Optimality of the Fully Discrete Filtered Backprojection Algorithm in 2D

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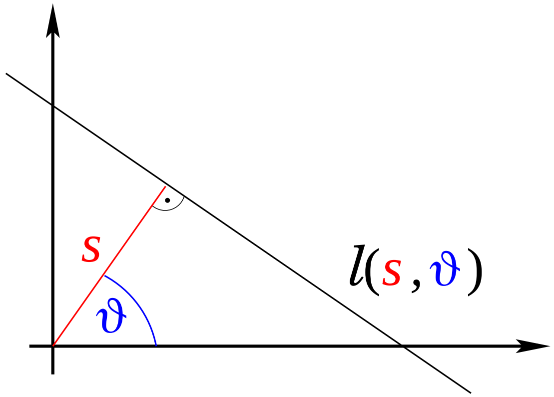
2D-Radon-Transform (parallel scanning geometry)

$$\mathbf{R}f(s,\vartheta) := \int_{l(s,\vartheta)\cap\Omega} f(x)\,\mathrm{d}\sigma(x)$$



tomographic inversion: $\mathbf{R}f(s,\vartheta) = g(s,\vartheta)$

$$\mathbf{R}\colon L^2(\Omega) \to L^2(Z), \quad Z = [-1,1]\times[0,2\pi]$$

Inversion formula

$$f = \frac{1}{4\pi}\mathbf{R}^*(\Lambda \otimes I)\mathbf{R}f$$

$\mathbf{R}^* : L^2(Z) \to L^2(\Omega)$ Backprojection operator

$$\mathbf{R}^* g(x) = \int_0^{2\pi} g(x^t\omega(\vartheta), \vartheta)\,\mathrm{d}\vartheta, \quad \omega(\vartheta) = (\cos\vartheta, \sin\vartheta)^t$$

$\Lambda : H^\alpha(\mathbb{R}) \to H^{\alpha-1}(\mathbb{R})$ Riesz potential

$$\widehat{\Lambda u}(\xi) = |\xi|\widehat{u}(\xi).$$

Filtered backprojection algorithm (FBA)

discrete Radon data $D = \{\mathbf{R}f(kh, jh_\vartheta) : k = -q, \ldots, q,\ j = 0, \ldots, 2p-1\},$

$$h = 1/q, \quad h_\vartheta = \pi/p$$

$$f_{\text{FBA}}(x) := \frac{1}{4\pi}\mathbf{R}^*_{h_\vartheta}(\mathrm{I}_h \Lambda E_h \otimes I)\mathbf{R}f(x)$$

R. & Faridani '03

where

$E_h,\ \mathrm{I}_h$ generalized interpolation operators

and

$$\mathbf{R}^*_{h_\vartheta} g(x) := h_\vartheta \sum_{j=0}^{2p-1} g(x^t\omega(\vartheta_j), \vartheta_j), \quad \vartheta_j = jh_\vartheta$$

Remark: The action of $\mathrm{I}_h \Lambda E_h$ can be implemented as a convolution (filtering) followed by an interpolation. The convolution kernel (reconstruction filter) depends on I_h and E_h.

Convergence of the semi discrete FBA

Theorem [R.& Faridani '03] Under reasonable assumptions on E_h and I_h there are $0 \le \alpha_{\min}(E_h, \mathrm{I}_h) < \alpha_{\max}(E_h, \mathrm{I}_h)$ such that

$$\left\| \frac{1}{4\pi}\mathbf{R}^*(\mathrm{I}_h \Lambda E_h \otimes I)\mathbf{R}f - f \right\|_{L^2(\Omega)} \lesssim h^{\alpha} \|f\|_{\alpha}$$

for $f \in H_0^{\alpha}(\Omega)$ where $\alpha_{\min} \le \alpha \le \alpha_{\max}$.

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for $f \in H_0^{\alpha}(\Omega)$ where $\alpha_{\min} \leq \alpha \leq \alpha_{\max}$.

Consequence

$$\left\| \frac{1}{4\pi} \mathbf{R}^*(\mathrm{I}_h \Lambda E_h \otimes I)\mathbf{R}f - f \right\|_{L^2(\Omega)} \lesssim h^{\min\{\alpha_{\max}, \alpha\}} \|f\|_{\alpha}, \quad \alpha > 0$$

$$\alpha_{\max} = \begin{cases} 3/2 : & \text{Shepp-Logan with piecewise constant interpol.} \\ 2 : & \text{Shepp-Logan with piecewise linear interpol.} \\ 5/2 : & \text{mod. Shepp-Logan with piecewise linear interpol.} \end{cases}$$

Error estimate for the fully discrete FBA, part I

$$\|f - f_{\mathsf{FBA}}\|_{L^2(\Omega)} \le \Big\| f - \frac{1}{4\pi}\mathbf{R}^*(\mathrm{I}_h \Lambda E_h \otimes I)\mathbf{R} f \Big\|_{L^2(\Omega)} + \Big\| (\mathbf{R}^* - \mathbf{R}^*_{h_\vartheta})(\mathrm{I}_h \Lambda E_h \otimes I)\mathbf{R} f \Big\|_{L^2(\Omega)}$$

$$\Big\| (\mathbf{R}^* - \mathbf{R}^*_{h_\vartheta})(\mathrm{I}_h \Lambda E_h \otimes I)\mathbf{R} f \Big\|_{L^2(\Omega)} \le \Big\| (\mathbf{R}^* - \mathbf{R}^*_{h_\vartheta})\big((\mathrm{I}_h \Lambda E_h - \Lambda) \otimes I\big)\mathbf{R} f \Big\|_{L^2(\Omega)} + \Big\| (\mathbf{R}^* - \mathbf{R}^*_{h_\vartheta})(\Lambda \otimes I)\mathbf{R} f \Big\|_{L^2(\Omega)}.$$

Error estimate for the fully discrete FBA, part II

Lemma If $f \in H_0^\alpha(\Omega)$, $\alpha \geq 1$, then $\left\|(\mathbf{R}^* - \mathbf{R}^*_{h_\vartheta})(\Lambda \otimes I)\mathbf{R}f\right\|_{L^2(\Omega)} \lesssim h_\vartheta^\alpha \|f\|_\alpha$.

Proof: Let $f \in \mathcal{C}_0^\infty(\Omega)$. We use a duality argument due to Natterer 1979:

$$\begin{aligned}
\left\|(\mathbf{R}^* - \mathbf{R}^*_{h_\vartheta})(\Lambda \otimes I)\mathbf{R}f\right\|_{L^2(\Omega)} &= \sup_{g \in \mathcal{C}_0^\infty(\Omega)} \frac{|\langle(\mathbf{R}^* - \mathbf{R}^*_{h_\vartheta})(\Lambda \otimes I)\mathbf{R}f, g\rangle_{L^2}|}{\|g\|_{L^2}} \\
&= \sup_{g \in \mathcal{C}_0^\infty(\Omega)} \frac{\left|\int_0^{2\pi} u(\vartheta)\,\mathrm{d}\vartheta - h_\vartheta \sum_{j=0}^{2p-1} u(\vartheta_j)\right|}{\|g\|_{L^2}} \\
&\lesssim h_\vartheta^{2k+1} \sup_{g \in \mathcal{C}_0^\infty(\Omega)} \frac{\int_0^{2\pi} |u^{(2k+1)}(\vartheta)|\mathrm{d}\vartheta}{\|g\|_{L^2}}
\end{aligned}$$

with $u(\vartheta) = \int_\Omega g(x)(\Lambda \otimes I)\mathbf{R}f(x^t\omega(\vartheta), \vartheta)\mathrm{d}x$. As $\int_0^{2\pi} |u^{(2k+1)}(\vartheta)|\mathrm{d}\vartheta \lesssim \|f\|_{2k+1}\|g\|_{L^2}$, $k \in \mathbb{N}_0$, we are done.

Error estimate for the fully discrete FBA, part III

Lemma Let $f \in H_0^\alpha(\Omega)$. There are $1 \le \alpha_{\min}(E_h, \mathrm{I}_h) < \alpha_{\max}(E_h, \mathrm{I}_h)$ such that

$$\Big\|(\mathbf{R}^* - \mathbf{R}^*_{h_\vartheta})\big((\mathrm{I}_h \Lambda E_h - \Lambda) \otimes I\big)\mathbf{R} f\Big\|_{L^2} \lesssim (h^\alpha + h_\vartheta^\alpha)\|f\|_\alpha, \quad \alpha_{\min} \le \alpha \le \alpha_{\max}.$$

Proof: Let $f \in \mathcal{C}_0^\infty(\Omega)$ and $\Psi = \big((\mathrm{I}_h \Lambda E_h - \Lambda) \otimes I\big)\mathbf{R} f$. Again by duality

$$\Big\|(\mathbf{R}^* - \mathbf{R}^*_{h_\vartheta})\Psi\Big\|_{L^2(\Omega)} = \sup_{g \in \mathcal{C}_0^\infty(\Omega)} \frac{\Big|\int_0^{2\pi} u(\vartheta)\,\mathrm{d}\vartheta - h_\vartheta \sum_{j=0}^{2p-1} u(\vartheta_j)\Big|}{\|g\|_{L^2}}$$

$$\lesssim h_\vartheta \sup_{g \in \mathcal{C}_0^\infty(\Omega)} \frac{\int_0^{2\pi} |u'(\vartheta)|\mathrm{d}\vartheta}{\|g\|_{L^2}}$$

where $u(\vartheta) := \displaystyle\int_\Omega g(x)\Psi(x^t\omega(\vartheta), \vartheta)\mathrm{d}x$. Since

$$\int_0^{2\pi} |u'(\vartheta)|\mathrm{d}\vartheta \lesssim \|g\|_{L^2}\|\Psi\|_{H^{(-1/2,1)}} \lesssim h^{\alpha-1}\|g\|_{L^2}\|f\|_\alpha$$

we are done.

Convergence of the fully discrete FBA

Theorem Under reasonable assumptions on E_h and I_h there are constants $1 \leq \alpha_{\min}(E_h, \mathrm{I}_h) < \alpha_{\max}(E_h, \mathrm{I}_h)$ such that

$$\left\| \frac{1}{4\pi} \mathbf{R}^*_{h_\vartheta} (\mathrm{I}_h \Lambda E_h \otimes I) \mathbf{R} f - f \right\|_{L^2(\Omega)} \lesssim (h^\alpha + h^\alpha_\vartheta) \|f\|_\alpha$$

for $f \in H^\alpha_0(\Omega)$ where $\alpha_{\min} \leq \alpha \leq \alpha_{\max}$.

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for $f \in H_0^\alpha(\Omega)$ where $\alpha_{\min} \leq \alpha \leq \alpha_{\max}$.

Consequence

$$\left\| \frac{1}{4\pi} \mathbf{R}^*_{h_\vartheta} (\mathrm{I}_h \Lambda E_h \otimes I) \mathbf{R} f - f \right\|_{L^2} \lesssim \left(h^{\min\{\alpha_{\mathrm{max}}, \alpha\}} + h_\vartheta^{\min\{\alpha_{\mathrm{max}}, \alpha\}} \right) \|f\|_\alpha, \quad \alpha \geq 1$$

$$\alpha_{\mathrm{max}} = \begin{cases} 3/2 : & \text{Shepp-Logan with piecewise constant interpol.} \\ 2 : & \text{Shepp-Logan with piecewise linear interpol.} \\ 5/2 : & \text{mod. Shepp-Logan with piecewise linear interpol.} \end{cases}$$

Remarks and Comments

- From an asymptotic point of view it is most efficient to choose $h = h_\vartheta$ yielding the well-known optimal sampling rate $p = \pi q$. Further, under the optimal sampling rate the convergence rate h^α as $h \to 0$ is optimal for density distributions in $H_0^\alpha(\Omega)$ (Natterer 1980).

- One can construct efficient filtered backprojection schemes with an arbitrarily large $\alpha_{\max}$. Of course, one would fully benefit from these filtered backprojection schemes if the searched-for density distributions are sufficiently smooth which is not the case in medical imaging.

- Unfortunately, our analysis does not cover the medical imaging situation where $f \in H_0^\alpha(\Omega)$ with $\alpha < 1/2$ but close to $1/2$.

A modified FBA

$$f_{\mathsf{FBA}}(x) = \frac{1}{4\pi}\mathbf{R}^*_{h_\vartheta}(\mathrm{I}_h \Lambda E_h \otimes I)\mathbf{R}f(x)$$

$$= \frac{1}{4\pi}\mathbf{R}^*_{h_\vartheta}(\mathrm{I}_h \otimes I)(\Lambda \otimes I)(E_h \otimes I)\mathbf{R}f(x)$$

$$f_{\mathsf{MFBA}}(x) := \frac{1}{4\pi}\mathbf{R}^*(\mathrm{I}_h \otimes I)(\Lambda \otimes I)(E_h \otimes T_{h_\vartheta})\mathbf{R}f(x)$$

$$= \frac{1}{4\pi}\mathbf{R}^*(\mathrm{I}_h \Lambda E_h \otimes T_{h_\vartheta})\mathbf{R}f(x)$$

T_{h_ϑ} piecewise linear interpolation

Remark: The evaluation of $f_{\mathsf{MFBA}}(x)$ can be organized as standard FBA with an additional multiplication of the filtered data by a sparse matrix.

Convergence of MFBA

Theorem Under reasonable assumptions on E_h and I_h there are constants $1/2 < \alpha_{\min}(E_h, \mathrm{I}_h) < \alpha_{\max}(E_h, \mathrm{I}_h)$ such that

$$\left\| \frac{1}{4\pi} \mathbf{R}^* (\mathrm{I}_h \Lambda E_h \otimes T_{h_\vartheta}) \mathbf{R} f - f \right\|_{L^2(\Omega)} \lesssim \left(h^{\min\{\alpha_{\max}, \alpha\}} + h_\vartheta^{\min\{\alpha_{\mathrm{T}}, \alpha\}} \right) \|f\|_\alpha$$

for $f \in H_0^\alpha(\Omega)$, $\alpha_{\min} \leq \alpha \leq \alpha_{\max}$ and any $\alpha_{\mathrm{T}} < 5/2$.

Convergence of MFBA

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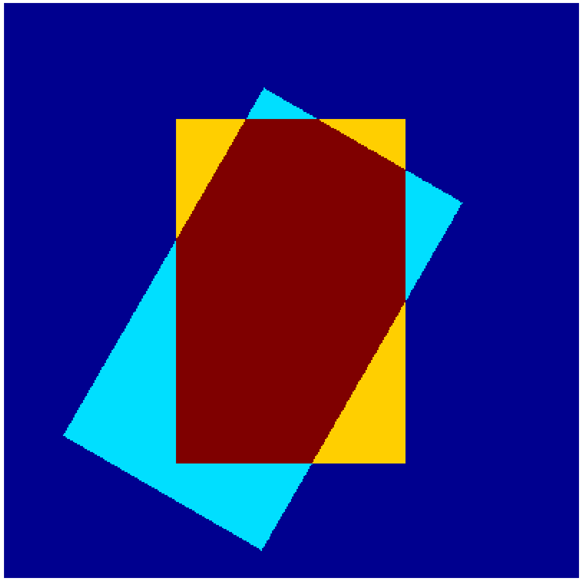
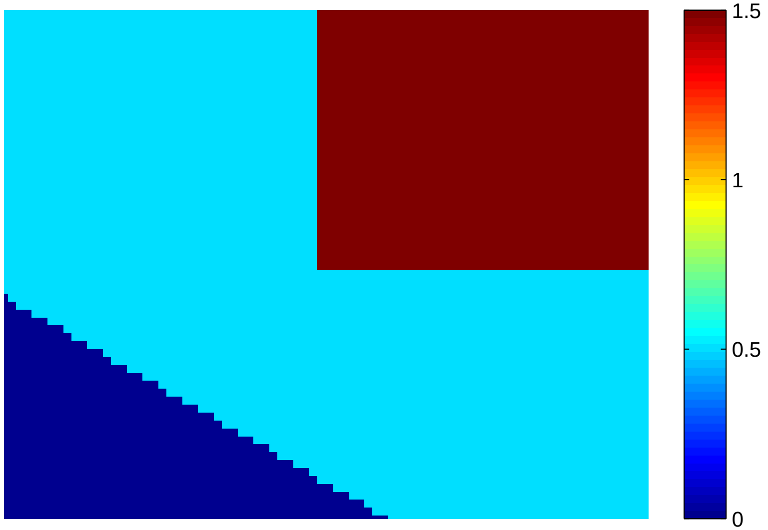
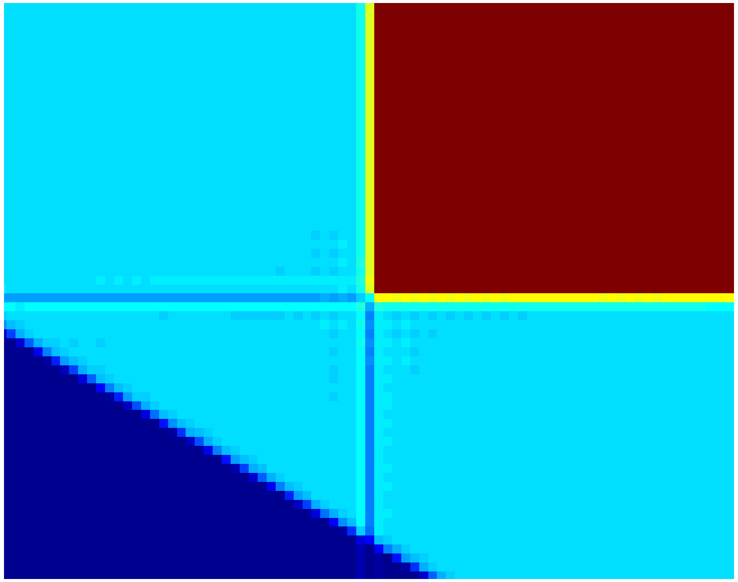
Numerical comparison I



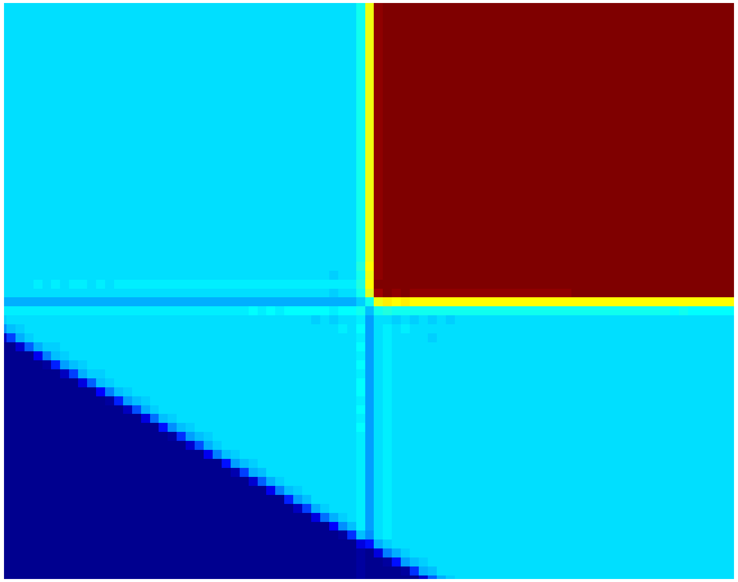


Numerical comparison II

$p = 600$, $q = 200$, Shepp-Logan filter with piecewise linear interpolation



FBA



MFBA