GLOBAL PROPERTIES OF GENERALIZED ORNSTEIN–UHLENBECK OPERATORS ON $L^p(\mathbb{R}^N, \mathbb{R}^N)$ WITH MORE THAN LINEARLY GROWING COEFFICIENTS

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ABSTRACT. We show that the realization A_p of the elliptic operator $\mathcal{A}u = \mathrm{div}(Q\nabla u) + F\cdot\nabla u + Vu$ in $L^p(\mathbb{R}^N, \mathbb{R}^N)$, $p \in [1, +\infty[$, generates a strongly continuous semigroup, and we determine its domain $D(A_p) = \{u \in W^{2,p}(\mathbb{R}^N, \mathbb{R}^N) : F\cdot\nabla u + Vu \in L^p(\mathbb{R}^N, \mathbb{R}^N)\}$ if $1 < p < +\infty$. The diffusion coefficients $Q = (q_{ij})$ are uniformly elliptic and bounded together with their first-order derivatives, the drift coefficients F can grow as $|x|\log|x|$, and V can grow logarithmically. Our approach relies on the Monniaux-Prüss theorem on the sum of non commuting operators. We also prove L^p-L^q estimates and, under somewhat stronger assumptions, we establish pointwise gradient estimates and smoothing of the semigroup in the spaces $W^{\alpha,p}(\mathbb{R}^N, \mathbb{R}^N)$, $\alpha \in [0, 1]$, where $1 < p < +\infty$.

1. INTRODUCTION

Elliptic operators $\mathcal{A} = \mathrm{Tr}(QD^2) + F\cdot\nabla$ with unbounded coefficients on $\mathbb{R}^N$ appear naturally in many branches of mathematics, such as probability and mathematical finance. For this reason, the interest in such operators has considerably grown in recent years. Under mild assumptions one can construct a semigroup $\{T(t)\}$ of bounded operators in $C_b(\mathbb{R}^N)$ which solves the parabolic equation corresponding to $\mathcal{A}$. In general, $\{T(t)\}$ is neither strongly continuous nor analytic in $C_b(\mathbb{R}^N)$, in sharp contrast to the case of bounded coefficients. (See [3], [17], [21], and the references therein.) Nevertheless, under suitable assumptions on the coefficients one can prove pointwise gradient estimates for the function $T(t)f$, see [2], [3], [16]. Such estimates are crucial for the investigations of the inhomogeneous elliptic and parabolic equations corresponding to $\mathcal{A}$, as discussed in, e.g., [3, Chapter 5]. In the prototypical case of the Ornstein Uhlenbeck Operator (and in related cases), there is an invariant probability measure μ for $\{T(t)\}$ (i.e, it holds $\int_{\mathbb{R}^N} T(t)f\,d\mu = \int_{\mathbb{R}^N} f\,d\mu$ for all $f \in C_b(\mathbb{R}^N)$). One can thus extend $T(t)$ to the weighted space $L^p(\mathbb{R}^N, \mu)$. Here, it can be shown that the semigroup on $L^p(\mathbb{R}^N, \mu)$ is strongly continuous and analytic and that its generator is the realization of $\mathcal{A}$ defined on the weighted Sobolev space $W^{2,p}(\mathbb{R}^N, \mu)$, $1 < p < +\infty$, see [5], [15], [20], and the references therein. The picture changes drastically if one works on the usual Lebesgue space $L^p(\mathbb{R}^N)$. As it was observed in [23], already the one dimensional operator $\mathcal{A}\varphi(x) = \varphi''(x) - \mathrm{sign}(x)|x|^{1+\varepsilon}\varphi'(x)$, $x \in \mathbb{R}$, does not generate a C_0–semigroup on $L^p(\mathbb{R})$, if $\varepsilon > 0$. One obtains much better results for operators having a dominating potential, see, e.g., the recent papers [4], [9], [19], [24]. Without a dominating potential, one has to require strong conditions on F; for instance linear growth of F, see [8], [18], or that F grows at most as $|x|\ln(1 + |x|)$, see [23]. It turns out that the domain of the generator of the semigroup on $L^p(\mathbb{R}^N)$ is the intersection of the domains of the diffusion and the drift part. (The

2000 *Mathematics Subject Classification.* 35J15, 35K45, 47A55, 47D06.

Key words and phrases. Systems of elliptic PDE's, unbounded coefficients, strongly continuous semigroups, L^p-L^q estimates, gradient L^p-estimates.

L.L. wishes to thank the Departments of Mathematics of the Technische Universität in Darmstadt and of the Martin-Luther Universität in Halle, where this paper originated, for the kind hospitality during his visit. He also acknowledges the financial support of the M.I.U.R. research projects Prin 2004 and 2006 "Kolmogorov equations".

semigroup is not analytic, see [26]). Such results can be used in the investigation of global regularity properties of the densities of the invariant measure (if such a measure exist), see [23].

In the study of Navier-Stokes equations with linearly growing initial data, systems of the form

$$\begin{aligned}(\mathcal{A}\varphi)(x) &= \operatorname{div}(Q(x)\nabla\varphi(x)) + F(x)\cdot\nabla\varphi(x) + V(x)\varphi(x) \\ &= \Big(\operatorname{div}(Q(x)\nabla\varphi_i(x)) + F(x)\cdot\nabla\varphi_i(x) + \sum_{j=1}^N v_{ij}(x)\varphi_j(x)\Big)_{i=1,\dots,N},\end{aligned} \tag{1.1}$$

appear naturally (where $x \in \mathbb{R}^N$ and $\varphi \in C_c^\infty(\mathbb{R}^N, \mathbb{R}^N)$), see [12], [13]. Here one perturbs a diagonal operator given as in [8], [18], [23], by nondiagonal potential terms, which are bounded in the setting of [12], [13]. For the applications to Navier-Stokes equations, it is crucial to have gradient estimates and a precise description of the domain of the realization A_p of $\mathcal{A}$ in $L^p(\mathbb{R}^N, \mathbb{R}^N)$, see [12], [13]. However, if one tries to go beyond linearly growing initial data, one is confronted with more than linearly growing drift coefficients and with unbounded potentials.

In this paper we consider systems of the type (1.1) where the diffusion coefficients are uniformly elliptic and bounded together with their first derivatives, ∇V is bounded and the quadratic forms corresponding to $(DF)Q$ and V are bounded. These assumptions allow for coefficients such that F grows like $|x|\ln(1+|x|)$ and V as $\ln(1+|x|)$, see Example 2.2. Our first main result Theorem 2.7 shows that the realization A_p of the operator $\mathcal{A}$ with domain

$$D(A_p) = \left\{u \in W^{2,p}(\mathbb{R}^N, \mathbb{R}^N) : F\cdot\nabla u + Vu \in L^p(\mathbb{R}^N, \mathbb{R}^N)\right\},$$

generates a strongly continuous, consistent semigroup $\{T_p(t)\} = \{T(t)\}$ on $L^p(\mathbb{R}^N, \mathbb{R}^N)$, $p \in]1, +\infty[$. We stress that here the crucial point is the characterization of the domain. Under slightly stronger assumptions on the drift coefficient F, we also show that test functions are a core for A_p. We then deal with the case when $p = 1$ and prove that $\{T_2(t)\}$ can be extended from $L^1(\mathbb{R}^N, \mathbb{R}^N) \cap L^2(\mathbb{R}^N, \mathbb{R}^N)$ to a C_0–semigroup on $L^1(\mathbb{R}^N, \mathbb{R}^N)$, using the results for $p > 1$. This semigroup is consistent with $\{T_p(t)\}$ for each $p \in]1, +\infty[$ and its generator coincides with A_p on a core. We can then show that the semigroup maps $L^p(\mathbb{R}^N, \mathbb{R}^N)$ into $L^q(\mathbb{R}^N, \mathbb{R}^N)$, $1 \le p \le q \le +\infty$, and establish a corresponding estimate in Theorem 4.2. In the last section we prove analogous norm estimates for $T(t)$ acting from $W^{\alpha,p}(\mathbb{R}^N, \mathbb{R}^N)$ to $W^{\beta,p}(\mathbb{R}^N, \mathbb{R}^N)$, where $0 \le \alpha \le \beta \le 1$ and $1 < p < +\infty$, under slightly stronger hypotheses. This is done in Theorem 5.8, which follows by interpolation from the pointwise gradient estimates

$$\begin{aligned}|(\nabla T(t)f)(x)|^p &\le C_p e^{\omega_p t}\Big(\tilde{T}(t)(|f|^2 + |\nabla f|^2)^{\frac{p}{2}}\Big)(x), \\ |(\nabla T(t)f)(x)|^p &\le \tilde{C}_p e^{\tilde{\omega}_p t} t^{-\frac{p}{2}} \tilde{T}(t)(|f|^p)(x),\end{aligned} \tag{1.2}$$

for all $x \in \mathbb{R}^N$ and $f \in C_c^\infty(\mathbb{R}^N, \mathbb{R}^N)$. The inequalities (1.2) are shown in Section 5. Here, $\{\tilde{T}(t)\}$ is the semigroup associated with the realization of the operator $\tilde{\mathcal{A}} = \operatorname{div}(Q\nabla) + F\cdot\nabla$ in $C_b(\mathbb{R}^N)$.

For $V = 0$, we proved Theorem 2.7 in [23] for the operator $\tilde{A}_p = \operatorname{div}(Q\nabla) + F\cdot\nabla$. However, for $V \ne 0$, the result cannot be obtained by perturbating $\tilde{A}_p$ by V since $D(A_p) \ne D(\tilde{A}_p) \cap D(V)$, in general, as seen in Example 2.2. In the present proof in the second section we follow the strategy of [23] in so far that we treat $\mathcal{A}$ as the sum of the diffusion part $\mathcal{A}_0 = \operatorname{div} Q\nabla$ and the lower order part $B = F\cdot\nabla + V$. However, the presence of the (non-diagonal) potential perturbation leads to various new difficulties throughout the present paper. Some of them are technical, but some are more fundamental: In contrast to the case $V = 0$, the group generated by B is not positive and $\mathcal{A}$ does not satisfy a maximum principle, in general. Both properties have been crucial for our previous works [11] and [23]. To show Theorem 2.7 we apply the Dore-Venni type theorem on sums of non commuting operators from [22]. The first step is the construction of the (unbounded) group generated by B on $L^p(\mathbb{R}^N, \mathbb{R}^N)$. Using a recent result from [10], we can check that B has bounded imaginary powers.

In view of the known properties of $\mathcal{A}_0$ it then remains to verify a (rather sophisticated) commutator estimate for the resolvents of $\mathcal{A}_0$ and B. The case $p=1$ and the L^p–L^q estimates can then be settled using methods from semigroup theory and the Nash inequality in Sections 3 and 4. The proofs of the pointwise gradient estimates (1.2) in the last section are quite demanding. The basic idea is to apply the maximum principle Proposition 5.4 for $\tilde{\mathcal{A}}$ to certain functions constructed from the data and the semigroups $\{T(t)\}$ and $\{\tilde{T}(t)\}$. The necessary regularity results for this procedure are proved in Theorem 5.3 which is based on the domain description from Theorem 2.7.

Notation. For $k \in \mathbb{N} \cup \{+\infty\}$, we denote by $C_b^k(\mathbb{R}^N)$ the space of all functions $f : \mathbb{R}^N \to \mathbb{R}$ which are continuously differentiable up to the k-th order. We endow $C_b(\mathbb{R}^N)$ with the sup norm $\|f\|_\infty$ and $C_b^k(\mathbb{R}^N)$ ($k \in \mathbb{N}$) with the norm $\|f\|_{C_b^k(\mathbb{R}^N)} = \sum_{|\alpha|\le k} \|D^\alpha f\|_\infty$. Moreover, by $C_c^k(\mathbb{R}^N)$ we denote the space of $f \in C_b^k(\mathbb{R}^N)$ having compact support. Similarly, we define the space $C_b^k(\mathbb{R}^N, \mathbb{R}^N)$ (resp. $C_c^\infty(\mathbb{R}^N, \mathbb{R}^N)$) of functions $f : \mathbb{R}^N \to \mathbb{R}^N$ such that each component f_j ($j = 1, \ldots, N$) belongs to $C_b^k(\mathbb{R}^N)$ (resp. to $C_c^\infty(\mathbb{R}^N)$). The space $C_b^k(\mathbb{R}^N, \mathbb{R}^N)$ is endowed with the norm $\|f\|_{C_b^k(\mathbb{R}^N,\mathbb{R}^N)} = \sum_{j=1}^N \|f_j\|_{C_b^k(\mathbb{R}^N)}$. For $p \in [1, +\infty]$, $L^p(\mathbb{R}^N, \mathbb{R}^N)$ is the space of $f : \mathbb{R}^N \to \mathbb{R}^N$ such that $f_j \in L^p(\mathbb{R}^N)$ for each $j \in \{1, \cdots, n\}$, equipped with the norm

$$\|f\|_p^p = \sum_{i=1}^N \int_{\mathbb{R}^N} |f_i|^p\, dx, \quad \text{if} \ \ 1 \le p < +\infty, \quad \text{resp.} \quad \|f\|_\infty = \max_{i=1,\ldots,N} \|f_i\|_\infty, \quad \text{if} \ \ p = +\infty.$$

In a similar way one defines the vector valued Sobolev spaces $W^{k,p}(\mathbb{R}^N, \mathbb{R}^N)$ and Slobodetskii spaces $W^{\theta,p}(\mathbb{R}^N, \mathbb{R}^N)$, $\theta \in]0, 1[$, and denote by $\|\cdot\|_{k,p}$ and $\|\cdot\|_{\theta,p}$ the corresponding norms. By DF we designate the Jacobian of a function $F \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ and by ∇u the vector $(D_1 u, \cdots, D_N u)$ for a function $u \in C^1(\mathbb{R}^N, X)$ and a Banach space X. The Euclidean scalar product in $\mathbb{R}^N$ is denoted by $x \cdot y$ or $\langle x, y\rangle$, and $|x|$ is the corresponding norm. Also when A is a matrix, we use the notation $|A|$ to denote its Euclidean norm. When there is no danger of confusion, we use the notation $\langle f, g\rangle$ for the duality pairing between $L^p(\mathbb{R}^N, \mathbb{R}^N)$ and $L^{p'}(\mathbb{R}^N, \mathbb{R}^N)$, where the conjugate exponent $p' \in [1, +\infty]$ of p is given by $1/p + 1/p' = 1$. The open ball centered at 0 with radius $R > 0$ is designated by $B(R)$. For every measurable set $E \subset \mathbb{R}^N$, we denote by χ_E the characteristic function of E.

2. Generation and determination of the domain in $L^p(\mathbb{R}^N, \mathbb{R}^N)$

In this section we want to show that the realization A_p of the operator

$$\mathcal{A}u = \operatorname{div}(Q\nabla u) + F \cdot \nabla u + V u, \qquad u \in C_c^\infty(\mathbb{R}^N, \mathbb{R}^N), \tag{2.1}$$

in $L^p(\mathbb{R}^N, \mathbb{R}^N)$, $p \in]1, +\infty[$, with the domain

$$D(A_p) = \left\{u \in W^{2,p}(\mathbb{R}^N, \mathbb{R}^N) : F \cdot \nabla u + V u \in L^p(\mathbb{R}^N, \mathbb{R}^N)\right\}, \tag{2.2}$$

is the generator of a strongly continuous semigroup $\{T_p(t)\}$ on $L^p(\mathbb{R}^N, \mathbb{R}^N)$. Equation (2.1) means that $(\mathcal{A}u)_j = \operatorname{div}(Q\nabla u_j) + F \cdot \nabla u_j + (V u)_j$ for $u = (u_1, \cdots, u_N)$ and $j = 1, \cdots, N$. Throughout the paper we assume that the following conditions are satisfied.

Hypothesis 2.1. (i) $Q \in C_b^1(\mathbb{R}^N, \mathbb{R}^{N\times N})$, *the matrices* $Q(x) = [q_{ij}(x)]$ *are symmetric for any* $x \in \mathbb{R}^N$, *and there exist constants* $\alpha_1, \alpha_2 > 0$ *such that*

$$\alpha_1 |\xi|^2 \le \sum_{i,j=1}^N q_{ij}(x)\xi_i\xi_j \le \alpha_2 |\xi|^2, \qquad x, \xi \in \mathbb{R}^N; \tag{2.3}$$

(ii) $F \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ *and there exist constants* $\beta_1, \beta_2 \ge 0$ *such that*

$$|\langle DF(x)Q(x)\xi, \xi\rangle| \le \beta_1 |\xi|^2, \qquad x, \xi \in \mathbb{R}^N, \tag{2.4}$$

$$|(F(x)\cdot\nabla Q(x))_{ij}| = \left|\sum_{h=1}^N F_h(x)D_hq_{ij}(x)\right| \le \beta_2, \qquad i,j=1,\cdots,N,\ x\in\mathbb{R}^N; \tag{2.5}$$

(iii) $V \in C^1(\mathbb{R}^N,\mathbb{R}^{N\times N})$, *$\nabla V$ is bounded, and there exists a constant β_3 such that*

$$|\langle V(x)\xi,\xi\rangle| \le \beta_3|\xi|^2, \qquad x,\xi\in\mathbb{R}^N. \tag{2.6}$$

Due to Theorem 2.4 of [23], Hypothesis 2.1(i) and (ii) imply that the (diagonal) operator $\tilde{A}_p u = \operatorname{div}(Q\nabla u)+F\cdot\nabla u$ with the domain $D(\tilde{A}_p)=\{u\in W^{2,p}(\mathbb{R}^N,\mathbb{R}^N) : F\cdot\nabla u\in L^p(\mathbb{R}^N,\mathbb{R}^N)\}$ generates a C_0–semigroup on $L^p(\mathbb{R}^N,\mathbb{R}^N)$, $1<p<+\infty$. Even if the potential V satisfies Hypothesis 2.1(iii), it can happen that $D(A_p)$ is neither contained in $D(\tilde{A}_p)$ nor in $D(V)$, as shown by the following example.

Example 2.2. Let $N=2$, $p=2$, $Q=I$, $F\in C^1(\mathbb{R}^2,\mathbb{R}^2)$, and $V\in C^1(\mathbb{R}^2,\mathbb{R}^{2\times 2})$ be such that

$$F(x)=\ln|x|\begin{pmatrix}-x_2\\ x_1\end{pmatrix} \qquad\text{and}\qquad V(x)=\ln|x|\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} \qquad \text{for}\ \ |x|\ge 1.$$

It is easy to verify Hypothesis 2.1 for these functions. Take a function $\phi\in C^1(\mathbb{R}_+)$ with $\phi=0$ on $[0,1]$ and $\phi(r)=(r^2\ln r)^{-1}$ for $r\ge 2$. Define $u(x)=\phi(|x|)\,(x_2,x_1)$ for $x\in\mathbb{R}^2$. A straightforward computation shows that $u\in W^{2,2}(\mathbb{R}^2,\mathbb{R}^2)$ and $F\cdot\nabla u(x)+V(x)u(x)=0$ for $|x|\ge 2$; hence $u\in D(A_2)$. However, $|Vu(x)|=1/|x|$ for $|x|\ge 2$ so that $Vu\notin L^2(\mathbb{R}^2,\mathbb{R}^2)$, $u\notin D(V)$, and $u\notin D(\tilde{A}_2)$. ◊

As remarked in [23], the estimates (2.3) and (2.4) yield that

$$|DF(x)Q(x)+Q(x)(DF(x))^*| \le 2\beta_1, \tag{2.7}$$

$$|\operatorname{div}F(x)| \le C_N\alpha_1^{-1}\beta_1 =: \nu, \tag{2.8}$$

for all $x,\xi\in\mathbb{R}^N$ and a constant $C_N>0$ depending only on N. We further recall Lemma 2.1 of [23].

Lemma 2.3. *There exists a global flow $\varphi\in C^1(\mathbb{R}\times\mathbb{R}^N,\mathbb{R}^N)$ such that $u(t)=\varphi(t,x)$ is the unique solution of the initial value problem*

$$u'(t)=F(u(t)), \quad t\in\mathbb{R}, \qquad u(0)=x, \tag{2.9}$$

for each given $x\in\mathbb{R}^N$. Moreover, for some constants $M>0$ and $\gamma\in\mathbb{R}$ we have

$$|\nabla\varphi(t,x)| \le Me^{\gamma|t|}, \qquad t\in\mathbb{R},\ x\in\mathbb{R}^N. \tag{2.10}$$

For each fixed $x\in\mathbb{R}^N$, we denote by $\mathcal{U}(\cdot,x)$ the fundamental solution of the Cauchy problem

$$v'(t)=V(\varphi(-t,x))v(t), \quad t\in\mathbb{R}, \qquad v(0)=I_N, \tag{2.11}$$

where I_N denotes the $N\times N$ identity matrix.

Lemma 2.4. *We have $\mathcal{U}\in C^1(\mathbb{R}\times\mathbb{R}^N,\mathbb{R}^{N\times N})$, and there exist constants $c>0$ and $\overline{\gamma}\in\mathbb{R}$ such*

$$|\mathcal{U}(t,x)| \le e^{\beta_3|t|}, \tag{2.12}$$

$$|D_k\mathcal{U}(t,x)| \le ce^{\overline{\gamma}|t|}, \tag{2.13}$$

for every $(t,x)\in\mathbb{R}\times\mathbb{R}^N$ and $k\in\{1,\cdots,N\}$. (Recall that β_3 is given by (2.6)).

Proof. It is clear that $\mathcal{U}\in C^1(\mathbb{R}\times\mathbb{R}^N,\mathbb{R}^{N\times N})$. We introduce the function $\Phi_\xi(t,x)=|\mathcal{U}(t,x)\xi|^2$ for $(t,x)\in\mathbb{R}\times\mathbb{R}^N$ and a fixed $\xi\in\mathbb{R}^N$. Using condition (2.6), we derive

$$|D_t\Phi_\xi(t,x)| = 2\,|\langle V(\varphi(-t,x))\mathcal{U}(t,x)\xi,\mathcal{U}(t,x)\xi\rangle| \le 2\beta_3\,\Phi_\xi(t,x),$$

for all $(t,x) \in \mathbb{R} \times \mathbb{R}^N$. Estimate (2.12) then follows from Gronwall's lemma. To verify (2.13), we notice that

$$\begin{aligned} D_t D_k \mathcal{U}(t,x) &= D_k D_t \mathcal{U}(t,x) = D_k[V(\varphi(-t,\cdot))\mathcal{U}(t,\cdot)]\,(x) \\ &= V(\varphi(-t,x))D_k\mathcal{U}(t,x) + \sum_{h=1}^N (D_k\varphi_h)(-t,x)\,(D_hV)(\varphi(-t,x))\,\mathcal{U}(t,x) \\ &=: V(\varphi(-t,x))D_k\mathcal{U}(t,x) + g(t,x), \end{aligned}$$

for all $(t,x) \in \mathbb{R} \times \mathbb{R}^N$. Since $D_k\mathcal{U}(0,x) = 0$ for all $x \in \mathbb{R}^N$, we deduce that

$$D_k\mathcal{U}(t,x) = \int_0^t \mathcal{U}(t-s,x)g(s,x)\,ds. \tag{2.14}$$

Inequalities (2.10) and (2.12) and Hypothesis 2.1(iii) imply that

$$\begin{aligned} |g(t,x)| &\le |\nabla\varphi(-t,x)|\,|(\nabla V)(\varphi(-t,x))|\,|\mathcal{U}(t,x)| \\ &\le M\|\nabla V\|_\infty\, e^{\gamma|t|}e^{\beta_3|t|} =: Ce^{\overline{\gamma}|t|}, \end{aligned} \tag{2.15}$$

for $\overline{\gamma} := \gamma + \beta_3$ and every $t \in \mathbb{R}$, where we may assume that $\gamma > 0$. From formulas (2.14), (2.12) and (2.15), it follows that

$$|D_k\mathcal{U}(t,x)| \le C\left|\int_0^t e^{\beta_3|t-s|}e^{\overline{\gamma}|s|}\,ds\right| = \frac{C}{\gamma}\left(e^{\overline{\gamma}|t|} - e^{\beta_3|t|}\right),$$

for $(t,x) \in \mathbb{R} \times \mathbb{R}^N$, i.e., (2.13) holds. □

We now introduce the operators $S_p(t)$, $t \in \mathbb{R}$, by setting

$$(S_p(t)f)(x) = \mathcal{U}(t,\varphi(t,x))f(\varphi(t,x)), \qquad x \in \mathbb{R}^N, \tag{2.16}$$

for $f \in L^p(\mathbb{R}^N,\mathbb{R}^N)$ and $p \in]1,+\infty[$. As we will see in the next proposition, $\{S_p(t) : t \in \mathbb{R}\}$ is the C_0–group generated by the lower order part of A_p. Given $\varphi \in C^1(\mathbb{R}^N,\mathbb{R}^N)$, we denote by $\operatorname{div}(\varphi F)$ the function with the components

$$(\operatorname{div}(\varphi F))_j = \operatorname{div}(\varphi_j F), \qquad j = 1,\dots,N.$$

Proposition 2.5. *$\{S_p(t),\ t \in \mathbb{R}\}$ is a strongly continuous group in $L^p(\mathbb{R}^N,\mathbb{R}^N)$ and*

$$\|S_p(t)\|_{\mathcal{L}(L^p(\mathbb{R}^N,\mathbb{R}^N))} \le e^{\nu_p|t|}, \qquad t \in \mathbb{R}, \tag{2.17}$$

where $p \in]1,+\infty[$, $\nu_p := \beta_3 + \frac{\nu}{p}$, and ν is given by (2.8). The generator of $\{S_p(t)\}$ is the operator B_p given by

$$\begin{aligned} D(B_p) = \big\{f \in L^p(\mathbb{R}^N,\mathbb{R}^N) : \exists\, g \in L^p(\mathbb{R}^N,\mathbb{R}^N) \text{ with } \langle g,\varphi\rangle = \langle f, V^*\varphi - \operatorname{div}(\varphi F)\rangle \\ \forall\,\varphi \in C_c^1(\mathbb{R}^N,\mathbb{R}^N)\big\}, \end{aligned} \tag{2.18}$$

and $B_pf = g$. In particular, if $f \in W^{1,p}(\mathbb{R}^N,\mathbb{R}^N)$ and $F\cdot\nabla f + Vf \in L^p(\mathbb{R}^N,\mathbb{R}^N)$, then $f \in D(B_p)$ and $B_pf = F\cdot\nabla f + Vf$. Moreover, $C_c^1(\mathbb{R}^N,\mathbb{R}^N)$ is a core of B_p. The adjoint of B_p on $L^{p'}(\mathbb{R}^N,\mathbb{R}^N)$, $1/p + 1/p' = 1$, is given by

$$\begin{aligned} D(B_p^*) = \big\{f \in L^{p'}(\mathbb{R}^N,\mathbb{R}^N) : \exists\, g \in L^{p'}(\mathbb{R}^N,\mathbb{R}^N) \text{ with } \langle\varphi,g\rangle = \langle V\varphi + \operatorname{div}(\varphi F), f\rangle \\ \forall\,\varphi \in C_c^1(\mathbb{R}^N,\mathbb{R}^N)\big\}, \end{aligned} \tag{2.19}$$

and $B_p^*f = g - (\operatorname{div} F)f$. In particular, if $f \in W^{1,p'}(\mathbb{R}^N,\mathbb{R}^N)$ and $-F\cdot\nabla f + V^*f \in L^{p'}(\mathbb{R}^N,\mathbb{R}^N)$, then $f \in D(B_p')$ and $B_p^*f = -F\cdot\nabla f + V^*f - (\operatorname{div} F)f$. Finally, the restriction of $\{S_p(t)\}$ to $W^{1,p}(\mathbb{R}^N,\mathbb{R}^N)$ is still a strongly continuous group.

Proof. Step 1. We prove that $\{S_p(t)\}$ is a strongly continuous group in $L^p(\mathbb{R}^N,\mathbb{R}^N)$. Let $f \in L^p(\mathbb{R}^N,\mathbb{R}^N)$, $t\in\mathbb{R}$, and $p\in]1,+\infty[$. Since (2.9) is an autonomous problem, we have $\varphi(r,\varphi(s,x)) = \varphi(r+s,x)$ for $r,s\in\mathbb{R}$ and $x\in\mathbb{R}^N$, so that $\varphi(t,\cdot)$ has the inverse $\varphi(-t,\cdot)$. Taking into account (2.10) and (2.12), we can thus estimate

$$\begin{aligned}\sum_{j=1}^N \int_{\mathbb{R}^N} |(S_p(t)f)_j|^p\,dx &= \sum_{j=1}^N \int_{\mathbb{R}^N} |(\mathcal{U}(t,\varphi(t,x))f(\varphi(t,x)))_j|^p\,dx\\ &\le c_1\int_{\mathbb{R}^N}\Big(\sum_{j=1}^N |(\mathcal{U}(t,y)f(y))_j|^2\Big)^{\frac{p}{2}}\,|\det(\nabla\varphi(-t,y))|\,dy\\ &\le c_2 e^{d|t|}\int_{\mathbb{R}^N}\Big(\sum_{i=1}^N |f_i(y)|^2\Big)^{\frac{p}{2}}\,dy \le c_3 e^{d|t|}\,\|f\|_p^p,\end{aligned}$$

for some constants $c_1,c_2,c_3,d\ge 0$. Hence, $S_p(t)$ is a bounded operator in $L^p(\mathbb{R}^N,\mathbb{R}^N)$. It further holds

$$\begin{aligned}(S_p(t)S_p(s)f)(x) &= \mathcal{U}(t,\varphi(t,x))(S_p(s)f)(\varphi(t,x)) = \mathcal{U}(t,\varphi(t,x))\mathcal{U}(s,\varphi(s,\varphi(t,x)))f(\varphi(s,\varphi(t,x)))\\ &= \mathcal{U}(t,\varphi(t,x))\mathcal{U}(s,\varphi(s+t,x))f(\varphi(s+t,x)),\end{aligned}\tag{2.20}$$

for all $s,t\in\mathbb{R}$ and $x\in\mathbb{R}^N$. On the other hand, the uniqueness of (2.11) implies that

$$\mathcal{U}(t,\varphi(-s,y))\mathcal{U}(s,y) = \mathcal{U}(s+t,y),\tag{2.21}$$

for every $y\in\mathbb{R}^N$. Inserting equation (2.21) with $y=\varphi(t+s,x)$ into formula (2.20), we derive that $\{S_p(t)\}$ is a group. It remains to show that $t\mapsto S_p(t)f$ is continuous at $t=0$ for each $f\in L^p(\mathbb{R}^N,\mathbb{R}^N)$. Of course, we can limit ourselves to consider the case when $f\in C_c^\infty(\mathbb{R}^N,\mathbb{R}^N)$. For such an f, estimate (2.12) yields

$$\begin{aligned}|(S_p(t)f)(x)-f(x)| &\le |\mathcal{U}(t,\varphi(t,x))(f(\varphi(t,x))-f(x))| + |\mathcal{U}(t,\varphi(t,x))f(x)-f(x)|\\ &\le e^{\beta_3|t|}|f(\varphi(t,x))-f(x)| + |\mathcal{U}(t,\varphi(t,x))f(x)-f(x)|,\end{aligned}\tag{2.22}$$

for all $t\in\mathbb{R}$ and $x\in\mathbb{R}^N$. Since f has compact support, both terms on the right-hand side of (2.22) vanish outside a compact set $H\subset\mathbb{R}^N$, uniformly with respect to $t\in[-1,1]$. (In fact, the first term vanishes outside $\varphi(-t,\operatorname{supp} f)\cup\operatorname{supp} f$. Since the function $(t,x)\mapsto\varphi(-t,x)$ is continuous in $\mathbb{R}\times\mathbb{R}^N$, there exists a compact set in $\mathbb{R}^N$ which contains $\varphi(-t,\operatorname{supp} f)$ for every $t\in[-1,1]$). As a result, both terms on the right-hand side of (2.22) converge to 0 as $t\to 0$ uniformly in $x\in\mathbb{R}^N$, and therefore they converge in $L^p(\mathbb{R}^N,\mathbb{R}^N)$.

Step 2. We next determine the generator B_p of $\{S_p(t)\}$, where $p\in]1,+\infty[$. Let $\tilde D_p$ be the space given by the right-hand side of (2.18) and set $\tilde B_p f = g$ for $f\in\tilde D_p$. As a first step, we prove that $C_c^1(\mathbb{R}^N,\mathbb{R}^N)$ is a core of B_p. For $f\in C_c^1(\mathbb{R}^N,\mathbb{R}^N)$, the function $u(t,x)=(S_p(t)f)(x)$ continuously differentiable in $\mathbb{R}\times\mathbb{R}^N$ and $D_t u(0,x) = F(x)\cdot\nabla f(x)+V(x)f(x)$ due to (2.16) and Lemmas 2.3 and 2.4. We even obtain that $S_p(\cdot)f\in C^1(\mathbb{R},L^p(\mathbb{R}^N,\mathbb{R}^N))$ and $S_p(t)f\in C_c^1(\mathbb{R}^N,\mathbb{R}^N)$, for any $t\in\mathbb{R}$, because f has compact support and $\varphi(t,\cdot)$ is bijective from $\mathbb{R}^N$ to $\mathbb{R}^N$ for any $t\in\mathbb{R}$. As a result, $C_c^1(\mathbb{R}^N,\mathbb{R}^N)\subset D(B_p)$,

$$B_p f = F\cdot\nabla f + Vf \qquad \text{for}\quad f\in C_c^1(\mathbb{R}^N,\mathbb{R}^N),\tag{2.23}$$

and $C_c^1(\mathbb{R}^N,\mathbb{R}^N)$ is a core of B_p by Proposition II.1.7 of [7]. Consequently, for a given $f\in D(B_p)$, there exist functions $f_n\in C_c^1(\mathbb{R}^N,\mathbb{R}^N)$, $n\in\mathbb{N}$, such that $f_n\to f$ and $B_pf_n\to B_pf$ in $L^p(\mathbb{R}^N,\mathbb{R}^N)$ as $n\to+\infty$. ¿From (2.23) we infer that

$$\langle B_pf,\varphi\rangle = \lim_{n\to+\infty}\langle B_pf_n,\varphi\rangle = \lim_{n\to+\infty}\langle f_n, V^*\varphi - \operatorname{div}(\varphi F)\rangle = \langle f, V^*\varphi-\operatorname{div}(\varphi F)\rangle,\tag{2.24}$$

for every $\varphi \in C_c^1(\mathbb{R}^N,\mathbb{R}^N)$. This means that $B_p \subset \tilde{B}_p$. To prove the other inclusion, we take a number ω larger than the growth bound $\omega_0(B_p)$ of $\{S_p(t)\}$ and a function $f \in \mathrm{Ker}\,(\tilde{B}_p - \omega I)$. Then we have

$$0 = \langle (\tilde{B}_p - \omega I)f, \varphi\rangle = \langle f, V^*\varphi - F\cdot\nabla\varphi - (\operatorname{div} F)\varphi - \omega\varphi\rangle, \tag{2.25}$$

for every $\varphi \in C_c^1(\mathbb{R}^N,\mathbb{R}^N)$. Since the functions $-F$ and V^* also satisfy Hypothesis 2.1, the above results show that the operator $\varphi \mapsto V^*\varphi - F\cdot\nabla\varphi$ with domain $C_c^1(\mathbb{R}^N,\mathbb{R}^N)$ has a closure C in $L^{p'}(\mathbb{R}^N,\mathbb{R}^N)$ which generates a C_0–group. Thanks to (2.8) and the bounded perturbation theorem, $C - \operatorname{div} F$ is also a generator in $L^{p'}(\mathbb{R}^N,\mathbb{R}^N)$ having the core $C_c^1(\mathbb{R}^N,\mathbb{R}^N)$. Fixing a sufficiently large $\omega > \omega_0(B_p)$, formula (2.25) now implies that $f = 0$; i.e., $\omega I - \tilde{B}_p$ is injective. On the other hand, $\omega I - B_p \subset \omega I - \tilde{B}_p$ and $\omega I - B_p$ is surjective, so that we deduce $B_p = \tilde{B}_p$. The second assertion concerning B_p is an immediate consequence of the formula (2.18). Moreover, the domain of C is given by the right-hand side of the equation (2.19). The identity (2.24) further shows that $B_p^* = C - \operatorname{div} F$ on $C_c^1(\mathbb{R}^N,\mathbb{R}^N)$, which is a core of $C - \operatorname{div} F$. Hence, the operator B_p^* coincides with $C - \operatorname{div} F$, as asserted.

Step 3. In order to show (2.17), we first assume that $p \in [2,+\infty[$. Take $u \in C_c^1(\mathbb{R}^N,\mathbb{R}^N)$ and $\lambda > \nu_p$, and set $f = \lambda u - B_p u$. Multiplying both the sides of this equation by $u|u|^{p-2}$ and integrating by parts, we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} f\cdot u|u|^{p-2}\,dx &= \lambda\int_{\mathbb{R}^N}|u|^p\,dx - \int_{\mathbb{R}^N}(Vu\cdot u)|u|^{p-2}\,dx - \int_{\mathbb{R}^N}(F\cdot\nabla u)\cdot u|u|^{p-2}\,dx\\ &= \lambda\int_{\mathbb{R}^N}|u|^p\,dx - \int_{\mathbb{R}^N}(Vu\cdot u)|u|^{p-2}\,dx + \frac{1}{p}\int_{\mathbb{R}^N}(\operatorname{div} F)|u|^p\,dx.\end{aligned}$$

Hölder's inequality and the estimates (2.6) and (2.8) then yield $(\lambda - \beta_3 - \nu/p)\,\|u\|_p^p \le \|u\|_p^{p-1}\,\|f\|_p$, so that

$$\|u\|_p \le \frac{1}{\lambda - \beta_3 - \frac{\nu}{p}}\|f\|_p. \tag{2.26}$$

If $p \in]1,2[$, we multiply both the sides of $\lambda u - B_p u = f$ by $u(|u|^2+\delta)^{\frac{p}{2}-1}$ for $\delta > 0$ and integrate by parts as above. Then, letting $\delta \to 0^+$ we again obtain (2.26). Since $C_c^1(\mathbb{R}^N,\mathbb{R}^N)$ is a core for B_p, the estimate (2.26) holds for all $u \in D(B_p)$ by approximation. Inequality (2.17) is then a consequence of the Lumer Phillips theorem and (2.26).

Step 4. To conclude the proof, we need to show that the restriction of $\{S_p(t)\}$ to $W^{1,p}(\mathbb{R}^N,\mathbb{R}^N)$ is again a strongly continuous semigroup. As in Step 1 we see that $t \mapsto S_p(t)f$ is continuous in $W^{1,p}(\mathbb{R}^N,\mathbb{R}^N)$ for any $f \in C_c^1(\mathbb{R}^N,\mathbb{R}^N)$. Moreover, equation (2.16) implies that

$$(D_h S_p(t)f)(x) = \sum_{k=1}^N (D_k\mathcal{U})(t,\varphi(t,x))\,D_h\varphi_k(t,x)f(\varphi(t,x)) + \sum_{k=1}^N (S_p(t)D_k f)(x)\,D_h\varphi_k(t,x),$$

for $h = 1,\dots,N$. Using (2.10), (2.13) and (2.17), we can then estimate $\|\nabla S_p(t)f\|_p \le ce^{d|t|}\|f\|_{1,p}$ for $t \in \mathbb{R}$ and some constants $c \ge 1$, where $d = \gamma + \max\{\overline{\gamma} + \gamma/p, \nu_p\}$. Now, the strong continuity of $\{S_p(t)\}$ in $W^{1,p}(\mathbb{R}^N,\mathbb{R}^N)$ follows by approximation. □

In order to apply the results from [22] to our problem, we introduce the operator

$$\mathcal{A}_0\varphi = \operatorname{div}(Q\nabla\varphi), \qquad \varphi \in C_c^\infty(\mathbb{R}^N,\mathbb{R}^N),$$

and we denote by A_0 its realization in $L^p(\mathbb{R}^N,\mathbb{R}^N)$ with the domain $D(A_0) = W^{2,p}(\mathbb{R}^N,\mathbb{R}^N)$, where $p \in]1,+\infty[$. We fix a number ω larger than the growth bounds of $\{S_p(t)\}$ on $L^p(\mathbb{R}^N,\mathbb{R}^N)$ and $W^{1,p}(\mathbb{R}^N,\mathbb{R}^N)$. (We can choose ω independent of p by the proof of Proposition 2.5). We then set

$$\hat{A}_0 = I - A_0, \qquad B_\omega = \omega I - B_p, \tag{2.27}$$

where $p \in]1,+\infty[$ is fixed. In the following lemma we compute the commutator of these two operators. The straightforward proof is omitted, see Lemma 2.3 of [23] for the case $V = 0$.

Lemma 2.6. *Assume that $Q, V \in C^2(\mathbb{R}^N, \mathbb{R}^{N\times N})$ and $F \in C^2(\mathbb{R}^N, \mathbb{R}^N)$. Then, we have*

$$[\hat{A}_0 B_\omega - B_\omega \hat{A}_0]\varphi = \mathrm{div}\{[Q(DF)^* + (DF)Q - (\mathrm{div}\, F)Q - (F\cdot\nabla)Q]\nabla\varphi\} + (\mathrm{div}\, F)A_0\varphi + \sum_{i,j=1}^N q_{ij}(D_iV)D_j\varphi + \sum_{i,j=1}^N D_i(q_{ij}(D_jV)\varphi),$$

for every $\varphi \in C_c^\infty(\mathbb{R}^N, \mathbb{R}^N)$, where $(F\cdot\nabla)Q\nabla\varphi = ((F\cdot\nabla)Q\nabla\varphi_1, \ldots, (F\cdot\nabla)Q\nabla\varphi_N)$ and $((F\cdot\nabla)Q\nabla\varphi_k)_j = \sum_{i,h=1}^N F_i(D_iq_{jh})D_h\varphi_k$ for all $j,k = 1,\ldots,N$.

We can now prove our main generation result.

Theorem 2.7. *Assume that Hypothesis 2.1 holds and let $p \in]1,+\infty[$. Then, the operator $A_p = \mathrm{div}(Q\nabla u) + F\cdot\nabla u + Vu$ with the domain $D(A_p) = \{u \in W^{2,p}(\mathbb{R}^N, \mathbb{R}^N) : F\cdot\nabla u + Vu \in L^p(\mathbb{R}^N, \mathbb{R}^N)\}$ generates a strongly continuous semigroup $\{T_p(t)\}$ in $L^p(\mathbb{R}^N, \mathbb{R}^N)$ and*

$$\|T_p(t)\|_{\mathcal{L}(L^p(\mathbb{R}^N,\mathbb{R}^N))} \le e^{\nu_p t}, \qquad t \ge 0, \tag{2.28}$$

where $\nu_p = \beta_3 + \frac{\nu}{p}$. Moreover, $\{T_p(t)\}$ is consistent, i.e., $\{T_p(t)\}$ and $\{T_q(t)\}$ coincide on $L^p(\mathbb{R}^N, \mathbb{R}^N) \cap L^q(\mathbb{R}^N, \mathbb{R}^N)$ for all $p, q \in]1,+\infty[$. The adjoint A_p^* is given by $A_p^* v = \mathrm{div}(Q\nabla v) - F\cdot\nabla v + V^*v - (\mathrm{div}\, F)v$ for $v \in D(A_p^*) = \{v \in W^{2,p'}(\mathbb{R}^N, \mathbb{R}^N) : -F\cdot\nabla v + V^*v \in L^{p'}(\mathbb{R}^N, \mathbb{R}^N)\}$.

Proof. We want to employ Corollary 2 of [22] in order to show that $A_p - \kappa I$ is invertible in $L^p(\mathbb{R}^N, \mathbb{R}^N)$ for each fixed $p \in]1,+\infty[$ and some $\kappa > 0$. To this purpose, we must work in complex Banach spaces. It is straightforward to extend the above results to the canonical complexifications of the spaces we have considered so far. Recall the definition of $\hat{A}_0$ and B_ω in (2.27). It is well known that

$$\|(\lambda + \hat{A}_0)^{-1}\|_{\mathcal{L}(L^p(\mathbb{R}^N,\mathbb{R}^N))} \le \frac{c}{|\lambda| + 1} \qquad \text{and} \qquad \|\hat{A}_0^{is}\|_{\mathcal{L}(L^p(\mathbb{R}^N,\mathbb{R}^N))} \le ce^{\theta_A |s|},$$

for a (fixed, but arbitrary) $\theta_A \in]0, \frac{\pi}{2}[$, $\lambda \in \mathbb{C}\setminus\{0\}$ with $|\arg\lambda| < \pi - \theta_A$, $s \in \mathbb{R}$ and a constant c not depending on λ and s. Proposition 2.5 shows that $-B_\omega$ generates an exponentially stable semigroup on $L^p(\mathbb{R}^N, \mathbb{C}^N)$ which is also a group. Taking also into account the functional calculus formulated in Theorem 3.6 of [10], we infer that there exists an angle $\theta_B > \frac{\pi}{2}$ such that $\theta_A + \theta_B < \pi$ and

$$\|(\mu + B_\omega)^{-1}\|_{\mathcal{L}(L^p(\mathbb{R}^N,\mathbb{R}^N))} \le \frac{c}{|\mu|} \qquad \text{and} \qquad \|B_\omega^{is}\|_{\mathcal{L}(L^p(\mathbb{R}^N,\mathbb{R}^N))} \le ce^{\theta_B|s|},$$

for $s \in \mathbb{R}$, $\mu \in \mathbb{C}\setminus\{0\}$ with $|\arg\mu| < \pi - \theta_B$, and a constant c not depending on μ and s. In order to apply Corollary 2 of [22] it remains to estimate the operator

$$C(\lambda,\mu) := \hat{A}_0(\lambda + \hat{A}_0)^{-1}[\hat{A}_0^{-1}(\mu + B_\omega)^{-1} - (\mu + B_\omega)^{-1}\hat{A}_0^{-1}],$$

on $L^p(\mathbb{R}^N, \mathbb{C}^N)$, for the above λ and μ. For this purpose, we use an approximation argument. We consider a family of mollifiers $\{\varrho_n,\ n \in \mathbb{N}\}$ and a function $\zeta \in C_c^\infty(\mathbb{R}^N, \mathbb{R})$ such that $\chi_{B(1)} \le \zeta \le \chi_{B(2)}$. We further introduce the operator T_n defined by

$$(T_n\varphi)(x) = \zeta(n^{-1}x)\int_{\mathbb{R}^N} \varrho_n(y)\varphi(x-y)dy, \qquad x \in \mathbb{R}^N,\ \ n \in \mathbb{N},$$

for a locally integrable function $\varphi : \mathbb{R}^N \to \mathbb{C}$. Clearly, $T_n\varphi \in C_c^\infty(\mathbb{R}^N)$ for all $n \in \mathbb{N}$, $T_n\varphi$ converges to φ in $W^{2,p}(\mathbb{R}^N)$ for every $\varphi \in W^{2,p}(\mathbb{R}^N)$, and $T_n\varphi$ converges locally in $C^1(\mathbb{R}^N)$ to φ for each $\varphi \in C^1(\mathbb{R}^N)$, as n tends to $+\infty$. We also set $\mathcal{T}_n f = (T_n f_1, \ldots, T_n f_N)$ for a locally integrable function $f : \mathbb{R}^N \to \mathbb{C}^N$, $F^{(k)} = \mathcal{T}_k F$, $q_{ij}^{(n)} = T_n(q_{ij})$ and $V_{ij}^{(n)} = T_n(V_{ij})$ for $i,j = 1,\ldots,N$ and $n \in \mathbb{N}$. The operators $\hat{A}_0^{(k)}$ and $B_\omega^{(k)}$ are defined as the operators $\hat{A}_0$ and B_ω replacing the coefficients Q, F, V by $Q^{(k)}$, $F^{(k)}$, $V^{(k)}$, respectively. For a given $f \in D(B_p)$, we set

$$\mathcal{V}_{k,n}(f) = \hat{A}_0^{-1}[\hat{A}_0^{(k)} B_\omega^{(k)} - B_\omega^{(k)} \hat{A}_0^{(k)}]\mathcal{T}_n \hat{A}_0^{-1} f,$$

for every $k, n \in \mathbb{N}$. Thanks to Lemma 2.6, we can write

$$\begin{aligned}\mathcal{V}_{k,n}(f) = \hat{A}_0^{-1} \operatorname{div}\Big\{&\big[Q^{(k)}(DF^{(k)})^* + (DF^{(k)})Q^{(k)} - (\operatorname{div} F^{(k)})Q^{(k)} - (F^{(k)}\cdot\nabla)Q^{(k)}\big]\nabla\mathcal{T}_n\hat{A}_0^{-1}f \\ &+ Q^{(k)}(\nabla V^{(k)})\mathcal{T}_n\hat{A}_0^{-1}f\Big\} + \hat{A}_0^{-1}(\operatorname{div} F^{(k)})A_0^{(k)}\mathcal{T}_n\hat{A}_0^{-1}f \\ &+ \hat{A}_0^{-1}\sum_{i,j=1}^N q_{ij}^{(k)}(D_iV^{(k)})D_j\mathcal{T}_n\hat{A}_0^{-1}f. \end{aligned} \tag{2.29}$$

We observe that $(\hat{A}_0^{-1/2})^*$ is bounded from $L^{p'}(\mathbb{R}^N,\mathbb{C}^N)$ to $W^{1,p'}(\mathbb{R}^N,\mathbb{C}^N)$. Therefore, the operator $D_k(\hat{A}_0^{-1/2})^*$ is bounded in $L^{p'}(\mathbb{R}^N,\mathbb{C}^N)$, and we can thus extend the operator $\hat{A}_0^{-1/2}\operatorname{div}$ defined on $C_c^\infty(\mathbb{R}^N,\mathbb{C}^N)$ to a bounded operator $S : L^p(\mathbb{R}^N,\mathbb{C}^N) \to L^p(\mathbb{R}^N,\mathbb{C}^N)$. Letting $k \to +\infty$ in (2.29) and recalling that $\mathcal{T}_n\hat{A}_0^{-1} \in C_c^\infty(\mathbb{R}^N,\mathbb{C}^N)$, it follows that $\mathcal{V}_{k,n}(f)$ converges to the function

$$\begin{aligned}\mathcal{V}_n(f) \;=\; &\hat{A}_0^{-1/2}S\Big\{\big[Q(DF)^* + (DF)Q - (\operatorname{div} F)Q - (F\cdot\nabla)Q\big]\nabla\mathcal{T}_n\hat{A}_0^{-1}f + Q(\nabla V)\mathcal{T}_n\hat{A}_0^{-1}f\Big\} \\ &+\hat{A}_0^{-1}(\operatorname{div} F)A_0\mathcal{T}_n\hat{A}_0^{-1}f + \hat{A}_0^{-1}\sum_{i,j=1}^N q_{ij}(D_iV)D_j\mathcal{T}_n\hat{A}_0^{-1}f.\end{aligned} \tag{2.30}$$

From Hypothesis 2.1 and estimates (2.7) and (2.8), we know that the maps $Q(DF)^* + (DF)Q$, $(\operatorname{div} F)Q$, $(F\cdot\nabla)Q$, Q, and ∇V are bounded on $\mathbb{R}^N$. Therefore, we can take the limit as $n \to +\infty$ in (2.30) and deduce that $\mathcal{V}_n(f)$ converges in $L^p(\mathbb{R}^N,\mathbb{C}^N)$ to the function

$$\begin{aligned}\mathcal{V}(f) = &\hat{A}_0^{-1/2}S\Big\{\big[Q(DF)^* + (DF)Q - (\operatorname{div} F)Q - (F\cdot\nabla)Q\big]\nabla\hat{A}_0^{-1}f + Q(\nabla V)\hat{A}_0^{-1}f\Big\} \\ &+ \hat{A}_0^{-1}\big[(\operatorname{div} F)A_0\hat{A}_0^{-1}f + \nabla V\cdot(Q\nabla(\hat{A}_0^{-1}f))\big].\end{aligned}$$

Moreover, there exists a constant $c \geq 0$ such that

$$\|\hat{A}_0^{1/2}\mathcal{V}(f)\|_p \leq c\,\|f\|_p, \qquad f \in D(B). \tag{2.31}$$

In order to relate the crucial estimate (2.31) with the operators $C(\lambda,\omega)$, we introduce the maps

$$\begin{aligned}C_{k,n}(\lambda,\mu) \;&=\; \hat{A}_0(\lambda+\hat{A}_0)^{-1}(\mu+B_\omega)^{-1}\hat{A}_0^{-1}\big[\hat{A}_0^{(k)}B_\omega^{(k)} - B_\omega^{(k)}\hat{A}_0^{(k)}\big]\mathcal{T}_n\hat{A}_0^{-1}(\mu+B_\omega)^{-1} \\ &=\; \hat{A}_0(\lambda+\hat{A}_0)^{-1}(\mu+B_\omega)^{-1}\mathcal{V}_{k,n}(\mu+B_\omega)^{-1},\end{aligned}$$

for $k,n \in \mathbb{N}$, $|\arg\lambda| < \pi-\theta_A$, and $|\arg\mu| < \pi-\theta_B$. Take $g \in L^p(\mathbb{R}^N,\mathbb{C}^N)$ and set $f = (\mu+B_\omega)^{-1}g \in D(B_p)$. The above results show that $C_{k,n}(\lambda,\omega)g$ converges to the function

$$\tilde{C}(\lambda,\mu)g := \hat{A}_0^{1/2}(\lambda+\hat{A}_0)^{-1}\big\{\hat{A}_0^{1/2}(\mu+B_\omega)^{-1}\hat{A}_0^{-1/2}\big\}\hat{A}_0^{1/2}\mathcal{V}(f), \tag{2.32}$$

letting first $k \to +\infty$ and then $n \to +\infty$. By Proposition 2.5, the operator $-B_\omega$ generates an exponentially stable semigroup on $D(\hat{A}_0^{1/2}) = W^{1,p}(\mathbb{R}^N,\mathbb{C}^N)$. The inequality (2.31) thus yields

$$\|\tilde{C}(\lambda,\mu)g\|_p \leq \frac{M}{|\lambda|^{1/2}|\mu|}\|f\|_p \leq \frac{M'}{|\lambda|^{1/2}|\mu|^2}\|g\|_p, \tag{2.33}$$

for all $\lambda,\mu \in \mathbb{C}\setminus\{0\}$ with $|\arg\lambda| < \pi-\theta_A$ and $|\arg\mu| < \pi-\theta_B$, and some constants $M, M' \geq 0$ independent of λ and μ. On the other hand, it holds

$$\langle\mathcal{V}_{k,n}(f),\hat{A}_0^*\varphi\rangle = \langle[\hat{A}_0^{(k)}B_\omega^{(k)} - B_\omega^{(k)}\hat{A}_0^{(k)}]\mathcal{T}_n\hat{A}_0^{-1}f,\varphi\rangle$$

$$= \langle \hat{A}_0^{(k)} \mathcal{T}_n \hat{A}_0^{-1} f, (V^{(k)})^* \varphi - \mathrm{div}(\varphi F^{(k)}) \rangle - \langle B_p^{(k)} \mathcal{T}_n \hat{A}_0^{-1} f, (\hat{A}_0^{(k)})^* \varphi \rangle,$$

for every $f \in D(B_p)$ and $\varphi \in C_c^\infty(\mathbb{R}^N, \mathbb{C}^N)$. Letting $k \to +\infty$, we derive

$$\langle \mathcal{V}_n(f), \hat{A}_0^* \varphi \rangle = \langle \hat{A}_0 \mathcal{T}_n \hat{A}_0^{-1} f, V^* \varphi - \mathrm{div}(\varphi F) \rangle - \langle F \cdot \nabla \mathcal{T}_n \hat{A}_0^{-1} f, \hat{A}_0^* \varphi \rangle - \langle V \mathcal{T}_n \hat{A}_0^{-1} f, \hat{A}_0^* \varphi \rangle.$$

Taking the limit as $n \to +\infty$, this equation and (2.19) yield

$$\begin{aligned} \langle \mathcal{V}(f), \hat{A}_0^* \varphi \rangle &= \langle f, V^* \varphi - \mathrm{div}(\varphi F) \rangle - \langle F \cdot \nabla \hat{A}_0^{-1} f, \hat{A}_0^* \varphi \rangle - \langle V \hat{A}_0^{-1} f, \hat{A}_0^* \varphi \rangle \\ &= \langle B_p f, \varphi \rangle - \langle F \cdot \nabla \hat{A}_0^{-1} f + V \hat{A}_0^{-1} f, \hat{A}_0^* \varphi \rangle. \end{aligned}$$

Setting $\psi = \hat{A}_0^* \varphi$, we obtain

$$\langle \mathcal{V}(f), \psi \rangle = \langle \hat{A}_0^{-1} B_p f, \psi \rangle - \langle F \cdot \nabla \hat{A}_0^{-1} f + V \hat{A}_0^{-1} f, \psi \rangle.$$

Since test functions are a core for $\hat{A}_0^*$, the set $\hat{A}_0^*(C_c^\infty(\mathbb{R}^N, \mathbb{R}^N))$ is dense in $L^{p'}(\mathbb{R}^n, \mathbb{C}^N)$. Proposition 2.5 thus shows that $\hat{A}_0^{-1} f$ belongs to $D(B_p)$ and that $\mathcal{V}(f) = \hat{A}_0^{-1} B_p f - B_p \hat{A}_0^{-1} f$, for each $f \in D(B_p)$. Inserting this equality into (2.32), we get

$$\begin{aligned} \tilde{C}(\lambda, \mu) g &= \hat{A}_0 (\lambda + \hat{A}_0)^{-1} (\mu + B_\omega)^{-1} [\hat{A}_0^{-1} B_p - B_p \hat{A}_0^{-1}] (\mu + B_\omega)^{-1} g \\ &= \hat{A}_0 (\lambda + \hat{A}_0)^{-1} (\mu + B_\omega)^{-1} [B_\omega \hat{A}_0^{-1} - \hat{A}_0^{-1} B_\omega] (\mu + B_\omega)^{-1} g \\ &= C(\lambda, \mu) f. \end{aligned}$$

Therefore, estimate (2.33) gives

$$\|C(\lambda, \mu)\|_{\mathcal{L}(L^p(\mathbb{R}^N, \mathbb{R}^N))} \le \frac{M'}{|\lambda|^{1/2} |\mu|^2},$$

for all $\lambda, \mu \in \mathbb{C} \setminus \{0\}$ with $|\arg \lambda| < \pi - \theta_A$ and $|\arg \mu| < \pi - \theta_B$. Observe that $D(A_0) \cap D(B_p) = D(A_p)$ by Proposition 2.5. Corollary 2 of [22] now shows that the operator $\kappa I + \hat{A}_0 + B_\omega = (\kappa + 1 + \omega) I - A_p$, with domain $D(A_p)$, is invertible in $L^p(\mathbb{R}^N, \mathbb{R}^N)$ for some $\kappa \ge 0$. Taking Proposition 2.5 into account, it is easy to see that $A_0 + B_p^*$ coincides with the operator $\mathrm{div}(Q\nabla) - F \cdot \nabla + V^* - (\mathrm{div}\, F) f$ on $W^{2,p'}(\mathbb{R}^N, \mathbb{R}^N) \cap D(B_p^*)$ and that it is a restriction of A_p^*. Moreover, $A_0 + B_p^*$ is a generator on $L^{p'}(\mathbb{R}^N, \mathbb{C}^N)$ by the above results and Proposition 2.5. Hence, $A_p^* = A_0 + B_p^*$ has the asserted representation. The remaining claims can be proved using the Trotter–Kato product formula, Proposition 2.5, and the dissipativity of A_0. □

Proposition 2.8. *Assume that Hypothesis* 2.1 *holds and that* $p \in]1, +\infty[$. *Denote the components of* $Q(x)^{-1}$ *by* $r_{ij}(x)$ $(i, j = 1, \dots, N)$. *The following assertions are true.*

(*a*) *If, additionally,*

$$\langle F(x), x \rangle \le c\, |x|^2 \tag{2.34}$$

for some constant $c > 0$ *and all* $x \in \mathbb{R}^N$, *then the operator* A_p *coincides with the maximal realization of* $\mathcal{A}$ *in* $L^p(\mathbb{R}^N, \mathbb{R}^N)$, *i.e.,*

$$D(A_p) = \{u \in L^p(\mathbb{R}^N, \mathbb{R}^N) \cap W^{2,p}_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{R}^N) : \mathcal{A} u \in L^p(\mathbb{R}^N, \mathbb{R}^N)\}. \tag{2.35}$$

(*b*) *If condition* (2.34) *holds with* $-F$ *instead of* F, *then the space* $C_c^\infty(\mathbb{R}^N, \mathbb{R}^N)$ *is a core for* A_p.

Proof. (a) It suffices to show the inclusion "$\supset$" in (2.35). We denote by $D_{\max,p}$ the space in the right-hand side of (2.35). Since $\lambda I - A_p$ is invertible for each $\lambda > \nu_p$ (cf. Theorem 2.7), we have to establish the injectivity of $\lambda I - \mathcal{A}$ on $D_{\max,p}$ for some $\lambda > \nu_p$. Let $\lambda v = \mathcal{A} v$ for some $\lambda > \nu_p$ and $v \in D_{\max,p}$. We fix a decreasing smooth function $\psi : [0, +\infty[\to \mathbb{R}$, with $\chi_{[0,1]} \le \psi \le \chi_{[0,2]}$, and we set $\vartheta_n(x) = \psi(|x|/n)$ for $x \in \mathbb{R}^N$ and $n \in \mathbb{N}$. Then, $\vartheta_n \in C_b^1(\mathbb{R}^N)$, $\chi_{B(n)} \le \vartheta_n \le \chi_{B(2n)}$, and $\|\nabla\vartheta_n\|_\infty \le C$ for every $n \in \mathbb{N}$. We first consider the case $p \ge 2$. If we multiply the equation $\lambda v - \mathcal{A}v = 0$ by $v|v|^{p-2}\vartheta_n^2$ and integrate by parts, we obtain

$$\begin{aligned}0 = \lambda \int_{\mathbb{R}^N} |v|^p \vartheta_n^2\, dx &+ \int_{\mathbb{R}^N} \sum_{h=1}^N \langle Q\nabla v_h, \nabla v_h\rangle\, |v|^{p-2}\vartheta_n^2\, dx + \frac{p-2}{4}\int_{\mathbb{R}^N} |v|^{p-4}\vartheta_n^2 \langle Q\nabla(|v|^2), \nabla(|v|^2)\rangle\, dx \\ &+ 2\int_{\mathbb{R}^N} |v|^{p-2}\vartheta_n \sum_{h=1}^N v_h \langle Q\nabla\vartheta_n, \nabla v_h\rangle\, dx + \frac{1}{p}\int_{\mathbb{R}^N} (\operatorname{div} F)|v|^p\vartheta_n^2\, dx - \int_{\mathbb{R}^N} |v|^{p-2}\vartheta_n^2\langle Vv, v\rangle\, dx \\ &+ \int_{\mathbb{R}^N} \frac{2\vartheta_n(x)}{pn\,|x|}\, \psi'(|x|/n)\langle F(x), x\rangle\, |v(x)|^p\, dx. \end{aligned} \tag{2.36}$$

Using the Cauchy–Schwarz and Young's inequality, we can estimate

$$\begin{aligned}\int_{\mathbb{R}^N} |v|^{p-2}\vartheta_n \sum_{h=1}^N v_h\langle Q\nabla\vartheta_n, \nabla v_h\rangle\, dx &\ge -\int_{\mathbb{R}^N} |v|^{p-2}\vartheta_n |Q^{\frac12}\nabla\vartheta_n| \sum_{h=1}^N |v_h|\, |Q^{\frac12}\nabla v_h|\, dx \\ &\ge -\sqrt{\alpha_2}\int_{\mathbb{R}^N} |v|^{p-1}\vartheta_n|\nabla\vartheta_n| \Big[\sum_{h=1}^N |Q^{\frac12}\nabla v_h|^2\Big]^{\frac12} dx \\ &\ge -C\sqrt{\alpha_2}\, \|v\|_p^{\frac{p}{2}} \Big[\int_{\mathbb{R}^N} |v|^{p-2}\vartheta_n^2 \sum_{h=1}^N \langle Q\nabla v_h, \nabla v_h\rangle\, dx\Big]^{\frac12} \\ &\ge -\frac{C^2\alpha_2}{2}\int_{\mathbb{R}^N} |v|^p\, dx - \frac12 \int_{\mathbb{R}^N} \sum_{h=1}^N \langle Q\nabla v_h, \nabla v_h\rangle\, |v|^{p-2}\vartheta_n^2\, dx. \end{aligned} \tag{2.37}$$

Concerning the last integral in (2.36) we note that its integrand vanishes if $|x| \notin [n, 2n]$ and that $\psi' \le 0$. Assumption (2.34) thus implies that this integral is larger than $-\overline{c}\,\|v\|_p^p$ for a constant $\overline{c} \ge 0$ independent of v and λ. Using also (2.6), we can thus derive from (2.36) and (2.8) that

$$0 \ge \Big(\lambda - \beta_3 - \frac{\nu}{p}\Big)\int_{\mathbb{R}^N} \vartheta_n^2 |v|^p\, dx - (C^2\alpha_2 + \overline{c})\|v\|_p^p.$$

Letting $n \to +\infty$ and choosing a sufficiently large $\lambda \ge 0$, we arrive at $v = 0$, so that $\lambda - \mathcal{A}$ is injective for such λ. If $p \in]1, 2[$, we multiply the equation $\lambda v - \mathcal{A}v = 0$ by $(|v|^2 + \delta)^{(p-2)/2} v\vartheta_n^2$, where $\delta > 0$. Integrating by parts, it follows

$$\begin{aligned}0 \ge \lambda\int_{\mathbb{R}^N} |v|^2\vartheta_n^2(|v|^2+\delta)^{\frac{p-2}{2}}\, dx &+ \int_{\mathbb{R}^N}\sum_{h=1}^N \langle Q\nabla v_h, \nabla v_h\rangle \vartheta_n^2 (|v|^2+\delta)^{\frac{p-4}{2}} \big((p-1)|v|^2 + \delta\big)\, dx \\ &+ 2\int_{\mathbb{R}^N}\sum_{h=1}^N v_h\vartheta_n (|v|^2+\delta)^{\frac{p-2}{2}} \langle Q\nabla v_h, \nabla\vartheta_n\rangle\, dx - \int_{\mathbb{R}^N}\sum_{i,h=1}^N \vartheta_n^2 F_i (D_i v_h) v_h (|v|^2+\delta)^{\frac{p-2}{2}}\, dx \\ &- \int_{\mathbb{R}^N} (|v|^2+\delta)^{\frac{p-2}{2}}\vartheta_n^2 \langle Vv, v\rangle\, dx. \end{aligned} \tag{2.38}$$

Arguing as in (2.37), we can show that the sum of the second and the third integral in (2.38) is larger than

$$-\frac{C^2\alpha_2}{p-1}\int_{\mathbb{R}^N} (|v|^2+\delta)^{\frac{p}{2}}\, dx. \tag{2.39}$$

Using the theorem of dominated convergence, we can now let $\delta \to 0^+$ in (2.39) and in the first, fourth and fifth integral of (2.38), obtaining

$$0 \geq (\lambda - \beta_3) \int_{\mathbb{R}^N} \vartheta_n^2 |v|^p \, dx - \int_{\mathbb{R}^N} \sum_{h=1}^N \langle F, \nabla v_h \rangle v_h \vartheta_n^2 |v|^{p-2} \, dx - \frac{C^2 \alpha_2}{p-1} \int_{\mathbb{R}^N} |v|^p \, dx.$$

On the other hand, integration by parts yields that

$$\begin{aligned} &- \int_{\mathbb{R}^N} \sum_{i,h=1}^N \vartheta_n^2 F_i (D_i v_h) v_h \, (|v|^2 + \delta)^{\frac{p-2}{2}} \, dx \\ &= \int_{\mathbb{R}^N} \vartheta_n^2 (\operatorname{div} F) \, |v|^2 (|v|^2 + \delta)^{\frac{p-2}{2}} \, dx + 2 \int_{\mathbb{R}^N} \vartheta_n \langle F, \nabla \vartheta_n \rangle \, |v|^2 (|v|^2 + \delta)^{\frac{p-2}{2}} \, dx \\ &\quad + \int_{\mathbb{R}^N} \sum_{h=1}^N \vartheta_n^2 \langle F, \nabla v_h \rangle v_h \, (|v|^2 + \delta)^{\frac{p-2}{2}} \, dx + (p-2) \int_{\mathbb{R}^N} \sum_{j=1}^N \vartheta_n^2 \langle F, \nabla v_j \rangle v_j \, (|v|^2 + \delta)^{\frac{p-2}{2}} \, dx \\ &\quad - (p-2) \int_{\mathbb{R}^N} \sum_{j=1}^N \vartheta_n^2 \langle F, \nabla v_j \rangle v_j \, \delta (|v|^2 + \delta)^{\frac{p-4}{2}} \, dx. \end{aligned}$$

So, we derive

$$\begin{aligned} &- \int_{\mathbb{R}^N} \sum_{h=1}^N \vartheta_n^2 \langle F, \nabla v_h \rangle v_h \, (|v|^2 + \delta)^{\frac{p-4}{2}} (p|v|^2 + 2\delta) \, dx \\ &= \int_{\mathbb{R}^N} \vartheta_n^2 (\operatorname{div} F) \, |v|^2 (|v|^2 + \delta)^{\frac{p-2}{2}} \, dx + 2 \int_{\mathbb{R}^N} \vartheta_n \langle F, \nabla \vartheta_n \rangle \, |v|^2 (|v|^2 + \delta)^{\frac{p-2}{2}} \, dx. \end{aligned}$$

As above, we can take the limit as $\delta \to 0^+$ in the integrals on the right-hand side. On the left-hand side, the function $\sum_{h=1}^N \vartheta_n^2 \langle F, \nabla v_h \rangle v_h \, (|v|^2 + \delta)^{\frac{p-4}{2}} (p|v|^2 + 2\delta)$ can be estimated from above by the function $c_n \vartheta_n^2 \, |\nabla v| \, |v|^{p-1}$ in the set $\{\delta \leq |v|^2\}$, and by the function $c_n \, \delta^{(p-1)/2} |\nabla v| \vartheta_n^2$ in the set $\{|v|^2 \leq \delta\}$. So, the theorem of dominated convergence applies, taking as a majorant the function $c_n \vartheta_n^2 \, (|\nabla v| \, |v|^{p-1} + |\nabla v|)$ for $\delta \in]0, 1]$, and it yields

$$- \int_{\mathbb{R}^N} \sum_{h=1}^N \langle F, \nabla v_h \rangle v_h \vartheta_n^2 |v|^{p-2} \, dx = \frac{1}{p} \int_{\mathbb{R}^N} (\operatorname{div} F) \vartheta_n^2 |v|^p \, dx + \frac{2}{p} \int_{\mathbb{R}^N} |v|^p \langle F, \nabla \vartheta_n \rangle \vartheta_n \, dx.$$

We can now conclude that $v = 0$ as in the case $p \geq 2$.

(b) Suppose that $\langle (\lambda - A_p)\phi, f \rangle = 0$ for some $f \in L^{p'}(\mathbb{R}^N, \mathbb{R}^N)$, $\lambda \geq 0$, and all $\phi \in C_c^\infty(\mathbb{R}^N, \mathbb{R}^N)$. Standard elliptic regularity, see e.g., [1], then yields that $f \in W^{2,p'}_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{R}^N)$. Integrating by parts we, thus, obtain $0 = \langle \phi, (\lambda - \mathcal{A}^*) f \rangle$ for the (formal) adjoint $\mathcal{A}^*$ of $\mathcal{A}$, i.e., $\lambda f = \mathcal{A}^* f$. Applying part (a) to the operator $\mathcal{A}^* = \operatorname{div}(QDu) - \langle F, \nabla u \rangle + V^* - \operatorname{div} F$, we see that f belongs to the kernel of $\lambda I - A_p^*$, and hence $f = 0$ for sufficiently large $\lambda \geq 0$. So assertion (b) has been established. □

3. The generation result in $L^1(\mathbb{R}^N, \mathbb{R}^N)$

Using the results in Section 2, we prove that we can associate a strongly continuous semigroup $\{T_1(t)\}$ with the realization of the operator $\mathcal{A}$ in $L^1(\mathbb{R}^N, \mathbb{R}^N)$.

Proposition 3.1. *Assume that Hypothesis* 2.1 *holds. Then, the restriction of the semigroup* $\{T_2(t)\}$ *to* $L^2(\mathbb{R}^N, \mathbb{R}^N) \cap L^1(\mathbb{R}^N, \mathbb{R}^N)$ *can be extended to a* C_0*–semigroup* $\{T_1(t)\}$ *on* $L^1(\mathbb{R}^N, \mathbb{R}^N)$ *which is consistent with* $\{T_p(t)\}$*,* $p \in]1, +\infty[$*, and satisfies*

$$\|T_1(t)\|_{\mathcal{L}(L^1(\mathbb{R}^N, \mathbb{R}^N))} \leq e^{\nu_1 t}, \qquad t \geq 0, \tag{3.1}$$

where $\nu_1 = \beta_3 + C_N\beta_1/\alpha_1$. *The generator* A_1 *of* $\{T_1(t)\}$ *has the cores* $D_p = \{u \in D(A_p) \cap L^1(\mathbb{R}^N,\mathbb{R}^N) : A_p u \in L^1(\mathbb{R}^N,\mathbb{R}^N)\}$ *and* $A_1 u = A_p u$ *for* $u \in D_p$, *for each* $p \in]1,+\infty[$. *In particular,* A_1 *extends* $\mathcal{A}$.

Proof. Let $f \in L^2(\mathbb{R}^N,\mathbb{R}^N) \cap L^1(\mathbb{R}^N,\mathbb{R}^N)$ and $r > 0$. Note that $f \in L^p(\mathbb{R}^N,\mathbb{R}^N)$ for $p \in]1,2[$. Due to Theorem 2.7, the semigroups $\{T_p(t)\}$ and $\{T_q(t)\}$ are consistent for $p,q \in]1,+\infty[$, so that we simply write $T(t)$ instead of $T_p(t)$. The theorem of dominated convergence and (2.28) yield

$$\|T(t)f\|_{L^1(B(r),\mathbb{R}^N)} = \lim_{p\to 1^+} \|T(t)f\|_{L^p(B(r),\mathbb{R}^N)} \le \lim_{p\to 1} e^{\nu_p t}\|f\|_p = e^{\nu_1 t}\|f\|_1 .$$

Thus we can extend $T(t)$ from $L^2(\mathbb{R}^N,\mathbb{R}^N) \cap L^1(\mathbb{R}^N,\mathbb{R}^N)$ to an operator $T_1(t)$ satisfying (3.1) for $t \ge 0$. It is clear that $\{T_1(t)\}$ is a semigroup which is consistent with $\{T(t)\}$. To show its strong continuity, we take $f \in C_c^\infty(\mathbb{R}^N,\mathbb{R}^N)$. Since $f \in D(A_p)$ for $p > 1$ by Theorem 2.7, we obtain

$$\begin{aligned} &T_1(t)f - f = T(t)f - f = \int_0^t T(s)\mathcal{A}f\,ds = \int_0^t T_1(s)\mathcal{A}f\,ds, \\ &\|T_1(t)f - f\|_{L^1(\mathbb{R}^N,\mathbb{R}^N)} \le ct, \end{aligned} \tag{3.2}$$

for all $t \in]0,1]$ and some constant $c > 0$ independent of t. Hence, $T_1(t)f \to f$ in $L^1(\mathbb{R}^N,\mathbb{R}^N)$ as $t \to 0^+$, and so $\{T_1(t)\}$ is strongly continuous. The semigroup $\{T_1(t)\}$ leaves invariant D_p, $1 < p < +\infty$, since it is consistent with $\{T(t)\}$. The space D_p is dense in $L^1(\mathbb{R}^N,\mathbb{R}^N)$ because it contains the test functions. Let $f \in D_p$. As in (3.2) we obtain that

$$T_1(t)f - f = \int_0^t T_1(s)A_p f\,ds.$$

Thus, $f \in D(A_1)$ and $A_1 = A_p f$. The last assertion now follows from Proposition II.1.7 of [7]. ☐

In view of the above results, in the rest of the paper we simply write $\{T(t)\}$ instead of $\{T_p(t)\}$.

4. L^p-L^q ESTIMATES

Under Hypothesis 2.1, we want to show that $T(t)$ maps $L^p(\mathbb{R}^N,\mathbb{R}^N)$ into $L^q(\mathbb{R}^N,\mathbb{R}^N)$ for all $t > 0$ and $1 \le p \le q \le +\infty$ and that there exist two constants $M > 0$ and $\omega \in \mathbb{R}$ such that

$$\|T(t)f\|_q \le M t^{-\frac{N}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} e^{\omega t}\|f\|_p, \qquad t > 0,\ f \in L^p(\mathbb{R}^N,\mathbb{R}^N). \tag{4.1}$$

The case $(p,q) = (1,+\infty)$ is the main step of the proof of (4.1), and it is treated in the next lemma.

Lemma 4.1. *The estimate* (4.1) *is true for* $(p,q) \in \{(1,2),(2,+\infty),(1,+\infty)\}$.

Proof. Step 1: $(p,q) = (1,2)$. If $u \in D(A_2)$, integrating by parts yields

$$\int_{\mathbb{R}^N} -\operatorname{div}(Q\nabla u)\cdot u\,dx \ge \alpha_1 \int_{\mathbb{R}^N} |\nabla u|^2\,dx.$$

As in Step 3 of the proof of Proposition 2.5, we obtain that

$$\int_{\mathbb{R}^N} (\lambda u - B_2 u)\cdot u\,dx \ge \Big(\lambda - \beta_3 - \frac{\nu}{2}\Big)\int_{\mathbb{R}^N} |u|^2\,dx.$$

Taking $\lambda \ge \lambda_0 > \beta_3 + \nu/2$, we infer

$$\|u\|_{1,2}^2 \le C_1 \int_{\mathbb{R}^N} u(\lambda u - A_2 u)dx, \tag{4.2}$$

for a positive constant $C_1 = C_1(\lambda_0)$ independent of u. For $0 \ne f \in C_c^\infty(\mathbb{R}^N,\mathbb{R}^N)$ and $\lambda \ge \lambda_0$, we set

$$v(t) = \|e^{-\lambda t}T(t)f\|_2^2, \qquad t \ge 0.$$

Let $\delta > 0$ be the supremum of $t_0 > 0$ such that $v(t) \neq 0$ for $0 \le t \le t_0$. For $t \in]0, \delta[$, we have

$$v'(t) = 2e^{-2\lambda t} \int_{\mathbb{R}^N} \langle T(t)f, (A_2 - \lambda)T(t)f \rangle dx, \qquad t > 0. \tag{4.3}$$

We recall Nash's inequality (see e.g., [6, Theorem 2.4.6]) which implies that

$$\|g\|_2^{2+4/N} \le C\|g\|_{1,2}^2 \, \|g\|_1^{4/N} \tag{4.4}$$

for a constant $C > 0$ and each $g \in W^{1,2}(\mathbb{R}^N, \mathbb{R}^N) \cap L^1(\mathbb{R}^N, \mathbb{R}^N)$. Since $T_2(t)f \in D(A_2) \cap L^1(\mathbb{R}^N, \mathbb{R}^N)$ by Proposition 3.1, we deduce from (4.3), (4.2) and (4.4) that

$$v'(t) \le -\frac{2}{C_1}\|e^{-\lambda t}T(t)f\|_{1,2}^2 \le -\frac{2}{CC_1}\|e^{-\lambda t}T(t)f\|_2^{2+4/N} \, \|e^{-\lambda t}T(t)f\|_1^{-4/N},$$

for every $t \in (0, \delta)$, and hence

$$\frac{d}{dt}(v(t)^{-2/N}) \ge \frac{4}{CC_1N}\|e^{-\lambda t}T(t)f\|_1^{-4/N} \ge M_1 e^{\frac{4}{N}(\lambda-\nu_1)t}\|f\|_1^{-4/N}, \tag{4.5}$$

where ν_1 is given by Theorem 3.1 and $M_1 := 4/(CC_1N)$. Integrating (4.5), we obtain

$$(v(t))^{-2/N} \ge M_1\|f\|_1^{-4/N} \int_0^t e^{\frac{4}{N}(\lambda-\nu_1)s}\, ds \ge M_1 t\|f\|_1^{-4/N},$$

$$v(t) = \|e^{-\lambda t}T(t)f\|_2^2 \le M_1^{-N/2}t^{-N/2}\|f\|_1^2 \tag{4.6}$$

for all $t \in]0, \delta[$ and $\lambda \ge \max\{\lambda_0, \nu_1\}$. If $\delta < +\infty$, the semigroup law shows that $v(t) = 0$ for $t \ge \delta$, i.e., (4.6) is true for all $t \ge 0$. By approximation, we arrive at

$$\|T(t)f\|_2 \le M_1^{-N/4}t^{-N/4}e^{\lambda t}\|f\|_1, \qquad t > 0,$$

for all $f \in L^1(\mathbb{R}^N, \mathbb{R}^N)$.

Step 2: $(p, q) = (2, +\infty)$. Based on the representation of A_2^* from Theorem 2.7, as in Step 1 we can establish the estimate

$$\|T(t)^*g\|_2 \le M_1^{-N/4}t^{-N/4}e^{\lambda t}\|g\|_1, \qquad t > 0,$$

for every $g \in L^1(\mathbb{R}^N, \mathbb{R}^N)$, $\lambda \ge \lambda_0'$ and some constant $\lambda_0' \ge \lambda_0$. By duality, this inequality leads to

$$\|T(t)f\|_\infty = \sup_{\|g\|_1 \le 1} \int_{\mathbb{R}^N} \langle f, T(t)^*g \rangle \, dx \le M_1^{-N/4}t^{-N/4}e^{\lambda t}\|f\|_2, \qquad t > 0,$$

for every $f \in C_c(\mathbb{R}^N, \mathbb{R}^N)$; i.e., estimate (4.1) holds for with $(p, q) = (2, +\infty)$.

Step 3: $(p, q) = (1, +\infty)$. Steps 1 and 2 imply that

$$\|T(t/2)T(t/2)f\|_\infty \le 2^{N/4}M_1^{-N/4}t^{-N/4}e^{\lambda t/2}\|T(t/2)f\|_2 \le 2^{N/2}M_1^{-N/2}t^{-N/2}e^{\lambda t}\|f\|_1,$$

for all $t > 0$ and $f \in L^1(\mathbb{R}^N, \mathbb{R}^N)$, where M_1 is as above. □

Theorem 4.2. *Assume that Hypothesis* 2.1 *is satisfied. Then, the inequality* (4.1) *holds for all* $1 \le p \le q \le +\infty$ *and some constants* $M \ge 1$ *and* $\omega \in \mathbb{R}$.

Proof. Lemma 4.1 and Proposition 3.1 show that

$$\|T(t)\|_{\mathcal{L}(L^1(\mathbb{R}^N,\mathbb{R}^N),L^\infty(\mathbb{R}^N,\mathbb{R}^N))} \le Mt^{-N/2}e^{\lambda t} \quad \text{and} \quad \|T(t)\|_{\mathcal{L}(L^1(\mathbb{R}^N,\mathbb{R}^N))} \le e^{\nu_1 t},$$

for all $t > 0$ and $f \in L^1(\mathbb{R}^N, \mathbb{R}^N)$ and some constants $\lambda \ge 0$ and $M \ge 1$. Set $\omega = \max\{\nu_1, \lambda\}$. The Riesz Thorin interpolation theorem, see e.g., Theorem 1.18.4 of [25], then yields

$$\|T(t)\|_{\mathcal{L}(L^1(\mathbb{R}^N,\mathbb{R}^N),L^q(\mathbb{R}^N,\mathbb{R}^N))} \le Me^{\omega t}t^{-\frac{N}{2}(1-\frac{1}{q})}, \qquad t > 0, \tag{4.7}$$

for all $q \in [1, +\infty]$. The assertion now follows by interpolation between (2.28) and (4.7). □

5. L^p-GRADIENT ESTIMATES

In this section, we prove L^p-gradient estimates for the spatial derivatives of the semigroup. We introduce the operator $\tilde{\mathcal{A}}$ defined on smooth functions $\psi : \mathbb{R}^N \to \mathbb{R}$ by

$$\tilde{\mathcal{A}}\psi = \mathrm{div}(Q\nabla\psi) + F \cdot \nabla\psi. \tag{5.1}$$

In addition to Hypothesis 2.1, we further assume the following conditions.

Hypothesis 5.1. (i) *$q_{ij} \in C^2_b(\mathbb{R}^N) \cap C^{2+\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$ for all $i,j = 1,\dots,N$ and some $\alpha \in]0,1[$;*
(ii) *there exist a function $\varphi \in C^2(\mathbb{R}^N)$ and a constant $\lambda \geq 0$ such that*

$$\lim_{|x|\to+\infty} \varphi(x) = +\infty, \qquad \sup_{\mathbb{R}^N} \tilde{\mathcal{A}}\varphi - \lambda\varphi < +\infty. \tag{5.2}$$

Remark 5.2. *Assume that Hypotheses* 2.1 *and* 5.1(*i*) *hold and that F satisfies* (2.34). *Take $\varphi \in C^2(\mathbb{R}^N)$ with $\varphi(x) = |x|$ for $|x| \geq 1$. The function φ then satisfies* (5.2) (*cf. the proof of Proposition* 2.8).

We begin with some preliminary results on local Hölder regularity.

Theorem 5.3. *Assume that Hypotheses* 2.1 *and* 5.1 *hold. Then, for each $f \in C^\infty_c(\mathbb{R}^N,\mathbb{R}^N)$, the function $u = T(\cdot)f$ belongs to $C^{1+\beta/2,2+\beta}_{\mathrm{loc}}([0,+\infty[\times\mathbb{R}^N,\mathbb{R}^N) \cap C([0,+\infty[;C^{1+\beta}_b(\mathbb{R}^N,\mathbb{R}^N))$ for each $\beta \in]0,1[$, and it satisfies*

$$\begin{cases} D_t u(t,x) = \mathcal{A}u(t,x), & t > 0, \quad x \in \mathbb{R}^N, \\ u(0,x) = f(x), & x \in \mathbb{R}^N. \end{cases} \tag{5.3}$$

Further, ∇u belongs to $C^{1+\alpha/2,2+\alpha}_{\mathrm{loc}}([0,+\infty[\times\mathbb{R}^N,\mathbb{R}^N)$, where α is given by Hypothesis 5.1(*i*).

Proof. Let $f \in C^\infty_c(\mathbb{R}^N,\mathbb{R}^N)$. By Theorem 2.7, we have $u = T(\cdot)f \in C([0,+\infty[;W^{2,p}(\mathbb{R}^N,\mathbb{R}^N))$ for every $p \in]1,+\infty[$. Sobolev's embedding theorem thus yields $u \in C([0,+\infty[;C^{1+\beta}_b(\mathbb{R}^N,\mathbb{R}^N))$ for all $\beta \in]0,1[$. We now turn our attention to the regularity of $D^2 u_k$ and $D_t u_k$ for $k = 1,\dots,N$. Set $\mathcal{A}_0\varphi = \mathrm{div}(QD\varphi)$ for smooth functions φ. Then u_k solves the Cauchy problem

$$\begin{cases} D_t w(t,x) = \mathcal{A}_0 w(t,x) + g_k(t,x), & t \geq 0, \quad x \in \mathbb{R}^N, \\ w(0,\cdot) = \vartheta(x) f_k(x), & x \in \mathbb{R}^N, \end{cases} \tag{5.4}$$

where $g_k = \langle F, \nabla u_k\rangle + \langle Vu, e_k\rangle$ and e_k is the k-th vector of the Euclidean basis of $\mathbb{R}^N$. Take $R > 0$ and a smooth function $\vartheta = \vartheta_R$ such that $\chi_{B(R)} \leq \vartheta \leq \chi_{B(2R)}$. As it is easily seen, the function $v_k = u_k\vartheta$ solves the Cauchy problem (5.4) with f_k and g_k replaced with the functions $f_k\vartheta$ and ψ_k, respectively, where

$$\psi_k := g_k\vartheta - u_k\mathcal{A}_0\vartheta - 2\langle Q\nabla u_k, \nabla\vartheta\rangle \ \in \ C([0,+\infty[;C^\beta_b(\mathbb{R}^N)),$$

for all $\beta \in]0,1[$. It is well known that the realization A_∞ of the operator $\mathcal{A}_0$ in $C_b(\mathbb{R}^N)$ generates an analytic semigroup $\{S_\infty(t)\}$ such that

$$D_{A_\infty}(\beta/2,\infty) = C^\beta_b(\mathbb{R}^N), \qquad D_{A_\infty}(1+\beta/2,\infty) = C^{2+\beta}_b(\mathbb{R}^N),$$

for all $\beta \in]0,1[$, with equivalence of the respective norms, see Section 3.1 of [14]. We further have

$$v_k(t,\cdot) = S_\infty(t)(f_k\vartheta) + \int_0^t S_\infty(t-s)\psi_k(s,\cdot)ds, \qquad t \geq 0. \tag{5.5}$$

Since $\vartheta f_k \in D_{A_\infty}(1+\beta/2,\infty)$ and $\psi_k \in C([0,+\infty[;D_{A_\infty}(\beta/2,\infty))$ for any β as above, we can apply [14, Corollary 4.3.9] obtaining that $D_t v_k$ is (bounded and) continuous on $[0,T]$ with values in

$C_b^\beta(\mathbb{R}^N)$, for each $T>0$. Moreover, since $\mathcal{A}_\infty(f_k\vartheta)+g_k(0,\cdot)\in D_{A_\infty}(\beta/2,\infty)$, the function $v_k(t,\cdot)$ belongs to $C_b^{2+\beta}(\mathbb{R}^N)$ for all $t>0$ and

$$\sup_{t\in[0,T]}\Big(\|v_k(t,\cdot)\|_{C_b^{2+\beta}(\mathbb{R}^N)}+\|D_tv_k(t,\cdot)\|_{C_b^\beta(\mathbb{R}^N)}\Big)<+\infty. \tag{5.6}$$

By interpolation we deduce that

$$\begin{aligned}\|v_k(t,\cdot)-v_k(s,\cdot)\|_{C_b^2(\mathbb{R}^N)}&\le C\,\|v_k(t,\cdot)-v_k(s,\cdot)\|_{C_b(\mathbb{R}^N)}^{\frac{\beta}{2+\beta}}\,\|v_k(t,\cdot)-v_k(s,\cdot)\|_{C_b^{2+\beta}(\mathbb{R}^N)}^{\frac{2}{2+\beta}}\\&\le C\Big(2\sup_{t\in[0,T]}\|v_k(t,\cdot)\|_{C_b^{2+\beta}(\mathbb{R}^N)}\Big)^{\frac{2}{2+\beta}}\|v_k(t,\cdot)-v_k(s,\cdot)\|_{C_b(\mathbb{R}^N)}^{\frac{\beta}{2+\beta}},\end{aligned} \tag{5.7}$$

for a constant $C>0$ and all $s,t\in[0,T]$ and $T>0$, see e.g., Proposition 1.1.3(ii) of [14]. Since $v_k\in C([0,+\infty[\,;C_b(\mathbb{R}^N))$, estimate (5.7) implies that $v_k\in C([0,+\infty[\,;C_b^2(\mathbb{R}^N))$. We next show that D^2v_k belongs to $C^{\beta/2}([0,T];C_b(\mathbb{R}^N))$ for all $T>0$. Observe that

$$\begin{aligned}|v_k(t,x_2)-v_k(s,x_2)-v_k(t,x_1)+v_k(s,x_1)|&\le\int_s^t|D_tv_k(r,x_2)-D_tv_k(r,x_1)|\,dr\\&\le|t-s|\,|x_2-x_1|^\beta\sup_{t\in[0,T]}\|D_tv_k(t,\cdot)\|_{C_b^\beta(\mathbb{R}^N)},\end{aligned} \tag{5.8}$$

$$|v_k(t,x)-v_k(s,x)|\le|t-s|\sup_{t\in[0,T]}\|D_tv_k(t,\cdot)\|_{C_b(\mathbb{R}^N)}, \tag{5.9}$$

for all $s,t\in[0,T]$, $T>0$, and $x,x_1,x_2\in\mathbb{R}^N$. The estimates (5.6), (5.8) and (5.9) yield $v_k\in\mathrm{Lip}([0,T];C_b^\beta(\mathbb{R}^N))$. Interpolating $C_b^2(\mathbb{R}^N)$ between $C_b^\beta(\mathbb{R}^N)$ and $C_b^{2+\beta}(\mathbb{R}^N)$ and arguing as in (5.7), we then conclude that $v_k\in C^{\beta/2}([0,T];C_b^2(\mathbb{R}^N))$, so that, in particular, $D_{ij}v_k\in C^{\beta/2}([0,T];C_b(\mathbb{R}^N))$ for all $T>0$ and $i,j=1,\dots,N$. We now prove that D_tv_k is $\beta/2$-Hölder continuous in $[0,T]$ with values in $C_b(\mathbb{R}^N)$. Since $R>0$ and $\beta\in]0,1[$ are arbitrary and $u_k\equiv v_k$ in $[0,T]\times B(R)$, from the results so far obtained, we deduce that $u_k\in C^{\beta/2}([0,T];C_b^1(B(2R)))$ and thus $\psi_k\in C^{\beta/2}([0,T];C_b(\mathbb{R}^N))$. Theorem 4.3.1(iii) of [14] now implies that $D_tv_k\in C^{\beta/2}([0,T];C_b(\mathbb{R}^N))$. Thus, $v_k\in C^{1+\beta/2,2+\beta}([0,T]\times\mathbb{R}^N)$ for all $T>0$. The first part of the assertion now follows. The last assertion can be deduced from classical estimates for solutions to nonhomogeneous parabolic equations on $\mathbb{R}^N$ with smooth coefficients and inhomogeneities. □

It is well known that, under Hypothesis 2.1 and 5.1, the realization $\tilde A$ of the operator $\tilde{\mathcal{A}}$ in $C_b(\mathbb{R}^N)$ with the maximal domain

$$D(\tilde A)=\Big\{u\in\bigcap_{1\le p<+\infty}W^{2,p}_{\mathrm{loc}}(\mathbb{R}^N)\cap C_b(\mathbb{R}^N):\ \tilde{\mathcal{A}}u\in C_b(\mathbb{R}^N)\Big\}, \tag{5.10}$$

generates a 'weak semigroup' $\{\tilde T(t)\}$ of contractions in $C_b(\mathbb{R}^N)$ (see e.g., Chapter 1 of [3] or [17]). Moreover, for any $f\in C_b(\mathbb{R}^N)$, the function $\tilde T(\cdot)f$ is the unique bounded classical solution to the Cauchy problem

$$\begin{cases}D_tu(t,x)=\tilde{\mathcal{A}}u(t,x), & t>0,\quad x\in\mathbb{R}^N,\\ u(0,x)=f(x), & x\in\mathbb{R}^N,\end{cases} \tag{5.11}$$

(i.e., it is the unique bounded function $u\in C([0,+\infty[\,\times\mathbb{R}^N)\cap C^{1,2}(]0,+\infty[\,\times\mathbb{R}^N)$ solving (5.11)). Here, the uniqueness is a consequence of the following generalized maximum principle, see e.g., Theorem 4.1.3 of [3].

Proposition 5.4. *Assume that Hypotheses* 2.1 *and* 5.1(*ii*) *are satisfied. Let* $u : [0,T]\times\mathbb{R}^N \to \mathbb{R}$ *be a bounded classical solution of the problem*

$$\begin{cases} D_t u(t,x) = \tilde{\mathcal{A}}u(t,x) + g(t,x), & t\in]0,T[, \quad x\in\mathbb{R}^N,\\ \quad u(0,x) = f(x), & \qquad\qquad x\in\mathbb{R}^N,\end{cases}$$

where $f\in C_b(\mathbb{R}^N)$ *and* $g\in C(]0,T[\times\mathbb{R}^N)$. *If* $g\le 0$ *then* $u\le\sup f^+$ *where* $f^+(x)=\max\{f(x),0\}$. *Similarly, if* $g\ge 0$, *then* $u\ge\inf f^-$, *where* $f^-(x)=\min\{f(x),0\}$.

In the following proposition, we collect some properties of the semigroup $\{\tilde{T}(t)\}$ needed below.

Proposition 5.5. *Assume that Hypotheses* 2.1 *and* 5.1 *hold. The following assertions are true.*

(i) *Let* $\{f_n\}$ *be a bounded sequence on* $C_b(\mathbb{R}^N)$ *converging locally uniformly in* $\mathbb{R}^N$ *to a function* $f\in C_b(\mathbb{R}^N)$ *as* $n\to+\infty$. *Then, the function* $\tilde{T}(\cdot)f_n$ *converges to* $\tilde{T}(\cdot)f$ *locally uniformly in* $[0,+\infty[\times\mathbb{R}^N$.

(ii) $\{\tilde{T}(t)\}$ *is a positivity preserving semigroup and* $\frac{d}{dt}\tilde{T}(t)f = \tilde{T}(t)\tilde{A}f$ *for all* $f\in D(\tilde{A})$ *and* $t\ge 0$.

(iii) *We have* $|\tilde{T}(t)(fg)|\le[\tilde{T}(t)(|f|^p)]^{\frac{1}{p}}\,[\tilde{T}(t)(|g|^{p'})]^{\frac{1}{p'}}$ *and* $|\tilde{T}(t)f|^p\le\tilde{T}(t)(|f|^p)$ *for all* $f,g\in C_b(\mathbb{R}^N)$, $t\ge 0$ *and* $p\in]1,+\infty[$.

(iv) *If* $\lambda\in\mathbb{R}$ *is such that*

$$\langle V(x)\xi,\xi\rangle\le\lambda|\xi|^2, \qquad x,\xi\in\mathbb{R}^N, \tag{5.12}$$

then, for all $p\in[1,+\infty[$ *and* $f\in L^p(\mathbb{R}^N)\cap C_b(\mathbb{R}^N)$, *it holds that*

$$|T(t)f|^p\le e^{\lambda pt}\tilde{T}(t)(|f|^p), \qquad t\in[0,+\infty[. \tag{5.13}$$

(v) *The restriction of the semigroup* $\{\tilde{T}(t)\}$ *to* $C_c^\infty(\mathbb{R}^N)$ *can be extended to a* C_0*-semigroup* $\{\tilde{T}_p(t)\}$ *on* $L^p(\mathbb{R}^N)$ *satisfying*

$$\|\tilde{T}_p(t)\|_{\mathcal{L}(L^p(\mathbb{R}^N))}\le e^{\sigma_p t}, \tag{5.14}$$

for all $p\in[1,+\infty[$, $t\ge 0$, *and constants* $\sigma_p\ge 0$.

Proof. For a proof of assertions (i) and (ii) we refer the reader to e.g., Propositions 2.2.9, 2.3.5, 2.3.6, 4.1.1 and Theorem 2.2.5 of [3].

(iii) Since there exists a positive function $G:]0,+\infty[\times\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R}$ such that

$$(\tilde{T}(t)f)(x)=\int_{\mathbb{R}^N}G(t,x,y)f(y)\,dy \qquad\text{and}\qquad \int_{\mathbb{R}^N}G(t,x,y)\,dy=1,$$

for all $x\in\mathbb{R}^N$, $t>0$, and $f\in C_b(\mathbb{R}^N)$ (see e.g., Theorem 2.2.5 of [3]), assertion (iii) easily follows from Hölder's inequality.

(iv) Because of (iii), we have to show assertion (iv) only for $p\in[1,2]$. So, let us fix some $p\in[1,2]$. For $f\in C_c^\infty(\mathbb{R}^N,\mathbb{R}^N)$ and $\delta>0$, we set $v_\delta(t,x)=(|e^{-\lambda t}(T(t)f)(x)|^2+\delta)^{p/2}$ for $(t,x)\in[0,+\infty[\times\mathbb{R}^N$. Theorem 5.3 shows that the function $(t,x)\mapsto e^{-\lambda t}(T(t)f)(x)$ is bounded on $[0,T]\times\mathbb{R}^N$ and that it belongs to $C^{1,2}([0,T]\times\mathbb{R}^N)$, for every $T>0$. Moreover, it is a classical solution of the Cauchy problem (5.3) with $\mathcal{A}$ replaced with $\mathcal{A}-\lambda I$. Therefore, $v_\delta\in C^{1,2}([0,+\infty[\times\mathbb{R}^N)$ and

$$\begin{aligned} D_t v_\delta(t,\cdot) &= p(|e^{-\lambda t}T(t)f|^2+\delta)^{\frac{p}{2}-1}\,\langle e^{-\lambda t}T(t)f,(\mathcal{A}-\lambda I)(e^{-\lambda t}T(t)f)\rangle\\ &= p(|e^{-\lambda t}T(t)f|^2+\delta)^{\frac{p}{2}-1}\sum_{k=1}^N e^{-\lambda t}(T(t)f)_k\tilde{\mathcal{A}}(e^{-\lambda t}T(t)f)_k\\ &\quad + p(|e^{-\lambda t}T(t)f|^2+\delta)^{\frac{p}{2}-1}\,\langle (V-\lambda I)e^{-\lambda t}T(t)f, e^{-\lambda t}T(t)f\rangle \end{aligned}$$

$$\leq p(|e^{-\lambda t}T(t)f|^2+\delta)^{\frac{p}{2}-1}\sum_{k=1}^N e^{-\lambda t}(T(t)f)_k\tilde{\mathcal{A}}(e^{-\lambda t}(T(t)f)_k), \tag{5.15}$$

for all $t \geq 0$. We further obtain

$$\begin{aligned}\tilde{\mathcal{A}}v_\delta(t,\cdot) &= p(|e^{-\lambda t}T(t)f|^2+\delta)^{\frac{p}{2}-1}e^{-2\lambda t}\sum_{k=1}^N(T(t)f)_k\tilde{\mathcal{A}}(T(t)f)_k \\ &\quad + p(|e^{-\lambda t}T(t)f|^2+\delta)^{\frac{p}{2}-1}e^{-2\lambda t}\sum_{k=1}^N\langle Q\nabla(T(t)f)_k,\nabla(T(t)f)_k\rangle \\ &\quad - p(2-p)(|e^{-\lambda t}T(t)f|^2+\delta)^{\frac{p}{2}-2}e^{-4\lambda t}\sum_{h,k=1}^N(T(t)f)_h(T(t)f)_k\langle Q\nabla(T(t)f)_h,\nabla(T(t)f)_k\rangle.\end{aligned} \tag{5.16}$$

In view of $p \leq 2$ and

$$\begin{aligned}&\Big|\sum_{h,k=1}^N(T(t)f)_h(T(t)f)_k\langle Q\nabla(T(t)f)_h,\nabla(T(t)f)_k\rangle\Big| \\ &\leq \sum_{h,k=1}^N|(T(t)f)_h|\,|(T(t)f)_k|\,\langle Q\nabla(T(t)f)_h,\nabla(T(t)f)_h\rangle^{\frac{1}{2}}\,\langle Q\nabla(T(t)f)_k,\nabla(T(t)f)_k\rangle^{\frac{1}{2}} \\ &= \left(\sum_{k=1}^N|(T(t)f)_k|\langle Q\nabla(T(t)f)_k,\nabla(T(t)f)_k\rangle^{\frac{1}{2}}\right)^2 \\ &\leq |T(t)f|^2\sum_{k=1}^N\langle Q\nabla(T(t)f)_k,\nabla(T(t)f)_k\rangle,\end{aligned}$$

equation (5.16) yields

$$\begin{aligned}\tilde{\mathcal{A}}v_\delta(t,\cdot) &\geq p(|e^{-\lambda t}T(t)f|^2+\delta)^{\frac{p}{2}-1}e^{-2\lambda t}\sum_{k=1}^N(T(t)f)_k\tilde{\mathcal{A}}(T(t)f)_k \\ &\quad + p(p-1)(|e^{-\lambda t}T(t)f|^2+\delta)^{\frac{p}{2}-1}e^{-2\lambda t}\sum_{k=1}^N\langle Q\nabla(T(t)f)_k,\nabla(T(t)f)_k\rangle \\ &\geq p(|e^{-\lambda t}T(t)f|^2+\delta)^{\frac{p}{2}-1}e^{-2\lambda t}\sum_{k=1}^N(T(t)f)_k\tilde{\mathcal{A}}(T(t)f)_k,\end{aligned} \tag{5.17}$$

for all $t \geq 0$. Set $w = v_\delta - \tilde{T}(\cdot)(v_\delta(0,\cdot))$. Combining (5.15) and (5.17), we derive that

$$\begin{cases} D_tw(t,x)-\tilde{\mathcal{A}}w(t,x)\leq 0, & t\geq 0,\quad x\in\mathbb{R}^N, \\ w(0,x)=0, & x\in\mathbb{R}^N.\end{cases}$$

Proposition 5.4 now implies that

$$(|e^{-\lambda t}T(t)f|^2+\delta)^{\frac{p}{2}} \leq \tilde{T}(t)\left((|f|^2+\delta)^{\frac{p}{2}}\right), \qquad t\geq 0.$$

Due to (i), we can let $\delta \to 0^+$ and obtain (5.13) for test functions f. For $f \in C_b(\mathbb{R}^N,\mathbb{R}^N) \cap L^p(\mathbb{R}^N,\mathbb{R}^N)$, there is a sequence $\{f_n\} \subset C_c^\infty(\mathbb{R}^N,\mathbb{R}^N)$ which is bounded in $C_b(\mathbb{R}^N,\mathbb{R}^N)$ and converges to f in $L^p(\mathbb{R}^N,\mathbb{R}^N)$ and locally uniformly on $\mathbb{R}^N$. Now, (iv) follows by approximation.

(v) Theorem 2.4 of [23] shows that the realization $\tilde{A}_p$ of the operator $\tilde{\mathcal{A}}$ in $L^p(\mathbb{R}^N)$, $p \in]1,+\infty[$, with the domain $D(\tilde{A}_p) = \{f \in W^{2,p}(\mathbb{R}^N) : F \cdot \nabla f \in L^p(\mathbb{R}^N)\}$, generates a consistent positive strongly continuous semigroup $\{\tilde{T}_p(t)\}$ satisfying (5.14). Arguing as in the proof of Theorem 5.3, one sees that $u = \tilde{T}_p f$ is a classical solution of (5.11) for each $f \in C_c^\infty(\mathbb{R}^N)$. So, Proposition 5.4 yields $\tilde{T}_p(\cdot)f = \tilde{T}(\cdot)f$ for test functions f. By approximation this equality holds for all $f \in C_b(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$. □

In the spirit of [11], we next show a proposition which is the main step towards the L^p-gradient estimates.

Proposition 5.6. *Assume that Hypotheses* 2.1 *and* 5.1 *are satisfied. Then, there exist constants* $C_p > 0$ *and* $\omega_p \in \mathbb{R}$ *such that*

$$|(\nabla T(t)f)(x)|^p \le C_p e^{\omega_p t}\left(\tilde{T}(t)(|f|^2 + |\nabla f|^2)^{\frac{p}{2}}\right)(x), \qquad x \in \mathbb{R}^N, \tag{5.18}$$

for all $t > 0$, $x \in \mathbb{R}^N$, $p \in]1,+\infty[$, *and* $f \in C_c^\infty(\mathbb{R}^N, \mathbb{R}^N)$.

Proof. Due to Proposition 5.5(iii), we can restrict ourselves to the case $p \in]1,2]$. For fixed $\varepsilon > 0$ and $p \in]1,2]$, we set

$$v(t,x) = \left(\sum_{k=1}^N (u_k(t,x))^2 + \sum_{k=1}^N \langle Q(x)\nabla u_k(t,x), \nabla u_k(t,x)\rangle + \varepsilon\right)^{\frac{p}{2}}, \qquad t \in [0,+\infty[,\ x \in \mathbb{R}^N, \tag{5.19}$$

where $u = T(\cdot)f$. The function v belongs to $C^{1,2}([0,+\infty[\times\mathbb{R}^N)$ and it is bounded in $[0,T] \times \mathbb{R}^N$ for all $T > 0$, by Theorem 5.3. We are going to prove that v satisfies

$$\begin{cases} D_t v(t,x) - \tilde{\mathcal{A}} v(t,x) \le \omega_p v(t,x), & t > 0, \quad x \in \mathbb{R}^N, \\ v(0,x) = g(x), & x \in \mathbb{R}^N, \end{cases} \tag{5.20}$$

where $g(x) := [|f(x)|^2 + \sum_{k=1}^N \langle Q(x)\nabla f_k(x), \nabla f_k(x)\rangle + \varepsilon]^{\frac{p}{2}}$ for all $x \in \mathbb{R}^N$. Since the function $(t,x) \mapsto e^{\omega_p t}(\tilde{T}(t)g)(x)$ solves the Cauchy problem associated with (5.20), Proposition 5.4 will then yield that $v(t,x) \le e^{\omega_p t}(\tilde{T}(t)g)(x)$ for all $t > 0$ and $x \in \mathbb{R}^N$. Taking the limit as $\varepsilon \to 0^+$, we will thus obtain

$$\Big(\sum_{k=1}^N \langle Q\nabla u_k(t,\cdot), \nabla u_k(t,\cdot)\rangle\Big)^{\frac{p}{2}} \le \tilde{T}(t)\Big(|f|^2 + \sum_{k=1}^N \langle Q\nabla f_k, \nabla f_k\rangle\Big)^{\frac{p}{2}},$$

for $t > 0$, and (5.18) will follow thanks to (2.3). So, let us prove (5.20). A long but straightforward computation gives

$$D_t v(t,x) = \tilde{\mathcal{A}} v(t,x) + \psi(t,x), \qquad (t,x) \in]0,+\infty[\times\mathbb{R}^N,$$

where $\psi = \psi_1 + \psi_2 + \psi_3 + \psi_4 + \psi_5$ with

$$\psi_1 = -pv^{1-\frac{2}{p}}\bigg(\sum_{i,j,k=1}^N q_{ij} D_i u_k D_j u_k + \sum_{i,j,k,l,m=1}^N q_{ij} q_{lm} D_{jl} u_k D_{im} u_k\bigg),$$

$$\psi_2 = pv^{1-\frac{2}{p}}\bigg(\langle Vu, u\rangle + \sum_{j,k,l,m=1}^N q_{lm} V_{kj} D_l u_j D_m u_k + \sum_{j,k,l,m=1}^N q_{lm} D_l V_{kj} u_j D_m u_k\bigg),$$

$$\psi_3 = pv^{1-\frac{2}{p}}\bigg(\sum_{j,k,l,m=1}^N q_{lm} D_l F_j D_j u_k D_m u_k - \frac{1}{2}\sum_{j,k,l,m=1}^N F_j D_j q_{lm} D_l u_k D_m u_k$$

$$-\frac{1}{2}\sum_{i,j,k,l,m=1}^N D_iq_{ij}D_jq_{lm}D_lu_kD_mu_k\Big),$$

$$\psi_4 = p(2-p)v^{1-\frac{4}{p}}\sum_{i,j=1}^N q_{ij}\Big(\sum_{k=1}^N u_kD_iu_k + \sum_{k,l,m=1}^N q_{lm}D_mu_kD_{il}u_k + \frac{1}{2}\sum_{k,l,m=1}^N D_iq_{lm}D_lu_kD_mu_k\Big)$$
$$\times\Big(\sum_{k=1}^N u_kD_ju_k + \sum_{k,l,m=1}^N q_{lm}D_mu_kD_{jl}u_k + \frac{1}{2}\sum_{k,l,m=1}^N D_jq_{lm}D_lu_kD_mu_k\Big),$$

$$\psi_5 = pv^{1-\frac{2}{p}}\Big(\sum_{i,j,k,l,m=1}^N q_{lm}D_lq_{ij}D_mu_kD_{ij}u_k + \sum_{i,j,k,l,m=1}^N q_{lm}D_{il}q_{ij}D_ju_kD_mu_k$$
$$-\frac{1}{2}\sum_{i,j,k,l,m=1}^N q_{ij}D_{ij}q_{lm}D_lu_kD_mu_k - 2\sum_{i,j,k,l,m=1}^N q_{ij}D_jq_{lm}D_mu_kD_{il}u_k\Big).$$

We first observe that

$$\begin{aligned}\sum_{i,j,k,l,m=1}^N q_{ij}(x)q_{lm}(x)D_{jl}u_k(t,x)D_{im}u_k(t,x) &= \sum_{k=1}^N \mathrm{Tr}(Q(x)D^2u_k(t,x)Q(x)D^2u_k(t,x))\\ &= \sum_{k=1}^N \mathrm{Tr}(Q^{\frac{1}{2}}(x)D^2u_k(t,x)Q(x)D^2u_k(t,x)Q^{\frac{1}{2}}(x))\\ &= \sum_{k=1}^N |Q^{\frac{1}{2}}(x)D^2u_k(t,x)Q^{\frac{1}{2}}(x)|^2,\end{aligned}$$

for all $(t,x)\in\,]0,+\infty[\times\mathbb{R}^N$. It follows that

$$\psi_1 = -pv^{1-\frac{2}{p}}\left(\sum_{k=1}^N |Q^{\frac{1}{2}}\nabla u_k|^2 + \sum_{k=1}^N |Q^{\frac{1}{2}}D^2u_kQ^{\frac{1}{2}}|^2\right). \tag{5.21}$$

The first and third term of ψ_2 can be estimated by means of Hypothesis 2.1(iii). For the second term, we set $V^{\mathrm{sim}} = \frac{1}{2}(V+V^*)$ and note that condition (2.6) implies that $|\langle V_{\mathrm{sim}}(x)\xi,\eta\rangle| \le \beta_3\,|\xi||\eta|$ for all $x,\xi,\eta\in\mathbb{R}^N$. Hence,

$$\begin{aligned}\Big|\sum_{j,k,l,m=1}^N q_{lm}V_{kj}D_lu_jD_mu_k\Big| &= \Big|\sum_{j,k,l,m=1}^N q_{lm}V_{kj}^{\mathrm{sim}}D_lu_jD_mu_k\Big|\\ &\le \sum_{l,m=1}^N |q_{lm}|\,|\langle V^{\mathrm{sim}}D_lu, D_mu\rangle|\\ &\le \beta_3\|Q\|_\infty\,|\nabla u|^2,\end{aligned}$$

so that

$$\begin{aligned}\psi_2 &\le pv^{1-\frac{2}{p}}\Big(\beta_3|u|^2 + \beta_3\|Q\|_\infty|\nabla u|^2 + \|Q\|_\infty\,\|\nabla V\|_\infty\,|u|\,|\nabla u|\Big)\\ &\le pv^{1-\frac{2}{p}}\Big\{\Big(\beta_3 + \frac{1}{2}\|Q\|_\infty\,\|\nabla V\|_\infty\Big)|u|^2 + \frac{1}{2}\|Q\|_\infty(2\beta_3+\|\nabla V\|_\infty)\,|\nabla u|^2\Big\}\\ &\le pv^{1-\frac{2}{p}}\Big\{\Big(\beta_3 + \frac{1}{2}\|Q\|_\infty\,\|\nabla V\|_\infty\Big)|u|^2 + \frac{1}{2\alpha_1}\|Q\|_\infty(2\beta_3+\|\nabla V\|_\infty)\,\langle Q\nabla u,\nabla u\rangle\Big\}.\end{aligned} \tag{5.22}$$

To treat ψ_3, we rewrite it in the more compact form

$$\psi_3 = pv^{1-\frac{2}{p}}\bigg\{\sum_{k=1}^N\langle DFQ\nabla u_k,\nabla u_k\rangle - \frac{1}{2}\sum_{k,l,m=1}^N D_l u_k D_m u_k \sum_{j=1}^N\Big(F_j+\sum_{i=1}^N D_i q_{ij}\Big)D_j q_{lm}\bigg\}.$$

Conditions (2.4) and (2.5) thus allow us to estimate

$$\psi_3 \le pv^{1-\frac{2}{p}}\Big(\beta_1+\frac{1}{2}\beta_2+\frac{\sqrt{N}}{2}\|\nabla Q\|_\infty^2\Big)|\nabla u|^2. \tag{5.23}$$

Let $\varepsilon>0$. Using the inequality

$$\langle Q(\xi+\eta),\xi+\eta\rangle \le (1+\varepsilon)\langle Q\xi,\xi\rangle + \Big(1+\frac{1}{\varepsilon}\Big)\langle Q\eta,\eta\rangle, \qquad \xi,\eta\in\mathbb{R}^N, \tag{5.24}$$

we deduce

$$\begin{aligned}\psi_4 \le{}& (1+\varepsilon)p(2-p)v^{1-\frac{4}{p}}\sum_{i,j=1}^N q_{ij}\Big(\sum_{k,l,m=1}^N q_{lm}D_m u_k D_{il}u_k\Big)\Big(\sum_{k,l,m=1}^N q_{lm}D_m u_k D_{jl}u_k\Big)\\ &+\Big(1+\frac{1}{\varepsilon}\Big)p(2-p)v^{1-\frac{4}{p}}\sum_{i,j=1}^N q_{ij}\Big(\sum_{k=1}^N u_k D_i u_k+\frac{1}{2}\sum_{k,l,m=1}^N D_i q_{lm}D_l u_k D_m u_k\Big)\\ &\qquad\times\Big(\sum_{k=1}^N u_k D_j u_k+\frac{1}{2}\sum_{k,l,m=1}^N D_j q_{lm}D_l u_k D_m u_k\Big)\\ =:{}&\psi_{41}+\psi_{42}.\end{aligned}$$

The Cauchy Schwarz inequality and the definition of v yield

$$\begin{aligned}\psi_{41} &= (1+\varepsilon)p(2-p)v^{1-\frac{4}{p}}\Big|\sum_{k=1}^N Q^{\frac{1}{2}}D^2u_k Q\nabla u_k\Big|^2\\ &\le (1+\varepsilon)p(2-p)v^{1-\frac{4}{p}}\sum_{k=1}^N |Q^{\frac{1}{2}}D^2u_kQ^{\frac{1}{2}}|^2\sum_{k=1}^N\langle Q\nabla u_k,\nabla u_k\rangle\\ &\le (1+\varepsilon)p(2-p)v^{1-\frac{2}{p}}\sum_{k=1}^N |Q^{\frac{1}{2}}D^2u_kQ^{\frac{1}{2}}|^2.\end{aligned} \tag{5.25}$$

Employing (5.24) with $\varepsilon=1$ and Hypothesis 2.1(i), we further calculate

$$\begin{aligned}\psi_{42} \le{}& 2\Big(1+\frac{1}{\varepsilon}\Big)p(2-p)v^{1-\frac{4}{p}}\sum_{h,k=1}^N\langle Q\nabla u_h,\nabla u_k\rangle u_h u_k\\ &+\frac{1}{2}\Big(1+\frac{1}{\varepsilon}\Big)p(2-p)v^{1-\frac{4}{p}}\sum_{h,k,l,m,p,r=1}^N\langle Q\nabla q_{lm},\nabla q_{pr}\rangle D_l u_h D_m u_h D_p u_k D_r u_k\\ \le{}& 2\Big(1+\frac{1}{\varepsilon}\Big)p(2-p)v^{1-\frac{4}{p}}|u|^2\sum_{k=1}^N\langle Q\nabla u_k,\nabla u_k\rangle\\ &+\frac{1}{2}\Big(1+\frac{1}{\varepsilon}\Big)p(2-p)v^{1-\frac{4}{p}}|\nabla u|^4\sum_{l,m=1}^N\langle Q\nabla q_{lm},\nabla q_{lm}\rangle\\ \le{}& \Big(1+\frac{1}{\varepsilon}\Big)\Big(2+\frac{1}{2\alpha_1^2}\|Q\|_\infty\|\nabla Q\|_\infty^2\Big)p(2-p)v.\end{aligned} \tag{5.26}$$

Estimates (5.25) and (5.26) lead to

$$\psi_4 \le \Big(1+\frac{1}{\varepsilon}\Big)\Big(2+\frac{1}{2\alpha_1^2}\|Q\|_\infty\,\|\nabla Q\|_\infty^2\Big)p(2-p)v+(1+\varepsilon)p(2-p)v^{1-\frac{2}{p}}\sum_{k=1}^N|Q^{\frac12}D^2u_kQ^{\frac12}|^2. \quad (5.27)$$

Using Young's inequality, ψ_5 is estimated in a straightforward way by

$$\begin{aligned}\psi_5 &\le 3pv^{1-\frac{2}{p}}\Big(\|Q\|_\infty\,\|\nabla Q\|_\infty\,|\nabla u|\,|D^2u|+\frac{\sqrt{N}}{2}\|Q\|_\infty\,\|D^2Q\|_\infty\,|\nabla u|^2\Big)\\ &\le 3pv^{1-\frac{2}{p}}\Big\{\varepsilon|D^2u|^2+\Big(\frac{1}{4\varepsilon}\|Q\|_\infty^2\,\|\nabla Q\|_\infty^2+\frac{\sqrt{N}}{2}\|Q\|_\infty\,\|D^2Q\|_\infty\Big)|\nabla u|^2\Big\}, \quad (5.28)\end{aligned}$$

for every $\varepsilon>0$. Estimates (5.21), (5.22), (5.23), (5.27) and (5.28) now imply that

$$\begin{aligned}\psi \le\; & p\Big\{-1+\frac{1}{2\alpha_1}\|Q\|_\infty\,(2\beta_3+\|\nabla V\|_\infty)+\frac{1}{\alpha_1}\Big(\beta_1+\frac12\beta_2+\frac{\sqrt{N}}{2}\,\|\nabla Q\|_\infty^2\Big)\\ &+\frac{3}{\alpha_1}\Big(\frac{1}{4\varepsilon}\|Q\|_\infty^2\,\|\nabla Q\|_\infty^2+\frac{\sqrt{N}}{2}\|Q\|_\infty\|D^2Q\|_\infty\Big)\Big\}v^{1-\frac{2}{p}}\sum_{k=1}^N\langle Q\nabla u_k,\nabla u_k\rangle\\ &+p\Big\{\beta_3+\frac12\|Q\|_\infty\|\nabla V\|_\infty+(2-p)\Big(1+\frac{1}{\varepsilon}\Big)\Big(2+\frac{1}{2\alpha_1^2}\|Q\|_\infty\,\|\nabla Q\|_\infty^2\Big)\Big\}v\\ &+p\left(1-p+(2-p)\varepsilon+\frac{c_N}{\alpha_1^2}\varepsilon\right)v^{1-\frac{2}{p}}\sum_{k=1}^N|Q^{\frac12}D^2u_kQ^{\frac12}|^2, \quad (5.29)\end{aligned}$$

for some positive constant c_N, depending only on N. Since $p>1$, we can choose $\varepsilon>0$ such that in the last term of (5.29) the factor in large brackets vanishes. Using once more the definition of v, we arrive at (5.20). $\square$

Using Proposition 5.6 we can now prove the second type of pointwise gradient estimates.

Proposition 5.7. *Assume that Hypotheses* 2.1 *and* 5.1(*i*) *are satisfied and that F fulfills* (2.34). *Then, there exist two constants $C_p>0$ and $\tilde\omega_p\in\mathbb{R}$ such that*

$$|(\nabla T(t)f)(x)|^p \le \tilde C_p e^{\tilde\omega_p t}t^{-\frac{p}{2}}\tilde T(t)(|f|^p)(x) \quad (5.30)$$

for all $t>0$, $x\in\mathbb{R}^N$, $f\in C_c^\infty(\mathbb{R}^N,\mathbb{R}^N)$, and $p\in\,]1,+\infty[\,$.

Proof. We first note that Hypothesis 5.1(ii) holds due to Remark 5.2. By rescaling, we can assume that the constant λ in (5.12) is nonpositive. As in the proof of Proposition 5.6 it suffices to consider the case $p\in\,]1,2]$. We fix $\delta,t>0$ and set

$$\Psi(s)=\tilde T(s)\big((|\vartheta_nT(t-s)f|^2+\delta)^{\frac{p}{2}}\big):=\tilde T(s)\big((|\vartheta_ng(s)|^2+\delta)^{\frac{p}{2}}\big), \qquad s\in[0,t],$$

where $\vartheta_n(x)=\vartheta(|x|/n)$ for all $x\in\mathbb{R}^N$, $n\in\mathbb{N}$, and $\vartheta:\mathbb{R}_+\to\mathbb{R}$ is a smooth decreasing function with $\chi_{[0,1]}\le\vartheta\le\chi_{[0,2]}$. The function $(|\vartheta_ng(s)|^2+\delta)^{\frac{p}{2}}$ belongs to $D(\tilde A)$ for each $s\in[0,t]$, by virtue of Theorem 5.3. A straightforward computation shows that

$$\begin{aligned}\tilde{\mathcal A}\Big((|\vartheta_ng|^2+\delta)^{\frac{p}{2}}\Big)=\;&p(|\vartheta_ng|^2+\delta)^{\frac{p}{2}-1}\vartheta_n^2(\tilde{\mathcal A}g\cdot g)+p(|\vartheta_ng|^2+\delta)^{\frac{p}{2}-1}\vartheta_n^2\sum_{k=1}^N\langle Q\nabla g_k,\nabla g_k\rangle\\ &+p(|\vartheta_ng|^2+\delta)^{\frac{p}{2}-1}|g|^2\langle Q\nabla\vartheta_n,\nabla\vartheta_n\rangle+p(|\vartheta_ng|^2+\delta)^{\frac{p}{2}-1}\vartheta_n|g|^2\tilde{\mathcal A}\vartheta_n\\ &+4p(|\vartheta_ng|^2+\delta)^{\frac{p}{2}-1}\vartheta_n\sum_{k=1}^N g_k\langle Q\nabla g_k,\nabla\vartheta_n\rangle\end{aligned}$$

$$
\begin{aligned}
&+p(p-2)(|\vartheta_n g|^2+\delta)^{\frac{p}{2}-2}\vartheta_n^2|g|^4\langle Q\nabla\vartheta_n,\nabla\vartheta_n\rangle\\
&+2p(p-2)(|\vartheta_n g|^2+\delta)^{\frac{p}{2}-2}\vartheta_n^3|g|^2\sum_{k=1}^N g_k\langle Q\nabla g_k,\nabla\vartheta_n\rangle\\
&+p(p-2)(|\vartheta_n g|^2+\delta)^{\frac{p}{2}-2}\vartheta_n^4\sum_{h,k=1}^N g_hg_k\langle Q\nabla g_h,\nabla g_k\rangle\\
=:{}& p(|\vartheta_n g|^2+\delta)^{\frac{p}{2}-1}\vartheta_n^2(\tilde{\mathcal{A}}g\cdot g)+p(|\vartheta_n g|^2+\delta)^{\frac{p}{2}-1}\vartheta_n^2\sum_{k=1}^N\langle Q\nabla g_k,\nabla g_k\rangle\\
&+p(p-2)(|\vartheta_n g|^2+\delta)^{\frac{p}{2}-2}\vartheta_n^4\sum_{h,k=1}^N g_hg_k\langle Q\nabla g_h,\nabla g_k\rangle+p\psi_n,
\end{aligned}
$$

in $[0,t]$. Moreover,

$$
\begin{aligned}
D_s(|\vartheta_nT(t-s)f|^2+\delta)^{\frac{p}{2}}) &= -p(|\vartheta_n g(s)|^2+\delta)^{\frac{p}{2}-1}\vartheta_n^2(\tilde{\mathcal{A}}g(s)\cdot g(s))\\
&\quad -p(|\vartheta_n g(s)|^2+\delta)^{\frac{p}{2}-1}\vartheta_n^2(Vg(s)\cdot g(s))\\
&\ge -p(|\vartheta_n g(s)|^2+\delta)^{\frac{p}{2}-1}\vartheta_n^2(\tilde{\mathcal{A}}g(s)\cdot g(s)),
\end{aligned}
$$

for every $s\in[0,t]$. Therefore,

$$
\begin{aligned}
\Psi'(s) &= \tilde{T}(s)\Big(\tilde{\mathcal{A}}\left((|\vartheta_n g(s)|^2+\delta)^{\frac{p}{2}}\right)+D_s\left((|\vartheta_nT(t-s)f|^2+\delta)^{\frac{p}{2}}\right)\Big)\\
&\ge p\tilde{T}(s)\bigg((|\vartheta_n g(s)|^2+\delta)^{\frac{p}{2}-1}\vartheta_n^2\sum_{k=1}^N\langle Q\nabla g_k(s),\nabla g_k(s)\rangle\bigg)\\
&\quad +p(p-2)\tilde{T}(s)\bigg((|\vartheta_n g(s)|^2+\delta)^{\frac{p}{2}-2}\vartheta_n^4\sum_{h,k=1}^N g_h(s)g_k(s)\langle Q\nabla g_h(s),\nabla g_k(s)\rangle\bigg)+p\tilde{T}(s)(\psi_n(s)),
\end{aligned}
$$

and, hence,

$$
\begin{aligned}
&\tilde{T}(t)\Big((|\vartheta_n f|^2+\delta)^{\frac{p}{2}}\Big)-(|\vartheta_nT(t)f|^2+\delta)^{\frac{p}{2}}\\
&\quad\ge p\int_0^t\tilde{T}(s)\bigg((|\vartheta_nT(t-s)f|^2+\delta)^{\frac{p}{2}-1}\vartheta_n^2\sum_{k=1}^N\langle Q\nabla(T(t-s)f)_k,\nabla(T(t-s)f)_k\rangle\bigg)ds\\
&\qquad +p(p-2)\int_0^t\tilde{T}(s)\bigg(\sum_{h,k=1}^N(T(t-s)f)_h(T(t-s)f)_k\langle Q\nabla(T(t-s)f)_h,\nabla(T(t-s)f)_k\rangle\\
&\qquad\qquad\times(|\vartheta_nT(t-s)f|^2+\delta)^{\frac{p}{2}-2}\vartheta_n^4\bigg)ds+p\int_0^t\tilde{T}(s)\psi_n(s)ds, \qquad (5.31)
\end{aligned}
$$

for any $s\in[0,t]$. Here and below, the integrals are meant in a pointwise sense. Thanks to (2.34), we have $F(x)\cdot\nabla\vartheta_n(x)\ge 2c\vartheta'(|x|/n)$ for $x\in\mathbb{R}^N$ and $n\in\mathbb{N}$. This shows that

$$
\begin{aligned}
h_n(s) &:= \tilde{T}(s)[(|\vartheta_n g(s)|^2+\delta)^{\frac{p}{2}-1}\vartheta_n|g(s)|^2F\cdot\nabla\vartheta_n]\\
&\ \ge\ 2c\tilde{T}(s)[(|\vartheta_n g(s)|^2+\delta)^{\frac{p}{2}-1}\vartheta_n|g(s)|^2\vartheta'(|\cdot|/n)].
\end{aligned}
$$

The function in square brackets on the right-hand side is uniformly bounded in $n\in\mathbb{N}$, for $(s,x)\in[0,t]\times\mathbb{R}^N$, and it tends to 0 uniform for (s,x) in compact sets as $n\to+\infty$. So, Theorem 5.3 implies that

$$\liminf_{n\to+\infty}\int_0^t h_n(s)\,ds \ge 0.$$

The other terms in ψ_n even converge to 0 uniformly in $[0,t]\times\mathbb{R}^N$ as $n\to+\infty$. Moreover, the function $(|\vartheta_n f|^2+\delta)^{p/2}$ and the other terms in the large brackets in the right-hand side of (5.31) are bounded in $[0,t]\times\mathbb{R}^N$ and converge uniformly in $[0,t]\times K$, for each compact set $K\subset\mathbb{R}^N$. Proposition 5.5(i) thus allows us to let $n\to+\infty$ in (5.31). Further, using Cauchy-Schwarz inequality and (2.3), we finally obtain

$$\begin{aligned}
&\tilde T(t)\Big((|f|^2+\delta)^{\frac p2}\Big)-(|T(t)f|^2+\delta)^{\frac p2}\\
&\quad\ge p\int_0^t \tilde T(s)\Big((|T(t-s)f|^2+\delta)^{\frac p2-1}\sum_{k=1}^N\langle Q\nabla(T(t-s)f)_k,\nabla(T(t-s)f)_k\rangle\Big)ds\\
&\qquad+p(p-2)\int_0^t\tilde T(s)\Big(\sum_{h,k=1}^N(T(t-s)f)_h(T(t-s)f)_k\langle Q\nabla(T(t-s)f)_h,\nabla(T(t-s)f)_k\rangle\\
&\qquad\qquad\times(|T(t-s)f|^2+\delta)^{\frac p2-2}\Big)ds\\
&\quad\ge p(p-1)\int_0^t\tilde T(s)\Big((|T(t-s)f|^2+\delta)^{\frac p2-1}\sum_{k=1}^N\langle Q\nabla(T(t-s)f)_k,\nabla(T(t-s)f)_k\rangle\Big)ds\\
&\quad\ge p(p-1)\alpha_1\int_0^t\tilde T(s)\Big((|T(t-s)f|^2+\delta)^{\frac p2-1}|\nabla T(t-s)f|^2\Big)ds. \qquad (5.32)
\end{aligned}$$

Applying Propositions 5.5(iii) and 5.6 and observing that $(1+t)^q\le 1+t^q$ for all $t\ge 0$ and $q\in[0,1]$, we deduce

$$\begin{aligned}
|\nabla T(t)f|^p &= |\nabla T(s)T(t-s)f|^p\\
&\le C_pe^{\omega_p s}\tilde T(s)(|\nabla T(t-s)f|^p)+C_pe^{\omega_p s}\tilde T(s)(|T(t-s)f|^p)\\
&= C_pe^{\omega_p s}\tilde T(s)\Big(|\nabla T(t-s)f|^p\,(|T(t-s)f|^2+\delta)^{-\beta}\,(|T(t-s)f|^2+\delta)^{\beta}\Big)\\
&\quad+C_pe^{\omega_p s}\tilde T(s)(|T(t-s)f|^p)\\
&\le C_pe^{\omega_p s}\Big[\tilde T(s)\Big(|\nabla T(t-s)f|^2\,(|T(t-s)f|^2+\delta)^{-\frac{2\beta}{p}}\Big)\Big]^{\frac p2}\\
&\quad\times\Big[\tilde T(s)\Big((|T(t-s)f|^2+\delta)^{\frac{2\beta}{2-p}}\Big)\Big]^{1-\frac p2}+C_pe^{\omega_p s}\tilde T(s)(|T(t-s)f|^p),
\end{aligned}$$

for all $\beta\in\mathbb{R}$ and $s\in[0,t[$. We now choose $\beta=p(2-p)/4$ and use Proposition 5.5(iii) and (iv) (with $\lambda=0$) and Young's inequality, arriving at

$$\begin{aligned}
|\nabla T(t)f|^p &\le C_pe^{\omega_p s}\Big[\tilde T(s)\Big(|\nabla T(t-s)f|^2\,(|T(t-s)f|^2+\delta)^{\frac p2-1}\Big)\Big]^{\frac p2}\\
&\quad\times\Big[\tilde T(s)\Big((|T(t-s)f|^2+\delta)^{\frac p2}\Big)\Big]^{1-\frac p2}+C_pe^{\omega_p s}\tilde T(t)(|f|^p)\\
&\le C_pe^{\omega_p s}\Big\{\frac p2\varepsilon^{\frac 2p}\tilde T(s)\Big(|\nabla T(t-s)f|^2\,(|T(t-s)f|^2+\delta)^{\frac p2-1}\Big)\\
&\quad+\Big(1-\frac p2\Big)\varepsilon^{\frac{2}{p-2}}\tilde T(s)\Big((|T(t-s)f|^2+\delta)^{\frac p2}\Big)\Big\}+C_pe^{\omega_p s}\tilde T(t)(|f|^p),
\end{aligned}$$

for each $\varepsilon > 0$. After multiplying by $e^{-\omega_p s}$, we integrate this inequality from 0 to t. Using also (5.32), we estimate

$$\begin{aligned}\frac{(1-e^{-\omega_p t})}{\omega_p}|\nabla T(t)f|^p \le\ & \frac{p}{2}\varepsilon^{\frac{2}{p}}C_p\int_0^t \tilde T(s)\left(|\nabla T(t-s)f|^2\left(|T(t-s)f|^2+\delta\right)^{\frac{p}{2}-1}\right)ds\\ &+C_p\left(1-\frac{p}{2}\right)\varepsilon^{\frac{2}{p-2}}\int_0^t \tilde T(s)\Big((|T(t-s)f|^2+\delta)^{\frac{p}{2}}\Big)ds + C_p t\tilde T(t)(|f|^p)\\ \le\ & \varepsilon^{\frac{2}{p}}\frac{C_p}{2\alpha_1(p-1)}\tilde T(t)\Big((|f|^2+\delta)^{\frac{p}{2}}\Big)+C_p t\tilde T(t)(|f|^p)\\ &+C_p\left(1-\frac{p}{2}\right)\varepsilon^{\frac{2}{p-2}}\int_0^t \tilde T(s)\Big((|T(t-s)f|^2+\delta)^{\frac{p}{2}}\Big)ds.\end{aligned}$$

In the limit $\delta \to 0^+$, this inequality yields

$$|\nabla T(t)f|^p \le \frac{\omega_p C_p}{(1-e^{-\omega_p t})}\left\{\left(\frac{\varepsilon^{\frac{2}{p}}}{2\alpha_1(p-1)}+t\right)\tilde T(t)(|f|^p)+\left(1-\frac{p}{2}\right)\varepsilon^{\frac{2}{p-2}}\int_0^t \tilde T(s)(|T(t-s)f|^p)ds\right\},$$

for all $t > 0$. Taking in account Proposition 5.5(iv) (with $\lambda = 0$), we conclude that

$$|\nabla T(t)f|^p \le \frac{\omega_p C_p}{1-e^{-\omega_p t}}\left\{\frac{\varepsilon^{\frac{2}{p}}}{2\alpha_1(p-1)}+\left[\left(1-\frac{p}{2}\right)\varepsilon^{\frac{2}{p-2}}+1\right]t\right\}\tilde T(t)(|f|^p),\qquad t>0.$$

The optimal choice $\varepsilon = \{\alpha_1 p(p-1)t\}^{\frac{p(2-p)}{4}}$ yields

$$|\nabla T(t)f|^p \le \frac{\omega_p C_p}{1-e^{-\omega_p t}}\left\{\frac{1}{(\alpha_1 p(p-1))^{\frac{p}{2}}}t^{1-\frac{p}{2}}+t\right\}\tilde T(t)(|f|^p),\qquad t>0.$$

This inequality implies the assertion. □

We can now prove the main result of this section.

Theorem 5.8. *Assume that Hypotheses* 2.1 *and* 5.1(*i*) *are satisfied and that F fulfills* (2.34)*. Let $0 \le \alpha \le \beta \le 1$ and $1 < p < +\infty$. Then, the function $T(t)f$ belongs to $W^{1,p}(\mathbb{R}^N,\mathbb{R}^N)$ for every $f \in L^p(\mathbb{R}^N)$ and $t > 0$, and there exist two constants $M_p > 0$ and $\hat\omega_p \in \mathbb{R}$ such that*

$$\|T(t)f\|_{\beta,p} \le M_p t^{-\frac{(\beta-\alpha)p}{2}}e^{\hat\omega_p t}\|f\|_{\alpha,p},\qquad t>0,\ \ f\in W^{\alpha,p}(\mathbb{R}^N,\mathbb{R}^N). \tag{5.33}$$

Proof. Integrating (5.18) and (5.30) in $\mathbb{R}^N$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^N}|\nabla T(t)f|^p dx &\le C_p e^{\omega_p t}\int_{\mathbb{R}^N}\tilde T(t)\big((|f|^2+|\nabla f|^2)^{\frac{p}{2}}\big)dx\\ &\le C_p\max\{2^{\frac{p}{2}-1},1\}e^{(\omega_p+\sigma_1)t}\int_{\mathbb{R}^N}(|f|^p+|\nabla f|^p)dx,\end{aligned}\tag{5.34}$$

$$\int_{\mathbb{R}^N}|\nabla T(t)f|^p dx \le \tilde C_p e^{\tilde\omega_p t}t^{-\frac{p}{2}}\int_{\mathbb{R}^N}\tilde T(t)(|f|^p)dx \le \tilde C_p e^{(\tilde\omega_p+\sigma_1)t}t^{-\frac{p}{2}}\int_{\mathbb{R}^N}|f|^p dx, \tag{5.35}$$

for all $t > 0$, $p \in]1,+\infty[$ and $f \in C_c^\infty(\mathbb{R}^N,\mathbb{R}^N)$. By density, (5.34), resp. (5.35), can be extended to every $f \in W^{1,p}(\mathbb{R}^N,\mathbb{R}^N)$, resp. $f \in L^p(\mathbb{R}^N,\mathbb{R}^N)$, so that (5.33) holds for $\alpha = 1, 0$ and $\beta = 1$ (taking also into account (2.28)). We recall that $W^{\theta,p}(\mathbb{R}^N,\mathbb{R}^N)$ is isomorphic to the real interpolation space of order $\theta \in (0,1)$ and exponent $p \in]1,+\infty[$, between $L^p(\mathbb{R}^N,\mathbb{R}^N)$ and $W^{1,p}(\mathbb{R}^N,\mathbb{R}^N)$, cf. [25, (2.4.2.16)]. The assertion now follows from (2.28), (5.34), and (5.35) by means of standard interpolation arguments. □

References

[1] S. Agmon: *The L_p approach to the Dirichlet problem,* Ann. Scuola Norm. Sup. Pisa **III-13** (1960), 405-448.

[2] M. Bertoldi, L. Lorenzi: *Estimates of the derivatives for parabolic operators with unbounded coefficients*, Trans. Amer. Math. Soc. **357** (2005), pp. 2627-2664.

[3] M. Bertoldi, L. Lorenzi: Analytical Methods for Markov Semigroups, *Pure and Applied Mathematics* **283**, Chapman Hall/CRC Press (2006).

[4] G. Cupini, S. Fornaro: *Maximal regularity in $L^p(\mathbb{R}^N)$ for a class of elliptic operators with unbounded coefficients*, Differential Integral Equations **17** (2004), pp. 259-296.

[5] G. Da Prato, V. Vespri: *Maximal L^p regularity for elliptic equations with unbounded coefficients.* Nonlinear Analysis **49** (2002), pp. 747-755.

[6] E.B. Davies: Heat Kernels and Spectral Theory, Cambridge Univ. Press (1989).

[7] K.J. Engel, R. Nagel: One-Parameter Semigroups for Linear Evolution Equations, *Graduate Text in Mathematics* **194**, Springer-Verlag, New York (2000).

[8] S. Fornaro, L. Lorenzi: *Generation results for elliptic operators with unbounded diffusion coefficients in L^p- and C_b-spaces*, Discr. Cont. Dyn. Syst. Series A, to appear.

[9] F. Gozzi, R. Monte, V. Vespri: *Generation of analytic semigroups and domain characterization for degenerate elliptic operators with unbounded coefficients arising in financial mathematics. I*, Differential Integral Equations **15** (2002), pp. 1085-1128.

[10] M. Haase: *A transference principle for general groups and fuctional calculus on UMD spaces*, preprint.

[11] M. Hieber, L. Lorenzi, A. Rhandi: *Uniform and L^p estimates of the derivatives for parabolic operators with unbounded coefficients in exterior domains*, preprint **425** Dipartimento di Matematica, Università di Parma, submitted.

[12] M. Hieber, A. Rhandi, O. Sawada: *On the Navier-Stokes flow with globally Lipschitz initial velocity*, submitted.

[13] M. Hieber, O. Sawada: *The Navier-Stokes equations in $\mathbb{R}^n$ with linearly growing initial data.*, Arch. Ration. Mech. Anal. **175** (2005), pp. 269-285.

[14] A. Lunardi: Analytic Semigroups and Optimal Regularity in Parabolic Problems, Birkhäuser (1995).

[15] A. Lunardi: *On the Ornstein-Uhlenbeck Operator in L^2 spaces with respect to invariant measures*, Trans. Amer. Math. Soc. **349** (1997), no. 1, pp. 155-169.

[16] A. Lunardi: *Schauder theorems for linear elliptic and parabolic problems with unbounded coefficients in $\mathbb{R}^n$*, Studia Math. **128** (1998), no. 2, pp. 171-198.

[17] G. Metafune, D. Pallara, M. Wacker: *Feller semigroups on $\mathbb{R}^N$*, Semigroup Forum **65** (2002), pp. 159-205.

[18] G. Metafune, D. Pallara, V. Vespri: *L^p-estimates for a class of elliptic operators with unbounded coefficients in $\mathbb{R}^N$*, Houston J. Math. **31** (2005), pp. 605-620.

[19] G. Metafune, D. Pallara, J. Prüss, R. Schnaubelt: *L^p-theory for elliptic operators on $\mathbb{R}^d$ with singular coefficients*, Z. Anal. Anwendungen **24** (2005), pp. 497-521.

[20] G. Metafune, D. Pallara, A. Rhandi, R. Schnaubelt: *L^p-regularity for elliptic operators with unbounded coefficients*, Adv. Differential Equations **10** (2005), pp. 1131-1164.

[21] G. Metafune, E. Priola: *Some classes of non-analytic Markov semigroups*, J. Math. Anal. Appl. **294** (2004), pp. 596-613.

[22] S. Monniaux, J Prüss: *A theorem of Dore-Venni type for noncommuting operators*, Trans. Amer. Math. Soc. **349** (1997), pp. 4787-4814.

[23] J. Prüss, A. Rhandi, R. Schnaubelt: *The domain of elliptic operators on $L^p(\mathbb{R}^d)$ with unbounded drift coefficients*, Houston J. Math. **32** (2006), pp. 563-576.

[24] R. Rabier: *Elliptic problems in $\mathbb{R}^N$ with unbounded coefficients*, Math. Z. **249** (2005), pp. 1-30.

[25] H. Triebel: Interpolation Theory, Function Spaces, Differential operators, North-Holland, Amsterdam (1978).

[26] J.M.A.M. van Neerven, J. Zabczyk: *Norm discontinuity of Ornstein-Uhlenbeck semigroups,* Semigroup Forum **59** (1999), pp. 389-403.

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