^2}{2}\right)\cdot\Phi\left(\frac{y-\rho y}{1-\rho^2}\right)\frac{dy}{dx} \tag{3-19}$$

So, using also (3-14), we ultimately obtain the surprisingly simple form

$$\frac{dy}{dx}=-\frac{\sigma_2}{\sigma_1} \quad \text{or} \quad y=\frac{\sigma_2}{\sigma_1}\cdot x+\text{const.} \tag{3-20}$$

Consequently, if we plot for a given α, according to (3-9), $y = \phi^{-1}(1-\alpha_2)$ as a function of $x = \phi^{-1}(1-\alpha_1)$, we have to determine only the pair of values (y, x) according to (3-20) for which pair the gradient of this function adopts the value $-\sigma_2/\sigma_1$. The advantage is that (3-9) is dependent solely on α and ρ which means that it can be represented as a one-parameter family of curves for a given value of α and the dependence on σ_2/σ_1, according to (3-20), plays a part only through the gradient.

a1\m1	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2
0.049	0.7708	0.7903	0.805	0.8145	0.8187	0.8173	0.8105	0.7984	0.7814	0.7597	0.7339	0.7045	0.6721	0.6371	0.6002	0.5619	0.5227	0.4831	0.4435	0.4044	0.3662
0.047	0.7036	0.7298	0.7517	0.7686	0.7801	0.786	0.786	0.7801	0.7685	0.7515	0.7296	0.7033	0.6733	0.6402	0.6046	0.5672	0.5285	0.4892	0.4498	0.4107	0.3724
0.045	0.6563	0.6863	0.7125	0.7341	0.7505	0.7613	0.7662	0.765	0.7578	0.7447	0.7262	0.7028	0.675	0.6437	0.6094	0.573	0.5349	0.496	0.4568	0.4177	0.3793
0.043	0.6191	0.6516	0.6807	0.7056	0.7258	0.7405	0.7493	0.752	0.7485	0.7389	0.7234	0.7027	0.6771	0.6476	0.6147	0.5792	0.5419	0.5034	0.4644	0.4254	0.3669
0.041	0.5882	0.6224	0.6537	0.6812	0.7042	0.7221	0.7343	0.7404	0.7402	0.7338	0.7212	0.703	0.6796	0.6518	0.6204	0.586	0.5494	0.5114	0.4727	0.4337	0.3951
0.039	0.5616	0.5971	0.63	0.6596	0.685	0.7056	0.7207	0.7299	0.7327	0.7292	0.7194	0.7036	0.6824	0.6565	0.6265	0.5932	0.5575	0.5201	0.4816	0.4427	0.4041
0.037	0.5382	0.5746	0.6089	0.6402	0.6676	0.6906	0.7083	0.7201	0.7258	0.725	0.7179	0.7046	0.6856	0.6615	0.6331	0.601	0.5662	0.5294	0.4913	0.4526	0.4139
0.035	0.5173	0.5545	0.5898	0.6224	0.6517	0.6767	0.6967	0.7111	0.7194	0.7213	0.7168	0.7059	0.6891	0.667	0.6401	0.6094	0.5756	0.5394	0.5017	0.4632	0.4246
0.033	0.4985	0.5361	0.5723	0.6061	0.6369	0.6637	0.6859	0.7026	0.7134	0.7179	0.7159	0.7075	0.693	0.6729	0.6478	0.6184	0.5857	0.5503	0.5131	0.4749	0.4363
0.03	0.4812	0.5193	0.5561	0.591	0.6231	0.6516	0.6757	0.6947	0.7079	0.7149	0.7154	0.7095	0.6973	0.6793	0.656	0.6282	0.5966	0.5621	0.5256	0.4877	0.4492
0.028	0.4654	0.5037	0.5411	0.5768	0.6101	0.6402	0.6661	0.6872	0.7027	0.7122	0.7153	0.7118	0.7021	0.6862	0.6649	0.6387	0.6085	0.5751	0.5392	0.5018	0.4634
0.025	0.4507	0.4892	0.5271	0.5636	0.5979	0.6293	0.657	0.68	0.6978	0.7098	0.7154	0.7146	0.7073	0.6938	0.6746	0.6503	0.6216	0.5893	0.5544	0.5175	0.4794
0.023	0.4371	0.4757	0.5139	0.551	0.5863	0.619	0.6483	0.6733	0.6933	0.7077	0.7159	0.7177	0.7131	0.7021	0.6852	0.663	0.636	0.6051	0.5712	0.535	0.4973
0.02	0.4243	0.4629	0.5014	0.5391	0.5753	0.6092	0.64	0.6668	0.689	0.7059	0.7168	0.7214	0.7196	0.7113	0.697	0.677	0.6521	0.6229	0.5903	0.555	0.5179
0.017	0.4123	0.4509	0.4897	0.5278	0.5648	0.5998	0.632	0.6607	0.6851	0.7044	0.7181	0.7257	0.7269	0.7216	0.7102	0.6929	0.6703	0.6431	0.6121	0.5781	0.5418
0.014	0.401	0.4396	0.4785	0.517	0.5547	0.5907	0.6244	0.6549	0.6814	0.7034	0.7199	0.7307	0.7352	0.7334	0.7252	0.7111	0.6914	0.6667	0.6377	0.6053	0.5702
0.011	0.3903	0.4288	0.4678	0.5067	0.545	0.582	0.617	0.6493	0.6781	0.7027	0.7224	0.7367	0.7449	0.747	0.7428	0.7325	0.7164	0.6949	0.6688	0.6387	0.6054
0.008	0.3801	0.4185	0.4576	0.4968	0.5357	0.5736	0.61	0.6441	0.6752	0.7027	0.7259	0.744	0.7567	0.7635	0.7642	0.7589	0.7476	0.7307	0.7086	0.6821	0.6518
0.004	0.3705	0.4087	0.4478	0.4873	0.5267	0.5655	0.6032	0.6392	0.6729	0.7036	0.7308	0.7536	0.7721	0.7852	0.7927	0.7946	0.7906	0.7809	0.7658	0.7457	0.7211

Table A.3.1: Numerical representation of the nondetection probability according to (3-11) for $\sigma_1 = 0.5$, $\sigma_2 = 1$, $\rho = 0.5$,. $\alpha = 0.05$ and $\mu = 2$. The saddle point lies at $(\alpha_1, \alpha_2) = (0.0184, 0.0363)$ and $(\mu_1, \mu_2) = (0.62, 1.38)$.

Using (3-17) and (3-20), we can write (3-20) also as

$$\frac{d\lambda_2}{d\lambda_1} = -1 \quad \text{or} \quad \lambda_2 + \lambda_1 = \text{const.} \tag{3-21}$$

which means that the significance points are determined by (3-5) and (3-20) and the respective interpretation in geometric terms is possible. However, according to (3-5), the dependence on σ_1 and σ_2 is more complicated here.

A.4. Intercomparison of the two Minimax Tests

As has been said in Section A.2, the Neyman-Pearson test is the best test for given values of μ_1 and μ_2. However, we wish to demonstrate in this section that the minimax test derived from the Neyman-Pearson test is not the better choice for any values of μ_1 and μ_2 compared to the minimax test based on the separate procedures.

To demonstrate this, we will consider the very simple special case

$$\sigma_1 = \sigma_2 = \sigma, \rho = 0 \,. \tag{4-1}$$

Then the probability of occurrence of error of the second kind according to the minimax test derived from the Neyman-Pearson test reads according to (2-16) for any values of μ_1, with $0 \le \mu_1 \le \mu$,

$$\beta^*_{NP}(\mu_1, \mu - \mu_1) = \Phi\left(\Phi^{-1}(1-\alpha) - \frac{\mu}{\sqrt{2}\cdot\sigma}\right) \tag{4-2}$$

which, as already said, depends solely on the sum μ.

The equations for the optimal values of α_1 and α_2 of the minimax test derived from the separate tests, according to (3-14) to (3-16) and (3-9), yield the solutions

$$1-\alpha_1 = 1-\alpha_2 = \sqrt{1-\alpha} \tag{4-3a}$$

$$\mu_1 = \mu_2 = \frac{\mu}{2} \tag{4-3b}$$

so that, with (3-10) and (3-8), the probability of the error of the second kind according to the minimax test derived from the separate test is for any values of μ_1, with $0 \le \mu_1 \le \mu$

$$\beta_G^*(\mu_1, \mu-\mu_1) = \Phi\left(\Phi^{-1}\left(\sqrt{1-\alpha}\right) - \frac{\mu_1}{\sigma}\right)\cdot\Phi\left(\Phi^{-1}\left(\sqrt{1-\alpha}\right) - \frac{\mu-\mu_1}{\sigma}\right). \qquad (4\text{-}4)$$

We will now consider the very special case

$$\mu = \sqrt{2}\cdot\sigma\cdot\Phi^{-1}(1-\alpha) \qquad (4\text{-}5)$$

which, for $\mu > 0$, implies that $\alpha < 0.5$. Then the following relation holds

$$\beta_{NP}^*\left(\frac{\mu_1}{\sigma}, \frac{\mu-\mu_1}{\sigma}\right) = \frac{1}{2} \qquad (4\text{-}6)$$

and also

$$\beta_G^*\left(\frac{\mu}{\sigma}, o\right) = \beta_G^*\left(o, \frac{\mu}{\sigma}\right) = \Phi\left(\Phi^{-1}\left(\sqrt{1-\alpha}\right) - \sqrt{2}\cdot\Phi^{-1}(1-\alpha)\right)\cdot\sqrt{1-\alpha}$$

$$\beta_G^*\left(\frac{\mu}{2\sigma}, \frac{\mu}{2\sigma}\right) = \left(\Phi\left(\Phi^{-1}\left(\sqrt{1-\alpha}\right) - \frac{1}{\sqrt{2}}\cdot\Phi^{-1}(1-\alpha)\right)\right).$$

In Fig. A.4.1, $ß^*_{NP}$ and $ß^*_G$ have been plotted versus μ_1 for the case (4-5) and for $\alpha = 0.05$.

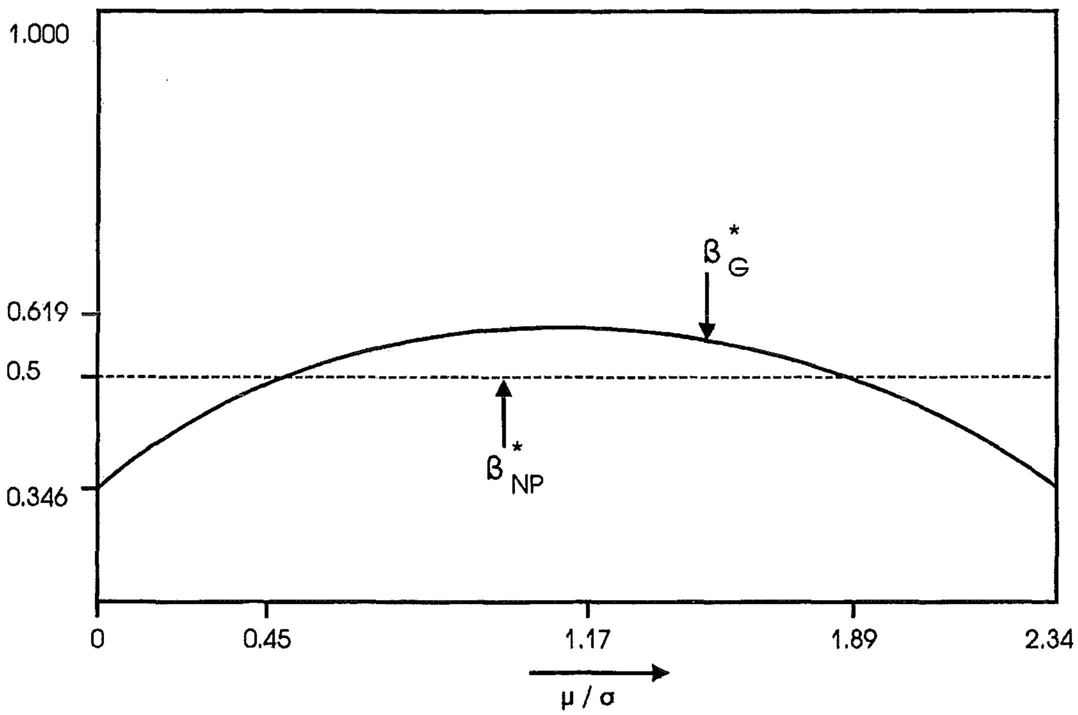

Fig. A.4.1: Probabilities of errors of the second kind according to (4-2) and (4-4) as a function of μ_1 for $\alpha = 0.05$.

We see that in the vicinity of $\mu/2$ the test based on the Neyman-Pearson test is better than the other test. This is plausible because this test, according to (2-13), is actually the best suited test for $\mu_1 = \mu/2$ so that it can be expected that this applies in the vicinity of $\mu/2$ as well. On the other hand we see that the Neymann-Pearson test may be worse than the separate test, if μ_1 is very different from $\mu/2$.

More numerical examples are given in [5].

A.5 Literature for the Annex

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