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Abstract

Green's function method and its application to verification of diffusion models of GASFLOW code

To validate the diffusion model and the aerosol particle model of the GASFLOW computer code, theoretical solutions of advection diffusion problems are developed by using the Green's function method. The work consists of a theory part and an application part. In the first part, the Green's functions of one-dimensional advection diffusion problems are solved in infinite, semi-infinite and finite domains with the Dirichlet, the Neumann and/or the Robin boundary conditions. Novel and effective image systems especially for the advection diffusion problems are made to find the Green's functions in a semi-infinite domain. Eigenfunction method is utilized to find the Green's functions in a bounded domain. In the case, key steps of a coordinate transform based on a concept of reversed time scale, a Laplace transform and an exponential transform are proposed to solve the Green's functions. Then the product rule of the multi-dimensional Green's functions is discussed in a Cartesian coordinate system. Based on the building blocks of one-dimensional Green's functions, the multi-dimensional Green's function solution can be constructed by applying the product rule. Green's function tables are summarized to facilitate the application of the Green's function. In the second part, the obtained Green's function solutions benchmark a series of validations to the diffusion model of gas species in continuous phase and the diffusion model of discrete aerosol particles in the GASFLOW code. Perfect agreements are obtained between the GASFLOW simulations and the Green's function solutions in case of the gas diffusion. Very good consistencies are found between the theoretical solutions of the advection diffusion equations and the numerical particle distributions in advective flows, when the drag force between the micron-sized particles and the conveying gas flow meets the Stokes' law about resistance. This situation is corresponding to a very small Reynolds number based on the particle diameter, with a negligible inertia effect of the particles. It is concluded that, both the gas diffusion model and the discrete particle diffusion model of GASFLOW can reproduce numerically the corresponding physics successfully. The Green's function tables containing the building blocks for multi-dimensional problems is hopefully able to facilitate the application of the Green's function method to the future work.

Kurzfassung

Die Green-Funktion und ihre Anwendung zur Überprüfung von GASFLOW-Diffusionsmodellen

Zur Überprüfung des vom GASFLOW-Rechenprogramm erstellten Diffusionsmodells und des Aerosolpartikelmodells werden mit Hilfe der Green-Funktion theoretische Lösungen von Advektions-/Diffusionsproblemen entwickelt. Die vorliegende Arbeit gliedert sich in einen theoretischen und einen praktischen Teil. Im ersten Teil werden die Green-Funktionen von eindimensionalen Advektions-/Diffusionsproblemen im unendlichen, quasi-unendlichen und im endlichen Bereich unter Dirichlet-, Neumann- und/oder Robin-Rahmenbedingungen gelöst. Zur Bestimmung der Green-Funktionen in einem quasi-unendlichen Bereich werden insbesondere für Advektions-/Diffusionsprobleme effektive Abbildungssysteme neu entwickelt. Die Eigenfunktionsmethode wird eingesetzt, um die Green-Funktionen in einem endlichen Bereich zu ermitteln. Zur Lösung der Green-Funktionen wird eine Koordinatentransformation auf der Grundlage einer umgekehrten Zeitskala, einer Laplace-Transformation und einer exponentiellen Transformation vorgeschlagen. Anschließend wird die Produktregel der mehrdimensionalen Green-Funktionen in einem kartesischen Koordinatensystem diskutiert. Mit Hilfe der Produktregel lässt sich die Lösung der mehrdimensionalen Green-Funktion aus den Bestandteilen der eindimensionalen Green-Funktionen ableiten. Die Green-Funktionstabellen werden zusammengefasst, um die Anwendung der Green-Funktion zu erleichtern. Im zweiten Teil der Arbeit werden die ermittelten Lösungen der Green-Funktion zur Validierung des in GASFLOW erstellten Diffusionsmodells für Gase in einer kontinuierlichen Phase und des Diffusionsmodells für diskrete Aerosolpartikel eingesetzt. Für die Gasdiffusion zeigen die GASFLOW-Simulationen eine perfekte Übereinstimmung mit den Lösungen der Green-Funktion. Sehr gute Übereinstimmungen finden sich ebenfalls zwischen den theoretischen Lösungen der Advektions-/Diffusionsgleichungen und den numerischen Partikelverteilungen in advektiven Strömungen, sofern die Zugkraft zwischen den Partikeln im Mikronbereich und der Gasströmung das Widerstandsgesetz von Stokes erfüllt. Dieser Fall entspricht einer sehr kleinen Reynoldszahl bezogen auf den Partikeldurchmesser, mit einem vernachlässigbaren Trägheitseffekt auf die Partikel. Es zeigt sich, dass sowohl das Gasdiffusionsmodell als auch das diskrete Partikeldiffusionsmodell in GASFLOW die entsprechende Physik hervorragend numerisch modellieren. Somit sollten die Green-Funktionstabellen mit den Bestandteilen für mehrdimensionale Probleme die Anwendung der Green-Funktion in Zukunft erleichtern.

CONTENTS

1. Introduction

Analytical solutions play import roles in validations and verifications for Computational Fluid Dynamic (CFD) computer codes, besides of experimental data. Diffusion problems with specified sources and /or boundary values are often encountered in both engineering and code validations. Green's function method (GFM) supplies a powerful tool to solve linear partial differential equations. A lot of applications of the GFM can be found in literatures. For example, a problem of solute transport in porous media is solved by using the GFM (Leij and Priesack et al., 2000); the GFM is applied to model analytically the non-aqueous phase liquid dissolution (Leij and van Genuchten, 2000); analytical solutions are made by utilizing the GFM for contamination transport from multi-dimensional sources in finite thickness aquifer (Park and Zhan, 2001); atmospheric diffusion equations with multiple sources and height-dependent wind speed and eddy diffusions are solved by using the GFM (Lin and Hildemann, 1996); a two dimensional analysis of advection diffusion problem is carried out by using the GFM to understand the effluent dispersion in shallow tidal waters (Kay, 1990). A systematic application of the GFM to heat conduction is formed as a monograph by Beck and Cole et al. (1992). A more mathematical monograph about Green's functions with applications is the work of Duffy (2001). This report is engaged in applying the GFM to solve a series of diffusion and advection diffusion equations (ADE) in various concerned domains like, infinite (free), semi-infinite and bounded multi- dimensional regions, no matter the involved background is about heat transfer, mass diffusion, particle dispersion and so on. These solutions are utilized to make comparisons with numerical simulations, and to benchmark computer code validations based on solving this sort of ADE problems. A secondary aim of this work is to supply a kind of handbook on Green's functions to facilitate readers to apply the GFM to their own problems.

In order to make it easier to understand, some necessary mathematical fundamentals are presented at the beginning, including the basic knowledge about Dirac Delta function, Heaviside function, Fourier transform and Laplace transform. Then the basic idea of the GFM is illustrated in a general sense. In the next part concentrations are focused on the detailed procedures to solve ADE problem with different boundary conditions. The results are applied, as examples, to validate the diffusion solver and dust motion (or particle model) of the GASFLOW computer code, which is widely used in nuclear industries and developed in Research Center Karlsruhe.

Some figures exported from Mathcad or self-made C programs could appear in the report to illustrate the results more explicitly.

2. Mathematical fundamentals

2.1 Dirac Delta function

Dirac function (or called, delta function) is used very often through the report. Therefore, the basic concepts like the definition and the properties of the function are presented here, although it is not the aim to repeat the content of math text book.

Delta function is defined as (James, 2002),

$\delta(\mathbf{x}) = 0$ unless $\mathbf{x} = 0$,

$\delta(0) = \infty$,

$$\int_{-\infty}^{\infty} \delta(\mathbf{x})\mathbf{dx} = 1. \qquad (2\text{-}1\text{-}1)$$

The following useful properties (Duffy, 2001; James, 2002) should be kept in mind,

$$\delta(\mathbf{x}-\mathbf{a}) = 0 \text{ unless } \mathbf{x} = \mathbf{a}, \qquad (2\text{-}1\text{-}2)$$

and the so-called shift theorem,

$$\int_{-\infty}^{\infty} \mathbf{f}(\mathbf{x})\delta(\mathbf{x}-\mathbf{a})\mathbf{dx} = \mathbf{f}(\mathbf{a}), \qquad (2\text{-}1\text{-}3)$$

and others,

$$\delta(\mathbf{ax}) = \frac{1}{|\mathbf{a}|}\delta(\mathbf{x}), \text{ specially } \delta(-\mathbf{x}) = \delta(\mathbf{x}), \qquad (2\text{-}1\text{-}4)$$

$$\mathbf{f}(\mathbf{x})\delta(\mathbf{x}-\mathbf{a}) = \mathbf{f}(\mathbf{a})\delta(\mathbf{x}-\mathbf{a}), \qquad (2\text{-}1\text{-}5)$$

$$\delta(\mathbf{x}) = -\mathbf{x}\delta'(\mathbf{x}), \qquad (2\text{-}1\text{-}6)$$

$$\mathbf{x}^{\mathbf{n}}\delta^{(\mathbf{m})}(\mathbf{x}) = 0 \text{ if } 0 \le \mathbf{m} < \mathbf{n}, \qquad (2\text{-}1\text{-}7)$$

$$\delta(\mathbf{x},\mathbf{y},\mathbf{z}) = \delta(\mathbf{x})\delta(\mathbf{y})\delta(\mathbf{z}), \qquad (2\text{-}1\text{-}8)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbf{f}(\mathbf{x},\mathbf{y},\mathbf{z})\delta(\mathbf{x}-\mathbf{a},\mathbf{y}-\mathbf{b},\mathbf{z}-\mathbf{c}) = \mathbf{f}(\mathbf{a},\mathbf{b},\mathbf{c}). \qquad (2\text{-}1\text{-}9)$$

2.2 Heaviside function

Heaviside function with another name of step function is defined as (Duffy, 2001),

$$\mathbf{H}(\mathbf{x}-\mathbf{a}) = \begin{cases} 1, \mathbf{x} > \mathbf{a}, \\ 0.5, \mathbf{x} = \mathbf{a}, \\ 0, \mathbf{x} < \mathbf{a}. \end{cases} \qquad (2\text{-}2\text{-}1)$$

(The function value at $\mathbf{x} = \mathbf{a}$ could be defined as other values as 1 or 0 in some literatures.)

Delta function is the derivative of step function,

$$\delta(x) = \frac{d}{dx} H(x) . \qquad (2\text{-}2\text{-}2)$$

2.3 Fourier transform

The Fourier transform is the natural extension of Fourier series to a function $f(x)$ of infinite period. It is defined in terms of a pair of integrals (James, 2002),

$$f(x) = \mathcal{F}^{-1}[F(\xi)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\xi) \exp(i\xi x) d\xi , \qquad (2\text{-}3\text{-}1)$$

and

$$F(\xi) = \mathcal{F}[f(x)] = \int_{-\infty}^{\infty} f(x) \exp(-i\xi x) dx . \qquad (2\text{-}3\text{-}2)$$

Equality (2-3-2) is the Fourier transform of $f(x)$, while (2-3-1) is the inverse Fourier transform.

Some properties and useful Fourier transforms of functions (James, 2002) are listed here, which are also used in the following sections.

$$\mathcal{F}[f'(x)] = i\xi F(\xi) , \qquad (2\text{-}4\text{-}1)$$

$$\mathcal{F}[\delta(x-a)] = \exp(-ia\xi) . \qquad (2\text{-}4\text{-}2)$$

2.4 Laplace transform

If a function $f(x)$ equals to 0 for $x < 0$, then the integral,

$$\mathcal{L}[f(x)] = F(s) = \int_{0}^{\infty} f(x) \exp(-sx) dx , \qquad (2\text{-}4\text{-}1)$$

is defined as the Laplace transform of $f(x)$ (Duffy, 2001). Some Laplace transforms, cited from the "Handbook of mathematics" by Bronshtein et al. (2003), are used in the report,

$$\mathcal{L}[H(x-a)] = \frac{\exp(-as)}{s}, s > 0 , \qquad (2\text{-}4\text{-}2$$

$$\mathcal{L}[\delta(x-a)] = \exp(-as) , \qquad (2\text{-}4\text{-}3)$$

$$\mathcal{L}[\exp(ax)] = \frac{1}{s-a}, s > a , \qquad (2\text{-}4\text{-}4)$$

$$\mathcal{L}[f'(x)] = sF(s) - f(0) , \qquad (2\text{-}4\text{-}5)$$

$$\mathcal{L}[f(x-a)H(x-a)] = \exp(-as)F(s) . \qquad (2\text{-}4\text{-}6)$$

2.5 Error function

Error function is defined as,

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_{0}^{x} \exp(-t^2) dt , \qquad (2\text{-}5\text{-}1)$$

and complementary error function as,

$$\mathbf{erfc(x)} = 1 - \mathbf{erf(x)} = \frac{2}{\sqrt{\pi}}\int_{\mathbf{x}}^{\infty}\mathbf{exp(-t^2)dt} \,. \qquad (2\text{-}5\text{-}2)$$

Some of the properties are listed here,

$$\mathbf{erf(\infty)} = 1 \,, \qquad (2\text{-}5\text{-}3)$$

$$\mathbf{erf(-x)} = -\mathbf{erf(x)} \,, \qquad (2\text{-}5\text{-}4)$$

$$\mathbf{erfc(-x)} = 2 - \mathbf{erfc(x)} \,. \qquad (2\text{-}5\text{-}5)$$

3. Green's function method

3.1 General partial differential equation

The Green's function was first proposed by George Green in 1830s and was named after him. The general ideal of the GFM is described in details in a monograph by M. D. Greenberg (1971). The convention of notations about the GFM in the monograph is followed by this report. However the formulation of the GFM is not simply repeated in the report, but oriented to the aim of this work. The theory about Green's function can be referred to the works of Stakgold (1998), Tychonov and Samarski (1967), Roach (1982) and so on.

In general, a linear partial differential equation (PDE) of second order, with two independent variables $\mathbf{x}$ and $\mathbf{y}$ (sometimes, $\mathbf{t}$) can be formulated as,

$$\mathbf{Lu} = \mathbf{Au}_{\mathbf{xx}} + 2\mathbf{Bu}_{\mathbf{xy}} + \mathbf{Cu}_{\mathbf{yy}} + \mathbf{Du}_{\mathbf{x}} + \mathbf{Eu}_{\mathbf{y}} + \mathbf{Fu} = \phi, \qquad (3\text{-}1\text{-}1)$$

where $\mathbf{L}$ is the original differential operator, $\mathbf{A,...F,\phi}$ are given functions of $\mathbf{x}$ and $\mathbf{y}$. Equation (3-1-1) holds over a prescribed region $\mathcal{R}$ with general linear boundary conditions,

$$\mathbf{B(u)} = \alpha\mathbf{u} + \beta\mathbf{u}_{\mathbf{n}} = \mathbf{f}, \qquad (3\text{-}1\text{-}2)$$

on the boundary curve $\mathcal{B}$ of the domain $\mathcal{R}$, where $\mathbf{u}_{\mathbf{n}}$ denotes the outward normal derivative $\frac{\partial \mathbf{u}}{\partial \mathbf{n}}$, and $\alpha, \beta, \mathbf{f}$ may be functions defined on $\mathcal{B}$, as shown in Figure 3-1-1.

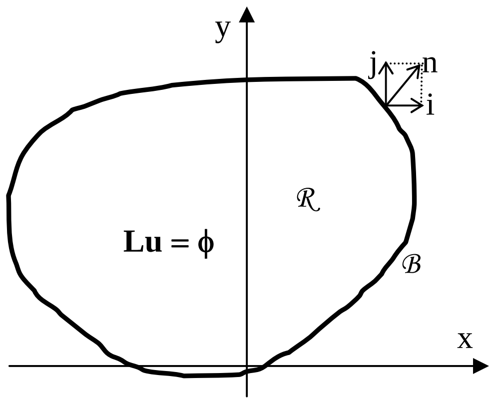


Figure 3-1-1 General boundary value problem

The GFM is applicable only when differential operator satisfies superposition principle. In fact, most operators encountered in engineering can meet the basic requirement, e.g., the diffusion equation, wave equation, Laplace equation, actually all the linear second order partial differential equations and so on. The general second order equation (3-1-1) is taken as an example to explain how the GFM works. First, multiply $\mathbf{Lu}$ by an arbitrary function $\mathbf{v}$, and make integration by parts,

$$\iint_{\mathcal{R}} \mathbf{vLu}d\sigma = \int_{\mathcal{B}} (\mathbf{Mi} + \mathbf{Nj}) \cdot \mathbf{n}ds + \iint_{\mathcal{R}} \mathbf{uL}^{*}\mathbf{v}d\sigma, \qquad (3\text{-}1\text{-}3)$$

where

$\mathbf{d\sigma}$ is the differential area on $\mathcal{R}$,

$\mathbf{ds}$ is the differential arc on $\mathcal{B}$, $\mathbf{dy} = \mathbf{n} \cdot \mathbf{ids}$, $\mathbf{dx} = \mathbf{n} \cdot \mathbf{jds}$,

$$\mathbf{L^* v = (Av)_{xx} + 2(Bv)_{xy} + (Cv)_{yy} - (Dv)_x - (Ev)_y + Fv}, \qquad (3\text{-}1\text{-}4)$$

$\mathbf{L^*}$ is the adjoint operator associated to $\mathbf{L}$,

$$\mathbf{M = Avu_x - u(Av)_x + 2vBu_y + Duv},$$

$$\mathbf{N = -2u(Bv)_x + Cvu_y - u(Cv)_y + Euv}. \qquad (3\text{-}1\text{-}5)$$

If $\mathbf{v = v(\xi,\eta)}$ is selected so smart that,

$$\mathbf{L^* v = \delta(\xi - x, \eta - y)}, \qquad (3\text{-}1\text{-}6)$$

then the second integration term of the right hand side of (3-1-3),

$$\iint \mathbf{uL^* v d\xi d\eta} = \iint \mathbf{u\delta(\xi - x, \eta - y) d\xi d\eta = u(x,y)}, \qquad (3\text{-}1\text{-}7)$$

is the solution of (3-1-1). The last step proceeds by using the property of delta function (2-1-9). This function $\mathbf{v(\xi,\eta;x,y)}$ is called Green's function, denoted as $\mathbf{G(\xi,\eta;x,y)}$, which is determined by,

$$\mathbf{L^* G = \delta(\xi - x, \eta - y)}. \qquad (3\text{-}1\text{-}8)$$

Substitute (3-1-7) into (3-1-3) with considering that $\mathbf{Lu} = \phi$, the expression of the solution is obtained as,

$$\mathbf{u(x,y)} = \iint_{\mathcal{R}} \mathbf{GLud\sigma} - \int_{\mathcal{B}} \mathbf{(Mi + Nj) \cdot nds} = \iint_{\mathcal{R}} \mathbf{G\phi d\sigma} - \int_{\mathcal{B}} \mathbf{(Mi + Nj) \cdot nds}. \qquad (3\text{-}1\text{-}9)$$

If the equation (3-1-8) about the Green's function $\mathbf{G}$ is solved successfully with certain boundary conditions, then the solution of the problem (3-1-1) is explicitly expressed by (3-1-9). This is the basic idea of the GFM.

3.2 One-dimensional advection diffusion equation

Based on the general idea of the GFM described in last subsection, this subsection is devoted to apply the method to solve the advection diffusion equation in a general sense. The differential operator of one dimensional advection diffusion problem is denoted as,

$$\mathbf{L} = \frac{\partial}{\partial \mathbf{t}} + \mathbf{V}\frac{\partial}{\partial \mathbf{x}} - \mathbf{D}\frac{\partial^2}{\partial^2 \mathbf{x}}, \qquad (3\text{-}2\text{-}1)$$

where $\mathbf{V}$ stands for the advection velocity, $\mathbf{D}$ for diffusion coefficient, both are assumed as constants for simplicity. The problem is formulated as,

$$\mathbf{Lu = u_t + Vu_x - Du_{xx}} = \phi\mathbf{(x,t)}, \qquad (3\text{-}2\text{-}2)$$

with certain boundary conditions. In order to show explicitly the procedure to integrate by parts, the domain is prescribed as a rectangular region, as shown in Figure 3-2-1, namely, $\mathbf{x_1 < x < x_2}$, $\mathbf{t_1 < t < t_2}$.

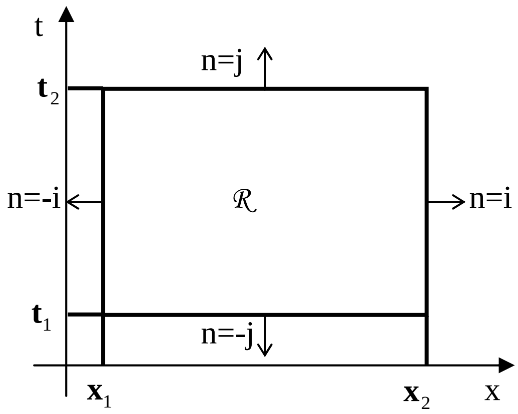


Figure 3-2-1 Rectangular region for advection diffusion problem

Integrate by parts for $\iint\limits_{\mathcal{R}} \mathbf{vLud\sigma}$,

$$\iint\limits_{\mathcal{R}} \mathbf{vLud\sigma} = \int_{x_1}^{x_2}\int_{t_1}^{t_2} (\mathbf{vu_t + Vvu_x - Dvu_{xx})dtdx}. \qquad (3\text{-}2\text{-}3)$$

The three terms in the brackets are calculated one by one,

$$\int_{x_1}^{x_2}\int_{t_1}^{t_2} (\mathbf{vu_t})\mathbf{dtdx} = \int_{x_1}^{x_2} \left(\mathbf{vu}\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \mathbf{uv_t dt}\right)\mathbf{dx} = \int_{x_1}^{x_2} \left(\mathbf{vu}\Big|_{t_1}^{t_2}\right)\mathbf{dx} - \iint\limits_{\mathcal{R}} \mathbf{uv_t d\sigma},$$

$$\int_{x_1}^{x_2}\int_{t_1}^{t_2} (\mathbf{Vvu_x})\mathbf{dtdx} = \mathbf{V}\int_{t_1}^{t_2} \left(\mathbf{vu}\Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \mathbf{uv_x dx}\right)\mathbf{dt} = \int_{t_1}^{t_2} \left(\mathbf{Vvu}\Big|_{x_1}^{x_2}\right)\mathbf{dt} - \iint\limits_{\mathcal{R}} \mathbf{Vuv_x d\sigma},$$

$$-\int_{x_1}^{x_2}\int_{t_1}^{t_2} (\mathbf{Dvu_{xx}})\mathbf{dtdx} = -\mathbf{D}\int_{t_1}^{t_2} \left[(\mathbf{vu_x - v_x u})\Big|_{x_1}^{x_2} + \int_{x_1}^{x_2} \mathbf{uv_{xx} dx}\right]\mathbf{dt}$$

$$= -\int_{t_1}^{t_2} \left[\mathbf{D(vu_x - v_x u)}\Big|_{x_1}^{x_2}\right]\mathbf{dt} - \iint\limits_{\mathcal{R}} \mathbf{Duv_{xx} d\sigma}.$$

The summation of the three equalities is

$$\iint\limits_{\mathcal{R}} \mathbf{vLud\sigma} = \int_{x_1}^{x_2} \left(\mathbf{vu}\Big|_{t_1}^{t_2}\right)\mathbf{dx} + \int_{t_1}^{t_2} \left[\mathbf{Vvu}\Big|_{x_1}^{x_2} - \mathbf{D(vu_x - v_x u)}\Big|_{x_1}^{x_2}\right]\mathbf{dt} + \iint\limits_{\mathcal{R}} (\mathbf{-uv_t - Vuv_x - Duv_{xx}})\mathbf{d\sigma}$$

$$= \int_{x_1}^{x_2} \left(\mathbf{vu}\Big|_{t_1}^{t_2}\right)\mathbf{dx} + \int_{t_1}^{t_2} \left[\mathbf{Vvu}\Big|_{x_1}^{x_2} - \mathbf{D(vu_x - v_x u)}\Big|_{x_1}^{x_2}\right]\mathbf{dt} + \iint\limits_{\mathcal{R}} \mathbf{uL^* v d\sigma}. \qquad (3\text{-}2\text{-}4)$$

Here $\mathbf{L^* v = -v_t - Vv_x - Dv_{xx}}$, and $\mathbf{L^*}$ is the adjoint differential operator associated to the $\mathbf{L}$ in (3-2-1), denoted as,

$$\mathbf{L^*} = -\frac{\partial}{\partial \mathbf{t}} - \mathbf{V}\frac{\partial}{\partial \mathbf{x}} - \mathbf{D}\frac{\partial^2}{\partial^2 \mathbf{x}}. \qquad (3\text{-}2\text{-}5)$$

If we choose the function of $\mathbf{v}$ as the Green's function $\mathbf{G(\xi,\tau;x,t)}$ to satisfy,

$$\mathbf{L^* G = \delta(\xi - x, \tau - t)}, \qquad (3\text{-}2\text{-}6)$$

with certain boundary conditions for $\mathbf{G}$. Then substitute $\mathbf{v}$ as $\mathbf{G}$ in (3-2-4),

$$\iint_{\mathcal{R}} \mathbf{GLu d\sigma} = \int_{\mathbf{x}_1}^{\mathbf{x}_2} (\mathbf{Gu}\Big|_{\mathbf{t}_1}^{\mathbf{t}_2})\mathbf{dx} + \int_{\mathbf{t}_1}^{\mathbf{t}_2} [\mathbf{VGu}\Big|_{\mathbf{x}_1}^{\mathbf{x}_2} - \mathbf{D}(\mathbf{Gu_x} - \mathbf{G_x u})\Big|_{\mathbf{x}_1}^{\mathbf{x}_2}]\mathbf{dt} + \iint_{\mathcal{R}} \mathbf{uL^* G d\sigma}\,. \qquad (3\text{-}2\text{-}7)$$

Applying the property of delta function and the equality of $\mathbf{Lu = \phi}$, then changing the integration dummy variables from $(\mathbf{x,t})$ to (ξ,τ), we have the general expression of solution of the problem (3-2-2),

$$\mathbf{u(x,t)} = \iint_{\mathcal{R}} \mathbf{G\phi d\xi d\tau} - \int_{\xi_1}^{\xi_2} (\mathbf{Gu}\Big|_{\tau_1}^{\tau_2})\mathbf{d\xi} - \int_{\tau_1}^{\tau_2} [\mathbf{VGu}\Big|_{\xi_1}^{\xi_2} - \mathbf{D}(\mathbf{Gu_\xi} - \mathbf{G_\xi u})\Big|_{\xi_1}^{\xi_2}]\mathbf{d\tau}\,. \qquad (3\text{-}2\text{-}8)$$

According to the GFM, solving the original problem (3-2-2) is transferred to solving the problem of (3-2-6) to seek the Green's function! The essential point of the GFM is that the boundary value problem governing $\mathbf{G}$ is in general somewhat simpler than the original one governing $\mathbf{u}$. It is worth to mention that the running variables in Green's functions are ξ,τ instead of $\mathbf{x,t}$. However the latter are active in the function of $\mathbf{u}$.

4. Green's function of advection diffusion equation

In this section, the GFM is applied to solve step by step the advection diffusion problems, with different boundary conditions.

4.1 Dirichlet boundary

Now we have to come to a specific problem to show the detailed techniques to apply the GFM, including the determination of the "adjoint" boundary conditions for Green's function.

Let's define a one-dimensional advection diffusion problem in a semi-infinite axis with an initial condition at $\mathbf{t} = 0$ and a boundary condition at $\mathbf{x} = 0$ and a source term of $\boldsymbol{\phi}(\mathbf{x},\mathbf{t})$. The problem is generally formulated as,

$$\mathbf{Lu} = \mathbf{u}_t + \mathbf{Vu}_x - \mathbf{Du}_{xx} = \boldsymbol{\phi}(\mathbf{x},\mathbf{t}),\ 0 < \mathbf{x} < \infty,\ 0 < \mathbf{t} < \infty, \qquad \text{(4-1-1a)}$$

$$\mathbf{u}(\mathbf{x},0) = \mathbf{f}(\mathbf{x}), \qquad \text{(4-1-1b)}$$

$$\mathbf{u}(0,\mathbf{t}) = \mathbf{h}(\mathbf{t}). \qquad \text{(4-1-1c)}$$

In the case, the function value is prescribed on the boundary. This type of condition is called a Dirichlet boundary or first type of boundary. According to (3-2-8), the solution of (4-1-1) can be expressed as,

$$\mathbf{u}(\mathbf{x},\mathbf{t}) = \int_0^\infty\int_0^\infty \mathbf{G}\boldsymbol{\phi}\mathbf{d}\boldsymbol{\xi}\mathbf{d}\boldsymbol{\tau} - \int_0^\infty [(\mathbf{Gu})\big|_{\tau=\infty} - (\mathbf{Gu})\big|_{\tau=0}]\mathbf{d}\boldsymbol{\xi} -$$

$$-\int_0^t [(\mathbf{VGu} - \mathbf{DGu}_\xi + \mathbf{DG}_\xi\mathbf{u})\big|_{\xi=\infty} - (\mathbf{VGu} - \mathbf{DGu}_\xi + \mathbf{DG}_\xi\mathbf{u})\big|_{\xi=0}]\mathbf{d}\boldsymbol{\tau}. \qquad \text{(4-1-2)}$$

In order to identify $\mathbf{G}\big|_{\tau=\infty}$, $\mathbf{G}\big|_{\xi=\infty}$ and $\mathbf{G}_\xi\big|_{\xi=\infty}$, the natural boundary conditions at infinite of Green's function have to be understood based on the physics that the differential equation describes. Let's take the advection diffusion equation as an example. The Green's function is determined by (3-2-6), and the adjoint operator $\mathbf{L}^*$ can be denoted in another way besides (3-2-5),

$$\mathbf{L}^* = \frac{\partial}{\partial(-\mathbf{t})} + (-\mathbf{V})\frac{\partial}{\partial \mathbf{x}} - \mathbf{D}\frac{\partial^2}{\partial^2 \mathbf{x}}. \qquad \text{(4-1-3)}$$

Comparing (4-1-3) to (3-2-1), it can be concluded that $\mathbf{L}^*$ is still an advection diffusion differential operator, with "reversed" time coordinate and "reversed" advection direction. The solution of the Equation (3-2-6), namely, the Green's function $\mathbf{G}(\boldsymbol{\xi},\boldsymbol{\tau};\mathbf{x},\mathbf{t})$ stands for, say, the subsequent heat distribution incurred by an instantaneous unit heat pulse at the location of $\mathbf{x}$ and at the moment of $\mathbf{t}$. Hereafter it can be concluded that $\mathbf{G}\big|_{\tau=\infty}$, $\mathbf{G}\big|_{\xi=\infty}$ and $\mathbf{G}_\xi\big|_{\xi=\infty}$ must be zero, because the unit heat pulse exerted at a finite location can not influence the distribution at an infinite far place or at infinite long time future. The nature features of $\mathbf{G}$, $\mathbf{G}\big|_{\tau=\infty}$, $\mathbf{G}\big|_{\xi=\infty} = 0$ and $\mathbf{G}_\xi\big|_{\xi=\infty} = 0$, are called homogenous boundary condition of Green's functions at infinite. By applying the homogeneous boundary conditions at infinite, (4-1-2) can be simplified as,

$$\mathbf{u(x,t)} = \int_0^\infty\int_0^\infty \mathbf{G}(\xi,\tau)\phi(\xi,\tau)\mathbf{d\xi d\tau} + \int_0^\infty \mathbf{G}(\xi,0)\mathbf{u}(\xi,0)\mathbf{d}\xi +$$

$$+\int_0^{\mathbf{t}} [\mathbf{VG}(0,\tau)\mathbf{u}(0,\tau) - \mathbf{DG}(0,\tau)\mathbf{u}_\xi(0,\tau) + \mathbf{DG}_\xi(0,\tau)\mathbf{u}(0,\tau)]\mathbf{d\tau}. \qquad (4\text{-}1\text{-}4)$$

Keep in mind that $\mathbf{u}(\xi,0) = \mathbf{f}(\xi)$ and $\mathbf{u}(0,\tau) = \mathbf{h}(\tau)$ according to (4-1-1b) and (4-1-1b), so, in (4-1-4), only the term "$\mathbf{DG}(0,\tau)\mathbf{u}_\xi(0,\tau)$" is unwelcome. In order to make it vanish, one can have $\mathbf{G}(0,\tau) = 0$ as a boundary condition for determining $\mathbf{G}$. Then (4-1-4) can be simplified further, based on the assumed conditions of $\mathbf{G}$,

$$\mathbf{u(x,t)} = \int_0^\infty\int_0^\infty \mathbf{G}(\xi,\tau)\phi(\xi,\tau)\mathbf{d\xi d\tau} + \int_0^\infty \mathbf{G}(\xi,0)\mathbf{f}(\xi)\mathbf{d}\xi + \int_0^{\mathbf{t}} [\mathbf{DG}_\xi(0,\tau)\mathbf{h}(\tau)]\mathbf{d\tau}. \qquad (4\text{-}1\text{-}5)$$

And the adjoint problem about $\mathbf{G}$ is formulated as,

$$\mathbf{L^*G} = -\mathbf{G}_\tau - \mathbf{VG}_\xi - \mathbf{DG}_{\xi\xi} = \delta(\xi - \mathbf{x}, \tau - \mathbf{t}),\ 0 < \xi < \infty,\ 0 < \tau < \infty, \qquad (4\text{-}1\text{-}6a)$$

$$\mathbf{G}(0,\tau) = 0. \qquad (4\text{-}1\text{-}6b)$$

4.2 Principle solution

Based on the basic principle of differential equations, the problem (4-1-6) can be divided into two problems defined on the infinite space. One is to solve only the inhomogeneous differential equation without considering any boundary conditions, the solution is called principle solution, say, $\mathbf{U}$; the other is to solve the homogeneous differential equation with the given boundary conditions, the solution is called regular solution, say, $\mathbf{g}$. Then the solution of the original problem (4-1-6) is $\mathbf{U}$ plus $\mathbf{g}$, namely,

$$\mathbf{G}(\xi,\tau;\mathbf{x},\mathbf{t}) = \mathbf{U}(\xi,\tau;\mathbf{x},\mathbf{t}) + \mathbf{g}(\xi,\tau;\mathbf{x},\mathbf{t}). \qquad (4\text{-}2\text{-}1)$$

The two problems are formulated as,

$$\mathbf{L^*U} = -\mathbf{U}_\tau - \mathbf{VU}_\xi - \mathbf{DU}_{\xi\xi} = \delta(\xi - \mathbf{x}, \tau - \mathbf{t}),\ -\infty < \xi < \infty,\ 0 < \tau < \infty, \qquad (4\text{-}2\text{-}2)$$

and,

$$\mathbf{L^*g} = -\mathbf{g}_\tau - \mathbf{Vg}_\xi - \mathbf{Dg}_{\xi\xi} = 0,\ -\infty < \xi < \infty,\ 0 < \tau < \infty, \qquad (4\text{-}2\text{-}3a)$$

$$\mathbf{g}(0,\tau) = (\mathbf{G} - \mathbf{U})\big|_{\xi=0}, \qquad (4\text{-}2\text{-}3b)$$

respectively.

The main task of this subsection is to solve problem (4-2-2) to obtain the principle solution $\mathbf{U}$.

In order to solve equation (4-2-2), Fourier transform is performed on the equation from ξ to ω. Multiply $\mathbf{exp(-i\omega\xi)d\xi}$ on both sides of equation (4-2-2), and integrate from $-\infty$ to ∞,

$$-\int_{-\infty}^\infty \mathbf{U}_\tau \mathbf{exp(-i\omega\xi)d\xi} - \mathbf{V}\int_{-\infty}^\infty \mathbf{U}_\xi \mathbf{exp(-i\omega\xi)d\xi} - \mathbf{D}\int_{-\infty}^\infty \mathbf{U}_{\xi\xi}\mathbf{exp(-i\omega\xi)d\xi}$$

$$= \int_{-\infty}^{\infty} \delta(\xi - x, \tau - t)\exp(-i\omega\xi)d\xi$$

$$= \delta(\tau - t)\int_{-\infty}^{\infty} \delta(\xi - x)\exp(-i\omega\xi)d\xi$$

$$= \delta(\tau - t)\exp(-i\omega x)\,. \qquad (4\text{-}2\text{-}4)$$

Define

$$\hat{U}(\omega,\tau) \equiv \int_{-\infty}^{\infty} U(\xi,\tau)\exp(-i\omega\xi)d\xi\,. \qquad (4\text{-}2\text{-}5)$$

Apply the properties about Fourier transform to the left hand side of (4-2-4), it is reduced to,

$$-\frac{d\hat{U}}{d\tau} - i\omega V\hat{U} + D\omega^2\hat{U} = \delta(\tau - t)\exp(-i\omega x)\text{, namely,}$$

$$-\frac{d\hat{U}}{d\tau} + (D\omega^2 - i\omega V)\hat{U} = \delta(\tau - t)\exp(-i\omega x)\,. \qquad (4\text{-}2\text{-}6)$$

According to definition of delta function, the right hand side of (4-2-6) is equal to zero for $\tau > t$ and $\tau < t$, namely,

$$-\frac{d\hat{U}}{d\tau} + (D\omega^2 - i\omega V)\hat{U} = 0\text{, if } \tau > t \text{ or } \tau < t\,. \qquad (4\text{-}2\text{-}7)$$

Separate variables between $\hat{U}$ and τ, then integrate (4-2-7),

$$\hat{U} = \begin{cases} A_1 \exp[(D\omega^2 - iV\omega)\tau], \text{if} \tau > t, \\ A_2 \exp[(D\omega^2 - iV\omega)\tau], \text{if} \tau < t, \end{cases} \qquad (4\text{-}2\text{-}8)$$

where A_1 and A_2 are undetermined integration constants. To determine them, make integration of (4-2-6) on τ from $t - 0$ to $t + 0$,

$$-\hat{U}\Big|_{t-0}^{t+0} + (D\omega^2 - i\omega V)\int_{t-0}^{t+0} \hat{U}d\tau = \exp(-i\omega x)\int_{t-0}^{t+0} \delta(\tau - t)d\tau\,. \qquad (4\text{-}2\text{-}9)$$

The second term on the left hand side of (4-2-9) is zero because the integrand $\hat{U}$ is finite anyhow and the interval is infinitesimal. Thus the singularity caused by the delta function must be embodied on the first term. Using the property of delta function again $\int_{t-0}^{t+0} \delta(\tau - t)d\tau = 1$ and substituting (4-2-8) into (4-2-9), it becomes as,

$$A_2 \exp[(D\omega^2 - iV\omega)t] - A_1 \exp[(D\omega^2 - iV\omega)t] = \exp(-i\omega x)\,. \qquad (4\text{-}2\text{-}10)$$

Remember that U hence $\hat{U}$ stands for the "heat" distribution along the REVERSED time coordinate, caused by the instantaneous "heat" pulse at the time $\tau = t$, therefore,

$\hat{U} = 0$, if $\tau > t$. Namely $A_1 = 0$. According to (4-2-10), we have,

$$\mathbf{A}_2 = \mathbf{exp[-i\omega x - (D\omega^2 - iV\omega)t]}. \tag{4-2-11}$$

Substitute $\mathbf{A}_1$ and $\mathbf{A}_2$ into (4-2-8), the solution of (4-2-6) is obtained as,

$$\hat{\mathbf{U}} = \mathbf{exp[-i\omega x - (D\omega^2 - iV\omega)(t-\tau)]H(t-\tau)}. \tag{4-2-12}$$

Make inverse Fourier transform of (4-2-12) from $\boldsymbol{\omega}$ to $\boldsymbol{\xi}$,

$$\mathbf{U(\xi,\tau)} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{\mathbf{U}}\,\mathbf{exp(i\omega\xi)d\omega}$$

$$= \frac{\mathbf{H(t-\tau)}}{2\pi}\int_{-\infty}^{\infty}\mathbf{exp[i\omega(\xi-x) - (D\omega^2 - iV\omega)(t-\tau)]d\omega}$$

$$= \frac{\mathbf{H(t-\tau)}}{2\pi}\int_{-\infty}^{\infty}\mathbf{exp}\left\{-\mathbf{D(t-\tau)}\left[\boldsymbol{\omega}^2 - \mathbf{i}\frac{\mathbf{(\xi-x)+V(t-\tau)}}{\mathbf{D(t-\tau)}}\boldsymbol{\omega}\right]\right\}\mathbf{d\omega}. \tag{4-2-13}$$

Let,

$$2\mathbf{c} = \mathbf{i}\frac{\mathbf{(\xi-x)+V(t-\tau)}}{\mathbf{D(t-\tau)}}, \text{ namely,} \tag{4-2-14}$$

$$\mathbf{c} = \mathbf{i}\frac{\mathbf{(\xi-x)+V(t-\tau)}}{2\mathbf{D(t-\tau)}}. \tag{4-2-15}$$

Substitute (4-2-14) into (4-2-13), continue to simplify,

$$\mathbf{U(\xi,\tau)} = \frac{\mathbf{H(t-\tau)}}{2\pi}\int_{-\infty}^{\infty}\mathbf{exp[-D(t-\tau)(\omega^2 - 2c\omega)]d\omega}$$

$$= \frac{\mathbf{H(t-\tau)}}{2\pi}\int_{-\infty}^{\infty}\mathbf{exp\{-D(t-\tau)[(\omega-c)^2 - c^2]\}d\omega}. \tag{4-2-16}$$

Let,

$\boldsymbol{\zeta} = \sqrt{\mathbf{D(t-\tau)}}\mathbf{(\omega-c)}$, then $\mathbf{d\omega} = \frac{1}{\sqrt{\mathbf{D(t-\tau)}}}\mathbf{d\zeta}$. Substitute them into (4-2-16),

$$\mathbf{U(\xi,\tau)} = \frac{\mathbf{H(t-\tau)}}{2\pi}\int_{-\infty}^{\infty}\mathbf{exp(-\zeta^2)\cdot exp[D(t-\tau)c^2]}\cdot\frac{1}{\sqrt{\mathbf{D(t-\tau)}}}\mathbf{d\zeta}$$

$$= \frac{\mathbf{H(t-\tau)}}{2\pi}\mathbf{exp[D(t-\tau)c^2]}\cdot\frac{1}{\sqrt{\mathbf{D(t-\tau)}}}\int_{-\infty}^{\infty}\mathbf{exp(-\zeta^2)d\zeta}. \tag{4-2-17}$$

Substitute (4-2-15) into (4-2-17) and use the equality $\int_{-\infty}^{\infty}\mathbf{exp(-\zeta^2)d\zeta} = \sqrt{\pi}$, then

$$\mathbf{U(\xi,\tau;x,t)} = \frac{\mathbf{H(t-\tau)}}{\sqrt{4\pi\mathbf{D(t-\tau)}}}\mathbf{exp}\left[-\frac{\mathbf{[\xi-x+V(t-\tau)]}^2}{4\mathbf{D(t-\tau)}}\right]. \tag{4-2-18}$$

This is the solution of (4-2-2), also the principle solution of (4-1-6).

4.3 Regular solution

Now it is the time to solve problem (4-2-3) to obtain the regular solution for problem (4-1-6). The boundary condition for **g** can be reformulated based on the obtained principle solution. According to (4-2-3b),

$$\mathbf{g}(0,\tau) = \mathbf{G}(0,\tau) - \mathbf{U}(0,\tau)\,. \tag{4-3-1}$$

According to (4-1-6b), $\mathbf{G}(0,\tau) = 0$. According to (4-2-18),

$$\mathbf{U}(0,\tau) = \frac{\mathbf{H}(\mathbf{t}-\tau)}{\sqrt{4\pi\mathbf{D}(\mathbf{t}-\tau)}}\mathbf{exp}\left[-\frac{[\mathbf{x}-\mathbf{V}(\mathbf{t}-\tau)]^2}{4\mathbf{D}(\mathbf{t}-\tau)}\right]. \tag{4-3-2}$$

So problem (4-2-3) can be reformulated as,

$$\mathbf{L}^*\mathbf{g} = -\mathbf{g}_\tau - \mathbf{V}\mathbf{g}_\xi - \mathbf{D}\mathbf{g}_{\xi\xi} = 0\,,\ -\infty < \xi < \infty\,,\ 0 < \tau < \infty\,, \tag{4-3-3a}$$

$$\mathbf{g}(0,\tau) = \frac{-\mathbf{H}(\mathbf{t}-\tau)}{\sqrt{4\pi\mathbf{D}(\mathbf{t}-\tau)}}\mathbf{exp}\left[-\frac{[\mathbf{x}-\mathbf{V}(\mathbf{t}-\tau)]^2}{4\mathbf{D}(\mathbf{t}-\tau)}\right]. \tag{4-3-3b}$$

Inspection approach is of much importance and is widely used to solve the regular solution in the GFM. It needs to understand the solutions in view of physics. Keep in mind that, in physics, the principle solution about the Green's function, $\mathbf{U}(\xi,\tau;\mathbf{x},\mathbf{t})$, of the advection diffusion equation describes the heat distribution at (ξ,τ) caused by a positive unit heat pulse at $(\mathbf{x},\mathbf{t})$. In the specific problem (4-1-6), the Green's function value is required to be zero at the origin. In order to satisfy the condition, an image of negative heat source can be designed at $(-\mathbf{x},\mathbf{t})$, as shown in Figure 4-3-1, to compensate the positive value being diffused from $(\mathbf{x},\mathbf{t})$. The strength of the negative source is unclear so far because of the reversed advection effects, but it can be assumed as γ times of unit and γ may vary with (ξ,τ), namely, $\gamma = \gamma(\xi,\tau)$. Based on the inspection, the solution of problem (4-3-3) is anticipated to be in the form of,

$$\mathbf{g}(\xi,\tau) = -\gamma(\xi,\tau)\mathbf{U}(\xi,\tau;-\mathbf{x},\mathbf{t})\,. \tag{4-3-4}$$

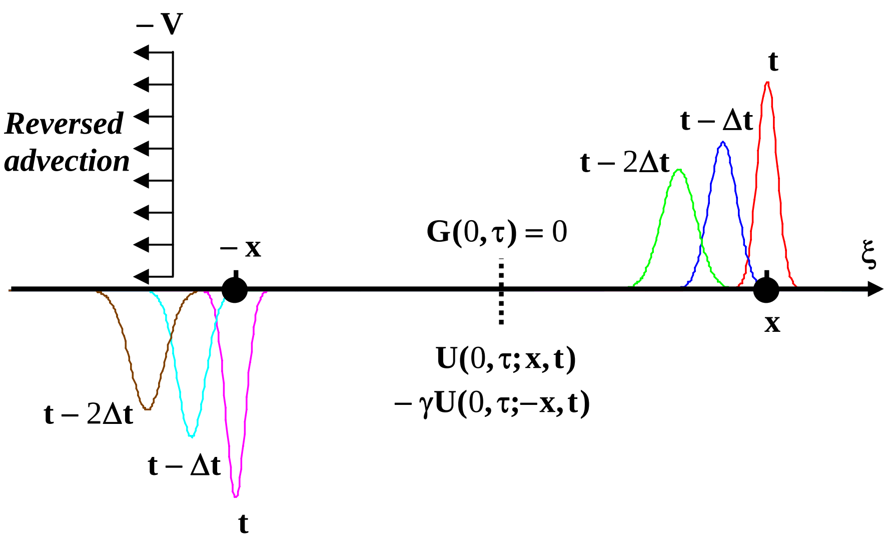


Figure 4-3-1 Schematic image system for advection diffusion problem with Dirichlet boundary condition

The boundary condition of $\mathbf{g}$ in (4-3-3b) is applied to determine $\gamma = \gamma(\xi, \tau)$. According (4-3-4) and (4-2-18), we have,

$$\mathbf{g}(0,\tau) = -\gamma(0,\tau)\mathbf{U}(0,\tau;-\mathbf{x},\mathbf{t}) = -\gamma(0,\tau)\frac{\mathbf{H}(\mathbf{t}-\tau)}{\sqrt{4\pi\mathbf{D}(\mathbf{t}-\tau)}}\mathbf{exp}\left[-\frac{[\mathbf{x}+\mathbf{V}(\mathbf{t}-\tau)]^2}{4\mathbf{D}(\mathbf{t}-\tau)}\right]$$

$$= -\gamma(0,\tau)\frac{\mathbf{H}(\mathbf{t}-\tau)}{\sqrt{4\pi\mathbf{D}(\mathbf{t}-\tau)}}\mathbf{exp}\left[-\frac{[\mathbf{x}-\mathbf{V}(\mathbf{t}-\tau)]^2+4\mathbf{Vx}(\mathbf{t}-\tau)}{4\mathbf{D}(\mathbf{t}-\tau)}\right]$$

$$= \frac{-\mathbf{H}(\mathbf{t}-\tau)}{\sqrt{4\pi\mathbf{D}(\mathbf{t}-\tau)}}\mathbf{exp}\left[-\frac{[\mathbf{x}-\mathbf{V}(\mathbf{t}-\tau)]^2}{4\mathbf{D}(\mathbf{t}-\tau)}\right]\cdot\mathbf{exp}\left(-\frac{\mathbf{Vx}}{\mathbf{D}}\right)\cdot\gamma(0,\tau)$$

$$= \mathbf{g}(0,\tau)\cdot\mathbf{exp}\left(-\frac{\mathbf{Vx}}{\mathbf{D}}\right)\cdot\gamma(0,\tau). \qquad (4\text{-}3\text{-}5)$$

The last step is based on (4-3-3b), therefore we have,

$$\gamma(0,\tau) = \mathbf{exp}\left(\frac{\mathbf{Vx}}{\mathbf{D}}\right). \qquad (4\text{-}3\text{-}6)$$

γ is a function defined on the (ξ,τ) plane, but the right hand side of (4-3-6) is nothing about ξ and τ at all, just a “constant” term. Thus it must be,

$$\gamma(\xi,\tau) \equiv \mathbf{exp}\left(\frac{\mathbf{Vx}}{\mathbf{D}}\right). \qquad (4\text{-}3\text{-}7)$$

So far the regular solution of $\mathbf{g}$ is obtained,

$\mathbf{g}(\xi,\tau;\mathbf{x},\mathbf{t}) = -\mathbf{exp}\left(\frac{\mathbf{Vx}}{\mathbf{D}}\right)\cdot\mathbf{U}(\xi,\tau;-\mathbf{x},\mathbf{t})$, namely,

$$\mathbf{g}(\xi,\tau;\mathbf{x},\mathbf{t}) = -\frac{\mathbf{H}(\mathbf{t}-\tau)}{\sqrt{4\pi\mathbf{D}(\mathbf{t}-\tau)}}\cdot\mathbf{exp}\left(\frac{\mathbf{Vx}}{\mathbf{D}}\right)\cdot\mathbf{exp}\left[-\frac{[\xi+\mathbf{x}+\mathbf{V}(\mathbf{t}-\tau)]^2}{4\mathbf{D}(\mathbf{t}-\tau)}\right]. \qquad (4\text{-}3\text{-}8)$$

The solution (4-3-8) is obtained by inspection approach. Is it really the solution of (4-3-3)? The answer is definitely positive. It is proofed as follows.

First, it is obvious that (4-3-8) satisfies the boundary condition of (4-3-3b). Then, it should be proofed to satisfy the differential equation of (4-3-3a) also.

Based on (4-3-8), we have,

$$\mathbf{g}_\xi = \mathbf{g}\cdot\left[-\frac{\xi+\mathbf{x}+\mathbf{V}(\mathbf{t}-\tau)}{2\mathbf{D}(\mathbf{t}-\tau)}\right], \qquad (4\text{-}3\text{-}9)$$

$$\mathbf{g}_{\xi\xi} = \mathbf{g}\cdot\left[\frac{[\xi+\mathbf{x}+\mathbf{V}(\mathbf{t}-\tau)]^2}{4\mathbf{D}^2(\mathbf{t}-\tau)^2}-\frac{1}{2\mathbf{D}(\mathbf{t}-\tau)}\right], \qquad (4\text{-}3\text{-}10)$$

$$\mathbf{g}_\tau = \mathbf{g}\cdot\left[\frac{1}{2(\mathbf{t}-\tau)}-\frac{(\xi+\mathbf{x})^2}{4\mathbf{D}(\mathbf{t}-\tau)^2}+\frac{\mathbf{V}^2}{4\mathbf{D}}\right]. \qquad (4\text{-}3\text{-}11)$$

Thus,

$$-\mathbf{g}_\tau - \mathbf{V}\mathbf{g}_\xi - \mathbf{D}\mathbf{g}_{\xi\xi} = -\mathbf{g}\cdot\{\frac{1}{2(\mathbf{t}-\tau)} - \frac{(\xi+\mathbf{x})^2}{4\mathbf{D}(\mathbf{t}-\tau)^2} + \frac{\mathbf{V}^2}{4\mathbf{D}} - \frac{\mathbf{V}(\xi+\mathbf{x})+\mathbf{V}^2(\mathbf{t}-\tau)}{2\mathbf{D}(\mathbf{t}-\tau)} +$$

$$+\frac{[\xi+\mathbf{x}+\mathbf{V}(\mathbf{t}-\tau)]^2}{4\mathbf{D}(\mathbf{t}-\tau)^2} - \frac{1}{2(\mathbf{t}-\tau)}\}. \qquad (4\text{-}3\text{-}12)$$

The sum of the terms in "{}" of (4-3-12) is zero by straightforward calculations. It does mean that the differential equation of (4-3-3a) is met also by (4-3-8). So far it is proofed that the solution (4-3-8) obtained by observation in view of physics is genuinely the solution of the problem (4-3-3).

Let's come back to the problem (4-1-6), both the principle and regular solutions are obtained. According to (4-2-1), (4-2-18) and (4-3-8), the Green's function being sought is expressed as,

$$\mathbf{G}(\xi,\tau;\mathbf{x},\mathbf{t}) = \frac{\mathbf{H}(\mathbf{t}-\tau)}{\sqrt{4\pi\mathbf{D}(\mathbf{t}-\tau)}}\left\{\mathbf{exp}\left[-\frac{[\xi-\mathbf{x}+\mathbf{V}(\mathbf{t}-\tau)]^2}{4\mathbf{D}(\mathbf{t}-\tau)}\right] - \mathbf{exp}\left[\frac{\mathbf{V}\mathbf{x}}{\mathbf{D}} - \frac{[\xi+\mathbf{x}+\mathbf{V}(\mathbf{t}-\tau)]^2}{4\mathbf{D}(\mathbf{t}-\tau)}\right]\right\} \qquad (4\text{-}3\text{-}13)$$

or in a succinct form,

$$\mathbf{G}(\xi,\tau;\mathbf{x},\mathbf{t}) = \mathbf{U}(\xi,\tau;\mathbf{x},\mathbf{t}) - \mathbf{exp}\left(\frac{\mathbf{V}\mathbf{x}}{\mathbf{D}}\right)\mathbf{U}(\xi,\tau;-\mathbf{x},\mathbf{t}), \qquad (4\text{-}3\text{-}14)$$

where $\mathbf{U}(\xi,\tau;\mathbf{x},\mathbf{t})$ is expressed in (4-2-18). The solution (4-3-13) or (4-3-14) is the Green's function in semi-infinite domain with a Dirichlet boundary condition. Once the Green's function is determined, then the solution of the original problem (4-1-1) can be obtained by the integration expression of (4-1-5).

To verify the analytical solution, a numerical scheme is established to solve numerically the 1D advection diffusion equation by using a C program. If it is purely a boundary value problem with the Dirichlet type, say, $\mathbf{u}(0,\mathbf{t}) = \mathbf{h}(\mathbf{t}) = \mathbf{sin}(0.4\pi\mathbf{t})$, and $\phi(\mathbf{x},\mathbf{t}) = 0$, $\mathbf{u}(\mathbf{x},0) = \mathbf{f}(\mathbf{x}) = 0$, then the analytical solution can be obtained based on (4-1-5) and (4-3-13) as,

$$\mathbf{u}(\mathbf{x},\mathbf{t}) = \int_0^{\mathbf{t}} \mathbf{D}\,\mathbf{sin}(0.4\pi\tau)\frac{1}{\sqrt{4\pi\mathbf{D}(\mathbf{t}-\tau)}}\{\mathbf{exp}\left[-\frac{[\mathbf{x}-\mathbf{V}(\mathbf{t}-\tau)]^2}{4\mathbf{D}(\mathbf{t}-\tau)}\right]\cdot\left[\frac{\mathbf{x}-\mathbf{V}(\mathbf{t}-\tau)}{2\mathbf{D}(\mathbf{t}-\tau)}\right] +$$

$$+\mathbf{exp}\left[\frac{\mathbf{V}\mathbf{x}}{\mathbf{D}} - \frac{[\mathbf{x}+\mathbf{V}(\mathbf{t}-\tau)]^2}{4\mathbf{D}(\mathbf{t}-\tau)}\right]\cdot\left[\frac{\mathbf{x}+\mathbf{V}(\mathbf{t}-\tau)}{2\mathbf{D}(\mathbf{t}-\tau)}\right]\}\mathbf{d}\tau.$$

By using Mathcad, the analytical solutions at different values of $\mathbf{t}$ are plotted as solid lines in Figure 4-3-2, while the symbols stand for the corresponding numerical solutions. The high agreement between them supplies a good verification to the Green's function obtained in this section.

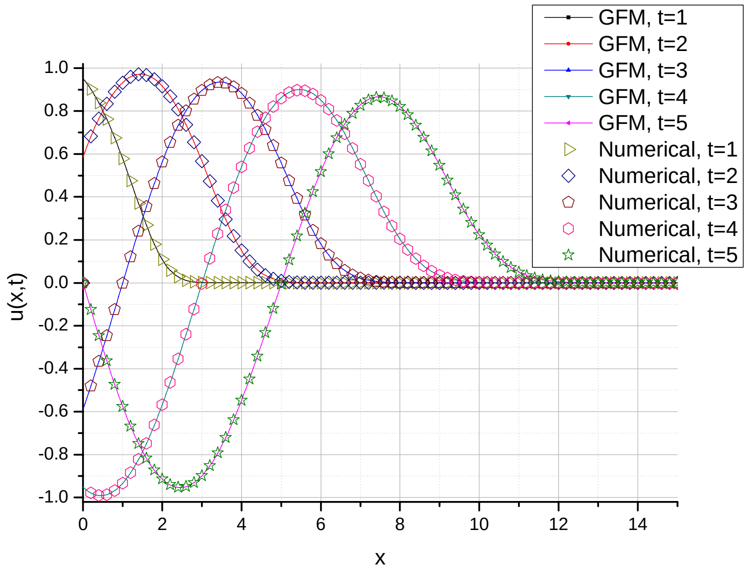


Figure 4-3-2 Verification of Green's function in semi-infinite domain with Dirichlet boundary condition

4.4 Neumann and Robin boundaries

The advection diffusion problem with a Dirichlet boundary condition is solved in last subsections. Actually, the boundary conditions can be given in another form in engineering, e.g., the derivative of the function instead of the function value, or an expression of combination of the derivative and the function values. The former is called a Neumann type or second type of boundary, the latter is called a Robin boundary, or a mixed type or third type of boundary in some literatures. The aim of this subsection is to solve the advection diffusion problem with more general Robin boundary conditions in a semi-infinite domain.

The formal problem is described as,

$$\mathbf{Lu = u_t + Vu_x - Du_{xx} = \phi(x,t)},\ 0 < \mathbf{x} < \infty,\ 0 < \mathbf{t} < \infty, \qquad (4\text{-}4\text{-}1a)$$

$$\mathbf{u(x,0) = f(x)}, \qquad (4\text{-}4\text{-}1b)$$

$$\mathbf{\alpha u(0,t) + \beta u_x(0,t) = h(t)},\ (\beta \neq 0). \qquad (4\text{-}4\text{-}1c)$$

If $\beta = 0$, it is actually a Dirichlet boundary, which is discussed in last subsections. So only the case of $\beta \neq 0$ is considered here. If $\alpha = 0$, it changes to a Neumann boundary. Namely, the Neumann boundary is a special case of the Robin boundary. The general expression of the solution is already given in (4-1-4),

$$\mathbf{u(x,t)} = \int_0^\infty\int_0^\infty \mathbf{G(\xi,\tau)\phi(\xi,\tau)d\xi d\tau} + \int_0^\infty \mathbf{G(\xi,0)u(\xi,0)d\xi} +$$

$$+\int_0^t \mathbf{[VG(0,\tau)u(0,\tau) - DG(0,\tau)u_\xi(0,\tau) + DG_\xi(0,\tau)u(0,\tau)]d\tau}. \qquad (4\text{-}1\text{-}4)$$

Rearrange (4-4-1c) as,

$$\mathbf{u_x(0,t)} = \frac{1}{\beta}\mathbf{[h(t) - \alpha u(0,t)]}, \text{ namely,} \qquad (4\text{-}4\text{-}2)$$

$$\mathbf{u}_\xi(0,\tau) = \frac{1}{\beta}[\mathbf{h}(\tau) - \alpha\mathbf{u}(0,\tau)]. \qquad (4\text{-}4\text{-}3)$$

Substitute (4-4-3) into (4-1-4),

$$\mathbf{u}(\mathbf{x},\mathbf{t}) = \int_0^\infty\int_0^\infty \mathbf{G}(\xi,\tau)\phi(\xi,\tau)\mathbf{d}\xi\mathbf{d}\tau + \int_0^\infty \mathbf{G}(\xi,0)\mathbf{u}(\xi,0)\mathbf{d}\xi +$$

$$+\int_0^{\mathbf{t}} \{\mathbf{VG}(0,\tau)\mathbf{u}(0,\tau) - \mathbf{DG}(0,\tau)\frac{1}{\beta}[\mathbf{h}(\tau) - \alpha\mathbf{u}(0,\tau)] + \mathbf{DG}_\xi(0,\tau)\mathbf{u}(0,\tau)\}\mathbf{d}\tau$$

$$= \int_0^\infty\int_0^\infty \mathbf{G}(\xi,\tau)\phi(\xi,\tau)\mathbf{d}\xi\mathbf{d}\tau + \int_0^\infty \mathbf{G}(\xi,0)\mathbf{u}(\xi,0)\mathbf{d}\xi +$$

$$+\int_0^{\mathbf{t}} \{[(\mathbf{V} + \frac{\alpha}{\beta}\mathbf{D})\mathbf{G}(0,\tau) + \mathbf{DG}_\xi(0,\tau)]\mathbf{u}(0,\tau) - \frac{\mathbf{D}}{\beta}\mathbf{G}(0,\tau)\mathbf{h}(\tau)\}\mathbf{d}\tau. \qquad (4\text{-}4\text{-}4)$$

In order that the unwelcome term "$[(\mathbf{V} + \frac{\alpha}{\beta}\mathbf{D})\mathbf{G}(0,\tau) + \mathbf{DG}_\xi(0,\tau)]\mathbf{u}(0,\tau)$" in (4-4-4) can vanish, Green's function has to satisfy the boundary condition,

$(\mathbf{V} + \frac{\alpha}{\beta}\mathbf{D})\mathbf{G}(0,\tau) + \mathbf{DG}_\xi(0,\tau) = 0$, namely, $(\alpha\mathbf{D} + \beta\mathbf{V})\mathbf{G}(0,\tau) + \beta\mathbf{DG}_\xi(0,\tau) = 0$.

If so, (4-4-4) can be further reduced to,

$$\mathbf{u}(\mathbf{x},\mathbf{t}) = \int_0^\infty\int_0^\infty \mathbf{G}(\xi,\tau)\phi(\xi,\tau)\mathbf{d}\xi\mathbf{d}\tau + \int_0^\infty \mathbf{G}(\xi,0)\mathbf{f}(\xi)\mathbf{d}\xi - \int_0^{\mathbf{t}} [\frac{\mathbf{D}}{\beta}\mathbf{G}(0,\tau)\mathbf{h}(\tau)]\mathbf{d}\tau. \qquad (4\text{-}4\text{-}5)$$

Therefore the boundary value problem for $\mathbf{G}(\xi,\tau)$ is formulated as,

$$\mathbf{L}^*\mathbf{G} = -\mathbf{G}_\tau - \mathbf{VG}_\xi - \mathbf{DG}_{\xi\xi} = \delta(\xi - \mathbf{x}, \tau - \mathbf{t}),\ 0 < \xi < \infty,\ 0 < \tau < \infty, \qquad (4\text{-}4\text{-}6a)$$

$$(\alpha\mathbf{D} + \beta\mathbf{V})\mathbf{G}(0,\tau) + \beta\mathbf{DG}_\xi(0,\tau) = 0. \qquad (4\text{-}4\text{-}6b)$$

Similar to solving the Dirichlet boundary problem, Green's function $\mathbf{G}$ is sought to be the sum of principle solution $\mathbf{U}$ and regular solution $\mathbf{g}$, namely,

$$\mathbf{G}(\xi,\tau;\mathbf{x},\mathbf{t}) = \mathbf{U}(\xi,\tau;\mathbf{x},\mathbf{t}) + \mathbf{g}(\xi,\tau;\mathbf{x},\mathbf{t}). \qquad (4\text{-}2\text{-}1)$$

The principle solution $\mathbf{U}(\xi,\tau;\mathbf{x},\mathbf{t})$ is obtained already in Section 4.2. By means of inspection approach again, the regular solution $\mathbf{g}(\xi,\tau;\mathbf{x},\mathbf{t})$ is constructed by an image system, as show in Figure 4-4-1. [*Similar image system is used to solve the normal diffusion problem without advection in reference (Greenberg, 1971). The image system depicted here is an extension of that one.*]

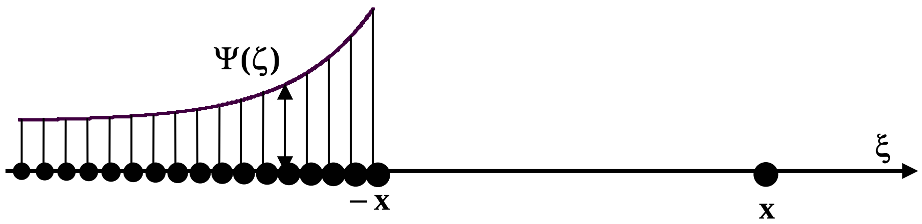


Figure 4-4-1 Schematic image system for advection diffusion problem with Robin boundary condition

Based on the layout of source image, the regular solution can be in the form of,

$$\mathbf{g}(\xi,\tau;\mathbf{x},\mathbf{t}) = \gamma\mathbf{U}(\xi,\tau;-\mathbf{x},\mathbf{t}) + \int_{-\infty}^{-\mathbf{x}} \psi(\zeta)\mathbf{U}(\xi,\tau;\zeta,\mathbf{t})\mathbf{d}\zeta\,. \tag{4-4-7}$$

Substitute (4-4-7) to (4-2-1),

$$\mathbf{G}(\xi,\tau;\mathbf{x},\mathbf{t}) = \mathbf{U}(\xi,\tau;\mathbf{x},\mathbf{t}) + \gamma\mathbf{U}(\xi,\tau;-\mathbf{x},\mathbf{t}) + \int_{-\infty}^{-\mathbf{x}} \psi(\zeta)\mathbf{U}(\xi,\tau;\zeta,\mathbf{t})\mathbf{d}\zeta\,. \tag{4-4-8}$$

Now let's apply the boundary condition (4-4-6b) to determine γ and $\psi(\zeta)$.

In terms of (4-4-8),

$$\begin{aligned}\mathbf{G}(0,\tau;\mathbf{x},\mathbf{t}) &= \mathbf{U}(0,\tau;\mathbf{x},\mathbf{t}) + \gamma\mathbf{U}(0,\tau;-\mathbf{x},\mathbf{t}) + \int_{-\infty}^{-\mathbf{x}} \psi(\zeta)\mathbf{U}(0,\tau;\zeta,\mathbf{t})\mathbf{d}\zeta \\ &= \mathbf{U}(0,\tau;\mathbf{x},\mathbf{t}) + \mathbf{U}(0,\tau;\mathbf{x},\mathbf{t})\,\mathbf{exp}\left(-\frac{\mathbf{Vx}}{\mathbf{D}}\right)\gamma + \int_{-\infty}^{-\mathbf{x}} \psi(\zeta)\mathbf{U}(0,\tau;\zeta,\mathbf{t})\mathbf{d}\zeta\,. \end{aligned} \tag{4-4-9}$$

So,

$$\begin{aligned}(\alpha\mathbf{D}+\beta\mathbf{V})\mathbf{G}(0,\tau;\mathbf{x},\mathbf{t}) &= \mathbf{U}(0,\tau;\mathbf{x},\mathbf{t})(\alpha\mathbf{D}+\beta\mathbf{V})\left[1+\mathbf{exp}\left(-\frac{\mathbf{Vx}}{\mathbf{D}}\right)\gamma\right] + \\ &+ \int_{-\infty}^{-\mathbf{x}} \mathbf{U}(0,\tau;\zeta,\mathbf{t})(\alpha\mathbf{D}+\beta\mathbf{V})\psi(\zeta)\mathbf{d}\zeta\,. \end{aligned} \tag{4-4-10}$$

On the other hand,

$$\begin{aligned}\mathbf{G}_\xi(0,\tau;\mathbf{x},\mathbf{t}) &= \mathbf{U}(0,\tau;\mathbf{x},\mathbf{t})\frac{\mathbf{x}-\mathbf{V}\tau}{2\mathbf{D}\tau} - \\ &- \mathbf{U}(0,\tau;\mathbf{x},\mathbf{t})\frac{\mathbf{x}-\mathbf{V}\tau}{2\mathbf{D}\tau}\cdot\mathbf{exp}\left(-\frac{\mathbf{Vx}}{\mathbf{D}}\right)\gamma - \mathbf{U}(0,\tau;\mathbf{x},\mathbf{t})\,\mathbf{exp}\left(-\frac{\mathbf{Vx}}{\mathbf{D}}\right)\gamma\frac{\mathbf{V}}{\mathbf{D}} + \\ &+ \mathbf{U}(0,\tau;\mathbf{x},\mathbf{t})\,\mathbf{exp}\left(-\frac{\mathbf{Vx}}{\mathbf{D}}\right)\cdot[-\psi(-\mathbf{x})] + \int_{-\infty}^{-\mathbf{x}} \mathbf{U}(0,\tau;\zeta,\mathbf{t})\psi'(\zeta)\mathbf{d}\zeta\,. \end{aligned} \tag{4-4-11}$$

Then,

$$\beta\mathbf{D}\mathbf{G}_\xi(0,\tau;\mathbf{x},\mathbf{t}) = \mathbf{U}(0,\tau;\mathbf{x},\mathbf{t})\frac{\mathbf{x}-\mathbf{V}\tau}{2\mathbf{D}\tau}\cdot\beta\mathbf{D}\left[1-\mathbf{exp}\left(-\frac{\mathbf{Vx}}{\mathbf{D}}\right)\gamma\right] -$$

$$-\mathbf{U}(0,\tau;\mathbf{x},\mathbf{t})\mathbf{exp}\left(-\frac{\mathbf{Vx}}{\mathbf{D}}\right)[\gamma\beta\mathbf{V}+\beta\mathbf{D}\psi(-\mathbf{x})]+\int_{-\infty}^{-\mathbf{x}}\mathbf{U}(0,\tau;\zeta,\mathbf{t})\beta\mathbf{D}\psi'(\zeta)\mathbf{d}\zeta\,. \qquad (4\text{-}4\text{-}12)$$

Equality (4-4-10) plus (4-4-12),

$$(\alpha\mathbf{D}+\beta\mathbf{V})\mathbf{G}(0,\tau;\mathbf{x},\mathbf{t})+\beta\mathbf{D}\mathbf{G}_\xi(0,\tau;\mathbf{x},\mathbf{t})=\mathbf{U}(0,\tau;\mathbf{x},\mathbf{t})\frac{\mathbf{x}-\mathbf{V}\tau}{2\mathbf{D}\tau}\cdot\Phi_1+$$

$$+\mathbf{U}(0,\tau;\mathbf{x},\mathbf{t})\cdot\Phi_2+\int_{-\infty}^{-\mathbf{x}}\mathbf{U}(0,\tau;\zeta,\mathbf{t})\cdot\Phi_3\mathbf{d}\zeta\,, \qquad (4\text{-}4\text{-}13)$$

where, Φ_1, Φ_2 and Φ_3 are temporary variables,

$$\Phi_1=\beta\mathbf{D}\left[1-\mathbf{exp}\left(-\frac{\mathbf{Vx}}{\mathbf{D}}\right)\gamma\right], \qquad (4\text{-}4\text{-}14)$$

$$\Phi_2=(\alpha\mathbf{D}+\beta\mathbf{V})\left[1+\mathbf{exp}\left(-\frac{\mathbf{Vx}}{\mathbf{D}}\right)\gamma\right]-\mathbf{exp}\left(-\frac{\mathbf{Vx}}{\mathbf{D}}\right)[\gamma\beta\mathbf{V}+\beta\mathbf{D}\Psi(-\mathbf{x})], \qquad (4\text{-}4\text{-}15)$$

$$\Phi_3=(\alpha\mathbf{D}+\beta\mathbf{V})\psi(\zeta)+\beta\mathbf{D}\psi'(\zeta)\,. \qquad (4\text{-}4\text{-}16)$$

According to boundary condition (4-4-6b), the left hand side of (4-4-13) is zero. Because equality (4-4-13) holds for any τ, it must have $\Phi_1=\Phi_2=\Phi_3=0$. This results in three equations about γ and $\psi(\zeta)$,

$$\beta\mathbf{D}\left[1-\mathbf{exp}\left(-\frac{\mathbf{Vx}}{\mathbf{D}}\right)\gamma\right]=0\,, \qquad (4\text{-}4\text{-}14a)$$

$$(\alpha\mathbf{D}+\beta\mathbf{V})\left[1+\mathbf{exp}\left(-\frac{\mathbf{Vx}}{\mathbf{D}}\right)\gamma\right]-\mathbf{exp}\left(-\frac{\mathbf{Vx}}{\mathbf{D}}\right)[\gamma\beta\mathbf{V}+\beta\mathbf{D}\Psi(-\mathbf{x})]=0\,, \qquad (4\text{-}4\text{-}15a)$$

$$(\alpha\mathbf{D}+\beta\mathbf{V})\psi(\zeta)+\beta\mathbf{D}\psi'(\zeta)=0\,. \qquad (4\text{-}4\text{-}16a)$$

Because $\beta\neq 0$ and obviously $\mathbf{D}\neq 0$, (4-4-14a) tells that,

$$\gamma=\mathbf{exp}\left(\frac{\mathbf{Vx}}{\mathbf{D}}\right). \qquad (4\text{-}4\text{-}17)$$

Based on (4-4-17), (4-4-15a) can be simplified as,

$$\Psi(-\mathbf{x})=\left(\frac{\mathbf{V}}{\mathbf{D}}+\frac{2\alpha}{\beta}\right)\mathbf{exp}\left(\frac{\mathbf{Vx}}{\mathbf{D}}\right). \qquad (4\text{-}4\text{-}18)$$

Separate variables of (4-4-16a) and integrate on both sides, we have,

$$\Psi(\zeta)=\mathbf{A}\,\mathbf{exp}\left[-\left(\frac{\mathbf{V}}{\mathbf{D}}+\frac{\alpha}{\beta}\right)\zeta\right], \qquad (4\text{-}4\text{-}19)$$

where $\mathbf{A}$ is an undetermined constant, and is determined by putting (4-4-18) and (4-4-19) together,

$$\Psi(\zeta)\Big|_{\zeta=-x} = \mathbf{A}\exp\left[\left(\frac{\mathbf{V}}{\mathbf{D}}+\frac{\alpha}{\beta}\right)\mathbf{x}\right] = \Psi(-\mathbf{x}) = \left(\frac{\mathbf{V}}{\mathbf{D}}+\frac{2\alpha}{\beta}\right)\exp\left(\frac{\mathbf{Vx}}{\mathbf{D}}\right).$$

Thus,

$$\mathbf{A} = \left(\frac{\mathbf{V}}{\mathbf{D}}+\frac{2\alpha}{\beta}\right)\exp\left(\frac{\mathbf{Vx}}{\mathbf{D}}\right)\exp\left[-\left(\frac{\mathbf{V}}{\mathbf{D}}+\frac{\alpha}{\beta}\right)\mathbf{x}\right]. \tag{4-4-20}$$

Substitute (4-4-20) into (4-4-19), we have,

$$\Psi(\zeta) = \left(\frac{\mathbf{V}}{\mathbf{D}}+\frac{2\alpha}{\beta}\right)\exp\left(\frac{\mathbf{Vx}}{\mathbf{D}}\right)\exp\left[-\left(\frac{\mathbf{V}}{\mathbf{D}}+\frac{\alpha}{\beta}\right)(\zeta+\mathbf{x})\right]. \tag{4-4-21}$$

Substitute (4-4-17) about γ and (4-4-21) about $\psi(\zeta)$ back into (4-4-8), the solution of Green's function is obtained as,

$$\mathbf{G}(\xi,\tau;\mathbf{x},\mathbf{t}) = \mathbf{U}(\xi,\tau;\mathbf{x},\mathbf{t}) + \exp\left(\frac{\mathbf{Vx}}{\mathbf{D}}\right)\mathbf{U}(\xi,\tau;-\mathbf{x},\mathbf{t}) +$$

$$+\left(\frac{\mathbf{V}}{\mathbf{D}}+\frac{2\alpha}{\beta}\right)\exp\left(\frac{\mathbf{Vx}}{\mathbf{D}}\right)\int_{-\infty}^{-\mathbf{x}}\exp\left[-\left(\frac{\mathbf{V}}{\mathbf{D}}+\frac{\alpha}{\beta}\right)(\zeta+\mathbf{x})\right]\mathbf{U}(\xi,\tau;\zeta,\mathbf{t})\mathbf{d}\zeta\,, \tag{4-4-22}$$

where $\mathbf{U}(\xi,\tau;\mathbf{x},\mathbf{t})$ is expressed in (4-2-18).

In (4-4-22), the last term contains an integral, which can be replaced in form of error function. If define for convenience,

$$\sigma = \frac{\alpha}{\beta}, \tag{4-4-23}$$

$$\tilde{\tau} = \mathbf{t} - \tau\,, \tag{4-4-24}$$

and substitute (4-2-18) into the last term of (4-4-22), denoted as $\mathbf{T}_3$, then have,

$$\mathbf{T}_3 = \left(\frac{\mathbf{V}}{\mathbf{D}}+\frac{2\alpha}{\beta}\right)\exp\left(\frac{\mathbf{Vx}}{\mathbf{D}}\right)\exp\left[-\left(\frac{\mathbf{V}}{\mathbf{D}}+\frac{\alpha}{\beta}\right)\mathbf{x}\right]\int_{-\infty}^{-\mathbf{x}}\exp\left[-\left(\frac{\mathbf{V}}{\mathbf{D}}+\frac{\alpha}{\beta}\right)\zeta\right]\mathbf{U}(\xi,\tau;\zeta,\mathbf{t})\mathbf{d}\zeta$$

$$= \left(\frac{\mathbf{V}}{\mathbf{D}}+2\sigma\right)\exp(-\sigma\mathbf{x})\frac{\mathbf{H}(\tilde{\tau})}{\sqrt{4\pi\mathbf{D}\tilde{\tau}}}\int_{-\infty}^{-\mathbf{x}}\exp\left[-\left(\frac{\mathbf{V}}{\mathbf{D}}+\sigma\right)\zeta\right]\exp\left[-\frac{[\xi-\zeta+\mathbf{V}\tilde{\tau}]^2}{4\mathbf{D}\tilde{\tau}}\right]\mathbf{d}\zeta$$

$$= \ldots$$

$$= \left(\frac{\mathbf{V}}{\mathbf{D}}+2\sigma\right)\exp(-\sigma\mathbf{x})\frac{\mathbf{H}(\tilde{\tau})}{\sqrt{4\pi\mathbf{D}\tilde{\tau}}}\cdot$$

$$\cdot\int_{-\infty}^{-\mathbf{x}}\exp\{-\frac{1}{4\mathbf{D}\tilde{\tau}}[(\zeta-\xi+\mathbf{V}\tilde{\tau}+2\sigma\mathbf{D}\tilde{\tau})^2+4(\xi-\sigma\mathbf{D}\tilde{\tau})(\mathbf{V}\tilde{\tau}+\sigma\mathbf{D}\tilde{\tau})]\}\mathbf{d}\zeta$$

$$= \left(\frac{\mathbf{V}}{\mathbf{D}}+2\sigma\right)\exp(-\sigma\mathbf{x})\frac{\mathbf{H}(\tilde{\tau})}{\sqrt{4\pi\mathbf{D}\tilde{\tau}}}\cdot\exp\left[-(\xi-\sigma\mathbf{D}\tilde{\tau})\left(\sigma+\frac{\mathbf{V}}{\mathbf{D}}\right)\right]\cdot$$

$$\cdot\int_{-\infty}^{-x}\exp\left[-\frac{(\zeta-\xi+V\tilde{\tau}+2\sigma D\tilde{\tau})^2}{4D\tilde{\tau}}\right]d\zeta. \qquad (4\text{-}4\text{-}25)$$

Assume that,

$$\chi=\frac{\zeta-\xi+V\tilde{\tau}+2\sigma D\tilde{\tau}}{\sqrt{4D\tilde{\tau}}}, \qquad (4\text{-}4\text{-}26)$$

and that,

$$-\chi_0=\left(\frac{\zeta-\xi+V\tilde{\tau}+2\sigma D\tilde{\tau}}{\sqrt{4D\tilde{\tau}}}\right)\Bigg|_{\zeta=-x}=\frac{-x-\xi+V\tilde{\tau}+2\sigma D\tilde{\tau}}{\sqrt{4D\tilde{\tau}}}, \qquad (4\text{-}4\text{-}27)$$

then

$$d\zeta=\sqrt{4D\tilde{\tau}}d\chi. \qquad (4\text{-}4\text{-}28)$$

Substitute (4-4-26), (4-4-27) and (4-4-28) into (4-4-25),

$$T_3=\left(\frac{V}{D}+2\sigma\right)\exp(-\sigma x)\frac{H(\tilde{\tau})}{\sqrt{4\pi D\tilde{\tau}}}\cdot\exp\left[-(\xi-\sigma D\tilde{\tau})\left(\sigma+\frac{V}{D}\right)\right]\sqrt{4D\tilde{\tau}}\int_{-\infty}^{-\chi_0}\exp(-\chi^2)d\chi$$

$$=\left(\frac{V}{2D}+\sigma\right)\exp(-\sigma x)H(\tilde{\tau})\exp\left[-(\xi-\sigma D\tilde{\tau})\left(\sigma+\frac{V}{D}\right)\right]\mathrm{erfc}(\chi_0)$$

$$=H(\tilde{\tau})\left(\frac{V}{2D}+\sigma\right)\exp(-\sigma x)\exp\left[-(\xi-\sigma D\tilde{\tau})\left(\sigma+\frac{V}{D}\right)\right]\mathrm{erfc}\left[\frac{\xi+x-(V+2\sigma D)\tilde{\tau}}{\sqrt{4D\tilde{\tau}}}\right].$$

(4-4-29)

Substitute the third term of (4-4-22) with (4-4-29) and replace $\tilde{\tau}$ by $(t-\tau)$,

$$G(\xi,\tau;x,t)=U(\xi,\tau;x,t)+\exp\left(\frac{Vx}{D}\right)U(\xi,\tau;-x,t)+$$

$$+H(t-\tau)\left(\frac{V}{2D}+\sigma\right)\exp(-\sigma x)\exp\left[-[\xi-\sigma D(t-\tau)]\left(\sigma+\frac{V}{D}\right)\right]\cdot$$

$$\cdot\,\mathrm{erfc}\left[\frac{\xi+x-(V+2\sigma D)(t-\tau)}{\sqrt{4D(t-\tau)}}\right]. \qquad (4\text{-}4\text{-}30)$$

The final solution of the Neumann or Robin boundary condition problem (4-4-1) can be obtained by substituting the Green's function into (4-4-5).

$$u(x,t)=\int_0^{\infty}\int_0^{\infty}G(\xi,\tau)\phi(\xi,\tau)d\xi d\tau+\int_0^{\infty}G(\xi,0)f(\xi)d\xi-\int_0^{t}[\frac{D}{\beta}G(0,\tau)h(\tau)]d\tau. \qquad (4\text{-}4\text{-}5)$$

Here give an example to show how the final solution looks like. To make it easier, the advection diffusion problem with only boundary value is considered, without source term, namely, $\phi(x,t)=0$ and without non-zero initial value, i.e., $f(x)=0$. Then (4-4-5) is reduced as,

$$\mathbf{u(x,t)} = -\int_0^t [\frac{\mathbf{D}}{\beta}\mathbf{G}(0,\tau)\mathbf{h}(\tau)]\mathbf{d}\tau\,. \qquad (4\text{-}4\text{-}31)$$

According to (4-4-30),

$$\mathbf{G}(0,\tau) = \mathbf{U}(0,\tau;\mathbf{x},\mathbf{t}) + \exp\left(\frac{\mathbf{Vx}}{\mathbf{D}}\right)\mathbf{U}(0,\tau;-\mathbf{x},\mathbf{t}) +$$

$$+\mathbf{H}(\mathbf{t}-\tau)\left(\frac{\mathbf{V}}{2\mathbf{D}}+\sigma\right)\exp(-\sigma\mathbf{x})\exp[(\mathbf{D}\sigma^2+\mathbf{V}\sigma)(\mathbf{t}-\tau)]\mathrm{erfc}\left[\frac{\mathbf{x}-(\mathbf{V}+2\sigma\mathbf{D})(\mathbf{t}-\tau)}{\sqrt{4\mathbf{D}(\mathbf{t}-\tau)}}\right]$$

$$= \frac{\mathbf{H}(\mathbf{t}-\tau)}{\sqrt{4\pi\mathbf{D}(\mathbf{t}-\tau)}}\left\{\exp\left[-\frac{[\mathbf{x}-\mathbf{V}(\mathbf{t}-\tau)]^2}{4\mathbf{D}(\mathbf{t}-\tau)}\right]+\exp\left(\frac{\mathbf{Vx}}{\mathbf{D}}\right)\exp\left[-\frac{[\mathbf{x}+\mathbf{V}(\mathbf{t}-\tau)]^2}{4\mathbf{D}(\mathbf{t}-\tau)}\right]\right\}+$$

$$+\mathbf{H}(\mathbf{t}-\tau)\left(\frac{\mathbf{V}}{2\mathbf{D}}+\sigma\right)\exp(-\sigma\mathbf{x})\exp[(\mathbf{D}\sigma^2+\mathbf{V}\sigma)(\mathbf{t}-\tau)]\mathrm{erfc}\left[\frac{\mathbf{x}-(\mathbf{V}+2\sigma\mathbf{D})(\mathbf{t}-\tau)}{\sqrt{4\mathbf{D}(\mathbf{t}-\tau)}}\right]$$

$$= \frac{2\mathbf{H}(\mathbf{t}-\tau)}{\sqrt{4\pi\mathbf{D}(\mathbf{t}-\tau)}}\exp\left[-\frac{[\mathbf{x}-\mathbf{V}(\mathbf{t}-\tau)]^2}{4\mathbf{D}(\mathbf{t}-\tau)}\right]+$$

$$+\mathbf{H}(\mathbf{t}-\tau)\left(\frac{\mathbf{V}}{2\mathbf{D}}+\sigma\right)\exp(-\sigma\mathbf{x})\exp[(\mathbf{D}\sigma^2+\mathbf{V}\sigma)(\mathbf{t}-\tau)]\mathrm{erfc}\left[\frac{\mathbf{x}-(\mathbf{V}+2\sigma\mathbf{D})(\mathbf{t}-\tau)}{\sqrt{4\mathbf{D}(\mathbf{t}-\tau)}}\right].$$

(4-4-32)

Substitute (4-4-32) into (4-4-31) and apply the commutative law about convolution, i.e., $\int_0^t \mathbf{f}_1(\tau)\cdot\mathbf{f}_2(\mathbf{t}-\tau)\mathbf{d}\tau = \int_0^t \mathbf{f}_1(\mathbf{t}-\tau)\cdot\mathbf{f}_2(\tau)\mathbf{d}\tau$, and note that $\mathbf{H}(\mathbf{t}-\tau)=1$ if $\tau<\mathbf{t}$, then we have,

$$\mathbf{u(x,t)} = -\frac{\mathbf{D}}{\beta}\int_0^t \frac{2}{\sqrt{4\pi\mathbf{D}\tau}}\exp\left[-\frac{(\mathbf{x}-\mathbf{V}\tau)^2}{4\mathbf{D}\tau}\right]\mathbf{h}(\mathbf{t}-\tau)\mathbf{d}\tau -$$

$$-\frac{\mathbf{D}}{\beta}\left(\frac{\mathbf{V}}{2\mathbf{D}}+\sigma\right)\exp(-\sigma\mathbf{x})\int_0^t\{\exp[(\mathbf{D}\sigma^2+\mathbf{V}\sigma)\tau]\mathrm{erfc}\left[\frac{\mathbf{x}-(\mathbf{V}+2\sigma\mathbf{D})\tau}{\sqrt{4\mathbf{D}\tau}}\right]\mathbf{h}(\mathbf{t}-\tau)\}\mathbf{d}\tau\,.$$

(4-4-33)

If the parameters are specified as, $\mathbf{D}=0.1,\mathbf{V}=2,\alpha=0,\beta=-1$, and $\mathbf{h(t)}=\sin(0.4\pi\mathbf{t})$, i.e., a Neumann boundary of $\mathbf{u}_\mathbf{x}(0,\mathbf{t})=-\sin(0.4\pi\mathbf{t})$, the analytical solutions at different time are shown in Figure 4-4-2 as solid lines. In the figure, the symbol lines are the computational results by solving the advection-differential equation numerically. The high consistency between the analytical and numerical solutions identifies the correctness of the solutions obtained by the GFM.

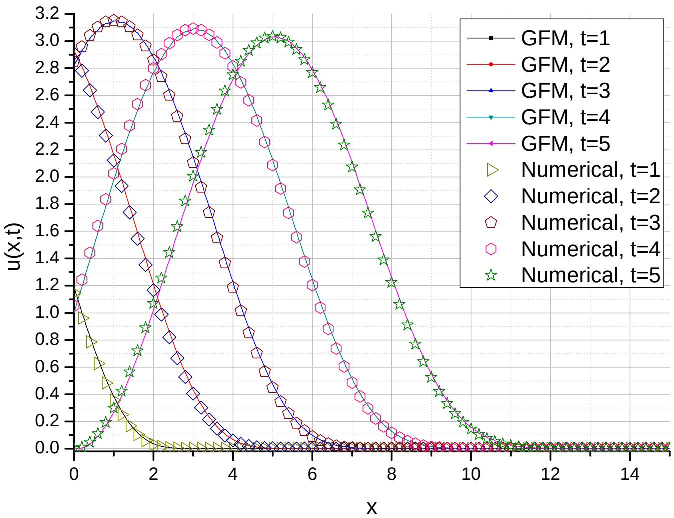


Figure 4-4-2 Verification of Green's function solution in semi-infinite domain with Neumann boundary

Similar to the example of the Neumann boundary, if a change is made only of $\alpha = 1$, then it becomes a Robin boundary, i.e., $\mathbf{u}(0,\mathbf{t}) - \mathbf{u}_{\mathbf{x}}(0,\mathbf{t}) = \mathbf{sin}(0.4\pi\mathbf{t})$. The comparison between the analytical solutions and the numerical solutions is shown in Figure 4-4-3. It verifies again the Green's function obtained in this section.

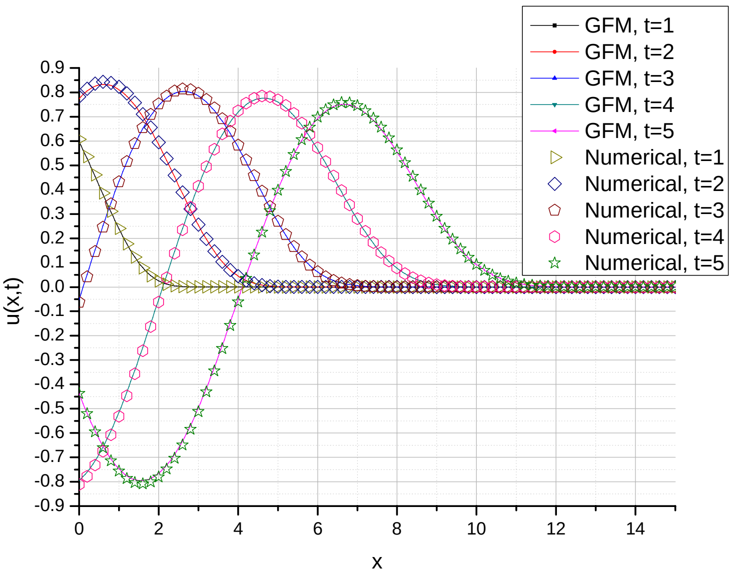


Figure 4-4-3 Verification of Green's function solution in semi-infinite domain with Robin boundary

5. Green's functions of finite domains

In Section 4, the principle solution $\mathbf{U}$ is actually the Green's function for the infinite domain. For the case of semi-infinite domain, the Green's function problems with the Dirichlet, the Neumann and the Robin boundary conditions are solved in last section. This section is devoted to solve the Green's function in a bounded finite domain. It is not enough for this kind of problem to apply only inspection approach. New approaches like the eigenfunction method have to be utilized to seek the Green's function.

5.1 Eigenfunction method

Eigenvalue problem for an ordinary differential equation has to be understood before stepping into the next subsection. Any linear second order inhomogeneous ordinary differential equation can be transformed as,

$$[\mathbf{pu'}]' + \mathbf{wu} = \mathbf{f}, \ \mathbf{a} \le \mathbf{x} \le \mathbf{b} \tag{5-1-1}$$

where $\mathbf{u} = \mathbf{u(x)}, \mathbf{p} = \mathbf{p(x)}, \mathbf{w} = \mathbf{w(x)}, \mathbf{f} = \mathbf{f(x)}$. Among the equations expressed as (5-1-1), the most commonly encountered in practice is in the form of,

$$[\mathbf{pu'}]' + (\mathbf{q} + \lambda\mathbf{r})\mathbf{u} = \mathbf{f}, \ \mathbf{a} \le \mathbf{x} \le \mathbf{b} \tag{5-1-2}$$

where, like others, $\mathbf{q} = \mathbf{q(x)}, \mathbf{r} = \mathbf{r(x)}$, λ is a parameter. (5-1-2) is called Sturm-Liouville equation. The eigenvalue problem is constructed by the homogenous Sturm-Liouville equation with linear homogeneous boundary conditions, i.e.,

$$[\mathbf{pu'}]' + (\mathbf{q} + \lambda\mathbf{r})\mathbf{u} = 0, \ \mathbf{a} \le \mathbf{x} \le \mathbf{b} \tag{5-1-3a}$$

$$\alpha_1\mathbf{u(a)} + \beta_1\mathbf{u'(a)} = 0, \tag{5-1-3b}$$

$$\alpha_2\mathbf{u(b)} + \beta_2\mathbf{u'(b)} = 0, \tag{5-1-3c}$$

where $\alpha_1^2 + \beta_1^2 \neq 0$ and $\alpha_2^2 + \beta_2^2 \neq 0$. Three distinct features of the regular Sturm-Liouville problem should be caught,

(i) a bounded finite interval $[\mathbf{a}, \mathbf{b}]$, where both $\mathbf{a}$ and $\mathbf{b}$ are neither $-\infty$ nor ∞;

(ii) $\mathbf{p}, \mathbf{p'}, \mathbf{q}, \mathbf{r}$ must be continuous;

(iii) $\mathbf{p} > 0, \mathbf{r} > 0$ must hold strictly for any $\mathbf{x} \in [\mathbf{a}, \mathbf{b}]$.

Based on the principle of differential equation, the homogeneous Sturm-Liouville problem (5-1-3) has nontrivial solutions only when λ equals to certain values. These nontrivial solutions are called eigenfunctions, denoted as $\varphi_\mathbf{n}(\mathbf{x})$, each of which is corresponding to an eigenvalue $\lambda_\mathbf{n}$. Furthermore, the functions $\varphi_\mathbf{n}(\mathbf{x})$, $(\mathbf{n} = 1, 2, 3, ...)$, are mutually orthogonal, namely,

$$\int_\mathbf{a}^\mathbf{b} \mathbf{r(x)}\varphi_\mathbf{m}(\mathbf{x})\varphi_\mathbf{n}(\mathbf{x})\mathbf{dx} = 0, \text{ if } \mathbf{m} \neq \mathbf{n}. \tag{5-1-4}$$

If they are normalized, then,

$$\int_\mathbf{a}^\mathbf{b} \mathbf{r(x)}\varphi_\mathbf{m}(\mathbf{x})\varphi_\mathbf{n}(\mathbf{x})\mathbf{dx} = 1, \text{ if } \mathbf{m} = \mathbf{n}. \tag{5-1-5}$$

The set of $\{\varphi_n(x)\}$ is a normalized orthogonal complete set in the sense of mean square convergence. For any function $f(x)$ defined on $[a,b]$, it can be expanded as Fourier series based on $\{\varphi_n(x)\}$, i.e.,

$$f(x) \sim \sum_{n=1}^{\infty} a_n \varphi_n(x), \; a_n = \int_a^b f(x)\varphi_n(x) r(x) dx, \tag{5-1-6}$$

and the series is absolutely and uniformly convergent to $f(x)$.

This is the basic idea of eigenfunction method. It should be noted that this paper work is devoted to explain the application of the GFM, not to form a mathematical textbook. Therefore any details about, say, singularity, existence and uniqueness and so forth, are not discussed fundamentally. Let's continue the eigenvalue problem. If it is still somehow abstract, an example may be helpful to understand. A simple problem is listed as follows,

$$\varphi'' + \lambda\varphi = 0, \; 0 \le x \le a, \tag{5-1-7a}$$

$$\varphi(0) = 0, \tag{5-1-7b}$$

$$\varphi(a) = 0. \tag{5-1-7c}$$

The problem satisfies strictly the three requirements of homogeneous Sturm-Liouville problem. It would be more straight if (5-1-7a) is rewritten as $[1 \cdot \varphi']' + (0 + \lambda \cdot 1)\varphi = 0$. It is quite easy to obtain the normalized eigenfunctions and corresponding eigenvalues of the problem,

$$\varphi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right), \; \lambda_n = \frac{n^2\pi^2}{a^2}, \; (n = 1,2,3,...). \tag{5-1-8}$$

It is actually the sine series of Fourier transform, which can be looked as, in a sense, a particular case of eigenfunction expansions.

5.2 Eigenfunction expansion

In the subsection, only a short example is discussed to explain how to use eigenfunction method to solve a Green's function-like problem with a delta function on the right hand side of an equation. For an instance,

$$g''(\xi;x) + k^2 g(\xi;x) = \delta(\xi - x), \; 0 \le \xi \le a \tag{5-2-1a}$$

$$g(0;x) = 0, \tag{5-2-1b}$$

$$g(a;x) = 0, \tag{5-2-1c}$$

where, like in Section 3 and Section 4, ξ is the running variable, x is looked as constant. It is known already that the associated eigenvalue problem of (5-2-1) is actually the one of (5-1-7), which solution is used directly here. According to the eigenfunction method, the both functions of $g(\xi;x)$ and $\delta(\xi - x)$ in (5-2-1a) can be expanded based on $\{\varphi_n(x)\}$ shown in (5-1-8),

$$g(\xi;x) = \sum_{n=1}^{\infty} a_n(x)\left[\sqrt{\frac{2}{a}} \sin\left(\frac{n\pi\xi}{a}\right)\right], \tag{5-2-2}$$

where, $\mathbf{a_n(x)}$ is to be determined.

Differentiate twice (5-2-2) on both sides,

$$\mathbf{g''(\xi;x)} = \sum_{\mathbf{n=1}}^{\infty}\left(-\frac{\mathbf{n}^2\pi^2}{\mathbf{a}^2}\right)\mathbf{a_n(x)}\left[\sqrt{\frac{2}{\mathbf{a}}}\,\mathbf{sin}\left(\frac{\mathbf{n}\pi\xi}{\mathbf{a}}\right)\right]. \qquad (5\text{-}2\text{-}3)$$

Expand the delta function $\delta(\xi - \mathbf{x})$ as,

$$\delta(\xi - \mathbf{x}) = \sum_{\mathbf{n=1}}^{\infty}\mathbf{b_n(x)}\left[\sqrt{\frac{2}{\mathbf{a}}}\,\mathbf{sin}\left(\frac{\mathbf{n}\pi\xi}{\mathbf{a}}\right)\right], \qquad (5\text{-}2\text{-}4)$$

where, according to (5-1-6) and noting that "$\mathbf{r(x)} = 1$" in the eigenvalue problem of (5-1-7) or "$\mathbf{r}(\xi) = 1$" here,

$$\mathbf{b_n(x)} = \int_0^{\mathbf{a}}\delta(\xi - \mathbf{x})\left[\sqrt{\frac{2}{\mathbf{a}}}\,\mathbf{sin}\left(\frac{\mathbf{n}\pi\xi}{\mathbf{a}}\right)\right]\mathbf{d}\xi = \sqrt{\frac{2}{\mathbf{a}}}\,\mathbf{sin}\left(\frac{\mathbf{n}\pi\mathbf{x}}{\mathbf{a}}\right). \qquad (5\text{-}2\text{-}5)$$

The last step proceeds by using the property of delta function (2-1-3).

Substitute (5-2-2), (5-2-3) and (5-2-4) into (5-2-1a),

$$\sum_{\mathbf{n=1}}^{\infty}\left(-\frac{\mathbf{n}^2\pi^2}{\mathbf{a}^2}\right)\mathbf{a_n(x)}\left[\sqrt{\frac{2}{\mathbf{a}}}\,\mathbf{sin}\left(\frac{\mathbf{n}\pi\xi}{\mathbf{a}}\right)\right] + \mathbf{k}^2\sum_{\mathbf{n=1}}^{\infty}\mathbf{a_n(x)}\left[\sqrt{\frac{2}{\mathbf{a}}}\,\mathbf{sin}\left(\frac{\mathbf{n}\pi\xi}{\mathbf{a}}\right)\right] =$$

$$= \sum_{\mathbf{n=1}}^{\infty}\mathbf{b_n(x)}\left[\sqrt{\frac{2}{\mathbf{a}}}\,\mathbf{sin}\left(\frac{\mathbf{n}\pi\xi}{\mathbf{a}}\right)\right].$$

Simplified as,

$$\sum_{\mathbf{n=1}}^{\infty}\left(\mathbf{k}^2 - \frac{\mathbf{n}^2\pi^2}{\mathbf{a}^2}\right)\mathbf{a_n(x)}\left[\sqrt{\frac{2}{\mathbf{a}}}\,\mathbf{sin}\left(\frac{\mathbf{n}\pi\xi}{\mathbf{a}}\right)\right] = \sum_{\mathbf{n=1}}^{\infty}\mathbf{b_n(x)}\left[\sqrt{\frac{2}{\mathbf{a}}}\,\mathbf{sin}\left(\frac{\mathbf{n}\pi\xi}{\mathbf{a}}\right)\right]. \qquad (5\text{-}2\text{-}6)$$

Equality (5-2-6) must hold for any ξ, therefore it must have,

$\left(\mathbf{k}^2 - \frac{\mathbf{n}^2\pi^2}{\mathbf{a}^2}\right)\mathbf{a_n(x)} = \mathbf{b_n(x)}$. Then,

$$\mathbf{a_n(x)} = \mathbf{b_n(x)}\frac{\mathbf{a}^2}{\mathbf{k}^2\mathbf{a}^2 - \mathbf{n}^2\pi^2} = \frac{\mathbf{a}^2}{\mathbf{k}^2\mathbf{a}^2 - \mathbf{n}^2\pi^2}\sqrt{\frac{2}{\mathbf{a}}}\,\mathbf{sin}\left(\frac{\mathbf{n}\pi\mathbf{x}}{\mathbf{a}}\right). \qquad (5\text{-}2\text{-}7)$$

In the last step $\mathbf{b_n(x)}$ is substituted by using (5-2-5). The solution of (5-2-1) is obtained by bringing (5-2-7) into (5-2-2),

$$\mathbf{g(\xi;x)} = \sum_{\mathbf{n=1}}^{\infty}\frac{2\mathbf{a}}{\mathbf{k}^2\mathbf{a}^2 - \mathbf{n}^2\pi^2}\,\mathbf{sin}\left(\frac{\mathbf{n}\pi\mathbf{x}}{\mathbf{a}}\right)\mathbf{sin}\left(\frac{\mathbf{n}\pi\xi}{\mathbf{a}}\right). \qquad (5\text{-}2\text{-}8)$$

5.3 Green's function of general advection diffusion problem

The general problem is formulated as,

$$\mathbf{Lu} = \mathbf{u_t} + \mathbf{Vu_x} - \mathbf{Du_{xx}} = \phi(\mathbf{x,t}),\ 0 < \mathbf{x} < \mathbf{a},\ 0 < \mathbf{t} < \infty, \qquad (5\text{-}3\text{-}1a)$$

$$\mathbf{u}(\mathbf{x},0)=\mathbf{f}(\mathbf{x}), \tag{5-3-1b}$$

$$\alpha_1\mathbf{u}(0,\mathbf{t})+\beta_1\mathbf{u}_\mathbf{x}(0,\mathbf{t})=\mathbf{h}_1(\mathbf{t}), \tag{5-3-1c}$$

$$\alpha_2\mathbf{u}(\mathbf{a},\mathbf{t})+\beta_2\mathbf{u}_\mathbf{x}(\mathbf{a},\mathbf{t})=\mathbf{h}_2(\mathbf{t}), \tag{5-3-1d}$$

where $\alpha_1^2+\beta_1^2\neq 0$ and $\alpha_2^2+\beta_2^2\neq 0$. The boundary conditions cover the Dirichlet, the Neumann and the Robin boundary conditions.

According to (3-2-8), the solution of (5-3-1) can be expressed as,

$$\mathbf{u}(\mathbf{x},\mathbf{t})=\int_0^\mathbf{t}\int_0^\mathbf{a}\mathbf{G}\phi\mathbf{d}\xi\mathbf{d}\tau+\int_0^\mathbf{a}[(\mathbf{G}\mathbf{u})\big|_{\tau=0}]\mathbf{d}\xi-$$

$$-\int_0^\mathbf{t}\{[(\mathbf{V}\mathbf{G}+\mathbf{D}\mathbf{G}_\xi)\mathbf{u}-\mathbf{D}\mathbf{G}\mathbf{u}_\xi]\big|_{\xi=\mathbf{a}}-[(\mathbf{V}\mathbf{G}+\mathbf{D}\mathbf{G}_\xi)\mathbf{u}-\mathbf{D}\mathbf{G}\mathbf{u}_\xi]\big|_{\xi=0}\}\mathbf{d}\tau. \tag{5-3-2}$$

Based on the different values of $\alpha_1,\alpha_2,\beta_1,\beta_2$ in (5-3-1c) and (5-3-1d), the boundary values of $\mathbf{u}$ or $\mathbf{u}_\mathbf{x}$ $(\mathbf{u}_\xi)$ can be brought into (5-3-2). The adjoint boundary conditions for $\mathbf{G}$ can be determined by vanishing of the unwelcome terms in (5-3-2). The boundary conditions for $\mathbf{G}$ obtained by this way must be homogeneous, and be linear combinations of $\mathbf{G}$ and/or $\mathbf{G}_\xi$ taking values on the boundary. Together with the governing equation of Green's function, the boundary value problem on $\mathbf{G}$ can be formulated as,

$$\mathbf{L}^*\mathbf{G}=-\mathbf{G}_\tau-\mathbf{V}\mathbf{G}_\xi-\mathbf{D}\mathbf{G}_{\xi\xi}=\delta(\xi-\mathbf{x},\tau-\mathbf{t}),\ 0<\xi<\mathbf{a},\ 0<\tau<\infty \tag{5-3-3a}$$

$$\tilde{\alpha}_1\mathbf{G}(0,\tau)+\tilde{\beta}_1\mathbf{G}_\xi(0,\tau)=0, \tag{5-3-3b}$$

$$\tilde{\alpha}_2\mathbf{G}(\mathbf{a},\tau)+\tilde{\beta}_2\mathbf{G}_\xi(\mathbf{a},\tau)=0. \tag{5-3-3c}$$

where $\tilde{\alpha}_1,\tilde{\beta}_1,\tilde{\alpha}_2,\tilde{\beta}_2$ are determined by the values of $\alpha_1,\beta_1,\alpha_2,\beta_2,\mathbf{V},\mathbf{D}$ and so on depending on different boundary conditions.

To handle the delta function in (5-3-3), it might be easier to appeal to Laplace transform. According to Section 4, $\mathbf{G}=0$ if $\tau>\mathbf{t}$; and $\mathbf{G}$ is a function if $\tau<\mathbf{t}$. In another word, $\mathbf{G}$ might be nonzero if $\tau<0$, although negative time does not mean anything in physics. Therefore, in order to utilize Laplace transform, (referring to the definition of Laplace transform in Section 2), it is necessary to reverse the time coordinate, namely, to assume that,

$$\tilde{\tau}=-\tau, \tag{5-3-4}$$

then to denote $\mathbf{G}$ as $\tilde{\mathbf{G}}$ in the transformed time coordinate,

$$\tilde{\mathbf{G}}(\xi,\tilde{\tau};\mathbf{x},\mathbf{t})=\mathbf{G}(\xi,-\tau;\mathbf{x},\mathbf{t}). \tag{5-3-5}$$

It is obvious that, after transformation, $\tilde{\mathbf{G}}=0$ if $\tilde{\tau}<\mathbf{t}$. By the treatment and considering that delta function is an even function, the problem (5-3-3) is transformed as,

$$\tilde{\mathbf{G}}_{\tilde{\tau}}-\mathbf{V}\tilde{\mathbf{G}}_\xi-\mathbf{D}\tilde{\mathbf{G}}_{\xi\xi}=\delta(\xi-\mathbf{x},\tilde{\tau}+\mathbf{t}),\ 0<\xi<\mathbf{a},\ 0<\tilde{\tau}<\infty \tag{5-3-6a}$$

$$\tilde{\alpha}_1\tilde{\mathbf{G}}(0,\tilde{\tau})+\tilde{\beta}_1\tilde{\mathbf{G}}_\xi(0,\tilde{\tau})=0\,, \tag{5-3-6b}$$

$$\tilde{\alpha}_2\tilde{\mathbf{G}}(\mathbf{a},\tilde{\tau})+\tilde{\beta}_2\tilde{\mathbf{G}}_\xi(\mathbf{a},\tilde{\tau})=0\,. \tag{5-3-6c}$$

It is worth to mention that the reversed time $\tilde{\tau}$ can be from $-\infty$ to ∞, in the light of the broadened time concept. However, it is still defined as $0<\tilde{\tau}<\infty$ in (5-3-6a) for symmetry, because $\tilde{\mathbf{G}}\equiv 0$ when $\tilde{\tau}<\mathbf{t}$ and $\mathbf{t}$ is a positive constant here. Actually if someone likes, it can be defined as $\mathbf{t}<\tilde{\tau}<\infty$.

So far Laplace transform can be performed on (5-3-6a) from $\tilde{\tau}$ to $\mathbf{s}$. Recovering the properties about Laplace transform in Section 2, it is easy to obtain,

$$\mathbf{s}\overline{\mathbf{G}}-\mathbf{V}\overline{\mathbf{G}}'-\mathbf{D}\overline{\mathbf{G}}''=\delta(\xi-\mathbf{x})\,\mathbf{exp(ts)}\,, \tag{5-3-7a}$$

$$\tilde{\alpha}_1\overline{\mathbf{G}}(0,\mathbf{s})+\tilde{\beta}_1\overline{\mathbf{G}}_\xi(0,\mathbf{s})=0\,, \tag{5-3-7b}$$

$$\tilde{\alpha}_2\overline{\mathbf{G}}(\mathbf{a},\mathbf{s})+\tilde{\beta}_2\overline{\mathbf{G}}_\xi(\mathbf{a},\mathbf{s})=0\,, \tag{5-3-7c}$$

where $\overline{\mathbf{G}}(\xi,\mathbf{s};\mathbf{x},\mathbf{t})\equiv \mathcal{L}[\tilde{\mathbf{G}}(\xi,\tilde{\tau};\mathbf{x},\mathbf{t})]=\int_0^\infty \tilde{\mathbf{G}}(\xi,\tilde{\tau};\mathbf{x},\mathbf{t})\,\mathbf{exp}(-\mathbf{s}\tilde{\tau})\mathbf{d}\tilde{\tau}$.

It comes to apply eigenfunction method to solve the problem of (5-3-7). However it is not in the form of regular Sturm-Liouville equation (5-1-2), which does not have the term of the first derivative. So we must make proper transformation on $\overline{\mathbf{G}}$. Assume that,

$$\overline{\mathbf{G}}(\xi)=\mathbf{g}(\xi)\psi(\xi)\,. \tag{5-3-8}$$

Substitute (5-3-8) into (5-3-7a) and arrange it as,

$$\mathbf{D}\psi\mathbf{g}''+(2\mathbf{D}\psi'+\mathbf{V}\psi)\mathbf{g}'+(\mathbf{D}\psi''+\mathbf{V}\psi'-\mathbf{s}\psi)\mathbf{g}=-\delta(\xi-\mathbf{x})\,\mathbf{exp(ts)}\,. \tag{5-3-9}$$

Since the coefficient of the first derivative $\mathbf{g}'$ is not preferred by the form of Sturm-Liouville, it is demanded that,

$$2\mathbf{D}\psi'+\mathbf{V}\psi=0\,. \tag{5-3-10}$$

By straightforward calculation without considering the integration coefficient, we have,

$$\psi(\xi)=\mathbf{exp}\left(-\frac{\mathbf{V}\xi}{2\mathbf{D}}\right). \tag{5-3-11}$$

In the light of (5-3-8) and (5-3-11), we have,

$$\overline{\mathbf{G}}(\xi)=\mathbf{g}(\xi)\,\mathbf{exp}\left(-\frac{\mathbf{V}\xi}{2\mathbf{D}}\right) \text{ and } \mathbf{g}(\xi)=\overline{\mathbf{G}}(\xi)\,\mathbf{exp}\left(\frac{\mathbf{V}\xi}{2\mathbf{D}}\right). \tag{5-3-12}$$

Substitute (5-3-11) back into (5-3-9) and simplify as,

$$\mathbf{g}''+\left(\frac{-\mathbf{V}^2-4\mathbf{Ds}}{4\mathbf{D}^2}\right)\mathbf{g}=-\frac{1}{\mathbf{D}}\mathbf{exp}\left(\frac{\mathbf{V}\xi}{2\mathbf{D}}+\mathbf{ts}\right)\delta(\xi-\mathbf{x}), \text{ namely,}$$

$$\mathbf{g}''+\left(\frac{-\mathbf{V}^2-4\mathbf{Ds}}{4\mathbf{D}^2}\right)\mathbf{g}=-\frac{1}{\mathbf{D}}\exp\left(\frac{\mathbf{Vx}}{2\mathbf{D}}+\mathbf{ts}\right)\delta(\xi-\mathbf{x}) \tag{5-3-13a}$$

In the exponential term on the right and side, ξ is replaced by $\mathbf{x}$ in the last step. This is the direct result by using the property of delta function (2-1-5). The boundary conditions (5-3-7b) and (5-3-7c) are transformed accordingly as,

$$\left(\tilde{\alpha}_1-\frac{\mathbf{V}}{2\mathbf{D}}\tilde{\beta}_1\right)\mathbf{g}(0)+\tilde{\beta}_1\mathbf{g}'(0)=0\,, \tag{5-3-13b}$$

$$\left(\tilde{\alpha}_2-\frac{\mathbf{V}}{2\mathbf{D}}\tilde{\beta}_2\right)\mathbf{g}(\mathbf{a})+\tilde{\beta}_2\mathbf{g}'(\mathbf{a})=0\,. \tag{5-3-13c}$$

Fortunately they are still homogeneous after transformation. Equation (5-3-13a) and boundary conditions (5-3-13b) and (5-3-13c) form the eigenvalue problem. The associated Sturm-Liouville problem to (5-3-13) is formulated as,

$$\varphi''+\lambda\varphi=0\,, \tag{5-3-14a}$$

$$\left(\tilde{\alpha}_1-\frac{\mathbf{V}}{2\mathbf{D}}\tilde{\beta}_1\right)\varphi(0)+\tilde{\beta}_1\varphi'(0)=0\,, \tag{5-3-14b}$$

$$\left(\tilde{\alpha}_2-\frac{\mathbf{V}}{2\mathbf{D}}\tilde{\beta}_2\right)\varphi(\mathbf{a})+\tilde{\beta}_2\varphi'(\mathbf{a})=0\,. \tag{5-3-14c}$$

It should be emphasized that the boundary conditions for the associated Sturm-Liouville problem (5-3-14) should be identical to the original problem (5-3-13). Assume that the solution of (5-3-14) is $\varphi_\mathbf{n}(\xi)$ corresponding to $\lambda_\mathbf{n}$ $(\mathbf{n}=1,2,3,...)$. The specific expressions or values depend on the specific boundary conditions. In view of eigenfunction method, all the function in (5-3-13a), $\mathbf{g}$, $\mathbf{g}''$ and $\delta(\xi-\mathbf{x})$ can be expanded as Fourier series on the complete set $\{\varphi_\mathbf{n}(\xi)\}$. Let's say,

$$\mathbf{g}(\xi;\mathbf{x})=\sum_{\mathbf{n}=1}^{\infty}\mathbf{a}_\mathbf{n}(\mathbf{x})\varphi_\mathbf{n}(\xi)\,. \tag{5-3-15}$$

Differentiate twice (5-3-15) on both sides,

$$\mathbf{g}''(\xi;\mathbf{x})=\sum_{\mathbf{n}=1}^{\infty}\mathbf{a}_\mathbf{n}(\mathbf{x})\varphi_\mathbf{n}''(\xi)\,. \tag{5-3-16}$$

Noting that $\varphi_\mathbf{n}(\xi)$ is a solution of (5-3-14), it must satisfy the equation (5-3-14a), namely,

$$\varphi_\mathbf{n}''(\xi)+\lambda_\mathbf{n}\varphi_\mathbf{n}(\xi)=0\text{, i.e., }\varphi_\mathbf{n}''(\xi)=-\lambda_\mathbf{n}\varphi_\mathbf{n}(\xi)\,. \tag{5-3-17}$$

Substitute (5-3-17) into (5-3-16),

$$\mathbf{g}''(\xi;\mathbf{x})=\sum_{\mathbf{n}=1}^{\infty}(-\lambda_\mathbf{n})\mathbf{a}_\mathbf{n}(\mathbf{x})\varphi_\mathbf{n}(\xi)\,. \tag{5-3-18}$$

$\delta(\xi-\mathbf{x})$ is expanded as,

$$\delta(\xi - x) = \sum_{n=1}^{\infty} b_n(x)\varphi_n(\xi), \tag{5-3-19}$$

where, according to (5-1-6) with considering that "$r(\xi) = 1$",

$$b_n(x) = \int_0^a \delta(\xi - x)\varphi_n(\xi)d\xi = \varphi_n(x). \tag{5-3-20}$$

Bring (5-3-20) into (5-3-19),

$$\delta(\xi - x) = \sum_{n=1}^{\infty} \varphi_n(x)\varphi_n(\xi) \tag{5-3-21}$$

Substituting (5-3-15), (5-3-18) and (5-3-21) to (5-3-13a), we have,

$$\sum_{n=1}^{\infty}(-\lambda_n)a_n(x)\varphi_n(\xi) + \left(\frac{-V^2 - 4Ds}{4D^2}\right)\sum_{n=1}^{\infty} a_n(x)\varphi_n(\xi) = -\frac{1}{D}\exp\left(\frac{Vx}{2D} + ts\right)\sum_{n=1}^{\infty}\varphi_n(x)\varphi_n(\xi)$$

$$\sum_{n=1}^{\infty}\left(\lambda_n + \frac{V^2 + 4Ds}{4D^2}\right)a_n(x)\varphi_n(\xi) = \sum_{n=1}^{\infty}\frac{1}{D}\exp\left(\frac{Vx}{2D} + ts\right)\varphi_n(x)\varphi_n(\xi) \tag{5-3-22}$$

Since the equality (5-3-22) holds for any ξ, it must have,

$\left(\lambda_n + \frac{V^2 + 4Ds}{4D^2}\right)a_n(x) = \frac{1}{D}\exp\left(\frac{Vx}{2D} + ts\right)\varphi_n(x)$, namely,

$$a_n(x) = \frac{4D}{4Ds + 4D^2\lambda_n + V^2}\exp\left(\frac{Vx}{2D} + ts\right)\varphi_n(x). \tag{5-3-23}$$

Substitute (5-3-23) into (5-3-15),

$$g(\xi; x) = \sum_{n=1}^{\infty}\frac{\exp(ts)}{s + D\lambda_n + \frac{V^2}{4D}}\exp\left(\frac{Vx}{2D}\right)\varphi_n(x)\varphi_n(\xi) \tag{5-3-24}$$

This is the solution of problem (5-3-13).

Depending on (5-3-12), the solution of problem (5-3-7) is obtained as,

$$\overline{G}(\xi, s) = \sum_{n=1}^{\infty}\frac{\exp(ts)}{s + D\lambda_n + \frac{V^2}{4D}}\exp\left[-\frac{V(\xi - x)}{2D}\right]\varphi_n(x)\varphi_n(\xi). \tag{5-3-25}$$

By making inverse Laplace transform on (5-3-25) from s to $\tilde{\tau}$, the solution of the problem (5-3-6) is obtained as,

$$\tilde{G}(\xi, \tilde{\tau}; x, t) = H(\tilde{\tau} + t)\sum_{n=1}^{\infty}\exp\left[-\left(D\lambda_n + \frac{V^2}{4D}\right)(\tilde{\tau} + t)\right]\exp\left[-\frac{V(\xi - x)}{2D}\right]\varphi_n(x)\varphi_n(\xi). \tag{5-3-26}$$

By recovering the original time coordinate based on (5-3-4), the solution of the problem (5-3-3) is obtained as,

$$\mathbf{G}(\xi,\tau;\mathbf{x},\mathbf{t})=\mathbf{H}(\mathbf{t}-\tau)\sum_{\mathbf{n}=1}^{\infty}\mathbf{exp}\left[-\left(\mathbf{D}\lambda_{\mathbf{n}}+\frac{\mathbf{V}^2}{4\mathbf{D}}\right)(\mathbf{t}-\tau)\right]\mathbf{exp}\left[-\frac{\mathbf{V}(\xi-\mathbf{x})}{2\mathbf{D}}\right]\varphi_{\mathbf{n}}(\mathbf{x})\varphi_{\mathbf{n}}(\xi),$$

(5-3-27)

where, the eigenfunctions and corresponding eigenvalues $\{\varphi_{\mathbf{n}}(\xi),\lambda_{\mathbf{n}}\}$ are determined by (5-3-14), i.e., the explicit expressions about $\{\varphi_{\mathbf{n}}(\xi),\lambda_{\mathbf{n}}\}$ are determined only by the given boundary conditions. Once the Green's function $\mathbf{G}(\xi,\tau;\mathbf{x},\mathbf{t})$ is solved, the final solution $\mathbf{u}(\mathbf{x},\mathbf{t})$ of the original general problem (5-3-1) can be obtained by substituting (5-3-27) into (5-3-2). If it is still abstract, we have to come to solve specific problems as examples.

5.4 Examples

For a one-dimensional problem, the concerned interval has two boundaries, on each of which one of three sorts of boundary conditions could be prescribed, the Dirichlet, the Neumann or the Robin boundary, noted as D, N and R, respectively. It is simple to know there are nine different combinations. Fortunately the Neumann type of boundary can be handled somehow as a special case of the Robin boundary with specifying $\alpha_{\mathbf{i}}=0$ and $\beta_{\mathbf{i}}=1$ $(\mathbf{i}=1\mathbf{or}2)$ in (5-3-1). In other words, combinations can be reduced by handling only two types of boundaries, the Dirichlet and the Robin boundaries. Therefore the four problems with D-D, D-R, R-D and R-R are solved as four examples.

5.4.1 Dirichlet- Dirichlet (D-D) boundaries

As the title says, Dirichlet boundary conditions are prescribed at both ends. The problem is described as,

$$\mathbf{Lu}=\mathbf{u}_{\mathbf{t}}+\mathbf{Vu}_{\mathbf{x}}-\mathbf{Du}_{\mathbf{xx}}=\phi(\mathbf{x},\mathbf{t}),\ 0<\mathbf{x}<\mathbf{a},\ 0<\mathbf{t}<\infty, \quad (5\text{-}4\text{-}1\text{-}1a)$$

$$\mathbf{u}(\mathbf{x},0)=\mathbf{f}(\mathbf{x}), \quad (5\text{-}4\text{-}1\text{-}1b)$$

$$\mathbf{u}(0,\mathbf{t})=\mathbf{h}_1(\mathbf{t}), \quad (5\text{-}4\text{-}1\text{-}1c)$$

$$\mathbf{u}(\mathbf{a},\mathbf{t})=\mathbf{h}_2(\mathbf{t}). \quad (5\text{-}4\text{-}1\text{-}1d)$$

According to (5-3-2), the solution can be expressed as,

$$\mathbf{u}(\mathbf{x},\mathbf{t})=\int_0^{\mathbf{t}}\int_0^{\mathbf{a}}\mathbf{G}\phi\mathbf{d}\xi\mathbf{d}\tau+\int_0^{\mathbf{a}}[(\mathbf{Gu})|_{\tau=0}]\mathbf{d}\xi-$$

$$-\int_0^{\mathbf{t}}\{[(\mathbf{VG}+\mathbf{DG}_{\xi})\mathbf{u}-\mathbf{DGu}_{\xi}]|_{\xi=\mathbf{a}}-[(\mathbf{VG}+\mathbf{DG}_{\xi})\mathbf{u}-\mathbf{DGu}_{\xi}]|_{\xi=0}\}\mathbf{d}\tau. \quad (5\text{-}3\text{-}2)$$

Based on the given boundary conditions, the terms of " $(-\mathbf{DGu}_{\xi})|_{\xi=0}$ " and " $(-\mathbf{DGu}_{\xi})|_{\xi=\mathbf{a}}$ " in (5-3-2) are unwelcome, so the adjoint boundary conditions for Green's function can be defined as, $\mathbf{G}|_{\xi=0}=0$ and $\mathbf{G}|_{\xi=\mathbf{a}}=0$. Together with the governing equation of $\mathbf{G}$, the Green's function problem is defined as,

$$\mathbf{L}^*\mathbf{G}=-\mathbf{G}_{\tau}-\mathbf{VG}_{\xi}-\mathbf{DG}_{\xi\xi}=\delta(\xi-\mathbf{x},\tau-\mathbf{t}),\ 0<\xi<\mathbf{a},\ 0<\tau<\infty \quad (5\text{-}4\text{-}1\text{-}2a)$$

$$\mathbf{G}(0,\tau)=0, \tag{5-4-1-2b}$$

$$\mathbf{G}(\mathbf{a},\tau)=0. \tag{5-4-1-2c}$$

On the other hand, by applying the boundary conditions of $\mathbf{G}$, the expression of solution (5-3-2) can be simplified as,

$$\mathbf{u}(\mathbf{x},\mathbf{t})=\int_0^{\mathbf{t}}\int_0^{\mathbf{a}}\mathbf{G}\phi\mathbf{d}\xi\mathbf{d}\tau+\int_0^{\mathbf{a}}\mathbf{G}(\xi,0)\mathbf{f}(\xi)\mathbf{d}\xi-\int_0^{\mathbf{t}}[\mathbf{DG}_\xi(\mathbf{a},\tau)\mathbf{h}_2(\tau)-\mathbf{DG}_\xi(0,\tau)\mathbf{h}_1(\tau)]\mathbf{d}\tau. \tag{5-4-1-3}$$

By comparing (5-4-1-2) to (5-3-3), it is obvious that,

$$\tilde{\alpha}_1=\tilde{\alpha}_2=1,\tilde{\beta}_1=\tilde{\beta}_2=0. \tag{5-4-1-4}$$

By substituting the above values into (5-3-14), the eigenvalue problem for this case is directly obtained as,

$$\varphi''+\lambda\varphi=0, \tag{5-4-1-5a}$$

$$\varphi(0)=0, \tag{5-4-1-5b}$$

$$\varphi(\mathbf{a})=0. \tag{5-4-1-5c}$$

The following steps are devoted to solve the eigenvalue problem.

The general solution of (5-4-1-5a) is in the form of,

$$\varphi(\xi)=\mathbf{A}\sin(\sqrt{\lambda}\xi)+\mathbf{B}\cos(\sqrt{\lambda}\xi), \tag{5-4-1-6}$$

where $\mathbf{A}$ and $\mathbf{B}$ are undetermined constants. By using (5-4-1-5b), we have,

$$\varphi(0)=\mathbf{A}\sin(0)+\mathbf{B}\cos(0)=\mathbf{B}=0.$$

Thus,

$$\varphi(\xi)=\mathbf{A}\sin(\sqrt{\lambda}\xi).$$

According to (5-4-1-4c),

$$\varphi(\mathbf{a})=\mathbf{A}\sin(\sqrt{\lambda}\mathbf{a})=0.$$

In order to obtain nontrivial solutions for (5-4-1-5a), λ must meet, $\sqrt{\lambda}\mathbf{a}=\mathbf{n}\pi$, namely,

$$\lambda_{\mathbf{n}}=\frac{\mathbf{n}^2\pi^2}{\mathbf{a}^2},\ \mathbf{n}=1,2,3,\ldots. \tag{5-4-1-7}$$

This is the eigenvalue of the problem (5-4-1-5). The corresponding eigenfunction is,

$$\varphi_{\mathbf{n}}(\xi)=\mathbf{A}\sin\left(\frac{\mathbf{n}\pi}{\mathbf{a}}\xi\right).$$

To determine $\mathbf{A}$, $\varphi_{\mathbf{n}}$ is normalized as follows,

$$\int_0^{\mathbf{a}}\mathbf{A}^2\sin^2\left(\frac{\mathbf{n}\pi}{\mathbf{a}}\xi\right)\mathbf{d}\xi=\frac{\mathbf{A}^2}{2}\int_0^{\mathbf{a}}\left[1-\cos\left(\frac{2\mathbf{n}\pi}{\mathbf{a}}\xi\right)\right]\mathbf{d}\xi=\frac{\mathbf{A}^2}{2}\cdot\mathbf{a}=1,$$

thus,

$$\mathbf{A}=\sqrt{\frac{2}{\mathbf{a}}}\,.$$

So the normalized eigenfunction is obtained as,

$$\boldsymbol{\varphi}_{\mathbf{n}}(\boldsymbol{\xi})=\sqrt{\frac{2}{\mathbf{a}}}\,\mathbf{sin}\left(\frac{\mathbf{n}\boldsymbol{\pi}}{\mathbf{a}}\boldsymbol{\xi}\right). \tag{5-4-1-8}$$

Based on (5-3-27), the Green's function can be obtained as,

$$\mathbf{G}(\boldsymbol{\xi},\boldsymbol{\tau};\mathbf{x},\mathbf{t})=\mathbf{H}(\mathbf{t}-\boldsymbol{\tau})\sum_{\mathbf{n}=1}^{\infty}\frac{2}{\mathbf{a}}\mathbf{exp}\left[-\mathbf{D}\left(\frac{\mathbf{n}^2\boldsymbol{\pi}^2}{\mathbf{a}^2}+\frac{\mathbf{V}^2}{4\mathbf{D}^2}\right)(\mathbf{t}-\boldsymbol{\tau})-\frac{\mathbf{V}}{2\mathbf{D}}(\boldsymbol{\xi}-\mathbf{x})\right]\cdot$$
$$\cdot\,\mathbf{sin}\left(\frac{\mathbf{n}\boldsymbol{\pi}}{\mathbf{a}}\mathbf{x}\right)\mathbf{sin}\left(\frac{\mathbf{n}\boldsymbol{\pi}}{\mathbf{a}}\boldsymbol{\xi}\right).$$

For convenience, $\hbar$ is introduced and defined as,

$$\hbar=\frac{\mathbf{V}}{2\mathbf{D}}\,. \tag{5-4-1-9}$$

The following $\hbar$ in the text shares the same definition as here. Then the Green's function can be expressed as,

$$\mathbf{G}(\boldsymbol{\xi},\boldsymbol{\tau};\mathbf{x},\mathbf{t})=\mathbf{H}(\mathbf{t}-\boldsymbol{\tau})\sum_{\mathbf{n}=1}^{\infty}\frac{2}{\mathbf{a}}\mathbf{exp}\left[-\mathbf{D}\left(\frac{\mathbf{n}^2\boldsymbol{\pi}^2}{\mathbf{a}^2}+\hbar^2\right)(\mathbf{t}-\boldsymbol{\tau})-\hbar(\boldsymbol{\xi}-\mathbf{x})\right]\cdot$$
$$\cdot\,\mathbf{sin}\left(\frac{\mathbf{n}\boldsymbol{\pi}}{\mathbf{a}}\mathbf{x}\right)\mathbf{sin}\left(\frac{\mathbf{n}\boldsymbol{\pi}}{\mathbf{a}}\boldsymbol{\xi}\right). \tag{5-4-1-10}$$

The final solution $\mathbf{u}(\mathbf{x},\mathbf{t})$ of the problem (5-4-1-1) with the Dirichlet- Dirichlet boundary conditions can be obtained by substituting the Green's function into (5-4-1-3).

5.4.2 Dirichlet- Robin (D-R) boundaries

As the title says, a Dirichlet boundary condition is prescribed at $\mathbf{x}=0$ and a Robin boundary at $\mathbf{x}=\mathbf{a}$. The problem is formulated as,

$$\mathbf{Lu}=\mathbf{u}_{\mathbf{t}}+\mathbf{Vu}_{\mathbf{x}}-\mathbf{Du}_{\mathbf{xx}}=\boldsymbol{\phi}(\mathbf{x},\mathbf{t}),\ 0<\mathbf{x}<\mathbf{a},\ 0<\mathbf{t}<\infty, \tag{5-4-2-1a}$$

$$\mathbf{u}(\mathbf{x},0)=\mathbf{f}(\mathbf{x}), \tag{5-4-2-1b}$$

$$\mathbf{u}(0,\mathbf{t})=\mathbf{h}_1(\mathbf{t}), \tag{5-4-2-1c}$$

$$\boldsymbol{\alpha}_2\mathbf{u}(\mathbf{a},\mathbf{t})+\boldsymbol{\beta}_2\mathbf{u}_{\mathbf{x}}(\mathbf{a},\mathbf{t})=\mathbf{h}_2(\mathbf{t}),\ (\boldsymbol{\beta}_2\neq 0). \tag{5-4-2-1d}$$

According to (5-3-2), the general solution can be expressed as,

$$\mathbf{u}(\mathbf{x},\mathbf{t})=\int_0^{\mathbf{t}}\int_0^{\mathbf{a}}\mathbf{G}\boldsymbol{\phi}\mathbf{d}\boldsymbol{\xi}\mathbf{d}\boldsymbol{\tau}+\int_0^{\mathbf{a}}[(\mathbf{Gu})|_{\boldsymbol{\tau}=0}]\mathbf{d}\boldsymbol{\xi}-$$

$$-\int_0^t \{[(\mathbf{VG}+\mathbf{DG}_\xi)\mathbf{u}-\mathbf{DGu}_\xi]\big|_{\xi=\mathbf{a}} - [(\mathbf{VG}+\mathbf{DG}_\xi)\mathbf{u}-\mathbf{DGu}_\xi]\big|_{\xi=0}\}\mathbf{d}\tau . \quad (5\text{-}3\text{-}2)$$

The boundary condition (5-4-2-1d) can be changed as,

$$\mathbf{u}_\mathbf{x}(\mathbf{a},\mathbf{t}) = \frac{1}{\beta_2}[\mathbf{h}_2(\mathbf{t})-\alpha_2\mathbf{u}(\mathbf{a},\mathbf{t})] \text{, i.e.,}$$

$$\mathbf{u}_\xi(\mathbf{a},\tau) = \frac{1}{\beta_2}[\mathbf{h}_2(\tau)-\alpha_2\mathbf{u}(\mathbf{a},\tau)] .$$

By using the equality, (5-3-2) is rearranged as,

$$\mathbf{u}(\mathbf{x},\mathbf{t}) = \int_0^t\int_0^a \mathbf{G}\phi\mathbf{d}\xi\mathbf{d}\tau + \int_0^a [(\mathbf{Gu})\big|_{\tau=0}]\mathbf{d}\xi -$$

$$-\int_0^t \{[(\mathbf{VG}+\sigma_2\mathbf{DG}+\mathbf{DG}_\xi)\mathbf{u}-\frac{\mathbf{D}}{\beta_2}\mathbf{Gh}_2]\big|_{\xi=\mathbf{a}} - [(\mathbf{VG}+\mathbf{DG}_\xi)\mathbf{u}-\mathbf{DGu}_\xi]\big|_{\xi=0}\}\mathbf{d}\tau \ ,$$

(5-4-2-2)

where, $\sigma_2 = \dfrac{\alpha_2}{\beta_2}$. (5-4-2-3)

(The following σ_2 has the same definition as here.)

According to the given boundary conditions, the terms of " $(-\mathbf{DGu}_\xi)\big|_{\xi=0}$ " and "$[(\mathbf{VG}+\sigma_2\mathbf{DG}+\mathbf{DG}_\xi)\mathbf{u}]\big|_{\xi=\mathbf{a}}$ " in (5-4-2-2) are unwelcome, so the adjoint boundary conditions for Green's function can be defined as, $\mathbf{G}\big|_{\xi=0}=0$ and $(\mathbf{VG}+\sigma_2\mathbf{DG}+\mathbf{DG}_\xi)\big|_{\xi=\mathbf{a}}=0$. Together with the governing equation of $\mathbf{G}$, the Green's function problem is defined as,

$\mathbf{L}^*\mathbf{G} = -\mathbf{G}_\tau - \mathbf{VG}_\xi - \mathbf{DG}_{\xi\xi} = \delta(\xi-\mathbf{x},\tau-\mathbf{t})$, $0<\xi<\mathbf{a}$, $0<\tau<\infty$ (5-4-2-4a)

$\mathbf{G}(0,\tau)=0$, (5-4-2-4b)

$(\mathbf{V}+\sigma_2\mathbf{D})\mathbf{G}(\mathbf{a},\tau)+\mathbf{DG}_\xi(\mathbf{a},\tau)=0$. (5-4-2-4c)

On the other hand, by applying the boundary conditions of $\mathbf{G}$, the expression about solution (5-4-2-2) can be simplified as,

$$\mathbf{u}(\mathbf{x},\mathbf{t}) = \int_0^t\int_0^a \mathbf{G}\phi\mathbf{d}\xi\mathbf{d}\tau + \int_0^a \mathbf{G}(\xi,0)\mathbf{f}(\xi)\mathbf{d}\xi - \int_0^t [-\frac{\mathbf{D}}{\beta_2}\mathbf{G}(\mathbf{a},\tau)\mathbf{h}_2(\tau) - \mathbf{DG}_\xi(0,\tau)\mathbf{h}_1(\tau)]\mathbf{d}\tau .$$

(5-4-2-5)

By comparing (5-4-2-4) to (5-3-3), it is obvious that,

$\tilde{\alpha}_1 = 1, \tilde{\alpha}_2 = (\mathbf{V}+\sigma_2), \tilde{\beta}_1 = 0, \tilde{\beta}_2 = \mathbf{D}$. (5-4-2-6)

By substituting the above values into (5-3-14), the eigenvalue problem for this case is obtained as,

$$\varphi'' + \lambda\varphi = 0,\tag{5-4-2-7a}$$

$$\varphi(0) = 0,\tag{5-4-2-7b}$$

$$(\hbar + \sigma_2)\varphi(\mathbf{a}) + \varphi'(\mathbf{a}) = 0,\tag{5-4-2-7c}$$

where, $\hbar$ is defined in (5-4-1-9) and $\hbar + \boldsymbol{\sigma}_2 \neq 0$.

The general solution of the eigenvalue problem is in the form of,

$$\varphi(\xi) = \mathbf{A}\sin(\sqrt{\lambda}\xi) + \mathbf{B}\cos(\sqrt{\lambda}\xi).\tag{5-4-1-6}$$

According to (5-4-2-7b),

$$\varphi(0) = \mathbf{A}\sin(0) + \mathbf{B}\cos(0) = \mathbf{B} = 0.$$

That is,

$\varphi(\xi) = \mathbf{A}\sin(\sqrt{\lambda}\xi)$, thus, $\varphi'(\xi) = \mathbf{A}\sqrt{\lambda}\cos(\sqrt{\lambda}\xi)$.

According to (5-4-2-7c),

$(\hbar + \sigma_2)\varphi(\mathbf{a}) + \varphi'(\mathbf{a}) = (\hbar + \sigma_2)\mathbf{A}\sin(\sqrt{\lambda}\mathbf{a}) + \mathbf{A}\sqrt{\lambda}\cos(\sqrt{\lambda}\mathbf{a}) = 0$, i.e.,

$$\tan(\mathbf{a}\sqrt{\lambda}) = \frac{-\sqrt{\lambda}}{\hbar + \boldsymbol{\sigma}_2}.\tag{5-4-2-8}$$

Assume that the $\mathbf{n}$ th positive root of the above equation is $\lambda_{\mathbf{n}}$, which is the eigenvalue of the problem. Then the eigenfunction is, accordingly,

$$\varphi_{\mathbf{n}}(\xi) = \mathbf{A}\sin(\sqrt{\lambda_{\mathbf{n}}}\xi).$$

To determine $\mathbf{A}$, $\varphi_{\mathbf{n}}$ is normalized as follows,

$$\int_0^{\mathbf{a}} \mathbf{A}^2\sin^2(\sqrt{\lambda_{\mathbf{n}}}\xi)d\xi = \frac{\mathbf{A}^2}{2}\int_0^{\mathbf{a}}[1 - \cos(2\sqrt{\lambda_{\mathbf{n}}}\xi)]d\xi = \frac{\mathbf{A}^2}{2}\left[\mathbf{a} - \frac{1}{2\sqrt{\lambda_{\mathbf{n}}}}\sin(2\sqrt{\lambda_{\mathbf{n}}}\mathbf{a})\right] =$$

$$= \frac{\mathbf{A}^2}{2}\left[\mathbf{a} - \frac{1}{\sqrt{\lambda_{\mathbf{n}}}}\sin(\sqrt{\lambda_{\mathbf{n}}}\mathbf{a})\cos(\sqrt{\lambda_{\mathbf{n}}}\mathbf{a})\right] = \frac{\mathbf{A}^2}{2}\left[\mathbf{a} + \frac{\hbar + \boldsymbol{\sigma}_2}{\lambda_{\mathbf{n}} + (\hbar + \boldsymbol{\sigma}_2)^2}\right] = 1$$

The last second step proceeds by using the equality of $\tan(\mathbf{a}\sqrt{\lambda_{\mathbf{n}}}) = \dfrac{-\sqrt{\lambda_{\mathbf{n}}}}{\hbar + \boldsymbol{\sigma}_2}$.

Thus,

$$\mathbf{A} = \sqrt{\frac{2[\lambda_{\mathbf{n}} + (\hbar + \boldsymbol{\sigma}_2)^2]}{\mathbf{a}[\lambda_{\mathbf{n}} + (\hbar + \boldsymbol{\sigma}_2)^2] + (\hbar + \boldsymbol{\sigma}_2)}}.$$

So the normalized eigenfunction is obtained as,

$$\varphi_{\mathbf{n}}(\xi) = \sqrt{\frac{2[\lambda_{\mathbf{n}} + (\hbar + \boldsymbol{\sigma}_2)^2]}{\mathbf{a}[\lambda_{\mathbf{n}} + (\hbar + \boldsymbol{\sigma}_2)^2] + (\hbar + \boldsymbol{\sigma}_2)}}\sin(\sqrt{\lambda_{\mathbf{n}}}\xi).\tag{5-4-2-9}$$

Based on (5-3-27), the Green’s function is obtained as,

$$G(\xi,\tau;x,t)=H(t-\tau)\sum_{n=1}^{\infty}\frac{2[\lambda_n+(\hbar+\sigma_2)^2]}{a[\lambda_n+(\hbar+\sigma_2)^2]+(\hbar+\sigma_2)}\cdot$$

$$\cdot\exp[-D(\lambda_n+\hbar^2)(t-\tau)-\hbar(\xi-x)]\sin(\sqrt{\lambda_n}x)\sin(\sqrt{\lambda_n}\xi)\,. \qquad (5\text{-}4\text{-}2\text{-}10)$$

The final solution about **u** of the problem (5-4-2-1) with the Dirichlet- Robin boundary conditions can be obtained by substituting the Green's function into (5-4-2-5).

5.4.3 Robin- Dirichlet (R-D) boundaries

Although the boundary conditions are quite similar to Example 5.4.2, it is still being put here to complete the boundary combinations. The problem is formulated as,

$$Lu=u_t+Vu_x-Du_{xx}=\phi(x,t)\,,\ 0<x<a\,,\ 0<t<\infty\,, \qquad (5\text{-}4\text{-}3\text{-}1a)$$

$$u(x,0)=f(x)\,, \qquad (5\text{-}4\text{-}3\text{-}1b)$$

$$\alpha_1 u(0,t)+\beta_1 u_x(0,t)=h_1(t)\,,\ (\beta_1\neq 0)\,, \qquad (5\text{-}4\text{-}3\text{-}1c)$$

$$u(a,t)=h_2(t)\,. \qquad (5\text{-}4\text{-}3\text{-}1d)$$

According to (5-3-2), the general solution can be expressed as,

$$u(x,t)=\int_0^t\int_0^a G\phi d\xi d\tau+\int_0^a[(Gu)|_{\tau=0}]d\xi-$$

$$-\int_0^t\{[(VG+DG_\xi)u-DGu_\xi]|_{\xi=a}-[(VG+DG_\xi)u-DGu_\xi]|_{\xi=0}\}d\tau\,. \qquad (5\text{-}3\text{-}2)$$

Based on the boundary condition (5-4-3-1c),

$$u_\xi(0,\tau)=\frac{1}{\beta_1}[h_1(\tau)-\alpha_1 u(0,\tau)]\,.$$

By using the equality, (5-3-2) is rearranged as,

$$u(x,t)=\int_0^t\int_0^a G\phi d\xi d\tau+\int_0^a[(Gu)|_{\tau=0}]d\xi-$$

$$-\int_0^t\{[(VG+DG_\xi)u-DGu_\xi]|_{\xi=a}-[(VG+\sigma_1 DG+DG_\xi)u-\frac{D}{\beta_1}Gh_1]|_{\xi=0}\}d\tau\ ,$$

(5-4-3-2)

where, $\sigma_1=\frac{\alpha_1}{\beta_1}$. (5-4-3-3)

(The following σ_1 has the same definition as here.)

By vanishings of the unwelcome terms in (5-4-3-2), the adjoint Green's function problem is formulated as,

$$L^*G=-G_\tau-VG_\xi-DG_{\xi\xi}=\delta(\xi-x,\tau-t)\,,\ 0<\xi<a\,,\ 0<\tau<\infty \qquad (5\text{-}4\text{-}3\text{-}4a)$$

$$(V+\sigma_1 D)G(0,\tau)+DG_\xi(0,\tau)=0\,, \qquad (5\text{-}4\text{-}3\text{-}4b)$$

$$\mathbf{G(a,\tau)} = 0\,. \tag{5-4-3-4c}$$

Therefore the expression (5-4-3-2) can be simplified as,

$$\mathbf{u(x,t)} = \int_0^{\mathbf{t}}\int_0^{\mathbf{a}} \mathbf{G\phi d\xi d\tau} + \int_0^{\mathbf{a}} \mathbf{G(\xi,}0\mathbf{)f(\xi)d\xi} - \int_0^{\mathbf{t}} \mathbf{[DG_{\xi}(a,\tau)h}_2\mathbf{(\tau)} + \frac{\mathbf{D}}{\beta_1}\mathbf{G(}0\mathbf{,\tau)h}_1\mathbf{(\tau)]d\tau}\,.$$

(5-4-3-5)

By comparing (5-4-3-4) to (5-3-3), we have,

$$\tilde{\alpha}_1 = \mathbf{(V+\sigma_2)},\tilde{\alpha}_2 = 1\mathbf{,}\tilde{\beta}_1 = \mathbf{D,}\tilde{\beta}_2 = 0\,. \tag{5-4-3-6}$$

By substituting the above values into (5-3-14), the corresponding eigenvalue problem is obtained as,

$$\varphi'' + \lambda\varphi = 0\,, \tag{5-4-3-7a}$$

$$(\hbar + \sigma_1)\varphi(0) + \varphi'(0) = 0\,, \tag{5-4-3-7b}$$

$$\varphi\mathbf{(a)} = 0\,, \tag{5-4-3-7c}$$

where $\hbar + \sigma_1 \neq 0$.

By solving the eigenvalue problem, the eigenvalues satisfies the equation of,

$$\mathbf{tan(a}\sqrt{\lambda}\mathbf{)} = \frac{\sqrt{\lambda}}{\hbar + \sigma_1}\,, \tag{5-4-3-8}$$

which $\mathbf{n}$ th positive root is denoted as $\lambda_{\mathbf{n}}$. The corresponding eigenfunction is,

$$\varphi_{\mathbf{n}}(\xi) = \sqrt{\frac{2[\lambda_{\mathbf{n}} + (\hbar+\sigma_1)^2]}{\mathbf{a}[\lambda_{\mathbf{n}} + (\hbar+\sigma_1)^2] - (\hbar+\sigma_1)}}\,\mathbf{sin}[\sqrt{\lambda_{\mathbf{n}}}(\xi - \mathbf{a})]\,. \tag{5-4-3-9}$$

Based on (5-3-27), the Green’s function is obtained as,

$$\mathbf{G(\xi,\tau;x,t) = H(t-\tau)}\sum_{\mathbf{n}=1}^{\infty}\frac{2[\lambda_{\mathbf{n}} + (\hbar+\sigma_1)^2]}{\mathbf{a}[\lambda_{\mathbf{n}} + (\hbar+\sigma_1)^2] - (\hbar+\sigma_1)}\,\cdot$$

$$\cdot\,\mathbf{exp[-D(\lambda_n} + \hbar^2\mathbf{)(t-\tau)} - \hbar(\xi - \mathbf{x})]\mathbf{sin}[\sqrt{\lambda_{\mathbf{n}}}(\mathbf{x}-\mathbf{a})]\mathbf{sin}[\sqrt{\lambda_{\mathbf{n}}}(\xi - \mathbf{a})]\,. \tag{5-4-3-10}$$

The final solution of the problem (5-4-3-1) with the Robin- Dirichlet boundary conditions can be obtained by substituting the Green’s function into (5-4-3-5).

5.4.4 Robin- Robin (R-R) boundaries

This problem might be the most complicated one. It is defined as,

$$\mathbf{Lu = u_t + Vu_x - Du_{xx} = \phi(x,t)},\ 0 < \mathbf{x} < \mathbf{a},\ 0 < \mathbf{t} < \infty\,, \tag{5-4-4-1a}$$

$$\mathbf{u(x,}0\mathbf{) = f(x)}\,, \tag{5-4-4-1b}$$

$$\alpha_1\mathbf{u(}0\mathbf{,t)} + \beta_1\mathbf{u_x(}0\mathbf{,t) = h}_1\mathbf{(t)},\ (\beta_1 \neq 0)\,, \tag{5-4-4-1c}$$

$$\alpha_2\mathbf{u(a,t)} + \beta_2\mathbf{u_x(a,t) = h}_2\mathbf{(t)},\ (\beta_2 \neq 0)\,. \tag{5-4-4-1d}$$

The general solution is in (5-3-2),

$$\mathbf{u(x,t)} = \int_0^t\int_0^a \mathbf{G\phi d\xi d\tau} + \int_0^a [\mathbf{(Gu)}|_{\tau=0}]\mathbf{d\xi} -$$

$$-\int_0^t \{[\mathbf{(VG+DG_\xi)u - DGu_\xi}]|_{\xi=a} - [\mathbf{(VG+DG_\xi)u - DGu_\xi}]|_{\xi=0}\}\mathbf{d\tau}\,. \quad \text{(5-3-2)}$$

Boundary conditions of (5-4-4-1c) and (5-4-4-1d) can be changed as,

$$\mathbf{u_x}(0,\mathbf{t}) = \frac{1}{\beta_1}[\mathbf{h}_1\mathbf{(t)} - \alpha_1\mathbf{u}(0,\mathbf{t})],\ \mathbf{u_x(a,t)} = \frac{1}{\beta_2}[\mathbf{h}_2\mathbf{(t)} - \alpha_2\mathbf{u(a,t)}],$$

respectively. By utilizing the equalities, (5-3-2) is rearranged as,

$$\mathbf{u(x,t)} = \int_0^t\int_0^a \mathbf{G\phi d\xi d\tau} + \int_0^a [\mathbf{(Gu)}|_{\tau=0}]\mathbf{d\xi} -$$

$$-\int_0^t \{[\mathbf{(VG+\sigma_2 DG+DG_\xi)u} - \frac{\mathbf{D}}{\beta_2}\mathbf{Gh}_2]|_{\xi=a} - [\mathbf{(VG+\sigma_1 DG+DG_\xi)u} - \frac{\mathbf{D}}{\beta_1}\mathbf{Gh}_1]|_{\xi=0}\}\mathbf{d\tau}$$

(5-4-4-2)

where, σ_1 and σ_2 are defined in (5-4-3-3) and (5-4-2-3), respectively. In this case, the unwelcome terms are " $[\mathbf{(VG+\sigma_1 DG+DG_\xi)u}]|_{\xi=0}$ " and " $[\mathbf{(VG+\sigma_2 DG+DG_\xi)u}]|_{\xi=a}$ " in (5-4-4-2). Thus the assumptions about Green's function's boundary conditions of $\mathbf{(VG+\sigma_1 DG+DG_\xi)}|_{\xi=0} = 0$ and $\mathbf{(VG+\sigma_2 DG+DG_\xi)}|_{\xi=a} = 0$ can make them disappear. So, the Green's function problem is defined as,

$$\mathbf{L^*G} = -\mathbf{G}_\tau - \mathbf{VG}_\xi - \mathbf{DG}_{\xi\xi} = \delta(\xi - \mathbf{x}, \tau - \mathbf{t}),\ 0 < \xi < \mathbf{a},\ 0 < \tau < \infty \quad \text{(5-4-4-3a)}$$

$$\mathbf{(V+\sigma_1 D)G}(0,\tau) + \mathbf{DG}_\xi(0,\tau) = 0, \quad \text{(5-4-4-3b)}$$

$$\mathbf{(V+\sigma_2 D)G(a,\tau)} + \mathbf{DG}_\xi\mathbf{(a,\tau)} = 0. \quad \text{(5-4-4-3c)}$$

By using (5-4-4-3b) and (5-4-4-3c), the expression of solution (5-4-4-2) becomes as,

$$\mathbf{u(x,t)} = \int_0^t\int_0^a \mathbf{G\phi d\xi d\tau} + \int_0^a \mathbf{G(\xi,}0\mathbf{)f(\xi)d\xi} + \int_0^t [\frac{\mathbf{D}}{\beta_2}\mathbf{G(a,\tau)h}_2\mathbf{(\tau)} - \frac{\mathbf{D}}{\beta_1}\mathbf{G}(0,\tau)\mathbf{h}_1\mathbf{(\tau)}]\mathbf{d\tau}\,.$$

(5-4-4-4)

By comparing (5-4-4-3) to (5-3-3), we have,

$$\tilde{\alpha}_1 = \mathbf{(V+\sigma_1 D)}, \tilde{\alpha}_2 = \mathbf{(V+\sigma_2 D)}, \tilde{\beta}_1 = \tilde{\beta}_2 = \mathbf{D}\,.$$

By applying these values into (5-3-14), the eigenvalue problem of this case is obtained as,

$$\varphi'' + \lambda\varphi = 0, \quad \text{(5-4-4-5a)}$$

$$(\hbar + \sigma_1)\varphi(0) + \varphi'(0) = 0, \quad \text{(5-4-4-5b)}$$

$$(\hbar + \sigma_2)\varphi(\mathbf{a}) + \varphi'(\mathbf{a}) = 0, \quad \text{(5-4-4-5c)}$$

where $(\hbar+\sigma_1)^2+(\hbar+\sigma_2)^2 \neq 0$.

The eigenvalue problem is solved as follows,

$\varphi(\xi)=\mathbf{A}\sin(\sqrt{\lambda}\xi)+\mathbf{B}\cos(\sqrt{\lambda}\xi)$, thus,

$\varphi'(\xi)=\mathbf{A}\sqrt{\lambda}\cos(\sqrt{\lambda}\xi)-\mathbf{B}\sqrt{\lambda}\sin(\sqrt{\lambda}\xi)$,

where $\mathbf{A}$ and $\mathbf{B}$ are undetermined constants. By using (5-4-4-5b), we have,

$(\hbar+\sigma_1)\varphi(0)+\varphi'(0)=(\hbar+\sigma_1)\mathbf{B}+\mathbf{A}\sqrt{\lambda}=0$.

By using (5-4-4-5c),

$$(\hbar+\sigma_2)\varphi(\mathbf{a})+\varphi'(\mathbf{a})=(\hbar+\sigma_2)\mathbf{A}\sin(\sqrt{\lambda}\mathbf{a})+(\hbar+\sigma_2)\mathbf{B}\cos(\sqrt{\lambda}\mathbf{a})+$$
$$+\mathbf{A}\sqrt{\lambda}\cos(\sqrt{\lambda}\mathbf{a})-\mathbf{B}\sqrt{\lambda}\sin(\sqrt{\lambda}\mathbf{a})=0.$$

Based on the above two equations, we have,

$$\tan(\mathbf{a}\sqrt{\lambda})=\frac{\sqrt{\lambda}(\sigma_2-\sigma_1)}{\lambda+(\hbar+\sigma_1)(\hbar+\sigma_2)}. \tag{5-4-4-6}$$

Assume that the $\mathbf{n}$th positive root of the above equation is $\lambda_{\mathbf{n}}$, $\mathbf{n}=1,2,3\ldots$. $\lambda_{\mathbf{n}}$ is the eigenvalue of the problem. Then the eigenfunction is, accordingly,

$\varphi_{\mathbf{n}}(\xi)=\tilde{\mathbf{A}}[\sqrt{\lambda_{\mathbf{n}}}\cos(\sqrt{\lambda_{\mathbf{n}}}\xi)-(\hbar+\sigma_1)\sin(\sqrt{\lambda_{\mathbf{n}}}\xi)]$,

where, $\tilde{\mathbf{A}}$ is a constant, which is determined by the normalization of $\varphi_{\mathbf{n}}(\xi)$.

$$\int_0^{\mathbf{a}}\varphi_{\mathbf{n}}^2(\xi)\mathbf{d}\xi=\int_0^{\mathbf{a}}\tilde{\mathbf{A}}^2[\sqrt{\lambda_{\mathbf{n}}}\cos(\sqrt{\lambda_{\mathbf{n}}}\xi)-(\hbar+\sigma_1)\sin(\sqrt{\lambda_{\mathbf{n}}}\xi)]^2\mathbf{d}\xi$$

$$=\tilde{\mathbf{A}}^2\int_0^{\mathbf{a}}[(\hbar+\sigma_1)^2\sin^2(\sqrt{\lambda_{\mathbf{n}}}\xi)+\lambda_{\mathbf{n}}\cos^2(\sqrt{\lambda_{\mathbf{n}}}\xi)-\sqrt{\lambda_{\mathbf{n}}}(\hbar+\sigma_1)\sin(2\sqrt{\lambda_{\mathbf{n}}}\xi)]\mathbf{d}\xi$$

$$=\frac{\tilde{\mathbf{A}}^2}{2}\int_0^{\mathbf{a}}\{(\hbar+\sigma_1)^2[1-\cos(2\sqrt{\lambda_{\mathbf{n}}}\xi)]+\lambda_{\mathbf{n}}[1+\cos(2\sqrt{\lambda_{\mathbf{n}}}\xi)]-2\sqrt{\lambda_{\mathbf{n}}}(\hbar+\sigma_1)\sin(2\sqrt{\lambda_{\mathbf{n}}}\xi)\}\mathbf{d}\xi$$

$$=\frac{\tilde{\mathbf{A}}^2}{2}\{[\lambda_{\mathbf{n}}+(\hbar+\sigma_1)^2]\mathbf{a}+[\lambda_{\mathbf{n}}-(\hbar+\sigma_1)^2]\int_0^{\mathbf{a}}\cos(2\sqrt{\lambda_{\mathbf{n}}}\xi)\mathbf{d}\xi-2\sqrt{\lambda_{\mathbf{n}}}(\hbar+\sigma_1)\int_0^{\mathbf{a}}\sin(2\sqrt{\lambda_{\mathbf{n}}}\xi)\mathbf{d}\xi\}$$

$$=\frac{\tilde{\mathbf{A}}^2}{2}\{[\lambda_{\mathbf{n}}+(\hbar+\sigma_1)^2]\mathbf{a}+[\lambda_{\mathbf{n}}-(\hbar+\sigma_1)^2]\frac{1}{\sqrt{\lambda_{\mathbf{n}}}}\sin(\sqrt{\lambda_{\mathbf{n}}}\mathbf{a})\cos(\sqrt{\lambda_{\mathbf{n}}}\mathbf{a})-2(\hbar+\sigma_1)\sin^2(\sqrt{\lambda_{\mathbf{n}}}\mathbf{a})\}$$

$$=\frac{\tilde{\mathbf{A}}^2}{2}\{[\lambda_{\mathbf{n}}+(\hbar+\sigma_1)^2]\mathbf{a}+[\lambda_{\mathbf{n}}-(\hbar+\sigma_1)^2]\frac{1}{\sqrt{\lambda_{\mathbf{n}}}}\frac{\sin^2(\sqrt{\lambda_{\mathbf{n}}}\mathbf{a})}{\tan(\sqrt{\lambda_{\mathbf{n}}}\mathbf{a})}-2(\hbar+\sigma_1)\sin^2(\sqrt{\lambda_{\mathbf{n}}}\mathbf{a})\}$$

Noting that,

$$\tan(\mathbf{a}\sqrt{\lambda_{\mathbf{n}}})=\frac{\sqrt{\lambda_{\mathbf{n}}}(\sigma_2-\sigma_1)}{\lambda_{\mathbf{n}}+(\hbar+\sigma_1)(\hbar+\sigma_2)}, \text{ thus,}$$

$$\sin^2(a\sqrt{\lambda_n}) = \frac{\lambda_n(\sigma_2-\sigma_1)^2}{[\lambda_n+(\hbar+\sigma_1)(\hbar+\sigma_2)]^2+\lambda_n(\sigma_2-\sigma_1)^2}$$

Substituting them into the integration of $\int_0^a \varphi_n^2(\xi)d\xi =$

$$= \frac{\tilde{A}^2}{2}\{[\lambda_n+(\hbar+\sigma_1)^2]a +$$

$$+\frac{(\sigma_2-\sigma_1)[\lambda_n^2+\lambda_n(\hbar+\sigma_1)(\sigma_2-\sigma_1)-(\hbar+\sigma_1)^3(\hbar+\sigma_2)]}{[\lambda_n+(\hbar+\sigma_1)(\hbar+\sigma_2)]^2+\lambda_n(\sigma_2-\sigma_1)^2} -$$

$$-\frac{2\lambda_n(\hbar+\sigma_1)(\sigma_2-\sigma_1)^2}{[\lambda_n+(\hbar+\sigma_1)(\hbar+\sigma_2)]^2+\lambda_n(\sigma_2-\sigma_1)^2}\}$$

$$= \frac{\tilde{A}^2}{2}\{[\lambda_n+(\hbar+\sigma_1)^2]a+\frac{(\sigma_2-\sigma_1)[\lambda_n^2-\lambda_n(\hbar+\sigma_1)(\sigma_2-\sigma_1)-(\hbar+\sigma_1)^3(\hbar+\sigma_2)]}{[\lambda_n+(\hbar+\sigma_1)(\hbar+\sigma_2)]^2+\lambda_n(\sigma_2-\sigma_1)^2}\}$$

$$= \frac{\tilde{A}^2}{2}\frac{\Psi_{2n}+\Psi_{3n}}{\Psi_{1n}} = 1,$$

where,

$$\Psi_{1n} = [\lambda_n+(\hbar+\sigma_1)(\hbar+\sigma_2)]^2+\lambda_n(\sigma_2-\sigma_1)^2,$$

$$\Psi_{2n} = a[\lambda_n+(\hbar+\sigma_1)^2]\cdot\Psi_{1n}, \tag{5-4-4-7}$$

$$\Psi_{3n} = (\sigma_2-\sigma_1)[\lambda_n^2-\lambda_n(\hbar+\sigma_1)(\sigma_2-\sigma_1)-(\hbar+\sigma_1)^3(\hbar+\sigma_2)].$$

Therefore,

$$\tilde{A} = \sqrt{\frac{2\Psi_{1n}}{\Psi_{2n}+\Psi_{3n}}}.$$

Then the normalized eigenfunction is,

$$\varphi_n(\xi) = \sqrt{\frac{2\Psi_{1n}}{\Psi_{2n}+\Psi_{3n}}}[\sqrt{\lambda_n}\cos(\sqrt{\lambda_n}\xi)-(\hbar+\sigma_1)\sin(\sqrt{\lambda_n}\xi)]. \tag{5-4-4-8}$$

Therefore, according to (5-3-27), the Green's function is,

$$G(\xi,\tau;x,t) = H(t-\tau)\sum_{n=1}^{\infty}\frac{2\Psi_{1n}}{\Psi_{2n}+\Psi_{3n}}\exp[-D(\lambda_n+\hbar^2)(t-\tau)-\hbar(\xi-x)]\cdot$$

$$\cdot[\sqrt{\lambda_n}\cos(\sqrt{\lambda_n}x)-(\hbar+\sigma_1)\sin(\sqrt{\lambda_n}x)][\sqrt{\lambda_n}\cos(\sqrt{\lambda_n}\xi)-(\hbar+\sigma_1)\sin(\sqrt{\lambda_n}\xi)], \tag{5-4-4-9}$$

then the solution of problem (5-4-4-1) with the Robin- Robin boundary conditions can be obtained by substituting the Green's function into (5-4-4-4).

As stated at the beginning of the subsection, the Neumann boundary condition can be treated as a special case of the Robin boundary. Among the boundary combinations, the Neumann- related are the D-N, N-D, N-N, N-R and R-N combinations. Although it is straightforward to obtain the solutions by shrinking from a Robin to a Neumann boundary, the Green's functions of the remaining boundary combinations are all solved in the following examples, but in a more succinct way.

5.4.5 Dirichlet- Neumann (D-N) boundaries

Based on the solved D-R problem (Example 5.4.2), it is specified as,

$\alpha_2 = 0$, namely, $\boldsymbol{\sigma}_2 = 0$, and $\boldsymbol{\beta}_2 = 1$.

The solution of D-N problem can be obtained by substituting the newly specified parameters into the equations or solutions of the D-R problem.

The D-N problem is described as,

$$\mathbf{Lu} = \mathbf{u}_t + \mathbf{Vu}_x - \mathbf{Du}_{xx} = \boldsymbol{\phi}(\mathbf{x},\mathbf{t}),\ 0 < \mathbf{x} < \mathbf{a},\ 0 < \mathbf{t} < \infty, \tag{5-4-5-1a}$$

$$\mathbf{u}(\mathbf{x},0) = \mathbf{f}(\mathbf{x}), \tag{5-4-5-1b}$$

$$\mathbf{u}(0,\mathbf{t}) = \mathbf{h}_1(\mathbf{t}), \tag{5-4-5-1c}$$

$$\mathbf{u}_x(\mathbf{a},\mathbf{t}) = \mathbf{h}_2(\mathbf{t}). \tag{5-4-5-1d}$$

The adjoint Green's function problem is formulated as, accordingly,

$$\mathbf{L}^*\mathbf{G} = -\mathbf{G}_\tau - \mathbf{VG}_\xi - \mathbf{DG}_{\xi\xi} = \boldsymbol{\delta}(\xi - \mathbf{x}, \tau - \mathbf{t}),\ 0 < \xi < \mathbf{a},\ 0 < \tau < \infty \tag{5-4-5-2a}$$

$$\mathbf{G}(0,\tau) = 0, \tag{5-4-5-2b}$$

$$\mathbf{VG}(\mathbf{a},\tau) + \mathbf{DG}_\xi(\mathbf{a},\tau) = 0. \tag{5-4-5-2c}$$

The corresponding eigenvalue problem is,

$$\varphi'' + \lambda\varphi = 0, \tag{5-4-5-3a}$$

$$\varphi(0) = 0, \tag{5-4-5-3b}$$

$$\hbar\varphi(\mathbf{a}) + \varphi'(\mathbf{a}) = 0, \tag{5-4-5-3c}$$

where $\hbar \neq 0$. The solutions are obtained by substituting $\boldsymbol{\sigma}_2 = 0$ into (5-4-2-8) and (5-4-2-9),

$$\tan(\mathbf{a}\sqrt{\lambda}) = \frac{-\sqrt{\lambda}}{\hbar + \boldsymbol{\sigma}_2} = -\frac{\sqrt{\lambda}}{\hbar}, \text{ and} \tag{5-4-5-4}$$

$$\varphi_n(\xi) = \sqrt{\frac{2[\lambda_n + (\hbar + \sigma_2)^2]}{\mathbf{a}[\lambda_n + (\hbar + \sigma_2)^2] + (\hbar + \sigma_2)}}\,\sin(\sqrt{\lambda_n}\,\xi) = \sqrt{\frac{2(\lambda_n + \hbar^2)}{\mathbf{a}(\lambda_n + \hbar^2) + \hbar}}\,\sin(\sqrt{\lambda_n}\,\xi). \tag{5-4-5-5}$$

Then the Green's function is,

$$\mathbf{G}(\xi,\tau;\mathbf{x},\mathbf{t}) = \mathbf{H}(\mathbf{t}-\tau)\sum_{n=1}^{\infty}\frac{2(\lambda_n + \hbar^2)}{\mathbf{a}(\lambda_n + \hbar^2) + \hbar}\cdot$$

$$\cdot\exp[-\mathbf{D}(\lambda_n + \hbar^2)(\mathbf{t}-\tau) - \hbar(\xi - \mathbf{x})]\sin(\sqrt{\lambda_n}\,\mathbf{x})\sin(\sqrt{\lambda_n}\,\xi). \tag{5-4-5-6}$$

The final solution about $\mathbf{u}$ is expressed as (5-4-2-5) with $\boldsymbol{\beta}_2 = 1$,

$$\mathbf{u(x,t)} = \int_0^t\int_0^a \mathbf{G\phi d\xi d\tau} + \int_0^a \mathbf{G(\xi,}0\mathbf{)f(\xi)d\xi} - \int_0^t \mathbf{[-DG(a,\tau)h}_2\mathbf{(\tau) - DG}_\xi(0,\tau)\mathbf{h}_1\mathbf{(\tau)]d\tau}.$$

(5-4-5-7)

5.4.6 Neumann- Dirichlet (N-D) boundaries

Based on the solved R-D problem (Example 5.4.3), it is specified as,

$\alpha_1 = 0$, namely, $\sigma_1 = 0$, and $\beta_1 = 1$.

The solution of N-D problem can be obtained by substituting the newly specified parameters into the equations or solutions of the R-D problem.

The N-D problem is described as,

$$\mathbf{Lu = u_t + Vu_x - Du_{xx} = \phi(x,t)},\ 0 < \mathbf{x} < \mathbf{a},\ 0 < \mathbf{t} < \infty, \qquad (5\text{-}4\text{-}6\text{-}1a)$$

$$\mathbf{u(x,}0\mathbf{) = f(x)}, \qquad (5\text{-}4\text{-}6\text{-}1b)$$

$$\mathbf{u_x}(0,\mathbf{t}) = \mathbf{h}_1\mathbf{(t)}, \qquad (5\text{-}4\text{-}6\text{-}1c)$$

$$\mathbf{u(a,t) = h}_2\mathbf{(t)}. \qquad (5\text{-}4\text{-}6\text{-}1d)$$

The adjoint Green's function problem is formulated as, accordingly,

$$\mathbf{L^*G = -G_\tau - VG_\xi - DG_{\xi\xi} = \delta(\xi - x, \tau - t)},\ 0 < \xi < \mathbf{a},\ 0 < \tau < \infty \qquad (5\text{-}4\text{-}6\text{-}2a)$$

$$\mathbf{VG}(0,\tau) + \mathbf{DG}_\xi(0,\tau) = 0, \qquad (5\text{-}4\text{-}6\text{-}2b)$$

$$\mathbf{G(a,\tau)} = 0. \qquad (5\text{-}4\text{-}6\text{-}2c)$$

The corresponding eigenvalue problem is,

$$\varphi'' + \lambda\varphi = 0, \qquad (5\text{-}4\text{-}6\text{-}3a)$$

$$\hbar\varphi(0) + \varphi'(0) = 0, \qquad (5\text{-}4\text{-}6\text{-}3b)$$

$$\varphi(\mathbf{a}) = 0, \qquad (5\text{-}4\text{-}6\text{-}3c)$$

where $\hbar \neq 0$. The solutions are obtained by substituting $\sigma_1 = 0$ into (5-4-3-8) and (5-4-3-9),

$$\mathbf{tan(a}\sqrt{\lambda}\mathbf{)} = \frac{\sqrt{\lambda}}{\hbar + \sigma_1} = \frac{\sqrt{\lambda}}{\hbar}, \text{ and} \qquad (5\text{-}4\text{-}6\text{-}4)$$

$$\varphi_{\mathbf{n}}(\xi) = \sqrt{\frac{2[\lambda_{\mathbf{n}} + (\hbar + \sigma_1)^2]}{\mathbf{a}[\lambda_{\mathbf{n}} + (\hbar + \sigma_1)^2] - (\hbar + \sigma_1)}}\,\mathbf{sin}[\sqrt{\lambda_{\mathbf{n}}}(\xi - \mathbf{a})] =$$

$$= \sqrt{\frac{2(\lambda_{\mathbf{n}} + \hbar^2)}{\mathbf{a}(\lambda_{\mathbf{n}} + \hbar^2) - \hbar}}\,\mathbf{sin}[\sqrt{\lambda_{\mathbf{n}}}(\xi - \mathbf{a})]. \qquad (5\text{-}4\text{-}6\text{-}5)$$

Then the Green's function is,

$$\mathbf{G(\xi,\tau;x,t) = H(t-\tau)}\sum_{\mathbf{n}=1}^{\infty}\frac{2(\lambda_{\mathbf{n}} + \hbar^2)}{\mathbf{a}(\lambda_{\mathbf{n}} + \hbar^2) - \hbar}\cdot$$

$$\cdot\exp[-D(\lambda_n+\hbar^2)(t-\tau)-\hbar(\xi-x)]\sin[\sqrt{\lambda_n}(x-a)]\sin[\sqrt{\lambda_n}(\xi-a)]. \quad (5\text{-}4\text{-}6\text{-}6)$$

The final solution about $\mathbf{u}$ is expressed as (5-4-3-5) with $\beta_1=1$,

$$\mathbf{u}(\mathbf{x},\mathbf{t})=\int_0^t\int_0^a \mathbf{G}\phi\mathbf{d}\xi\mathbf{d}\tau+\int_0^a \mathbf{G}(\xi,0)\mathbf{f}(\xi)\mathbf{d}\xi-\int_0^t[\mathbf{DG}_\xi(\mathbf{a},\tau)\mathbf{h}_2(\tau)+\mathbf{DG}(0,\tau)\mathbf{h}_1(\tau)]\mathbf{d}\tau. \quad (5\text{-}4\text{-}6\text{-}7)$$

5.4.7 Neumann- Neumann (N-N) boundaries

Based on the solved R-R problem (Example 5.4.4), it is specified as,

$\alpha_1=\alpha_2=0$, namely, $\sigma_1=\sigma_2=0$, and $\beta_1=\beta_2=1$.

The solution of N-N problem can be obtained by substituting the newly specified parameters into the R-R problem.

The N-N problem is described as,

$$\mathbf{Lu}=\mathbf{u}_t+\mathbf{Vu}_x-\mathbf{Du}_{xx}=\phi(\mathbf{x},\mathbf{t}),\ 0<\mathbf{x}<\mathbf{a},\ 0<\mathbf{t}<\infty, \quad (5\text{-}4\text{-}7\text{-}1a)$$

$$\mathbf{u}(\mathbf{x},0)=\mathbf{f}(\mathbf{x}), \quad (5\text{-}4\text{-}7\text{-}1b)$$

$$\mathbf{u}_x(0,\mathbf{t})=\mathbf{h}_1(\mathbf{t}), \quad (5\text{-}4\text{-}7\text{-}1c)$$

$$\mathbf{u}_x(\mathbf{a},\mathbf{t})=\mathbf{h}_2(\mathbf{t}). \quad (5\text{-}4\text{-}7\text{-}1d)$$

The adjoint Green's function problem is formulated as, accordingly,

$$\mathbf{L}^*\mathbf{G}=-\mathbf{G}_\tau-\mathbf{VG}_\xi-\mathbf{DG}_{\xi\xi}=\delta(\xi-\mathbf{x},\tau-\mathbf{t}),\ 0<\xi<\mathbf{a},\ 0<\tau<\infty \quad (5\text{-}4\text{-}7\text{-}2a)$$

$$\mathbf{VG}(0,\tau)+\mathbf{DG}_\xi(0,\tau)=0, \quad (5\text{-}4\text{-}7\text{-}2b)$$

$$\mathbf{VG}(\mathbf{a},\tau)+\mathbf{DG}_\xi(\mathbf{a},\tau)=0. \quad (5\text{-}4\text{-}7\text{-}2c)$$

The corresponding eigenvalue problem is,

$$\varphi''+\lambda\varphi=0, \quad (5\text{-}4\text{-}7\text{-}3a)$$

$$\hbar\varphi(0)+\varphi'(0)=0, \quad (5\text{-}4\text{-}7\text{-}3b)$$

$$\hbar\varphi(\mathbf{a})+\varphi'(\mathbf{a})=0, \quad (5\text{-}4\text{-}7\text{-}3c)$$

where $\hbar\neq 0$. The solutions are obtained by substituting $\sigma_1=\sigma_2=0$ into (5-4-4-6), (5-4-4-7) and (5-4-4-8),

$$\tan(\mathbf{a}\sqrt{\lambda})=\frac{\sqrt{\lambda}(\sigma_2-\sigma_1)}{\lambda+(\hbar+\sigma_1)(\hbar+\sigma_2)}=0\text{, thus,}$$

$$\lambda_n=\frac{\mathbf{n}^2\pi^2}{\mathbf{a}^2},\ \mathbf{n}=1,2,3\ldots, \quad (5\text{-}4\text{-}7\text{-}4)$$

then,

$$\Psi_{1n}=[\lambda_n+(\hbar+\sigma_1)(\hbar+\sigma_2)]^2+\lambda_n(\sigma_2-\sigma_1)^2=(\lambda_n+\hbar^2)^2,$$

$$\Psi_{2n}=\mathbf{a}[\lambda_n+(\hbar+\sigma_1)^2]\cdot\Psi_{1n}=\mathbf{a}(\lambda_n+\hbar^2)\cdot\Psi_{1n}, \quad (5\text{-}4\text{-}7\text{-}5)$$

$$\Psi_{3n} = (\sigma_2 - \sigma_1)[\lambda_n^2 - \lambda_n(\hbar + \sigma_1)(\sigma_2 - \sigma_1) - (\hbar + \sigma_1)^3(\hbar + \sigma_2)] = 0,$$

thus,

$$\varphi_n(\xi) = \sqrt{\frac{2\Psi_{1n}}{\Psi_{2n} + \Psi_{3n}}}[\sqrt{\lambda_n}\cos(\sqrt{\lambda_n}\xi) - (\hbar + \sigma_1)\sin(\sqrt{\lambda_n}\xi)] =$$

$$= \sqrt{\frac{2}{a(\lambda_n + \hbar^2)}}[\frac{n\pi}{a}\cos(\sqrt{\lambda_n}\xi) - \hbar\sin(\sqrt{\lambda_n}\xi)] =$$

$$= \sqrt{\frac{2a}{a^2\hbar^2 + n^2\pi^2}}\left[\frac{n\pi}{a}\cos\left(\frac{n\pi}{a}\xi\right) - \hbar\sin\left(\frac{n\pi}{a}\xi\right)\right]. \qquad (5\text{-}4\text{-}7\text{-}6)$$

Then the Green's function is,

$$G(\xi,\tau;x,t) = H(t-\tau)\sum_{n=1}^{\infty}\frac{2a}{a^2\hbar^2 + n^2\pi^2}\exp\left[-D\left(\frac{n^2\pi^2 + a^2\hbar^2}{a^2}\right)(t-\tau) - \hbar(\xi - x)\right]\cdot$$

$$\cdot\left[\frac{n\pi}{a}\cos\left(\frac{n\pi x}{a}\right) - \hbar\sin\left(\frac{n\pi x}{a}\right)\right]\left[\frac{n\pi}{a}\cos\left(\frac{n\pi\xi}{a}\right) - \hbar\sin\left(\frac{n\pi\xi}{a}\right)\right]. \qquad (5\text{-}4\text{-}7\text{-}7)$$

The final solution about $\mathbf{u}$ is obtained on (5-4-4-4) with $\beta_1 = \beta_2 = 1$,

$$u(x,t) = \int_0^t\int_0^a G\phi\, d\xi d\tau + \int_0^a G(\xi,0)f(\xi)d\xi + \int_0^t [DG(a,\tau)h_2(\tau) - DG(0,\tau)h_1(\tau)]d\tau. \qquad (5\text{-}4\text{-}7\text{-}8)$$

5.4.8 Neumann- Robin (N-R) boundaries

Based on the solved R-R problem (Example 5.4.4), it is specified as,

$\alpha_1 = 0$, namely, $\sigma_1 = 0$, and $\beta_1 = 1$.

The solution of N-R problem can be obtained by substituting the newly specified parameters into the R-R problem.

The N-R problem is described as,

$$Lu = u_t + Vu_x - Du_{xx} = \phi(x,t),\ 0 < x < a,\ 0 < t < \infty, \qquad (5\text{-}4\text{-}8\text{-}1a)$$

$$u(x,0) = f(x), \qquad (5\text{-}4\text{-}8\text{-}1b)$$

$$u_x(0,t) = h_1(t), \qquad (5\text{-}4\text{-}8\text{-}1c)$$

$$\alpha_2 u(a,t) + \beta_2 u_x(a,t) = h_2(t),\ (\beta_2 \neq 0). \qquad (5\text{-}4\text{-}8\text{-}1d)$$

The adjoint Green's function problem is formulated as, accordingly,

$$L^*G = -G_\tau - VG_\xi - DG_{\xi\xi} = \delta(\xi - x, \tau - t),\ 0 < \xi < a,\ 0 < \tau < \infty \qquad (5\text{-}4\text{-}8\text{-}2a)$$

$$VG(0,\tau) + DG_\xi(0,\tau) = 0, \qquad (5\text{-}4\text{-}8\text{-}2b)$$

$$(V + \sigma_2 D)G(a,\tau) + DG_\xi(a,\tau) = 0. \qquad (5\text{-}4\text{-}8\text{-}2c)$$

The corresponding eigenvalue problem is,

$$\varphi''+\lambda\varphi = 0, \tag{5-4-8-3a}$$

$$\hbar\varphi(0) + \varphi'(0) = 0, \tag{5-4-8-3b}$$

$$(\hbar + \sigma_2)\varphi(\mathbf{a}) + \varphi'(\mathbf{a}) = 0, \tag{5-4-8-3c}$$

where $\hbar^2 + (\hbar + \sigma_2)^2 \neq 0$. The solutions are obtained by substituting $\sigma_1 = 0$ into (5-4-4-6), (5-4-4-7) and (5-4-4-8),

$$\tan(\mathbf{a}\sqrt{\lambda}) = \frac{\sqrt{\lambda}(\sigma_2 - \sigma_1)}{\lambda + (\hbar + \sigma_1)(\hbar + \sigma_2)} = \frac{\sqrt{\lambda}\sigma_2}{\lambda + \hbar(\hbar + \sigma_2)}, \tag{5-4-8-4}$$

$$\Psi_{1\mathbf{n}} = [\lambda_\mathbf{n} + (\hbar + \sigma_1)(\hbar + \sigma_2)]^2 + \lambda_\mathbf{n}(\sigma_2 - \sigma_1)^2 = [\lambda_\mathbf{n} + \hbar(\hbar + \sigma_2)]^2 + \lambda_\mathbf{n}\sigma_2^2,$$

$$\Psi_{2\mathbf{n}} = \mathbf{a}[\lambda_\mathbf{n} + (\hbar + \sigma_1)^2] \cdot \Psi_{1\mathbf{n}} = \mathbf{a}(\lambda_\mathbf{n} + \hbar^2) \cdot \Psi_{1\mathbf{n}}, \tag{5-4-8-5}$$

$$\Psi_{3\mathbf{n}} = (\sigma_2 - \sigma_1)[\lambda_\mathbf{n}^2 - \lambda_\mathbf{n}(\hbar + \sigma_1)(\sigma_2 - \sigma_1) - (\hbar + \sigma_1)^3(\hbar + \sigma_2)] =$$

$$= \sigma_2[\lambda_\mathbf{n}^2 - \lambda_\mathbf{n}\hbar\sigma_2 - \hbar^3(\hbar + \sigma_2)],$$

thus,

$$\varphi_\mathbf{n}(\xi) = \sqrt{\frac{2\Psi_{1\mathbf{n}}}{\Psi_{2\mathbf{n}} + \Psi_{3\mathbf{n}}}}[\sqrt{\lambda_\mathbf{n}}\cos(\sqrt{\lambda_\mathbf{n}}\xi) - (\hbar + \sigma_1)\sin(\sqrt{\lambda_\mathbf{n}}\xi)] =$$

$$= \sqrt{\frac{2\Psi_{1\mathbf{n}}}{\Psi_{2\mathbf{n}} + \Psi_{3\mathbf{n}}}}[\sqrt{\lambda_\mathbf{n}}\cos(\sqrt{\lambda_\mathbf{n}}\xi) - \hbar\sin(\sqrt{\lambda_\mathbf{n}}\xi)]. \tag{5-4-8-6}$$

Then the Green's function is,

$$\mathbf{G}(\xi,\tau;\mathbf{x},\mathbf{t}) = \mathbf{H}(\mathbf{t}-\tau)\sum_{\mathbf{n}=1}^{\infty}\frac{2\Psi_{1\mathbf{n}}}{\Psi_{2\mathbf{n}} + \Psi_{3\mathbf{n}}}\exp[-\mathbf{D}(\lambda_\mathbf{n} + \hbar^2)(\mathbf{t}-\tau) - \hbar(\xi - \mathbf{x})]\cdot$$

$$\cdot[\sqrt{\lambda_\mathbf{n}}\cos(\sqrt{\lambda_\mathbf{n}}\mathbf{x}) - \hbar\sin(\sqrt{\lambda_\mathbf{n}}\mathbf{x})][\sqrt{\lambda_\mathbf{n}}\cos(\sqrt{\lambda_\mathbf{n}}\xi) - \hbar\sin(\sqrt{\lambda_\mathbf{n}}\xi)], \tag{5-4-8-7}$$

The final solution about $\mathbf{u}$ is obtained on (5-4-4-4) with $\beta_1 = 1$,

$$\mathbf{u}(\mathbf{x},\mathbf{t}) = \int_0^\mathbf{t}\int_0^\mathbf{a}\mathbf{G}\phi\mathbf{d}\xi\mathbf{d}\tau + \int_0^\mathbf{a}\mathbf{G}(\xi,0)\mathbf{f}(\xi)\mathbf{d}\xi + \int_0^\mathbf{t}[\frac{\mathbf{D}}{\beta_2}\mathbf{G}(\mathbf{a},\tau)\mathbf{h}_2(\tau) - \mathbf{D}\mathbf{G}(0,\tau)\mathbf{h}_1(\tau)]\mathbf{d}\tau. \tag{5-4-8-8}$$

5.4.9 Robin- Neumann (R-N) boundaries

Based on the solved R-R problem (Example 5.4.4), it is specified as,

$\alpha_2 = 0$, namely, $\sigma_2 = 0$, and $\beta_2 = 1$.

The solution of R-N problem can be obtained by substituting the newly specified parameters into the R-R problem, and by somewhat transform or simplification.

The R-N problem is described as,

$$\mathbf{Lu} = \mathbf{u}_\mathbf{t} + \mathbf{Vu}_\mathbf{x} - \mathbf{Du}_{\mathbf{xx}} = \phi(\mathbf{x},\mathbf{t}),\ 0 < \mathbf{x} < \mathbf{a},\ 0 < \mathbf{t} < \infty, \tag{5-4-9-1a}$$

$$\mathbf{u}(\mathbf{x},0)=\mathbf{f}(\mathbf{x}),\tag{5-4-9-1b}$$

$$\alpha_1\mathbf{u}(0,\mathbf{t})+\beta_1\mathbf{u}_\mathbf{x}(0,\mathbf{t})=\mathbf{h}_1(\mathbf{t}),\ (\beta_1\neq 0),\tag{5-4-9-1c}$$

$$\mathbf{u}_\mathbf{x}(\mathbf{a},\mathbf{t})=\mathbf{h}_2(\mathbf{t}).\tag{5-4-9-1d}$$

The adjoint Green's function problem is formulated as, accordingly,

$$\mathbf{L}^*\mathbf{G}=-\mathbf{G}_\tau-\mathbf{V}\mathbf{G}_\xi-\mathbf{D}\mathbf{G}_{\xi\xi}=\delta(\xi-\mathbf{x},\tau-\mathbf{t}),\ 0<\xi<\mathbf{a},\ 0<\tau<\infty\tag{5-4-9-2a}$$

$$(\mathbf{V}+\sigma_1\mathbf{D})\mathbf{G}(0,\tau)+\mathbf{D}\mathbf{G}_\xi(0,\tau)=0,\tag{5-4-9-2b}$$

$$\mathbf{V}\mathbf{G}(\mathbf{a},\tau)+\mathbf{D}\mathbf{G}_\xi(\mathbf{a},\tau)=0.\tag{5-4-9-2c}$$

The corresponding eigenvalue problem is,

$$\varphi''+\lambda\varphi=0,\tag{5-4-9-3a}$$

$$(\hbar+\sigma_1)\varphi(0)+\varphi'(0)=0,\tag{5-4-9-3b}$$

$$\hbar\varphi(\mathbf{a})+\varphi'(\mathbf{a})=0,\tag{5-4-9-3c}$$

where $(\hbar+\sigma_1)^2+\hbar^2\neq 0$. The solutions are obtained by substituting $\sigma_2=0$ into (5-4-4-6), (5-4-4-7) and (5-4-4-8),

$$\tan(\mathbf{a}\sqrt{\lambda})=\frac{\sqrt{\lambda}(\sigma_2-\sigma_1)}{\lambda+(\hbar+\sigma_1)(\hbar+\sigma_2)}=\frac{-\sqrt{\lambda}\sigma_1}{\lambda+\hbar(\hbar+\sigma_1)},\tag{5-4-9-4}$$

$$\Psi_{1\mathbf{n}}=[\lambda_\mathbf{n}+(\hbar+\sigma_1)(\hbar+\sigma_2)]^2+\lambda_\mathbf{n}(\sigma_2-\sigma_1)^2=[\lambda_\mathbf{n}+\hbar(\hbar+\sigma_1)]^2+\lambda_\mathbf{n}\sigma_1^2,$$

$$\Psi_{2\mathbf{n}}=\mathbf{a}[\lambda_\mathbf{n}+(\hbar+\sigma_1)^2]\cdot\Psi_{1\mathbf{n}},\tag{5-4-9-5}$$

$$\Psi_{3\mathbf{n}}=(\sigma_2-\sigma_1)[\lambda_\mathbf{n}^2-\lambda_\mathbf{n}(\hbar+\sigma_1)(\sigma_2-\sigma_1)-(\hbar+\sigma_1)^3(\hbar+\sigma_2)]=$$

$$=-\sigma_1[\lambda_\mathbf{n}^2+\lambda_\mathbf{n}\sigma_1(\hbar+\sigma_1)-\hbar(\hbar+\sigma_1)^3],$$

thus,

$$\varphi_\mathbf{n}(\xi)=\sqrt{\frac{2\Psi_{1\mathbf{n}}}{\Psi_{2\mathbf{n}}+\Psi_{3\mathbf{n}}}}[\sqrt{\lambda_\mathbf{n}}\cos(\sqrt{\lambda_\mathbf{n}}\xi)-(\hbar+\sigma_1)\sin(\sqrt{\lambda_\mathbf{n}}\xi)].\tag{5-4-9-6}$$

Then the Green's function is,

$$\mathbf{G}(\xi,\tau;\mathbf{x},\mathbf{t})=\mathbf{H}(\mathbf{t}-\tau)\sum_{\mathbf{n}=1}^{\infty}\frac{2\Psi_{1\mathbf{n}}}{\Psi_{2\mathbf{n}}+\Psi_{3\mathbf{n}}}\exp[-\mathbf{D}(\lambda_\mathbf{n}+\hbar^2)(\mathbf{t}-\tau)-\hbar(\xi-\mathbf{x})]\cdot$$

$$\cdot[\sqrt{\lambda_\mathbf{n}}\cos(\sqrt{\lambda_\mathbf{n}}\mathbf{x})-(\hbar+\sigma_1)\sin(\sqrt{\lambda_\mathbf{n}}\mathbf{x})][\sqrt{\lambda_\mathbf{n}}\cos(\sqrt{\lambda_\mathbf{n}}\xi)-(\hbar+\sigma_1)\sin(\sqrt{\lambda_\mathbf{n}}\xi)],\tag{5-4-9-7}$$

The final solution about $\mathbf{u}$ is obtained on (5-4-4-4) with $\beta_2=1$,

$$\mathbf{u}(\mathbf{x},\mathbf{t})=\int_0^\mathbf{t}\int_0^\mathbf{a}\mathbf{G}\phi\mathbf{d}\xi\mathbf{d}\tau+\int_0^\mathbf{a}\mathbf{G}(\xi,0)\mathbf{f}(\xi)\mathbf{d}\xi+\int_0^\mathbf{t}[\mathbf{D}\mathbf{G}(\mathbf{a},\tau)\mathbf{h}_2(\tau)-\frac{\mathbf{D}}{\beta_1}\mathbf{G}(0,\tau)\mathbf{h}_1(\tau)]\mathbf{d}\tau.\tag{5-4-9-8}$$

Example 5.4.1 through 5.4.9 summarize all possible boundary conditions for the advection diffusion problems in a finite interval. In principle, the Green's functions in the Example 5.4.1 through 5.4.9 can be shrunk into the solutions for the corresponding problems without advection, simply by applying the equalities, $\mathbf{V} = 0$ and/or $\hbar = 0$. However some exceptional cases are listed as follows. (The numberings of the sub-titles and equations in the exceptional examples use the same as those in the corresponding advection problems, but plus a prime.)

5.4.10 Dirichlet- Neumann (D-N) boundaries without advection

Based on Example 5.4.5, the corresponding D-N problem without advection is described as,

$$\mathbf{u_t} - \mathbf{D}\mathbf{u_{xx}} = \phi(\mathbf{x},\mathbf{t}),\ 0 < \mathbf{x} < \mathbf{a},\ 0 < \mathbf{t} < \infty, \tag{5-4-5-1a'}$$

$$\mathbf{u}(\mathbf{x},0) = \mathbf{f}(\mathbf{x}), \tag{5-4-5-1b'}$$

$$\mathbf{u}(0,\mathbf{t}) = \mathbf{h}_1(\mathbf{t}), \tag{5-4-5-1c'}$$

$$\mathbf{u_x}(\mathbf{a},\mathbf{t}) = \mathbf{h}_2(\mathbf{t}). \tag{5-4-5-1d'}$$

The adjoint Green's function problem is formulated as, accordingly,

$$-\mathbf{G}_\tau - \mathbf{D}\mathbf{G}_{\xi\xi} = \delta(\xi - \mathbf{x}, \tau - \mathbf{t}),\ 0 < \xi < \mathbf{a},\ 0 < \tau < \infty \tag{5-4-5-2a'}$$

$$\mathbf{G}(0,\tau) = 0, \tag{5-4-5-2b'}$$

$$\mathbf{G}_\xi(\mathbf{a},\tau) = 0. \tag{5-4-5-2c'}$$

The corresponding eigenvalue problem is,

$$\varphi'' + \lambda\varphi = 0, \tag{5-4-5-3a'}$$

$$\varphi(0) = 0, \tag{5-4-5-3b'}$$

$$\varphi'(\mathbf{a}) = 0. \tag{5-4-5-3c'}$$

The solutions are easily obtained as,

$$\lambda_\mathbf{n} = \frac{(2\mathbf{n}-1)^2\pi^2}{4\mathbf{a}^2},\ \mathbf{n} = 1,2,3\ldots,\ \text{and,} \tag{5-4-5-4'}$$

$$\varphi_\mathbf{n}(\xi) = \sqrt{\frac{2}{\mathbf{a}}}\,\mathbf{sin}\left[\frac{(2\mathbf{n}-1)\pi}{2\mathbf{a}}\xi\right]. \tag{5-4-5-5'}$$

Then the Green's function is,

$$\mathbf{G}(\xi,\tau;\mathbf{x},\mathbf{t}) = \mathbf{H}(\mathbf{t}-\tau)\sum_{\mathbf{n}=1}^{\infty}\frac{2}{\mathbf{a}}\mathbf{exp}\left[-\frac{(2\mathbf{n}-1)^2\pi^2}{4\mathbf{a}^2}\mathbf{D}(\mathbf{t}-\tau)\right]\cdot$$
$$\cdot\,\mathbf{sin}\left[\frac{(2\mathbf{n}-1)\pi}{2\mathbf{a}}\mathbf{x}\right]\mathbf{sin}\left[\frac{(2\mathbf{n}-1)\pi}{2\mathbf{a}}\xi\right]. \tag{5-4-5-6'}$$

The final solution about $\mathbf{u}$ is expressed as,

$$\mathbf{u}(\mathbf{x},\mathbf{t}) = \int_0^{\mathbf{t}}\int_0^{\mathbf{a}}\mathbf{G}\phi\mathbf{d}\xi\mathbf{d}\tau + \int_0^{\mathbf{a}}\mathbf{G}(\xi,0)\mathbf{f}(\xi)\mathbf{d}\xi - \int_0^{\mathbf{t}}[-\mathbf{D}\mathbf{G}(\mathbf{a},\tau)\mathbf{h}_2(\tau) - \mathbf{D}\mathbf{G}_\xi(0,\tau)\mathbf{h}_1(\tau)]\mathbf{d}\tau.$$

(5-4-5-7')

5.4.11 Neumann- Dirichlet (N-D) boundaries without advection

Based on Example 5.4.6, the corresponding N-D problem without advection is described as,

$$\mathbf{u_t - Du_{xx} = \phi(x,t)},\ 0 < \mathbf{x} < \mathbf{a},\ 0 < \mathbf{t} < \infty, \tag{5-4-6-1a'}$$

$$\mathbf{u(x,}0\mathbf{) = f(x)}, \tag{5-4-6-1b'}$$

$$\mathbf{u_x}(0,\mathbf{t}) = \mathbf{h}_1\mathbf{(t)}, \tag{5-4-6-1c'}$$

$$\mathbf{u(a,t) = h}_2\mathbf{(t)}. \tag{5-4-6-1d'}$$

The adjoint Green's function problem is formulated as, accordingly,

$$-\mathbf{G}_\tau - \mathbf{DG}_{\xi\xi} = \delta(\xi - \mathbf{x}, \tau - \mathbf{t}),\ 0 < \xi < \mathbf{a},\ 0 < \tau < \infty \tag{5-4-6-2a'}$$

$$\mathbf{G}_\xi(0,\tau) = 0, \tag{5-4-6-2b'}$$

$$\mathbf{G(a,}\tau) = 0. \tag{5-4-6-2c'}$$

The corresponding eigenvalue problem is,

$$\varphi'' + \lambda\varphi = 0, \tag{5-4-6-3a'}$$

$$\varphi'(0) = 0, \tag{5-4-6-3b'}$$

$$\varphi(\mathbf{a}) = 0. \tag{5-4-6-3c'}$$

The solutions are obtained as,

$$\lambda_\mathbf{n} = \frac{(2\mathbf{n}-1)^2\pi^2}{4\mathbf{a}^2},\ \mathbf{n} = 1,2,3\ldots, \text{ and,} \tag{5-4-6-4'}$$

$$\varphi_\mathbf{n}(\xi) = \sqrt{\frac{2}{\mathbf{a}}}\mathbf{cos}\left[\frac{(2\mathbf{n}-1)\pi}{2\mathbf{a}}\xi\right]. \tag{5-4-6-5'}$$

Then the Green's function is,

$$\mathbf{G}(\xi,\tau;\mathbf{x},\mathbf{t}) = \mathbf{H}(\mathbf{t}-\tau)\sum_{\mathbf{n}=1}^{\infty}\frac{2}{\mathbf{a}}\mathbf{exp}\left[-\frac{(2\mathbf{n}-1)^2\pi^2}{4\mathbf{a}^2}\mathbf{D}(\mathbf{t}-\tau)\right]\cdot$$

$$\cdot\,\mathbf{cos}\left[\frac{(2\mathbf{n}-1)\pi}{2\mathbf{a}}\mathbf{x}\right]\mathbf{cos}\left[\frac{(2\mathbf{n}-1)\pi}{2\mathbf{a}}\xi\right]. \tag{5-4-6-6'}$$

The final solution about $\mathbf{u}$ is expressed as,

$$\mathbf{u}(\mathbf{x},\mathbf{t}) = \int_0^{\mathbf{t}}\int_0^{\mathbf{a}}\mathbf{G}\phi\mathbf{d}\xi\mathbf{d}\tau + \int_0^{\mathbf{a}}\mathbf{G}(\xi,0)\mathbf{f}(\xi)\mathbf{d}\xi - \int_0^{\mathbf{t}}[\mathbf{DG}_\xi(\mathbf{a},\tau)\mathbf{h}_2(\tau) + \mathbf{DG}(0,\tau)\mathbf{h}_1(\tau)]\mathbf{d}\tau.$$

(5-4-6-7')

5.4.12 Neumann- Neumann (N-N) boundaries without advection

Based on Example 5.4.7, the corresponding N-N problem without advection is described as,

$$\mathbf{u_t - Du_{xx} = \phi(x,t)},\ 0 < \mathbf{x} < \mathbf{a},\ 0 < \mathbf{t} < \infty, \tag{5-4-7-1a'}$$

$$\mathbf{u(x,}0\mathbf{) = f(x)}, \tag{5-4-7-1b'}$$

$$\mathbf{u_x}(0,\mathbf{t}) = \mathbf{h}_1(\mathbf{t}), \tag{5-4-7-1c'}$$

$$\mathbf{u_x(a,t) = h}_2\mathbf{(t)}. \tag{5-4-7-1d'}$$

The adjoint Green's function problem is formulated as, accordingly,

$$-\mathbf{G}_\tau - \mathbf{DG}_{\xi\xi} = \delta(\xi - \mathbf{x}, \tau - \mathbf{t}),\ 0 < \xi < \mathbf{a},\ 0 < \tau < \infty \tag{5-4-7-2a'}$$

$$\mathbf{G}_\xi(0,\tau) = 0, \tag{5-4-7-2b'}$$

$$\mathbf{G}_\xi(\mathbf{a},\tau) = 0. \tag{5-4-7-2c'}$$

The corresponding eigenvalue problem is,

$$\varphi'' + \lambda\varphi = 0, \tag{5-4-7-3a'}$$

$$\varphi'(0) = 0, \tag{5-4-7-3b'}$$

$$\varphi'(\mathbf{a}) = 0. \tag{5-4-7-3c'}$$

The solutions are obtained as,

$$\lambda_0 = 0,\ \lambda_\mathbf{n} = \frac{\mathbf{n}^2\pi^2}{\mathbf{a}^2},\ \mathbf{n} = 1,2,3,\ldots,\ \text{and,} \tag{5-4-7-4'}$$

$$\varphi_0(\xi) = \frac{1}{\sqrt{\mathbf{a}}},\ \varphi_\mathbf{n}(\xi) = \sqrt{\frac{2}{\mathbf{a}}}\cos\left(\frac{\mathbf{n}\pi}{\mathbf{a}}\xi\right). \tag{5-4-7-5'}$$

In this case, the corresponding solution to the eigenvalue of zero is non-trivial, so it must be an element of the orthogonal complete set.

Then the Green's function is,

$$\mathbf{G(\xi,\tau;x,t) = H(t-\tau)}\left\{\frac{1}{\mathbf{a}} + \sum_{\mathbf{n}=1}^{\infty}\frac{2}{\mathbf{a}}\mathbf{exp}\left[-\frac{\mathbf{n}^2\pi^2}{\mathbf{a}^2}\mathbf{D(t-\tau)}\right]\mathbf{cos}\left(\frac{\mathbf{n}\pi}{\mathbf{a}}\mathbf{x}\right)\mathbf{cos}\left(\frac{\mathbf{n}\pi}{\mathbf{a}}\xi\right)\right\}. \tag{5-4-7-6'}$$

The final solution about $\mathbf{u}$ is obtained as,

$$\mathbf{u(x,t)} = \int_0^{\mathbf{t}}\int_0^{\mathbf{a}}\mathbf{G\phi d\xi d\tau} + \int_0^{\mathbf{a}}\mathbf{G(\xi,}0\mathbf{)f(\xi)d\xi} + \int_0^{\mathbf{t}}[\mathbf{DG(a,\tau)h}_2(\tau) - \mathbf{DG}(0,\tau)\mathbf{h}_1(\tau)]\mathbf{d\tau}. \tag{5-4-7-7'}$$

6. Multi-dimensional Green's function

All the content in Section 3, 4 and 5 are about solving one-dimensional advection diffusion problem. This section is contributed to solve the problem in multi-dimensions, say, three-dimension (3D).

6.1 Direct solution

Actually, the extension of the GFM from one dimension to higher dimensions, (even from lower order to higher orders), is entirely straightforward. The advection diffusion problem in 3D is generally formulated as,

$$\mathbf{Lu} = \mathbf{u_t} + \mathbf{Vu_x} - \mathbf{D}(\mathbf{u_{xx}} + \mathbf{u_{yy}} + \mathbf{u_{zz}}) = \phi(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{t}), \qquad \text{(6-1-1a)}$$

$$\mathbf{Bu} = \alpha\mathbf{u} + \beta\mathbf{u_n} = \mathbf{f}, \qquad \text{(6-1-1b)}$$

where, (6-1-1a) holds in a given domain $\mathcal{R}$, which could be infinite or finite, and (6-1-1b) on the boundary $\mathcal{B}$ of $\mathcal{R}$. Here the advection direction is defined in the $\mathbf{x}$-direction without losing any general sense. The following steps are just to repeat the idea or procedure of the GFM to solve multi-dimensional problems.

First, multiply Green's function $\mathbf{G}(\xi, \eta, \zeta, \tau)$ on both sides of (6-1-1a), and integrate by parts,

$$\iiiint \mathbf{GLu}d\xi d\eta d\zeta d\tau = \mathbf{Boundary\_Terms} + \iiiint \mathbf{uL^*G}d\xi d\eta d\zeta d\tau, \qquad \text{(6-1-2)}$$

where,

$$\mathbf{L} = \frac{\partial}{\partial \mathbf{t}} + \mathbf{V}\frac{\partial}{\partial \mathbf{x}} - \mathbf{D}\nabla^2, \text{ and,} \qquad \text{(6-1-3)}$$

$$\mathbf{L^*} = -\frac{\partial}{\partial \mathbf{t}} - \mathbf{V}\frac{\partial}{\partial \mathbf{x}} - \mathbf{D}\nabla^2. \qquad \text{(6-1-4)}$$

Then, $\mathbf{G}$ is required to satisfy,

$$\mathbf{L^*G} = \delta(\xi - \mathbf{x}, \eta - \mathbf{y}, \zeta - \mathbf{z}, \tau - \mathbf{t}), \qquad \text{(6-1-5)}$$

with homogeneous boundary conditions that can make the unwelcome boundary terms in (6-1-2) vanish. They are unwelcome because they contain boundary values being not prescribed. Therefore the remaining terms are only those containing the prescribed boundary values. In terms of (6-1-1a), (6-1-2) and (6-1-5), the solution can be expressed as,

$$\mathbf{u}(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{t}) = \iiiint \mathbf{G}\phi d\xi d\eta d\zeta d\tau - \mathbf{Remaining\_Boundary\_Terms}. \qquad \text{(6-1-6)}$$

If the domain $\mathcal{R}$ is rectangular and is bounded in $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ space by,

$[\mathbf{x}_1, \mathbf{x}_2] \times [\mathbf{y}_1, \mathbf{y}_2] \times [\mathbf{z}_1, \mathbf{z}_2]$ or by $[\xi_1, \xi_2] \times [\eta_1, \eta_2] \times [\zeta_1, \zeta_2]$,

in (ξ, η, ζ) space, and if the time is define as $0 < \mathbf{t} < \infty$, then the "boundary terms" in (6-1-2), created by integration by parts, can be expressed explicitly, i.e.,

$$\mathbf{u}(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{t}) = \int_0^{\mathbf{t}}\int_{\zeta_1}^{\zeta_2}\int_{\eta_1}^{\eta_2}\int_{\xi_1}^{\xi_2} \mathbf{G}\phi d\xi d\eta d\zeta d\tau + \int_{\zeta_1}^{\zeta_2}\int_{\eta_1}^{\eta_2}\int_{\xi_1}^{\xi_2} (\mathbf{Gu})\big|_{\tau=0} d\xi d\eta d\zeta +$$

$$+\int_0^t\int_{\zeta_1}^{\zeta_2}\int_{\eta_1}^{\eta_2}[\mathbf{D}(\mathbf{G}\mathbf{u}_\xi-\mathbf{G}_\xi\mathbf{u})-\mathbf{V}\mathbf{G}\mathbf{u}]\Big|_{\xi_1}^{\xi_2}\mathbf{d}\eta\mathbf{d}\zeta\mathbf{d}\tau+$$

$$+\int_0^t\int_{\zeta_1}^{\zeta_2}\int_{\xi_1}^{\xi_2}[\mathbf{D}(\mathbf{G}\mathbf{u}_\eta-\mathbf{G}_\eta\mathbf{u})]\Big|_{\eta_1}^{\eta_2}\mathbf{d}\xi\mathbf{d}\zeta\mathbf{d}\tau+$$

$$+\int_0^t\int_{\eta_1}^{\eta_2}\int_{\xi_1}^{\xi_2}[\mathbf{D}(\mathbf{G}\mathbf{u}_\zeta-\mathbf{G}_\zeta\mathbf{u})]\Big|_{\zeta_1}^{\zeta_2}\mathbf{d}\xi\mathbf{d}\eta\mathbf{d}\tau\,, \qquad (6\text{-}1\text{-}7)$$

where, both the welcome and unwelcome terms are included. (In the process of the integration by parts, a property of Green's function, $\mathbf{G}\big|_{\tau=\infty}=0$ is applied.) As mentioned, the unwelcome terms can be eliminated by applying the homogeneous boundary conditions of the Green's function. In case of two-dimension, (6-1-7) is reduced as,

$$\mathbf{u}(\mathbf{x},\mathbf{y},\mathbf{t})=\int_0^t\int_{\eta_1}^{\eta_2}\int_{\xi_1}^{\xi_2}\mathbf{G}\phi\mathbf{d}\xi\mathbf{d}\eta\mathbf{d}\tau+\int_{\eta_1}^{\eta_2}\int_{\xi_1}^{\xi_2}(\mathbf{G}\mathbf{u})\Big|_{\tau=0}\mathbf{d}\xi\mathbf{d}\eta+$$

$$+\int_0^t\int_{\eta_1}^{\eta_2}[\mathbf{D}(\mathbf{G}\mathbf{u}_\xi-\mathbf{G}_\xi\mathbf{u})-\mathbf{V}\mathbf{G}\mathbf{u}]\Big|_{\xi_1}^{\xi_2}\mathbf{d}\eta\mathbf{d}\tau+\int_0^t\int_{\xi_1}^{\xi_2}[\mathbf{D}(\mathbf{G}\mathbf{u}_\eta-\mathbf{G}_\eta\mathbf{u})]\Big|_{\eta_1}^{\eta_2}\mathbf{d}\xi\mathbf{d}\tau\,. \quad (6\text{-}1\text{-}8)$$

The expressions of (6-1-7) and (6-1-8) are useful to make multi-dimensional solutions by using the GFM.

Green's function equation (6-1-5) with its boundary conditions can be solved directly by using image system based on inspection, integral transform, or by using eigenfunction method, like in the case of one dimension. The extension from one dimension to multi-dimension is completely straightforward. However, efforts are not taken to that direction in this paper, but to apply a so-called "product rule" to create multi-dimensional Green's function based on the one-dimensional Green's functions we have solved already in last sections.

6.2 Product solution

Based on the approach of product solution, the multi-dimensional Green's function can be obtained easily by multiplying one-dimensional Green's functions together, if the boundary conditions of Green's function are homogeneous. In other words, the problem of solving multi-dimensional Green's function can be degraded to several one-dimensional problems. It is a big advantage and a lovely feature of the GFM. On the other hand, it must be emphasized that the approach of product solution is not universal. It is applicable only to Cartesian coordinate systems and cylindrical systems in a more limited sense. The approach does NOT apply to spherical coordinate system. Fortunately the latter two systems are not involved in this paper. This subsection is devoted to explain the principle of product solution based on the example of the advection diffusion problem.

Let's consider a cubic volume in space of (ξ,η,ζ) bounded by $\xi=\xi_i$, $\eta=\eta_i$, $\zeta=\zeta_i$, where $\mathbf{i}=1,2$ and ξ_i,η_i,ζ_i can be finite constants or infinite, the 3D Green's function problem defined on the cube with general linear homogeneous boundary conditions is formulated as,

$$-\mathbf{G}_{\tau}-\mathbf{V}\mathbf{G}_{\xi}-\mathbf{D}(\mathbf{G}_{\xi\xi}+\mathbf{G}_{\eta\eta}+\mathbf{G}_{\zeta\zeta})=\delta(\xi-\mathbf{x},\eta-\mathbf{y},\zeta-\mathbf{z},\tau-\mathbf{t}), \quad (6\text{-}2\text{-}1a)$$

$$(\alpha_{1i}\mathbf{G}+\beta_{1i}\mathbf{G}_{\xi})\big|_{\xi=\xi_i}=0, \quad (6\text{-}2\text{-}1b)$$

$$(\alpha_{2i}\mathbf{G}+\beta_{2i}\mathbf{G}_{\eta})\big|_{\eta=\eta_i}=0, \quad (6\text{-}2\text{-}1c)$$

$$(\alpha_{3i}\mathbf{G}+\beta_{3i}\mathbf{G}_{\zeta})\big|_{\zeta=\zeta_i}=0,\ (\mathbf{i}=1,2), \quad (6\text{-}2\text{-}1d)$$

where, as usual in the GFM, ξ,η,ζ,τ are running variables and $\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{t}$ are constants. Let's recover the physical meaning of $\mathbf{G}$ in (6-2-1a). It stands for the heat distribution, caused by a unit heat pulse released at $(\mathbf{x},\mathbf{y},\mathbf{z})$ and at $\mathbf{t}$, along a reversed time scale and with a reversed advection direction. (The latter reversed nature is not concerned here.) If the time is reversed and shifted by $\mathbf{t}$, the physical meaning of Green's function can be re-formulated as: it describes the heat distribution, initiated by a unit heat pulse released at an initial time of zero, along a time axis with normal direction. In other words, the original Green's function problem with point source (6-2-1) is equivalent to an initial value problem without source term. By using mathematical language, the transform about the time coordinate is defined as,

$$\tilde{\tau}=-\tau+\mathbf{t}. \quad (6\text{-}2\text{-}2)$$

Denoting the Green's function as $\tilde{\mathbf{G}}$ in the new time coordinate, the problem (6-2-1) is transformed as,

$$\tilde{\mathbf{G}}_{\tilde{\tau}}-\mathbf{V}\tilde{\mathbf{G}}_{\xi}-\mathbf{D}(\tilde{\mathbf{G}}_{\xi\xi}+\tilde{\mathbf{G}}_{\eta\eta}+\tilde{\mathbf{G}}_{\zeta\zeta})=0, \quad (6\text{-}2\text{-}3a)$$

$$\tilde{\mathbf{G}}\big|_{\tilde{\tau}=0}=\delta(\xi-\mathbf{x},\eta-\mathbf{y},\zeta-\mathbf{z}), \quad (6\text{-}2\text{-}3b)$$

$$(\alpha_{1i}\tilde{\mathbf{G}}+\beta_{1i}\tilde{\mathbf{G}}_{\xi})\big|_{\xi=\xi_i}=0, \quad (6\text{-}2\text{-}3c)$$

$$(\alpha_{2i}\tilde{\mathbf{G}}+\beta_{2i}\tilde{\mathbf{G}}_{\eta})\big|_{\eta=\eta_i}=0, \quad (6\text{-}2\text{-}3d)$$

$$(\alpha_{3i}\tilde{\mathbf{G}}+\beta_{3i}\tilde{\mathbf{G}}_{\zeta})\big|_{\zeta=\zeta_i}=0,\ (\mathbf{i}=1,2). \quad (6\text{-}2\text{-}3e)$$

The problem (6-2-3) can be solved in the way to assume that,

$$\tilde{\mathbf{G}}(\xi,\eta,\zeta,\tilde{\tau})=\tilde{\mathbf{X}}(\xi,\tilde{\tau})\cdot\tilde{\mathbf{Y}}(\eta,\tilde{\tau})\cdot\tilde{\mathbf{Z}}(\zeta,\tilde{\tau}). \quad (6\text{-}2\text{-}4)$$

By substituting (6-2-4) into (6-2-3a) and rearranging, we have,

$$(\tilde{\mathbf{X}}_{\tilde{\tau}}-\mathbf{V}\tilde{\mathbf{X}}_{\xi}-\mathbf{D}\tilde{\mathbf{X}}_{\xi\xi})\tilde{\mathbf{Y}}\tilde{\mathbf{Z}}+(\tilde{\mathbf{Y}}_{\tilde{\tau}}-\mathbf{D}\tilde{\mathbf{Y}}_{\eta\eta})\tilde{\mathbf{X}}\tilde{\mathbf{Z}}+(\tilde{\mathbf{Z}}_{\tilde{\tau}}-\mathbf{D}\tilde{\mathbf{Z}}_{\zeta\zeta})\tilde{\mathbf{X}}\tilde{\mathbf{Y}}=0, \quad (6\text{-}2\text{-}5)$$

which can be satisfied if,

$$\tilde{\mathbf{X}}_{\tilde{\tau}}-\mathbf{V}\tilde{\mathbf{X}}_{\xi}-\mathbf{D}\tilde{\mathbf{X}}_{\xi\xi}=0, \quad (6\text{-}2\text{-}6a)$$

$$\tilde{\mathbf{Y}}_{\tilde{\tau}}-\mathbf{D}\tilde{\mathbf{Y}}_{\eta\eta}=0, \quad (6\text{-}2\text{-}7a)$$

$$\tilde{\mathbf{Z}}_{\tilde{\tau}}-\mathbf{D}\tilde{\mathbf{Z}}_{\zeta\zeta}=0. \quad (6\text{-}2\text{-}8a)$$

Substitute (6-2-4) into the initial condition of (6-2-3b),

$$\tilde{\mathbf{G}}(\xi,\eta,\zeta,0)=\tilde{\mathbf{X}}(\xi,0)\cdot\tilde{\mathbf{Y}}(\eta,0)\cdot\tilde{\mathbf{Z}}(\zeta,0)=\delta(\xi-\mathbf{x},\eta-\mathbf{y},\zeta-\mathbf{z})=$$

$$= \delta(\xi - x)\delta(\eta - y)\delta(\zeta - z), \tag{6-2-9}$$

where, the last step is the result by using the property of delta function (2-1-8). Equality (6-2-9) can be satisfied if,

$$\tilde{X}\big|_{\tilde{\tau}=0} = \delta(\xi - x), \tag{6-2-6b}$$

$$\tilde{Y}\big|_{\tilde{\tau}=0} = \delta(\eta - y), \tag{6-2-7b}$$

$$\tilde{Z}\big|_{\tilde{\tau}=0} = \delta(\zeta - z). \tag{6-2-8b}$$

Then, substitute (6-2-4) into the boundary condition of (6-2-3c),

$$\tilde{Y}\tilde{Z}(\alpha_{1i}\tilde{X} + \beta_{1i}\tilde{X}_\xi)\big|_{\xi=\xi_i} = 0, \tag{6-2-10}$$

where, $i = 1,2$. Clearly, equality (6-2-10) can be satisfied if,

$$(\alpha_{1i}\tilde{X} + \beta_{1i}\tilde{X}_\xi)\big|_{\xi=\xi_i} = 0. \tag{6-2-6c}$$

Similar equalities can be obtained by substituting (6-2-4) into (6-2-3d) and (6-2-3e), respectively,

$$(\alpha_{2i}\tilde{Y} + \beta_{2i}\tilde{Y}_\eta)\big|_{\eta=\eta_i} = 0, \tag{6-2-7c}$$

$$(\alpha_{3i}\tilde{Z} + \beta_{3i}\tilde{Z}_\zeta)\big|_{\zeta=\zeta_i} = 0, \tag{6-2-8c}$$

where, $i = 1,2$. Therefore, the initial value problem (6-2-3) is divided into three initial value problems,

$$\begin{cases} \tilde{X}_{\tilde{\tau}} - V\tilde{X}_\xi - D\tilde{X}_{\xi\xi} = 0 \\ \tilde{X}\big|_{\tilde{\tau}=0} = \delta(\xi - x) \\ (\alpha_{1i}\tilde{X} + \beta_{1i}\tilde{X}_\xi)\big|_{\xi=\xi_i} = 0 \\ (i = 1,2) \end{cases} \qquad \begin{cases} \tilde{Y}_{\tilde{\tau}} - D\tilde{Y}_{\eta\eta} = 0 \\ \tilde{Y}\big|_{\tilde{\tau}=0} = \delta(\eta - y) \\ (\alpha_{2i}\tilde{Y} + \beta_{2i}\tilde{Y}_\eta)\big|_{\eta=\eta_i} = 0 \\ (i = 1,2) \end{cases} \qquad \begin{cases} \tilde{Z}_{\tilde{\tau}} - D\tilde{Z}_{\zeta\zeta} = 0 \\ \tilde{Z}\big|_{\tilde{\tau}=0} = \delta(\zeta - z) \\ (\alpha_{3i}\tilde{Z} + \beta_{3i}\tilde{Z}_\zeta)\big|_{\zeta=\zeta_i} = 0 \\ (i = 1,2) \end{cases}$$

(6-2-6) (6-2-7) (6-2-8)

Based on the splitting procedure described above, the product solution of the three problems must satisfy the equation and the initial value and the boundary conditions of (6-2-3). That is to say, the product solution must be the solution of (6-2-3). The problems (6-2-6), (6-2-7) and (6-2-8) can be inversely transformed from $\tilde{\tau}$ backwards to τ based on (6-2-2), and the initial value problems are re-formulated as point source problems, respectively,

$$\begin{cases} -X_\tau - VX_\xi - DX_{\xi\xi} = \delta(\xi - x, \tau - t) \\ (\alpha_{1i}X + \beta_{1i}X_\xi)\big|_{\xi=\xi_i} = 0 \\ (i = 1,2) \end{cases} \tag{6-2-11}$$

$$\begin{cases} -\mathbf{Y}_\tau - \mathbf{D}\mathbf{Y}_{\eta\eta} = \delta(\eta - \mathbf{y}, \tau - \mathbf{t}) \\ (\alpha_{2\mathbf{i}}\mathbf{Y} + \beta_{2\mathbf{i}}\mathbf{Y}_\eta)\Big|_{\eta=\eta_i} = 0 \\ (\mathbf{i} = 1,2) \end{cases} \tag{6-2-12}$$

$$\begin{cases} -\mathbf{Z}_\tau - \mathbf{D}\mathbf{Z}_{\zeta\zeta} = \delta(\zeta - \mathbf{z}, \tau - \mathbf{t}) \\ (\alpha_{3\mathbf{i}}\mathbf{Z} + \beta_{3\mathbf{i}}\mathbf{Z}_\zeta)\Big|_{\zeta=\zeta_i} = 0 \\ (\mathbf{i} = 1,2) \end{cases} \tag{6-2-13}$$

where, $\mathbf{X}(\xi,\tau) = \tilde{\mathbf{X}}(\xi,\tilde{\tau})$, $\mathbf{Y}(\eta,\tau) = \tilde{\mathbf{Y}}(\eta,\tilde{\tau})$, $\mathbf{Z}(\zeta,\tau) = \tilde{\mathbf{Z}}(\zeta,\tilde{\tau})$. Look at the three problems, they are so familiar to us and appear many times in last sections. They are all one-dimensional Green's function problem. The problems of (6-2-12) and (6-2-13) are looked as special cases of (6-2-11) with the advection velocity $\mathbf{V}$ equal to zero. It is obvious that the transform on time coordinate does not affect at all the conclusion that, the product of the one-dimensional Green's functions determined by (6-2-11), (6-2-12) and (6-2-13) are genuinely the solution of the multi-dimensional Green's function of the problem (6-2-1), namely,

$$\mathbf{G}(\xi,\eta,\zeta,\tau;\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{t}) = \mathbf{X}(\xi,\tau;\mathbf{x},\mathbf{t})\mathbf{Y}(\eta,\tau;\mathbf{y},\mathbf{t})\mathbf{Z}(\zeta,\tau;\mathbf{z},\mathbf{t}). \tag{6-2-14}$$

6.3 Green's function tables

Only for convenience, the Green's functions for different domains and different boundary conditions are summarized in tables. The tables are constructed completely based on the solutions obtained in the previous sections. Without losing general sense, $\mathbf{x}$ is selected as the advection direction and the directional Green's functions are listed in Table 6-3-1. Accordingly, $\mathbf{y}$ and $\mathbf{z}$ are the transverse directions, which Green's functions are summarized in Table 6-3-2 and Table 6-3-3, respectively.

Table 6-3-1 List of Green's functions in advection direction, x or ξ *

I. Infinite domain $(-\infty,\infty)$			
B.C. of $\mathbf{u}$	B.C. of $\mathbf{G}$	$\mathbf{G}$	Solution term related to B.C.
none	none	*1*	none
II. Semi-infinite domain $(0,\infty)$			
B.C. of $\mathbf{u}$	B.C. of $\mathbf{G}$	$\mathbf{G}$	Solution term related to B.C.
D	$\mathbf{G}(0,\tau)=0$	*2*	$+\int_0^{t}[\mathbf{D}\mathbf{G}_\xi(0,\tau)\mathbf{h}(\tau)]\mathbf{d}\tau$
N	$\mathbf{V}\mathbf{G}(0,\tau)+\mathbf{D}\mathbf{G}_\xi(0,\tau)=0$	*3*	$-\int_0^{t}[\mathbf{D}\mathbf{G}(0,\tau)\mathbf{h}(\tau)]\mathbf{d}\tau$
R	$(\alpha\mathbf{D}+\beta\mathbf{V})\mathbf{G}(0,\tau)+\beta\mathbf{D}\mathbf{G}_\xi(0,\tau)=0$	*4*	$-\int_0^{t}[\frac{\mathbf{D}}{\beta}\mathbf{G}(0,\tau)\mathbf{h}(\tau)]\mathbf{d}\tau$
III. Finite domain $(0,\mathbf{a})$			
B.C. of $\mathbf{u}$	B.C. of $\mathbf{G}$	$\mathbf{G}$	Solution term related to B.C.
D-D	$\mathbf{G}(0,\tau)=0$ $\mathbf{G}(\mathbf{a},\tau)=0$	*5*	$-\int_0^{t}[\mathbf{D}\mathbf{G}_\xi(\mathbf{a},\tau)\mathbf{h}_2(\tau)-\mathbf{D}\mathbf{G}_\xi(0,\tau)\mathbf{h}_1(\tau)]\mathbf{d}\tau$
D-N	$\mathbf{G}(0,\tau)=0$ $\mathbf{V}\mathbf{G}(\mathbf{a},\tau)+\mathbf{D}\mathbf{G}_\xi(\mathbf{a},\tau)=0$	*6*	$-\int_0^{t}[-\mathbf{D}\mathbf{G}(\mathbf{a},\tau)\mathbf{h}_2(\tau)-\mathbf{D}\mathbf{G}_\xi(0,\tau)\mathbf{h}_1(\tau)]\mathbf{d}\tau$
D-R	$\mathbf{G}(0,\tau)=0$ $(\mathbf{V}+\sigma_2\mathbf{D})\mathbf{G}(\mathbf{a},\tau)+\mathbf{D}\mathbf{G}_\xi(\mathbf{a},\tau)=0$	*7*	$+\int_0^{t}[\frac{\mathbf{D}}{\beta_2}\mathbf{G}(\mathbf{a},\tau)\mathbf{h}_2(\tau)+\mathbf{D}\mathbf{G}_\xi(0,\tau)\mathbf{h}_1(\tau)]\mathbf{d}\tau$
N-D	$\mathbf{V}\mathbf{G}(0,\tau)+\mathbf{D}\mathbf{G}_\xi(0,\tau)=0$ $\mathbf{G}(\mathbf{a},\tau)=0$	*8*	$-\int_0^{t}[\mathbf{D}\mathbf{G}_\xi(\mathbf{a},\tau)\mathbf{h}_2(\tau)+\mathbf{D}\mathbf{G}(0,\tau)\mathbf{h}_1(\tau)]\mathbf{d}\tau$
N-N	$\mathbf{V}\mathbf{G}(0,\tau)+\mathbf{D}\mathbf{G}_\xi(0,\tau)=0$ $\mathbf{V}\mathbf{G}(\mathbf{a},\tau)+\mathbf{D}\mathbf{G}_\xi(\mathbf{a},\tau)=0$	*9*	$+\int_0^{t}[\mathbf{D}\mathbf{G}(\mathbf{a},\tau)\mathbf{h}_2(\tau)-\mathbf{D}\mathbf{G}(0,\tau)\mathbf{h}_1(\tau)]\mathbf{d}\tau$
N-R	$\mathbf{V}\mathbf{G}(0,\tau)+\mathbf{D}\mathbf{G}_\xi(0,\tau)=0$ $(\mathbf{V}+\sigma_2\mathbf{D})\mathbf{G}(\mathbf{a},\tau)+\mathbf{D}\mathbf{G}_\xi(\mathbf{a},\tau)=0$	*10*	$+\int_0^{t}[\frac{\mathbf{D}}{\beta_2}\mathbf{G}(\mathbf{a},\tau)\mathbf{h}_2(\tau)-\mathbf{D}\mathbf{G}(0,\tau)\mathbf{h}_1(\tau)]\mathbf{d}\tau$
R-D	$(\mathbf{V}+\sigma_1\mathbf{D})\mathbf{G}(0,\tau)+\mathbf{D}\mathbf{G}_\xi(0,\tau)=0$ $\mathbf{G}(\mathbf{a},\tau)=0$	*11*	$-\int_0^{t}[\mathbf{D}\mathbf{G}_\xi(\mathbf{a},\tau)\mathbf{h}_2(\tau)+\frac{\mathbf{D}}{\beta_1}\mathbf{G}(0,\tau)\mathbf{h}_1(\tau)]\mathbf{d}\tau$

R-N	$(\mathbf{V}+\sigma_1\mathbf{D})\mathbf{G}(0,\tau)+\mathbf{D}\mathbf{G}_\xi(0,\tau)=0$ $\mathbf{V}\mathbf{G}(\mathbf{a},\tau)+\mathbf{D}\mathbf{G}_\xi(\mathbf{a},\tau)=0$	*12*	$+\int_0^t [\mathbf{D}\mathbf{G}(\mathbf{a},\tau)\mathbf{h}_2(\tau)-\frac{\mathbf{D}}{\beta_1}\mathbf{G}(0,\tau)\mathbf{h}_1(\tau)]\mathbf{d}\tau$
R-R	$(\mathbf{V}+\sigma_1\mathbf{D})\mathbf{G}(0,\tau)+\mathbf{D}\mathbf{G}_\xi(0,\tau)=0$ $(\mathbf{V}+\sigma_2\mathbf{D})\mathbf{G}(\mathbf{a},\tau)+\mathbf{D}\mathbf{G}_\xi(\mathbf{a},\tau)=0$	*13*	$+\int_0^t [\frac{\mathbf{D}}{\beta_2}\mathbf{G}(\mathbf{a},\tau)\mathbf{h}_2(\tau)-\frac{\mathbf{D}}{\beta_1}\mathbf{G}(0,\tau)\mathbf{h}_1(\tau)]\mathbf{d}\tau$

*i D means a Dirichlet boundary condition. It is defined as,

$\mathbf{u}(0,\mathbf{t})=\mathbf{h}(\mathbf{t})$, or $\mathbf{u}(0,\mathbf{t})=\mathbf{h}_1(\mathbf{t})$, $\mathbf{u}(\mathbf{a},\mathbf{t})=\mathbf{h}_2(\mathbf{t})$.

*ii N means a Neumann boundary condition. It is defined as,

$\mathbf{u}_\mathbf{x}(0,\mathbf{t})=\mathbf{h}(\mathbf{t})$, or $\mathbf{u}_\mathbf{x}(0,\mathbf{t})=\mathbf{h}_1(\mathbf{t})$, $\mathbf{u}_\mathbf{x}(\mathbf{a},\mathbf{t})=\mathbf{h}_2(\mathbf{t})$.

*iii R means a Robin boundary condition. It is defined as,

$\alpha\mathbf{u}(0,\mathbf{t})+\beta\mathbf{u}_\mathbf{x}(0,\mathbf{t})=\mathbf{h}(\mathbf{t})$, or,

$\alpha_1\mathbf{u}(0,\mathbf{t})+\beta_1\mathbf{u}_\mathbf{x}(0,\mathbf{t})=\mathbf{h}_1(\mathbf{t})$, $\alpha_2\mathbf{u}(\mathbf{a},\mathbf{t})+\beta_2\mathbf{u}_\mathbf{x}(\mathbf{a},\mathbf{t})=\mathbf{h}_2(\mathbf{t})$.

*iv $\sigma=\frac{\alpha}{\beta}$, $\sigma_1=\frac{\alpha_1}{\beta_1}$, $\sigma_2=\frac{\alpha_2}{\beta_2}$, $\hbar=\frac{\mathbf{V}}{2\mathbf{D}}$

*v Green's functions 1 through 13 are given as,

1[I] $\frac{\mathbf{H}(\mathbf{t}-\tau)}{\sqrt{4\pi\mathbf{D}(\mathbf{t}-\tau)}}\mathbf{exp}\left[-\frac{[\xi-\mathbf{x}+\mathbf{V}(\mathbf{t}-\tau)]^2}{4\mathbf{D}(\mathbf{t}-\tau)}\right]$

2[D] $\frac{\mathbf{H}(\mathbf{t}-\tau)}{\sqrt{4\pi\mathbf{D}(\mathbf{t}-\tau)}}\left\{\mathbf{exp}\left[-\frac{[\xi-\mathbf{x}+\mathbf{V}(\mathbf{t}-\tau)]^2}{4\mathbf{D}(\mathbf{t}-\tau)}\right]-\mathbf{exp}\left[\frac{\mathbf{Vx}}{\mathbf{D}}-\frac{[\xi+\mathbf{x}+\mathbf{V}(\mathbf{t}-\tau)]^2}{4\mathbf{D}(\mathbf{t}-\tau)}\right]\right\}$

3[N] $\frac{\mathbf{H}(\mathbf{t}-\tau)}{\sqrt{4\pi\mathbf{D}(\mathbf{t}-\tau)}}\left\{\mathbf{exp}\left[-\frac{[\xi-\mathbf{x}+\mathbf{V}(\mathbf{t}-\tau)]^2}{4\mathbf{D}(\mathbf{t}-\tau)}\right]+\mathbf{exp}\left[\frac{\mathbf{Vx}}{\mathbf{D}}-\frac{[\xi+\mathbf{x}+\mathbf{V}(\mathbf{t}-\tau)]^2}{4\mathbf{D}(\mathbf{t}-\tau)}\right]\right\}+$

$$+\mathbf{H}(\mathbf{t}-\tau)\left(\frac{\mathbf{V}}{2\mathbf{D}}\right)\mathbf{exp}\left(-\frac{\mathbf{V}}{\mathbf{D}}\xi\right)\cdot\mathbf{erfc}\left[\frac{\xi+\mathbf{x}-\mathbf{V}(\mathbf{t}-\tau)}{\sqrt{4\mathbf{D}(\mathbf{t}-\tau)}}\right]$$

4[R] $\frac{\mathbf{H}(\mathbf{t}-\tau)}{\sqrt{4\pi\mathbf{D}(\mathbf{t}-\tau)}}\left\{\mathbf{exp}\left[-\frac{[\xi-\mathbf{x}+\mathbf{V}(\mathbf{t}-\tau)]^2}{4\mathbf{D}(\mathbf{t}-\tau)}\right]+\mathbf{exp}\left[\frac{\mathbf{Vx}}{\mathbf{D}}-\frac{[\xi+\mathbf{x}+\mathbf{V}(\mathbf{t}-\tau)]^2}{4\mathbf{D}(\mathbf{t}-\tau)}\right]\right\}+$

$$+\mathbf{H}(\mathbf{t}-\tau)\left(\frac{\mathbf{V}}{2\mathbf{D}}+\sigma\right)\mathbf{exp}(-\sigma\mathbf{x})\mathbf{exp}\left[-[\xi-\sigma\mathbf{D}(\mathbf{t}-\tau)]\left(\sigma+\frac{\mathbf{V}}{\mathbf{D}}\right)\right]\cdot$$

$$\cdot\mathbf{erfc}\left[\frac{\xi+\mathbf{x}-(\mathbf{V}+2\sigma\mathbf{D})(\mathbf{t}-\tau)}{\sqrt{4\mathbf{D}(\mathbf{t}-\tau)}}\right]$$

5[D-D] $\mathbf{H}(\mathbf{t}-\tau)\sum_{\mathbf{n}=1}^{\infty}\frac{2}{\mathbf{a}}\mathbf{exp}\left[-\mathbf{D}\left(\frac{\mathbf{n}^2\pi^2}{\mathbf{a}^2}+\hbar^2\right)(\mathbf{t}-\tau)-\hbar(\xi-\mathbf{x})\right]\mathbf{sin}\left(\frac{\mathbf{n}\pi}{\mathbf{a}}\mathbf{x}\right)\mathbf{sin}\left(\frac{\mathbf{n}\pi}{\mathbf{a}}\xi\right)$

$$6[\text{D-N}]\ \mathbf{H(t-\tau)}\sum_{n=1}^{\infty}\frac{2(\lambda_n+\hbar^2)}{\mathbf{a}(\lambda_n+\hbar^2)+\hbar}\mathbf{exp[-D}(\lambda_n+\hbar^2)\mathbf{(t-\tau)}-\hbar(\xi-\mathbf{x})]\cdot$$

$$\cdot\sin(\sqrt{\lambda_n}\mathbf{x})\sin(\sqrt{\lambda_n}\xi),$$

where, $\tan(\mathbf{a}\sqrt{\lambda}) = -\frac{\sqrt{\lambda}}{\hbar}$

$$7[\text{D-R}]\ \mathbf{H(t-\tau)}\sum_{n=1}^{\infty}\frac{2[\lambda_n+(\hbar+\sigma_2)^2]}{\mathbf{a}[\lambda_n+(\hbar+\sigma_2)^2]+(\hbar+\sigma_2)}\cdot$$

$$\cdot\mathbf{exp[-D}(\lambda_n+\hbar^2)\mathbf{(t-\tau)}-\hbar(\xi-\mathbf{x})]\sin(\sqrt{\lambda_n}\mathbf{x})\sin(\sqrt{\lambda_n}\xi),$$

where, $\tan(\mathbf{a}\sqrt{\lambda}) = \frac{-\sqrt{\lambda}}{\hbar+\sigma_2}$

$$8[\text{N-D}]\ \mathbf{H(t-\tau)}\sum_{n=1}^{\infty}\frac{2(\lambda_n+\hbar^2)}{\mathbf{a}(\lambda_n+\hbar^2)-\hbar}\mathbf{exp[-D}(\lambda_n+\hbar^2)\mathbf{(t-\tau)}-\hbar(\xi-\mathbf{x})]\cdot$$

$$\cdot\sin[\sqrt{\lambda_n}(\mathbf{x}-\mathbf{a})]\sin[\sqrt{\lambda_n}(\xi-\mathbf{a})],$$

where, $\tan(\mathbf{a}\sqrt{\lambda}) = \frac{\sqrt{\lambda}}{\hbar}$

$$9[\text{N-N}]\ \mathbf{H(t-\tau)}\sum_{n=1}^{\infty}\frac{2\mathbf{a}}{\mathbf{a}^2\hbar^2+\mathbf{n}^2\pi^2}\mathbf{exp}\left[-\mathbf{D}\left(\frac{\mathbf{n}^2\pi^2+\mathbf{a}^2\hbar^2}{\mathbf{a}^2}\right)\mathbf{(t-\tau)}-\hbar(\xi-\mathbf{x})\right]\cdot$$

$$\cdot\left[\frac{\mathbf{n}\pi}{\mathbf{a}}\cos\left(\frac{\mathbf{n}\pi\mathbf{x}}{\mathbf{a}}\right)-\hbar\sin\left(\frac{\mathbf{n}\pi\mathbf{x}}{\mathbf{a}}\right)\right]\left[\frac{\mathbf{n}\pi}{\mathbf{a}}\cos\left(\frac{\mathbf{n}\pi\xi}{\mathbf{a}}\right)-\hbar\sin\left(\frac{\mathbf{n}\pi\xi}{\mathbf{a}}\right)\right]$$

$$10[\text{N-R}]\ \mathbf{H(t-\tau)}\sum_{n=1}^{\infty}\frac{2\Psi_{1n}}{\Psi_{2n}+\Psi_{3n}}\mathbf{exp[-D}(\lambda_n+\hbar^2)\mathbf{(t-\tau)}-\hbar(\xi-\mathbf{x})]\cdot$$

$$\cdot[\sqrt{\lambda_n}\cos(\sqrt{\lambda_n}\mathbf{x})-\hbar\sin(\sqrt{\lambda_n}\mathbf{x})][\sqrt{\lambda_n}\cos(\sqrt{\lambda_n}\xi)-\hbar\sin(\sqrt{\lambda_n}\xi)],$$

where, $\tan(\mathbf{a}\sqrt{\lambda}) = \frac{\sqrt{\lambda}\sigma_2}{\lambda+\hbar(\hbar+\sigma_2)},$

$$\Psi_{1n} = [\lambda_n+\hbar(\hbar+\sigma_2)]^2+\lambda_n\sigma_2^2,$$

$$\Psi_{2n} = \mathbf{a}(\lambda_n+\hbar^2)\cdot\Psi_{1n},$$

$$\Psi_{3n} = \sigma_2[\lambda_n^2-\lambda_n\hbar\sigma_2-\hbar^3(\hbar+\sigma_2)]$$

$$11[\text{R-D}]\ \mathbf{H(t-\tau)}\sum_{n=1}^{\infty}\frac{2[\lambda_n+(\hbar+\sigma_1)^2]}{\mathbf{a}[\lambda_n+(\hbar+\sigma_1)^2]-(\hbar+\sigma_1)}\cdot$$

$$\cdot\mathbf{exp[-D}(\lambda_n+\hbar^2)\mathbf{(t-\tau)}-\hbar(\xi-\mathbf{x})]\sin[\sqrt{\lambda_n}(\mathbf{x}-\mathbf{a})]\sin[\sqrt{\lambda_n}(\xi-\mathbf{a})],$$

where, $\mathbf{tan(a\sqrt{\lambda})} = \dfrac{\sqrt{\lambda}}{\hbar + \sigma_1}$

$$12\text{[R-N]}\ \mathbf{H(t-\tau)}\sum_{\mathbf{n}=1}^{\infty}\frac{2\Psi_{1\mathbf{n}}}{\Psi_{2\mathbf{n}}+\Psi_{3\mathbf{n}}}\mathbf{exp[-D(\lambda_n}+\hbar^2\mathbf{)(t-\tau)}-\hbar\mathbf{(\xi-x)]}\cdot$$

$$\cdot\mathbf{[\sqrt{\lambda_n}\cos(\sqrt{\lambda_n}x)-(}\hbar+\sigma_1\mathbf{)\sin(\sqrt{\lambda_n}x)][\sqrt{\lambda_n}\cos(\sqrt{\lambda_n}\xi)-(}\hbar+\sigma_1\mathbf{)\sin(\sqrt{\lambda_n}\xi)]},$$

where, $\mathbf{tan(a\sqrt{\lambda})} = \dfrac{-\sqrt{\lambda}\sigma_1}{\lambda + \hbar(\hbar + \sigma_1)}$,

$$\Psi_{1\mathbf{n}} = [\lambda_{\mathbf{n}} + \hbar(\hbar + \sigma_1)]^2 + \lambda_{\mathbf{n}}\sigma_1^2,$$

$$\Psi_{2\mathbf{n}} = \mathbf{a}[\lambda_{\mathbf{n}} + (\hbar + \sigma_1)^2]\cdot\Psi_{1\mathbf{n}},$$

$$\Psi_{3\mathbf{n}} = -\sigma_1[\lambda_{\mathbf{n}}^2 + \lambda_{\mathbf{n}}\sigma_1(\hbar + \sigma_1) - \hbar(\hbar + \sigma_1)^3]$$

$$13\text{[R-R]}\ \mathbf{H(t-\tau)}\sum_{\mathbf{n}=1}^{\infty}\frac{2\Psi_{1\mathbf{n}}}{\Psi_{2\mathbf{n}}+\Psi_{3\mathbf{n}}}\mathbf{exp[-D(\lambda_n}+\hbar^2\mathbf{)(t-\tau)}-\hbar\mathbf{(\xi-x)]}\cdot$$

$$\cdot\mathbf{[\sqrt{\lambda_n}\cos(\sqrt{\lambda_n}x)-(}\hbar+\sigma_1\mathbf{)\sin(\sqrt{\lambda_n}x)][\sqrt{\lambda_n}\cos(\sqrt{\lambda_n}\xi)-(}\hbar+\sigma_1\mathbf{)\sin(\sqrt{\lambda_n}\xi)]},$$

where, $\mathbf{tan(a\sqrt{\lambda})} = \dfrac{\sqrt{\lambda}(\sigma_2 - \sigma_1)}{\lambda + (\hbar + \sigma_1)(\hbar + \sigma_2)}$,

$$\Psi_{1\mathbf{n}} = [\lambda_{\mathbf{n}} + (\hbar + \sigma_1)(\hbar + \sigma_2)]^2 + \lambda_{\mathbf{n}}(\sigma_2 - \sigma_1)^2,$$

$$\Psi_{2\mathbf{n}} = \mathbf{a}[\lambda_{\mathbf{n}} + (\hbar + \sigma_1)^2]\cdot\Psi_{1\mathbf{n}},$$

$$\Psi_{3\mathbf{n}} = (\sigma_2 - \sigma_1)[\lambda_{\mathbf{n}}^2 - \lambda_{\mathbf{n}}(\hbar + \sigma_1)(\sigma_2 - \sigma_1) - (\hbar + \sigma_1)^3(\hbar + \sigma_2)].$$

Table 6-3-2 List of Green's functions in transverse direction, y or η *

I. Infinite domain $(-\infty,\infty)$			
B.C. of $\mathbf{u}$	B.C. of $\mathbf{G}$	$\mathbf{G}$	Solution term related to B.C.
none	none	*1*	none
II. Semi-infinite domain $(0,\infty)$			
B.C. of $\mathbf{u}$	B.C. of $\mathbf{G}$	$\mathbf{G}$	Solution term related to B.C.
D	$\mathbf{G}(0,\tau)=0$	*2*	$+\int_0^t [\mathbf{DG}_\eta(0,\tau)\mathbf{h}(\tau)]\mathbf{d}\tau$
N	$\mathbf{G}_\eta(0,\tau)=0$	*3*	$-\int_0^t [\mathbf{DG}(0,\tau)\mathbf{h}(\tau)]\mathbf{d}\tau$
R	$\alpha\mathbf{G}(0,\tau)+\beta\mathbf{G}_\eta(0,\tau)=0$	*4*	$-\int_0^t [\frac{\mathbf{D}}{\beta}\mathbf{G}(0,\tau)\mathbf{h}(\tau)]\mathbf{d}\tau$
III. Finite domain $(0,\mathbf{b})$			
B.C. of $\mathbf{u}$	B.C. of $\mathbf{G}$	$\mathbf{G}$	Solution term related to B.C.
D-D	$\mathbf{G}(0,\tau)=0$ $\mathbf{G}(\mathbf{b},\tau)=0$	*5*	$-\int_0^t [\mathbf{DG}_\eta(\mathbf{b},\tau)\mathbf{h}_2(\tau)-\mathbf{DG}_\eta(0,\tau)\mathbf{h}_1(\tau)]\mathbf{d}\tau$
D-N	$\mathbf{G}(0,\tau)=0$ $\mathbf{G}_\eta(\mathbf{b},\tau)=0$	*6*	$-\int_0^t [-\mathbf{DG}(\mathbf{b},\tau)\mathbf{h}_2(\tau)-\mathbf{DG}_\eta(0,\tau)\mathbf{h}_1(\tau)]\mathbf{d}\tau$
D-R	$\mathbf{G}(0,\tau)=0$ $\sigma_2\mathbf{G}(\mathbf{b},\tau)+\mathbf{G}_\eta(\mathbf{b},\tau)=0$	*7*	$+\int_0^t [\frac{\mathbf{D}}{\beta_2}\mathbf{G}(\mathbf{b},\tau)\mathbf{h}_2(\tau)+\mathbf{DG}_\eta(0,\tau)\mathbf{h}_1(\tau)]\mathbf{d}\tau$
N-D	$\mathbf{G}_\eta(0,\tau)=0$ $\mathbf{G}(\mathbf{b},\tau)=0$	*8*	$-\int_0^t [\mathbf{DG}_\eta(\mathbf{b},\tau)\mathbf{h}_2(\tau)+\mathbf{DG}(0,\tau)\mathbf{h}_1(\tau)]\mathbf{d}\tau$
N-N	$\mathbf{G}_\eta(0,\tau)=0$ $\mathbf{G}_\eta(\mathbf{b},\tau)=0$	*9*	$+\int_0^t [\mathbf{DG}(\mathbf{b},\tau)\mathbf{h}_2(\tau)-\mathbf{DG}(0,\tau)\mathbf{h}_1(\tau)]\mathbf{d}\tau$
N-R	$\mathbf{G}_\eta(0,\tau)=0$ $\sigma_2\mathbf{G}(\mathbf{b},\tau)+\mathbf{G}_\eta(\mathbf{b},\tau)=0$	*10*	$+\int_0^t [\frac{\mathbf{D}}{\beta_2}\mathbf{G}(\mathbf{b},\tau)\mathbf{h}_2(\tau)-\mathbf{DG}(0,\tau)\mathbf{h}_1(\tau)]\mathbf{d}\tau$
R-D	$\sigma_1\mathbf{G}(0,\tau)+\mathbf{G}_\eta(0,\tau)=0$ $\mathbf{G}(\mathbf{b},\tau)=0$	*11*	$-\int_0^t [\mathbf{DG}_\eta(\mathbf{b},\tau)\mathbf{h}_2(\tau)+\frac{\mathbf{D}}{\beta_1}\mathbf{G}(0,\tau)\mathbf{h}_1(\tau)]\mathbf{d}\tau$

R-N	$\sigma_1 \mathbf{G}(0,\tau)+\mathbf{G}_\eta(0,\tau)=0$ $\mathbf{G}_\eta(\mathbf{b},\tau)=0$	*12*	$+\int_0^t [\mathbf{D}\mathbf{G}(\mathbf{b},\tau)\mathbf{h}_2(\tau)-\frac{\mathbf{D}}{\beta_1}\mathbf{G}(0,\tau)\mathbf{h}_1(\tau)]\mathbf{d}\tau$
R-R	$\sigma_1 \mathbf{G}(0,\tau)+\mathbf{G}_\eta(0,\tau)=0$ $\sigma_2 \mathbf{G}(\mathbf{b},\tau)+\mathbf{G}_\eta(\mathbf{b},\tau)=0$	*13*	$+\int_0^t [\frac{\mathbf{D}}{\beta_2}\mathbf{G}(\mathbf{b},\tau)\mathbf{h}_2(\tau)-\frac{\mathbf{D}}{\beta_1}\mathbf{G}(0,\tau)\mathbf{h}_1(\tau)]\mathbf{d}\tau$

*i D means a Dirichlet boundary condition. It is defined as,

$\mathbf{u}(0,\mathbf{t})=\mathbf{h}(\mathbf{t})$, or $\mathbf{u}(0,\mathbf{t})=\mathbf{h}_1(\mathbf{t})$, $\mathbf{u}(\mathbf{b},\mathbf{t})=\mathbf{h}_2(\mathbf{t})$.

*ii N means a Neumann boundary condition. It is defined as,

$\mathbf{u}_y(0,\mathbf{t})=\mathbf{h}(\mathbf{t})$, or $\mathbf{u}_y(0,\mathbf{t})=\mathbf{h}_1(\mathbf{t})$, $\mathbf{u}_y(\mathbf{b},\mathbf{t})=\mathbf{h}_2(\mathbf{t})$.

*iii R means a Robin boundary condition. It is defined as,

$\alpha\mathbf{u}(0,\mathbf{t})+\beta\mathbf{u}_y(0,\mathbf{t})=\mathbf{h}(\mathbf{t})$, or,

$\alpha_1\mathbf{u}(0,\mathbf{t})+\beta_1\mathbf{u}_y(0,\mathbf{t})=\mathbf{h}_1(\mathbf{t})$, $\alpha_2\mathbf{u}(\mathbf{b},\mathbf{t})+\beta_2\mathbf{u}_y(\mathbf{b},\mathbf{t})=\mathbf{h}_2(\mathbf{t})$.

*iv $\sigma=\frac{\alpha}{\beta}$, $\sigma_1=\frac{\alpha_1}{\beta_1}$, $\sigma_2=\frac{\alpha_2}{\beta_2}$.

*v Green's functions 1 through 8 are given as,

1[I] $\frac{\mathbf{H}(\mathbf{t}-\tau)}{\sqrt{4\pi\mathbf{D}(\mathbf{t}-\tau)}}\mathbf{exp}\left[-\frac{(\eta-\mathbf{y})^2}{4\mathbf{D}(\mathbf{t}-\tau)}\right]$

2[D] $\frac{\mathbf{H}(\mathbf{t}-\tau)}{\sqrt{4\pi\mathbf{D}(\mathbf{t}-\tau)}}\left\{\mathbf{exp}\left[-\frac{(\eta-\mathbf{y})^2}{4\mathbf{D}(\mathbf{t}-\tau)}\right]-\mathbf{exp}\left[-\frac{(\eta+\mathbf{y})^2}{4\mathbf{D}(\mathbf{t}-\tau)}\right]\right\}$

3[N] $\frac{\mathbf{H}(\mathbf{t}-\tau)}{\sqrt{4\pi\mathbf{D}(\mathbf{t}-\tau)}}\left\{\mathbf{exp}\left[-\frac{(\eta-\mathbf{y})^2}{4\mathbf{D}(\mathbf{t}-\tau)}\right]+\mathbf{exp}\left[-\frac{(\eta+\mathbf{y})^2}{4\mathbf{D}(\mathbf{t}-\tau)}\right]\right\}$

4[R] $\frac{\mathbf{H}(\mathbf{t}-\tau)}{\sqrt{4\pi\mathbf{D}(\mathbf{t}-\tau)}}\left\{\mathbf{exp}\left[-\frac{(\eta-\mathbf{y})^2}{4\mathbf{D}(\mathbf{t}-\tau)}\right]+\mathbf{exp}\left[-\frac{[\eta+\mathbf{y}]^2}{4\mathbf{D}(\mathbf{t}-\tau)}\right]\right\}+$

$$+\mathbf{H}(\mathbf{t}-\tau)\sigma\,\mathbf{exp}\big[-\sigma[\eta+\mathbf{y}-\sigma\mathbf{D}(\mathbf{t}-\tau)]\big]\cdot\mathbf{erfc}\left[\frac{\eta+\mathbf{y}-2\sigma\mathbf{D}(\mathbf{t}-\tau)}{\sqrt{4\mathbf{D}(\mathbf{t}-\tau)}}\right]$$

5[D-D] $\mathbf{H}(\mathbf{t}-\tau)\sum_{\mathbf{n}=1}^{\infty}\frac{2}{\mathbf{b}}\mathbf{exp}\left[-\mathbf{D}\left(\frac{\mathbf{n}^2\pi^2}{\mathbf{b}^2}\right)(\mathbf{t}-\tau)\right]\mathbf{sin}\left(\frac{\mathbf{n}\pi}{\mathbf{b}}\mathbf{y}\right)\mathbf{sin}\left(\frac{\mathbf{n}\pi}{\mathbf{b}}\eta\right)$

6[D-N] $\mathbf{H}(\mathbf{t}-\tau)\sum_{\mathbf{n}=1}^{\infty}\frac{2}{\mathbf{b}}\mathbf{exp}\left[-\frac{(2\mathbf{n}-1)^2\pi^2}{4\mathbf{b}^2}\mathbf{D}(\mathbf{t}-\tau)\right]\mathbf{sin}\left[\frac{(2\mathbf{n}-1)\pi}{2\mathbf{b}}\mathbf{y}\right]\mathbf{sin}\left[\frac{(2\mathbf{n}-1)\pi}{2\mathbf{b}}\eta\right]$

7[D-R] $\mathbf{H}(\mathbf{t}-\tau)\sum_{\mathbf{n}=1}^{\infty}\frac{2(\lambda_\mathbf{n}+\sigma_2^2)}{\mathbf{b}(\lambda_\mathbf{n}+\sigma_2^2)+\sigma_2}\mathbf{exp}[-\mathbf{D}\lambda_\mathbf{n}(\mathbf{t}-\tau)]\mathbf{sin}(\sqrt{\lambda_\mathbf{n}}\mathbf{y})\mathbf{sin}(\sqrt{\lambda_\mathbf{n}}\eta)$, where, $\tan(b\sqrt{\lambda}) = -\frac{\sqrt{\lambda}}{\sigma_2}$

8[N-D] $\mathbf{H(t-\tau)}\sum_{n=1}^{\infty}\frac{2}{b}\exp\left[-\frac{(2n-1)^2\pi^2}{4b^2}D(t-\tau)\right]\cos\left[\frac{(2n-1)\pi}{2b}y\right]\cos\left[\frac{(2n-1)\pi}{2b}\eta\right]$

9[N-N] $\mathbf{H(t-\tau)}\left\{\frac{1}{b}+\sum_{n=1}^{\infty}\frac{2}{b}\exp\left[-\frac{n^2\pi^2}{b^2}D(t-\tau)\right]\cos\left(\frac{n\pi}{b}y\right)\cos\left(\frac{n\pi}{b}\eta\right)\right\}$

10[N-R] $\mathbf{H(t-\tau)}\sum_{n=1}^{\infty}\frac{2(\lambda_n+\sigma_2^2)}{b(\lambda_n+\sigma_2^2)+\sigma_2}\exp[-D\lambda_n(t-\tau)]\cos(\sqrt{\lambda_n}y)\cos(\sqrt{\lambda_n}\eta)$,

where, $\tan(b\sqrt{\lambda}) = \frac{\sigma_2}{\sqrt{\lambda}}$

11[R-D] $\mathbf{H(t-\tau)}\sum_{n=1}^{\infty}\frac{2(\lambda_n+\sigma_1^2)}{b(\lambda_n+\sigma_1^2)-\sigma_1}\exp[-D\lambda_n(t-\tau)]\cdot$

$\cdot\sin[\sqrt{\lambda_n}(y-b)]\sin[\sqrt{\lambda_n}(\eta-b)]$,

where, $\tan(b\sqrt{\lambda}) = \frac{\sqrt{\lambda}}{\sigma_1}$

12[R-N] $\mathbf{H(t-\tau)}\sum_{n=1}^{\infty}\frac{2(\lambda_n+\sigma_1^2)}{b(\lambda_n+\sigma_1^2)-\sigma_1}\exp[-D\lambda_n(t-\tau)]\cdot$

$\cdot\cos[\sqrt{\lambda_n}(y-b)]\cos[\sqrt{\lambda_n}(\eta-b)]$,

where, $\tan(b\sqrt{\lambda}) = -\frac{\sigma_1}{\sqrt{\lambda}}$

13[R-R] $\mathbf{H(t-\tau)}\sum_{n=1}^{\infty}\frac{2\Psi_{1n}}{\Psi_{2n}+\Psi_{3n}}\exp[-D\lambda_n(t-\tau)]\cdot$

$\cdot[\sqrt{\lambda_n}\cos(\sqrt{\lambda_n}y)-\sigma_1\sin(\sqrt{\lambda_n}y)][\sqrt{\lambda_n}\cos(\sqrt{\lambda_n}\eta)-\sigma_1\sin(\sqrt{\lambda_n}\eta)]$,

where, $\tan(b\sqrt{\lambda}) = \frac{\sqrt{\lambda}(\sigma_2-\sigma_1)}{\lambda+\sigma_1\sigma_2}$,

$\Psi_{1n} = (\lambda_n+\sigma_1\sigma_2)^2+\lambda_n(\sigma_2-\sigma_1)^2$,

$\Psi_{2n} = b(\lambda_n+\sigma_1^2)\cdot\Psi_{1n}$,

$\Psi_{3n} = (\sigma_2-\sigma_1)[\lambda_n^2-\lambda_n\sigma_1(\sigma_2-\sigma_1)-\sigma_1^3\sigma_2]$.

Table 6-3-3 List of Green's functions in transverse direction, z or ζ *

I. Infinite domain $(-\infty,\infty)$			
B.C. of **u**	B.C. of **G**	**G**	Solution term related to B.C.
none	none	*1*	none
II. Semi-infinite domain $(0,\infty)$			
B.C. of **u**	B.C. of **G**	**G**	Solution term related to B.C.
D	$\mathbf{G}(0,\tau)=0$	*2*	$+\int_0^t [\mathbf{D}\mathbf{G}_\zeta(0,\tau)\mathbf{h}(\tau)]d\tau$
N	$\mathbf{G}_\zeta(0,\tau)=0$	*3*	$-\int_0^t [\mathbf{D}\mathbf{G}(0,\tau)\mathbf{h}(\tau)]d\tau$
R	$\alpha\mathbf{G}(0,\tau)+\beta\mathbf{G}_\zeta(0,\tau)=0$	*4*	$-\int_0^t [\frac{\mathbf{D}}{\beta}\mathbf{G}(0,\tau)\mathbf{h}(\tau)]d\tau$
III. Finite domain $(0,\mathbf{c})$			
B.C. of **u**	B.C. of **G**	**G**	Solution term related to B.C.
D-D	$\mathbf{G}(0,\tau)=0$ $\mathbf{G}(\mathbf{c},\tau)=0$	*5*	$-\int_0^t [\mathbf{D}\mathbf{G}_\zeta(\mathbf{c},\tau)\mathbf{h}_2(\tau)-\mathbf{D}\mathbf{G}_\zeta(0,\tau)\mathbf{h}_1(\tau)]d\tau$
D-N	$\mathbf{G}(0,\tau)=0$ $\mathbf{G}_\zeta(\mathbf{c},\tau)=0$	*6*	$-\int_0^t [-\mathbf{D}\mathbf{G}(\mathbf{c},\tau)\mathbf{h}_2(\tau)-\mathbf{D}\mathbf{G}_\zeta(0,\tau)\mathbf{h}_1(\tau)]d\tau$
D-R	$\mathbf{G}(0,\tau)=0$ $\sigma_2\mathbf{G}(\mathbf{c},\tau)+\mathbf{G}_\zeta(\mathbf{c},\tau)=0$	*7*	$+\int_0^t [\frac{\mathbf{D}}{\beta_2}\mathbf{G}(\mathbf{c},\tau)\mathbf{h}_2(\tau)+\mathbf{D}\mathbf{G}_\zeta(0,\tau)\mathbf{h}_1(\tau)]d\tau$
N-D	$\mathbf{G}_\zeta(0,\tau)=0$ $\mathbf{G}(\mathbf{c},\tau)=0$	*8*	$-\int_0^t [\mathbf{D}\mathbf{G}_\zeta(\mathbf{c},\tau)\mathbf{h}_2(\tau)+\mathbf{D}\mathbf{G}(0,\tau)\mathbf{h}_1(\tau)]d\tau$
N-N	$\mathbf{G}_\zeta(0,\tau)=0$ $\mathbf{G}_\zeta(\mathbf{c},\tau)=0$	*9*	$+\int_0^t [\mathbf{D}\mathbf{G}(\mathbf{c},\tau)\mathbf{h}_2(\tau)-\mathbf{D}\mathbf{G}(0,\tau)\mathbf{h}_1(\tau)]d\tau$
N-R	$\mathbf{G}_\zeta(0,\tau)=0$ $\sigma_2\mathbf{G}(\mathbf{c},\tau)+\mathbf{G}_\zeta(\mathbf{c},\tau)=0$	*10*	$+\int_0^t [\frac{\mathbf{D}}{\beta_2}\mathbf{G}(\mathbf{c},\tau)\mathbf{h}_2(\tau)-\mathbf{D}\mathbf{G}(0,\tau)\mathbf{h}_1(\tau)]d\tau$
R-D	$\sigma_1\mathbf{G}(0,\tau)+\mathbf{G}_\zeta(0,\tau)=0$ $\mathbf{G}(\mathbf{c},\tau)=0$	*11*	$-\int_0^t [\mathbf{D}\mathbf{G}_\zeta(\mathbf{c},\tau)\mathbf{h}_2(\tau)+\frac{\mathbf{D}}{\beta_1}\mathbf{G}(0,\tau)\mathbf{h}_1(\tau)]d\tau$

R-N	$\sigma_1 G(0,\tau)+G_\zeta(0,\tau)=0$ $G_\zeta(c,\tau)=0$	*12*	$+\int_0^t [DG(c,\tau)h_2(\tau)-\frac{D}{\beta_1}G(0,\tau)h_1(\tau)]d\tau$
R-R	$\sigma_1 G(0,\tau)+G_\zeta(0,\tau)=0$ $\sigma_2 G(c,\tau)+G_\zeta(c,\tau)=0$	*13*	$+\int_0^t [\frac{D}{\beta_2}G(c,\tau)h_2(\tau)-\frac{D}{\beta_1}G(0,\tau)h_1(\tau)]d\tau$

*i D means a Dirichlet boundary condition. It is defined as,

$u(0,t)=h(t)$, or $u(0,t)=h_1(t)$, $u(c,t)=h_2(t)$.

*ii N means a Neumann boundary condition. It is defined as,

$u_z(0,t)=h(t)$, or $u_z(0,t)=h_1(t)$, $u_z(c,t)=h_2(t)$.

*iii R means a Robin boundary condition. It is defined as,

$\alpha u(0,t)+\beta u_z(0,t)=h(t)$, or,

$\alpha_1 u(0,t)+\beta_1 u_z(0,t)=h_1(t)$, $\alpha_2 u(c,t)+\beta_2 u_z(c,t)=h_2(t)$.

*iv $\sigma=\frac{\alpha}{\beta}$, $\sigma_1=\frac{\alpha_1}{\beta_1}$, $\sigma_2=\frac{\alpha_2}{\beta_2}$.

*v Green's functions 1 through 8 are given as,

1[I] $\frac{H(t-\tau)}{\sqrt{4\pi D(t-\tau)}}\exp\left[-\frac{(\zeta-z)^2}{4D(t-\tau)}\right]$

2[D] $\frac{H(t-\tau)}{\sqrt{4\pi D(t-\tau)}}\left\{\exp\left[-\frac{(\zeta-z)^2}{4D(t-\tau)}\right]-\exp\left[-\frac{(\zeta+z)^2}{4D(t-\tau)}\right]\right\}$

3[N] $\frac{H(t-\tau)}{\sqrt{4\pi D(t-\tau)}}\left\{\exp\left[-\frac{(\zeta-z)^2}{4D(t-\tau)}\right]+\exp\left[-\frac{(\zeta+z)^2}{4D(t-\tau)}\right]\right\}$

4[R] $\frac{H(t-\tau)}{\sqrt{4\pi D(t-\tau)}}\left\{\exp\left[-\frac{(\zeta-z)^2}{4D(t-\tau)}\right]+\exp\left[-\frac{[\zeta+z]^2}{4D(t-\tau)}\right]\right\}+$

$$+H(t-\tau)\sigma\exp[-\sigma[\zeta+z-\sigma D(t-\tau)]]\cdot \mathrm{erfc}\left[\frac{\zeta+z-2\sigma D(t-\tau)}{\sqrt{4D(t-\tau)}}\right]$$

5[D-D] $H(t-\tau)\sum_{n=1}^{\infty}\frac{2}{c}\exp\left[-D\left(\frac{n^2\pi^2}{c^2}\right)(t-\tau)\right]\sin\left(\frac{n\pi}{c}z\right)\sin\left(\frac{n\pi}{c}\zeta\right)$

6[D-N] $H(t-\tau)\sum_{n=1}^{\infty}\frac{2}{c}\exp\left[-\frac{(2n-1)^2\pi^2}{4c^2}D(t-\tau)\right]\sin\left[\frac{(2n-1)\pi}{2c}z\right]\sin\left[\frac{(2n-1)\pi}{2c}\zeta\right]$

7[D-R] $H(t-\tau)\sum_{n=1}^{\infty}\frac{2(\lambda_n+\sigma_2^2)}{c(\lambda_n+\sigma_2^2)+\sigma_2}\exp[-D\lambda_n(t-\tau)]\sin(\sqrt{\lambda_n}z)\sin(\sqrt{\lambda_n}\zeta)$, where, $\tan(c\sqrt{\lambda}) = -\frac{\sqrt{\lambda}}{\sigma_2}$

8[N-D] $H(t-\tau)\sum_{n=1}^{\infty}\frac{2}{c}\exp\left[-\frac{(2n-1)^2\pi^2}{4c^2}D(t-\tau)\right]\cdot$

$$\cdot\cos\left[\frac{(2n-1)\pi}{2c}z\right]\cos\left[\frac{(2n-1)\pi}{2c}\zeta\right]$$

9[N-N] $H(t-\tau)\left\{\frac{1}{c}+\sum_{n=1}^{\infty}\frac{2}{c}\exp\left[-\frac{n^2\pi^2}{c^2}D(t-\tau)\right]\cos\left(\frac{n\pi}{c}z\right)\cos\left(\frac{n\pi}{c}\zeta\right)\right\}$

10[N-R] $H(t-\tau)\sum_{n=1}^{\infty}\frac{2(\lambda_n+\sigma_2^2)}{c(\lambda_n+\sigma_2^2)+\sigma_2}\exp[-D\lambda_n(t-\tau)]\cos(\sqrt{\lambda_n}z)\cos(\sqrt{\lambda_n}\zeta)$,

where, $\tan(c\sqrt{\lambda}) = \frac{\sigma_2}{\sqrt{\lambda}}$

11[R-D] $H(t-\tau)\sum_{n=1}^{\infty}\frac{2(\lambda_n+\sigma_1^2)}{c(\lambda_n+\sigma_1^2)-\sigma_1}\exp[-D\lambda_n(t-\tau)]\cdot$

$$\cdot\sin[\sqrt{\lambda_n}(z-c)]\sin[\sqrt{\lambda_n}(\zeta-c)]$$

where, $\tan(c\sqrt{\lambda}) = \frac{\sqrt{\lambda}}{\sigma_1}$

12[R-N] $H(t-\tau)\sum_{n=1}^{\infty}\frac{2(\lambda_n+\sigma_1^2)}{c(\lambda_n+\sigma_1^2)-\sigma_1}\exp[-D\lambda_n(t-\tau)]\cdot$

$$\cdot\cos[\sqrt{\lambda_n}(z-c)]\cos[\sqrt{\lambda_n}(\zeta-c)],$$

where, $\tan(c\sqrt{\lambda}) = -\frac{\sigma_1}{\sqrt{\lambda}}$

13[R-R] $H(t-\tau)\sum_{n=1}^{\infty}\frac{2\Psi_{1n}}{\Psi_{2n}+\Psi_{3n}}\exp[-D\lambda_n(t-\tau)]\cdot$

$$\cdot[\sqrt{\lambda_n}\cos(\sqrt{\lambda_n}z)-\sigma_1\sin(\sqrt{\lambda_n}z)][\sqrt{\lambda_n}\cos(\sqrt{\lambda_n}\zeta)-\sigma_1\sin(\sqrt{\lambda_n}\zeta)],$$

where, $\tan(c\sqrt{\lambda}) = \frac{\sqrt{\lambda}(\sigma_2-\sigma_1)}{\lambda+\sigma_1\sigma_2}$,

$$\Psi_{1n} = (\lambda_n+\sigma_1\sigma_2)^2+\lambda_n(\sigma_2-\sigma_1)^2,$$

$$\Psi_{2n} = c(\lambda_n+\sigma_1^2)\cdot\Psi_{1n},$$

$$\Psi_{3n} = (\sigma_2-\sigma_1)[\lambda_n^2-\lambda_n\sigma_1(\sigma_2-\sigma_1)-\sigma_1^3\sigma_2].$$

7. Applications

7.1 Validation of gas diffusion model of GASFLOW

GASFLOW is a multi-dimensional fluid dynamics field computer code, which is widely applied in flow and safety analyses in nuclear industry (Travis et al., 1998). The original ideal of the work about Green's functions is to validate the diffusion solver of GASFLOW based on the theoretical solutions obtained by using the GFM. To exclude any adverse effects such as buoyancy and chemistry, a problem of nitrogen diffusing into an advective air flow is designed to verify purely the diffusion solver. The advection diffusion problems in 1D, 2D and 3D are simulated, respectively, by using GASFLOW; the computational results are compared with the Green's function solutions accordingly.

7.1.1 One-dimensional advection diffusion problem

A 1D channel is designed to contain a uniform advective air flow, as shown in Figure 7-1-1-1. At one end of the channel, nitrogen is injected into the air flow. Another end is assumed to be far enough. The pressure is 0.1013 MPa and the temperature 298.15 K. By defining the mass fraction of nitrogen as the solved variable $\mathbf{u} = \mathbf{u(x,t)}$, the advection diffusion process can be formulated as,

$$\mathbf{u_t} + \mathbf{Vu_x} - \mathbf{Du_{xx}} = 0,\ 0 < \mathbf{x} < \infty,\ 0 < \mathbf{t} < \infty, \qquad (7\text{-}1\text{-}1\text{-}1a)$$

$$\mathbf{u(x,}0\mathbf{)} = 0, \qquad (7\text{-}1\text{-}1\text{-}1b)$$

$$\mathbf{u(}0,\mathbf{t)} = 1. \qquad (7\text{-}1\text{-}1\text{-}1c)$$

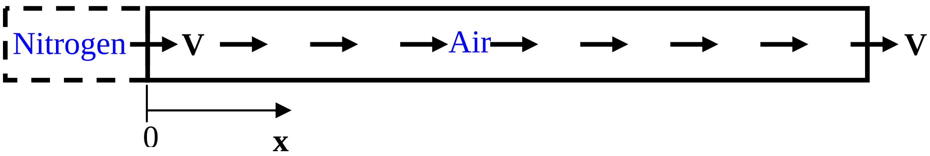


Figure 7-1-1-1 One-dimensional advective diffusion of nitrogen into air flow

This is a boundary value problem in a Dirichlet type, defined in a semi-infinite domain, without a source term and with a zero initial value. By looking up the Green's function in Table 6-3-1, the solution is simply expressed as,

$$\mathbf{u(x,t)} = \int_0^{\mathbf{t}} \frac{1}{\sqrt{16\pi\mathbf{D(t-\tau)}^3}} \cdot \{[\mathbf{x} - \mathbf{V(t-\tau)}]\exp\left[-\frac{[\mathbf{x}-\mathbf{V(t-\tau)}]^2}{4\mathbf{D(t-\tau)}}\right] +$$

$$+[\mathbf{x}+\mathbf{V(t-\tau)}]\exp\left[\frac{\mathbf{Vx}}{\mathbf{D}} - \frac{[\mathbf{x}+\mathbf{V(t-\tau)}]^2}{4\mathbf{D(t-\tau)}}\right]\}\mathbf{d\tau},$$

and simplified as, by using the property of convolution,

$$\mathbf{u(x,t)} = \int_0^{\mathbf{t}} \frac{1}{\sqrt{16\pi\mathbf{D\tau}^3}} \left\{(\mathbf{x}-\mathbf{V\tau})\exp\left[-\frac{(\mathbf{x}-\mathbf{V\tau})^2}{4\mathbf{D\tau}}\right] + (\mathbf{x}+\mathbf{V\tau})\exp\left[\frac{\mathbf{Vx}}{\mathbf{D}} - \frac{(\mathbf{x}+\mathbf{V\tau})^2}{4\mathbf{D\tau}}\right]\right\}\mathbf{d\tau}\ .$$

(7-1-1-2)

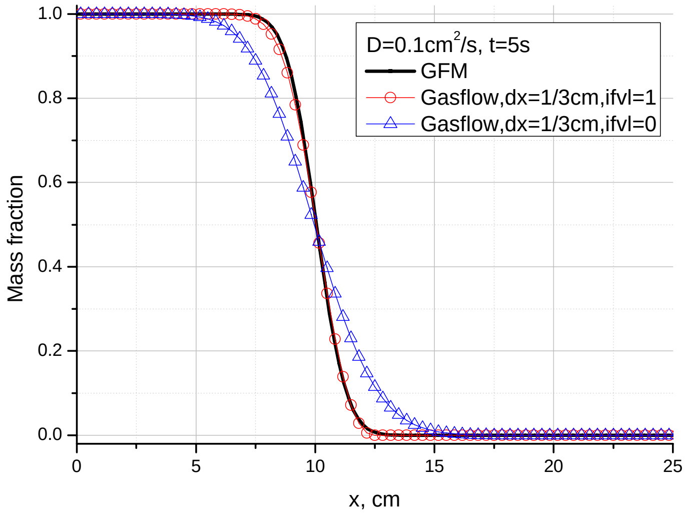
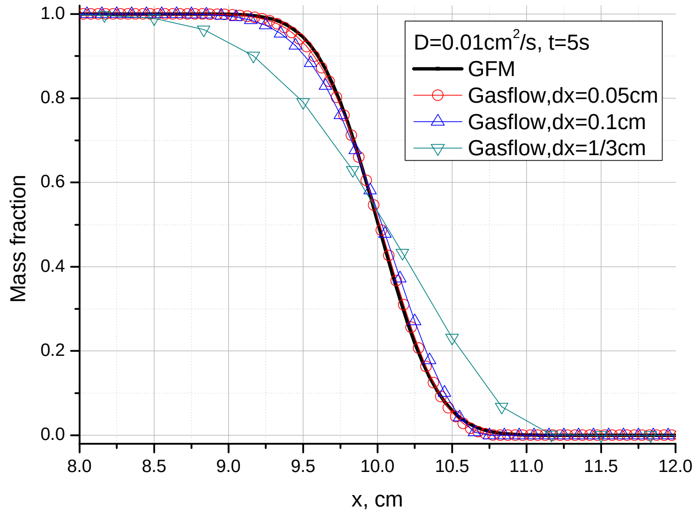


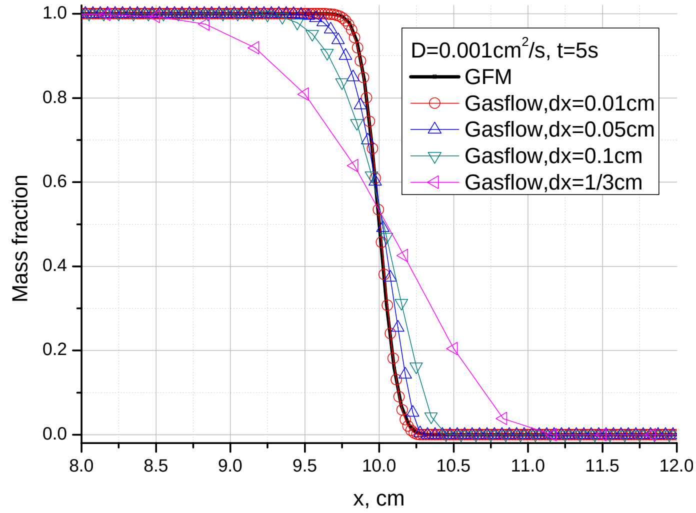
Figure 7-1-1-2 One-dimensional advective diffusion with $\mathbf{D} = 0.1\,\mathbf{cm^2/s}$ and different ifvl options

If $\mathbf{D} = 0.1\,\mathrm{cm^2/s}$ and $\mathbf{V} = 2\,\mathrm{cm/s}$, the theoretical nitrogen mass fraction distribution at $\mathbf{t} = 5\,\mathrm{s}$ is shown as the thicker solid line in Figure 7-1-1-2. The same problem is simulated by using GASFLOW. In the code, a van Leer option (**ifvl**) is set to choose the first- or second-order of the numerical schemes for discretizing of the advection term in space: "0" means the first-order, "1" the second-order. The simulation results by using both options are plotted in Figure 7-1-1-2 for comparison with the Green's function. The figure shows that the second order result is better than the first-order, and has a good agreement with the Green's function solution. Two important aspects about the propagating front of nitrogen are reflected in Figure 7-1-1-2: the advection distance and the diffusion length. The former is about 10 cm, equal to the product of the advection velocity (2 cm/s) and the time (5 s), the latter is about 5 cm, which depends on the diffusion coefficient and the time.

The sensitivity of the simulation results on cell sizes are performed for the cases of $\mathbf{D} = 0.01\,\mathrm{cm^2/s}$ and $\mathbf{D} = 0.001\,\mathrm{cm^2/s}$, and the results together with the corresponding Green's function solutions are shown in Figure 7-1-1-3 (a) and (b), respectively. The advection velocity is not changed and the second-order scheme about advection ($\mathbf{ifvl} = 1$) is applied in the GASFLOW simulations. It is the same as here if no special words are given in following text. Figure 7-1-1-3 indicates that the smaller the diffusion coefficient is, the sharper the nitrogen front, and the smaller cell size is needed to reproduce the leading front. The convergence of the simulating results to the theoretical solutions is presented from coarse grids to refined grids in both (a) and (b).



(a)



(b)

Figure 7-1-1-3 One-dimensional advective diffusion with different cell sizes

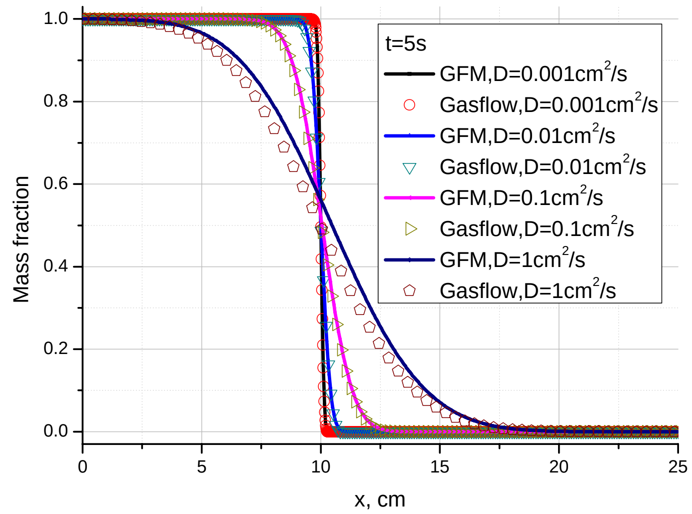


Figure 7-1-1-4 One-dimensional advective diffusion with different diffusion coefficients

The comparisons between the GASFLOW simulations and the Green's function solutions are also made for different diffusion coefficients ranging from 0.001 cm^2/s to 1 cm^2/s, as shown in Figure 7-1-1-4. Perfect agreements are obtained between the GFM and GASFLOW in the different cases except that of $\mathbf{D} = 1$cm^2/s, which has a slight deviation, but within an acceptable limit. Figure 7-1-1-4 clearly manifests that a bigger $\mathbf{D}$ results in a wider nitrogen front, or a bigger diffusion distance.

7.1.2 Two-dimensional advection diffusion problem

A similar two-dimensional advection diffusion problem of nitrogen into an air flow is designed to validate the performance of GASFLOW. As shown in Figure 7-1-2-1, nitrogen is released only from the interval $[\mathbf{y}_0, \mathbf{y}_1]$ to the advective air flow. The mass fraction of nitrogen is defined as the unknown variable, $\mathbf{u}$, to be solved. The problem can be modeled as a semi-infinite ($\mathbf{x}$ direction) and infinite ($\mathbf{y}$ direction) 2D advection diffusion problem with a Dirichlet boundary condition:

$$\mathbf{u}_t + \mathbf{V}\mathbf{u}_x - \mathbf{D}(\mathbf{u}_{xx} + \mathbf{u}_{yy}) = 0,\ 0 < \mathbf{x}, \mathbf{y} < \infty,\ 0 < \mathbf{t} < \infty, \qquad \text{(7-1-2-1a)}$$

$$\mathbf{u}(\mathbf{x}, \mathbf{y}, 0) = 0, \qquad \text{(7-1-2-1b)}$$

$$\mathbf{u}(0, \mathbf{y}, \mathbf{t}) = \begin{cases} 1, & \mathbf{if}, \mathbf{y} \in [\mathbf{y}_0, \mathbf{y}_1], \\ 0, & \mathbf{otherwise.} \end{cases} \qquad \text{(7-1-2-1c)}$$

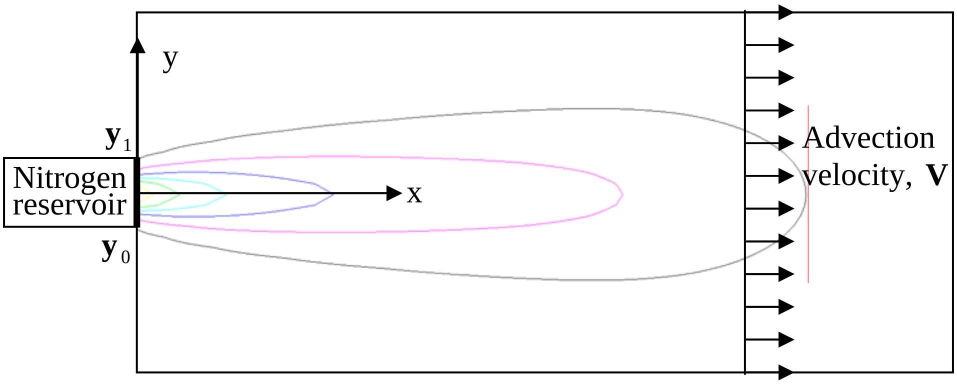


Figure 7-1-2-1 Two-dimensional advective diffusion of nitrogen into air flow

According to the GFM, the general solution for a 2D problem is expressed as (6-1-8),

$$\mathbf{u}(\mathbf{x},\mathbf{y},\mathbf{t}) = \int_0^{\mathbf{t}}\int_{\eta_1}^{\eta_2}\int_{\xi_1}^{\xi_2} \mathbf{G}\phi \mathbf{d}\xi \mathbf{d}\eta \mathbf{d}\tau + \int_{\eta_1}^{\eta_2}\int_{\xi_1}^{\xi_2} (\mathbf{G}\mathbf{u})\Big|_{\tau=0} \mathbf{d}\xi \mathbf{d}\eta +$$

$$+ \int_0^{\mathbf{t}}\int_{\eta_1}^{\eta_2} [\mathbf{D}(\mathbf{G}\mathbf{u}_\xi - \mathbf{G}_\xi \mathbf{u}) - \mathbf{V}\mathbf{G}\mathbf{u}]\Big|_{\xi_1}^{\xi_2} \mathbf{d}\eta \mathbf{d}\tau + \int_0^{\mathbf{t}}\int_{\xi_1}^{\xi_2} [\mathbf{D}(\mathbf{G}\mathbf{u}_\eta - \mathbf{G}_\eta \mathbf{u})]\Big|_{\eta_1}^{\eta_2} \mathbf{d}\xi \mathbf{d}\tau. \qquad \text{(6-1-8)}$$

In the application, $(\xi_1, \xi_2) = (0, \infty)$, $(\eta_1, \eta_2) = (-\infty, \infty)$. By substituting them into (6-1-8), it becomes as,

$$\mathbf{u}(\mathbf{x},\mathbf{y},\mathbf{t}) = \int_0^{\mathbf{t}}\int_{-\infty}^{\infty}\int_0^{\infty} \mathbf{G}\phi \mathbf{d}\xi \mathbf{d}\eta \mathbf{d}\tau + \int_{-\infty}^{\infty}\int_0^{\infty} (\mathbf{G}\mathbf{u})\Big|_{\tau=0} \mathbf{d}\xi \mathbf{d}\eta +$$

$$+ \int_0^{\mathbf{t}}\int_{-\infty}^{\infty} [\mathbf{D}(\mathbf{G}\mathbf{u}_\xi - \mathbf{G}_\xi \mathbf{u}) - \mathbf{V}\mathbf{G}\mathbf{u}]\Big|_{\xi=0}^{\xi=\infty} \mathbf{d}\eta \mathbf{d}\tau + \int_0^{\mathbf{t}}\int_0^{\infty} [\mathbf{D}(\mathbf{G}\mathbf{u}_\eta - \mathbf{G}_\eta \mathbf{u})]\Big|_{\eta=-\infty}^{\eta=\infty} \mathbf{d}\xi \mathbf{d}\tau \qquad \text{.(7-1-2-2)}$$

In the case, $\phi = 0$ and $\mathbf{u}\big|_{\tau=0} = 0$ according to (7-1-2-1a) and (7-1-2-1b), respectively, then (7-1-2-2) is reduced as,

$$\mathbf{u(x,y,t)} = \int_0^{\mathbf{t}}\int_{-\infty}^{\infty}[\mathbf{D(Gu_\xi - G_\xi u) - VGu}]\Big|_{\xi=0}^{\xi=\infty}\mathbf{d\eta d\tau} + \int_0^{\mathbf{t}}\int_0^{\infty}[\mathbf{D(Gu_\eta - G_\eta u)}]\Big|_{\eta=-\infty}^{\eta=\infty}\mathbf{d\xi d\tau}\,. \tag{7-1-2-3}$$

Considering the homogenous boundary conditions of the Green's function at infinite, namely,

$$\mathbf{G}\big|_{\xi=\infty} = 0,\ \mathbf{G}\big|_{\eta=\pm\infty} = 0,\ \mathbf{(G_\xi)}\big|_{\xi=\infty} = 0,\ \mathbf{(G_\eta)}\big|_{\eta=\pm\infty} = 0\,, \tag{7-1-2-4}$$

the expression of solution (7-1-2-3) is reduced further as,

$$\mathbf{u(x,y,t)} = -\int_0^{\mathbf{t}}\int_{-\infty}^{\infty}[\mathbf{D(Gu_\xi - G_\xi u) - VGu}]\Big|_{\xi=0}\mathbf{d\eta d\tau}\,, \tag{7-1-2-5}$$

where $\mathbf{u}\big|_{\xi=0}$ is prescribed in (7-1-2-1c) as a Dirichlet boundary and $\mathbf{(u_\xi)}\big|_{\xi=0}$ is not prescribed. Therefore the term $\mathbf{(Gu_\xi)}\big|_{\xi=0}$ in (7-1-2-5) is unwelcome, then the additional boundary condition of $\mathbf{G}$ must be $\mathbf{G}\big|_{\xi=0} = 0$. By applying the boundary condition, (7-1-2-5) is changed as,

$$\mathbf{u(x,y,t)} = \int_0^{\mathbf{t}}\int_{-\infty}^{\infty}[\mathbf{D(G_\xi u)}]\Big|_{\xi=0}\mathbf{d\eta d\tau}\,. \tag{7-1-2-6}$$

Based on the product rule of Green's function, we have,

$$\mathbf{G(\xi,\eta,\tau) = X(\xi,\tau)Y(\eta,\tau)}\,, \tag{7-1-2-7}$$

thus,

$$\frac{\partial}{\partial\xi}\mathbf{G(\xi,\eta,\tau)} = \frac{\partial}{\partial\xi}\mathbf{X(\xi,\tau)\cdot Y(\eta,\tau)}\,. \tag{7-1-2-8}$$

By substituting (7-1-2-8) into (7-1-2-6), we have,

$$\mathbf{u(x,y,t)} = \int_0^{\mathbf{t}}\int_{-\infty}^{\infty}[\mathbf{D}\cdot\frac{\partial}{\partial\xi}\mathbf{X}(0,\tau)\cdot\mathbf{Y(\eta,\tau)\cdot u}(0,\eta,\tau)]\mathbf{d\eta d\tau} =$$

$$= \int_0^{\mathbf{t}}\{\mathbf{D}\cdot\frac{\partial}{\partial\xi}\mathbf{X}(0,\tau)\cdot\int_{-\infty}^{\infty}[\mathbf{Y(\eta,\tau)u}(0,\eta,\tau)]\mathbf{d\eta}\}\mathbf{d\tau}\,. \tag{7-1-2-9}$$

By using the Green's function tables, Table 6-3-1 and Table 6-3-2, we can find the proper directional Green's functions according the domain and the boundary condition,

$$\mathbf{X(\xi,\tau)} = \frac{\mathbf{H(t-\tau)}}{\sqrt{4\pi\mathbf{D(t-\tau)}}}\left\{\mathbf{exp}\left[-\frac{[\mathbf{\xi - x + V(t-\tau)}]^2}{4\mathbf{D(t-\tau)}}\right] - \mathbf{exp}\left[\frac{\mathbf{Vx}}{\mathbf{D}} - \frac{[\mathbf{\xi + x + V(t-\tau)}]^2}{4\mathbf{D(t-\tau)}}\right]\right\},$$

$$\mathbf{Y(\eta,\tau)} = \frac{\mathbf{H(t-\tau)}}{\sqrt{4\pi\mathbf{D(t-\tau)}}}\mathbf{exp}\left[-\frac{(\mathbf{\eta - y})^2}{4\mathbf{D(t-\tau)}}\right].$$

Thus,

$$\frac{\partial}{\partial \xi}\mathbf{X}(0,\tau)=\frac{\mathbf{H}(\mathbf{t}-\tau)}{\sqrt{4\pi\mathbf{D}(\mathbf{t}-\tau)}}\{\mathbf{exp}\left[-\frac{[\mathbf{x}-\mathbf{V}(\mathbf{t}-\tau)]^2}{4\mathbf{D}(\mathbf{t}-\tau)}\right]\frac{[\mathbf{x}-\mathbf{V}(\mathbf{t}-\tau)]}{2\mathbf{D}(\mathbf{t}-\tau)}+$$

$$+\mathbf{exp}\left[\frac{\mathbf{Vx}}{\mathbf{D}}-\frac{[\mathbf{x}+\mathbf{V}(\mathbf{t}-\tau)]^2}{4\mathbf{D}(\mathbf{t}-\tau)}\right]\frac{[\mathbf{x}+\mathbf{V}(\mathbf{t}-\tau)]}{2\mathbf{D}(\mathbf{t}-\tau)}\}.$$

By bringing them into (7-1-2-9) and applying (7-1-2-1c), the solution becomes as,

$$\mathbf{u}(\mathbf{x},\mathbf{y},\mathbf{t})=\int_0^{\mathbf{t}}\frac{\mathbf{D}}{\sqrt{4\pi\mathbf{D}(\mathbf{t}-\tau)}}\cdot\{\mathbf{exp}\left[-\frac{[\mathbf{x}-\mathbf{V}(\mathbf{t}-\tau)]^2}{4\mathbf{D}(\mathbf{t}-\tau)}\right]\frac{[\mathbf{x}-\mathbf{V}(\mathbf{t}-\tau)]}{2\mathbf{D}(\mathbf{t}-\tau)}+$$

$$+\mathbf{exp}\left[\frac{\mathbf{Vx}}{\mathbf{D}}-\frac{[\mathbf{x}+\mathbf{V}(\mathbf{t}-\tau)]^2}{4\mathbf{D}(\mathbf{t}-\tau)}\right]\frac{[\mathbf{x}+\mathbf{V}(\mathbf{t}-\tau)]}{2\mathbf{D}(\mathbf{t}-\tau)}\}\cdot\int_{\mathbf{y}_0}^{\mathbf{y}_1}\frac{1}{\sqrt{4\pi\mathbf{D}(\mathbf{t}-\tau)}}\mathbf{exp}\left[-\frac{(\eta-\mathbf{y})^2}{4\mathbf{D}(\mathbf{t}-\tau)}\right]\mathbf{d}\eta\mathbf{d}\tau.$$

Being simplified as,

$$\mathbf{u}(\mathbf{x},\mathbf{y},\mathbf{t})=\int_0^{\mathbf{t}}\frac{1}{\sqrt{64\pi\mathbf{D}(\mathbf{t}-\tau)^3}}\cdot\{[\mathbf{x}-\mathbf{V}(\mathbf{t}-\tau)]\mathbf{exp}\left[-\frac{[\mathbf{x}-\mathbf{V}(\mathbf{t}-\tau)]^2}{4\mathbf{D}(\mathbf{t}-\tau)}\right]+$$

$$+[\mathbf{x}+\mathbf{V}(\mathbf{t}-\tau)]\mathbf{exp}\left[\frac{\mathbf{Vx}}{\mathbf{D}}-\frac{[\mathbf{x}+\mathbf{V}(\mathbf{t}-\tau)]^2}{4\mathbf{D}(\mathbf{t}-\tau)}\right]\}\cdot\left[\mathbf{erfc}\left(\frac{\mathbf{y}-\mathbf{y}_1}{\sqrt{4\mathbf{D}(\mathbf{t}-\tau)}}\right)-\mathbf{erfc}\left(\frac{\mathbf{y}-\mathbf{y}_0}{\sqrt{4\mathbf{D}(\mathbf{t}-\tau)}}\right)\right]\mathbf{d}\tau$$

in a further step, as,

$$\mathbf{u}(\mathbf{x},\mathbf{y},\mathbf{t})=\int_0^{\mathbf{t}}\frac{1}{\sqrt{64\pi\mathbf{D}\tau^3}}\cdot\left\{(\mathbf{x}-\mathbf{V}\tau)\,\mathbf{exp}\left[-\frac{(\mathbf{x}-\mathbf{V}\tau)^2}{4\mathbf{D}\tau}\right]+(\mathbf{x}+\mathbf{V}\tau)\,\mathbf{exp}\left[\frac{\mathbf{Vx}}{\mathbf{D}}-\frac{(\mathbf{x}+\mathbf{V}\tau)^2}{4\mathbf{D}\tau}\right]\right\}\cdot$$

$$\cdot\left[\mathbf{erfc}\left(\frac{\mathbf{y}-\mathbf{y}_1}{\sqrt{4\mathbf{D}\tau}}\right)-\mathbf{erfc}\left(\frac{\mathbf{y}-\mathbf{y}_0}{\sqrt{4\mathbf{D}\tau}}\right)\right]\mathbf{d}\tau. \qquad (7\text{-}1\text{-}2\text{-}10)$$

This is the final solution of the nitrogen mass fraction distribution in the advective air flow. If $\mathbf{D}=0.1\,\mathrm{cm}^2/\mathrm{s}$, $\mathbf{V}=2$ cm/s, $\mathbf{y}_0=-0.5$ cm, $\mathbf{y}_1=0.5$ cm, a 3D view of the nitrogen mass fraction distribution at $\mathbf{t}=10\,\mathrm{s}$ is shown in Figure 7-1-2-2, which is drawn based on (7-1-2-10) by using Mathcad, a mathematics compute program.

On the other hand, GASFLOW is applied to simulate the advection diffusion problem accordingly, with a grid size of 0.2 cm. Figure 7-1-2-3 presents the mass fraction distribution along the central line of the plume in the advection direction at different times of 2s, 4s, 6s, 8s and 10s, to depict the developing front of the diffusing concentration.

The mass fraction distributions in the transverse direction at $\mathbf{x}=10$ cm and at different times are shown in Figure 7-1-2-4, which shows a Gaussian distribution with a growing amplitude on time. In the light of the figure, the upstream diffusion front arrives here at about 4 s, and grows up to be mature at about 6 s. The fastest growing occurs at about 5 s, simply because the advection front arrives here at this moment and dominates the mixing process. The high consistency between the theoretical solutions and the numerical solutions manifests that GASFLOW can reproduce numerically the advection diffusion process in a quite satisfactory way.

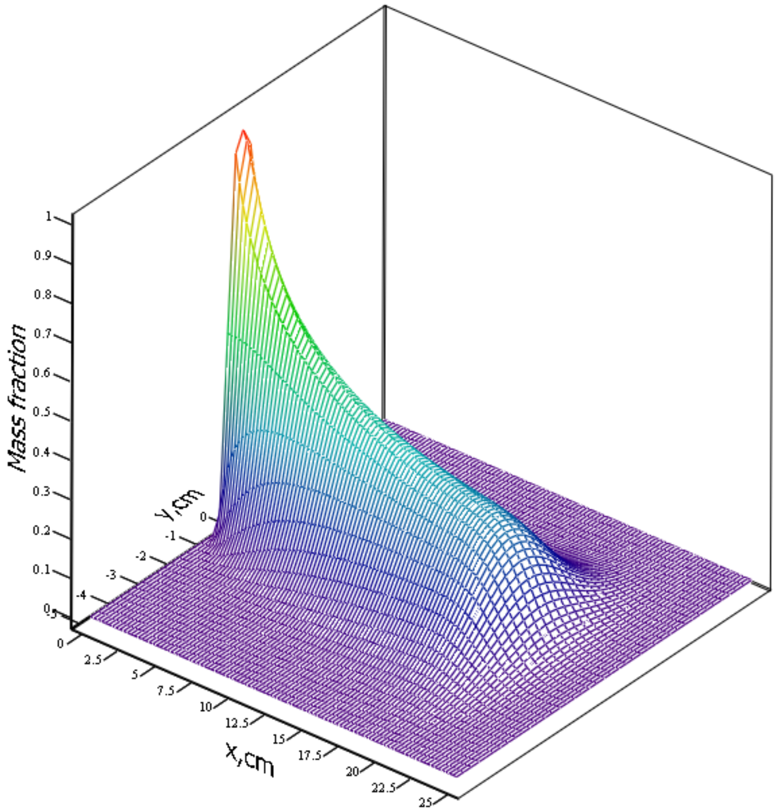


Figure 7-1-2-2 Three-dimensional view of advection diffusion plume of nitrogen at $t = 10$ s

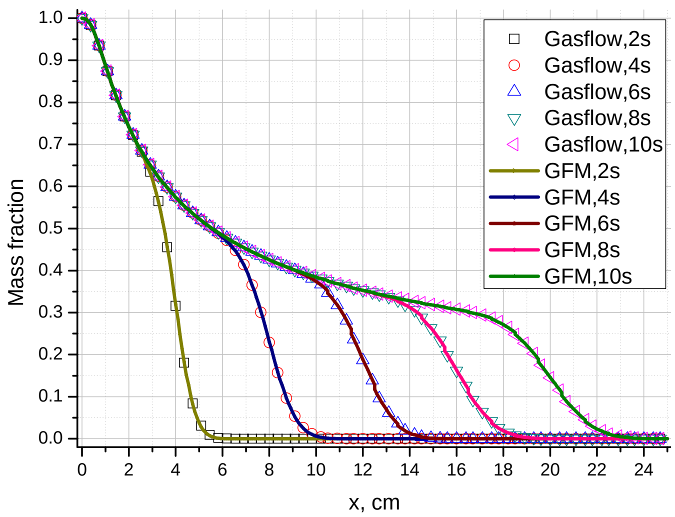


Figure 7-1-2-3 Mass fraction distributions in advection direction ($y = 0$ cm) at different times (two-dimensional problem)

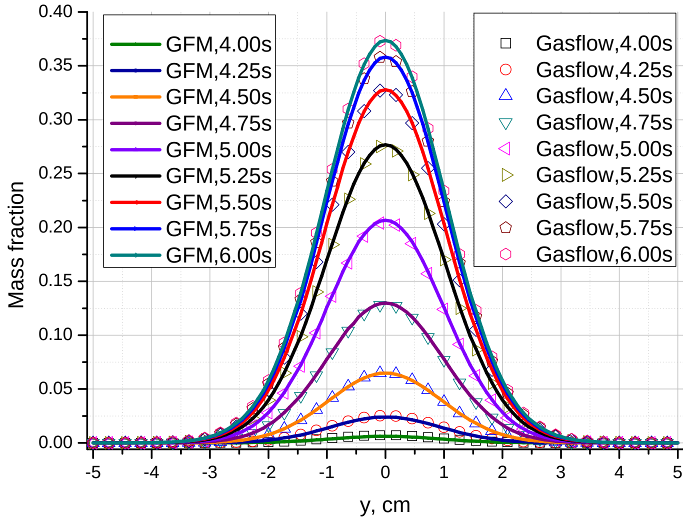


Figure 7-1-2-4 Mass fraction distributions in transverse direction ($x = 10$ cm) at different times (two-dimensional problem)

7.1.3 Three-dimensional advection diffusion problem

The three-dimensional problem of nitrogen diffusion into an air flow is depicted schematically in Figure 7-1-3-1. It is in principle a 3D boundary value problem. The governing equation about the nitrogen mass fraction distribution is,

$$\mathbf{u_t + Vu_x - D(u_{xx} + u_{yy} + u_{zz}) = 0,\ 0 < x, y, z < \infty,\ 0 < t < \infty,} \tag{7-1-3-1a}$$

$$\mathbf{u(x, y, z, 0) = 0,} \tag{7-1-3-1b}$$

$$\mathbf{u(0, y, z, t)} = \begin{cases} 1, & \textbf{if} \quad \mathbf{y \in [y_0, y_1]} \quad \textbf{and} \quad \mathbf{z \in [z_0, z_1],} \\ 0, & \textbf{otherwise.} \end{cases} \tag{7-1-3-1c}$$

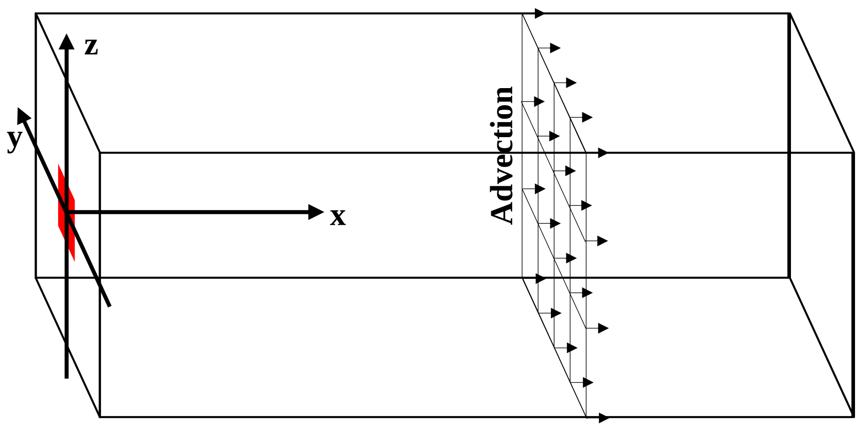


Figure 7-1-3-1 Three-dimensional advective diffusion of nitrogen into air flow

In the light of the GFM, the solution expression is,

$$\mathbf{u(x, y, z, t)} = \int_0^t \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathbf{[D(G_\xi u)]}\Big|_{\xi=0} \mathbf{d\eta\, d\zeta\, d\tau} =$$

$$= \int_0^t \mathbf{DX_\xi(0, \tau)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathbf{Y(\eta, \tau) Z(\zeta, \tau) u(0, \eta, \zeta, \tau) d\eta\, d\zeta\, d\tau} =$$

$$= \int_0^t DX_\xi(0,\tau) \int_{z_0}^{z_1}\int_{y_0}^{y_1} Y(\eta,\tau) Z(\zeta,\tau)\cdot 1 d\eta d\zeta d\tau = \int_0^t DX_\xi(0,\tau) \int_{y_0}^{y_1} Y(\eta,\tau) d\eta \int_{z_0}^{z_1} Z(\zeta,\tau) d\zeta d\tau \, .$$

(7-1-3-2)

By looking up the directional Green's functions $\mathbf{X, Y, Z}$ from Table 6-3-1, Table 6-3-2 and Table 6-3-3, respectively, and substituting them into (7-1-3-2) and applying the communication law of convolution, we have,

$$u(x,y,z,t) = \int_0^t \frac{1}{\sqrt{16\pi D\tau^3}} \cdot \left\{ (x - V\tau)\exp\left[-\frac{(x-V\tau)^2}{4D\tau} \right] + (x+V\tau)\exp\left[\frac{Vx}{D} - \frac{(x+V\tau)^2}{4D\tau} \right] \right\} \cdot$$

$$\cdot \int_{y_0}^{y_1} \frac{1}{\sqrt{4\pi D\tau}} \exp\left[-\frac{(\eta - y)^2}{4D\tau} \right] d\eta \cdot \int_{z_0}^{z_1} \frac{1}{\sqrt{4\pi D\tau}} \exp\left[-\frac{(\zeta - z)^2}{4D\tau} \right] d\zeta d\tau =$$

$$= \int_0^t \frac{1}{\sqrt{256\pi D\tau^3}} \cdot \left\{ (x - V\tau)\exp\left[-\frac{(x-V\tau)^2}{4D\tau} \right] + (x+V\tau)\exp\left[\frac{Vx}{D} - \frac{(x+V\tau)^2}{4D\tau} \right] \right\} \cdot$$

$$\cdot \left[\operatorname{erfc}\left(\frac{y - y_1}{\sqrt{4D\tau}} \right) - \operatorname{erfc}\left(\frac{y - y_0}{\sqrt{4D\tau}} \right) \right] \cdot \left[\operatorname{erfc}\left(\frac{z - z_1}{\sqrt{4D\tau}} \right) - \operatorname{erfc}\left(\frac{z - z_0}{\sqrt{4D\tau}} \right) \right] d\tau \, . \quad (7\text{-}1\text{-}3\text{-}3)$$

This is the formula the nitrogen mass fraction should satisfy. By using Mathcad, the mass fraction distributions along the $\mathbf{x}$-axis at $\mathbf{t} = 2$s, 4s, 6s and 10**s** are plotted as solid lines in Figure 7-1-3-2, where $\mathbf{D} = 0.1\,\text{cm}^2/\text{s}$, $\mathbf{V} = 2$ cm/s, $\mathbf{y}_0 = \mathbf{z}_0 = -0.5$ cm, $\mathbf{y}_1 = \mathbf{z}_1 = 0.5$ cm. In the figure, the symbols stand for the GASFLOW simulation with a grid size of 0.2 cm. The mass fraction distributions at transverse directions ($\mathbf{y}$ and $\mathbf{z}$) at $\mathbf{x} = 10$ cm and at $\mathbf{t} = 6$ s are also compared with the theoretical solution in Figure 7-1-3-3. Perfect agreements are obtained between the GFM and GASFLOW in both figures.

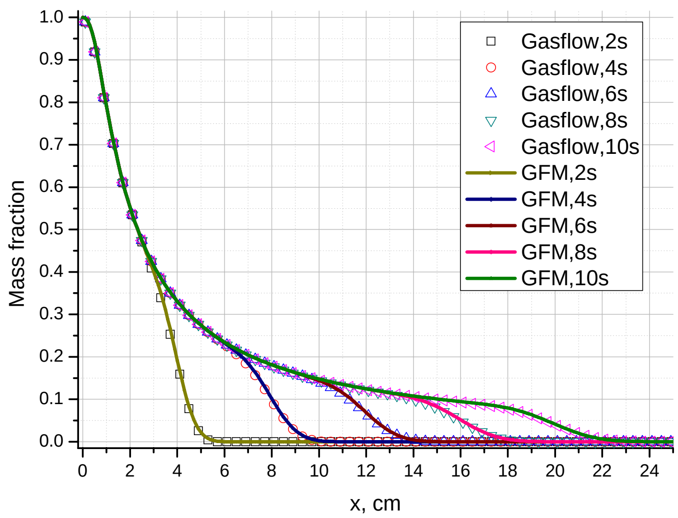


Figure 7-1-3-2 Mass fraction distributions in advection direction ($\mathbf{y} = \mathbf{z} = 0$ cm) at different times (three-dimensional problem)

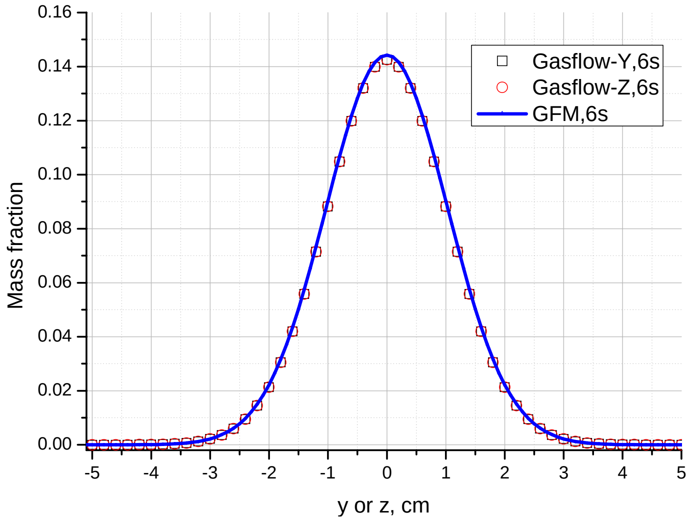


Figure 7-1-3-3 Mass fraction distributions in transverse directions ($x = 10$ cm) at $t = 6$ s (three-dimensional problem)

7.2 Validation of particle mobilization model of GASFLOW

A discrete Lagrangian particle transport model is developed in the GASFLOW code. The mean motion of each simulating particle is governed by particle momentum equations according to the Lagrangian approach. A stochastic turbulent particle diffusion model is developed accordingly to describe the particle diffusion caused by particle concentration gradients and the turbulence of the conveying gas flow (Travis et al., 1998). In terms of the model, the particle turbulent fluctuations on the mean motion are described by a diffusion velocity, $\vec{\mathbf{U}}_{\mathbf{diff}}$, which is determined by,

$$\vec{\mathbf{U}}_{\mathbf{diff}} = \sqrt{\frac{4\mathbf{D}}{\Delta\mathbf{t}}}[\pm\mathbf{erf}^{-1}(\zeta_1)\vec{\mathbf{i}} \pm \mathbf{erf}^{-1}(\zeta_2)\vec{\mathbf{j}} \pm \mathbf{erf}^{-1}(\zeta_3)\vec{\mathbf{k}}], \qquad (7\text{-}2\text{-}0\text{-}1)$$

where $\mathbf{D}$ is the particle diffusion coefficient, $\Delta\mathbf{t}$ is the time step of the numerical scheme, $\zeta_1, \zeta_2, \zeta_3$ are three random numbers within $[0,1]$, the sign "$\pm$" is also randomly determined, $\vec{\mathbf{i}}, \vec{\mathbf{j}}, \vec{\mathbf{k}}$ stand for unit vectors in the three coordinate directions, respectively, $\mathbf{erf}^{-1}(\cdot)$ is the inversed error function (Xu et al., 2007). The factor of the turbulence is accounted in the coefficient $\mathbf{D}$. In principle, the stochastic representation (7-2-0-1) satisfies the Fick's law about diffusion. In other word, the particle distribution, say, the concentration should converge to the continuous solution by solving the diffusion equations analytically, if the particle sample number is sufficiently big. This subsection is contributed to verify the particle model of GASFLOW fundamentally and systematically, based on the developed Green's function solutions in Section 6. Since particles could be released instantaneously or continuously, from ideal points, two-dimensional areas or three-dimensional cubes, into stagnant or advective flows, a series of diffusion problems are discussed in the following subsections.

7.2.1 Point source in stagnant flow

In case of stagnant flows, particles are driven to move only by the concentration gradients. Since the accompanying fluid is quiescent, it is a general diffusion problem without advection, namely, "$\mathbf{V} = 0$". The diffusion model of GASFLOW is verified in case of 1D, 2D and 3D, with instantaneous and continuous particle releases, respectively.

I. Instantaneous point source

Mathematically, a one- or multi-dimensional Dirac delta function is applied to formulate the instantaneous particle release as a point source term with a constant coefficient to stand for the strength of the particle source. It can be assumed that the particles are released instantaneously at $\mathbf{t}=0$, and at the very center (origin) of an infinite 1D pipe, a 2D square or a 3D cube. The 1D, 2D and 3D problems are formulated as, respectively,

$$\mathbf{u_t - D(u_{xx}) = Q_1\delta(x,t)},\ -\infty < \mathbf{x} < \infty,\ 0 < \mathbf{t} < \infty, \tag{7-2-1-1}$$

$$\mathbf{u_t - D(u_{xx} + u_{yy}) = Q_2\delta(x,y,t)},\ -\infty < \mathbf{x,y} < \infty,\ 0 < \mathbf{t} < \infty, \tag{7-2-1-2}$$

$$\mathbf{u_t - D(u_{xx} + u_{yy} + u_{zz}) = Q_3\delta(x,y,z,t)},\ -\infty < \mathbf{x,y,z} < \infty,\ 0 < \mathbf{t} < \infty, \tag{7-2-1-3}$$

where $\mathbf{Q_N(N = 1,2,3)}$ is the total number of the released particles, or called "source strength". By applying the GFM, and noting the properties of delta function and the product rule about Green's functions, the solutions can be expressed as,

$$\mathbf{u(x,t) = Q_1\int_0^t\int_{-\infty}^{\infty} G(\xi,\tau;x,t)\delta(\xi,\tau)d\xi d\tau = Q_1\int_0^t\left[\int_{-\infty}^{\infty} X(\xi,\tau;x,t)\delta(\xi)d\xi\right]\delta(\tau)d\tau} =$$

$$= \mathbf{Q_1\int_0^t X(0,\tau;x,t)\delta(\tau)d\tau = Q_1X(0,0;x,t)}, \text{ analogously,} \tag{7-2-1-4}$$

$$\mathbf{u(x,y,t) = Q_2\int_0^t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(\xi,\eta,\tau;x,y,t)\delta(\xi,\eta,\tau)d\xi d\eta d\tau} =$$

$$= \mathbf{Q_2\int_0^t\left[\int_{-\infty}^{\infty} X(\xi,\tau;x,t)\delta(\xi)d\xi\right]\left[\int_{-\infty}^{\infty} Y(\eta,\tau;y,t)\delta(\eta)d\eta\right]\delta(\tau)d\tau} =$$

$$= \mathbf{Q_2\int_0^t X(0,\tau;x,t)Y(0,\tau;y,t)\delta(\tau)d\tau = Q_2X(0,0;x,t)Y(0,0;y,t)}, \tag{7-2-1-5}$$

$$\mathbf{u(x,y,z,t) = Q_3\int_0^t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(\xi,\eta,\zeta,\tau;x,y,z,t)\delta(\xi,\eta,\zeta,\tau)d\xi d\eta d\zeta d\tau} =$$

$$= \mathbf{Q_3\int_0^t\left[\int_{-\infty}^{\infty} X(\xi,\tau;x,t)\delta(\xi)d\xi\right]\left[\int_{-\infty}^{\infty} Y(\eta,\tau;y,t)\delta(\eta)d\eta\right]\left[\int_{-\infty}^{\infty} Z(\zeta,\tau;z,t)\delta(\zeta)d\zeta\right]\delta(\tau)d\tau} =$$

$$= \mathbf{Q_3\int_0^t X(0,\tau;x,t)Y(0,\tau;y,t)Z(0,\tau;z,t)\delta(\tau)d\tau = Q_3X(0,0;x,t)Y(0,0;y,t)Z(0,0;z,t)},$$

(7-2-1-6)

where $\mathbf{X,Y,Z}$ stand for the directional Green's functions. By looking up the Green's functions in Table 6-3-2 or Table 6-3-3, the solutions can be obtained explicitly for the 1D, 2D and 3D, respectively,

$$\mathbf{u(x,t)} = \mathbf{Q}_1 \frac{1}{\sqrt{4\pi \mathbf{Dt}}} \mathbf{exp}\left(-\frac{\mathbf{x}^2}{4\mathbf{Dt}}\right), \tag{7-2-1-7}$$

$$\mathbf{u(x,y,t)} = \mathbf{Q}_2 \left(\frac{1}{\sqrt{4\pi \mathbf{Dt}}}\right)^2 \mathbf{exp}\left(-\frac{\mathbf{x}^2}{4\mathbf{Dt}}\right) \cdot \mathbf{exp}\left(-\frac{\mathbf{y}^2}{4\mathbf{Dt}}\right), \tag{7-2-1-8}$$

$$\mathbf{u(x,y,z,t)} = \mathbf{Q}_3 \left(\frac{1}{\sqrt{4\pi \mathbf{Dt}}}\right)^3 \mathbf{exp}\left(-\frac{\mathbf{x}^2}{4\mathbf{Dt}}\right) \cdot \mathbf{exp}\left(-\frac{\mathbf{y}^2}{4\mathbf{Dt}}\right) \cdot \mathbf{exp}\left(-\frac{\mathbf{z}^2}{4\mathbf{Dt}}\right). \tag{7-2-1-9}$$

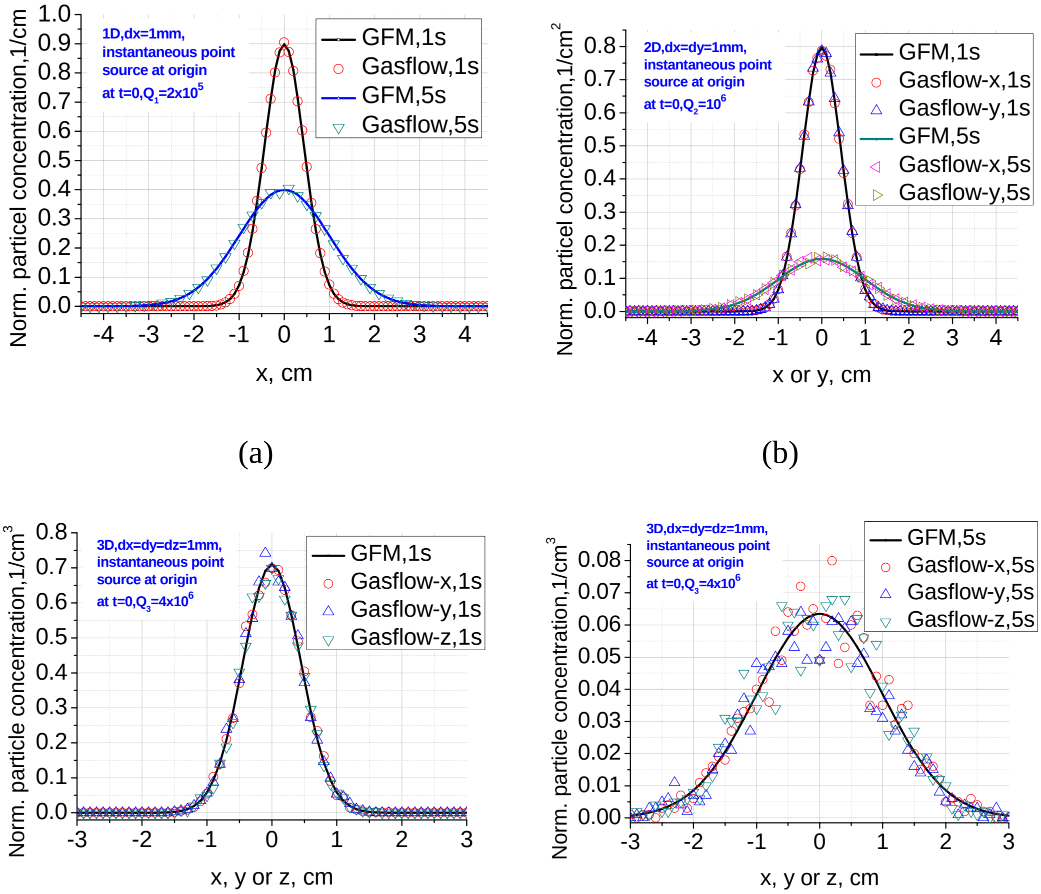
If the distance to the origin is denoted as $\mathbf{r}$, the above three formulas can be united into one expression by noting that $\mathbf{r}^2 = \mathbf{x}^2$, $\mathbf{r}^2 = \mathbf{x}^2 + \mathbf{y}^2$ or $\mathbf{r}^2 = \mathbf{x}^2 + \mathbf{y}^2 + \mathbf{z}^2$,

$$\mathbf{u(r,t)} = \frac{\mathbf{Q_N}}{\sqrt{(4\pi \mathbf{Dt})^{\mathbf{N}}}} \mathbf{exp}\left(-\frac{\mathbf{r}^2}{4\mathbf{Dt}}\right), \tag{7-2-1-10}$$

where, $\mathbf{t} \neq 0$, $\mathbf{N}$ denotes the number of dimensions.

The three diffusion problems are simulated numerically by using GASFLOW. The computational results are compared with the above theoretical solutions in Figure 7-2-1-1. In the GASFLOW simulations, the key parameters are specified as follows: the particle diameter, $\mathbf{d_p} = 5 \times 10^{-4}$ cm, the particle density, $\rho_\mathbf{p} = 1$g/cm^3, the gas density $\rho_\mathbf{g} = 1.179 \times 10^{-3}$ g/cm^3, the gas dynamic viscosity, $\mu_\mathbf{g} = 10^{-2}$ g/cms, thus, the particle Reynolds number $\mathbf{Re_p} = \rho_\mathbf{g} \mathbf{d_p} \left|\vec{\mathbf{U}}_\mathbf{g} - \vec{\mathbf{U}}_\mathbf{p}\right| / \mu_\mathbf{g} << 1$, where $\vec{\mathbf{U}}_\mathbf{g}, \vec{\mathbf{U}}_\mathbf{p}$ denote the velocities of the gas and the particle, respectively. In this case, the drag force of the conveying gas on the tiny particles satisfies the Stokes' law, and the Stokes coefficient $\alpha_s = 18\mu_\mathbf{g} / \left(\rho_\mathbf{p} \mathbf{d}_\mathbf{p}^2\right) = 7.2 \times 10^5$ s^{-1}. The particle diffusion coefficient is specified as $\mathbf{D} = 0.1$cm^2/s. The particle sample numbers are specified as 2×10^5, 10^6, and 4×10^6 for 1D, 2D and 3D simulations, respectively. The particle concentration is defined as the particle number in unit length, unit area or unit volume in cases of 1D, 2D and 3D, respectively. The normalized particle concentration is the particle concentration divided by the total particle number in that problem. Accordingly, the $\mathbf{Q_N}$ in the Green's function solutions is assumed equal to unit for normalization. The cell size is 0.1 cm in all of these simulations.

According to Figure 7-2-1-1, the particle concentrations are in Gaussian distributions with decaying amplitudes on time, as expected in view of physics. Good agreements are obtained between the analytical solutions and the simulations. In the case of 3D (c) and (d), some random deviations exist between the simulation points and the theoretical curve. However they are completely statistical effects, which should vanish if the particle sample number is sufficiently big. It can be concluded that the diffusion model of GASFLOW can work properly in case of diffusion from instantaneous sources in stagnant flows.



(a) (b) (c) (d)

Figure 7-2-1-1 Particle diffusion from instantaneous point source in quiescent flows in one-, two- and three-dimension

II. Continuous point source

In case of continuous point source, let's take the 3D problem as an example to show how to create the Green's function solution. The formulation of the diffusion problem is described as,

$$\mathbf{u_t - D(u_{xx} + u_{yy} + u_{zz}) = q_3\delta(x, y, z)}\,, \ -\infty < \mathbf{x, y, z} < \infty,\ 0 < \mathbf{t} < \infty\,. \qquad (7\text{-}2\text{-}1\text{-}11)$$

Based on the GFM, the solution should be in the form of,

$$\mathbf{u(x,y,z,t) = q_3 \int_0^t \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(\xi,\eta,\zeta,\tau;x,y,z,t)\delta(\xi,\eta,\zeta)d\xi d\eta d\zeta d\tau =}$$

$$\mathbf{= q_3 \int_0^t \left[\int_{-\infty}^{\infty} X(\xi,\tau;x,t)\delta(\xi)d\xi\right]\left[\int_{-\infty}^{\infty} Y(\eta,\tau;y,t)\delta(\eta)d\eta\right]\left[\int_{-\infty}^{\infty} Z(\zeta,\tau;z,t)\delta(\zeta)d\zeta\right]d\tau =}$$

$$\mathbf{= q_3 \int_0^t X(0,\tau;x,t)Y(0,\tau;y,t)Z(0,\tau;z,t)d\tau}\,. \qquad (7\text{-}2\text{-}1\text{-}12)$$

The directional Green's functions are the same as in the instantaneous case. By substituting them into the above expression, we have,

$$\mathbf{u(x,y,z,t)} = \mathbf{q}_3 \int_0^{\mathbf{t}} \left(\frac{1}{\sqrt{4\pi \mathbf{D(t-\tau)}}} \right)^3 \mathbf{exp}\left[-\frac{\mathbf{x}^2 + \mathbf{y}^2 + \mathbf{z}^2}{4\mathbf{D(t-\tau)}} \right] \mathbf{d}\tau \, .$$

If the distance to the origin is defined as $\mathbf{r}$, by considering the property about convolution, we have,

$$\mathbf{u(x,y,z,t)} = \mathbf{q}_3 \int_0^{\mathbf{t}} \frac{1}{\sqrt{(4\pi \mathbf{D}\tau)^3}} \mathbf{exp}\left(-\frac{\mathbf{r}^2}{4\mathbf{D}\tau} \right) \mathbf{d}\tau = \frac{\mathbf{q}_3}{4\pi \mathbf{D}|\mathbf{r}|} \mathbf{erfc}\left(\frac{|\mathbf{r}|}{\sqrt{4\mathbf{Dt}}} \right), \qquad (7\text{-}2\text{-}1\text{-}13)$$

where $\mathbf{r}^2 = \mathbf{x}^2 + \mathbf{y}^2 + \mathbf{z}^2 \neq 0$ and $\mathbf{t} \neq 0$. Analogously, the solution in the 2D case is expressed as,

$$\mathbf{u(x,y,t)} = \mathbf{q}_2 \int_0^{\mathbf{t}} \frac{1}{\sqrt{(4\pi \mathbf{D}\tau)^2}} \mathbf{exp}\left(-\frac{\mathbf{r}^2}{4\mathbf{D}\tau} \right) \mathbf{d}\tau = \frac{-\mathbf{q}_2}{4\pi \mathbf{D}} \mathbf{Ei}\left(-\frac{\mathbf{r}^2}{4\mathbf{Dt}} \right), \qquad (7\text{-}2\text{-}1\text{-}14)$$

where $\mathbf{r}^2 = \mathbf{x}^2 + \mathbf{y}^2 \neq 0$, $\mathbf{t} \neq 0$, $\mathbf{Ei}(\cdot)$ is the function of exponential integral (Bronshtein et al., 2003), defined as $\mathbf{Ei}(\chi) \equiv \int_{-\infty}^{\chi} \frac{1}{\gamma} \mathbf{exp}(\gamma) \mathbf{d}\gamma$, $(\chi < 0)$.

The solution in the 1D case is obtained similarly by integrations,

$$\mathbf{u(r,t)} = \mathbf{q}_1 \int_0^{\mathbf{t}} \frac{1}{\sqrt{(4\pi \mathbf{D}\tau)}} \mathbf{exp}\left(-\frac{\mathbf{r}^2}{4\mathbf{D}\tau} \right) \mathbf{d}\tau =$$

$$= \mathbf{q}_1 \left[\sqrt{\frac{\mathbf{t}}{\pi \mathbf{D}}} \mathbf{exp}\left(\frac{-\mathbf{r}^2}{4\mathbf{Dt}} \right) - \frac{|\mathbf{r}|}{2\mathbf{D}} \mathbf{erfc}\left(\frac{|\mathbf{r}|}{\sqrt{4\mathbf{Dt}}} \right) \right], \qquad (7\text{-}2\text{-}1\text{-}15)$$

where $\mathbf{r} = \mathbf{x} \neq 0$ and $\mathbf{t} \neq 0$. The $\mathbf{q_N}\ (\mathbf{N} = 1,2,3)$ in (7-2-1-11) through (7-2-1-15) stands for the released particle number per unit time.

The Green's function solutions (7-2-1-15), (7-2-1-14) and (7-2-1-13) are represented as solid lines in Figure 7-2-1-2 for the different dimensions, respectively. The $\mathbf{q_N}$ is specified as one for normalization. The corresponding diffusion problems with continuous point sources are simulated by using GASFLOW. The parameters about the continuous sources are specified as: for 1D, the total particle number 5×10^4, the total injection time 10s, the injection interval time 2×10^{-4} s; for 2D, the total particle number 10^6, the total injection time 10s, the injection interval time 2×10^{-4} s; for 3D, the total particle number 9.6×10^5, the total injection time 4s, the injection interval time 2.5×10^{-4} s. The other parameters are the same as the case of the instantaneous source.

Figure 7-2-1-2 manifests a good consistency about the particle distributions between the Green's function solutions and the GASFLOW simulations. It verifies that the model about the continuous particle source in GASFLOW performs in a proper manner. It is worth to mention that, the particle concentration at the origin is infinite theoretically. The distribution at the vicinity of the origin is not of interest to be concerned in reality, as can be seen in (b) and (c) of Figure 7-2-1-2.

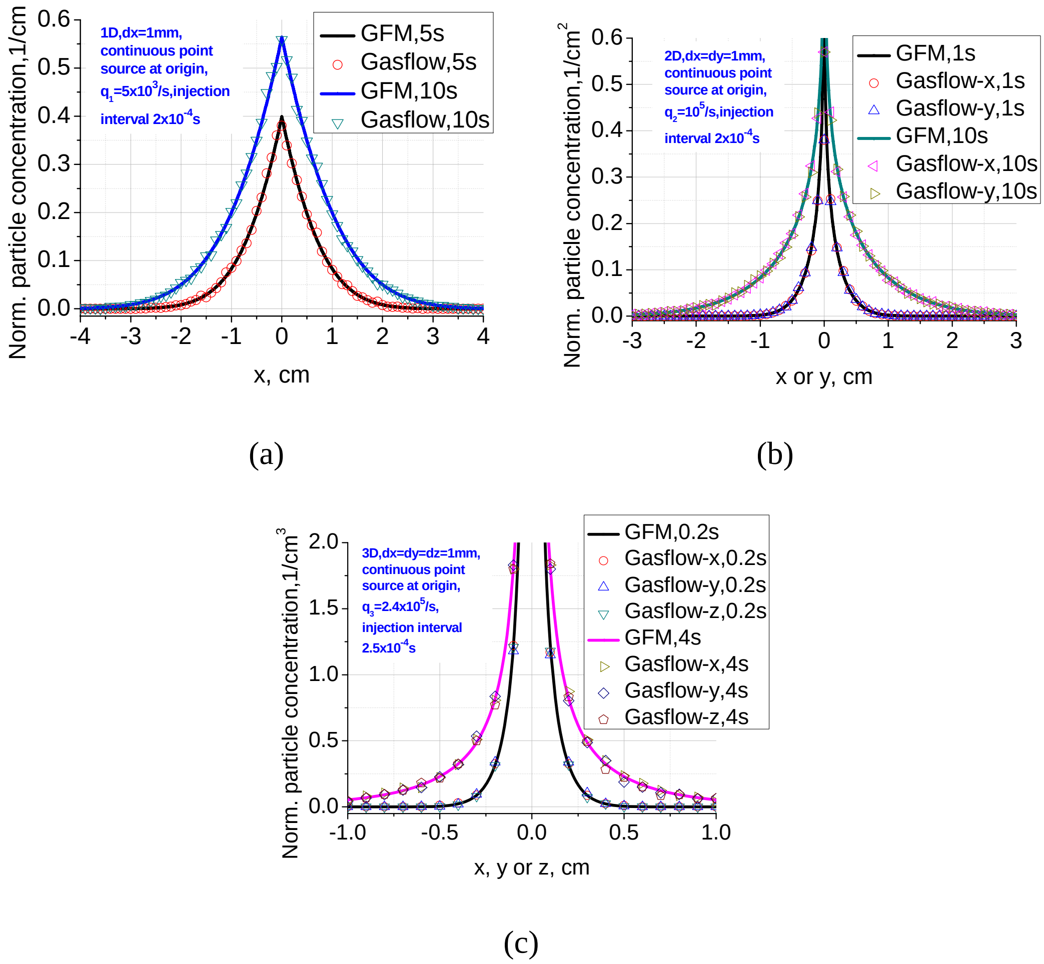


Figure 7-2-1-2 Particle diffusion from continuous point source in quiescent flows in one-, two- and three-dimension

7.2.2 Point source in advective flow

This subsection is contributed to solve or to simulate the diffusion from an instantaneous or continuous point source to a uniform advective flow. From here on, the advective flow velocity is assumed to be equal to $\mathbf{V} = 2$ cm/s as an example, if no additional words are given.

I. Instantaneous point source

The particles are released only once at the time $\mathbf{t} = 0$ s, then they are transported by the accompanying advective gas flow and start to diffuse. The mathematical expressions for the advection diffusion problems are similar to those in the case of stagnant flow in Section 7.2.1, except the advection term,

$$\mathbf{u_t + Vu_x - D(u_{xx}) = Q_1\delta(x,t)},\ -\infty < \mathbf{x} < \infty,\ 0 < \mathbf{t} < \infty, \qquad \text{(7-2-2-1)}$$

$$\mathbf{u_t + Vu_x - D(u_{xx} + u_{yy}) = Q_2\delta(x,y,t)},\ -\infty < \mathbf{x,y} < \infty,\ 0 < \mathbf{t} < \infty, \qquad \text{(7-2-2-2)}$$

$$\mathbf{u_t + Vu_x - D(u_{xx} + u_{yy} + u_{zz}) = Q_3\delta(x,y,z,t)},\ -\infty < \mathbf{x,y,z} < \infty,\ 0 < \mathbf{t} < \infty,$$

(7-2-2-3)

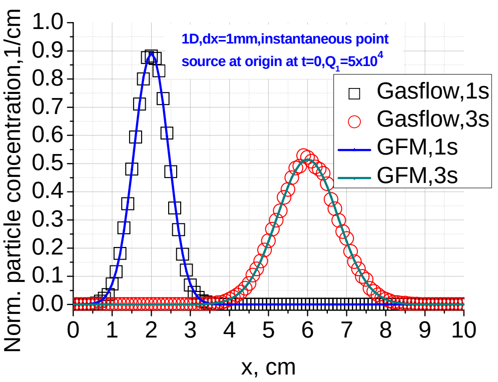
where the $\mathbf{u}$ denotes the particle concentration. The expressions of the solutions are also similar to those listed in (7-2-1-4) through (7-2-1-6), except that the Green's functions in $\mathbf{x}$-direction (advection direction) are replaced by the advective Green's functions fund in Table 6-3-1. By the replacements, the solutions for 1D, 2D and 3D can be obtained as,

$$\mathbf{u(x,t)} = \frac{\mathbf{Q_1}}{\sqrt{4\pi \mathbf{Dt}}} \mathbf{exp}\left[-\frac{\mathbf{(x-Vt)^2}}{\mathbf{4Dt}} \right], \tag{7-2-2-4}$$

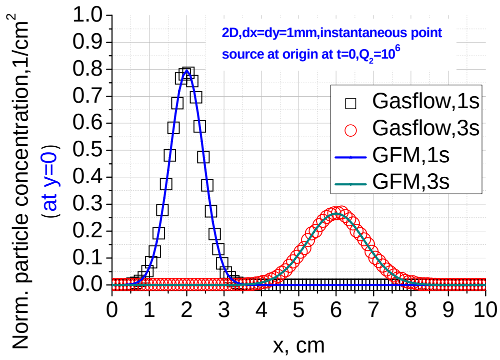
$$\mathbf{u(x,y,t)} = \frac{\mathbf{Q_2}}{\sqrt{(4\pi \mathbf{Dt})^2}} \mathbf{exp}\left[-\frac{\mathbf{(x-Vt)^2 + y^2}}{\mathbf{4Dt}} \right], \tag{7-2-2-5}$$

$$\mathbf{u(x,y,z,t)} = \frac{\mathbf{Q_3}}{\sqrt{(4\pi \mathbf{Dt})^3}} \mathbf{exp}\left[-\frac{\mathbf{(x-Vt)^2 + y^2 + z^2}}{\mathbf{4Dt}} \right]. \tag{7-2-2-6}$$

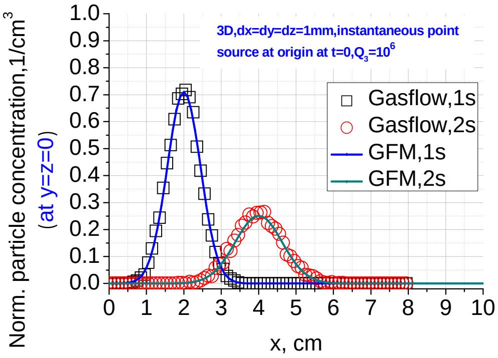
If $\mathbf{V} = 2$ cm/s, $\mathbf{D} = 0.1$ cm^2/s and $\mathbf{Q_N} = 1, (\mathbf{N} = 1,2,3)$, for normalization, the theoretical particle concentrations at different times are shown in Figure 7-2-2-1 as solid lines, in 1D, 2D and 3D, respectively. It presents propagating Gaussian distributions along the $\mathbf{x}$-direction and with decaying amplitudes on time, clearly due to the advective flow and the diffusion of the particles themselves. Accordingly the GASFLOW simulations are performed and the results are shown as symbols in Figure 7-2-2-1. The specifications about particles are the same as in Section 7.2.1. The released particle numbers are 5×10^4, 10^6 and 10^6 in 1D, 2D and 3D, respectively. It is obvious that the GASFLOW models reproduce the particle behaviors in a way following the analytical solutions.



(a)



(b)



(c)

Figure 7-2-2-1 Particle diffusion from instantaneous point source in advective flows in one-, two- and three-dimension

II. Continuous point source

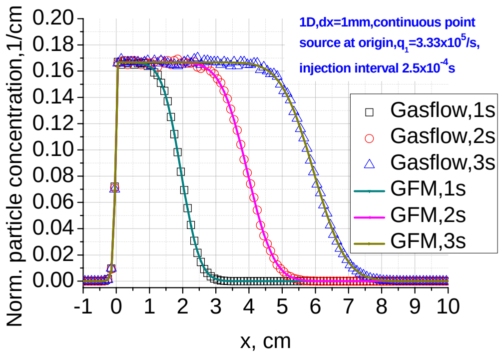
By applying the GFM, the theoretical particle concentrations caused by continuous point sources in infinite domains with advective flows can be simply obtained as,

$$\mathbf{u(x,t)} = \int_0^t \frac{\mathbf{q}_1}{\sqrt{4\pi \mathbf{D}\tau}} \mathbf{exp}\left[-\frac{(\mathbf{x}-\mathbf{V}\tau)^2}{4\mathbf{D}\tau}\right] \mathbf{d}\tau\,, \tag{7-2-2-7}$$

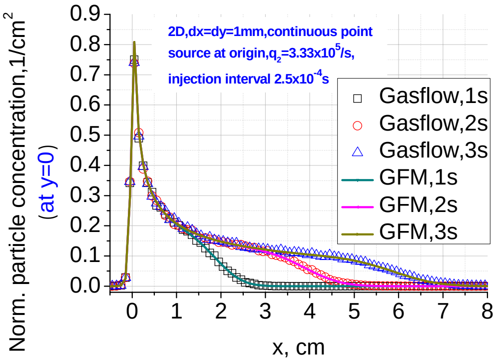
$$\mathbf{u(x,y,t)} = \int_0^t \frac{\mathbf{q}_2}{\sqrt{(4\pi \mathbf{D}\tau)^2}} \mathbf{exp}\left[-\frac{(\mathbf{x}-\mathbf{V}\tau)^2+\mathbf{y}^2}{4\mathbf{D}\tau}\right] \mathbf{d}\tau\,, \tag{7-2-2-8}$$

$$\mathbf{u(x,y,z,t)} = \int_0^t \frac{\mathbf{q}_3}{\sqrt{(4\pi \mathbf{D}\tau)^3}} \mathbf{exp}\left[-\frac{(\mathbf{x}-\mathbf{V}\tau)^2+\mathbf{y}^2+\mathbf{z}^2}{4\mathbf{D}\tau}\right] \mathbf{d}\tau\,, \tag{7-2-2-9}$$

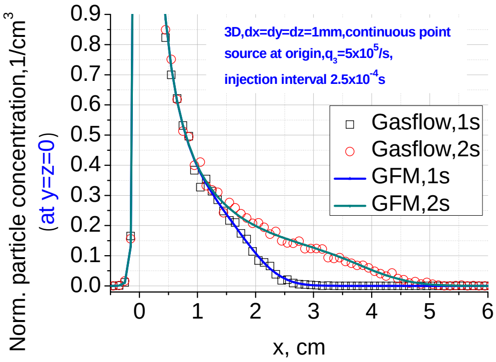
for 1D, 2D and 3D, respectively, where $\mathbf{q}_\mathbf{N}$ is the particle release rate. The comparisons between the GFM solutions and the GASFLOW simulations are presented in Figure 7-2-2-2. In the simulations the continuous sources are defined as: the total particle number is 10^6 and the injection interval time 2.5×10^{-4} s for 1D, 2D and 3D, the total injection time is 3 s for 1D and 2D, but 2 s for 3D. The other parameters are the same as the previous cases. The numerical simulations agree well with the corresponding Green's function solutions according to the figure, which also shows that, a minor part of particles diffuse from the origin (release place) backward to the upstream while the major are transported away along the advection direction. In Figure 7-2-2-2 (c), the concentration in the neighborhood of the origin is not shown. It is not concerned and is infinite at the origin theoretically.



(a)



(b)

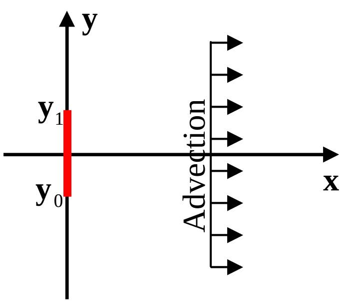


(c)

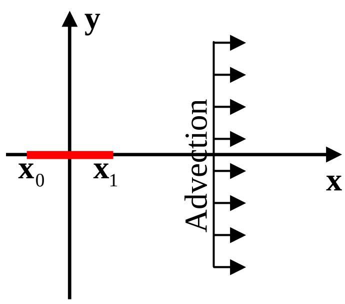
Figure 7-2-2-2 Particle diffusion from continuous point source in advective flows in one-, two- and three-dimension

7.2.3 Line source in two-dimensional advective flow

The particle source could be a line, for an instance, in the case that the dust is released from a tiny gap. In the subsection, two cases of continuous line sources are considered, depending on the line source distributed in the advection direction or in a transverse direction. The two cases are shown as (I) and (II) schematically in Figure 7-2-3-1.



(I)



(II)

Figure 7-2-3-1 Two cases of line sources in infinite two-dimensional domain

I. Line source in transverse direction

By appealing to a one-dimensional delta function, the mathematical formulation about the particle concentration of this case can be described as,

$$\mathbf{u_t} + \mathbf{V}\mathbf{u_x} - \mathbf{D}(\mathbf{u_{xx}} + \mathbf{u_{yy}}) = \begin{cases} \mathbf{q_l}\delta(\mathbf{x}), & \textbf{if} \quad \mathbf{y} \in [\mathbf{y}_0, \mathbf{y}_1], \\ 0, & \textbf{otherwise}, \end{cases} \tag{7-2-3-1}$$

where $-\infty < \mathbf{x}, \mathbf{y} < \infty$, $0 < \mathbf{t} < \infty$, $\mathbf{q_l}$ is the strength of the line source, i.e., the released particle number in unit time and from unit length. This is purely a source problem. Based on equality (6-1-8), the solution can be expressed as,

$$\mathbf{u}(\mathbf{x},\mathbf{y},\mathbf{t}) = \int_0^{\mathbf{t}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbf{G}(\xi,\eta,\tau;\mathbf{x},\mathbf{y},\mathbf{t})\phi(\xi,\eta,\tau)\mathbf{d}\xi\mathbf{d}\eta\mathbf{d}\tau = \int_0^{\mathbf{t}}\int_{\mathbf{y}_0}^{\mathbf{y}_1}\int_{-\infty}^{\infty} \mathbf{q_l}\mathbf{G}(\xi,\eta,\tau;\mathbf{x},\mathbf{y},\mathbf{t})\delta(\xi)\mathbf{d}\xi\mathbf{d}\eta\mathbf{d}\tau =$$

$$= \int_0^{\mathbf{t}}\int_{\mathbf{y}_0}^{\mathbf{y}_1} \mathbf{q_l}\mathbf{G}(0,\eta,\tau;\mathbf{x},\mathbf{y},\mathbf{t})\mathbf{d}\eta\mathbf{d}\tau = \int_0^{\mathbf{t}} \mathbf{q_l}\mathbf{X}(0,\tau;\mathbf{x},\mathbf{t})\int_{\mathbf{y}_0}^{\mathbf{y}_1}\mathbf{Y}(\eta,\tau;\mathbf{y},\mathbf{t})\mathbf{d}\eta\mathbf{d}\tau\,. \tag{7-2-3-2}$$

By looking up the Green's functions in Table 6-3-1 and Table 6-3-2 and substituting them into (7-2-3-2), the solution becomes, by noting the communication law of convolution,

$$\mathbf{u}(\mathbf{x},\mathbf{y},\mathbf{t}) = \int_0^{\mathbf{t}} \frac{\mathbf{q_l}}{\sqrt{4\pi\mathbf{D}\tau}}\mathbf{exp}\left[-\frac{(\mathbf{x}-\mathbf{V}\tau)^2}{4\mathbf{D}\tau}\right]\int_{\mathbf{y}_0}^{\mathbf{y}_1}\frac{1}{\sqrt{4\pi\mathbf{D}\tau}}\mathbf{exp}\left[-\frac{(\eta-\mathbf{y})^2}{4\mathbf{D}\tau}\right]\mathbf{d}\eta\mathbf{d}\tau =$$

$$= \int_0^{\mathbf{t}} \frac{\mathbf{q_l}}{\sqrt{16\pi\mathbf{D}\tau}}\mathbf{exp}\left[-\frac{(\mathbf{x}-\mathbf{V}\tau)^2}{4\mathbf{D}\tau}\right]\left[\mathbf{erfc}\left(\frac{\mathbf{y}-\mathbf{y}_1}{\sqrt{4\mathbf{D}\tau}}\right) - \mathbf{erfc}\left(\frac{\mathbf{y}-\mathbf{y}_0}{\sqrt{4\mathbf{D}\tau}}\right)\right]\mathbf{d}\tau\,. \tag{7-2-3-3}$$

The analytical particle concentrations at 1s, 2s and 3s along $\mathbf{x}$-axis after normalization are denoted as solid lines in Figure 7-2-3-2, where $\mathbf{D} = 0.1$ cm^2/s, $\mathbf{V} = 2$ cm/s, $\mathbf{q_l} = 1$/cms, $\mathbf{y}_0 = -0.5$ cm, $\mathbf{y}_1 = 0.5$ cm. If the total particle injection time is $\mathbf{T}$, the normalized particle concentration is defined as the particle concentration (particle/cm^2) divided by the total particle number, $\mathbf{q_l}\mathbf{l_{ength}}\mathbf{T}$, namely,

$$\mathbf{u_{norm}}(\mathbf{x},\mathbf{y},\mathbf{t}) = \frac{\mathbf{u}(\mathbf{x},\mathbf{y},\mathbf{t})}{\mathbf{q_l}\mathbf{l_{ength}}\mathbf{T}}\,. \tag{7-2-3-4}$$

In this case, $\mathbf{q_l} = 1$ /cms, $\mathbf{l_{ength}} = \mathbf{y}_1 - \mathbf{y}_0 = 1$ cm, $\mathbf{T} = 3$ s, thus $\mathbf{u_{norm}} = \mathbf{u}/3$. The corresponding concentrations in the GASFLOW simulation are represented by different symbols in Figure 7-2-3-2. To be normalized, the particle concentration (particle/cm^2) is also divided by the total particle number, which is indicated in the description of the figure. It is clear that good agreements between the two solutions are obtained in this case.

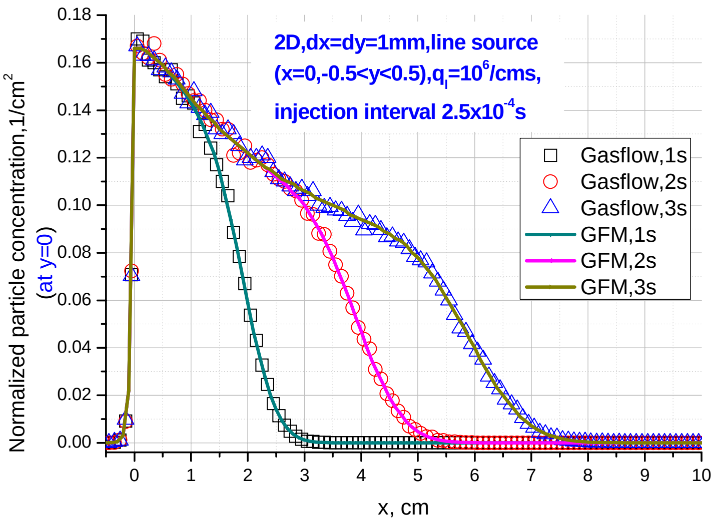


Figure 7-2-3-2 Advection diffusion from continuous particle line source distributed in transverse (y) direction in two-dimensional domain

II. Line source in advection direction

Analogously, the mathematical formulation about the particle concentration of this case can be described as,

$$\mathbf{u_t + Vu_x - D(u_{xx} + u_{yy}) = \begin{cases} q_l\delta(y), & \text{if} \quad x \in [x_0, x_1], \\ 0, & \text{otherwise}, \end{cases}} \qquad (7\text{-}2\text{-}3\text{-}4)$$

where $-\infty < \mathbf{x, y} < \infty$, $0 < \mathbf{t} < \infty$, $\mathbf{q_l}$ is the strength of the line source. Similar to the last case, the Green's function solution can be expressed as,

$$\mathbf{u(x,y,t) = \int_0^t \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(\xi,\eta,\tau;x,y,t)\phi(\xi,\eta,\tau)d\xi d\eta d\tau = \int_0^t \int_{x_0}^{x_1} \int_{-\infty}^{\infty} q_l G(\xi,\eta,\tau;x,y,t)\delta(\eta)d\eta d\xi d\tau =}$$

$$\mathbf{= \int_0^t \int_{x_0}^{x_1} q_l G(\xi,0,\tau;x,y,t)d\xi d\tau = \int_0^t q_l \int_{x_0}^{x_1} X(\xi,\tau;x,t)d\xi \cdot Y(0,\tau;y,t)d\tau} \,. \qquad (7\text{-}2\text{-}3\text{-}5)$$

By substituting the Green's functions into (7-2-3-5), the solution becomes,

$$\mathbf{u(x,y,t) = \int_0^t \int_{x_0}^{x_1} \frac{1}{\sqrt{4\pi D\tau}} \exp\left[-\frac{(\xi - x + V\tau)^2}{4D\tau}\right] d\xi \cdot \frac{q_l}{\sqrt{4\pi D\tau}} \exp\left[-\frac{y^2}{4D\tau}\right] d\tau =}$$

$$\mathbf{= \int_0^t \frac{q_l}{\sqrt{16\pi D\tau}} \left[erfc\left(\frac{x - V\tau - x_1}{\sqrt{4D\tau}}\right) - erfc\left(\frac{x - V\tau - x_0}{\sqrt{4D\tau}}\right)\right] \exp\left[-\frac{y^2}{4D\tau}\right] d\tau} \,. (7\text{-}2\text{-}3\text{-}6)$$

The normalized theoretical concentration distributions and the corresponding GASFLOW simulations are compared in Figure 7-2-3-3, where, $\mathbf{x_0} = -0.5\,\text{cm}$ and $\mathbf{x_1} = 0.5\,\text{cm}$. The concentrations are along the line of, as an example, $\mathbf{y} = 0.1\,\text{cm}$, to avoid the singularity of the theoretical solution on the source line ($\mathbf{x_0 < x < x_1}$ and $\mathbf{y} = 0$), where the concentration is infinite mathematically. Again good consistency between the theory and the simulation is found in the figure.

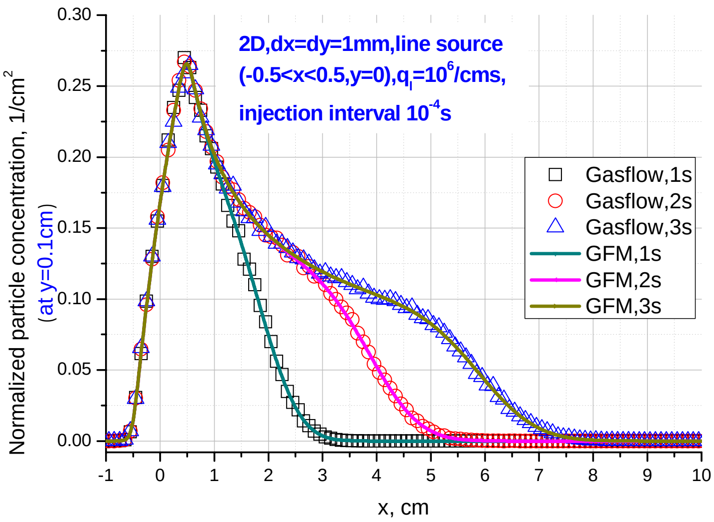
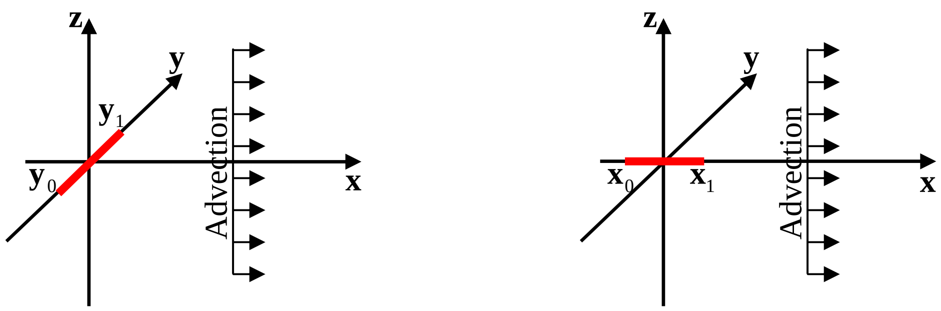


Figure 7-2-3-3 Advection diffusion from continuous particle line source distributed in advection (x) direction in two-dimensional domain

7.2.4 Line source in three-dimensional advective flow

Like in 2D, two kinds of line sources are considered. One is in a transverse direction, the other in the advection direction, as shown in Figure 7-2-4-1.



(I) (II)

Figure 7-2-4-1 Two cases of line sources in infinite three-dimensional domain

I. Line source in transverse direction

By using a two-dimensional delta function, the mathematical equation of the particle concentration is formulated as,

$$\mathbf{u}_t + \mathbf{V}\mathbf{u}_x - \mathbf{D}(\mathbf{u}_{xx} + \mathbf{u}_{yy} + \mathbf{u}_{zz}) = \begin{cases} \mathbf{q}_l\delta(\mathbf{x},\mathbf{z}), & \textbf{if} \quad \mathbf{y} \in [\mathbf{y}_0, \mathbf{y}_1], \\ 0, & \textbf{otherwise}, \end{cases} \qquad (7\text{-}2\text{-}4\text{-}1)$$

where $-\infty < \mathbf{x},\mathbf{y},\mathbf{z} < \infty$, $0 < \mathbf{t} < \infty$. In terms of the GFM, Green's functions are like building blocks, and the higher dimensional solutions can be created easily on the basis of lower dimensional ones. Let's take this problem as an example. Based on the equality (7-2-3-2), the 3D solution can be obtained simply by using the product rule and appending a factor of the Green's function in $\mathbf{z}$ -direction, and by utilizing the property of delta function,

$$\mathbf{u(x,y,z,t)} = \int_0^t \mathbf{q_l X(0,\tau;x,t)} \int_{y_0}^{y_1} \mathbf{Y(\eta,\tau;y,t)d\eta Z(0,\tau;z,t)d\tau}. \quad (7\text{-}2\text{-}4\text{-}2)$$

By bringing the Green's functions into the above equality and making a little simplification, the solution becomes,

$$\mathbf{u(x,y,z,t)} = \int_0^t \frac{\mathbf{q_l}}{8\pi \mathbf{D}\tau} \mathbf{exp}\left[-\frac{(\mathbf{x}-\mathbf{V}\tau)^2+\mathbf{z}^2}{4\mathbf{D}\tau}\right]\left[\mathbf{erfc}\left(\frac{\mathbf{y}-\mathbf{y}_1}{\sqrt{4\mathbf{D}\tau}}\right)-\mathbf{erfc}\left(\frac{\mathbf{y}-\mathbf{y}_0}{\sqrt{4\mathbf{D}\tau}}\right)\right]\mathbf{d}\tau.$$

(7-2-4-3)

The normalized solution and the numerical simulation are presented in Figure 7-2-4-2 for comparison, where $\mathbf{y}_0 = -0.5\,\mathrm{cm}$ and $\mathbf{y}_1 = 0.5\,\mathrm{cm}$. It is obvious that GASFLOW can reproduce numerically the particle diffusion in a good way.

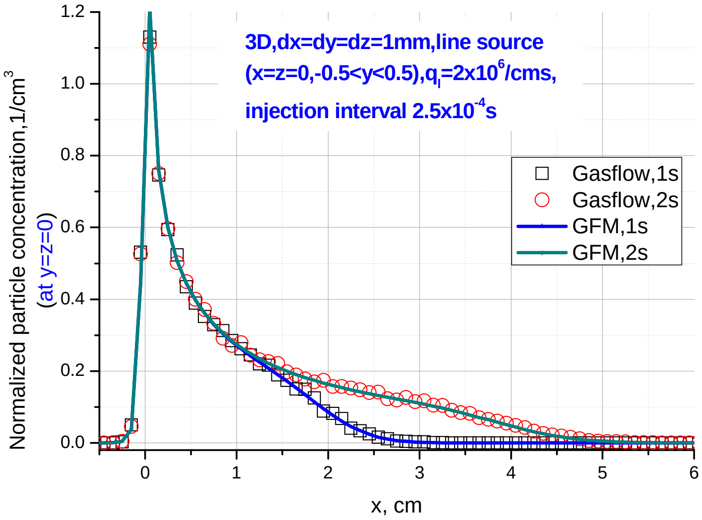


Figure 7-2-4-2 Advection diffusion from continuous particle line source distributed in transverse ($\mathbf{y}$) direction in three-dimensional domain

II. Line source in advection direction

Similarly, the solution of this case can be obtained based on the two-dimensional solution (7-2-3-5) or (7-2-3-6). The equation and its solution are listed here for completeness.

$$\mathbf{u_t + Vu_x - D(u_{xx} + u_{yy} + u_{zz})} = \begin{cases} \mathbf{q_l \delta(y,z)}, & \mathbf{if} \quad \mathbf{x} \in [\mathbf{x}_0, \mathbf{x}_1], \\ 0, & \mathbf{otherwise}, \end{cases} \quad (7\text{-}2\text{-}4\text{-}4)$$

where $-\infty < \mathbf{x,y,z} < \infty$, $0 < \mathbf{t} < \infty$. The solution is,

$$\mathbf{u(x,y,z,t)} = \int_0^t \frac{\mathbf{q_l}}{8\pi \mathbf{D}\tau}\left[\mathbf{erfc}\left(\frac{\mathbf{x}-\mathbf{V}\tau-\mathbf{x}_1}{\sqrt{4\mathbf{D}\tau}}\right)-\mathbf{erfc}\left(\frac{\mathbf{x}-\mathbf{V}\tau-\mathbf{x}_0}{\sqrt{4\mathbf{D}\tau}}\right)\right]\mathbf{exp}\left[-\frac{\mathbf{y}^2+\mathbf{z}^2}{4\mathbf{D}\tau}\right]\mathbf{d}\tau.$$

(7-2-4-5)

If $\mathbf{x}_0 = -0.5\,\text{cm}$ and $\mathbf{x}_1 = 0.5\,\text{cm}$, the theoretical curves and the GASFLOW simulation points are shown in Figure 7-2-4-3, where the concentrations distribute along the line of $\mathbf{y} = \mathbf{z} = 0.1\,\text{cm}$ instead of the $\mathbf{x}$-axis. According to the figure, the simulating points fit the analytical curves in a satisfactory way.

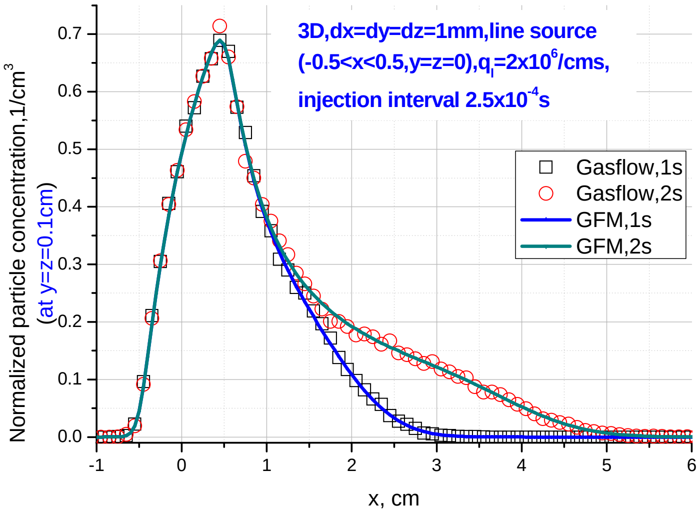


Figure 7-2-4-3 Advection diffusion from continuous particle line source distributed in advection ($\mathbf{x}$) direction in three-dimensional domain

7.2.5 Area source in two-dimensional advective flow

As shown in Figure 7-2-5-1, a square of particle source is released in a 2D plane with advective flow. The particle diffusion problem is formulated as,

$$\mathbf{u}_t + \mathbf{V}\mathbf{u}_x - \mathbf{D}(\mathbf{u}_{xx} + \mathbf{u}_{yy}) = \begin{cases} \mathbf{q}_a, & \textbf{if} \quad \mathbf{x} \in [\mathbf{x}_0, \mathbf{x}_1] \quad \textbf{and} \quad \mathbf{y} \in [\mathbf{y}_0, \mathbf{y}_1], \\ 0, & \textbf{otherwise}, \end{cases} \tag{7-2-5-1}$$

where $-\infty < \mathbf{x}, \mathbf{y} < \infty$, $0 < \mathbf{t} < \infty$, $\mathbf{q}_a$ denotes the released particle number in unit area and in unit time.

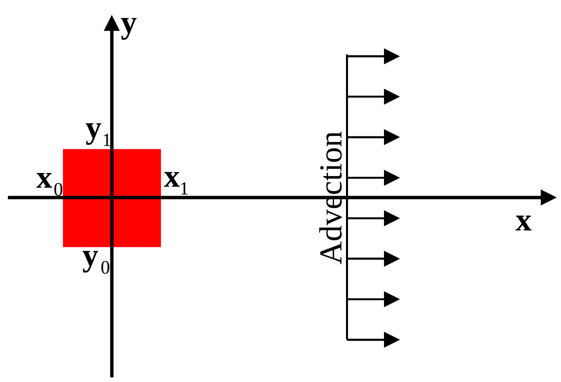


Figure 7-2-5-1 Area source in infinite two-dimensional plane

According to the GFM, the solution is expressed as, based on equality (6-1-8),

$$\mathbf{u(x,y,t)} = \int_0^t \int_{y_0}^{y_1} \int_{x_0}^{x_1} \mathbf{q_a G(\xi,\eta,\tau;x,y,t) d\xi d\eta d\tau} = \int_0^t \mathbf{q_a} \int_{x_0}^{x_1} \mathbf{X(\xi,\tau;x,t) d\xi} \int_{y_0}^{y_1} \mathbf{Y(\eta,\tau;y,t) d\eta d\tau}\,. \tag{7-2-5-2}$$

By substituting the Green's functions found in Table (6-3-1) and Table (6-3-2), the solution is obtained as,

$$\mathbf{u(x,y,t)} = \int_0^t \frac{\mathbf{q_a}}{4} \left[\mathbf{erfc}\left(\frac{\mathbf{x - V\tau - x_1}}{\sqrt{4\mathbf{D}\tau}} \right) - \mathbf{erfc}\left(\frac{\mathbf{x - V\tau - x_0}}{\sqrt{4\mathbf{D}\tau}} \right) \right] \cdot$$

$$\cdot \left[\mathbf{erfc}\left(\frac{\mathbf{y - y_1}}{\sqrt{4\mathbf{D}\tau}} \right) - \mathbf{erfc}\left(\frac{\mathbf{y - y_0}}{\sqrt{4\mathbf{D}\tau}} \right) \right] \mathbf{d\tau}\,. \tag{7-2-5-3}$$

The mathematical solutions are compared with the GASFLOW simulations at $\mathbf{t} = 5\,\mathrm{s}$ and $\mathbf{t} = 10\,\mathrm{s}$ in Figure 7-2-5-2, where $\mathbf{x}_0 = \mathbf{y}_0 = -0.5\,\mathrm{cm}$ and $\mathbf{x}_1 = \mathbf{y}_1 = 0.5\,\mathrm{cm}$. In this case the sensitivity of the numerical simulations on the grid sizes is analyzed. According to the figure, a rough grid size of 1 cm is too big, while a grid with a cell size of 0.2 cm is refined enough to reproduce the theoretical solution. The three simulations show the converging of the numerical simulations to the Green's function solution from a bigger cell size to a smaller one.

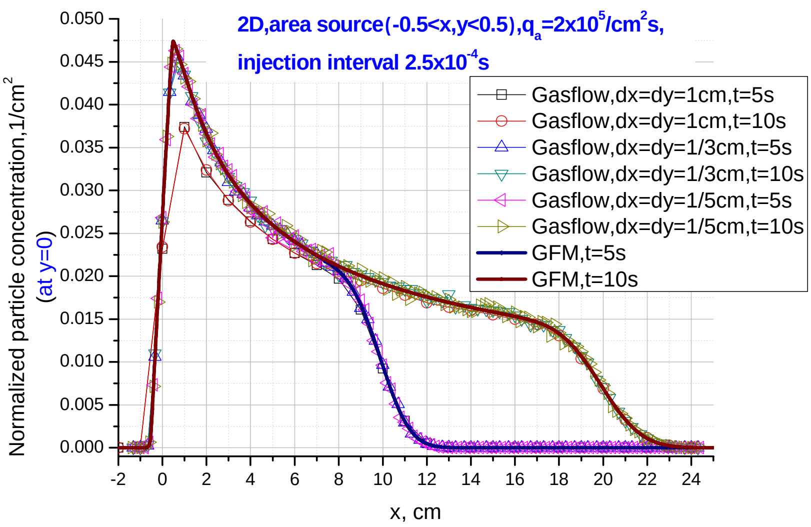
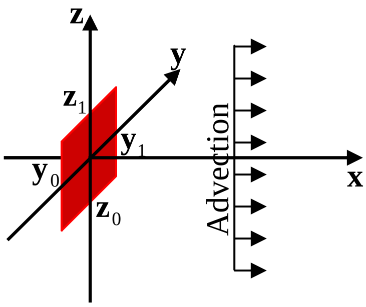


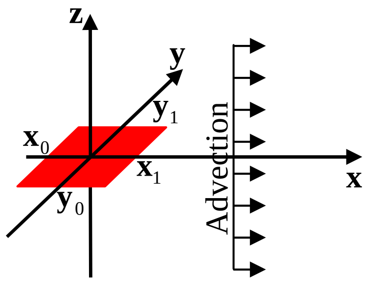
Figure 7-2-5-2 Advection diffusion from continuous particle area source in two-dimensional domain

7.2.6 Area source in three-dimensional advective flow

Two cases of area sources are studied in 3D domain, as shown in Figure 7-2-6-1, depending on the spatial relationship between the area and the advection direction.



(I)



(II)

Figure 7-2-6-1 Area source in infinite three-dimensional domain

I. Area source perpendicular to advection direction

The governing equation about the particle concentration is given as,

$$\mathbf{u_t + Vu_x - D(u_{xx} + u_{yy} + u_{zz}) = \begin{cases} q_a\delta(x), & \text{if} \quad y \in [y_0, y_1] \quad \text{and} \quad z \in [z_0, z_1], \\ 0, & \text{otherwise,} \end{cases}}$$

(7-2-6-1)

where $-\infty < \mathbf{x, y, z} < \infty$, $0 < \mathbf{t} < \infty$, $\mathbf{q_a}$ denotes the released particle number in unit area and in unit time.

According to the GFM and applying the property of delta function, the solution is in the form as,

$$\mathbf{u(x,y,z,t) = \int_0^t q_a X(0,\tau;x,t) \int_{y_0}^{y_1} Y(\eta,\tau;y,t)d\eta \int_{z_0}^{z_1} Z(\zeta,\tau;z,t)d\zeta d\tau =}$$

$$= \int_0^t \frac{\mathbf{q_a}}{\sqrt{64\pi \mathbf{D}\tau}} \mathbf{exp}\left[-\frac{(\mathbf{x} - \mathbf{V}\tau)^2}{4\mathbf{D}\tau}\right]\left[\mathbf{erfc}\left(\frac{\mathbf{y} - \mathbf{y}_1}{\sqrt{4\mathbf{D}\tau}}\right) - \mathbf{erfc}\left(\frac{\mathbf{y} - \mathbf{y}_0}{\sqrt{4\mathbf{D}\tau}}\right)\right] \cdot$$

$$\cdot \left[\mathbf{erfc}\left(\frac{\mathbf{z} - \mathbf{z}_1}{\sqrt{4\mathbf{D}\tau}}\right) - \mathbf{erfc}\left(\frac{\mathbf{z} - \mathbf{z}_0}{\sqrt{4\mathbf{D}\tau}}\right)\right] \mathbf{d}\tau \qquad (7\text{-}2\text{-}6\text{-}2)$$

The solid curves based on the formula, as shown in Figure 7-2-6-2, stand for the normalized particle concentrations along the $\mathbf{x}$-axis at $\mathbf{t} = 1\,\mathrm{s}$ and $\mathbf{t} = 2\,\mathrm{s}$, respectively, if $\mathbf{y}_0 = \mathbf{z}_0 = -0.5\,\mathrm{cm}$ and $\mathbf{y}_1 = \mathbf{z}_1 = 0.5\,\mathrm{cm}$. The different symbols are the GASFLOW simulating points, which are coincident with the theoretical curves except slight deviations on some points caused by statistical effects.

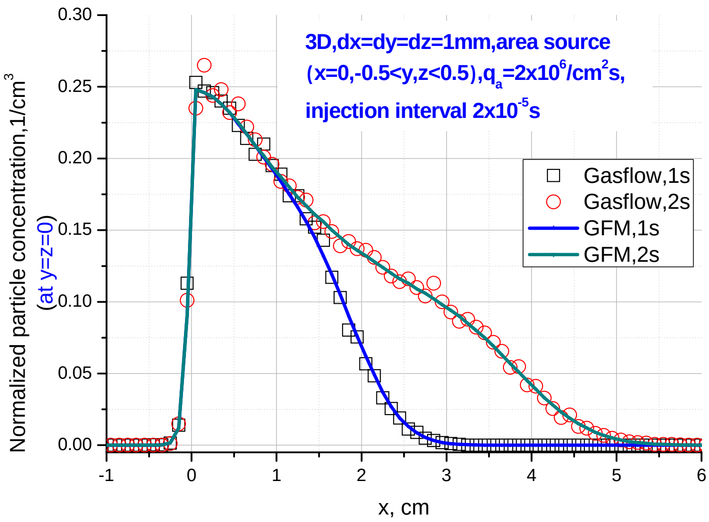


Figure 7-2-6-2 Advection diffusion from continuous particle area source distributed in transverse ($y-z$) plane in three-dimensional domain

II. Area source parallel to advection direction

For completeness, the equation and the solution are directly given as,

$$\mathbf{u_t + Vu_x - D(u_{xx} + u_{yy} + u_{zz}) = \begin{cases} q_a\delta(z), & \text{if} \quad x \in [x_0, x_1] \quad \text{and} \quad y \in [y_0, y_1], \\ 0, & \text{otherwise}, \end{cases}} \tag{7-2-6-3}$$

where $-\infty < \mathbf{x, y, z} < \infty$, $0 < \mathbf{t} < \infty$. The solution can be obtained simply,

$$\mathbf{u(x,y,z,t) = \int_0^t q_a \int_{x_0}^{x_1} X(\xi,\tau;x,t)d\xi \int_{y_0}^{y_1} Y(\eta,\tau;y,t)d\eta Z(0,\tau;z,t)d\tau =}$$

$$= \int_0^t \frac{\mathbf{q_a}}{\sqrt{64\pi \mathbf{D}\tau}} \left[\mathbf{erfc}\left(\frac{\mathbf{x - V}\tau - \mathbf{x_1}}{\sqrt{4\mathbf{D}\tau}} \right) - \mathbf{erfc}\left(\frac{\mathbf{x - V}\tau - \mathbf{x_0}}{\sqrt{4\mathbf{D}\tau}} \right) \right] \cdot$$

$$\cdot \left[\mathbf{erfc}\left(\frac{\mathbf{y - y_1}}{\sqrt{4\mathbf{D}\tau}} \right) - \mathbf{erfc}\left(\frac{\mathbf{y - y_0}}{\sqrt{4\mathbf{D}\tau}} \right) \right] \mathbf{exp}\left[-\frac{\mathbf{z}^2}{4\mathbf{D}\tau} \right] \mathbf{d}\tau \, . \tag{7-2-6-4}$$

As an example, the concentration distributions along the line of $\mathbf{y = z} = 0.1\,\text{cm}$ at different times are compared between the theory and the GASFLOW calculations, as shown in Figure 7-2-6-2, where $\mathbf{x_0 = y_0} = -0.5\,\text{cm}$ and $\mathbf{x_1 = y_1} = 0.5\,\text{cm}$. The figure shows good consistency between the solid lines and the simulating symbols expect slight statistical deviations at some numerical points.

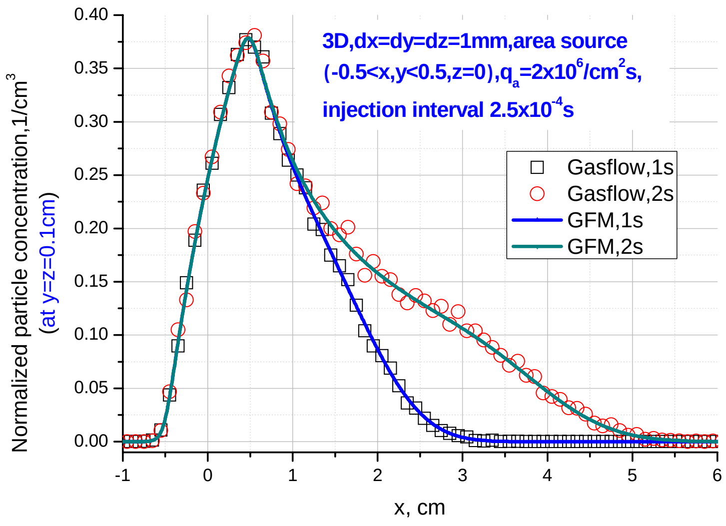


Figure 7-2-6-3 Advection diffusion from continuous particle area source distributed in $x-y$ plane in three-dimensional domain

7.2.7 Volumetric source in three-dimensional advective flow

As the last case, the advection diffusion from a cube of particle source in a 3D domain is considered, as shown in Figure 7-2-7-1.

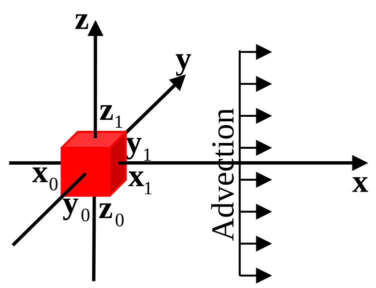


Figure 7-2-7-1 Volume source in infinite three-dimensional domain

The mathematical description of the problem is,

$$\mathbf{u_t + Vu_x - D(u_{xx} + u_{yy} + u_{zz}) = \begin{cases} q_v, & \text{if} \quad x \in [x_0, x_1], \quad y \in [y_0, y_1] \quad \text{and} \quad z \in [z_0, z_1], \\ 0, & \text{otherwise}, \end{cases}}$$

(7-2-7-1)

where $-\infty < \mathbf{x, y, z} < \infty$, $0 < \mathbf{t} < \infty$, $\mathbf{q_v}$ stands for the number of particles released in unit volume and in unit time. The solution is expressed as, according to the GFM,

$$\mathbf{u(x,y,z,t) = \int_0^t q_v \int_{x_0}^{x_1} X(\xi,\tau;x,t)d\xi \int_{y_0}^{y_1} Y(\eta,\tau;y,t)d\eta \int_{z_0}^{z_1} Z(\zeta,\tau;z,t)d\zeta d\tau =}$$

$$\mathbf{= \int_0^t \frac{q_v}{8}\left[\operatorname{erfc}\left(\frac{x - V\tau - x_1}{\sqrt{4D\tau}}\right) - \operatorname{erfc}\left(\frac{x - V\tau - x_0}{\sqrt{4D\tau}}\right)\right].}$$

$$\cdot\left[\mathbf{erfc}\left(\frac{\mathbf{y}-\mathbf{y}_1}{\sqrt{4\mathbf{D}\tau}}\right)-\mathbf{erfc}\left(\frac{\mathbf{y}-\mathbf{y}_0}{\sqrt{4\mathbf{D}\tau}}\right)\right]\left[\mathbf{erfc}\left(\frac{\mathbf{z}-\mathbf{z}_1}{\sqrt{4\mathbf{D}\tau}}\right)-\mathbf{erfc}\left(\frac{\mathbf{z}-\mathbf{z}_0}{\sqrt{4\mathbf{D}\tau}}\right)\right]\mathbf{d}\tau\,. \quad (7\text{-}2\text{-}7\text{-}2)$$

Figure 7-2-7-2 depicts the comparison between the Green's function solution and the GASFLOW simulations, while $\mathbf{x}_0 = \mathbf{y}_0 = \mathbf{z}_0 = -0.5\,\text{cm}$ and $\mathbf{x}_1 = \mathbf{y}_1 = \mathbf{z}_1 = 0.5\,\text{cm}$. As can be seen from the figure, the concentrations along the $\mathbf{x}$-axis, i.e., the advection direction, agree to each other between the theory and the numerical calculation at the different times.

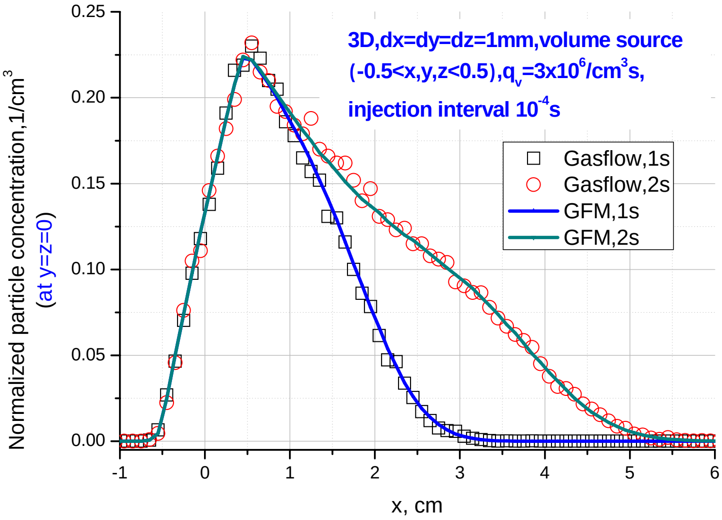


Figure 7-2-7-2 Advection diffusion from continuous particle volume source in three-dimensional domain

8. Conclusions

The Green's function method has been applied to solve the one- and multi-dimensional advection diffusion partial differential equations in infinite, semi-infinite and finite domains with the Dirichlet (first type), the Neumann (second type) and/or the Robin (third type) boundary conditions. A novel image system (Figure 4-3-1) for an advection diffusion problem is created to solve the Green's function solution in the case of semi-infinite domain with the Dirichlet boundary condition, and an extended image system (Figure 4-4-1) is made in the case with the Neumann or the Robin boundary condition. The obtained Green's functions are proofed to be the right solutions mathematically. The eigenfunction method is utilized to solve the Green's function when the advection diffusion problem is defined in a finite domain. In solving the problem, a coordinate transform based on a "reversed" time scale, a Laplace transform and an exponential transform (details in Section 5.3) are performed sequentially to adapt the original Green's function problem to a standard Sturm-Liouville problem, which is necessary to apply the eigenfunction method. The Green's function in the case is found to be a sum of an infinite sequence of eigenfunctions with certain coefficients, which vary on the prescribed boundary conditions of the bounded domain. The advection diffusion Green's function problems are solved with nine different boundary condition combinations. A small library of one-dimensional Green's functions for advection diffusion problems in different domains with different boundaries is summarized in form of tables (Table 6-3-1 through 6-3-3), to supply "building blocks" to construct the multi-dimensional Green's function solutions of any other linear advection diffusion problems, based on the product rule of the Green's function method in a Cartesian coordinate system. Initial value problems and source problems are not formulated separately, because an arbitrary initial value and/or an arbitrary source distribution are default configurations in the boundary value problems presented in the report.

The mathematical Green's function solutions have been utilized to validate the two diffusion models of gas species in a continuous phase and aerosol particles in a discrete phase in the GASFLOW computer code. As for the gas diffusion model, it is found that the second-order van Leer numerical scheme can produce more accurate results than the first-order. The validation calculations indicate that the diffusion equations are solved numerically in a right way in GASFLOW, therefore, that high consistencies between the GASFLOW simulations and the Green's function solutions are obtained in one- and multi-dimensional cases. The particle model in GASFLOW is verified systematically in cases of 1D, 2D and 3D, diffusing from instantaneous and/or continuous point, line, area and/or volume sources of particles in prescribed advective and/or stagnant flows. The numerical solutions about the normalized particle concentration distributions in the GASFLOW simulations are compared to the corresponding Green's function solutions. It is very interesting that agreements are obtained between the numerical solutions about the diffusion of discrete particles and the mathematical solutions about the diffusion of continuous media, if the drag force of the conveying gas flow on the moving particles satisfies the Stokes' law for resistance. The assumption is true when the Reynolds number based on the particle diameter in microns is much less than unit so that the particle inertia effects can be neglected. The series of validations manifest that the particle diffusion model can reproduce numerically the physical process of the particle movement. Meanwhile, the high consistencies of the comparisons between the numerical simulations and the

Green’s function solutions have also proved the correctness of the particle transport model in GASFLOW apart from the particle diffusion model itself.

The Green’s function solutions of the advection diffusion problems accommodate a host of benchmark test cases for validating CFD computer codes like GASFLOW. It should be mentioned that the Green’s function method can do much more than what it does in the report. The Green’s function method is a powerful mathematical tool and being widely used in many other fields like neutron transport, wave propagation and so on.

Acknowledgments

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