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On Green's Theorem

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In this note we give a proof of Green's Theorem, sometimes also called Gauß's Theorem for the plane. If Green's Theorem is presented in beginners' courses on calculus it is often proved for regions enclosed by the graphs of two functions, and for more general regions a decomposition argument is used, see [1] 5.15, or [2] 8.1. The following proof uses polar coordinates, and therefore works directly for circles, ellipses and other convex regions, but also for certain nonconvex regions such as starlike regions, for example.

Let $R : [a, b] \to (0, \infty)$ be continuously differentiable such that $R(a) = R(b)$, $\varphi : [a, b] \to [0, 2\pi]$ continuously differentiable, increasing and onto, and let B be the set in the plane described by

$$B = \{(x, y) | x = r \cos \varphi(t), y = r \sin \varphi(t), 0 \le r \le R(t), a \le t \le b\}.$$

Since $R(a) = R(b)$, ∂B is contained in the closed curve Γ, parametrized by

$$\gamma(t) = (R(t) \cos \varphi(t), R(t) \sin \varphi(t)) \quad (t \in [a, b]).$$

Note that using the function φ allows B to have edges.

Theorem. If $u, v : B \to \mathbb{R}$ are continuously differentiable, then

$$\int_B (u_x(x, y) + v_y(x, y)) d(x, y) = \int_\Gamma -v(x, y)dx + u(x, y)dy.$$

Proof. Set $\eta(r, t) = u(r \cos \varphi(t), r \sin \varphi(t))$. Differentiating with respect to r and t and eliminating u_y from the resulting equations leads to

$$(1) \quad u_x(r \cos \varphi(t), r \sin \varphi(t)) r\varphi'(t) = r\varphi'(t) \cos \varphi(t)\ \eta_r(r, t) - \sin \varphi(t)\ \eta_t(r, t).$$

In turn, by change of variables, equation (1), and partial integration with respect to r we get

$$\int_B u_x(x, y) d(x, y) = \int_a^b \int_0^{R(t)} u_x(r \cos \varphi(t), r \sin \varphi(t)) r\varphi'(t) drdt$$

$$= \int_a^b \int_0^{R(t)} (r\varphi'(t) \cos \varphi(t) \eta_r(r, t) - \sin \varphi(t) \eta_t(r, t)) drdt$$

$$= \int_a^b R(t)\varphi'(t) \cos \varphi(t)\ \eta(R(t), t) dt$$

$$- \int_a^b \int_0^{R(t)} (\varphi'(t) \cos \varphi(t)\ \eta(r, t) + \sin \varphi(t)\ \eta_t(r, t)) drdt = (*).$$

Using

$$\frac{d}{dt}\Big(\int_0^{R(t)} \sin\varphi(t)\ \eta(r,t)dr\Big)$$

$$= R'(t)\sin\varphi(t)\ \eta(R(t),t) + \int_0^{R(t)} \big(\cos\varphi(t)\ \varphi'(t)\eta(r,t) + \sin\varphi(t)\ \eta_t(r,t)\big)dr$$

we may continue

$$(*) = \int_a^b \Big\{\varphi'(t)R(t)\cos\varphi(t)\ \eta(R(t),t) + R'(t)\sin\varphi(t)\ \eta(R(t),t)$$

$$-\frac{d}{dt}\big(\sin\varphi(t)\int_0^{R(t)} \eta(r,t)dr\big)\Big\}dt = (**);$$

but

$$\int_a^b \Big\{\frac{d}{dt}\big(\sin\varphi(t)\int_0^{R(t)} \eta(r,t)dr\big)\Big\}dt = 0,$$

so

$$(**) = \int_a^b \eta(R(t),t)\big(R(t)\sin\varphi(t)\big)' dt$$

$$= \int_\Gamma u(x,y)dy.$$

Analoguously (or by considering $u(x,y) := v(y,x)$),

$$\int_B v_y(x,y)d(x,y) = -\int_\Gamma v(x,y)dx,$$

and the theorem is proved.
We remark that a similar proof works in higher dimensions if B is described in an obvious way by means of spherical coordinates.

References

[1] Apostol, Tom M.: Calculus. Vol. II: Calculus of several variables with applications to probability and vector analysis. New York and London: Blaisdell Publishing Company, a division of Random House. XV, 525 p. (1962).

[2] Walter, Wolfgang: Analysis 2. Berlin, Heidelberg und New York: Springer, 397 p. (1994).

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