Some remarks on an interval arithmetic version of the Lemke algorithm

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Abstract. We consider an interval arithmetic version of the Lemke algorithm for solving the linear complementarity problem where the given data are not exactly known but can only be enclosed in intervals.

Keywords: Linear complementarity problem, P-matrix, error control
Mathematics Subject Classification (2000): 90C33

1 Introduction

Given $M \in \mathbb{R}^{n\times n}$ and $q \in \mathbb{R}^n$ the linear complementarity problem (LCP) is to find two vectors ω and z such that

$$\omega = q + Mz \geq o, \quad z \geq o, \quad \omega^{\mathrm{T}} z = 0, \tag{1}$$

or to show that no such vectors exist. It is well-known that the LCP has a unique solution for any q if, and only if, M is a P-matrix, where M is called a P-matrix if all principal minors of M are positive. In order to find the unique solution for the case that M is a P-matrix, one can use the Lemke algorithm:

Theorem 1.1 *Let $M \in \mathbb{R}^{n\times n}$ be a P-matrix and $q \in \mathbb{R}^n$ be arbitrary. Then, the lexicographic Lemke algorithm applied to M and q results in the unique solution of the corresponding LCP.*

For a proof we refer to [4], and we want to emphasize that it is possible that the Lemke algorithm cycles if the minimum ratio rule is not used in the lexicographical sense (see [3]).

In [7], we considered (1) where M was a P-matrix of the form

$$M = A^{-1}B. \tag{2}$$

However, only in rare cases the entries of A^{-1} can be represented by a computer (e.g., if $\mathbb{R} \ni A = 3$, then A^{-1} is not representable by a floating point number). Therefore, the Lemke algorithm applied on a computer to M from (2) and any q will, in general, not consider the original problem.

In [7], we used an interval arithmetic version of the Lemke algorithm for solving (1) where the input data M and q are not given exactly but can be enclosed in intervals. In this paper we want to give the details of this algorithm, some counterexamples concerning its feasibility, and some further computational and experimental studies.

Throughout the paper we assume that the reader is familiar with the Lemke algorithm and corresponding notations such as nonbasic variables, ray termination, complementary pivot rule, and so on. Otherwise we refer to [4].

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2 Preliminaries

2.1 Interval arithmetic

We consider compact intervals $[\underline{a}, \overline{a}] := \{x \in \mathbb{R} : \underline{a} \leq x \leq \overline{a}\}$ and denote the set of all such intervals by $\mathbf{IR}$. We also write $[a]$ instead of $[\underline{a}, \overline{a}]$. Let $[a]$, $[b] \in \mathbf{IR}$, then the basic operations are defined as

$$\left.\begin{array}{rcl} [a]+[b] & = & [\underline{a}+\underline{b}, \overline{a}+\overline{b}], \quad [a]-[b] = [\underline{a}-\overline{b}, \overline{a}-\underline{b}], \\ {[a]\cdot[b]} & = & [\min\{\underline{a}\underline{b}, \underline{a}\overline{b}, \overline{a}\underline{b}, \overline{a}\overline{b}\}, \max\{\underline{a}\underline{b}, \underline{a}\overline{b}, \overline{a}\underline{b}, \overline{a}\overline{b}\}], \\ {[a]/[b]} & = & [\underline{a}, \overline{a}] \cdot [\frac{1}{\overline{b}}, \frac{1}{\underline{b}}] \quad \text{if } 0 \notin [b]. \end{array}\right\} \tag{3}$$

Furthermore, we consider matrices with intervals as elements; i.e., $([a_{ij}]) = ([\underline{a}_{ij}, \overline{a}_{ij}])$. We also write

$$[\underline{A}, \overline{A}] := \{A \in \mathbb{R}^{n\times n} : \underline{A} \leq A \leq \overline{A}\}. \tag{4}$$

By $\mathbf{IR}^{n\times n}$ we denote the set of all these so-called interval matrices (4). We also write $[A]$ instead of $[\underline{A}, \overline{A}]$. The set of interval vectors with n components is constructed in the same way and is denoted by $\mathbf{IR}^n$. For an introduction to interval computations we refer to [1].

2.2 PASCAL-XSC

The big advantage of interval arithmetic gets obvious using a computer. The tool we have used in this study is PASCAL-XSC ([2]). It extends the programming language PASCAL by the variable type *interval*. The operations (3) are predefined and the bounds of $[c] = [a]*[b]$ with $* \in \{+, -, \cdot, /\}$ are automatically rounded in the correct direction. For example, $\underline{a} + \underline{b}$ and $\overline{a} + \overline{b}$ are rounded downwardly and upwardly, respectively, to the next machine number.

As a consequence, if $\underline{a}$, $\underline{b}$, $\overline{a}$, $\overline{b}$ are machine numbers we have $a * b \in [a] * [b] = [\underline{c}, \overline{c}]$ for all real (not necessarily machine) numbers $a \in [\underline{a}, \overline{a}]$, $b \in [\underline{b}, \overline{b}]$, where $\underline{c}$ and $\overline{c}$ are machine numbers. Hence, we have a rigorous error control by the computer using PASCAL-XSC.

3 The necessary condition for feasibility

Given an interval matrix $[M] \in \mathbf{IR}^{n\times n}$ and an interval vector $[q] \in \mathbf{IR}^n$ using an interval arithmetic version of the Lemke algorithm we want to determine an interval vector

$$[x] = \begin{pmatrix} [\omega] \\ [z] \end{pmatrix} \in \mathbf{IR}^{2n} \tag{5}$$

with $[\omega], [z] \in \mathbf{IR}^n$ satisfying:

$$\left.\begin{array}{c} \text{For each } M \in [M] \text{ and for each } q \in [q] \text{ there exist } \omega \in [\omega] \text{ and } z \in [z] \text{ with} \\ \omega - Mz = q, \quad \omega^{\mathrm{T}} z = 0, \quad \omega \geq o, z \geq o. \end{array}\right\} \tag{6}$$

If the Lemke algorithm using the lexico minimum ratio rule does not end in ray termination, a solution of the LCP will be achieved. At the same time, the nonbasic variables are set to zero. So, if an interval version of the Lemke algorithm will give (5) and (6), then the interval vector (5) will fulfill

$$[\omega_i] = 0 \quad \text{or} \quad [z_i] = 0, \quad i = 1, ..., n. \tag{7}$$

This means, that an interval version of the Lemke algorithm can result in an interval vector (5) that fulfills (6) and (7) only if the interval matrix $[M]$ and the interval vector $[q]$ fulfill the following property:

$$\left.\begin{array}{l} \text{For each } M \in [M] \text{ and each } q \in [q] \text{ the LCP defined by} \\ M \text{ and } q \text{ has a solution } x = \begin{pmatrix} \omega \\ z \end{pmatrix} \in \mathbb{R}^{2n}, \text{ where independent} \\ \text{of } M \text{ and } q \text{ always the same } n \text{ components of } x \text{ are zero.} \end{array}\right\} \tag{8}$$

(8) is called basic stable. In the following example we will show that the fact, that any $M \in [M]$ is a P-matrix, is not sufficient to conclude (8) even for the case that $[q] = q \in \mathbb{R}^n$. (Finding a counterexample for (8) for the case $[q] \in \mathbf{IR}^n$ is easy: Choose $[M] = 1$ and $[q] = [-1, 1]$). Furthermore, also the assumption, that the lenghts of the interval entries of $[M]$ and $[q]$ are arbitrary small, is not sufficient to conclude (8).

Example 3.1 *Let*

$$[M] = \begin{pmatrix} 1 & [0, \frac{1}{2}] & [0, \frac{1}{2}] \\ [0, \frac{1}{2}] & 1 & [0, \frac{1}{2}] \\ [0, \frac{1}{2}] & [0, \frac{1}{2}] & 1 \end{pmatrix} \quad \textit{and} \quad q = \begin{pmatrix} -3 \\ -2 \\ -1 \end{pmatrix}.$$

In [5], it was shown that any $M \in [M]$ is regular. Since the identity belongs to $[M]$ it follows by Theorem 1.2 in [6] that any $M \in [M]$ is even a P-matrix. However, we will show that $[M]$ and $[q]$ are not basic stable concerning (8). In fact, the LCP defined by q and

$$M_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in [M]$$

has the unique solution

$$\omega = o, \quad z = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \tag{9}$$

whereas the LCP defined by q and

$$M_2 = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix} \in [M]$$

has the unique solution

$$\omega = \begin{pmatrix} 0 \\ 0 \\ \frac{2}{3} \end{pmatrix}, \quad z = \begin{pmatrix} \frac{8}{3} \\ \frac{2}{3} \\ 0 \end{pmatrix}. \tag{10}$$

Now, let $\varepsilon > 0$ be arbitrary. Then, there exists $m > 0$ such that $\frac{1}{m} \cdot \frac{1}{2} < \varepsilon$. We define $[\tilde{M}] := \frac{1}{m}[M]$ and $\tilde{q} := \frac{1}{m}q$. Then, the length of any interval entry of $[\tilde{M}]$ is less than ε. Moreover, it is easy to see that (ω, z) is a solution of the LCP defined by M and q if, and only if, $(\frac{1}{m} \cdot \omega, z)$ is a solution of the LCP defined by $\frac{1}{m} \cdot M$ and $\frac{1}{m} \cdot q$. Using again (9) and (10) we see that also $[\tilde{M}]$ and $\tilde{q}$ are not basic stable.

4 The interval Lemke algorithm

In this section we define the interval Lemke algorithm. This is done in the same way as the Gaussian algorithm was extended to the interval Gaussian algorithm (see [1]). The interval Gaussian algorithm is applied to an interval matrix $[A]$ and an interval vector $[b]$ in order to include any solution of $Ax = b$ where $A \in [A]$ and $b \in [b]$.

In [5], it was shown that the interval Gaussian algorithm may fail due to division by an interval containing 0 even for the case that any $A \in [A]$ is regular. Therefore, it was no surprise that it is possible that the interval Lemke algorithm fails even if any $M \in [M]$ is a P-matrix (see Example 3.1).

In contrast to the interval Gaussian algorithm it seems hard to find classes of interval matrices $[M]$ and interval vectors $[q]$ such that the feasibility of the interval Lemke algorithm is guaranteed theoretically. E.g., if we assume that any $M \in [M]$ is a K-matrix we cannot guarantee the feasibility of the interval Lemke algorithm (see Remark 4.3).

4.1 The definition of the interval Lemke algorithm

Let $[M] \in \mathbf{IR}^{n\times n}$, $[q] \in \mathbf{IR}^n$ and I denote the $n \times n$ identity. In the sequel we denote by $[M]_{i:}$ the i-th row and by $[M]_{:j}$ the j-th column of $[M]$. We define $[A^{(0)}] := (I \vdots -[M] \vdots -e) \in \mathbf{IR}^{n\times 2n+1}$ and $[q^{(0)}] := [q]$, where e denotes the vector with 1 in each component.

Case 1: If $\underline{q} \geq o$, then the algorithm terminates with $[\omega] := [q]$ and $[z] := o$.

Case 2: It holds that $\underline{q} \not\geq o$. Then, we choose the smallest $i_0 \in \{1, ..., n\}$ with $\underline{q}_{i_0} := \min_{1\leq i\leq n} \{\underline{q}_i\}$ and define $[A^{(1)}] := (E \vdots [M'] \vdots I_{:i_0}) \in \mathbf{IR}^{n\times 2n+1}$ and $[q^{(1)}] \in \mathbf{IR}^n$ by

$$[q^{(1)}_{i_0}] = -[q_{i_0}] \quad \text{and} \quad [q^{(1)}_i] = [q_i] - [q_{i_0}],\ i = 1, ..., n,\ i \neq i_0,$$

and

$$\begin{aligned} E_{:i} &= I_{:i}, & i &= 1, ..., n,\ i \neq i_0, & E_{:i_0} &= -e, \\ [M']_{i:} &= -[M]_{i:} + [M]_{i_0:}, & i &= 1, ..., n,\ i \neq i_0, & [M']_{i_0:} &= [M]_{i_0:}. \end{aligned}$$

The basic variables are $('\omega_1', ..,' \omega_{i_0-1}',' z_0',' \omega_{i_0+1}', ..,' \omega_n')$. Let

$$[a^{(k)}_{i_0 j_0}], \quad i_0 \in \{1, ..., n\}, \quad j_0 \in \{1, ..., 2n+1\}$$

be the k-th pivot element. Then, using (3) we define

$$
\begin{aligned}
[q_{i_0}^{(k+1)}] &:= [q_{i_0}^{(k)}]/[a_{i_0 j_0}^{(k)}], && \\
[q_{i}^{(k+1)}] &:= [q_{i}^{(k)}] - [q_{i_0}^{(k+1)}]\cdot[a_{i j_0}^{(k)}] && i = 1,...,n,\ i \neq i_0, \\
[a_{i_0 j}^{(k+1)}] &:= [a_{i_0 j}^{(k)}]/[a_{i_0 j_0}^{(k)}] && j = 1,...,2n+1,\ j \neq j_0, \\
[a_{i_0 j_0}^{(k+1)}] &:= 1, && \\
[a_{ij}^{(k+1)}] &:= [a_{ij}^{(k)}] - [a_{i_0 j}^{(k+1)}]\cdot[a_{i j_0}^{(k)}] && i = 1,...,n,\ i \neq i_0, \\
& && j = 1,...,2n+1,\ j \neq j_0, \\
[a_{i j_0}^{(k+1)}] &:= 0 && i = 1,...,n,\ i \neq i_0.
\end{aligned}
$$

The determination of j_0 is done by the complementary pivot rule, whereas the determination of i_0 is done as follows. Let j_0 be determined (by the complementary pivot rule). Then, the $(k+1)$-th pivot column is

$$
[s] = [\underline{s}, \overline{s}] := \begin{pmatrix} [a_{1j_0}^{(k)}] \\ \vdots \\ [a_{nj_0}^{(k)}] \end{pmatrix} \in \mathbf{IR}^n.
$$

If $\underline{s} \leq o$, then the algorithm terminates with the message that the algorithm cannot find a solution. On the other hand, if $\underline{s} \not\leq o$, we choose i_0 as the smallest index such that

$$
\min_{1\leq i\leq n} \left\{ \underline{q}_i^{(k)}/\overline{s}_i \,,\ \underline{s}_i > 0 \right\} = \underline{q}_{i_0}^{(k)}/\overline{s}_{i_0}.
$$

If $'z_0'$ has become nonbasic, then for $i = 1,...,n$ we define:

$$
[\omega_i] := \left\{ \begin{array}{ll} 0 & \text{if } '\omega_i' \text{ is nonbasic,} \\ & \\ [q_j^{(k)}] & \text{if } '\omega_i' \text{ is the } j\text{-th basic variable,} \end{array} \right\} \tag{11}
$$

and

$$
[z_i] := \left\{ \begin{array}{ll} 0 & \text{if } 'z_i' \text{ is nonbasic,} \\ & \\ [q_j^{(k)}] & \text{if } 'z_i' \text{ is the } j\text{-th basic variable.} \end{array} \right\} \tag{12}
$$

However, if there exists some $i \in \{1,..,n\}$ with $\underline{q}_i^{(k)} < 0$, then we cannot guarantee (6) and the algorithm fails (see Remark 4.2).

Theorem 4.1 *Let $[M] \in \mathbf{IR}^{n\times n}$ and $[q] \in \mathbf{IR}^n$ be given. Then, if the interval Lemke algorithm is applied to $[M]$ and $[q]$, there are two cases:*

1. *The interval Lemke algorithm fails due to one of the following reasons:*
   - (a) *The interval Lemke algorithm cycles.*
   - (b) *$'z_0'$ has become nonbasic, but (6) cannot be guaranteed.*
   - (c) *The interval Lemke algorithm breaks down, since there is a pivot column $[s] = [\underline{s}, \overline{s}]$ satisfying $\underline{s} \leq o$.*

2. The interval Lemke algorithm is feasible and ends with an interval vector

$$ILA([M],[q]) := \begin{pmatrix} [\omega] \\ [z] \end{pmatrix} \in \mathbf{IR}^{2n},\ [\omega],[z] \in \mathbf{IR}^n,$$

that fulfills (6) and (7).

Proof: Assume that the first k steps of the interval Lemke algorithm were feasible and let $[a^{(0)}_{i_0^{(0)} j_0^{(0)}}], [a^{(1)}_{i_0^{(1)} j_0^{(1)}}], \ldots, [a^{(k)}_{i_0^{(k)} j_0^{(k)}}]$ be the pivot elements; i.e.,

$$(i_0^{(0)} j_0^{(0)}), (i_0^{(1)} j_0^{(1)}), \ldots, (i_0^{(k)} j_0^{(k)}) \tag{13}$$

are the corresponding pivot indices. If $A^{(0)} \in [A^{(0)}]$ (i.e., $M \in [M]$) and $q^{(0)} \in [q]$ are arbitrarily chosen, then after k Gauss-Jordan steps applied to $A^{(0)}$ and $q^{(0)}$ using the pivot indices (13) we get

$$(\, A^{(0)} \,|\, q^{(0)} \,) \rightsquigarrow (\, A^{(k)} \,|\, q^{(k)} \,),$$

and it holds that

$$A^{(k)} \in [A^{(k)}] \quad \text{and} \quad q^{(k)} \in [q^{(k)}] \tag{14}$$

due to the so-called inclusion monotonicity (see [1]). If $'z_0'$ has become nonbasic, we define

$$\omega_i := \begin{cases} 0 & \text{if } '\omega_i' \text{ is nonbasic}, \\ q_j^{(k)} & \text{if } '\omega_i' \text{ is the } j\text{-th basic variable}, \end{cases}$$

$$z_i := \begin{cases} 0 & \text{if } 'z_i' \text{ is nonbasic}, \\ q_j^{(k)} & \text{if } 'z_i' \text{ is the } j\text{-th basic variable}, \end{cases}$$

and $z_0 := 0$. Then, we have due to (11),(12), and (14) that $\omega \in [\omega]$, $z \in [z]$, and

$$q^{(0)} = A^{(0)} \begin{pmatrix} \omega \\ z \\ z_0 \end{pmatrix} = \omega - Mz - z_0 \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \omega - Mz.$$

If $\underline{q}^{(k)} \geq o$ is satisfied, then it follows $\omega \geq \underline{\omega} \geq o$ and $z \geq \underline{z} \geq o$. Finally, we have $\omega^{\mathrm{T}} z = 0$ due to the complementary pivot rule. □

Remark 4.1 *It is not easy to see how the lexico minimum ratio rule can be extended to the interval case, so it was omitted in our implementation. As a consequence, the interval Lemke algorithm can cycle. For example, if the interval Lemke algorithm is applied to*

$$[M] = M = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix} \quad \textit{and} \quad [q] = q = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}, \tag{15}$$

then the algorithm cycles (see [3]). However, following the idea of perturbing q (see [4]), if we apply the interval Lemke algorithm to

$$[M] = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix} \quad \textit{and} \quad [q] = \begin{pmatrix} -1 \\ -1 \\ [-1.000000001, -1] \end{pmatrix},$$

we get the following basic variables

$$('z_0','\omega_2','\omega_3'), ('z_0','z_1','\omega_3'), ('z_0','z_1','z_2'), ('z_3','z_1','z_2'),$$

and then

$$\begin{pmatrix} [\omega] \\ [z] \end{pmatrix} = \begin{pmatrix} [0.000000000000000E+000, 0.000000000000000E+000] \\ [0.000000000000000E+000, 0.000000000000000E+000] \\ [0.000000000000000E+000, 0.000000000000000E+000] \\ [3.333333333333333E-001, 3.333333337777779E-001] \\ [3.333333329999999E-001, 3.333333334444445E-001] \\ [3.333333333333333E-001, 3.333333334444445E-001] \end{pmatrix},$$

which includes rigorously the unique solution $\omega = o$, $z = (\frac{1}{3}\,\frac{1}{3}\,\frac{1}{3})^{\mathrm{T}}$ of the LCP defined by (15).

Remark 4.2 *Consider*

$$M_1 = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 0 & 3 & 1 \end{pmatrix} \in \begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ [0,1] & 3 & [0,1] \end{pmatrix} = [M] \quad \text{and} \quad q = \begin{pmatrix} -6 \\ -5 \\ -4 \end{pmatrix}.$$

M_1 is a P-matrix. Therefore, the LCP defined by M_1 and q has a unique solution; namely

$$\omega = o\,, \quad z = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \tag{16}$$

Let

$$M_2 = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 1 & 3 & 0 \end{pmatrix} \in [M].$$

The LCP defined by M_2 and q has exactly two solutions; namely

$$\omega = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad z = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \omega = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}, \quad z = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}.$$

If we apply the interval Lemke algorithm to $[M]$ and q, we get after 3 steps

$$(\,[A^{(3)}]\,\big|\,[q^{(3)}]\,) = \left(\begin{array}{ccccccc|c} 0 & -\frac{1}{5} & 0 & 0 & 1 & 0 & \frac{1}{5} & 1 \\ -3 & \frac{2}{5} & 0 & 1 & 0 & 3 & \frac{3}{5} & 4 \\ [-1,0] & [-\frac{6}{5},\frac{2}{5}] & 1 & 0 & 0 & [-1,3] & \frac{3}{5} & [-1,3] \end{array}\right);$$

i.e., $'z_0'$ has become nonbasic and one might get the idea of setting

$$[\omega] = \begin{pmatrix} 0 \\ 0 \\ [0,3] \end{pmatrix} \quad \text{and} \quad [z] = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}.$$

However, the solution (16) is not included. So, $[\omega]$ and $[z]$ do not fulfill (6), and the interval Lemke algorithm fails. That's why the test $\underline{q}^{(k)} \geq o$ is necessary, if $'z_0'$ has become nonbasic after the k-th *step.*

Remark 4.3 *The condition (8) is not sufficient to conclude that the interval Lemke algorithm is feasible. Let*

$$[M] = \begin{pmatrix} 2 & [-1,0] & 0 \\ [-1,0] & 2 & [-1,0] \\ 0 & [-1,0] & 2 \end{pmatrix} \quad \textit{and} \quad q = \begin{pmatrix} -6 \\ -5 \\ -4 \end{pmatrix}.$$

Any $M \in [M]$ is a so-called K-matrix. Therefore, any $M \in [M]$ is a P-matrix satisfying $M^{-1} \geq O$ (see [4]) and the LCP defined by q and any $M \in [M]$ has the unique solution $\omega = o$, $z = -M^{-1}q > o$. So, (8) is fulfilled. However, after 3 steps of the interval Lemke algorithm we get as the pivot column

$$[\underline{s}, \overline{s}] := [A^{(3)}]_{:6} = \begin{pmatrix} [-\frac{5}{6}, 27] \\ [-\frac{27}{2}, \frac{1}{6}] \\ [-9, -\frac{1}{2}] \end{pmatrix},$$

and the interval Lemke algorithm fails, since $\underline{s} \leq o$.

5 Numerical examples

The main disadvantage of the interval Lemke algorithm is the overestimation that occurs using interval arithmetic. For example, concerning (3) we have $[1,2] - [1,2] = [-1,1]$ and not 0 as one might expect. Note also that the length of the resulting interval is the sum of the length of the two intervals. As a consequence, the lengths of the interval entries of $(\,[A^{(k)}] \,\big|\, [q^{(k)}]\,)$ increase from step to step and one might expect that the bigger the dimension n of the LCP the smaller the lengths of the interval entries of $[M]$ and $[q]$ have to be to gain the feasibility of the interval Lemke algorithm. The following two examples will comment on this.

Example 5.1 *We consider $[q] = ([q_i]) \in \mathbf{IR}^n$ with $[q_i] = -7 + i + [-\varepsilon, \varepsilon]$ and the symmetric tridiagonal interval matrix $[M] = ([m_{ij}]) \in \mathbf{IR}^{n \times n}$ with*

$$[m_{ii}] = [2.1, 2.1 + \varepsilon] \quad \textit{and} \quad [m_{ii+1}] = [m_{ii-1}] = -1 + [-\varepsilon, \varepsilon].$$

For small $\varepsilon > 0$ any $M \in [M]$ is a K-matrix and is therefore a P-matrix. We applied the interval Lemke algorithm to $[M]$ and $[q]$ for different n and for different ε, see Table 1. It shows the results as one has expected with the exception of $n \geq 9$. But this is due to the choice of $[q]$. For $\varepsilon = 10^{-6}$

$n \backslash \varepsilon$	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}
3	Y					
4	N	Y				
5	N	N	Y			
6	N	N	Y			
7	N	N	Y			
8	N	N	N	Y		
9	N	N	N	N	N	Y
10	N	N	N	N	N	Y
50	N	N	N	N	N	Y
100	N	N	N	N	N	Y
500	N	N	N	N	N	Y

Example 5.1

$n \backslash \varepsilon$	10^{-1}	10^{-2}	10^{-3}	10^{-4}
3	Y			
4	Y			
5	N	Y		
6	N	Y		
7	N	N	Y	
8	N	Y		
9	N	N	N	Y
10	N	N	N	Y
11	N	N	Y	
12	N	N	Y	
13	N	N	Y	

Example 5.2

Table 1: Y denotes the feasibility of the interval Lemke algorithm, whereas N denotes failure.

and any $n \geq 9$ we had

$$[z] = \begin{pmatrix} [9.803134504968790E+000, 9.839544971955811E+000] \\ [1.459550945858491E+001, 1.465407328464603E+001] \\ [1.585587395836163E+001, 1.592553441767262E+001] \\ [1.470968315524597E+001, 1.478166113820489E+001] \\ [1.204162631257776E+001, 1.210876148586848E+001] \\ [8.584039803008465E+000, 8.640406782035580E+000] \\ [4.990086667602106E+000, 5.030843916478347E+000] \\ [1.899013162623603E+000, 1.920480665319872E+000] \\ [0.000000000000000E+000, 0.000000000000000E+000] \\ \vdots \\ [0.000000000000000E+000, 0.000000000000000E+000] \end{pmatrix}.$$

For fixed $n \geq 9$ we decreased ε and got

ε	10^{-6}	10^{-8}	10^{-10}	10^{-12}	10^{-14}	10^{-16}
$\max_i\{\overline{z}_i - \underline{z}_i\}$	$7.1 \cdot 10^{-2}$	$7.1 \cdot 10^{-4}$	$7.1 \cdot 10^{-6}$	$7.1 \cdot 10^{-8}$	$7.4 \cdot 10^{-10}$	$1.9 \cdot 10^{-11}$

Example 5.2 *We consider $[q] = ([q_i]) \in \mathbf{IR}^n$ with*

$$[q_i] = \begin{cases} i/17 + [-\varepsilon, \varepsilon], & \text{if } i \text{ is even}, \\ -i/17 + [-\varepsilon, \varepsilon], & \text{if } i \text{ is odd}, \end{cases}$$

and $[M] = ([m_{ij}]) \in \mathbf{IR}^{n \times n}$ with $[m_{ii}] = n + 1 + [-\varepsilon, \varepsilon]$ and $[m_{ij}] = 0.5 + [-\varepsilon, \varepsilon]$, $i \neq j$. For small $\varepsilon > 0$ any $M \in [M]$ is diagonally dominant and is therefore a P-matrix. In Table 1, one can see the results of the interval Lemke algorithm with respect to alternating n and ε. Concerning $n = 7$, $n = 8$ and $n = 10$, $n = 11$ we see that it is not necessarily better to decrease the dimension. In *passing, for* $\varepsilon = 10^{-3}$ *and* $n = 11$ *we had*

$$[z] = \begin{pmatrix} [0.000000000000000E+000, 0.000000000000000E+000] \\ [0.000000000000000E+000, 0.000000000000000E+000] \\ [8.583546723631532E-003, 9.324597746631119E-003] \\ [0.000000000000000E+000, 0.000000000000000E+000] \\ [1.861234190085976E-002, 1.976279660412602E-002] \\ [0.000000000000000E+000, 0.000000000000000E+000] \\ [2.874216690277989E-002, 3.009733484473740E-002] \\ [0.000000000000000E+000, 0.000000000000000E+000] \\ [3.892222017236857E-002, 4.037974652487774E-002] \\ [0.000000000000000E+000, 0.000000000000000E+000] \\ [4.912734164899482E-002, 5.063604738311572E-002] \end{pmatrix}.$$

For $n = 11$ *we decreased* ε *and got*

ε	10^{-3}	10^{-5}	10^{-7}	10^{-9}	10^{-11}	10^{-13}	10^{-15}
$\max_i\{\overline{z}_i - \underline{z}_i\}$	$1.5 \cdot 10^{-3}$	$1.5 \cdot 10^{-5}$	$1.5 \cdot 10^{-7}$	$1.5 \cdot 10^{-9}$	$1.5 \cdot 10^{-11}$	$1.5 \cdot 10^{-13}$	$2.9 \cdot 10^{-15}$

If $\varepsilon = 0$*, using PASCAL-XSC on a computer the entries of* $[q]$ *are still intervals and during the elimination process the entries are rounded outwardly with machine precision. We wonder for which dimension the interval Lemke algorithm fails. The smallest* n*, for which this happened, was* 107*.*

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