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Rumpel, T.; Schweizerhof, K.

Finite element analysis of liquid filled structures under gravity loading

The large deformation analysis of membrane or shell structures loaded and/or supported by liquid can be based on a finite element description for the structure only. Then in statics the effects in the liquid have to be considered by using the equations of state for the liquid, the information about the current volume and the current shape of the structure. The interaction with the structure is then modeled by a pressure resulting from the liquid always acting normal to the current wetted structural part. This description can be easily used to model the filling process without all the difficulties involved with standard discretization procedures. In addition the consistent derivation of the nonlinear formulation and the linearization for a Newton type scheme results in a particular formulation which can be cast into a very efficient solution procedure based on a sequential application of the Sherman-Morrison formula.

1. Governing equations

The mathematical description of static liquid structure interaction can be based on the principle of stationarity for the total potential energy δW of a liquid in an elastic structure and additional equations describing the physical behaviour of the liquid. The variation of the elastic potential of the structure is specified by $\delta^{el}V$; $\delta^{\ell}V$ denotes the variation of a compressible gravity potential which interferes between structure and fluid; $\delta^{ex}\Pi$ is the virtual work of external forces acting on the structure, see also [1].

$$\delta W \quad = \quad \delta^{el}V + \delta^{\ell}V - \delta^{ex}\Pi = 0 \tag{1}$$

The interaction term between liquid and structure is described by a body fixed pressure force ${}^{\ell}p\ {}^{*}\mathbf{n}$, with a non-normalized normal vector ${}^{*}\mathbf{n} = \mathbf{x},_{\xi} \times \mathbf{x},_{\eta}$ and the pressure level ${}^{\ell}p$, see equation (2). $\mathbf{x},_{\xi}$, $\mathbf{x},_{\eta}$ denote covariant unit vectors on the wetted surface of the structure. The pressure acts normal to the covariant surface element $d\xi d\eta$ along the virtual displacement $\delta\mathbf{u}$. Therefore a variational expression of a follower force is given.

$$\delta^{\ell}V \quad = \quad \int_{\eta}\int_{\xi} {}^{\ell}p\ {}^{*}\mathbf{n}\cdot\delta\mathbf{u}\ d\xi d\eta \tag{2}$$

The hydrostatic pressure law ${}^{\ell}p$ for compressible liquids can be derived from a variational analysis of the gravity potential and the virtual work expression of the pressure resulting from Hooke's law ${}^{h}p = -K\frac{v-V}{V}$ by respecting mass conservation for the liquid. For technical applications an uniform density and compression distribution throughout the volume v of the liquid can be assumed. K denotes the bulk modulus and V the undeformed volume of the liquid.

$${}^{\ell}p \quad = \quad {}^{c}p - {}^{x}p - {}^{h}p = \rho(v)\mathbf{g}\cdot(\mathbf{c}-\mathbf{x}) - {}^{h}p \tag{3}$$

With $\mathbf{g}$ as gravity and ρ as density, the pressure at the center $\mathbf{c}$ of the liquid is given by: ${}^{c}p = \rho(v)\mathbf{g}\cdot\mathbf{c}$ and resp. the pressure at an arbitrary point $\mathbf{x}$ on the wetted structure by: ${}^{x}p = \rho(v)\mathbf{g}\cdot\mathbf{x}$. In the view of a mesh-free representation of the liquid, the constitutive equations are described by the shape and the volume of the structure via a boundary integral representation. The volume v and the center $\mathbf{c}$ of the liquid can be computed via:

$$v = \frac{1}{3}\int_{\eta}\int_{\xi}\mathbf{x}\cdot{}^{*}\mathbf{n}\ d\xi d\eta \quad \text{and} \quad \mathbf{c} = \frac{1}{4v}\int_{\eta}\int_{\xi}\mathbf{x}\ \mathbf{x}\cdot{}^{*}\mathbf{n}\ d\xi d\eta. \tag{4}$$

2. Linearization, discretization and solution algorithm

A large deformation analysis of the structure including the liquid can be performed using a Newton type scheme by applying a Taylor series expansion on the total potential energy and the constraint equations. The discretization with standard FE-shell elements results in a symmetric displacement formulation. This implies that the proposed model is conservative. The system of equations with the elastic and load stiffness matrix ${}^{el,\ell}\mathbf{K}$, the residual of internal, external and interaction forces ${}^{\ell}\mathbf{F}$, the nodal displacement vector $\mathbf{d}$, a volume pressure gradient ${}^{\ell}\alpha$ and two rank-one vectors $\mathbf{a}$ and $\mathbf{b}$ results in:

$$[{}^{el,\ell}\mathbf{K} + {}^{\ell}\alpha\mathbf{a}\otimes\mathbf{a} + \mathbf{b}\otimes\mathbf{a} + \mathbf{a}\otimes\mathbf{b}]\mathbf{d} \quad = \quad {}^{\ell}\mathbf{F}. \tag{5}$$

This can be interpreted as a symmetric rank-three update of the matrix ${}^{el,\ell}\mathbf{K}$ coupling all wetted degrees of freedom together. Applying the Sherman - Morrison formula an efficient solution can be computed by two additional forwardbackward substitutions:

$$\mathbf{d}_1 = {}^{el,\ell}\,\mathbf{K}^{-1}\;{}^{\ell}\mathbf{F}, \quad \mathbf{d}_2 \;=\; {}^{el,\ell}\mathbf{K}^{-1}\;\mathbf{a}, \quad \mathbf{d}_3 = {}^{el,\ell}\,\mathbf{K}^{-1}\;\mathbf{b}. \tag{6}$$

The nodal displacement vector for one iteration step is given by a linear combination of the three auxiliary solution vectors: $\mathbf{d} = \mathbf{d}(\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3)$.

3. Numerical example

An elastic cylindrical vessel (weightless, elastic modulus $E = 21 \cdot 10^{10} \frac{N}{m^2}$, Poisson's ratio $\nu = 0.3$) with a very thin wall - close to a membrane - is completely filled with water (density $\rho = 1000 \frac{kg}{m^3}$, bulk modulus $K = 0.5 \cdot 10^9 \frac{N}{m^2}$). In a first load step the vessel is pressurized by 1 bar at the top of the vessel indicating the weight of the plate. In a second step the structure is loaded by a given displacement u_{ext} of the loading plate, see figure (1b).

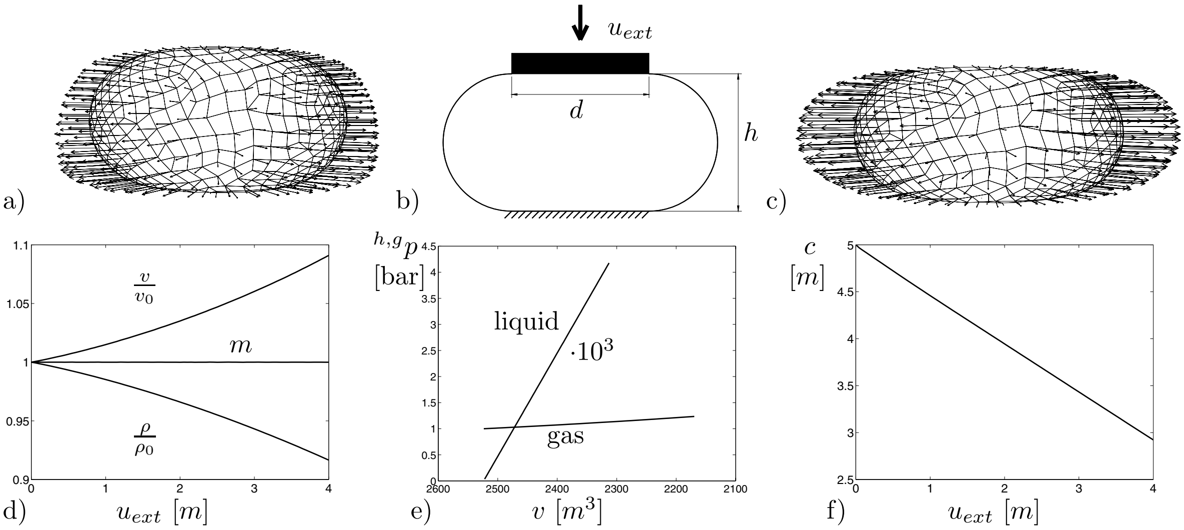

Figure 1: a) radial displacement vectors - first load step; b) height $h = 10m$, diameter $d = 10m$, piston diplacement $u_{ext} = 4m$; c) radial displacement vectors - final load step; d) mass conservation vs. piston displacement u_{ext}; e) liquid pressure ${}^{h}p$ / gas pressure ${}^{g}p$ vs. fluid volume v; f) location of center c of volume vs. piston displacement u_{ext}

Due to the large deformation with a maximum of $u_{ext} = 4m$ density and volume change according to the conservation of mass, see figure (1) d). The decrease of the volume implicates an increase in the pressure level ${}^{h}p$ in the liquid. For comparison only a gas filling is considered too, see figure (1) e). A provisional result is the position of the center c of the volume, which changes with the displacement of the piston, see figure (1) f).

4. Conclusion

The proposed approach to describe the interaction of liquids with strongly deforming structures by state equations has several advantages. First, the mesh-free analysis of the liquid avoids remeshing procedures in large deformation analysis. Second, no contact models between liquid and structure have to be considered. Third, stability investigations can be carried out taking the specific decomposition of the stiffness matrix of the coupled problem into account. Fourth, the solution of the coupled equation can be efficiently performed based on the subsequent use of the Sherman - Morrison formula involving only the triangular decomposition of the structural matrix. Summarizing all, the computational effort is significant lower and better adjusted than in conventional fully discretized methods.

5. References

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T. Rumpel, Institut für Technische Mechanik, Universität Karlsruhe (TH), D-76128 Karlsruhe
K. Schweizerhof, Institut für Mechanik, Universität Karlsruhe (TH), D-76128 Karlsruhe