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Robust estimates for the approximation of the dynamic consolidation problem

Martin Sauter and Christian Wieners

Abstract. We consider stable discretizations in time and space for the linear dynamic consolidation problem describing the wave propagation in a porous solid skeleton which is fully saturated with an incompressible fluid. Introducing the hydrostatic pressure, the flow problem is described by Darcy's law. In particular, we discuss the case of nearly impermeable solids which requires inf-sup stable discretizations in space for the limiting saddle point problem. Together with an (implicit) Newmark discretization in time we derive convergence estimates for the fully discrete scheme which are robust with respect to the coupling parameter of fluid and solid.

1. Introduction

We consider the following system of partial differential equations

$$\rho\,\ddot{\mathbf{u}}(t) - \operatorname{div}\big(\boldsymbol{\sigma}(t) - p(t)\boldsymbol{I}\big) = \rho\,\boldsymbol{b}(t)\,, \tag{1a}$$

$$\operatorname{div}\big(\dot{\boldsymbol{u}}(t) - \kappa\,\nabla p(t)\big) = 0\,, \tag{1b}$$

in $[0,T]\times\Omega\subset\mathbb{R}\times\mathbb{R}^d$ extending Biot's quasi-static consolidation problem by an inertial term (the classical quasi-static model is obtained by neglecting the acceleration term). The consolidation problem describes the dynamics of a system composed of a materially incompressible porous solid which is saturated by an incompressible viscous pore fluid. After imposing a constitutive equation relating the Cauchy stress and the solid displacement, the primary variables are the displacement vector of the solid skeleton $\boldsymbol{u}$ and the hydrostatic pressure p. In the above equations, we already inserted Darcy's law, describing the difference of the pore fluid velocity and the skeleton velocity (the seepage velocity $\boldsymbol{w}$), which is obtained by $\boldsymbol{w}(t) = -\kappa\,\nabla p(t)$. Here, κ is the hydraulic conductivity. Phenomenological, the second equation corresponds to the conservation of fluid content where $\operatorname{div}\dot{\boldsymbol{u}}$ represents the additional fluid content due to the dilation of the structure. The first equation describes the conservation of momentum. Thereby, $\boldsymbol{\sigma}(t) - p(t)\boldsymbol{I}$ is the effective stress, which is decomposed into the Cauchy stress $\boldsymbol{\sigma}(t)$ of the solid structure and the additional stress due to the fluid pressure given by $-p(t)\boldsymbol{I}$. Finally, $\boldsymbol{b}(t)$ denotes the volume forces (e. g., gravity forces).

The qualitative analysis of (1) is well established and a governing framework for the consolidation problem is now given by the the system of poro-elasticity. We refer to **[23]** and the references therein for a recent survey article on related problems and links to other applications, e.g. thermo-elasticity. It also addresses extensions in regard to plastic deformations of the solid skeleton. Finite element approximations of the quasi-static problem have been considered by various authors, see e.g. **[12, 14]** and the references therein. Approximations of a different fully dynamic consolidation problem have been considered by Santos et. al. **[18, 19]**.

In this work, we focus our attention on the behavior of the system for varying $\kappa = \frac{k}{\eta} > 0$ and κ might as well be tensor-valued. In the above relation, k represents the permeability of the pores and η the viscosity of the fluid. Of particular interest is the behavior for very small κ. This can

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be interpreted in two ways: either the pores are almost impermeable or the fluid is highly viscous. Freely spoken, both cases imply that the fluid does no longer flow through the porous solid, and the two-phase problem reduces to a single-phase problem for the displacement of the solid-fluid mixture. Then, p can be interpreted as the Lagrange multiplier of the constraint $\operatorname{div} \dot{\boldsymbol{u}} = 0$. However, we do not consider the case of vanishing κ.

In particular, we aim for robust discretizations in time and space for the full problem. Therefore, we combine two methods which are robust for its own: in time, the Newmark scheme is applied (which is unconditionally stable for a suitable choice of parameters) and in space an inf-sup stable saddle point discretization provides robust estimates. Our main result in Section 5 proves convergence estimates in space and time independent of κ and without mesh-dependent restrictions on Δt. Numerically, this was already observed in [**21**].

Here, we restrict ourselves to the basic linear model. The full numerical analysis of convergence in time and space for nonlinear models is more involved, see, e. g., [**4**] for a first order method in time applied to a dynamic model in finite elasticity. The extension of the numerical analysis to other nonlinear applications, e. g., models in poro-plasticity, is not done so far (see [**6, 25**] and the references therein for numerical simulations).

The paper is organized as follows. We start with a precise formulation of the dynamic consolidation model, and we review the main properties of the problem. Then, in Section 3 the Newmark scheme is introduced, its properties and some variants are discussed. In Section 4, based on an inf-sup stable discretization in space, a robust semi-discrete estimate is derived, and finally, in Section 5 estimates for the full discretization are considered. We close with some remarks on discrete energy conservation properties of the scheme.

2. The continuous problem

In this section we briefly recall the full equations of the consolidation problem, and we summarize the basic analytical properties. It does not contain new results; it serves for the clear definition of the problem and for the introduction of the notation.

2.1. Equations and boundary conditions. We assume that $[0,T] \subset \mathbb{R}$ is a finite time interval and $\Omega \subset \mathbb{R}^d$ is a bounded domain with Lipschitz boundary $\Gamma = \partial\Omega$. We consider disjoint boundary decompositions $\Gamma = \Gamma_c \cup \Gamma_t$ corresponding to Dirichlet (*c*lamped) and Neumann (*t*raction) boundary conditions for the solid, and for the pressure $\Gamma = \Gamma_d \cup \Gamma_f$, corresponding to Dirichlet (*d*rainage) and Neumann (*f*lux) boundary conditions. We assume that the Dirichlet parts have positive measure $\operatorname{meas}_{d-1}(\Gamma_c), \operatorname{meas}_{d-1}(\Gamma_d)$. Depending on the boundary, we use the spaces $\boldsymbol{X} = \{\boldsymbol{w} \in H^1(\Omega, \mathbb{R}^d) : \boldsymbol{w} = 0 \text{ on } \Gamma_c\}$ for the displacements and $Q = \{q \in H^1(\Omega) : q = 0 \text{ on } \Gamma_d\}$ for the pressure. Furthermore, define $\boldsymbol{H} = L_2(\Omega, \mathbb{R}^d)$ and $H = L_2(\Omega, \mathbb{R})$.

In Ω, the following data are given: a density distribution $\rho \in L_\infty(\Omega, \mathbb{R})$ with $\rho(x) \geq \rho_0 > 0$ a.e. in Ω, a permeability tensor $\kappa \in L_\infty(\Omega, \mathbb{R}^{d\times d})$ which is uniformly positive definite, i. e., $\xi^T \kappa(x) \xi \geq \kappa_0 |\xi|^2$ for $\xi \in \mathbb{R}^d$, and a volume force $\boldsymbol{b} \in L_2(0, T; \boldsymbol{H})$.

Finally, the system (1) needs to be closed by a constitutive equation describing the stress-strain relation. In the following, we only consider the linear and isotropic case where the relation between the Cauchy stress $\boldsymbol{\sigma}$ and the linearized Green-St. Venant strain $\boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T)$ is given by Hooke's law

$$\boldsymbol{\sigma}(t) = \mathbb{C} : \boldsymbol{\varepsilon}(\boldsymbol{u}(t)) = 2\mu\, \boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \lambda \operatorname{tr} \boldsymbol{\varepsilon}(\boldsymbol{u}(t)) \boldsymbol{I}\,. \tag{2}$$

Here, $\mathbb{C}$ denotes the fourth order elasticity tensor and μ and λ are the Lamé constants.

Now, the full system is given by the differential equations and the boundary conditions

$$\begin{aligned} \rho\, \ddot{\boldsymbol{u}}(t) - \operatorname{div}\big(\boldsymbol{\sigma}(t) - p(t)\boldsymbol{I}\big) - \rho\, \boldsymbol{b}(t) &= 0 \quad \text{in } \Omega\,, && (3a)\\ \operatorname{div}\big(\dot{\boldsymbol{u}}(t) - \kappa\, \nabla p(t)\big) &= 0 \quad \text{in } \Omega\,, && (3b)\\ p(t) &= 0 \quad \text{on } \Gamma_d\,, && (3c)\\ \kappa \nabla p(t)\cdot \boldsymbol{n} &= 0 \quad \text{on } \Gamma_f\,, && (3d)\\ \boldsymbol{u}(t) &= 0 \quad \text{on } \Gamma_c\,, && (3e)\\ \big(\boldsymbol{\sigma}(t) - p(t)\boldsymbol{I}\big)\boldsymbol{n} &= 0 \quad \text{on } \Gamma_t\,, && (3f) \end{aligned}$$

subject to initial values

$$\boldsymbol{u}(0) = \boldsymbol{u}_0 \qquad \text{and} \qquad \dot{\boldsymbol{u}}(0) = \boldsymbol{v}_0. \tag{4}$$

2.2. Norms and operators. We define the following bilinear forms and operators:

$$\begin{aligned} &A : \boldsymbol{X} \to \boldsymbol{X}'\,, && \langle A\boldsymbol{u}, \boldsymbol{w}\rangle_{\boldsymbol{X}'\times \boldsymbol{X}} = a(\boldsymbol{u},\boldsymbol{w}) := \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) : \mathbb{C} : \boldsymbol{\varepsilon}(\boldsymbol{w})\, dx\,,\\ &B : \boldsymbol{X} \to Q'\,, && \langle B\boldsymbol{u}, p\rangle_{Q'\times Q} = b(\boldsymbol{u},p) := -\int_\Omega p \operatorname{div} \boldsymbol{u}\, dx\,,\\ &B' : Q \to \boldsymbol{X}'\,, && \langle B'p, \boldsymbol{u}\rangle_{\boldsymbol{X}'\times \boldsymbol{X}} = b(\boldsymbol{u},p)\,,\\ &C_\kappa : Q \to Q'\,, && \langle C_\kappa p, q\rangle_{Q'\times Q} = c_\kappa(p,q) := \int_\Omega (\kappa\, \nabla p)\cdot \nabla q\, dx\,,\\ &M : \boldsymbol{H} \to \boldsymbol{H}'\,, && \langle M\boldsymbol{u}, \boldsymbol{w}\rangle_{\boldsymbol{H}'\times \boldsymbol{H}} = m(\boldsymbol{u},\boldsymbol{w}) := \int_\Omega \rho\, \boldsymbol{u}\cdot \boldsymbol{w}\, dx\,. \end{aligned}$$

The bilinear form $a(\cdot,\cdot)$ defines an inner product on $\boldsymbol{X}$ due to Korn's second inequality. Since $a(\cdot,\cdot)$ is symmetric, the operator A is self-adjoint. We define a norm on $\boldsymbol{X}$ by setting

$$\|\boldsymbol{u}\|_{\boldsymbol{X}} = \sqrt{a(\boldsymbol{u},\boldsymbol{u})}\,.$$

Similarly, the operator C_κ is self-adjoint and defines an inner product on Q, but the associated norm is not appropriate in our context since this would result in parameter dependent estimates. Therefore, we define the parameter dependent norm

$$\|p\|_Q = \sqrt{\|p\|_\kappa^2 + \|p\|_S^2}\,, \tag{5}$$

with $\|p\|_\kappa = \sqrt{c_\kappa(p,p)}$ and $\|p\|_S = \|A^{-1}B'p\|_{\boldsymbol{X}}$, which now allows for the robust estimate

$$\sup_{\boldsymbol{0}\neq \boldsymbol{u}\in \boldsymbol{X}} \frac{b(\boldsymbol{u},p)}{\|\boldsymbol{u}\|_{\boldsymbol{X}}} = \|B'p\|_{\boldsymbol{X}'} = \|A^{-1}B'p\|_{\boldsymbol{X}} \le \|p\|_Q\,. \tag{6}$$

Finally, we use the weighted norm $\|\boldsymbol{u}\|_{\boldsymbol{H}} = \sqrt{m(\boldsymbol{u},\boldsymbol{u})}$ on $\boldsymbol{H}$. Note that for $\boldsymbol{u}\in \boldsymbol{X}$, we can estimate

$$\|\boldsymbol{u}\|_{\boldsymbol{H}} \le C_\rho\, \|\boldsymbol{u}\|_{\boldsymbol{X}}$$

2.3. Weak formulation. The finite element discretization of (3) is based on the weak formulation: find $\boldsymbol{u}\in C^2(0,T;\boldsymbol{V})$ and $p(t)\in Q$ such that

$$\begin{aligned} m(\ddot{\boldsymbol{u}}(t),\boldsymbol{w}) + a(\boldsymbol{u}(t),\boldsymbol{w}) + b(\boldsymbol{w},p(t)) &= m(\boldsymbol{b}(t),\boldsymbol{w}) && \boldsymbol{w}\in \boldsymbol{X}\,, && (7a)\\ b(\dot{\boldsymbol{u}}(t),q) - c_\kappa(p(t),q) &= 0 && q\in Q\,. && (7b) \end{aligned}$$

Using the introduced notation, this can be rewritten as an operator equation in $\boldsymbol{X}'\times Q'$:

$$\begin{aligned} M\ddot{\boldsymbol{u}}(t) + A\boldsymbol{u}(t) + B'p(t) &= \ell(t)\,, && (8a)\\ B\dot{\boldsymbol{u}}(t) - C_\kappa p(t) &= 0\,, && (8b) \end{aligned}$$

where $\ell(t)\in \boldsymbol{X}'$ is defined by $\langle \ell(t),\boldsymbol{w}\rangle = \int_\Omega \rho\, \boldsymbol{b}(t)\cdot \boldsymbol{w}\, dx$.

2.4. Existence and uniqueness of a solution. Eliminating the pressure in (8) by the second equation gives $p(t) = C_\kappa^{-1} B\dot{\boldsymbol{u}}(t)$, and substitution into the first equation yields the wave equation with damping

$$M\ddot{\boldsymbol{u}}(t) + D_\kappa \dot{\boldsymbol{u}}(t) + A\boldsymbol{u}(t) = \ell(t)\,, \tag{9}$$

where $D_\kappa = B' C_\kappa^{-1} B\colon \boldsymbol{X} \longrightarrow \boldsymbol{X}'$. Since C_κ is strongly monotone, so is the inverse C_κ^{-1} and consequently, also D_κ is monotone: $\langle D_\kappa \boldsymbol{v}, \boldsymbol{v}\rangle \geq 0$. Standard analysis applies to this problem (see, e. g., [**22**, Prop. I.6.1] for a proof in a modified setting with homogeneous right-hand side).

THEOREM 1. *For $\ell \in C^0(0,T;\boldsymbol{X}')$ the second order differential equation* (9) *with initial values* (4) *has a unique solution $\boldsymbol{u} \in C^2(0,T;\boldsymbol{H}) \cap C^1(0,T;\boldsymbol{X})$.*

As a consequence, we obtain $p \in C^1(0,T;Q)$. In case of smooth right-hand sides and initial values, higher regularity can be studied, cf. [**7**, Sect. 7.2.3].

Note that the operator D_κ is not bounded independently of κ, so that the pressure elimination is not suitable for a robust discretization.

2.5. Monotonicity. Introducing the velocity $\boldsymbol{v}(t) = \dot{\boldsymbol{u}}(t)$ we consider an extended system. We define the product space

$$\mathcal{W} = \boldsymbol{X} \times \boldsymbol{X} \times Q$$

and the operators $\mathcal{A}, \mathcal{B}\colon \mathcal{W} \longrightarrow \mathcal{W}'$ by $\mathcal{A}[\boldsymbol{u},\boldsymbol{v},p] = (-A\boldsymbol{v}, A\boldsymbol{u}+B'p, -B\boldsymbol{v}+C_\kappa p)$ and by $\mathcal{B}[\boldsymbol{u},\boldsymbol{v},p] = (A\boldsymbol{u}, M\boldsymbol{v}, 0)$, and the linear form $\mathcal{L} \in \mathcal{W}'$ by $\mathcal{L}(t)[\boldsymbol{u},\boldsymbol{v},p] = \langle \ell(t), \boldsymbol{v}\rangle$.

Together with $A\boldsymbol{v}(t) = A\dot{\boldsymbol{u}}(t)$ this rewrites (8) as an implicit evolution equation $\mathcal{W}'$

$$\partial_t\big(\mathcal{B}\,w(t)\big) + \mathcal{A}\,w(t) = \mathcal{L}(t)\,, \qquad \mathcal{B}w(0) = (A\boldsymbol{u}_0, M\boldsymbol{v}_0, 0)\,, \tag{10}$$

for $w(t) := (\boldsymbol{u}(t), \boldsymbol{v}(t), p(t))$.

LEMMA 2. *The operators $\mathcal{A}$ and $\mathcal{B}$ are monotone.*

PROOF. This follows from $\langle \mathcal{B}w, w\rangle = a(\boldsymbol{u},\boldsymbol{u}) + (\boldsymbol{v},\boldsymbol{v})_{\boldsymbol{H}} \geq 0$ and

$$\langle \mathcal{A}w, w\rangle = c_\kappa(p,p) - b(\boldsymbol{v},p) + b(\boldsymbol{v},p) + a(\boldsymbol{u},\boldsymbol{v}) - a(\boldsymbol{v},\boldsymbol{u}) = c_\kappa(p,p) \geq 0,$$

due to the symmetry of $a(\cdot,\cdot)$ and the positivity of $c_\kappa(\cdot,\cdot)$. □

Note that $\mathcal{B}$ is singular, so that this has the structure of a differential algebraic equation. Thus, the application of semi-group theory requires a suitable factorization technique [**22**, Ch. IV]. This technique is applied in [**24**], where existence and uniqueness of a closely related system is investigated. In the following, monotonicity plays a major role in the estimates.

2.6. Energy estimates. The kinetic and potential energy of the wave equation

$$\mathcal{E}_{\text{wave}}(\boldsymbol{u},\boldsymbol{v}) = \tfrac{1}{2}m(\boldsymbol{v},\boldsymbol{v}) + \tfrac{1}{2}a(\boldsymbol{u},\boldsymbol{u})$$

is extended by the damping term, which defines the free energy by

$$\mathcal{E}\big(\boldsymbol{u},p,t\big) = \mathcal{E}_{\text{wave}}(\boldsymbol{u}(t),\dot{\boldsymbol{u}}(t)) + \int_0^t c_\kappa(p(s),p(s))\,ds\,,$$

and the external work is given by

$$\mathcal{E}_{\text{ext}}\big(\boldsymbol{v},t\big) = \int_0^t \langle \ell(s), \boldsymbol{v}(s)\rangle\,ds\,.$$

The initial energy is denoted by $\mathcal{E}_0 = \mathcal{E}_{\text{wave}}(\boldsymbol{u}_0,\boldsymbol{v}_0)$.

THEOREM 3. *Assume* $\boldsymbol{b} \in L^2(0,T;\boldsymbol{H})$, *and let* $(\boldsymbol{u},p) \in H^1(0,T;\boldsymbol{X}) \times L_2(0,T;Q)$ *be a solution of* (7) *for initial values* $(\boldsymbol{u}_0,\boldsymbol{v}_0) \in \boldsymbol{X} \times \boldsymbol{X}$. *Then we have energy conservation in the form*

$$\mathcal{E}(\boldsymbol{u},p,t) = \mathcal{E}_0 + \mathcal{E}_{\text{ext}}(\dot{\boldsymbol{u}},t), \qquad t \in [0,T]\,. \tag{11a}$$

and an a priori estimate

$$\mathcal{E}(\boldsymbol{u},p,t) \le C(T)\Big(\mathcal{E}_0 + \|\boldsymbol{b}\|^2_{L^2(0,T;\boldsymbol{H})}\Big)\,, \tag{11b}$$

where $C(T)$ *is a constant only depending on time* T.

PROOF. Inserting $\boldsymbol{w} = \dot{\boldsymbol{u}}(t)$ and $q = p(t)$ in (7) yields $b(\dot{\boldsymbol{u}}(t),p(t)) = c_\kappa(p(t),p(t))$ and

$$m(\ddot{\boldsymbol{u}}(t),\dot{\boldsymbol{u}}(t)) + a(\boldsymbol{u}(t),\dot{\boldsymbol{u}}(t)) + c_\kappa(p(t),p(t)) = \langle \ell(t),\dot{\boldsymbol{u}}(t)\rangle\,.$$

This gives (11a) by

$$\begin{aligned}\mathcal{E}_{\text{wave}}(\boldsymbol{u}(t),\boldsymbol{v}(t)) - \mathcal{E}_{\text{wave}}(\boldsymbol{u}(0),\boldsymbol{v}(0)) &= \int_0^t \Big(m(\ddot{\boldsymbol{u}}(s),\dot{\boldsymbol{u}}(s)) + a(\boldsymbol{u}(s),\dot{\boldsymbol{u}}(s))\Big)\,ds\\ &= \int_0^t \Big(\langle \ell(s),\dot{\boldsymbol{u}}(s)\rangle - c_\kappa(p(s),p(s))\Big)\,ds\ .\end{aligned}$$

We obtain from Young's inequality

$$\begin{aligned}\mathcal{E}_{\text{ext}}(\dot{\boldsymbol{u}},t) = \int_0^t m(\boldsymbol{b}(s),\boldsymbol{v}(s))\,ds &\le \int_0^t \|\boldsymbol{b}(s)\|_{\boldsymbol{H}}\|\dot{\boldsymbol{u}}(s)\|_{\boldsymbol{H}}\,ds\\ &\le \tfrac12\int_0^t \|\boldsymbol{b}(s)\|^2_{\boldsymbol{H}}\,ds + \tfrac12\int_0^t \|\dot{\boldsymbol{u}}(s)\|^2_{\boldsymbol{H}}\,ds\ ,\end{aligned}$$

which gives

$$\mathcal{E}(\boldsymbol{u},p,t) \le \mathcal{E}_0 + \tfrac12\int_0^t \|\boldsymbol{b}(s)\|^2_{\boldsymbol{H}}\,ds + \int_0^t \mathcal{E}(\boldsymbol{u},p,s)\,ds\,.$$

Then, (11b) is directly obtained from Gronwall's Lemma [**11**, Lem. A.4.12]. □

As an immediate consequence in case of vanishing external work ($\ell \equiv 0$), it follows that energy is conserved (i. e., $\mathcal{E}(\boldsymbol{u},p,t) \equiv \mathcal{E}_0$), and the wave energy is dissipative in the form

$$\frac{\partial}{\partial t}\mathcal{E}_{\text{wave}}(\boldsymbol{u}(t),\dot{\boldsymbol{u}}(t)) = -c_\kappa(p(t),p(t)) \le 0\,. \tag{12}$$

2.7. Lagrange principle. Finally, we introduce the corresponding Lagrange principle. For the generalized coordinates $\boldsymbol{q} = (\boldsymbol{u},p)$ define the Lagrangian by

$$L(\boldsymbol{q},\dot{\boldsymbol{q}},t) = \tfrac12\langle M\dot{\boldsymbol{u}}(t),\dot{\boldsymbol{u}}(t)\rangle - \tfrac12\langle A\boldsymbol{u}(t),\boldsymbol{u}(t)\rangle + \int_0^t \langle C_\kappa p(s),p(s)\rangle\,ds - \langle B\boldsymbol{u}(t),p(t)\rangle + \langle \ell(t),\dot{\boldsymbol{u}}(t)\rangle\ .$$

The corresponding Euler-Lagrange equation (obtained by the first variation of the action integral) $\dfrac{\partial}{\partial t}\dfrac{\partial L}{\partial \dot{\boldsymbol{q}}} = \dfrac{\partial L}{\partial \boldsymbol{q}}$ yields the equation (8) in integrated form

$$\begin{aligned}M\ddot{\boldsymbol{u}}(t) + A\boldsymbol{u}(t) + B'p(t) &= \ell(t)\,,\\ B\boldsymbol{u}(t) - \int_0^t C_\kappa p(s)\,ds &= 0\,.\end{aligned}$$

3. Discretization in time

In many cases physical restrictions require very small time steps for hyperbolic equations, and then explicit time stepping methods a favorable. Here, we consider applications in solid mechanics where in particular stability requirements would lead—in case of fine meshes—to unrealistic small time steps. Thus, one aims for time discretizations which are unconditionally stable. This can be easily obtained by enlarging the system by an equation for the velocity and the application of stable implicit Runge-Kutta schemes. However, the realization of such schemes is numerically expensive, and therefore in many cases Newmark schemes are applied: their realization has the same structure as implicit schemes for the quasi-static case, and it is quite easy to extend quasi-static applications to the full dynamic problem.

3.1. The Newmark scheme. We consider the Newmark discretization in time for the system (7), see [**17**, Ch. 6.5]. Therefore, let $0 = t_0 < t_1 < t_2 < \cdots t_N = T$ be a time series, and let $\Delta t_n = t_n - t_{n-1}$ be the time increment. Set $\ell^n = \ell(t_n)$.

Starting with $\boldsymbol{u}^0 = \boldsymbol{u}_0$ and $\boldsymbol{v}^0 = \boldsymbol{v}_0$ we compute for $n = 0, 1, 2, ..., N$ the acceleration vector $\boldsymbol{a}^n$, the velocity vector $\boldsymbol{v}^n$, the displacement vector $\boldsymbol{u}^n$, and the pressure p^n satisfying

$$m(\boldsymbol{a}^n, \boldsymbol{w}) + a(\boldsymbol{u}^n, \boldsymbol{w}) + b(\boldsymbol{w}, p^n) = \langle \ell^n, \boldsymbol{w} \rangle\,, \qquad \boldsymbol{w} \in \boldsymbol{X}\,, \tag{13a}$$

$$b(\boldsymbol{v}^n, q) - c_\kappa(p^n, q) = 0\,, \qquad q \in Q\,, \tag{13b}$$

and for $n > 0$

$$\boldsymbol{v}^n = \boldsymbol{v}^{n-1} + \frac{\Delta t_n}{2}\Big(\boldsymbol{a}^n + \boldsymbol{a}^{n-1}\Big)\,, \tag{14a}$$

$$\boldsymbol{u}^n = \boldsymbol{u}^{n-1} + \Delta t_n \boldsymbol{v}^{n-1} + \frac{(\Delta t_n)^2}{4}\Big(\boldsymbol{a}^n + \boldsymbol{a}^{n-1}\Big)\,. \tag{14b}$$

Algorithmically, one can solve a saddle point problem for $(\boldsymbol{u}^n, p^n)$

$$\frac{4}{(\Delta t_n)^2} m(\boldsymbol{u}^n, \boldsymbol{w}) + a(\boldsymbol{u}^n, \boldsymbol{w}) + b(\boldsymbol{w}, p^n) = \langle \boldsymbol{r}^n, \boldsymbol{w} \rangle\ , \qquad \boldsymbol{w} \in \boldsymbol{X}\,,$$

$$b(\boldsymbol{u}^n, q) - \frac{\Delta t_n}{2}\, c_\kappa(p^n, q) = \langle r^n, q \rangle\ , \qquad q \in Q\,,$$

with

$$\langle \boldsymbol{r}^n, \boldsymbol{w} \rangle = \langle \ell^n, \boldsymbol{w} \rangle + \frac{4}{(\Delta t_n)^2} m(\boldsymbol{u}^{n-1}, \boldsymbol{w}) + \frac{4}{\Delta t_n} m(\boldsymbol{v}^{n-1}, \boldsymbol{w}) + m(\boldsymbol{a}^{n-1}, \boldsymbol{w})\,, \qquad \boldsymbol{w} \in \boldsymbol{X}\,,$$

$$\langle r^n, q \rangle = b(\boldsymbol{u}^{n-1}, q) + \frac{\Delta t_n}{2}\, b(\boldsymbol{v}^{n-1}, q)\,, \qquad q \in Q\,,$$

and then recover $\boldsymbol{v}^n$ and $\boldsymbol{a}^n$ from

$$\boldsymbol{v}^n = \frac{2}{\Delta t_n}\Big(\boldsymbol{u}^n - \boldsymbol{u}^{n-1}\Big) - \boldsymbol{v}^{n-1}\,, \tag{17a}$$

$$\boldsymbol{a}^n = \frac{4}{(\Delta t_n)^2}\Big(\boldsymbol{u}^n - \boldsymbol{u}^{n-1}\Big) - \frac{4}{\Delta t_n}\boldsymbol{v}^{n-1} - \boldsymbol{a}^{n-1}\,. \tag{17b}$$

3.2. The Newmark discretization as a Nyström method. For the system

$$\dot{\boldsymbol{u}}(t) = \boldsymbol{v}(t)\,, \qquad \dot{\boldsymbol{v}}(t) = \boldsymbol{a}(t, \boldsymbol{u}, \dot{\boldsymbol{u}})\,,$$

an s-stage Nyström method is given by

$$\boldsymbol{k}_i = \boldsymbol{a}\Big(t_{n-1} + c_i\Delta t_n,\, \boldsymbol{u}^{n-1} + c_i\Delta t_n \boldsymbol{v}^{n-1} + (\Delta t_n)^2 \sum_{j=1}^{s} \overline{a}_{ij}\boldsymbol{k}_j,\, \boldsymbol{v}^{n-1} + \Delta t_n \sum_{j=1}^{s} a_{ij}\boldsymbol{k}_j\Big)\,,$$

$$\boldsymbol{u}^n = \boldsymbol{u}^{n-1} + \Delta t_n \boldsymbol{v}^{n-1} + (\Delta t_n)^2 \sum_{i=1}^{s} \overline{b}_i \boldsymbol{k}_i\,, \qquad \boldsymbol{v}^n = \boldsymbol{v}^{n-1} + \Delta t_n \sum_{i=1}^{s} b_i \boldsymbol{k}_i\,.$$

Following [**9**, Ch. II.14], a Nyström method is equivalent to a Runge-Kutta method, if the coefficients $\overline{a}_{ij}$ and $\overline{b}_j$ fulfill the relations

$$\overline{a}_{ij} = \sum_{k=1}^{s} a_{ik} a_{kj} \,, \qquad \overline{b}_i = \sum_{k=1} = b_k a_{ki} \,. \tag{18}$$

The Newmark method as it is defined above is a Nyström method ($s = 2$ stages) with

$$c = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \overline{a} = \begin{bmatrix} 0 & 0 \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}, \quad a = \begin{bmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \quad \overline{b} = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \end{bmatrix}, \quad b = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}.$$

A short computation evaluating (18) shows that this corresponds to the trapezoidal rule. This equivalence is useful for characterizing the Newmark method since all properties can be traced back to the trapezoidal rule. In particular, we can directly conclude that the Newmark method is A-stable and second order accurate.

The trapezoidal rule is symmetric in the sense of [**8**, Ch. 5], since the coefficients a_{ij}, b_i fulfill

$$a_{s+1-i,s+1-j} + a_{ij} = b_j \qquad \text{for all} \;\; 1 \leq i,j \leq s \,.$$

Moreover, in the context of linear problems, the concepts of symmetry and symplecticity coincide [**8**, Ch. VI.4.2], and hence, quadratic invariants are conserved. In particular—in regard to the linear dynamic consolidation problem—this implies the energy conservation by the Newmark method. This fact is also reflected in the close relationship of the trapezoidal rule with the midpoint rule. For linear problems, both methods are equivalent, and since the midpoint rule is symplectic, so is the trapezoidal rule for linear problems. The trapezoidal rule is also referred to as conjugate symplectic to the midpoint rule [**20**, Ch. 14.3], since it is the result of a change of variables in the midpoint rule.

Remark 4. *The general form of the Newmark method, as given in* [**17**, Ch. 6.5]*, is parameterized by β and γ such that*

$$\begin{aligned} \boldsymbol{v}^n &= \boldsymbol{v}^{n-1} + \Delta t_n \Big(\gamma \boldsymbol{a}^n + (\frac{1}{2} - \gamma) \boldsymbol{a}^{n-1} \Big) \,, \\ \boldsymbol{u}^n &= \boldsymbol{u}^{n-1} + \Delta t_n \boldsymbol{v}^{n-1} + (\Delta t_n)^2 \Big(\beta \boldsymbol{a}^n + (\frac{1}{4} - \beta) \boldsymbol{a}^{n-1} \Big) \,. \end{aligned}$$

The equivalent Nyström method is given by

$$c = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \overline{a} = \begin{bmatrix} 0 & 0 \\ \frac{1}{2} - \beta & \beta \end{bmatrix}, \quad a = \begin{bmatrix} 0 & 0 \\ 1-\gamma & \gamma \end{bmatrix}, \quad \overline{b} = \begin{bmatrix} \frac{1}{2} - \beta \\ \beta \end{bmatrix}, \quad b = \begin{bmatrix} 1-\gamma \\ \gamma \end{bmatrix}.$$

Suitable combinations of the parameters allow for controllable stability properties, and certain methods are retained in the general formulation, e.g. for $\gamma = \frac{1}{4}$ and $\beta = 0$, the leap-frog (or Störmer-Verlet) algorithm is recovered.

Since our fully discrete estimate in the next section is restricted to the basic Newmark scheme ($\gamma = \frac{1}{2}$, $\beta = \frac{1}{4}$), we do not analyse the variants here.

Remark 5. *In the engineering literature, the Newmark method can be traced back to* [**15**]*. Based on the Newmark method, further extensions were proposed: the HHT-α method of Hilber et. al.* [**10**] *and the WBZ-α method of Wood et. al.* [**26**] *were combined to the generalized-α method by Chung and Hulbert* [**5**]*, yielding a four parameter method allowing for adjustable numerical dissipation while retaining second order accuracy and unconditional stability in the linear regime. As special cases, the trapezoidal rule and the midpoint rule are recovered.*

4. Discretization in space

Since we aim for robust estimates with respect to κ, a full control of the acceleration and the pressure is required. This is obtained by employing an inf-sup stable finite element discretization. Then, in order to prepare the structure of the proof for the full estimate in the next section, we consider a semidiscrete estimate.

Note that for the wave equation both, semi-discrete and fully discrete estimates for the displacements, are considered in [**13**, Th. 13.2], but for a robust estimate with respect to κ (based on the inf-sup stability) substantial extentions are required.

4.1. The elliptic projection in space. Let $h \in (0, h_0)$ be a discretization parameter and let $\boldsymbol{X}_h \times Q_h \subset \boldsymbol{X} \times Q$ be a stable finite element discretization such that

$$\sup_{\boldsymbol{u}_h \in \boldsymbol{X}_h} \frac{b(\boldsymbol{u}_h, p_h)}{\|\boldsymbol{u}_h\|_{\boldsymbol{X}}} \geq \beta \sup_{\boldsymbol{u} \in \boldsymbol{X}} \frac{b(\boldsymbol{u}, p_h)}{\|\boldsymbol{u}\|_{\boldsymbol{X}}}, \qquad p_h \in Q_h\,, \tag{19}$$

where β is independent of the discretization parameter h. In particular, (6) implies

$$\sup_{\boldsymbol{u}_h \in \boldsymbol{X}_h} \frac{b(\boldsymbol{u}_h, p_h)}{\|\boldsymbol{u}_h\|_{\boldsymbol{X}}} \leq \frac{1}{\beta} \|p_h\|_Q, \qquad p_h \in Q_h\,. \tag{20}$$

For the discrete system, standard theory for linear ODE's apply, see [**1**, Th. II.9.5]:

THEOREM 6. *For $\ell \in C^k(0,T;\boldsymbol{X}')$ the second order semidiscrete equation*

$$m(\ddot{\boldsymbol{u}}_h(t), \boldsymbol{w}_h) + a(\boldsymbol{u}_h(t), \boldsymbol{w}_h) + b(\boldsymbol{w}_h, p_h(t)) = \langle \ell(t), \boldsymbol{w}_h\rangle \qquad \boldsymbol{w}_h \in \boldsymbol{X}_h\,, \tag{21a}$$

$$b(\dot{\boldsymbol{u}}(t), q_h) - c_\kappa(p_h(t), q_h) = 0 \qquad q_h \in Q_h\,, \tag{21b}$$

with initial values

$$(\boldsymbol{u}_h(0), \dot{\boldsymbol{u}}_h(0)) = (\boldsymbol{u}_{h,0}, \boldsymbol{v}_{h,0}) \in \boldsymbol{X}_h \times \boldsymbol{X}_h\,,$$

has a unique solution $(\boldsymbol{u}_h, p_h) \in C^{k+2}(0,T;\boldsymbol{X}_h) \times C^{k+1}(0,T;Q_h)$.

For the semidiscrete analysis we introduce a coupled elliptic projection

$$(I_h, J_h) \colon \boldsymbol{X} \times Q \longrightarrow \boldsymbol{X}_h \times Q_h$$

by solving the saddle point problem

$$a(I_h(\boldsymbol{u},p), \boldsymbol{w}_h) + b(\boldsymbol{w}_h, J_h(\boldsymbol{u},p)) = a(\boldsymbol{u}, \boldsymbol{w}_h) + b(\boldsymbol{w}_h, p) \qquad \boldsymbol{w}_h \in \boldsymbol{X}_h\,, \tag{22a}$$

$$b(I_h(\boldsymbol{u},p), q_h) - c_\kappa(J_h(\boldsymbol{u},p), q_h) = b(\boldsymbol{u}, q_h) - c_\kappa(p, q_h) \qquad q_h \in Q_h\,. \tag{22b}$$

Elaborate stability investigations related with such saddle point problems have been performed in the context of penalized saddle point problems. In particular, we obtain quasi-optimal estimates uniformly in κ, see [**2, 3**].

THEOREM 7. *A constant $C > 0$ depending only on β exists such that*

$$\|\boldsymbol{u} - I_h(\boldsymbol{u},p)\|_{\boldsymbol{X}} + \|p - J_h(\boldsymbol{u},p)\|_Q \leq C \inf_{(\boldsymbol{w}_h, q_h) \in \boldsymbol{X}_h \times Q_h} \big(\|\boldsymbol{u} - \boldsymbol{w}_h\|_{\boldsymbol{X}} + \|p - q_h\|_Q\big).$$

Moreover, if $\boldsymbol{u} \in H^2(\Omega, \mathbb{R}^3)$ and $p \in H^2(\Omega)$, and if the mesh is sufficiently regular, we have

$$\|\boldsymbol{u} - I_h(\boldsymbol{u},p)\|_{\boldsymbol{X}} + \|p - J_h(\boldsymbol{u},p)\|_Q \leq C(\boldsymbol{u},p)\, h\,,$$

with $C(\boldsymbol{u},p)$ independent of κ.

4.2. A semidiscrete estimate in space. The elliptic projection allows for decompositions

$$\boldsymbol{u}(t) - \boldsymbol{u}_h(t) = \boldsymbol{\varrho}_{\boldsymbol{u}}(t) + \boldsymbol{\theta}_{\boldsymbol{u}}(t)\,, \quad \boldsymbol{\varrho}_{\boldsymbol{u}}(t) = \boldsymbol{u}(t) - I_h(\boldsymbol{u}(t), p(t))\,,\ \boldsymbol{\theta}_{\boldsymbol{u}}(t) = I_h(\boldsymbol{u}(t), p(t)) - \boldsymbol{u}_h(t)\,,$$

$$p(t) - p_h(t) = \varrho_p(t) + \theta_p(t)\,, \qquad \varrho_p(t) = p(t) - J_h(\boldsymbol{u}(t), p(t))\,,\ \theta_p(t) = J_h(\boldsymbol{u}(t), p(t)) - p_h(t)\,,$$

where first terms will we estimated by Theorem 7; the second term (in discrete spaces) is estimated in Theorem 8 below.

Temporal derivatives are defined accordingly, e. g., $\dot{\boldsymbol{\theta}}_{\boldsymbol{u}}(t) = I_h(\dot{\boldsymbol{u}}(t), \dot{p}(t)) - \dot{\boldsymbol{u}}_h(t)$.

THEOREM 8. *If $\boldsymbol{u} \in H^3(0,T,\boldsymbol{X})$ and $p \in H^3(0,T;Q)$, the estimate*

$$\|\theta_p(t)\|_Q^2 + \|\ddot{\boldsymbol{\theta}}_{\boldsymbol{u}}(t)\|_{\boldsymbol{H}}^2 + \|\dot{\boldsymbol{\theta}}_{\boldsymbol{u}}(t)\|_{\boldsymbol{X}}^2 + \|\boldsymbol{\theta}_{\boldsymbol{u}}(t)\|_{\boldsymbol{X}}^2$$
$$\lesssim \|\ddot{\boldsymbol{\theta}}_{\boldsymbol{u}}(0)\|_{\boldsymbol{H}}^2 + \|\dot{\boldsymbol{\theta}}_{\boldsymbol{u}}(0)\|_{\boldsymbol{X}}^2 + \|\boldsymbol{\theta}_{\boldsymbol{u}}(0)\|_{\boldsymbol{X}}^2 + \|\theta_p(0)\|_Q^2 + \|\varrho_p(t)\|_Q^2 + \|\boldsymbol{\varrho}_{\boldsymbol{u}}\|_{H^3(0,t;\boldsymbol{X})}^2$$

holds, where the constant contained in $\lesssim$ is independent of κ.

PROOF. The construction of the projection implies

$$a(\boldsymbol{\varrho_u}(t), \boldsymbol{w}_h) + b(\boldsymbol{w}_h, \varrho_p(t)) = 0, \qquad \boldsymbol{w}_h \in \boldsymbol{X}_h\,, \tag{23a}$$

$$b(\boldsymbol{\varrho_u}(t), q_h) - c_\kappa(\varrho_p(t), q_h) = 0, \qquad q_h \in Q_h\,, \tag{23b}$$

and $b(\dot{\boldsymbol{\varrho}}_{\boldsymbol{u}}(t), q_h) - c_\kappa(\dot\varrho_p(t), q_h) = 0$. Subtracting (7) and (21) and inserting (23) gives

$$m(\ddot{\boldsymbol{\theta}}_{\boldsymbol{u}}(t), \boldsymbol{w}_h) + a(\boldsymbol{\theta_u}(t), \boldsymbol{w}_h) + b(\boldsymbol{w}_h, \theta_p(t)) + m(\ddot{\boldsymbol{\varrho}}_{\boldsymbol{u}}(t), \boldsymbol{w}_h) = 0 \qquad \boldsymbol{w}_h \in \boldsymbol{X}_h, \tag{24a}$$

$$b(\dot{\boldsymbol{\theta}}_{\boldsymbol{u}}(t), q_h) - c_\kappa(\theta_p(t), q_h) + b(\dot{\boldsymbol{\varrho}}_{\boldsymbol{u}}(t) - \boldsymbol{\varrho_u}(t), q_h) = 0 \qquad q_h \in Q_h\,. \tag{24b}$$

Choosing $\boldsymbol{w}_h = \dot{\boldsymbol{\theta}}_{\boldsymbol{u}}(t)$ and $q_h = \theta_p(t)$ in (24) yields together

$$m(\ddot{\boldsymbol{\theta}}_{\boldsymbol{u}}(t), \dot{\boldsymbol{\theta}}_{\boldsymbol{u}}(t)) + a(\boldsymbol{\theta_u}(t), \dot{\boldsymbol{\theta}}_{\boldsymbol{u}}(t)) + c_\kappa(\theta_p(t), \theta_p(t)) = -m(\ddot{\boldsymbol{\varrho}}_{\boldsymbol{u}}(t), \dot{\boldsymbol{\theta}}_{\boldsymbol{u}}(t)) + b(\dot{\boldsymbol{\varrho}}_{\boldsymbol{u}}(t) - \boldsymbol{\varrho_u}(t), \theta_p(t))\,,$$

and integrating results in

$$\begin{aligned}
&\|\dot{\boldsymbol{\theta}}_{\boldsymbol{u}}(t)\|_{\boldsymbol{H}}^2 + \|\boldsymbol{\theta_u}(t)\|_{\boldsymbol{X}}^2 + 2\int_0^t \|\theta_p(s)\|_\kappa^2\,ds \\
&\quad = \|\dot{\boldsymbol{\theta}}_{\boldsymbol{u}}(0)\|_{\boldsymbol{H}}^2 + \|\boldsymbol{\theta_u}(0)\|_{\boldsymbol{X}}^2 + 2\int_0^t b(\dot{\boldsymbol{\varrho}}_{\boldsymbol{u}}(s) - \boldsymbol{\varrho_u}(t), \theta_p(s))\,ds - 2\int_0^t m(\ddot{\boldsymbol{\varrho}}_{\boldsymbol{u}}(s), \dot{\boldsymbol{\theta}}_{\boldsymbol{u}}(s))\,ds \\
&\quad \le C_1(\boldsymbol{u}, p) + \int_0^t \|\theta_p(s)\|_S^2\,ds + \int_0^t \|\dot{\boldsymbol{\theta}}_{\boldsymbol{u}}(s)\|_{\boldsymbol{H}}^2\,ds\,,
\end{aligned} \tag{25}$$

with $C_1(\boldsymbol{u}, p) = \|\dot{\boldsymbol{\theta}}_{\boldsymbol{u}}(0)\|_{\boldsymbol{H}}^2 + \|\boldsymbol{\theta_u}(0)\|_{\boldsymbol{X}}^2 + \int_0^t \|\dot{\boldsymbol{\varrho}}_{\boldsymbol{u}}(s) - \boldsymbol{\varrho_u}(s)\|_{\boldsymbol{X}}^2\,ds + \int_0^t \|\ddot{\boldsymbol{\varrho}}_{\boldsymbol{u}}(s)\|_{\boldsymbol{H}}^2\,ds.$

Differentiating (23) and (24) in time analogously yields

$$\begin{aligned}
&\|\ddot{\boldsymbol{\theta}}_{\boldsymbol{u}}(t)\|_{\boldsymbol{H}}^2 + \|\dot{\boldsymbol{\theta}}_{\boldsymbol{u}}(t)\|_{\boldsymbol{X}}^2 + 2\int_0^t \|\dot\theta_p(s)\|_\kappa^2\,ds \\
&\quad = \|\ddot{\boldsymbol{\theta}}_{\boldsymbol{u}}(0)\|_{\boldsymbol{H}}^2 + \|\dot{\boldsymbol{\theta}}_{\boldsymbol{u}}(0)\|_{\boldsymbol{X}}^2 + 2\int_0^t b(\ddot{\boldsymbol{\varrho}}_{\boldsymbol{u}}(s) - \dot{\boldsymbol{\varrho}}_{\boldsymbol{u}}(s), \dot\theta_p(s))\,ds - 2\int_0^t m(\partial_t^3\boldsymbol{\varrho_u}(s), \ddot{\boldsymbol{\theta}}_{\boldsymbol{u}}(s))\,ds\,.
\end{aligned}$$

Using integration by parts and Young's inequality, we insert the estimate

$$\begin{aligned}
&\int_0^t b(\ddot{\boldsymbol{\varrho}}_{\boldsymbol{u}}(s) - \dot{\boldsymbol{\varrho}}_{\boldsymbol{u}}(s), \dot\theta_p(s))\,ds \\
&\quad = b(\ddot{\boldsymbol{\varrho}}_{\boldsymbol{u}}(t) - \dot{\boldsymbol{\varrho}}_{\boldsymbol{u}}(t), \theta_p(t)) - b(\ddot{\boldsymbol{\varrho}}_{\boldsymbol{u}}(0) - \dot{\boldsymbol{\varrho}}_{\boldsymbol{u}}(0), \theta_p(0)) - \int_0^t b(\partial_t^3\boldsymbol{\varrho_u}(s), \theta_p(s))\,ds \\
&\quad \le \frac{1}{2\eta}\|\ddot{\boldsymbol{\varrho}}_{\boldsymbol{u}}(t) - \dot{\boldsymbol{\varrho}}_{\boldsymbol{u}}(t)\|_{\boldsymbol{X}}^2 + \frac{\eta}{2}\|\theta_p(t)\|_S^2 + \frac12\|\ddot{\boldsymbol{\varrho}}_{\boldsymbol{u}}(0) - \dot{\boldsymbol{\varrho}}_{\boldsymbol{u}}(0)\|_{\boldsymbol{X}}^2 + \frac12\|\theta_p(0)\|_S^2 \\
&\qquad + \frac12\int_0^t \|\partial_t^3\boldsymbol{\varrho_u}(s)\|_{\boldsymbol{X}}^2\,ds + \frac12\int_0^t \|\theta_p(s)\|_S^2\,ds\,,
\end{aligned}$$

which gives together

$$\|\ddot{\boldsymbol{\theta}}_{\boldsymbol{u}}(t)\|_{\boldsymbol{H}}^2 + \|\dot{\boldsymbol{\theta}}_{\boldsymbol{u}}(t)\|_{\boldsymbol{X}}^2 \le \eta\|\theta_p(t)\|_S^2 + C_2(\boldsymbol{u}, p) + \int_0^t \|\theta_p(s)\|_S^2\,ds + \int_0^t \|\ddot{\boldsymbol{\theta}}_{\boldsymbol{u}}(s)\|_{\boldsymbol{H}}^2\,ds\,, \tag{26}$$

with

$$\begin{aligned}
C_2(\boldsymbol{u}, p) &= \|\ddot{\boldsymbol{\theta}}_{\boldsymbol{u}}(0)\|_{\boldsymbol{H}}^2 + \|\dot{\boldsymbol{\theta}}_{\boldsymbol{u}}(0)\|_{\boldsymbol{X}}^2 + \frac{1}{\eta}\|\ddot{\boldsymbol{\varrho}}_{\boldsymbol{u}}(t) - \dot{\boldsymbol{\varrho}}_{\boldsymbol{u}}(t)\|_{\boldsymbol{X}}^2 + \|\ddot{\boldsymbol{\varrho}}_{\boldsymbol{u}}(0) - \dot{\boldsymbol{\varrho}}_{\boldsymbol{u}}(0)\|_{\boldsymbol{X}}^2 + \|\theta_p(0)\|_S^2 \\
&\qquad + \int_0^t \|\partial_t^3\boldsymbol{\varrho_u}(s)\|_{\boldsymbol{X}}^2\,ds + \int_0^t \|\partial_t^3\boldsymbol{\varrho_u}(s)\|_{\boldsymbol{H}}^2\,ds\,.
\end{aligned}$$

Estimating $b(\boldsymbol{w}_h, \theta_p(t))$ in (24a) gives

$$\|\theta_p(t)\|_S \le C_\rho\big(\|\ddot{\boldsymbol{\theta}}_{\boldsymbol{u}}(t)\|_{\boldsymbol{H}} + \|\ddot{\boldsymbol{\varrho}}_{\boldsymbol{u}}(t)\|_{\boldsymbol{H}}\big) + \|\boldsymbol{\theta_u}(t)\|_{\boldsymbol{X}}\,, \tag{27}$$

and choosing $q_h = \theta_p(t)$ in (24b) results in

$$\|\theta_p(t)\|_\kappa^2 \le \|\theta_p(t)\|_S^2 + \|\dot{\boldsymbol{\theta}}_{\boldsymbol{u}}(t)\|_{\boldsymbol{X}}^2 + \|\dot{\boldsymbol{\varrho}}_{\boldsymbol{u}}(t) - \boldsymbol{\varrho_u}(t)\|_{\boldsymbol{X}}^2\,. \tag{28}$$

Together we obtain

$$\|\theta_p(t)\|_Q^2 \le 3C_\rho^2\big(\|\ddot{\boldsymbol{\theta}}_{\boldsymbol{u}}(t)\|_{\boldsymbol{H}}^2 + \|\ddot{\boldsymbol{\varrho}}_{\boldsymbol{u}}(t)\|_{\boldsymbol{H}}^2\big) + 3\|\boldsymbol{\theta}_{\boldsymbol{u}}(t)\|_{\boldsymbol{X}}^2 + \|\dot{\boldsymbol{\theta}}_{\boldsymbol{u}}(t)\|_{\boldsymbol{X}}^2 + \|\dot{\boldsymbol{\varrho}}_{\boldsymbol{u}}(t) - \boldsymbol{\varrho}_{\boldsymbol{u}}(t)\|_{\boldsymbol{X}}^2\,,$$

and inserting (25) and (26) and choosing $\eta > 0$ sufficiently small combines to

$$\begin{aligned}\|\theta_p(t)\|_Q^2 &+ \|\ddot{\boldsymbol{\theta}}_{\boldsymbol{u}}(t)\|_{\boldsymbol{H}}^2 + \|\dot{\boldsymbol{\theta}}_{\boldsymbol{u}}(t)\|_{\boldsymbol{X}}^2 + \|\boldsymbol{\theta}_{\boldsymbol{u}}(t)\|_{\boldsymbol{X}}^2 \\ &\lesssim C_3(\boldsymbol{u},p) + \int_0^t \Big(\|\theta_p(s)\|_S^2 + \|\ddot{\boldsymbol{\theta}}_{\boldsymbol{u}}(s)\|_{\boldsymbol{H}}^2\,ds + \|\dot{\boldsymbol{\theta}}_{\boldsymbol{u}}(s)\|_{\boldsymbol{H}}^2\Big)\,ds\,,\end{aligned}$$

with $C_3(\boldsymbol{u},p) = C_1(\boldsymbol{u},p) + C_2(\boldsymbol{u},p) + \|\dot{\boldsymbol{\varrho}}_{\boldsymbol{u}}(t) - \boldsymbol{\varrho}_{\boldsymbol{u}}(t)\|_{\boldsymbol{X}}^2 + \|\ddot{\boldsymbol{\varrho}}_{\boldsymbol{u}}(t)\|_{\boldsymbol{H}}^2$. Now, applying Gronwall's lemma [**11**, Lem. A.4.12] and Sobolev's embedding theorem [**7**, Th. 5.9.2] give the assertion. □

Corollary 9. *If $\boldsymbol{u} \in H^3(0,T,H^2(\Omega,\mathbb{R}^3))$ and $p \in H^3(0,T;H^2(\Omega))$, we have in case of exact initial values*

$$\|\ddot{\boldsymbol{u}}(t) - \ddot{\boldsymbol{u}}_h(t)\|_{\boldsymbol{H}} + \|\dot{\boldsymbol{u}}(t) - \dot{\boldsymbol{u}}_h(t)\|_{\boldsymbol{X}} + \|\boldsymbol{u}(t) - \boldsymbol{u}_h(t)\|_{\boldsymbol{X}} + \|p(t) - p_h(t)\|_Q \lesssim C(\boldsymbol{u})\ h\ .$$

Proof. The desired estimate is obtained from

$$\begin{aligned}\|\ddot{\boldsymbol{u}}(t) - \ddot{\boldsymbol{u}}_h(t)\|_{\boldsymbol{H}}^2 &+ \|\dot{\boldsymbol{u}}(t) - \dot{\boldsymbol{u}}_h(t)\|_{\boldsymbol{X}}^2 + \|\boldsymbol{u}(t) - \boldsymbol{u}_h(t)\|_{\boldsymbol{X}}^2 + \|p(t) - p_h(t)\|_Q^2 \\ &\le \|\ddot{\boldsymbol{\theta}}_{\boldsymbol{u}}(t)\|_{\boldsymbol{H}}^2 + \|\dot{\boldsymbol{\theta}}_{\boldsymbol{u}}(t)\|_{\boldsymbol{X}}^2 + \|\boldsymbol{\theta}_{\boldsymbol{u}}(t)\|_{\boldsymbol{X}}^2 + \|\theta_p(t)\|_Q^2 \\ &\quad + \|\ddot{\boldsymbol{\varrho}}_{\boldsymbol{u}}(t)\|_{\boldsymbol{H}}^2 + \|\dot{\boldsymbol{\varrho}}_{\boldsymbol{u}}(t)\|_{\boldsymbol{X}}^2 + \|\boldsymbol{\varrho}_{\boldsymbol{u}}(t)\|_{\boldsymbol{X}}^2 + \|\varrho_p(t)\|_Q^2\end{aligned}$$

and inserting Theorem 7 for the estimate of the interpolation error. □

5. Convergence in space and time

5.1. The Newmark discretization as a finite difference method. For simplicity, we restrict ourselves to the case of uniform time step size $\Delta t_n \equiv \Delta t$.

Furthermore, let $\Delta \boldsymbol{w}^n = \boldsymbol{w}^n - \boldsymbol{w}^{n-1}$ and $\Delta^2 \boldsymbol{w}^n = \Delta(\Delta \boldsymbol{w}^n) = \boldsymbol{w}^n - 2\boldsymbol{w}^{n-1} + \boldsymbol{w}^{n-2}$ be the first and second finite difference, let $\partial_{\Delta t}\boldsymbol{w}^n = (\Delta t)^{-1}\Delta \boldsymbol{w}^n$ and $\partial_{\Delta t}^k \boldsymbol{w}^n = \partial_{\Delta t}(\partial_{\Delta t}^{k-1}\boldsymbol{w}^n)$ be the difference quotients in time, and we introducing the averaged values $\{\boldsymbol{w}^n\} = \frac12(\boldsymbol{w}^n + \boldsymbol{w}^{n-1})$ and $\{\{\boldsymbol{w}^n\}\} = \frac12(\{\boldsymbol{w}^n\} + \{\boldsymbol{w}^{n-1}\}) = \frac14(\boldsymbol{w}^n + 2\boldsymbol{w}^{n-1} + \boldsymbol{w}^{n-2})$.

For time-continuous quantities, the above operators have to be interpreted as evaluations at times $t = t_n$, e.g. $\{\boldsymbol{u}(t_n)\} = \frac12\big(\boldsymbol{u}(t_n) + \boldsymbol{u}(t_{n-1})\big)$.

The fully discrete scheme is defined as follows: for given initial values $\boldsymbol{u}_h^0, \boldsymbol{v}_h^0 \in \boldsymbol{X}_h$ compute approximations $\boldsymbol{u}_h^n, \boldsymbol{v}_h^n, \boldsymbol{a}_h^n \in \boldsymbol{X}_h$ and $p_h^n \in Q_h$ satisfying

$$m(\boldsymbol{a}_h^n, \boldsymbol{w}_h) + a(\boldsymbol{u}_h^n, \boldsymbol{w}_h) + b(\boldsymbol{w}_h, p_h^n) = \langle \ell^n, \boldsymbol{w}_h\rangle \qquad \boldsymbol{w}_h \in \boldsymbol{X}_h\,, \tag{29a}$$

$$b(\boldsymbol{v}_h^n, q_h) - c_\kappa(p_h^n, q_h) = 0 \qquad q_h \in Q_h\,, \tag{29b}$$

for $n = 0, 1, 2, ..., N$, and for $n > 0$ (according to (17))

$$\boldsymbol{v}_h^n = \frac{2}{\Delta t_n}\Delta \boldsymbol{u}_h^n - \boldsymbol{v}_h^{n-1}\,, \qquad \boldsymbol{a}_h^n = \frac{4}{(\Delta t_n)^2}\Delta \boldsymbol{u}_h^n - \frac{4}{\Delta t_n}\boldsymbol{v}_h^{n-1} - \boldsymbol{a}_h^{n-1}\,, \tag{30}$$

For $n > 1$, we observe the identities $\{\boldsymbol{v}_h^n\} = \partial_{\Delta t}\boldsymbol{u}_h^n$ and $\{\{\boldsymbol{a}_h^n\}\} = \partial_{\Delta t}^2 \boldsymbol{u}_h^n = \{\partial_{\Delta t}\boldsymbol{v}_h^n\}$.

In analogy to Sect. 4.2 we define

$$\boldsymbol{\varrho}_{\boldsymbol{u}}^n := \boldsymbol{u}(t_n) - I_h(\boldsymbol{u}(t_n), p(t_n)) \in \boldsymbol{X}\,, \qquad \boldsymbol{\theta}_{\boldsymbol{u}}^n := I_h(\boldsymbol{u}(t_n), p(t_n)) - \boldsymbol{u}_h^n \in \boldsymbol{X}_h\,,$$

$$\varrho_p^n := p(t_n) - J_h(\boldsymbol{u}(t_n), p(t_n)) \in Q\,, \qquad \theta_p^n := J_h(\boldsymbol{u}(t_n), p(t_n)) - p_h^n \in Q_h\,,$$

and we set

$$\boldsymbol{\xi}_{\boldsymbol{u}}^n := \boldsymbol{u}(t_n) - \boldsymbol{u}_h^n = \boldsymbol{\varrho}_{\boldsymbol{u}}^n + \boldsymbol{\theta}_{\boldsymbol{u}}^n\ , \qquad \xi_p^n := p(t_n) - p_h^n = \varrho_p^n + \theta_p^n\ .$$

The corresponding quantities for the velocities and the acceleration are denoted by $\boldsymbol{\varrho}_{\boldsymbol{v}}^n$, $\boldsymbol{\theta}_{\boldsymbol{v}}^n$, $\boldsymbol{\xi}_{\boldsymbol{v}}^n$ and $\boldsymbol{\varrho}_{\boldsymbol{a}}^n$, $\boldsymbol{\theta}_{\boldsymbol{a}}^n$, $\boldsymbol{\xi}_{\boldsymbol{a}}^n$, respectively.

The estimate in the fully discrete case follows the lines of the semi-discrete estimate, where the Gronwall Lemma is replaced by the discrete version [**16**, Lemma 1.4.2], and the integration by parts is also replaced by a discrete counterpart.

LEMMA 10. *Assume $C_0 > 0$, and that the sequence (a_n) is non-negative. If the sequence (ϕ_n) satisfies $\phi_0 \le C_0$ and $\phi_n \le C_0 + \sum_{i=0}^{n-1} a_i\phi_i$ for $n \ge 1$, then $\phi_n \le C_0 \exp\Big(\sum_{i=0}^{n-1} a_i\Big)$ holds.*

LEMMA 11. *We have for an arbitrary bilinear form $B(\cdot,\cdot)$ and sequences $\boldsymbol{u}^n$, $\boldsymbol{v}^n$*

$$B(\{\boldsymbol{u}^n\}, \partial_{\Delta t}\boldsymbol{v}^n) = -B(\partial_{\Delta t}\boldsymbol{u}^n, \{\boldsymbol{v}^n\}) + \partial_{\Delta t} B(\boldsymbol{u}^n, \boldsymbol{v}^n)\,,$$

If additionally, the bilinear form is symmetric we have

$$2\Delta t\, B(\{\boldsymbol{u}^n\}, \partial_{\Delta t}\boldsymbol{u}^n) = B(\boldsymbol{u}^n, \boldsymbol{u}^n) - B(\boldsymbol{u}^{n-1}, \boldsymbol{u}^{n-1})\,. \tag{31}$$

PROOF. It holds

$$\begin{aligned}
&B(\{\boldsymbol{u}^n\}, \partial_{\Delta t}\boldsymbol{v}^n) + B(\partial_{\Delta t}\boldsymbol{u}^n, \{\boldsymbol{v}^n\}) \\
&\quad = \frac{1}{2\Delta t}\Big(B(\boldsymbol{u}^n + \boldsymbol{u}^{n-1}, \boldsymbol{v}^n - \boldsymbol{v}^{n-1}) + B(\boldsymbol{u}^n - \boldsymbol{u}^{n-1}, \boldsymbol{v}^n + \boldsymbol{v}^{n-1})\Big) \\
&\quad = \frac{1}{2\Delta t}\Big(2\, B(\boldsymbol{u}^n, \boldsymbol{v}^n) - 2\, B(\boldsymbol{u}^{n-1}, \boldsymbol{v}^{n-1})\Big) = \partial_{\Delta t} B(\boldsymbol{u}^n, \boldsymbol{v}^n)\,.
\end{aligned}$$

The second statement immediately follows. □

The error in space will be estimated in terms of $\boldsymbol{\varrho}_{\boldsymbol{u}}^n$.

LEMMA 12. *We have for $n > 1$*

$$\Delta t\, \|\{\partial_{\Delta t}\boldsymbol{\varrho}_{\boldsymbol{u}}^n\}\|_{\boldsymbol{X}}^2 \lesssim \|\dot{\boldsymbol{\varrho}}_{\boldsymbol{u}}\|_{L_2(t_{n-2},t_n,\boldsymbol{X})}^2 \le 2\Delta t\, \|\dot{\boldsymbol{\varrho}}_{\boldsymbol{u}}\|_{L_\infty(t_{n-2},t_n,\boldsymbol{X})}^2\,, \tag{32a}$$

$$\Delta t\, \|\partial_{\Delta t}^2\boldsymbol{\varrho}_{\boldsymbol{u}}^n\|_{\boldsymbol{X}}^2 \lesssim \|\ddot{\boldsymbol{\varrho}}_{\boldsymbol{u}}\|_{L_2(t_{n-2},t_n,\boldsymbol{X})}^2 \le 2\Delta t\, \|\ddot{\boldsymbol{\varrho}}_{\boldsymbol{u}}\|_{L_\infty(t_{n-2},t_n,\boldsymbol{X})}^2\,, \tag{32b}$$

and for $n > 2$

$$\Delta t\, \|\{\partial_{\Delta t}^2\boldsymbol{\varrho}_{\boldsymbol{u}}^n\}\|_{\boldsymbol{X}}^2 \lesssim \|\ddot{\boldsymbol{\varrho}}_{\boldsymbol{u}}\|_{L_2(t_{n-3},t_n,\boldsymbol{X})}^2 \le 3\Delta t\, \|\ddot{\boldsymbol{\varrho}}_{\boldsymbol{u}}\|_{L_\infty(t_{n-3},t_n,\boldsymbol{X})}^2\,, \tag{32c}$$

$$\Delta t\, \|\partial_{\Delta t}^3\boldsymbol{\varrho}_{\boldsymbol{u}}^n\|_{\boldsymbol{X}}^2 \lesssim \|\partial_t^3\boldsymbol{\varrho}_{\boldsymbol{u}}\|_{L_2(t_{n-3},t_n,\boldsymbol{X})}^2\,. \tag{32d}$$

Moreover, it holds for $n > 1$

$$\Delta t\, \|\{\{\boldsymbol{\varrho}_{\boldsymbol{u}}^n\}\}\|_{\boldsymbol{X}}^2 \lesssim \|\boldsymbol{\varrho}_{\boldsymbol{u}}\|_{L_2(t_{n-2},t_n,\boldsymbol{X})}^2 + (\Delta t)^4 \|\ddot{\boldsymbol{\varrho}}_{\boldsymbol{u}}\|_{L_2(t_{n-2},t_n,\boldsymbol{X})}^2\,. \tag{32e}$$

PROOF. We only show (32a) since the others follow similarly. Taylor expansions gives

$$\boldsymbol{\varrho}_{\boldsymbol{u}}^n = \boldsymbol{\varrho}_{\boldsymbol{u}}^{n-1} + \int_{t_{n-1}}^{t_n} \dot{\boldsymbol{\varrho}}_{\boldsymbol{u}}(s)\, ds\,, \qquad \text{and} \qquad \boldsymbol{\varrho}_{\boldsymbol{u}}^{n-2} = \boldsymbol{\varrho}_{\boldsymbol{u}}^{n-1} - \int_{t_{n-2}}^{t_{n-1}} \dot{\boldsymbol{\varrho}}_{\boldsymbol{u}}(s)\, ds\,,$$

and hence $\{\partial_{\Delta t}\boldsymbol{\varrho}_{\boldsymbol{u}}^n\} = (2\Delta t)^{-1}\int_{t_{n-2}}^{t_n} \dot{\boldsymbol{\varrho}}_{\boldsymbol{u}}(s)\, ds$. Applying $\|\cdot\|_{\boldsymbol{X}}^2$ and Hölder's inequality gives

$$\begin{aligned}
\|\{\partial_{\Delta t}\boldsymbol{\varrho}_{\boldsymbol{u}}^n\}\|_{\boldsymbol{X}}^2 &= (2\Delta t)^{-2}\Big\|\int_{t_{n-2}}^{t_n} \dot{\boldsymbol{\varrho}}_{\boldsymbol{u}}(s)\, ds\Big\|_{\boldsymbol{X}}^2 \\
&\le (2\Delta t)^{-2}\int_{t_{n-2}}^{t_n} 1^2\, ds \int_{t_{n-2}}^{t_n} \|\dot{\boldsymbol{\varrho}}_{\boldsymbol{u}}(s)\|_{\boldsymbol{X}}^2\, ds = (2\Delta t)^{-1}\|\dot{\boldsymbol{\varrho}}_{\boldsymbol{u}}\|_{L_2(t_{n-2},t_n,\boldsymbol{X})}^2\,,
\end{aligned}$$

which is the first assertion. The estimate (32e) is the achieved by using the error estimate of the trapezoidal quadrature rule. □

Summing up suitable quantities (as they will be used in the proof of the following theorem), for

$$S^k := \|\partial_{\Delta t}^2 \boldsymbol{\varrho}_{\boldsymbol{u}}^k\|_{\boldsymbol{H}}^2 + \|\{\partial_{\Delta t}\boldsymbol{\varrho}_{\boldsymbol{u}}^k\}\|_{\boldsymbol{X}}^2 + \|\{\{\boldsymbol{\varrho}_{\boldsymbol{u}}^k\}\}\|_{\boldsymbol{X}}^2 + \|\partial_{\Delta t}^2 \boldsymbol{\varrho}_{\boldsymbol{u}}^{k-1}\|_{\boldsymbol{H}}^2 + \|\{\partial_{\Delta t}\boldsymbol{\varrho}_{\boldsymbol{u}}^{k-1}\}\|_{\boldsymbol{X}}^2 \\ + \|\partial_{\Delta t}^3 \boldsymbol{\varrho}_{\boldsymbol{u}}^k\|_{\boldsymbol{H}}^2 + \|\{\{\boldsymbol{\varrho}_{\boldsymbol{u}}^{k-1}\}\}\|_{\boldsymbol{X}}^2 + \|\partial_{\Delta t}^3 \boldsymbol{\varrho}_{\boldsymbol{u}}^k\|_{\boldsymbol{X}}^2 + \|\{\partial_{\Delta t}^2 \boldsymbol{\varrho}_{\boldsymbol{u}}^k\}\|_{\boldsymbol{X}}^2 \tag{33a}$$

this Lemma immediately gives the bound

$$\Delta t \sum_{k=3}^{n} S^k \lesssim (1 + (\Delta t)^4)\|\boldsymbol{\varrho}_{\boldsymbol{u}}\|_{H^3(0,t_n;\boldsymbol{X})}^2 . \tag{33b}$$

The error in time is estimated by comparison with finite differences in the following Lemma.

LEMMA 13. *We have for* $n > 1$

$$\{\{\boldsymbol{\xi}_{\boldsymbol{a}}^n\}\} = \partial_{\Delta t}^2 \boldsymbol{\xi}_{\boldsymbol{u}}^n + \boldsymbol{\delta}_1^n \qquad \textit{with} \quad \Delta t\,\|\boldsymbol{\delta}_1^n\|_{\boldsymbol{H}}^2 \le C(\Delta t)^4 \|\partial_t^4 \boldsymbol{u}\|_{L_2(t_{n-2},t_n;\boldsymbol{H})}^2 , \tag{34a}$$

$$\{\{\boldsymbol{\xi}_{\boldsymbol{v}}^n\}\} = \{\partial_{\Delta t} \boldsymbol{\xi}_{\boldsymbol{u}}^n\} + \boldsymbol{\delta}_2^n \qquad \textit{with} \quad \Delta t\,\|\boldsymbol{\delta}_2^n\|_{\boldsymbol{X}}^2 \le C(\Delta t)^4 \|\partial_t^3 \boldsymbol{u}\|_{L_2(t_{n-2},t_n;\boldsymbol{X})}^2 , \tag{34b}$$

$$\{\partial_{\Delta t}\boldsymbol{\xi}_{\boldsymbol{v}}^n\} = \partial_{\Delta t}^2 \boldsymbol{\xi}_{\boldsymbol{u}}^n + \boldsymbol{\delta}_3^n \qquad \textit{with} \quad \|\boldsymbol{\delta}_3^n\|_{\boldsymbol{X}}^2 \le C(\Delta t)^4 \|\partial_t^4 \boldsymbol{u}\|_{L_\infty(t_{n-2},t_n;\boldsymbol{X})}^2 , \tag{34c}$$

and for $n > 2$

$$\{\{\partial_{\Delta t}\boldsymbol{\xi}_{\boldsymbol{a}}^n\}\} = \partial_{\Delta t}^3 \boldsymbol{\xi}_{\boldsymbol{u}}^n + \boldsymbol{\delta}_4^n \qquad \textit{with} \quad \Delta t\,\|\boldsymbol{\delta}_4^n\|_{\boldsymbol{H}}^2 \le C(\Delta t)^4 \|\partial_t^5 \boldsymbol{u}\|_{L_2(t_{n-3},t_n;\boldsymbol{H})}^2 , \tag{34d}$$

$$\{\partial_{\Delta t}^2\boldsymbol{\xi}_{\boldsymbol{v}}^n\} = \partial_{\Delta t}^3 \boldsymbol{\xi}_{\boldsymbol{u}}^n + \boldsymbol{\delta}_5^n \qquad \textit{with} \quad \Delta t\,\|\boldsymbol{\delta}_5^n\|_{\boldsymbol{X}}^2 \le C(\Delta t)^4 \|\partial_t^5 \boldsymbol{u}\|_{L_2(t_{n-3},t_n;\boldsymbol{X})}^2 . \tag{34e}$$

PROOF. Again, we only show the first estimate since the others may be obtained likewise. By Taylor expansion, we find that

$$\{\{\ddot{\boldsymbol{u}}(t_n)\}\} = \partial_{\Delta t}^2 \boldsymbol{u}(t_n) + \int_{t_{n-1}}^{t_n} (t_n - s)\Big(\frac{1}{4} - \frac{1}{6(\Delta t)^2}(t_n - s)^2\Big)\,\partial_t^4 \boldsymbol{u}(s)\,ds \\ + \int_{t_{n-2}}^{t_{n-1}} (s - t_{n-2})\Big(\frac{1}{4} - \frac{1}{6(\Delta t)^2}(s - t_{n-2})^2\Big)\,\partial_t^4 \boldsymbol{u}(s)\,ds ,$$

and inserting $\{\{\boldsymbol{a}_h^n\}\} = \partial_{\Delta t}^2 \boldsymbol{u}_h^n$ gives together with Hölder's inequality

$$\begin{aligned} \|\boldsymbol{\delta}_1^n\|_{\boldsymbol{H}}^2 &= \|\{\{\ddot{\boldsymbol{u}}(t_n)\}\} - \partial_{\Delta t}^2 \boldsymbol{u}(t_n)\|_{\boldsymbol{H}}^2 \\ &\le 2\Big\|\int_{t_{n-1}}^{t_n} (t_n - s)\Big(\frac{1}{4} - \frac{1}{6(\Delta t)^2}(t_n - s)^2\Big)\partial_t^4 \boldsymbol{u}(s)\,ds\Big\|_{\boldsymbol{H}}^2 \\ &\quad + 2\Big\|\int_{t_{n-2}}^{t_{n-1}} (s - t_{n-2})\Big(\frac{1}{4} - \frac{1}{6(\Delta t)^2}(s - t_{n-2})^2\Big)\partial_t^4 \boldsymbol{u}(s)\,ds\Big\|_{\boldsymbol{H}}^2 \\ &\le 2\int_{t_{n-1}}^{t_n} \Big((t_n - s)\Big(\frac{1}{4} - \frac{1}{6(\Delta t)^2}(t_n - s)^2\Big)\Big)^2\,ds \int_{t_{n-1}}^{t_n} \|\partial_t^4 \boldsymbol{u}(s)\|_{\boldsymbol{H}}^2\,ds \\ &\quad + 2\int_{t_{n-2}}^{t_{n-1}} \Big((s - t_{n-2})\Big(\frac{1}{4} - \frac{1}{6(\Delta t)^2}(s - t_{n-2})^2\Big)\Big)^2\,ds \int_{t_{n-2}}^{t_{n-1}} \partial_t^4\|\boldsymbol{u}(s)\|_{\boldsymbol{H}}^2\,ds \\ &= \frac{41}{2520}(\Delta t)^3 \|\partial_t^4 \boldsymbol{u}\|_{L_2(t_{n-2},t_n;\boldsymbol{H})}^2 . \end{aligned}$$

□

Again, we summarize for later use: the term

$$T^k := \|\boldsymbol{\delta}_1^k\|_{\boldsymbol{H}}^2 + \|\boldsymbol{\delta}_2^k\|_{\boldsymbol{X}}^2 + \|\boldsymbol{\delta}_1^{k-1}\|_{\boldsymbol{H}}^2 + \|\boldsymbol{\delta}_2^{k-1}\|_{\boldsymbol{X}}^2 + \|\boldsymbol{\delta}_4^k\|_{\boldsymbol{H}}^2 + \|\boldsymbol{\delta}_5^k\|_{\boldsymbol{X}}^2 , \tag{35a}$$

can be estimated by

$$\Delta t \sum_{k=3}^{n} T^k \lesssim (\Delta t)^4 \|\boldsymbol{u}\|_{H^5(0,t_n;\boldsymbol{X})}^2 . \tag{35b}$$

5.2. A fully discrete estimate in time and space. Having finished the preparations, we present a fully discrete estimate. As indicated, the approach is akin to the spatially discrete estimate of the last section.

THEOREM 14. *If* $\boldsymbol{u} \in H^5(0,T,\boldsymbol{X})$ *and* $p \in H^3(0,T;Q)$, *the estimate*

$$\begin{aligned}&\|\partial^2_{\Delta t}\boldsymbol{\theta}^n_{\boldsymbol{u}}\|^2_{\boldsymbol{H}} + \|\{\partial_{\Delta t}\boldsymbol{\theta}^n_{\boldsymbol{u}}\}\|^2_{\boldsymbol{X}} + \|\{\{\boldsymbol{\theta}^n_{\boldsymbol{u}}\}\}\|^2_{\boldsymbol{X}} + \|\{\{\theta^n_p\}\}\|^2_Q\\ &\qquad\lesssim C(\boldsymbol{u}^0_h, \boldsymbol{u}^1_h, p^1_h) + (\Delta t)^4\|\boldsymbol{u}\|^2_{H^5(0,T;\boldsymbol{X})} + \|\boldsymbol{\varrho}_{\boldsymbol{u}}\|^2_{H^3(0,T;\boldsymbol{X})}\end{aligned}$$

holds with constants independent of κ, Δt, *and* h.

PROOF. The construction of the projection (22a) implies

$$a(\boldsymbol{\varrho}^n_{\boldsymbol{u}}, \boldsymbol{w}_h) + b(\boldsymbol{w}_h, \varrho^n_p) = 0 \qquad \boldsymbol{w}_h \in \boldsymbol{X}_h\,, \tag{36a}$$
$$b(\boldsymbol{\varrho}^n_{\boldsymbol{u}}, q_h) - c_\kappa(\varrho^n_p, q_h) = 0 \qquad q_h \in Q_h\,. \tag{36b}$$

Subtracting (7) for $t = t_n$ and inserting (29) gives

$$m(\boldsymbol{\theta}^n_{\boldsymbol{a}}, \boldsymbol{w}_h) + a(\boldsymbol{\theta}^n_{\boldsymbol{u}}, \boldsymbol{w}_h) + b(\boldsymbol{w}_h, \theta^n_p) + m(\boldsymbol{\varrho}^n_{\boldsymbol{a}}, \boldsymbol{w}_h) = 0\,, \qquad \boldsymbol{w}_h \in \boldsymbol{X}_h, \tag{37a}$$
$$b(\boldsymbol{\theta}^n_{\boldsymbol{v}}, q_h) - c_\kappa(\theta^n_p, q_h) + b(\boldsymbol{\varrho}^n_{\boldsymbol{v}} - \boldsymbol{\varrho}^n_{\boldsymbol{u}}, q_h) = 0\,, \qquad q_h \in Q_h\,. \tag{37b}$$

Averaging and inserting $\boldsymbol{w}_h = \{\partial_{\Delta t}\boldsymbol{\theta}^n_{\boldsymbol{u}}\}$ and $q_h = \{\{\theta^n_p\}\}$ in (37), for $n > 1$, together yields

$$\begin{aligned}&m(\{\{\boldsymbol{\theta}^n_{\boldsymbol{a}} + \boldsymbol{\varrho}^n_{\boldsymbol{a}}\}\}, \{\partial_{\Delta t}\boldsymbol{\theta}^n_{\boldsymbol{u}}\}) + a(\{\{\boldsymbol{\theta}^n_{\boldsymbol{u}}\}\}, \{\partial_{\Delta t}\boldsymbol{\theta}^n_{\boldsymbol{u}}\}) + c_\kappa(\{\{\theta^n_p\}\}, \{\{\theta^n_p\}\})\\ &\qquad = b(\{\{\boldsymbol{\theta}^n_{\boldsymbol{v}} + \boldsymbol{\varrho}^n_{\boldsymbol{v}} - \boldsymbol{\varrho}^n_{\boldsymbol{u}}\}\} - \{\partial_{\Delta t}\boldsymbol{\theta}^n_{\boldsymbol{u}}\}, \{\{\theta^n_p\}\})\,,\end{aligned}$$

and inserting (34) gives

$$\begin{aligned}&m(\partial^2_{\Delta t}\boldsymbol{\theta}^n_{\boldsymbol{u}}, \{\partial_{\Delta t}\boldsymbol{\theta}^n_{\boldsymbol{u}}\}) + a(\{\{\boldsymbol{\theta}^n_{\boldsymbol{u}}\}\}, \{\partial_{\Delta t}\boldsymbol{\theta}^n_{\boldsymbol{u}}\}) + c_\kappa(\{\{\theta^n_p\}\}, \{\{\theta^n_p\}\})\\ &\quad = m(\partial^2_{\Delta t}\boldsymbol{\varrho}^n_{\boldsymbol{u}} + \boldsymbol{\delta}^n_1, \{\partial_{\Delta t}\boldsymbol{\theta}^n_{\boldsymbol{u}}\}) + b(\{\partial_{\Delta t}\boldsymbol{\varrho}^n_{\boldsymbol{u}}\} + \boldsymbol{\delta}^n_2 - \{\{\boldsymbol{\varrho}^n_{\boldsymbol{u}}\}\}, \{\{\theta^n_p\}\})\\ &\quad \le \|\partial^2_{\Delta t}\boldsymbol{\varrho}^n_{\boldsymbol{u}} + \boldsymbol{\delta}^n_1\|_{\boldsymbol{H}}\|\{\partial_{\Delta t}\boldsymbol{\theta}^n_{\boldsymbol{u}}\}\|_{\boldsymbol{H}} + \|\{\partial_{\Delta t}\boldsymbol{\varrho}^n_{\boldsymbol{u}}\} + \boldsymbol{\delta}^n_2 - \{\{\boldsymbol{\varrho}^n_{\boldsymbol{u}}\}\}\|_{\boldsymbol{X}}\|\{\{\theta^n_p\}\}\|_S\\ &\quad \le \frac{1}{2\eta_1}\Big(\|\partial^2_{\Delta t}\boldsymbol{\varrho}^n_{\boldsymbol{u}} + \boldsymbol{\delta}^n_1\|^2_{\boldsymbol{H}} + \|\{\partial_{\Delta t}\boldsymbol{\varrho}^n_{\boldsymbol{u}}\} + \boldsymbol{\delta}^n_2 - \{\{\boldsymbol{\varrho}^n_{\boldsymbol{u}}\}\}\|^2_{\boldsymbol{X}}\Big) + \frac{\eta_1}{2}\Big(\|\{\partial_{\Delta t}\boldsymbol{\theta}^n_{\boldsymbol{u}}\}\|^2_{\boldsymbol{H}} + \|\{\{\theta^n_p\}\}\|^2_S\Big)\,,\end{aligned}$$

where $\eta_1 \in (0,1)$ will be fixed later. From (31) we obtain

$$\begin{aligned}&\|\partial_{\Delta t}\boldsymbol{\theta}^n_{\boldsymbol{u}}\|^2_{\boldsymbol{H}} + \|\{\boldsymbol{\theta}^n_{\boldsymbol{u}}\}\|^2_{\boldsymbol{X}}\\ &\quad = \|\partial_{\Delta t}\boldsymbol{\theta}^{n-1}_{\boldsymbol{u}}\|^2_{\boldsymbol{H}} + \|\{\boldsymbol{\theta}^{n-1}_{\boldsymbol{u}}\}\|^2_{\boldsymbol{X}} + 2\Delta t\Big(m(\partial^2_{\Delta t}\boldsymbol{\theta}^n_{\boldsymbol{u}}, \{\partial_{\Delta t}\boldsymbol{\theta}^n_{\boldsymbol{u}}\}) + a(\{\{\boldsymbol{\theta}^n_{\boldsymbol{u}}\}\}, \partial_{\Delta t}\{\boldsymbol{\theta}^n_{\boldsymbol{u}}\})\Big)\\ &\quad \le \|\partial_{\Delta t}\boldsymbol{\theta}^{n-1}_{\boldsymbol{u}}\|^2_{\boldsymbol{H}} + \|\{\boldsymbol{\theta}^{n-1}_{\boldsymbol{u}}\}\|^2_{\boldsymbol{X}} + \frac{\Delta t}{\eta_1}\Big(\|\partial^2_{\Delta t}\boldsymbol{\varrho}^n_{\boldsymbol{u}} + \boldsymbol{\delta}^n_1\|^2_{\boldsymbol{H}} + \|\{\partial_{\Delta t}\boldsymbol{\varrho}^n_{\boldsymbol{u}}\} + \boldsymbol{\delta}^n_2 - \{\{\boldsymbol{\varrho}^n_{\boldsymbol{u}}\}\}\|^2_{\boldsymbol{X}}\Big)\\ &\qquad + \eta_1\Delta t\Big(\|\{\partial_{\Delta t}\boldsymbol{\theta}^n_{\boldsymbol{u}}\}\|^2_{\boldsymbol{H}} + \|\{\{\theta^n_p\}\}\|^2_S\Big)\,.\end{aligned}$$

Summing up and using (33) and (35) gives

$$\begin{aligned}\|\partial_{\Delta t}\boldsymbol{\theta}^n_{\boldsymbol{u}}\|^2_{\boldsymbol{H}} + \|\{\boldsymbol{\theta}^n_{\boldsymbol{u}}\}\|^2_{\boldsymbol{X}} &\le C_1 + \frac{\Delta t}{\eta_1}\sum_{k=3}^{n}(S^k + T^k) + \Delta t\sum_{k=2}^{n-1}\Big(\|\{\partial_{\Delta t}\boldsymbol{\theta}^k_{\boldsymbol{u}}\}\|^2_{\boldsymbol{H}} + \|\{\{\theta^k_p\}\}\|^2_S\Big)\\ &\quad + \eta_1\Delta t\Big(\|\{\partial_{\Delta t}\boldsymbol{\theta}^n_{\boldsymbol{u}}\}\|^2_{\boldsymbol{H}} + \|\{\{\theta^n_p\}\}\|^2_S\Big)\,,\end{aligned} \tag{38}$$

with $C_1 = \|\partial_{\Delta t}\boldsymbol{\theta}^1_{\boldsymbol{u}}\|^2_{\boldsymbol{H}} + \|\{\boldsymbol{\theta}^1_{\boldsymbol{u}}\}\|^2_{\boldsymbol{X}}$.
Again using (37), we obtain for $n > 2$

$$\begin{aligned}&m(\{\{\partial_{\Delta t}\boldsymbol{\theta}^n_{\boldsymbol{a}}\}\}, \boldsymbol{w}_h) + a(\{\{\partial_{\Delta t}\boldsymbol{\theta}^n_{\boldsymbol{u}}\}\}, \boldsymbol{w}_h) + b(\boldsymbol{w}_h, \{\{\partial_{\Delta t}\theta^n_p\}\}) + m(\{\{\partial_{\Delta t}\boldsymbol{\varrho}^n_{\boldsymbol{a}}\}\}, \boldsymbol{w}_h) = 0\,,\\ &\qquad b(\{\{\partial_{\Delta t}\boldsymbol{\theta}^n_{\boldsymbol{v}}\}\}, q_h) - c_\kappa(\{\{\partial_{\Delta t}\theta^n_p\}\}, q_h) + b(\{\{\partial_{\Delta t}\boldsymbol{\varrho}^n_{\boldsymbol{v}} - \partial_{\Delta t}\boldsymbol{\varrho}^n_{\boldsymbol{u}}\}\}, q_h) = 0\,,\end{aligned}$$

and inserting $\boldsymbol{w}_h = \{\partial^2_{\Delta t}\boldsymbol{\theta}^n_{\boldsymbol{u}}\}$ and $q_h = \{\{\partial_{\Delta t}\theta^n_p\}\}$ gives

$$\begin{aligned}&m(\{\{\partial_{\Delta t}\boldsymbol{\xi}^n_{\boldsymbol{a}}\}\}, \{\partial^2_{\Delta t}\boldsymbol{\theta}^n_{\boldsymbol{u}}\}) + a(\{\{\partial_{\Delta t}\boldsymbol{\theta}^n_{\boldsymbol{u}}\}\}, \{\partial^2_{\Delta t}\boldsymbol{\theta}^n_{\boldsymbol{u}}\}) + b(\{\partial^2_{\Delta t}\boldsymbol{\theta}^n_{\boldsymbol{u}}\}, \{\{\partial_{\Delta t}\theta^n_p\}\}) = 0\,,\\ &b(\{\{\partial_{\Delta t}\boldsymbol{\theta}^n_{\boldsymbol{v}}\}\}, \{\{\partial_{\Delta t}\theta^n_p\}\}) - c_\kappa(\{\{\partial_{\Delta t}\theta^n_p\}\}, \{\{\partial_{\Delta t}\theta^n_p\}\}) + b(\{\{\partial_{\Delta t}\boldsymbol{\varrho}^n_{\boldsymbol{v}} - \partial_{\Delta t}\boldsymbol{\varrho}^n_{\boldsymbol{u}}\}\}, \{\{\partial_{\Delta t}\theta^n_p\}\}) = 0\,,\end{aligned}$$

and we obtain

$$\begin{aligned} &m(\{\{\partial_{\Delta t}\boldsymbol{\xi}_{\boldsymbol{a}}^n\}\},\{\partial_{\Delta t}^2\boldsymbol{\theta}_{\boldsymbol{u}}^n\}) + a(\{\{\partial_{\Delta t}\boldsymbol{\theta}_{\boldsymbol{u}}^n\}\},\{\partial_{\Delta t}^2\boldsymbol{\theta}_{\boldsymbol{u}}^n\}) + c_\kappa(\{\{\partial_{\Delta t}\theta_p^n\}\},\{\{\partial_{\Delta t}\theta_p^n\}\}) \\ &\qquad = b(\partial_{\Delta t}\{\{\boldsymbol{\xi}_{\boldsymbol{v}}^n - \boldsymbol{\varrho}_{\boldsymbol{u}}^n\}\} - \{\partial_{\Delta t}^2\boldsymbol{\theta}_{\boldsymbol{u}}^n\},\{\{\partial_{\Delta t}\theta_p^n\}\})\,. \end{aligned}$$

Inserting (34) and using Lemma 11 gives

$$\begin{aligned} &m(\partial_{\Delta t}^3\boldsymbol{\xi}_{\boldsymbol{u}}^n + \boldsymbol{\delta}_4^n,\{\partial_{\Delta t}^2\boldsymbol{\theta}_{\boldsymbol{u}}^n\}) + a(\{\{\partial_{\Delta t}\boldsymbol{\theta}_{\boldsymbol{u}}^n\}\},\{\partial_{\Delta t}^2\boldsymbol{\theta}_{\boldsymbol{u}}^n\}) + c_\kappa(\{\{\partial_{\Delta t}\theta_p^n\}\},\{\{\partial_{\Delta t}\theta_p^n\}\}) \\ &\qquad = b(\partial_{\Delta t}\{\{\boldsymbol{\xi}_{\boldsymbol{v}}^n\}\} - \partial_{\Delta t}\{\{\boldsymbol{\varrho}_{\boldsymbol{u}}^n\}\} - \{\partial_{\Delta t}^2\boldsymbol{\xi}_{\boldsymbol{u}}^n\} + \{\partial_{\Delta t}^2\boldsymbol{\varrho}_{\boldsymbol{u}}^n\},\partial_{\Delta t}\{\{\theta_p^n\}\}) \\ &\qquad = -b(\partial_{\Delta t}^2\{\boldsymbol{\xi}_{\boldsymbol{v}}^n\} - \partial_{\Delta t}^2\{\boldsymbol{\varrho}_{\boldsymbol{u}}^n\} - \partial_{\Delta t}^3\boldsymbol{\xi}_{\boldsymbol{u}}^n + \partial_{\Delta t}^3\boldsymbol{\varrho}_{\boldsymbol{u}}^n,\{\{\{\theta_p^n\}\}\}) \\ &\qquad\quad + \partial_{\Delta t}b(\partial_{\Delta t}\{\boldsymbol{\xi}_{\boldsymbol{v}}^n\} - \partial_{\Delta t}\{\boldsymbol{\varrho}_{\boldsymbol{u}}^n\} - \partial_{\Delta t}^2\boldsymbol{\xi}_{\boldsymbol{u}}^n + \partial_{\Delta t}^2\boldsymbol{\varrho}_{\boldsymbol{u}}^n,\{\{\theta_p^n\}\}) \\ &\qquad = -b(\boldsymbol{\delta}_5^n + \partial_{\Delta t}^3\boldsymbol{\varrho}_{\boldsymbol{u}}^n - \{\partial_{\Delta t}^2\boldsymbol{\varrho}_{\boldsymbol{u}}^n\},\{\{\{\theta_p^n\}\}\}) + \partial_{\Delta t}b(\boldsymbol{\delta}_3^n + \partial_{\Delta t}^2\boldsymbol{\varrho}_{\boldsymbol{u}}^n - \{\partial_{\Delta t}\boldsymbol{\varrho}_{\boldsymbol{u}}^n\},\{\{\theta_p^n\}\})\,. \end{aligned}$$

Now we set $\boldsymbol{\eta}_1^n = \partial_{\Delta t}^3\boldsymbol{\varrho}_{\boldsymbol{u}}^n + \boldsymbol{\delta}_4^n$, $\boldsymbol{\eta}_2^n = \boldsymbol{\delta}_5^n + \partial_{\Delta t}^3\boldsymbol{\varrho}_{\boldsymbol{u}}^n - \{\partial_{\Delta t}^2\boldsymbol{\varrho}_{\boldsymbol{u}}^n\}$, $\boldsymbol{\eta}_3^n = \boldsymbol{\delta}_3^n + \partial_{\Delta t}^2\boldsymbol{\varrho}_{\boldsymbol{u}}^n - \{\partial_{\Delta t}\boldsymbol{\varrho}_{\boldsymbol{u}}^n\}$. From (31) we obtain

$$\begin{aligned} \|\partial_{\Delta t}^2\boldsymbol{\theta}_{\boldsymbol{u}}^n\|_{\boldsymbol{H}}^2 + \|\{\partial_{\Delta t}\boldsymbol{\theta}_{\boldsymbol{u}}^n\}\|_{\boldsymbol{X}}^2 &= \|\partial_{\Delta t}^2\boldsymbol{\theta}_{\boldsymbol{u}}^{n-1}\|_{\boldsymbol{H}}^2 + \|\{\partial_{\Delta t}\boldsymbol{\theta}_{\boldsymbol{u}}^{n-1}\}\|_{\boldsymbol{X}}^2 \\ &\quad + 2\Delta t\Big(m(\partial_{\Delta t}^3\boldsymbol{\theta}_{\boldsymbol{u}}^n,\{\partial_{\Delta t}^2\boldsymbol{\theta}_{\boldsymbol{u}}^n\}) + a(\{\{\partial_{\Delta t}\boldsymbol{\theta}_{\boldsymbol{u}}^n\}\},\partial_{\Delta t}^2\{\boldsymbol{\theta}_{\boldsymbol{u}}^n\})\Big) \\ &\le \|\partial_{\Delta t}^2\boldsymbol{\theta}_{\boldsymbol{u}}^{n-1}\|_{\boldsymbol{H}}^2 + \|\{\partial_{\Delta t}\boldsymbol{\theta}_{\boldsymbol{u}}^{n-1}\}\|_{\boldsymbol{X}}^2 + 2\Delta t\,\|\boldsymbol{\eta}_1^n\|_{\boldsymbol{H}}\|\{\partial_{\Delta t}^2\boldsymbol{\theta}_{\boldsymbol{u}}^n\}\|_{\boldsymbol{H}} \\ &\quad + 2\Delta t\,b(\boldsymbol{\eta}_2^n,\{\{\{\theta_p^n\}\}\}) + 2\Delta t\,\partial_{\Delta t}b(\boldsymbol{\eta}_3^n,\{\{\theta_p^n\}\})\,. \end{aligned}$$

Summing up and using $\|\{\{\{\theta_p^n\}\}\}\|_S \le \frac{1}{2}\|\{\{\theta_p^{n-1}\}\}\|_S + \frac{1}{2}\|\{\{\theta_p^n\}\}\|_S$ gives

$$\begin{aligned} &\|\partial_{\Delta t}^2\boldsymbol{\theta}_{\boldsymbol{u}}^n\|_{\boldsymbol{H}}^2 + \|\{\partial_{\Delta t}\boldsymbol{\theta}_{\boldsymbol{u}}^n\}\|_{\boldsymbol{X}}^2 \\ &\quad \le \|\partial_{\Delta t}^2\boldsymbol{\theta}_{\boldsymbol{u}}^2\|_{\boldsymbol{H}}^2 + \|\{\partial_{\Delta t}\boldsymbol{\theta}_{\boldsymbol{u}}^2\}\|_{\boldsymbol{X}}^2 + 2\Delta t\sum_{k=3}^n\Big(\|\boldsymbol{\eta}_1^k\|_{\boldsymbol{H}}\|\{\partial_{\Delta t}^2\boldsymbol{\theta}_{\boldsymbol{u}}^k\}\|_{\boldsymbol{H}} + \|\boldsymbol{\eta}_2^k\|_{\boldsymbol{X}}\|\{\{\{\theta_p^k\}\}\}\|_S\Big) \\ &\qquad + 2\,b(\boldsymbol{\eta}_3^n,\{\{\theta_p^n\}\}) - 2\,b(\boldsymbol{\eta}_3^2,\{\{\theta_p^2\}\}) \\ &\quad \le C_2 + \frac{\Delta t}{\eta_2}\sum_{k=3}^n(S^k + T^k) + \frac{1}{\eta_3}\|\boldsymbol{\eta}_3^n\|_{\boldsymbol{X}}^2 + \Delta t\sum_{k=3}^{n-1}\Big(\|\partial_{\Delta t}^2\boldsymbol{\theta}_{\boldsymbol{u}}^k\|_{\boldsymbol{H}}^2 + \|\{\{\theta_p^k\}\}\|_S^2\Big) \\ &\qquad + \eta_2\Delta t\Big(\|\partial_{\Delta t}^2\boldsymbol{\theta}_{\boldsymbol{u}}^n\|_{\boldsymbol{H}}^2 + \|\{\{\theta_p^n\}\}\|_S^2\Big) + \eta_3\|\{\{\theta_p^n\}\}\|_S^2\,, \end{aligned} \tag{39}$$

with $C_2 = \|\partial_{\Delta t}^2\boldsymbol{\theta}_{\boldsymbol{u}}^2\|_{\boldsymbol{H}}^2 + \|\{\partial_{\Delta t}\boldsymbol{\theta}_{\boldsymbol{u}}^2\}\|_{\boldsymbol{X}}^2 + \|\boldsymbol{\eta}_3^2\|_{\boldsymbol{X}}^2 + \|\{\{\theta_p^2\}\}\|_S^2$. Estimating $b(\boldsymbol{w}_h,\theta_p^n)$ in (37a) gives

$$\|\{\{\theta_p^n\}\}\|_S \le C_\rho\|\{\{\boldsymbol{\xi}_{\boldsymbol{a}}^n\}\}\|_{\boldsymbol{H}} + \|\{\{\boldsymbol{\theta}_{\boldsymbol{u}}\}\}\|_{\boldsymbol{X}}\,,$$

and choosing $q_h = \{\{\theta_p^n\}\}$ in (37b) results in

$$\|\{\{\theta_p^n\}\}\|_\kappa^2 \le \|\{\{\theta_p^n\}\}\|_S^2 + \|\{\{\boldsymbol{\xi}_{\boldsymbol{v}}^n\}\}\|_{\boldsymbol{X}}^2 + \|\{\{\boldsymbol{\varrho}_{\boldsymbol{u}}^n\}\}\|_{\boldsymbol{X}}^2\,.$$

Together we obtain

$$\|\{\{\theta_p^n\}\}\|_Q^2 \lesssim \|\partial_{\Delta t}^2\boldsymbol{\theta}_{\boldsymbol{u}}^n\|_{\boldsymbol{H}}^2 + \|\{\partial_{\Delta t}\boldsymbol{\theta}_{\boldsymbol{u}}^n\}\|_{\boldsymbol{X}}^2 + \|\{\{\boldsymbol{\theta}_{\boldsymbol{u}}^n\}\}\|_{\boldsymbol{X}}^2 + S^n + T^n\,. \tag{40}$$

and combining (38), (39), (40) and choosing $\eta_1,\eta_2,\eta_3 \in (0,1)$ sufficiently small, we obtain

$$\begin{aligned} &\|\{\{\theta_p^n\}\}\|_Q^2 + \|\partial_{\Delta t}^2\boldsymbol{\theta}_{\boldsymbol{u}}^n\|_{\boldsymbol{H}}^2 + \|\{\partial_{\Delta t}\boldsymbol{\theta}_{\boldsymbol{u}}^n\}\|_{\boldsymbol{X}}^2 + \|\{\{\boldsymbol{\theta}_{\boldsymbol{u}}^n\}\}\|_{\boldsymbol{X}}^2 \\ &\qquad \lesssim C_1 + C_2 + \|\boldsymbol{\eta}_3^n\|_{\boldsymbol{X}}^2 + \Delta t\sum_{k=3}^n\Big(T^k + S^k\Big) \\ &\qquad\quad + \Delta t\sum_{k=3}^{n-1}\Big(\|\{\{\theta_p^n\}\}\|_Q^2 + \|\partial_{\Delta t}^2\boldsymbol{\theta}_{\boldsymbol{u}}^n\|_{\boldsymbol{H}}^2 + \|\{\partial_{\Delta t}\boldsymbol{\theta}_{\boldsymbol{u}}^n\}\|_{\boldsymbol{X}}^2 + \|\{\{\boldsymbol{\theta}_{\boldsymbol{u}}^n\}\}\|_{\boldsymbol{X}}^2\Big)\,. \end{aligned}$$

Now, we estimate $\|\boldsymbol{\eta}_3^n\|_{\boldsymbol{X}}^2$ by Lemmata 12 and 13 and by Sobolev's embedding theorem [**7**, Th. 5.9.2]

$$\|\boldsymbol{\eta}_3^n\|_{\boldsymbol{X}}^2 \lesssim (\Delta t)^4\|\boldsymbol{u}\|_{W^{4,\infty}(0,T;\boldsymbol{X})}^2 + \|\boldsymbol{\varrho}_{\boldsymbol{u}}\|_{W^{2,\infty}(0,T;\boldsymbol{X})}^2 \lesssim (\Delta t)^4\|\boldsymbol{u}\|_{H^5(0,T;\boldsymbol{X})}^2 + \|\boldsymbol{\varrho}_{\boldsymbol{u}}\|_{H^3(0,T;\boldsymbol{X})}^2$$

Together with (33) and (35) follows

$$\begin{aligned} &\|\{\{\theta_p^n\}\}\|_Q^2 + \|\partial_{\Delta t}^2 \boldsymbol{\theta}_{\boldsymbol{u}}^n\|_{\boldsymbol{H}}^2 + \|\{\partial_{\Delta t}\boldsymbol{\theta}_{\boldsymbol{u}}^n\}\|_{\boldsymbol{X}}^2 + \|\{\{\boldsymbol{\theta}_{\boldsymbol{u}}^n\}\}\|_{\boldsymbol{X}}^2 \\ &\quad \lesssim C_1 + C_2 + (\Delta t)^4 \|\boldsymbol{u}\|_{H^5(0,T;\boldsymbol{X})}^2 + \|\boldsymbol{\varrho}_{\boldsymbol{u}}\|_{H^3(0,T;\boldsymbol{X})}^2 \\ &\qquad + \Delta t \sum_{k=3}^{n-1} \Big(\|\{\{\theta_p^n\}\}\|_Q^2 + \|\partial_{\Delta t}^2 \boldsymbol{\theta}_{\boldsymbol{u}}^n\|_{\boldsymbol{H}}^2 + \|\{\partial_{\Delta t}\boldsymbol{\theta}_{\boldsymbol{u}}^n\}\|_{\boldsymbol{X}}^2 + \|\{\{\boldsymbol{\theta}_{\boldsymbol{u}}^n\}\}\|_{\boldsymbol{X}}^2 \Big). \end{aligned}$$

Finally, applying the discrete Gronwall's lemma (Lemma 10) gives for $t_n \leq T$

$$\|\{\{\theta_p^n\}\}\|_Q^2 + \|\partial_{\Delta t}^2 \boldsymbol{\theta}_{\boldsymbol{u}}^n\|_{\boldsymbol{H}}^2 + \|\{\partial_{\Delta t}\boldsymbol{\theta}_{\boldsymbol{u}}^n\}\|_{\boldsymbol{X}}^2 + \|\{\{\boldsymbol{\theta}_{\boldsymbol{u}}^n\}\}\|_{\boldsymbol{X}}^2 \lesssim C_3 + (\Delta t)^4 \|\boldsymbol{u}\|_{H^5(0,T;\boldsymbol{X})}^2 + \|\boldsymbol{\varrho}_{\boldsymbol{u}}\|_{H^3(0,T;\boldsymbol{X})}^2 ,$$

with a larger constant depending on $\exp(T)$ and $C_3 = C_1 + C_2$. □

Together with Theorem 7 and Lemma 12, the final estimate follows.

Corollary 15. *If $\boldsymbol{u} \in H^3(0,T,H^2(\Omega,\mathbb{R}^3)) \cap H^5(0,T;\boldsymbol{X})$ and $p \in H^3(0,T;H^2(\Omega,\mathbb{R}) \cap Q)$ we have in case of sufficiently accurate initial values and sufficiently accurate first steps*

$$\begin{aligned} &\|\{\{\ddot{\boldsymbol{u}}(t_n) - \boldsymbol{a}_h^n\}\}\|_{\boldsymbol{H}} + \|\{\{\dot{\boldsymbol{u}}(t_n) - \boldsymbol{v}_h^n\}\}\|_{\boldsymbol{X}} + \|\{\{\boldsymbol{u}(t_n) - \boldsymbol{u}_h^n\}\}\|_{\boldsymbol{X}} + \|\{\{p(t_n) - p_h^n\}\}\|_Q \\ &\qquad \leq C(\boldsymbol{u},p)\Big(h + (\Delta t)^2\Big) \end{aligned}$$

with $C(\boldsymbol{u},p)$ independent of κ, Δt and h.

5.3. A discrete energy estimate. Lemma 11 directly yields the time-discrete analog to (11) and (12). For this purpose, we define for $n \geq 0$

$$\mathcal{E}_{\mathrm{wave},h}^n(\boldsymbol{u}_h^n, \boldsymbol{v}_h^n) = \frac{1}{2} m(\boldsymbol{v}_h^n, \boldsymbol{v}_h^n) + \frac{1}{2} a(\boldsymbol{u}_h^n, \boldsymbol{v}_h^n), \tag{41a}$$

$$\mathcal{E}_h^n(\boldsymbol{u}_h^n, \boldsymbol{v}_h^n, p_h^n) = \mathcal{E}_{\mathrm{wave},h}^n(\boldsymbol{u}_h^n, \boldsymbol{v}_h^n) + \Delta t \sum_{i=1}^n c_\kappa(\{p_h^i\}, \{p_h^i\}). \tag{41b}$$

Let $\boldsymbol{u}_h^n, \boldsymbol{v}_h^n, \boldsymbol{a}_h^n \in \boldsymbol{X}_h$ and $p_h^n \in Q_h$ be a solution of (29) with homogeneous right-hand side $\ell \equiv 0$ and initial values $(\boldsymbol{u}_h^0, \boldsymbol{v}_h^0) \in \boldsymbol{X}_h \times \boldsymbol{X}_h$. Then, for $n > 0$, we have energy conservation in the form

$$\begin{aligned} \mathcal{E}_h^n(\boldsymbol{u}_h^n, \boldsymbol{v}_h^n, p_h^n) &= \mathcal{E}_{\mathrm{wave},h}^0(\boldsymbol{u}_h^0, \boldsymbol{v}_h^0). \\ \partial_{\Delta t}\mathcal{E}_{\mathrm{wave},h}^n(\boldsymbol{u}_h^0, \boldsymbol{v}_h^0) &= -c_\kappa(\{p_h^n\}, \{p_h^n\}) \leq 0. \end{aligned}$$

In particular, in the undamped case $\kappa = 0$, energy would be conserved exactly.

A discrete interpretation of the Lagrange principle in Sect. 2.7 is not obvious.

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