First published in:

PAMM · Proc. Appl. Math. Mech. **3**, 509–510 (2003) / **DOI** 10.1002/pamm.200310524

Götz Alefeld, Uwe Schäfer

On the optimal solution of interval linear complementarity problems

1. The linear complementarity problem

The linear complementarity problem, abbreviated LCP, is to find a vector z such that

$$q + Mz \geq o, \quad z \geq o, \quad (q + Mz)^{\mathrm{T}} z = 0, \tag{1}$$

or to show that no such vector z exists. The inequalities appearing in (1) and in the sequel are meant componentwise and o denotes the zero vector. For a detailed introduction to the LCP we refer to [3].

The present paper is concerned with the case, where the given matrix M and the given vector q are not exactly known but can be enclosed in an interval matrix $[M]$ and an interval vector $[q]$, respectively. This generalization arises, for example, from free boundary problems by discretizing the problem without neglecting the discretization error ([4]).

2. Interval linear complementarity problems

We consider compact intervals $[\underline{a}, \overline{a}] := \{x \in \mathbb{R} : \underline{a} \leq x \leq \overline{a}\}$ and denote the set of all such intervals by $\mathbf{IR}$. We also write $[a]$ instead of $[\underline{a}, \overline{a}]$. Furthermore, we consider matrices with an interval in each of its elements; i.e., $[\underline{A}, \overline{A}] = ([a_{ij}]) = ([\underline{a}_{ij}, \overline{a}_{ij}])$. We also write $[\underline{A}, \overline{A}] := \{A \in \mathbb{R}^{n \times n} : \underline{A} \leq A \leq \overline{A}\}$. By $\mathbf{IR}^{n \times n}$ we denote the set of all these so-called interval matrices. We also write $[A]$ instead of $[\underline{A}, \overline{A}]$. The set of interval vectors with n components is constructed in the same way and denoted by $\mathbf{IR}^n$. For an introduction to interval computations we refer to [2].

Let $[q] \in \mathbf{IR}^n$ and $[M] \in \mathbf{IR}^{n \times n}$ be given. Then, we are interested in the set

$$\Sigma([M], [q]) := \left\{ z \in \mathbb{R}^n : z \geq o,\ q + Mz \geq o,\ (q + Mz)^{\mathrm{T}} z = 0,\ M \in [M],\ q \in [q] \right\}. \tag{2}$$

To describe $\Sigma([M], [q])$ we consider the auxiliary set

$$AUX([M], [q]) := \left\{ \begin{pmatrix} \omega \\ z \end{pmatrix} \in \mathbb{R}^{2n} : \omega - Mz = q,\ \omega \geq o,\ z \geq o,\ \omega^{\mathrm{T}} z = 0,\ M \in [M],\ q \in [q] \right\}. \tag{3}$$

Lemma 1 ([5]). *Let $[M] \in \mathbf{IR}^{n \times n}$ and $[q] \in \mathbf{IR}^n$. Then, $z \in \Sigma([M], [q])$ iff there exists $\omega \in \mathbb{R}^n$ such that $\binom{\omega}{z} \in AUX([M], [q])$.*

Theorem 1 ([5]). *Let $[M] \in \mathbf{IR}^{n \times n}$, I the $n \times n$ identity matrix, $[q] \in \mathbf{IR}^n$. Then,*

$$\begin{pmatrix} \omega \\ z \end{pmatrix} \in AUX([M], [q]) \Leftrightarrow \left(I \vdots - [M] \right) \cdot \begin{pmatrix} \omega \\ z \end{pmatrix} \cap [q] \neq \emptyset \text{ and } \omega^{\mathrm{T}} z = 0,\ \omega \geq o,\ z \geq o.$$

We mention the following equivalence:

$$[a] \cap [b] \neq \emptyset \Leftrightarrow \underline{a} \leq \overline{b} \text{ and } \underline{b} \leq \overline{a}, \quad \text{if } [a], [b] \in \mathbf{IR}. \tag{4}$$

Let $\begin{pmatrix} \omega \\ z \end{pmatrix} \in AUX([M], [q])$. Then we have via Theorem 1 and (4) (used componentwise)

$$\omega_i - \sum_{j=1}^{n} \overline{m}_{ij} \cdot z_j \leq \overline{q}_i \text{ and } \underline{q}_i \leq \omega_i - \sum_{j=1}^{n} \underline{m}_{ij} \cdot z_j, \quad \text{for all } i \in \{1, ..., n\}, \tag{5}$$

since $z_j \geq 0$. Due to the complementarity we have to consider 2^n cases then.

Example 1. Let

$$[M] = \begin{pmatrix} [\frac{1}{8}, 1] & [-\frac{1}{4}, -\frac{1}{5}] \\ [-\frac{1}{4}, -\frac{1}{10}] & 1 \end{pmatrix} \text{ and } [q] = \begin{pmatrix} [-3, -1] \\ [1, 2] \end{pmatrix}.$$

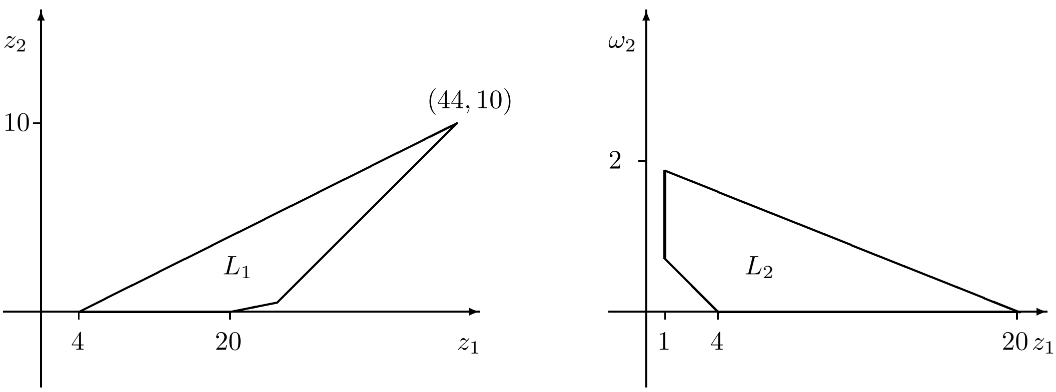
(5) leads to four inequalities:

$$\text{(I) } \omega_1 - (z_1 - \tfrac{1}{5} z_2) \le -1\,; \qquad \text{(II) } -3 \le \omega_1 - (\tfrac{1}{8} z_1 - \tfrac{1}{4} z_2)\,;$$
$$\text{(III) } \omega_2 - (-\tfrac{1}{10} z_1 + z_2) \le 2\,; \qquad \text{(IV) } 1 \le \omega_2 - (-\tfrac{1}{4} z_1 + z_2)\,.$$

We consider four cases: 1. $\omega_1 = 0$, $\omega_2 = 0$; 2. $\omega_1 = 0$, $z_2 = 0$; 3. $z_1 = 0$, $\omega_2 = 0$; 4. $z_1 = 0$, $z_2 = 0$.
The cases 3. and 4. cannot give a contribution to $AUX([M],[q])$ due to (I). Considering the cases 1. and 2. we get

$$AUX([M],[q]) = \left\{ \begin{pmatrix} 0 \\ 0 \\ z_1 \\ z_2 \end{pmatrix} : (z_1, z_2) \in L_1 \right\} \cup \left\{ \begin{pmatrix} 0 \\ \omega_2 \\ z_1 \\ 0 \end{pmatrix} : (z_1, \omega_2) \in L_2 \right\},$$

where L_1 and L_2 are illustrated in the following figure.



By Lemma 1 we get $\Sigma([M],[q]) = L_1 \cup \begin{pmatrix} [1, 20] \\ 0 \end{pmatrix}$.

3. The optimal solution of an interval linear complementarity problem

The optimal solution of an interval linear complementarity problem is the narrowest interval vector that includes $\Sigma([M],[q])$. Concerning Example 1 the optimal solution is $\begin{pmatrix} [1, 44] \\ [0, 10] \end{pmatrix}$. In [1], we have shown that the total step method, the single step method and the symmetric single step method are convergent to the optimal solution for the case that all $M \in [M]$ are M(inkowski) matrices. For the case that all $M \in [M]$ are H-matrices with positive diagonal entries the above mentioned methods are also convergent to an interval vector that includes $\Sigma([M],[q])$, but this inclusion is not necessarily optimal.

4. References

1 Alefeld, G.; Schäfer, U.: Iterative methods for linear complementarity problems with interval data. Computing (to appear).
2 Alefeld, G.; Herzberger, J.: Introduction to interval computations. Academic Press, 1983.
3 Cottle, R.W.; Pang, J,S.; Stone, R.E.: The linear complementarity problem. Academic Press, 1992.
4 Schäfer, U.: An enclosure method for free boundary problems based on a linear complementarity problems with interval data. Numer. Funct. Anal. Optim., **22** (2001), 991–1011.
5 Schäfer, U.: Das lineare Komplementaritätsproblem mit Intervalleinträgen. Dissertation, Universität Karlsruhe, 1999.

Prof. Dr. Götz Alefeld, Dr. Uwe Schäfer., Institut für Angewandte Mathematik, Universität Karlsruhe, 76128 Karlsruhe, Germany