Invariant transports of stationary random measures and mass-stationarity

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Abstract

We introduce and discuss balancing invariant transports of stationary random measures on a Polish Abelian group. The first main result is an associated fundamental invariance property of Palm measures, derived from a generalization of Neveu's exchange formula. The second main result is a simple sufficient and necessary criterion for the existence of balancing invariant transports. We then introduce (in a non-stationary setting) the concept of mass-stationarity with respect to a random measure, formalizing the intuitive idea that the origin is a typical location in the mass. The third main result of the paper is that a measure is a Palm measure if and only if it is mass-stationary.

1 Introduction

We consider (jointly) stationary random measures on a Polish Abelian group $\mathbf{G}$, for instance $\mathbf{G} = \mathbb{R}^d$. A *transport* T is a Markovian kernel (depending on both ω in the underlying sample space Ω and a location $s \in \mathbf{G}$) that reshapes the mass of a random measure ξ. The transport is *invariant* if the rule, moving the mass from one location s to another location, is the same for all s. It is *balancing* if the resulting new shape is a given random measure η and, in particular, it is *measure-preserving* if the new shape is the same as the original one: $\eta = \xi$. Sometimes the transport T can be reduced to a *transport-map* τ (depending on $\omega \in \Omega$) which maps each location s to a new location $\tau(s)$. In fact we might think of a transport as the conditional distribution of a randomized transport-map.

The aim of the paper is to treat three interwoven aspects of invariant transports: basic invariance properties of Palm measures are presented in Section 4, a general existence result in Section 5, and an intrinsic characterization of Palm measures – *mass-stationarity* – in Sections 6-8.

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Invariance properties (Section 4)
It is a fundamental and classical theorem that the Palm distribution of a stationary point process on the line is invariant under shifts to the next point to the right. In fact there is an essentially unique correspondence between such stationary point processes and stationary sequences of interpoint distances, see Theorem 11.4 in [9] and the references given there. In higher dimension the situation is more complicated. Mecke [14] found an intrinsic characterization of Palm measures using an integral equation, see (2.7). Geman and Horowitz [3] showed that a stationary random measure is distributionally invariant under shifts from the neutral element $0 \in \mathbf{G}$ to $\tau(0)$, if τ is an invariant transport-map preserving Haar measure. Independently, Mecke [15] extended this result to general Palm measures, see Remark 4.9. For point processes and using *bijective point-shifts*, this fact was rediscovered in [19], see also [4] for a short proof. Holroyd and Peres [8] considered (in case $\mathbf{G} = \mathbb{R}^d$) invariant transports T balancing Lebesgue measure and an ergodic point process η of intensity 1. Theorem 16 in that paper shows that the stationary distribution is then transformed into the Palm distribution of η, if the origin is shifted to a "randomized" location with distribution $T(0, \cdot)$. An analogous result holds for discrete groups.

Neveu's [16] well-known exchange formula (see Remark 4.3) is an apparently quite different property of Palm measures. Extending the concept of balancing transports to balancing *quasi-transports* (the kernel T need not be Markovian), we will generalize Neveu's result in our Theorem 4.2. This is then actually the key for obtaining the general invariance property of Theorem 4.1, containing all the invariance results mentioned above. Another crucial idea for Theorem 4.1 is that any balancing invariant quasi-transport has an *inverse* invariant transport.

Existence (Section 5)
Liggett [12] presented the following (surprising?) result. Consider a doubly-infinite sequence of i.i.d. coin tosses. Move the origin to a head as follows. If there is a head at the origin, stay there. If there is a tail at the origin, move to the right counting heads and tails until you have more heads than tails. Then you are at a head. Now remove that head. Then the rest of the coin tosses are i.i.d. (so you have found an extra head!). This procedure thus gives an explicit *shift-coupling* of an i.i.d. sequence ξ and its Palm version: they are the same up to a shift of the origin. Note that if Liggett's rule is applied to all locations then it generates an invariant transport-map τ, transporting counting measure on the integers to the Bernoulli (1/2) random measure with intensity 1. Liggett also treated a general Bernoulli parameter p and the Poisson process on the line.

Triggered by Liggett's paper the case where $\mathbf{G} = \mathbb{R}^d$ (or $\mathbf{G} = \mathbb{Z}^d$), ξ is Lebesgue measure, and η is an ergodic point process of intensity 1 has received considerable attention in recent years (see [6], [8], [11], [1]). Holroyd and Peres [8] presented the following beautiful transport-map of Lebesgue measure to the Poisson process with intensity 1. Partition space by associating to each point a region of space of exact size 1. For this purpose place a small ball around each point of the Poisson process and expand the balls simultaneously until they reach size 1. If a ball hits space that has already been allocated to another point, let it continue to grow under the allocated region until it finds space that has not yet been allocated. In this way $\mathbb{R}^d$ is partitioned into (not necessarily connected) regions of size 1, each containing one point of the Poisson process. Now transport each location to the point of its region. This reshapes Lebesgue measure into the Poisson process in an invariant way. In particular, if we shift the origin to the point of its region we obtain a shift-coupling of the stationary Poisson process and its Palm version. In fact, Holroyd and Peres treated the case when η is an ergodic stationary point process.

Actually, an abstract group-coupling result from [18] implies that shift-coupings exist in the above cases. In Section 5 we shall apply that result to stationary random measures with finite intensities to prove that there exists an invariant balancing transport if and only if the two random measures have the same intensity conditional on the invariant σ-field.

Mass-Stationarity (Sections 6-8)
Mass-stationarity of a random measure ξ means, informally, that the origin is a typical location in the mass, just like stationarity means that the origin is a typical location in space. Mass-stationarity is an extension of the concept of point-stationarity introduced in Thorisson [19] in the case of simple point processes on $\mathbb{R}^d$. The formal definition of point-stationarity in that paper can be loosely phrased as follows: a simple point-process ξ is point-stationary if its distribution is invariant under bijective point-shifts against any independent stationary background (the shifts were allowed to depend not only on ξ but also on any independent stationary random field obtained by extending the underlying probability space). The main result of [19] was that point-stationarity is a characterizing property of Palm versions of stationary point processes.

The question whether the independent stationary background could be removed from the definition of point-stationarity inspired considerable research activity, see [7], [2], [21], [11]. Finally, Heveling and Last ([4], [5]) showed that this can be done, that is, no external randomization is needed.

In the point process case, bijective point-shifts generate measure-preserving transport-maps. Thus it is natural to attempt to extend the above definition to random measures by demanding that the distribution of ξ be invariant under shifts induced by invariant measure-preserving transport-maps. It turns out, however, that here external randomization would be needed, see Example 8.1. One could add an independent stationary background to the condition, but it is still an open problem whether it would then characterize Palm versions of stationary random measures.

Instead of using the above condition as a definition, we use the following condition, which in the point process case was proved in [19] to be equivalent to point-stationarity: a random measure ξ is mass-stationary if its distribution is invariant under shifting the origin by first placing a set at random around it and then relocating the origin to a uniform position in the mass on the set. (Actually we have to include the random position of the origin in the set.) This is equivalent to distributional invariance under a certain class of quasi-transports, see Theorem 8.2. Section 8 concludes with four open problems.

2 Preliminaries on stationary random measures

We choose to work in the abstract setting of a flow acting on the underlying sample space (see [3], [15], [16]), and with σ-finite measures rather than with probability measures, see Remark 2.7.

We consider a topologial Abelian group $\mathbf{G}$ that is assumed to be a locally compact, second countable Hausdorff space with Borel σ-field $\mathcal{G}$. On $\mathbf{G}$ there exists an invariant measure λ, that is unique up to normalization. A measure μ on $\mathbf{G}$ is *locally finite* if it is finite on compact sets. We denote by $\mathbf{M}$ the set of all locally finite measures on $\mathbf{G}$, and by $\mathcal{M}$ the cylindrical σ-field on $\mathbf{M}$ which is generated by the evaluation functionals $\mu \mapsto \mu(B)$, $B \in \mathcal{G}$. The *support* $\operatorname{supp} \mu$ of a measure $\mu \in \mathbf{M}$ is the smallest closed set $F \subset \mathbf{G}$ such that $\mu(\mathbf{G} \setminus F) = 0$. By $\mathbf{N} \subset \mathbf{M}$ we denote the measurable set of all (simple) *counting measures* on $\mathbf{G}$, i.e. the set of all those $\mu \in \mathbf{M}$ with discrete support and $\mu\{s\} := \mu(\{s\}) \in \{0, 1\}$ for all $s \in \mathbf{G}$. We can and will identify $\mathbf{N}$ with the class of all *locally finite* subsets of $\mathbf{G}$, where a set is called locally finite if its intersection with any compact set is finite.

In this paper we mostly work on a σ-finite measure space $(\Omega, \mathcal{F}, \mathbb{P})$ (but see also Remark 2.7). However, we will consider several measures on $(\Omega, \mathcal{F})$. A *random measure* on $\mathbf{G}$ is a measurable mapping $\xi : \Omega \to \mathbf{M}$ and a (simple) *point process* on $\mathbf{G}$ is measurable mapping $\xi : \Omega \to \mathbf{N}$. A random measure ξ can also be regarded as a *kernel* from Ω to $\mathbf{G}$. Accordingly we write $\xi(\omega, B)$ instead of $\xi(\omega)(B)$. If ξ is a random measure, then the mapping $(\omega, s) \mapsto \mathbf{1}\{s \in \operatorname{supp} \xi(\omega)\}$ is measurable.

We assume that $(\Omega, \mathcal{F})$ is equipped with a *measurable flow* $\theta_s : \Omega \to \Omega$, $s \in \mathbf{G}$. This is a family of measurable mappings such that $(\omega, s) \mapsto \theta_s\omega$ is measurable, θ_0 is the identity on Ω and

$$\theta_s \circ \theta_t = \theta_{s+t}, \quad s, t \in \mathbf{G}, \tag{2.1}$$

where 0 denotes the neutral element in $\mathbf{G}$ and $\circ$ denotes composition. A random measure ξ on $\mathbf{G}$ is called *invariant* (or *flow-adapted*) if

$$\xi(\theta_s\omega, B - s) = \xi(\omega, B), \quad \omega \in \Omega,\ s \in \mathbf{G},\ B \in \mathcal{G}. \tag{2.2}$$

A measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ is called *stationary* if it is invariant under the flow, i.e.

$$\mathbb{P} \circ \theta_s = \mathbb{P}, \quad s \in \mathbf{G},$$

where θ_s is interpreted as a mapping from $\mathcal{F}$ to $\mathcal{F}$ in the usual way:

$$\theta_s A := \{\theta_s\omega : \omega \in A\}, \quad A \in \mathcal{F},\ s \in \mathbf{G}.$$

Because of the next examples we may think of $\theta_s\omega$ as of ω *shifted* by $-s$.

Example 2.1. Consider the measurable space $(\mathbf{M}, \mathcal{M})$ and define for $\mu \in \mathbf{M}$ and $s \in \mathbf{G}$ the measure $\theta_s\mu$ by $\theta_s\mu(B) := \mu(B + s)$, $B \in \mathcal{G}$. Then $\{\theta_s : s \in \mathbf{G}\}$ is a measurable flow and the identity ξ on $\mathbf{M}$ is an invariant random measure. A stationary probability measure on $(\mathbf{M}, \mathcal{M})$ can be interpreted as the distribution of a *stationary random measure*.

Remark 2.2. Since a random measure ξ is a random element in $\mathbf{M}$, we can rewrite the invariance condition (2.2) as

$$\xi(\theta_s\omega) = \theta_s\xi(\omega), \quad \omega \in \Omega, \tag{2.3}$$

where we use θ_s, $s \in \mathbf{G}$, to denote both the abstract flow and the specific flow defined in Example 2.1. Therefore (2.2) is also referred to as *flow-covariance*. We follow here the terminology of [10]. A similar remark applies to invariant transports, to be defined below.

Example 2.3. Consider some measurable space $(E, \mathcal{E})$ and the product space $E^{\mathbf{G}}$. The shift operator $\theta_s : E^{\mathbf{G}} \to E^{\mathbf{G}}$ is defined by $\theta_s w(t) := w(s+t)$, $t \in \mathbf{G}$. Consider a path space $\mathbf{W} \subset E^{\mathbf{G}}$ that is invariant under shifts and that is equipped with a σ-field $\mathcal{W}$ rendering all mappings $w \mapsto w(s)$, $s \in \mathbf{G}$, measurable. As in [20], assume that $(s, w) \mapsto \theta_s w$ is measurable with respect to $\mathcal{G} \otimes \mathcal{W}$ and $\mathcal{W}$. We can then define $\Omega := \mathbf{M} \times \mathbf{W}$, $\mathcal{F} := \mathcal{M} \otimes \mathcal{W}$ and (abusing notation) $\theta_s \omega := (\theta_s \mu, \theta_s w)$ for $\omega = (\mu, w) \in \mathbf{M} \times \mathbf{W}$ and $s \in \mathbf{G}$. This is a flow, and the projection $\xi(\omega) := \mu$ defines an invariant random measure. A stationary probability measure on $(\Omega, \mathcal{F})$ can be interpreted as the joint distribution of a random measure and the E-valued stochastic process Y on $\mathbf{G}$ (defined by $Y(\mu, w) := w$) which are *jointly stationary*.

Example 2.4. Assume that $(\Omega, \mathcal{F}) = (\mathbf{M} \times \mathbf{M}, \mathcal{M} \otimes \mathcal{M})$ and (again abusing notation) $\theta_s \omega = (\theta_s \mu, \theta_s \nu)$ for $\omega = (\mu, \nu) \in \mathbf{M} \times \mathbf{M}$. Both projections define invariant random measures. A stationary probability measure on $(\Omega, \mathcal{F})$ can be interpreted as the joint distribution of two jointly stationary random measures.

Example 2.5. Assume that $(\Omega, \mathcal{F}) = (\mathbf{G}, \mathcal{G})$ and $\theta_s \omega := \omega + s$. Then the Haar measure $\mathbb{P} := \lambda$ is stationary. If $\mathbb{P}'$ is a probability measure on $\mathbf{G}$, then $\xi(\omega, B) := \mathbb{P}'(B + \omega)$ defines an invariant random measure.

Example 2.6. Let $\mathbf{G} = \mathbb{R}$ and Ω be the space of all continuous functions $w : \mathbb{R} \to \mathbb{R}^d$. Take $\mathcal{F}$ as the σ-field rendering the projections measurable. Define θ_s by $\theta_s w(t) := w(t+s)$, $t \in \mathbb{R}$. Let $\mathbb{P}'$ be the distribution of a standard Brownian motion, starting at $0 \in \mathbb{R}^d$. Then $\mathbb{P} := \int \mathbb{P}'(\{w : x + w \in \cdot\})\, dx$ is a σ-finite stationary measure on $(\Omega, \mathcal{F})$. (Here $x + w$, denotes the function $t \mapsto x + w(t)$.)

Let $\mathbb{P}$ be a stationary σ-finite measure on $(\Omega, \mathcal{F})$ and ξ an invariant random measure on $\mathbf{G}$. Then ξ is stationary in the usual sense, i.e. $\mathbb{P}(\xi \in \cdot) = \mathbb{P}(\theta_s \xi \in \cdot)$ for all $s \in \mathbf{G}$, where we have used the notation of Example 2.1. Let $B \in \mathcal{G}$ be a set with positive and finite Haar measure $\lambda(B)$. The measure

$$\mathbb{P}_\xi(A) := \lambda(B)^{-1} \iint \mathbf{1}\{\theta_s \omega \in A, s \in B\}\, \xi(\omega, ds)\, \mathbb{P}(d\omega), \quad A \in \mathcal{F}, \tag{2.4}$$

is called the *Palm measure* of ξ (with respect to $\mathbb{P}$), see [14]. This measure is σ-finite. As the definition (2.4) is independent of B, we have the *refined Campbell theorem*

$$\iint f(\theta_s \omega, s)\, \xi(\omega, ds)\, \mathbb{P}(d\omega) = \iint f(\omega, s)\, ds\, \mathbb{P}_\xi(d\omega)$$

for all measurable $f : \Omega \times \mathbf{G} \to [0, \infty)$, where ds refers to integration with respect to the Haar measure λ. Using a standard convention in probability theory, we write this as

$$\mathbb{E}\left[\int f(\theta_s, s)\, \xi(ds)\right] = \mathbb{E}_{\mathbb{P}_\xi}\left[\int f(\theta_0, s)\, ds\right], \tag{2.5}$$

where $\mathbb{E}$ and $\mathbb{E}_{\mathbb{P}_\xi}$ denote integration with respect to $\mathbb{P}$ and $\mathbb{P}_\xi$, respectively. If the *intensity* $\mathbb{P}_\xi(\Omega)$ of ξ is positive and finite, then the normalized Palm measure

$$\mathbb{P}^0_\xi := \mathbb{P}_\xi(\Omega)^{-1} \mathbb{P}_\xi$$

is called *Palm probability measure* of ξ (w.r.t. $\mathbb{P}$). Note that $\mathbb{P}_\xi$ and $\mathbb{P}^0_\xi$ are both defined on the underlying space $(\Omega, \mathcal{F})$. The stationary measure $\mathbb{P}$ can be recovered from the Palm measure $\mathbb{P}_\xi$ using a measurable function $\tilde{h} : \mathbf{M}\times\mathbf{G} \to [0,\infty)$ satisfying $\int \tilde{h}(\mu, s)\,\mu(ds) = 1$, whenever $\mu \in \mathbf{M}$ is not the null measure. For one example of such a function we refer to [14]. We then have the *inversion formula*

$$\mathbb{E}_{\mathbb{P}}[\mathbf{1}\{\xi(\mathbf{G}) > 0\}f] = \mathbb{E}_{\mathbb{P}_\xi}\left[\int \tilde{h}(\xi \circ \theta_{-s}, s) f(\theta_{-s})\, ds\right], \tag{2.6}$$

for all measurable $f : \Omega \to [0,\infty)$, see Satz 2.4 in [14]. This is a direct consequence of the refined Campbell theorem (2.5).

Let ξ be an invariant random measure and $\mathbb{Q}$ be a σ-finite measure on $(\Omega, \mathcal{F})$ satisfying $\mathbb{Q}(\xi(\mathbf{G}) = 0) = 0$. Satz 2.5 in [14] says that $\mathbb{Q}$ is a Palm measure of ξ if and only if

$$\mathbb{E}_{\mathbb{Q}}\left[\int g(\theta_s, -s)\,\xi(ds)\right] = \mathbb{E}_{\mathbb{Q}}\left[\int g(\theta_0, s)\,\xi(ds)\right] \tag{2.7}$$

holds for any measurable $g : \Omega\times\mathbf{G} \to [0,\infty)$. Mecke has proved his result in the canonical framework of Example 2.1. But his proof applies in our more general framework as well. The necessity of (2.7) is a special case of Neveu's exchange formula, see Remark 4.3. To prove that (2.7) is also sufficient for $\mathbb{Q}$ to be a Palm measure one can use the function $\tilde{h}$ in (2.6) to define a σ-finite measure $\mathbb{P}$ by

$$\mathbb{P}(A) := \mathbb{E}_{\mathbb{Q}}\left[\int \tilde{h}(\xi \circ \theta_{-s}, s)\mathbf{1}\{\theta_{-s} \in A\}\, ds\right], \quad A \in \mathcal{F}.$$

It can be shown as in [14], that (2.7) implies stationarity of $\mathbb{P}$ and $\mathbb{Q} = \mathbb{P}_\xi$.

Remark 2.7. We would like to mention two reasons (other than just generality), why we are not assuming the stationary measure $\mathbb{P}$ to be a probability measure. First, some of the fundamental results can be more easily stated this way. An example is the one-to-one correspondence between $\mathbb{P}$ and the Palm measure $\mathbb{P}_\xi$ (see [14]). Otherwise, extra technical integrability assumptions are required (see e.g. Theorem 11.4 in [9]). A second reason is that in some applications it is the Palm probability measure that has a probabilistic interpretation (see e.g. [22]). The latter can be defined whenever the (stationary) intensity is positive and finite, see Example 3.2 for a simple illustration of this fact.

3 Invariant transports and quasi-transports

A *transport* (on $\mathbf{G}$) is a kernel T from $\Omega \times \mathbf{G}$ to $\mathbf{G}$ which is Markovian, i.e. which has $T(\omega, s, \mathbf{G}) = 1$ for all $(\omega, s) \in \Omega\times\mathbf{G}$. We think of $T(\omega, s, \cdot)$ as redistributing a (potential) unit mass at s within $\mathbf{G}$. A transport T is called *invariant* if

$$T(\theta_t\omega, s - t, B - t) = T(\omega, s, B), \quad s, t \in \mathbf{G},\, \omega \in \Omega,\, B \in \mathcal{G}. \tag{3.1}$$

This is equivalent to $T(\theta_t\omega, 0, B - t) = T(\omega, t, B)$ for all t, ω, and B. Quite often we use the short-hand notation $T(s, \cdot) := T(\theta_0, s, \cdot)$.

A *quasi-transport* on $\mathbf{G}$ is just a kernel T from $\Omega \times \mathbf{G}$ to $\mathbf{G}$, where it is always understood that $T(\omega, s, \cdot)$ is locally finite. *Invariance* of T is defined as above. If ξ is an invariant random measure on $\mathbf{G}$ and $\eta := \int T(\omega, s, \cdot)\, \xi(\omega, ds)$ is locally finite for each $\omega \in \Omega$, then η is again an invariant random measure. Our interpretation is, that T transports ξ to η in an invariant way. If $\tilde{T}$ is kernel from Ω to $\mathbf{G}$ such that $\tilde{T}(\omega, \cdot)$ is locally finite for all $\omega \in \Omega$, then $T(\omega, s, B) := \tilde{T}(\theta_s\omega, B - s)$ defines an invariant quasi-transport T on $\mathbf{G}$.

Let ξ and η be two invariant random measures on $\mathbf{G}$. A quasi-transport T on $\mathbf{G}$ is called (ξ, η)-*balancing* if

$$\int T(\omega, s, \cdot)\, \xi(\omega, ds) = \eta(\omega, \cdot) \tag{3.2}$$

holds for all $\omega \in \Omega$. In case $\xi = \eta$ we also say that T is ξ-*preserving.* If $\mathbb{Q}$ is a measure on $(\Omega, \mathcal{F})$ such that (3.2) holds for $\mathbb{Q}$-a.e. $\omega \in \Omega$ then we say that T is $\mathbb{Q}$-a.e. (ξ, η)-balancing.

Example 3.1. A measurable mapping $\tau : \Omega \times \mathbf{G} \to \mathbf{G}$ is called *transport-map.* Then $T(s, \cdot) := \delta_{\tau(s)}$ is a transport. This transport is invariant if and only if τ is *invariant* in the sense that

$$\tau(\theta_t\omega, s - t) = \tau(\omega, s) - t, \quad s, t \in \mathbf{G},\, \omega \in \Omega.$$

This is equivalent to $\tau(\theta_t\omega, 0) = \tau(\omega, t) - t$ for all ω, t. Writing $\pi := \tau(0) := \tau(\theta_0, 0)$, we can express this as $\tau(s) = \pi \circ \theta_s + s$. Any measurable mapping $\pi : \Omega \to \mathbf{G}$ can be used to generate a transport-map this way. We interpret a transport-map as a rule that is transporting, given $\omega \in \Omega$, an actual unit of mass close to s to a new location $\tau(\omega, s)$. Let ξ and η be two random measures on $\mathbf{G}$. The transport $T(s, \cdot) := \delta_{\tau(s)}$ is (ξ, η)-balancing if and only if

$$\int \mathbf{1}\{\tau(s) \in \cdot\}\, \xi(ds) = \eta. \tag{3.3}$$

In case $\xi = \eta$ we also say that τ is ξ-*preserving.*

Example 3.2. Consider the setting of Example 2.5. Letting $\mathbb{P}'$ and ξ as in that example, we obtain from an easy calculation that $\mathbb{P}_\xi = \mathbb{P}'$.

Now let K be a Markovian kernel from $\mathbf{G}$ to $\mathbf{G}$ and define $\mathbb{P}'' := \int K(s, \cdot)\, \mathbb{P}'(ds)$. The probability measure $\iint \mathbf{1}\{(s,t) \in \cdot\} K(s, dt)\, \mathbb{P}'(ds)$ is a *coupling* of $\mathbb{P}'$ and $\mathbb{P}''$. In the Monge-Kantorovich mass transportation theory (see e.g. [17]) K is interpreted as transporting the mass distribution $\mathbb{P}'$ to $\mathbb{P}''$. Let η be the invariant random measure $\eta(\omega, B) := \mathbb{P}''(B+\omega)$, and define an invariant transport T by $T(\omega, s, B) := K(\omega+s, B+\omega)$. Then

$$\int T(\omega, s, B)\xi(\omega, ds) = \int K(s, B + \omega)\mathbb{P}'(ds) = \mathbb{P}''(B + \omega) = \eta(\omega, B),$$

i.e. T is (ξ, η)-balancing. Conversely, if $\mathbb{P}'$ and $\mathbb{P}''$ are given, and T is a (ξ, η)-balancing transport (with ξ and η defined as before), then $T(0, s, B)$ is a Markovian kernel transporting $\mathbb{P}'$ to $\mathbb{P}''$.

4 Invariance properties of Palm measures

In this section we fix a stationary σ-finite measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$. We shall establish fundamental relationships between invariant balancing quasi-transports and Palm measures. Special cases of our first theorem have been known for a long time, see the references below. We were partly motivated by Theorem 16 in the recent paper [8], dealing with (λ, η)-balancing transports in case $\mathbf{G} = \mathbb{R}^d$ and η is an ergodic point process, see Example 4.12. (Note that [8] is using a different terminology.)

Theorem 4.1. *Consider two invariant random measures ξ and η on $\mathbf{G}$ and an invariant quasi-transport T. Then T is $\mathbb{P}$-a.e. (ξ, η)-balancing iff*

$$\mathbb{E}_{\mathbb{P}_\xi}\left[\int f(\theta_t)\, T(0, dt)\right] = \mathbb{E}_{\mathbb{P}_\eta}[f] \tag{4.4}$$

holds for all measurable $f : \Omega \to [0, \infty]$.

For simplicity we will refer to (4.4) in case $\xi = \eta$ as *invariance* of $\mathbb{P}_\xi$ *under* T. One implication of Theorem 4.1 will be a consequence of the following result of considerable independent interest.

Theorem 4.2. *Consider two invariant random measures ξ and η on $\mathbf{G}$ and let T and T^* be invariant quasi-transports satisfying*

$$\iint \mathbf{1}\{(s,t) \in \cdot\}\, T(\omega, s, dt)\, \xi(\omega, ds) = \iint \mathbf{1}\{(s,t) \in \cdot\}\, T^*(\omega, t, ds)\, \eta(\omega, dt) \tag{4.5}$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$. Then we have for any measurable function $h : \Omega \times \mathbf{G} \to [0, \infty)$ that

$$\mathbb{E}_{\mathbb{P}_\xi}\left[\int h(\theta_t, -t)\, T(0, dt)\right] = \mathbb{E}_{\mathbb{P}_\eta}\left[\int h(\theta_0, t)\, T^*(0, dt)\right]. \tag{4.6}$$

Proof. Let $B \in \mathcal{G}$ satisfy $\lambda(B) = 1$ and take a measurable $h : \Omega \times \mathbb{R}^d \to [0, \infty)$. From the definition (2.4) of $\mathbb{P}_\xi$ and (2.1) we obtain

$$\begin{aligned}\mathbb{E}_{\mathbb{P}_\xi}\left[\int h(\theta_t, -t)\, T(\theta_0, 0, dt)\right] &= \mathbb{E}_{\mathbb{P}}\left[\iint \mathbf{1}\{s \in B\} h(\theta_{s+t}, -t)\, T(\theta_s, 0, dt)\, \xi(ds)\right] \\ &= \mathbb{E}_{\mathbb{P}}\left[\iint \mathbf{1}\{s \in B\} h(\theta_t, -t+s)\, T(\theta_0, s, dt)\, \xi(ds)\right],\end{aligned}$$

where we have used the invariance (3.1) to get the second equation. Now we can apply assumption (4.5) to get

$$\begin{aligned}\mathbb{E}_{\mathbb{P}_\xi}\left[\int h(\theta_t, -t)\, T(\theta_0, 0, dt)\right] &= \mathbb{E}_{\mathbb{P}}\left[\iint \mathbf{1}\{s \in B\} h(\theta_t, s-t)\, T^*(\theta_0, t, ds)\, \eta(dt)\right] \\ &= \mathbb{E}_{\mathbb{P}}\left[\iint \mathbf{1}\{t+s \in B\} h(\theta_t, s)\, T^*(\theta_t, 0, ds)\, \eta(dt)\right],\end{aligned}$$

where we have again used (3.1), this time for the transport T^*. By the refined Campbell theorem (2.5),

$$\mathbb{E}_{\mathbb{P}_\xi}\left[\int h(\theta_t,-t)\,T(\theta_0,0,dt)\right] = \mathbb{E}_{\mathbb{P}_\eta}\left[\iint \mathbf{1}\{t+s\in B\}h(\theta_0,s)\,T^*(\theta_0,0,ds)\,dt\right]$$
$$= \mathbb{E}_{\mathbb{P}_\eta}\left[\int h(\theta_0,s)\,T^*(\theta_0,0,ds)\right],$$

where we have used Fubini's theorem and $\lambda(B)=1$ for the final equation. □

Remark 4.3. One possible choice of T and T^* in (4.5) is $T(s,\cdot):=\eta$ and $T^*(s,\cdot):=\xi$. Then (4.6) is Neveu's exchange formula, cf. [16].

For some purposes the following version of Theorem 4.2 might be more convenient.

Corollary 4.4. *Under the assumption of Theorem 4.2 we have*

$$\mathbb{E}_{\mathbb{P}_\xi}\left[g\int f(\theta_t)\,T(0,dt)\right] = \mathbb{E}_{\mathbb{P}_\eta}\left[f\int g(\theta_s)\,T^*(0,ds)\right] \tag{4.7}$$

for all measurable functions $f,g:\Omega\to[0,\infty)$.

Proof. Apply (4.6) with $h(\omega,s):=f(\omega)g(\theta_s\omega)$. □

To apply Corollary 4.4 we need the following consequence of a result in [10].

Lemma 4.5. *Under the hypothesis of Theorem 4.1 there is an invariant transport* T^* *on* $\mathbf{G}$ *such that* (4.5) *holds for* $\mathbb{P}$*-a.e.* $\omega\in\Omega$.

Proof. Consider the following measure M on $\Omega\times\mathbf{G}\times\mathbf{G}$:

$$M:=\iiint \mathbf{1}\{(\omega,s,t)\in\cdot\}\,T(\omega,s,dt)\,\xi(\omega,ds)\,\mathbb{P}(d\omega).$$

Stationarity of $\mathbb{P}$, (2.2), and (3.1) easily imply that

$$\int \mathbf{1}\{(\theta_r\omega,s-r,t-r)\in\cdot\}\,M(d(\omega,s,t)) = M,\quad r\in\mathbf{G}.$$

Moreover, as (3.2) is assumed to hold for $\mathbb{P}$-a.e. ω we have

$$M':=\int \mathbf{1}\{(\omega,t)\in\cdot\}\,M(d(\omega,s,t)) = \iint \mathbf{1}\{(\omega,t)\in\cdot\}\,\eta(\omega,dt)\,\mathbb{P}(d\omega). \tag{4.8}$$

This is a σ-finite measure on $\Omega\times\mathbf{G}$. We can now apply Theorem 3.5 in Kallenberg [10] to obtain an invariant transport T^* satisfying

$$M=\iint \mathbf{1}\{(\omega,s,t)\in\cdot\}\,T^*(\omega,t,ds)\,M'(d(\omega,t)).$$

(In fact the theorem yields an invariant kernel T', satisfying this equation. But in our specific situation we have $T'(\omega,t,\mathbf{G})=1$ for M'-a.e. (ω,t), so that T' can be modified in an obvious way to yield the desired T^*.) Recalling the definition of M and the second equation in (4.8) we get that

$$\iiint \mathbf{1}\{(\omega, s, t) \in \cdot\}\, T(\omega, s, dt)\, \xi(\omega, ds)\, \mathbb{P}(d\omega)$$
$$= \iiint \mathbf{1}\{(\omega, s, t) \in \cdot\}\, T^*(\omega, t, ds)\, \eta(\omega, dt)\, \mathbb{P}(d\omega)$$

and hence the assertion of the lemma. □

Proof of Theorem 4.1. If T is $\mathbb{P}$-a.e. (ξ, η)-balancing, then (4.4) follows from Lemma 4.5, Theorem 4.2, and Corollary 4.4. Conversely, assume that (4.4) holds. Let $f : \Omega \times \mathbf{G} \to [0, \infty)$ be measurable. Using the refined Campbell theorem for η and (4.4), we get

$$\mathbb{E}\left[\int f(\theta_s, s)\, \eta(ds)\right] = \mathbb{E}_{\mathbb{P}_\xi}\left[\iint f(\theta_t, s)\, ds\, T(\theta_0, 0, dt)\right]$$
$$= \mathbb{E}_{\mathbb{P}_\xi}\left[\iint f(\theta_t, s + t)\, T(\theta_0, 0, dt)\, ds\right],$$

where we have used invariance of λ and Fubini's theorem for the latter equation. By the refined Campbell theorem for ξ and invariance of T we get that the last term equals

$$\mathbb{E}\left[\iint f(\theta_{s+t}, s + t)\, T(\theta_s, 0, dt)\, \xi(ds)\right] = \mathbb{E}\left[\iint f(\theta_t, t)\, T(\theta_0, s, dt)\, \xi(ds)\right].$$

Now we combine the latter equations and apply them with $f(\omega, s) := g(\theta_{-s}\omega, s)$, where $g : \Omega \times \mathbf{G} \to [0, \infty)$ is measurable. This yields

$$\mathbb{E}\left[\int g(\theta_0, s)\, \eta(ds)\right] = \mathbb{E}\left[\iint g(\theta_0, t)\, T(\theta_0, s, dt)\, \xi(ds)\right].$$

Using this with $g := \mathbf{1}_{A \times B}$, for $A \in \mathcal{F}$ and $B \in \mathcal{G}$ gives

$$\mathbb{E}[\mathbf{1}_A \eta(B)] = \mathbb{E}\left[\mathbf{1}_A \int T(\theta_0, s, B)\, \xi(ds)\right]$$

and hence $\eta(B) = \int T(\theta_0, s, B)\, \xi(ds)$ $\mathbb{P}$-a.e. Since $\mathcal{G}$ is countably generated, this concludes the proof of the theorem. □

If T is a (ξ, η)-balancing invariant quasi-transport and T^* is an invariant transport satisfying (4.5) for all $\omega \in \Omega$, then we call T^* an *inverse* transport of T. If T is $\mathbb{P}$-a.e. (ξ, η)-balancing and T^* is an invariant transport satisfying (4.5) for $\mathbb{P}$-a.e. $\omega \in \Omega$, then we call T^* a $\mathbb{P}$-a.e. *inverse* transport of T. If T is a transport, then T^* is actually $\mathbb{P}$-a.e. (η, ξ)-balancing. For completeness we state the following consequence of Theorem 4.2 and Lemma 4.5 explicitly.

Theorem 4.6. *Consider invariant random measures ξ and η on $\mathbf{G}$ and let T be a $\mathbb{P}$-a.e. (ξ, η)-balancing invariant quasi-transport on $\mathbf{G}$. Then there exists a $\mathbb{P}$-a.e. inverse transport T^* of T and (4.6) holds.*

Example 4.7. Let ξ be an invariant (simple) point process on $\mathbf{G}$. A *point-transport for* ξ is a transport $\tau : \Omega \times \mathbf{G} \to \mathbf{G}$ such that $\tau(s) \in \operatorname{supp} \xi$ whenever $s \in \operatorname{supp} \xi$. Such a point-transport is called *bijective* if $s \mapsto \tau(s)$ is a bijection on $\operatorname{supp} \xi$ whenever $\xi(\mathbf{G}) > 0$. Clearly this is equivalent to the fact that τ is ξ-preserving, see [20] and [4] for more details. Consider a bijective invariant point-transport τ for ξ. There is an *inverse* point-transport τ^*, i.e. a bijective point-transport satisfying $\tau(\omega, \tau^*(\omega, s)) = \tau^*(\omega, \tau(\omega, s)) = s$ for all $(\omega, s) \in \Omega \times \mathbf{G}$ such that $s \in \operatorname{supp} \xi(\omega)$. It can easily be checked that τ^* may be chosen invariant (let $\tau^*(s) := s$ for $s \notin \operatorname{supp} \xi$) and that the invariant transports

$$T(s, \cdot) := \delta_{\tau(s)}, \quad T^*(s, \cdot) := \delta_{\tau^*(s)}, \quad s \in \mathbf{G},$$

satisfy (4.5) with $\eta := \xi$. Therefore we obtain from (4.7) that

$$\mathbb{E}_{\mathbb{P}_\xi}[g f(\theta_\tau)] = \mathbb{E}_{\mathbb{P}_\xi}[f g(\theta_{\tau^*})], \tag{4.9}$$

where the *point-shift* $\theta_\tau : \Omega \to \Omega$ is defined by

$$\theta_\tau(\omega) := \theta_{\tau(\omega,0)}(\omega), \quad \omega \in \Omega. \tag{4.10}$$

Taking $g \equiv 1$ yields $\mathbb{E}_{\mathbb{P}_\xi}[f(\theta_\tau)] = \mathbb{E}_{\mathbb{P}_\xi}[f]$, i.e. the invariance of $\mathbb{P}_\xi$ under the point-shift θ_τ. This is Theorem 3.1 in [4] (cf. also Theorem 9.4.1 in [20]). In fact this result can also be derived from a more general result in [15].

The results in the previous example can be generalized to invariant random measures ξ using the transport-maps defined in Example 3.1. To do so, we consider an invariant transport-map τ which is $\mathbb{P}$-a.e. ξ-preserving. Let T^* be an inverse transport of $T(s, \cdot) := \delta_{\tau(s)}$. Then (4.7) says that

$$\mathbb{E}_{\mathbb{P}_\xi}[g f(\theta_\tau)] = \mathbb{E}_{\mathbb{P}_\xi}\left[f \int g(\theta_s)\, T^*(0, ds)\right]. \tag{4.11}$$

In particular we obtain that $\mathbb{P}_\xi(\theta_\tau \in \cdot) = \mathbb{P}_\xi$, where the *mass-shift* θ_τ is again defined by (4.10). For simplicity (and in accordance with the terminology introduced after Theorem 4.1) we will refer to this as *invariance* of $\mathbb{P}_\xi$ *under* τ. Theorem 4.1 implies the following proposition.

Proposition 4.8. *Let ξ be an invariant random measure and τ be an invariant transport-map. Then τ is $\mathbb{P}$-a.e. ξ-preserving iff $\mathbb{P}_\xi$ is invariant under τ.*

Remark 4.9. One implication of Proposition 4.8 is (essentially) a consequence of Satz 4.3 in [15]. The special case $\xi = \lambda$ was treated in [3].

Proposition 4.8 can also be formulated for two invariant random measures:

Proposition 4.10. *Consider two invariant random measures ξ and η and let τ be an invariant transport-map. Then τ is $\mathbb{P}$-a.e. (ξ, η)-balancing iff*

$$\mathbb{P}_\xi(\theta_\tau \in A) = \mathbb{P}_\eta(A), \quad A \in \mathcal{F}. \tag{4.12}$$

The *invariant* σ-field $\mathcal{I} \subset \mathcal{F}$ is the class of all sets $A \in \mathcal{F}$ satisfying $\theta_s A = A$ for all $s \in \mathbf{G}$. Let ξ be an invariant random measure with finite intensity and define

$$\hat{\xi} := \mathbb{E}[\xi(B)|\mathcal{I}], \tag{4.13}$$

where $B \in \mathcal{G}$ has $\lambda(B) = 1$ and the conditional expectation is defined as for probability measures. (Stationarity implies that this definition is $\mathbb{P}$-a.e. independent of the choice of B.) If $\mathbb{P}$ is a probability measure and $\mathbf{G} = \mathbb{R}^d$, then $\hat{\xi}$ is called *sample intensity* of ξ, see [13] and [9]. Assuming that $\mathbb{P}(\hat{\xi} = 0) = 0$, we define the *modified Palm probability measure* $\mathbb{P}^*_\xi$ (see [13], [20], [11]) by

$$\mathbb{P}^*_\xi(A) := \mathbb{E}_{\mathbb{P}_\xi}[\hat{\xi}^{-1}\mathbf{1}_A], \quad A \in \mathcal{F}. \tag{4.14}$$

By this definition and $\hat{\xi} \circ \theta_s = \hat{\xi}$, $s \in \mathbf{G}$, we have

$$\mathbb{P}^*_\xi(A) = \mathbb{E}\left[\hat{\xi}^{-1} \int \mathbf{1}\{\theta_s \in A, s \in B\}\,\xi(ds)\right] = \mathbb{P}_{\xi'}(A), \quad A \in \mathcal{F}, \tag{4.15}$$

where the invariant random measure ξ' is defined by $\xi' := \hat{\xi}^{-1}\xi$ if $0 < \hat{\xi} < \infty$ and is the null measure, otherwise. Using (4.15), we obtain the following version of Theorem 4.1.

Theorem 4.11. *Consider two invariant random measures ξ and η with finite intensities such that $\mathbb{P}(\hat{\xi} = 0) = \mathbb{P}(\hat{\eta} = 0) = 0$ and let T be an invariant quasi-transport. Define $\xi' := \hat{\xi}^{-1}\xi$ and $\eta' := \hat{\eta}^{-1}\eta$. Then T is $\mathbb{P}$-a.e. (ξ', η')-balancing, iff*

$$\mathbb{E}_{\mathbb{P}^*_\xi}\left[\int f(\theta_t)\,T(0, dt)\right] = \mathbb{E}_{\mathbb{P}^*_\eta}[f] \tag{4.16}$$

holds for all measurable $f : \Omega \to [0, \infty]$.

Example 4.12. Let η be an invariant random measure with finite intensity and such that $\mathbb{P}(\hat{\eta} = 0) = 0$. Consider the invariant random measure $\xi := \hat{\eta}\lambda$ and let T be an invariant quasi-transport. Then T is $\mathbb{P}$-a.e. (ξ, η)-balancing iff T is $\mathbb{P}$-a.e. (λ, η')-balancing, where $\eta' := \hat{\eta}^{-1}\eta$. By Theorem 4.11 this is equivalent to

$$\mathbb{E}\left[\int \mathbf{1}\{\theta_t \in A\}\,T(0, dt)\right] = \mathbb{P}^*_\eta(A), \quad A \in \mathcal{F}. \tag{4.17}$$

In case $\mathbf{G} = \mathbb{R}^d$, η is a point process, T is Markovian, and $\mathbb{P}$ is an ergodic probability measure, this boils down to Theorem 16 in [8].

Example 4.13. Consider Example 4.12 in case η is a point process and the transport is generated by an invariant transport-map τ satisfying $\lambda(\{s \in \mathbf{G} : \tau(s) \notin \operatorname{supp}\eta\}) = 0$. Clearly, τ is $\mathbb{P}$-a.e. (λ, η')-balancing iff

$$\lambda(\{s \in \mathbf{G} : \tau(s) = t\}) = \hat{\eta}^{-1}, \quad t \in \operatorname{supp}\eta, \tag{4.18}$$

holds $\mathbb{P}$-a.e. By Theorem 4.11 this is then equivalent to

$$\mathbb{P}(\theta_\tau \in A) = \mathbb{P}^*_\eta(A), \quad A \in \mathcal{F}. \tag{4.19}$$

The special case $\mathbf{G} = \mathbb{R}^d$ is Theorem 9.1 in [11], a slight generalization of Theorem 13 in [8]. It is quite remarkable that invariant transport-maps satisfying (4.18) do exist in case $\mathbf{G} = \mathbb{R}^d$, see Theorem 1 in [8] (and Theorem 10.1 in [11] for the non-ergodic case). We also refer to the discussions in the introduction and Remark 5.2. Theorem 20 in [8] shows that the situation is different for discrete groups.

Remark 4.14. Relations (4.12) and (4.19) are examples of *group-coupling*, see [18], the term "group-coupling" is from [9]. Actually, the relation (4.4) can also be seen as group-coupling by extending the underlying space $(\Omega, \mathcal{F}, \mathbb{P}_\xi)$ to support a random element γ in $\mathbf{G}$ such that the conditional distribution of γ given $\mathcal{F}$ is T. Then (4.4) can be rewritten as (4.12) with θ_τ replaced by θ_γ. A similar remark applies to (4.17).

5 Balancing invariant transports always exist

Again we fix a stationary σ-finite measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$. Our aim is to establish a necessary and sufficient condition for the existence of balancing invariant transports.

Theorem 5.1. *Let ξ and η be invariant random measures with positive and finite intensities. Then there exists a $\mathbb{P}$-a.e. (ξ, η)-balancing invariant transport if and only if*

$$\mathbb{E}[\xi(B)|\mathcal{I}] = \mathbb{E}[\eta(B)|\mathcal{I}] \quad \mathbb{P} - a.e. \tag{5.1}$$

for some $B \in \mathcal{G}$ satisfying $0 < \lambda(B) < \infty$.

Proof. Let $B \in \mathcal{G}$ satisfy $0 < \lambda(B) < \infty$. For any $A \in \mathcal{I}$ we have from (2.5) that

$$\lambda(B)\mathbb{P}_\xi(A) = \mathbb{E}[\mathbf{1}_A \xi(B)], \quad \lambda(B)\mathbb{P}_\eta(A) = \mathbb{E}[\mathbf{1}_A \eta(B)]. \tag{5.2}$$

If T is a $\mathbb{P}$-a.e. (ξ, η)-balancing invariant transport, then Theorem 4.1 implies the equality $\mathbb{P}_\eta(A) = \mathbb{P}_\xi(A)$ and thus $\mathbb{E}[\mathbf{1}_A \eta(B)] = \mathbb{E}[\mathbf{1}_A \xi(B)]$. This entails (5.1)

Let us now assume that (5.1) holds for some $B \in \mathcal{G}$ satisfying $0 < \lambda(B) < \infty$. Since $\mathbb{E}[\xi(\cdot)]$ and $\mathbb{E}[\eta(\cdot)]$ are multiples of λ, ξ and η have the same intensities. We assume without loss of generality that these intensities are equal to 1. From (5.2) and conditioning we obtain that $\mathbb{P}_\xi = \mathbb{P}_\eta$ on $\mathcal{I}$. The group-coupling result in Thorisson [18] (see also Theorem 10.28 in Kallenberg [9]) implies the existence of random elements δ and δ' in Ω and ρ in $\mathbf{G}$, all defined on some probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, such that δ has distribution $\mathbb{P}_\xi$, δ' has distribution $\mathbb{P}_\eta$, and $\delta'(\tilde{\omega}) = \theta_{\rho(\tilde{\omega})}\delta(\tilde{\omega})$ for $\tilde{\mathbb{P}}$-a.e. $\tilde{\omega} \in \tilde{\Omega}$. Let $\tilde{T}(\omega, \cdot)$, $\omega \in \Omega$, be a regular version of the conditional distribution $\tilde{\mathbb{P}}(\rho \in \cdot|\delta = \omega)$. Then we have for any $A \in \mathcal{F}$ that

$$\begin{aligned}\mathbb{P}_\eta(A) = \tilde{\mathbb{P}}(\delta' \in A) = \tilde{\mathbb{P}}(\theta_\rho\delta \in A) = \mathbb{E}_{\tilde{\mathbb{P}}}\left[\int \mathbf{1}\{\theta_s\delta \in A\}\, \tilde{T}(\delta, ds)\right] \\ = \mathbb{E}_{\mathbb{P}_\xi}\left[\int \mathbf{1}\{\theta_s \in A\}\, \tilde{T}(\theta_0, ds)\right].\end{aligned} \tag{5.3}$$

Using $\tilde{T}$ as a generator, we now define an invariant transport $T(\omega, s, B) := \tilde{T}(\theta_s\omega, B - s)$. Then (5.3) implies (4.4), and Theorem 4.1 yields that T is $\mathbb{P}$-a.e. (ξ, η)-balancing. □

Remark 5.2. The above proof doesn't provide a method for actually constructing balancing invariant transports. As mentioned in the introduction, explicit constructions of invariant transport-maps, in case $\xi = \lambda$ and η is an ergodic point process of intensity 1, have been presented by Liggett [12] and Holroyd and Peres [8]. Note that (4.12) means in this case that $\mathbb{P}(\theta_\tau \in \cdot) = \mathbb{P}_\eta$. The construction of balancing invariant transports or transport-maps in other cases is an interesting topic for further research.

Remark 5.3. Let ξ and η be invariant random measures with finite intensities such that $\mathbb{P}(\hat{\xi} = 0) = \mathbb{P}(\hat{\eta} = 0) = 0$. Then the invariant random measure $\xi' := \hat{\xi}^{-1}\xi$ and $\eta' := \hat{\eta}^{-1}\eta$ satisfy (5.2), see also Theorem 4.11.

6 From stationarity to mass-stationarity

We consider a stationary σ-finite measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ together with an invariant random measure ξ. We will discuss a nice invariance property of the Palm measure $\mathbb{P}_\xi$, that will turn out to be sufficient for a σ-finite measure on $(\Omega, \mathcal{F})$ to be a Palm measure of ξ.

Let $C \in \mathcal{G}$ be relatively compact and define an invariant transport T_C by

$$T_C(t, B) := \xi(C + t)^{-1}\xi(B \cap (C + t)), \quad t \in \mathbf{G}, B \in \mathcal{G}, \tag{6.1}$$

if $\xi(C + t) > 0$, and by letting $T_C(t, \cdot)$ equal some fixed probability measure, otherwise. In the former case $T_C(t, \cdot)$ is just governing a $\mathbf{G}$-valued stochastic experiment that picks a point uniformly in the mass of ξ in $C + t$. If $\lambda(C) > 0$ we also define the uniform distribution λ_C on $\mathbf{G}$ by $\lambda_C(B) := \lambda(B \cap C)/\lambda(C)$. The interior of a set $C \subset \mathbf{G}$ is denoted by $\operatorname{int} G$.

Theorem 6.1. *If $C \in \mathcal{G}$ is relatively compact with $\lambda(C) > 0$ and $\lambda(C \setminus \operatorname{int} C) = 0$, then*

$$\mathbb{E}_{\mathbb{P}_\xi}\left[\iint \mathbf{1}\{(\theta_s, s + r) \in A\}\, T_C(-r, ds)\, \lambda_C(dr)\right] = \mathbb{P}_\xi \otimes \lambda_C(A), \quad A \in \mathcal{F} \otimes \mathcal{G}. \tag{6.2}$$

Proof. Let $C \in \mathcal{G}$ be relatively compact with $\lambda(C) > 0$ and $\lambda(C \setminus \operatorname{int} C) = 0$. Then $\lambda(\operatorname{int} C) > 0$ and we have for λ-a.e. $r \in C$ that $r \in \operatorname{int} C$. For $r \in \operatorname{int} C$ and $t \in \mathbf{G}$ we have $t \in \operatorname{int}(C - r + t)$. If, in addition, $t \in \operatorname{supp} \xi$, then

$$\xi(C + t - r) \geq \xi(\operatorname{int}(C - r + t)) > 0.$$

By definition (6.1) of T (and using the above fact) we have for all $B, D \in \mathcal{G}$ and $t \in \operatorname{supp} \xi$ that

$$\begin{aligned}
&\iint \mathbf{1}\{s \in B, s - t + r \in D\}\, T_C(t - r, ds)\, \lambda_C(dr) \\
&\quad = \iint \mathbf{1}\{s \in B, s - t + r \in D\}\mathbf{1}\{s \in C + t - r\}\xi(C + t - r)^{-1}\lambda_C(dr)\, \xi(ds) \\
&\quad = \lambda(C)^{-1} \iint \mathbf{1}\{s \in B, r + s \in D\}\mathbf{1}\{r + s \in C, r + t \in C\}\xi(C - r)^{-1}\, dr\, \xi(ds),
\end{aligned}$$

where the second equation comes from a change of variables. It follows that

$$\begin{aligned}\iiint &\mathbf{1}\{s \in B, s-t+r \in D\}\, T_C(t-r, ds)\, \lambda_C(dr)\, \xi(dt)\\ =&\lambda(C)^{-1} \iint \mathbf{1}\{s \in B, r+s \in D, r+s \in C\}\, dr\, \xi(ds)\\ =&\xi(B)\lambda_C(D). \qquad (6.3)\end{aligned}$$

Equation (6.3) implies that the invariant quasi-transport

$$T_{C,D}(t,\cdot) := \iint \mathbf{1}\{s \in \cdot, s-t+r \in D\}\, T_C(t-r, ds)\, \lambda_C(dr) \qquad (6.4)$$

is (ξ,η)-balancing, where $\eta := \lambda_C(D)\xi$. (Invariance of $T_{C,D}$ is a quick consequence of the same property of T_C.) Since $\mathbb{P}_\eta = \lambda_C(D)\mathbb{P}_\xi$ we get from Theorem 4.1

$$\mathbb{E}_{\mathbb{P}_\xi}\left[\iint \mathbf{1}\{\theta_s \in A', s+r \in D\}\, T_C(-r, ds)\, \lambda_C(dr)\right] = \lambda_C(D)\mathbb{P}_\xi(A'), \quad A' \in \mathcal{F}.$$

This is (6.2) for measurable product sets, implying (6.2) for general $A \in \mathcal{F} \otimes \mathcal{G}$. □

Let C be as in Theorem 6.1 and assume that $0 \in \operatorname{int} C$. Then $\xi(C+t) > 0$ for all $t \in \operatorname{supp}\xi$ and one might think (at least at first glance) that $\mathbb{P}_\xi$ is invariant under T_C. The following simple example (other examples can be based on the Poisson process) shows that this is wrong. The reason is that T_C is not $\mathbb{P}$-a.e. ξ-preserving, see Theorem 4.1.

Example 6.2. Consider the group $\mathbf{G} = \{0,1,2\}$ with addition modulo 3. Let ξ_0, ξ_1, ξ_2 be independent Bernoulli (1/2) random variables. The distribution $\mathbb{P}$ of the point process $\xi_0\delta_0 + \xi_1\delta_1 + \xi_2\delta_2$ is stationary. Let $\mathbb{Q}$ be the Palm probability measure $\mathbb{P}^0_\xi$, defined in the setting of Example 2.1. Since λ is (a multiple of) the counting measure, we can take $B := \{0\}$ in (2.4) to see that $\xi\{1\}$ and $\xi\{2\}$ are independent Bernoulli (1/2) under $\mathbb{Q}$. (Of course we have $\mathbb{Q}(\xi\{0\} = 1) = 1$.) Consider the set $C := \{0,1\}$ and the event $A := \{\xi\{1\} = 1\}$, where we recall that ξ is the identity on $\Omega = \mathbf{M}$. Then we obtain from a trivial calculation that

$$\mathbb{E}_\mathbb{Q}\left[\int \mathbf{1}\{\theta_s \in A\}\, T_C(0, ds)\right] = \mathbb{E}_\mathbb{Q}\left[\int \mathbf{1}\{\xi\{1+s\} = 1\}\, T_C(0, ds)\right] = \frac{3}{8}.$$

Since $\mathbb{Q}(A) = 1/2$, $\mathbb{Q}$ is not invariant under T_C.

Remark 6.3. Consider the measure space $(\Omega, \mathcal{F}, \mathbb{Q})$, where $\mathbb{Q} := \mathbb{P}_\xi$. Let C be as assumed in (6.2). Extend the space $(\Omega, \mathcal{F}, \mathbb{Q})$, so as to carry random elements U, V in $\mathbf{G}$ such that θ_0 and U are independent, U has distribution λ_C, and the conditional distribution of V given (θ_0, U) is uniform in the mass of ξ on $C-U$. (The mappings θ_s, $s \in \mathbf{G}$, are extended, so that they still take values in the original space Ω.) Then (6.2) can be written as

$$(\theta_V, U+V) \overset{d}{=} (\theta_0, U). \qquad (6.5)$$

In the case of simple point processes on $\mathbb{R}^d$, this is (essentially) the property that was proved in Thorisson ([20], Theorem 9.5.1) to be equivalent to point-stationarity.

We are going to call the property established in Theorem 6.1 *mass-stationarity.*

7 From mass-stationarity back to stationarity

Let ξ be an invariant random measure on $\mathbf{G}$. In contrast to the previous sections we do not fix a stationary measure on $(\Omega, \mathcal{F})$. Instead we consider here a σ-finite measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ as a candidate for a Palm measure $\mathbb{P}_\xi$ of ξ w.r.t. some stationary measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$. We will establish that the following condition is necessary and sufficient for the existence of such a $\mathbb{P}$. Recall the definitions of T_C and λ_C preceding Theorem 6.1. The boundary of a set $C \subset \mathbf{G}$ is denoted by ∂C.

Definition 7.1. The σ-finite measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ is called *mass-stationary for* ξ if $\mathbb{Q}(\xi(\mathbf{G}) = 0) = 0$ and

$$\mathbb{E}_{\mathbb{Q}}\left[\iint \mathbf{1}\{(\theta_s, s+r) \in A\}\, T_C(-r, ds)\, \lambda_C(dr)\right] = \mathbb{Q} \otimes \lambda_C(A), \quad A \in \mathcal{F} \otimes \mathcal{G}, \tag{7.1}$$

holds for all relatively compact sets $C \in \mathcal{G}$ with $\lambda(C) > 0$ and $\lambda(\partial C) = 0$.

Remark 7.2. Extending the measure space $(\Omega, \mathcal{F}, \mathbb{Q})$ as in Remark 6.3, mass-stationarity of $\mathbb{Q}$ is then equivalent to assuming (6.5) for all pairs (U, V) considered there.

Theorem 7.3. *Let $\mathbb{Q}$ be a σ-finite measure on $(\Omega, \mathcal{F})$ and ξ an invariant random measure on $\mathbf{G}$. Then there exists a σ-finite stationary measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ such that $\mathbb{Q} = \mathbb{P}_\xi$ iff $\mathbb{Q}$ is mass-stationary for ξ.*

Remark 7.4. The inversion formula (2.6) implies that the measure $\mathbb{Q}$ in Theorem 7.3 determines $\mathbb{P}$.

Proof of Theorem 7.3. In view of Theorem 6.1 it remains to prove only one implication. So we assume that $\mathbb{Q}$ is mass-stationary for ξ. For simplicity we can then also assume that $\operatorname{supp} \xi \neq \emptyset$ everywhere on Ω. We will show the Mecke equation (2.7).

Let $C \in \mathcal{G}$ be a relatively compact set with $\lambda(C) > 0$ and $\lambda(\partial C) = 0$. Mass-stationarity of $\mathbb{Q}$ implies for any measurable $f : \Omega \to [0, \infty)$ and any $D \in \mathcal{G}$ that

$$\mathbb{E}_{\mathbb{Q}}\left[\iint f(\theta_s)\mathbf{1}\{s + r \in D\}\, T_C(-r, ds)\, \lambda_C(dr)\right] = \lambda_C(D)\mathbb{E}_{\mathbb{Q}}[f].$$

By definition (6.1) of T_C this means that

$$\mathbb{E}_{\mathbb{Q}}\left[\iint f(\theta_s)\mathbf{1}\{s + r \in D\}\mathbf{1}\{s + r \in C, r \in C\}\xi(C - r)^{-1}\, dr\, \xi(ds)\right] = \lambda(D \cap C)\mathbb{E}_{\mathbb{Q}}[f],$$

where we recall the first paragraph of the proof of Theorem 6.1. A change of variables and Fubini's theorem give

$$\int_D \mathbf{1}\{r \in C\}\mathbb{E}_{\mathbb{Q}}\left[\int f(\theta_s)\mathbf{1}\{r - s \in C\}\xi(C - r + s)^{-1}\, \xi(ds)\right] dr = \mathbb{E}_{\mathbb{Q}}[f] \int_D \mathbf{1}\{r \in C\}\, dr.$$

As $D \in \mathcal{G}$ is arbitrary, this shows that

$$\mathbb{E}_{\mathbb{Q}}\left[\int f(\theta_s)\mathbf{1}\{r - s \in C\}\xi(C - r + s)^{-1}\, \xi(ds)\right] = \mathbb{E}_{\mathbb{Q}}[f]$$

holds for λ-a.e. $r \in C$. Applying this with f replaced by $f(\tilde{f} \circ \xi)$, where $\tilde{f} : \mathbf{M} \to [0, \infty)$ is measurable, we obtain that

$$\mathbb{E}_{\mathbb{Q}}\left[\int f(\theta_s)\tilde{f}(\xi \circ \theta_s)\mathbf{1}\{r - s \in C\}\xi(C - r + s)^{-1}\,\xi(ds)\right] = \mathbb{E}_{\mathbb{Q}}[f\tilde{f}(\xi)] \tag{7.2}$$

for λ-a.e. $r \in C$. By separability of $\mathbf{M}$ (see e.g. Theorem A2.3 in [9]) and a monotone class argument we can choose the corresponding null set $C' \in \mathcal{G}$ independently of $\tilde{f}$. Applying (7.2) with $r \in C \setminus C'$ and $\tilde{f}(\mu) := \mu(C - r)$, $\mu \in \mathbf{M}$, gives

$$\mathbb{E}_{\mathbb{Q}}\left[\int f(\theta_s)\mathbf{1}\{r - s \in C\}\,\xi(ds)\right] = \mathbb{E}_{\mathbb{Q}}[f\xi(C - r)] \quad \lambda-\text{a.e.}\, r \in C. \tag{7.3}$$

Let $B_n \subset \mathbf{G}$, $n \in \mathbb{N}$, be an increasing sequence of compact sets satisfying $\lambda(\partial B_n) = 0$ and $\cup_n B_n = \mathbf{G}$. (Such a sequence can be constructed with the help of a metric generating the topology on $\mathbf{G}$. For any $s \in \mathbf{G}$ there is a compact and non-empty ball centred at s whose boundary has λ-measure 0. Let B^*_n, $n \in \mathbb{N}$, be an increasing sequence of compact sets with union $\mathbf{G}$. Then B^*_n is contained in the union $\tilde{B}_n$ of finitely many of the above balls. The sequence $\tilde{B}_1 \cup \ldots \cup \tilde{B}_n$, $n \in \mathbb{N}$, has the desired properties.) Fix $n \in \mathbb{N}$ and assume temporarily that

$$\mathbb{E}_{\mathbb{Q}}[f\xi(B_n - B_n)] < \infty, \tag{7.4}$$

where $B_n - B_n := \{r - r' : r, r' \in B_n\}$. Since $(r, r') \mapsto r - r'$ is continuous, $B_n - B_n$ is again compact. Then we have for all measurable $C' \subset B_n$ and $r \in B_n$ that

$$\mathbb{E}_{\mathbb{Q}}[f\xi(C' - r)] \leq \mathbb{E}_{\mathbb{Q}}[f\xi(B_n - B_n)] < \infty.$$

Assume now that $C \subset B_n$ is satisfying the assumptions made in (7.2) and let $C_0 := B_n \setminus C$. Applying (7.3) to B_n yields

$$\mathbb{E}_{\mathbb{Q}}\left[\int f(\theta_s)\mathbf{1}\{r - s \in C_0\}\,\xi(ds)\right] + \mathbb{E}_{\mathbb{Q}}\left[\int f(\theta_s)\mathbf{1}\{r - s \in C\}\,\xi(ds)\right]$$
$$= \mathbb{E}_{\mathbb{Q}}[f\xi(C_0 - r)] + \mathbb{E}_{\mathbb{Q}}[f\xi(C - r)] \tag{7.5}$$

for λ-a.e. $r \in B_n$. Since $\partial C_0 \subset \partial B_n \cup \partial(\mathbf{G} \setminus C) = \partial B_n \cup \partial C$, we have $\lambda(\partial C_0) = 0$. Hence we can apply (7.3) to C_0, to obtain that the respective first summands in (7.5) coincide λ-a.e. $r \in C_0$. Therefore the respective second summands coincide λ-a.e. $r \in C_0$. Combining this with (7.3), gives

$$\mathbb{E}_{\mathbb{Q}}\left[\int f(\theta_s)\mathbf{1}\{r - s \in C\}\,\xi(ds)\right] = \mathbb{E}_{\mathbb{Q}}[f\xi(C - r)] \quad \lambda-\text{a.e.}\, r \in B_n. \tag{7.6}$$

Integrating (7.6) over a measurable set $D \subset B_n$, using (on both sides) Fubini's theorem and a change of variables gives

$$\mathbb{E}_{\mathbb{Q}}\left[\iint f(\theta_s)\mathbf{1}\{r + s \in D, r \in C\}\,\xi(ds)\,dr\right] = \mathbb{E}_{\mathbb{Q}}\left[f \iint \mathbf{1}\{r - s \in D, r \in C\}\,\xi(ds)\,dr\right].$$

As both sides are finite measures in C (the right-hand side is bounded by $\mathbb{E}_{\mathbb{Q}}[f\xi(B_n - B_n)]$) and the class $\mathcal{G}' := \{C \in \mathcal{G} : C \subset B_n, \lambda(\partial C) = 0\}$ is stable under intersections and generates $\mathcal{G} \cap B_n$, we obtain this equation even for all measurable $C \subset B_n$. (To check that $\sigma(\mathcal{G}') = \mathcal{G} \cap B_n$, it is sufficient to show for any non-empty open $U \subset G$ that there is a non-empty open $U' \subset U$ such that $U' \cap B_n \in \mathcal{G}'$. This can be achieved with an open ball U' having $\lambda(\partial U') = 0$.) Reversing the above steps, we obtain (7.6) for all measurable $C \subset B_n$. Since $\mathcal{G}$ is countably generated, we can choose the corresponding null-sets independently of C. This means that there is a measurable set $B'_n \subset B_n$ such that $\lambda(B_n \setminus B'_n) = 0$ and

$$\mathbb{E}_{\mathbb{Q}}\left[\int f(\theta_s)\mathbf{1}\{r - s \in C\}\,\xi(ds)\right] = \mathbb{E}_{\mathbb{Q}}[f\xi(C - r)], \quad r \in B'_n,\, C \in \mathcal{G} \cap B_n. \tag{7.7}$$

Still keeping $n \in \mathbb{N}$ fixed in (7.7), we now lift the assumption (7.4) on $f : \Omega \to [0, \infty)$. If $\mathbb{E}_{\mathbb{Q}}[f] < \infty$, we can apply (7.7) with f replaced by $f\mathbf{1}\{\xi(B_n - B_n) \le m\}$ and then let $m \to \infty$. For general f we decompose Ω into measurable sets $D_m \uparrow \Omega$ with $\mathbb{Q}(D_m) < \infty$, apply the previous result to $\mathbf{1}_{D_m} \min\{f, k\}$, and let $m, k \to \infty$. Then (7.7) still holds for all $r \in B''_n \in \mathcal{G}$, where $B''_n \subset B_n$ such that $\lambda(B_n \setminus B''_n) = 0$. For notational simplicity we assume $B''_n = B'_n$.

In the final step of the proof we would like to take the limit in (7.7) as $n \to \infty$. First we can assume without loss of generality that $\lambda(B_1) > 0$. Let $r_0 \in B_1 \setminus B^*$, where B^* is the λ-null set $\cup_n B_n \setminus B'_n$. Then $r_0 \in B'_n$ for all $n \ge 1$. Take an arbitrary $\tilde{C} \in \mathcal{G}$. Applying (7.7) to $C := \tilde{C} \cap B_n$ and letting $n \to \infty$, yields

$$\mathbb{E}_{\mathbb{Q}}\left[\int f(\theta_s)\mathbf{1}\{-s \in C'\}\,\xi(ds)\right] = \mathbb{E}_{\mathbb{Q}}\left[\int f\mathbf{1}\{s \in C'\}\,\xi(ds)\right], \tag{7.8}$$

for $C' = \tilde{C} - r_0$ and hence for any $C' \in \mathcal{G}$. The measure $\mathbb{E}_{\mathbb{Q}}[\int \mathbf{1}\{(\theta_0, s) \in \cdot\}\,\xi(ds)]$ is finite on measurable product sets of the form $\{\mu \in D : \mu(B) \le k\} \times B$, where $\mathbb{Q}(D) < \infty$, B is compact, and $k \in \mathbb{N}$. Since $\Omega \times \mathbf{G}$ is the monotone union of countably many of such sets, it is now straightforward to proceed from (7.8) to the full Mecke equation (2.7). $\square$

8 Discussion of mass-stationarity

We consider the setting of Section 7 and assume that $\mathbb{Q}(\xi(\mathbf{G}) = 0) = 0$. The first intrinsic characterization of Palm measures was given by Mecke [14], see (2.7). Our present proof of Theorem 7.3 is making essential use of his result. In case of simple point processes on $\mathbb{R}^d$, Thorisson [19] has found a more explicit characterization, using bijective point-maps as discussed in Example 4.7. However, that paper had to use randomizations, allowing for independent stationary background processes. Finally, Heveling and Last ([4], [5]) showed that randomization is actually not needed. They showed, that $\mathbb{Q}$ is a Palm measure of ξ if and only if $\mathbb{Q}$ is invariant under all $\sigma(\xi)\otimes\mathcal{G}$-measurable invariant bijective point-transports τ for ξ, i.e.

$$\mathbb{Q}(\theta_\tau \in A) = \mathbb{Q}(A), \quad A \in \mathcal{F}, \tag{8.1}$$

see (4.10). At this stage one might be tempted to guess that (8.1) is characterizing Palm measures also for general ξ. The following example shows that in general, randomization is needed to define mass-stationarity. We will construct a probability measure $\mathbb{Q}$ and an invariant random measure ξ satisfying (8.1) for all ξ-preserving invariant transport-maps τ, see Example 3.1. Still $\mathbb{Q}$ will be no Palm measure of ξ. The construction applies to any Abelian group as considered in this paper.

Example 8.1. We establish the setting of Example 2.4. Let Π denote the distribution of a stationary Poisson process with intensity 1, considered as probability measure on $(\mathbf{M}, \mathcal{M})$. It is well-known that the associated Palm probability measure (defined in the framework of Example 2.1) is given by $\Pi^0 = \int \mathbf{1}\{\mu + \delta_0 \in \cdot\}\Pi(d\mu)$. Let $\mathbb{Q} := \Pi^0 \otimes \Pi$ and $c > 0$ be an irrational number. Define an invariant random measure ξ on $\mathbf{G}$ by $\xi := \xi_1 + c\xi_2$, where ξ_1 and ξ_2 are the projections of Ω onto the first and second component, respectively. Let τ be a ξ-preserving transport, i.e.

$$\int \mathbf{1}\{\tau(s) = t\}\, \xi_1(ds) + c \int \mathbf{1}\{\tau(s) = t\}\, \xi_2(ds) = \xi\{t\}, \quad t \in \mathbf{G}.$$

Since c is irrational, this can only hold if τ is ξ_1-preserving. As it can be straightforwardly checked that $\mathbb{Q}$ is the Palm measure of ξ_1, Theorem 4.1 implies (8.1).

We now show that $\mathbb{Q}$ is not mass-stationary for ξ. Therefore and by Theorem 7.3 it cannot be a Palm measure of ξ. Consider a set $C \in \mathcal{G}$ as in Definition 7.1 and place it at random around the origin. This random set will contain a ξ_2-point with positive probability; this point will in turn be chosen with positive probability as a new origin. Thus if $\mathbb{Q}$ was mass-stationary for ξ, it should also have mass c at 0 with positive probability. But this is not the case.

Instead of working with general invariant transports or quasi-transports, we define mass-stationarity by (7.1). This property has the advantage of allowing for the direct probabilistic interpretation (6.5), at least if $\mathbb{Q}$ is a probability measure. To see how it is related to invariance under quasi-transports, we let $C \in \mathcal{G}$ be as in Definition 7.1 and $D \in \mathcal{G}$ with $\lambda(C \cap D) > 0$. Define the invariant quasi-transport $T'_{C,D} := \lambda_C(D)^{-1} T_{C,D}$, where $T_{C,D}$ is given by (6.3). As noted at (6.3), we have that $T'_{C,D}$ is ξ-preserving. (We cannot claim that $T'_{C,D}(\omega, s, \mathbf{G}) = 1$ for all $(\omega, s) \in \Omega \times \mathbf{G}$.) Mass-stationarity of $\mathbb{Q}$ is equivalent to assuming invariance of $\mathbb{Q}$ under all these quasi-transports. Now, Theorem 7.1 and Theorem 4.1 yield the following result.

Theorem 8.2. *The measure $\mathbb{Q}$ is mass-stationary for ξ iff it is invariant under ξ-preserving invariant quasi-transports T such that $T(\cdot, \cdot, \mathbf{G})$ is bounded.*

We finish this section with some open problems related to mass-stationarity. A kernel T from $\Omega \times \mathbf{G}$ to $\mathbf{G}$ is called ξ-*measurable* if $T(\cdot, \cdot, B)$ is $\sigma(\xi) \otimes \mathcal{G}$-measurable for all $B \in \mathcal{G}$.

Problem 8.3. Assume that $\mathbb{Q}$ is invariant under ξ-preserving and ξ-measurable invariant transports. Is $\mathbb{Q}$ mass-stationary for ξ?

The condition in Problem 8.3 implies that of the following problem.

Problem 8.4. Assume that

$$\mathbb{E}_{\mathbb{Q}} \left[\iint \mathbf{1}\{\theta_s \in A\}\, T_C(-r, ds)\, \lambda_C(dr) \right] = \mathbb{Q}(A), \quad A \in \mathcal{F}, \tag{8.2}$$

holds for all C as in Definition 7.1. Is $\mathbb{Q}$ mass-stationary for ξ?

The counterexample in Example 8.1 arises because mass-atoms of relatively prime size cannot be mapped into each other in a measure-preserving way. But what about diffuse random measures?

Problem 8.5. Assume that ξ is diffuse and that (8.1) holds for all ξ-preserving (and $\sigma(\xi) \otimes \mathcal{G}$-measurable) transport-maps τ. Is $\mathbb{Q}$ mass-stationary for ξ?

If the answer to Problem 8.5 is negative, we might attempt to introduce a stationary independent background:

Problem 8.6. Let θ_t, $t \in \mathbf{G}$, and ξ be defined on $(\Omega, \mathcal{F}, \mathbb{Q})$. Introduce a *stationary independent background* as follows: let θ'_t, $t \in \mathbf{G}$, be another flow defined on a space $(\Omega', \mathcal{F}', \mathbb{Q}')$, where $\mathbb{Q}'$ is stationary under the flow, and consider the joint flow on $(\Omega, \mathcal{F}, \mathbb{Q}) \otimes (\Omega', \mathcal{F}', \mathbb{Q}')$ with ξ defined in the natural way on this extended space. Assume that ξ is diffuse and that $\mathbb{Q} \otimes \mathbb{Q}'$ is invariant under invariant ξ-preserving (and $\sigma(\xi) \otimes \mathcal{G}$-measurable) transport-maps for all such stationary independent backgrounds. Is $\mathbb{Q}$ mass-stationary for ξ?

Remark 8.7. In Problem 8.6 we might only assume that $\mathbb{Q}$ – and not $\mathbb{Q} \otimes \mathbb{Q}'$ – is invariant. Remark 3.2 in Thorisson [19] claims that the answer to this question is positive in the case of simple point processes on $\mathbb{R}^d$. This claim is a mistake stemming from the author's forgetting that the argument in Section 4.4 in [19] relies on the joint invariance. Similarly, Lemma 4.1 in [19] needs to be corrected by adding the background in (4.11). Same applies to Remark 9.3.2 and Lemma 9.4.1 in [20].

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