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A Note on a Nonlinear Model of a Piezoelectric Rod

If piezoceramics are excited by weak electric fields a nonlinear behavior can be observed, if the excitation frequency is close to a resonance frequency of the system. To derive a theoretical model nonlinear constitutive equations are used, to describe the longitudinal oscillations of a slender piezoceramic rod near the first resonance frequency. Hamilton's principle is used to receive a variational principle for the piezoelectric rod. Introducing a Rayleigh Ritz ansatz with the eigenfunctions of the linearized system to approximate the exact solution leads to nonlinear ordinary differential equations. These equations are approximated with the method of harmonic balance. Finally it is possible to calculate the amplitudes of the displacements numerically. As a result it is shown, that the Duffing type nonlinearities found in measurements can be described with this model.

1. Theoretical Model

The model of the piezoceramic rod is shown in figure 1. A coordinate system is introduced such that the 3-axis is aligned with the poling direction. The left and right end of the rod are stress-free (mechanical boundary conditions) and the rod is excited by a harmonic electric voltage $\varphi(t)$ (electrical boundary condition). The thickness to length ratio is assumed to be small, thus simple rod theory is used. Furthermore, the influence of the electrodes is neglected and the cross sectional area is assumed to be constant in the theoretical model. To derive a variational principle for the piezoceramic, it is convenient to use HAMILTON's principle for dielectric continua

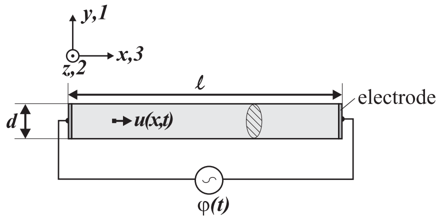

Figure 1: Model of the piezoceramic rod

$$\delta \int_{t_1}^{t_2} L(u,\varphi)\,\mathrm{d}t + \int_{t_1}^{t_2} \delta \mathrm{W}(\mathrm{u},\varphi)\,\mathrm{d}t = 0, \text{ with } L = \int_V (T-H)\,\mathrm{d}V. \tag{1}$$

In equation (1) H denotes the electric enthalpy density and T the kinetic energy density, which are given by

$$T = \frac{1}{2}\rho \dot{u}^2, \quad H = \frac{1}{2}c_0 S_3^2 + \frac{1}{3}c_1 S_3^3 + \frac{1}{4}c_2 S_3^4 - e_0 S_3 E_3 - \frac{1}{2}e_1 S_3^2 E_3 - \frac{1}{3}e_2 S_3^3 E_3 - \frac{1}{2}\epsilon_{33}^S E_3^2. \tag{2}$$

The field quantities are the strain S_3 and the electric field E_3. In the second equation terms of higher order are introduced to describe the nonlinear behavior of the piezoceramics. The stiffness constants are denoted by c_0, c_1, and c_2, additionally the abbreviations $e_0 = c_0 d_0$, $e_1 = c_0 d_1 + c_1 d_0$, $e_2 = c_0 d_2 + c_2 d_0 + c_1 d_1$ are introduced. The piezoelectric constants are named d_0, d_1 and d_2. The nonlinear constitutive equations can be obtained by the partial derivatives of the electric enthalpy density with respect to the strain S_3 and the electric field E_3

$$T_3 = \frac{\partial H}{\partial S_3} = c_0 S_3 + c_1 S_3^2 + c_2 S_3^3 - e_0 E_3 - e_1 S_3 E_3 - e_2 S_3^2 E_3, \tag{3}$$

$$D_3 = -\frac{\partial H}{\partial E_3} = e_0 S_3 + \frac{1}{2}e_1 S_3^2 + \frac{1}{3}e_2 S_3^3 + \epsilon_{33}^S E_3. \tag{4}$$

In equation (3) and (4) T_3 denotes the tension in longitudinal direction and D_3 the dielectric displacement. It is obvious that the stiffness and the piezoelectric coupling depend in a nonlinear way on the strain. Due to the mechanical and electrical boundary conditions the virtual work δW in Hamilton's principle (1) reduces to the material damping. From the second nonlinear constitutive equation (4) a relation for the electric field is derived. Applying the field equation $E = -\mathrm{grad}\,\varphi$ to the electric field, it is possible to receive the dielectric displacement depending on the electric potential, the displacement and its derivatives. Introducing the variations of equation (2), the electric field, the electric displacement and their variations into Hamilton's principle (1) an equation of the form

$$A \int_{t_1}^{t_2} \int_0^l f\left(\varphi\,, \ddot{u}\,, \dot{u}_{,x}\,, \delta u\,, \delta u_{,x}\right) \mathrm{d}x\,\mathrm{d}t = 0 \tag{5}$$

is obtained. It is not possible to find an exact solution for this equation. Therefore, a RAYLEIGH-RITZ ansatz is used to get an approximate nonlinear ordinary differential equation. In this paper only the first eigenfunction of the linearized problem is used as an approximation for the exact solution. For this simplified case a nonlinear ordinary differential equation is obtained

$$C_m\ddot{p} + C_d\dot{p} + C_5p^5 + C_4p^4 + C_3p^3 + (C_{2a}\varphi + C_{2b})p^2 + (C_{1a}\varphi + C_{1b})p = C_\varphi\varphi, \tag{6}$$

with the coefficients C_i which depend on material properties and dimensions of the rod. It can be seen that the ODE is inhomogeneous and of 5th order. Furthermore there is a parametric excitation by $C_{2a}\varphi$ and $C_{1a}\varphi$ due to the time dependence of the excitation voltage $\varphi(t)$. Solutions of this nonlinear differential equation may be calculated by the method of harmonic balance. As in experiments only the fundamental and the second harmonic frequency are observed, an ansatz of such a form is used to approximate the exact solution of (6). Additional a harmonic excitation voltage $\varphi(t) = \hat{\varphi}\cos\Omega t$ is assumed. Finally, a solution for the displacement amplitudes can be obtained numerically and two transfer functions are defined. The intention is to receive a relation between the amplitudes of the first and second harmonic oscillation to the amplitude of the excitation voltage.

2. Results

To calculate the influence of the nonlinear parameters in the constitutive equations (3) and (4) the first transfer function is used. Certain material parameters are chosen and the electric field strength is approximately 120 V/mm, considered in the range of low level signals. The nonlinear material parameters c_1, c_2, d_1 and d_2 are varied and their influence on the transfer function is investigated. An excitation near the first resonance frequency shows a nonlinear behavior in form of DUFFING type nonlinearities. From measurements it is known that the transfer curve inclines to the left near the first resonance frequency. To investigate the influence of the nonlinear parameters, each parameter is varied, while the others are set to zero. The value of the inner damping is constant d= 0.45 Ns for all simulations. It can be shown, that c_1, d_1, positive c_2 and positive d_2 lead to curves $V_1(\Omega)$, which incline to the right. For values of c_2 and d_2, which are negative, the curves $V_1(\Omega)$ incline to the left. This is the behavior which is observed in measurements.

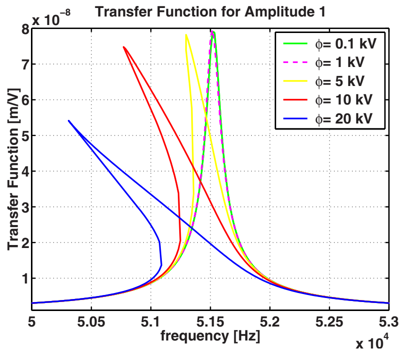

Figure 2: Transfer functions $V_1(\Omega)$ for varying $\hat{\varphi}$

Figure 2 shows the influence on the transfer curve for varying excitation voltages. In this simulation the piezoelectric constant is d_2= -10^{-7} m/V and the other nonlinear parameters are set to zero. The typical DUFFING type nonlinearities can be observed. The curves incline to the left and the jump phenomena occurs. Moreover the difference at which the curves achieve their maximum can be seen. This behavior can be found in measurements, too.

3. Conclusions

In this paper a nonlinear model of a piezoceramic rod is presented. With the electric enthalpy density and kinetic energy density HAMILTON's principle for dielectric media is used to derive an integral form into which a RITZ ansatz is introduced. The resulting nonlinear ordinary differential equation is approximated by the method of harmonic balance. The DUFFING type nonlinearities, as they can be observed in measurements, can be modelled with the presented nonlinear constitutive equations.

4. References

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