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On The Equivalence Of Addition Formula Of Lattice Paths And The Kolmogorov Equation Of Birth And Death Processes*

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Abstract

We establish the equivalence of the addition formula of weighted lattice paths and the Kolmogorov equation of birth and death processes and provide suitable illustrations.

1 Introduction

The enumeration of lattice paths consists of sequences of lattice points in the plane, beginning at the origin with steps restricted to $(1, 1), (1, 0), (1, -1)$ and in which every point has a non-negative altitude. By using a simple decomposition, the enumerator for these paths with respect to steps of each type at each altitude is given by the continued fraction of Jacobi (called J-fraction) [2].

Approximations employing continued fractions (CFs) often provide a good representation for transcendental functions. CFs arise naturally in calculations of random walk probabilities, and are strongly connected with combinatorial configurations equivalent to lattice paths.

Birth and death processes (BDPs) are widely used to model a variety of applications including computer communications, production management, neutron propagation, chemical reactions, epidemics, population dynamics and so on. Parthasarathy and Lenin [4] have obtained closed form transient solutions of system size probabilities of BDPs by means of CFs.

Goulden and Jackson [2] have given a addition formula as follows: Let $a(x) = \sum_{i\geq 0} a_i\, x^i/i!$ and $\hat{a}(x) = \sum_{i\geq 0} a_i\, x^i$. If $a(x)$ has the property that

$$a(x + y) = \sum_{m\geq 0} p_m\, f_m(x) f_m(y),$$

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where p_m is independent of x and y, and

$$f_m(x) = \frac{x^m}{m!} + q_{m+1}\frac{x^{m+1}}{(m+1)!} + \mathcal{O}(x^{m+2})$$

then $a(x)$ has an addition formula with parameters $\{(p_m, q_{m+1})|m \geq 0\}$. The formal power series $\widehat{f}(x)$ has an addition formula with parameters $\{(p_m, q_{m+1})|\ m \geq 0\}$ if and only if

$$\frac{1}{x}\hat{f}\left(\frac{1}{x}\right) = \frac{1}{x-(q_1-q_0)-}\ \frac{p_1 p_0^{-1}}{x-(q_2-q_1)-}\ \frac{p_2 p_1^{-1}}{x-(q_3-q_2)-}\cdots. \tag{1}$$

where we use the notation for CFs

$$\frac{a_1}{b_1-}\ \frac{a_2}{b_2-}\ \frac{a_3}{b_3-}\ \cdots = \frac{a_1}{b_1-\frac{a_2}{b_2-\frac{a_3}{b_3-\cdots}}}.$$

They have given a combinatorial proof of this theorem by considering the generating function for paths of non-negative length from altitude 0 to altitude 0, with respect to rises and levels.

It is well known that the transition function $P_{mn}(t) = \mathrm{P}(X(t) = n|X(0) = m)$ of BDPs satisfies the Kolmogorov equation:

$$P_{mn}(t+u) = \sum_{k\in\mathcal{N}} P_{mk}(t)P_{kn}(u).$$

This equation states that in order to move from state m to state n in time $t+u$, $X(t)$ first moves to some state k in time t and then from k to n in the remaining time u.

In this paper, we present the equivalence of the addition formula of lattice paths and the Chapman-Kolmogorov equation of birth and death processes. We give examples to illustrate our result.

2 Birth and Death Processes

A BDP is a continuous time Markov chain $\{X(t), t \geq 0\}$ defined on a probability space $(\Omega, \mathcal{F}, P)$ with stationary transition probability $P_{mn}(t)$ satisfying the forward Kolmogorov differential equations:

$$\begin{array}{ll} P'_{m0}(t) & = -a_0 P_{m0}(t) + a_1 P_{m1}(t) \\ P'_{mn}(t) & = a_{2n-2}P_{m,n-1}(t) - (a_{2n-1}+a_{2n})P_{mn}(t) + a_{2n+1}P_{m,n+1}(t), \end{array} \tag{2}$$

for $n = 1, 2, 3, \ldots$. The parameters a_{2n} $(n \geq 0)$ and a_{2n-1} $(n \geq 1)$ are respectively called the birth and death rates (instead of the usual λ_n and μ_n). The reason for this notation will become apparent in the sequel.

Let us take Laplace transforms

$$\hat{P}_{mn}(s) = \int_0^\infty e^{-st}P_{mn}(t)dt, \quad n = 0, 1, 2, \ldots,$$

for $\mathrm{Re}(s) > 0$ of the system of equations given by (2). We assume that $m = 0$, and in that case $\hat{P}_{00}(s)$ simplifies to the expression

$$\hat{P}_{00}(s) = \frac{1}{s + a_0 - a_1 \frac{\hat{P}_{01}(s)}{\hat{P}_{00}(s)}},$$

and

$$\begin{aligned} \frac{\hat{P}_{0n}(s)}{\hat{P}_{0,n-1}(s)} &= \frac{a_{2n-2}}{s + a_{2n-1} + a_{2n} - a_{2n+1} \frac{\hat{P}_{0,n+1}(s)}{\hat{P}_{0n}(s)}} \\ &= \frac{a_{2n-2}}{s + a_{2n-1} + a_{2n} -} \cdot \frac{a_{2n} a_{2n+1}}{s + a_{2n+1} + a_{2n+2} -} \cdot \frac{a_{2n+2} a_{2n+3}}{s + a_{2n+3} + a_{2n+4} -} \cdots \end{aligned} \tag{3}$$

Using (3),

$$\hat{P}_{00}(s) = \frac{1}{s + a_0 -} \; \frac{a_0 a_1}{s + a_1 + a_2 -} \; \frac{a_2 a_3}{s + a_3 + a_4 -} \cdots . \tag{4}$$

We shall now give a fundamental correspondence between the addition formula and the Kolmogorov equation of a birth-death process. For this, we first refer to the following theorem of Parthasarathy and Sudhesh [5].

THEOREM 1. If $X(0) = m$, then the power series representation for the state probabilities $P_{mn}(t),\ n = 0, 1, 2, \ldots$, corresponding to a state-dependent BDP with birth and death rates respectively a_{2n} and a_{2n+1}, $(0 \leq n < \infty)$ are given by

$$P_{mn}(t) = \begin{cases} \frac{M_m}{M_n} \sum_{r=0}^{\infty} (-1)^r \frac{t^{r+m-n}}{(r+m-n)!} \sum_{k=0}^{\min(r,n)} (-1)^k \phi_k(2n+1) A(r-k, 2m), \\ \qquad\qquad n = 0, 1, 2, \ldots, m \\ \frac{L_{n-1}}{L_{m-1}} \sum_{r=0}^{\infty} (-1)^r \frac{t^{r+n-m}}{(r+n-m)!} \sum_{k=0}^{\min(r,m)} (-1)^k \phi_k(2m+1) A(r-k, 2n), \\ \qquad\qquad n = m, m+1, \ldots . \end{cases} \tag{5}$$

where $L_{-1} = 1,\ L_{n-1} = a_0 a_2 a_4 \cdots a_{2n-2},\ M_0 = 1,\ M_n = a_1 a_3 a_5 \cdots a_{2n-1},\ \phi_0(n) = 1,\ \phi_r(n) = 0$ if $r \geq [\frac{n+1}{2}]$, for $r \geq 1,\ n = 1, 2, 3, \ldots$,

$$\phi_r(n+1) = \sum_{i_1=0}^{n-2r} a_{i_1} \sum_{i_2=i_1+2}^{n-2r+2} a_{i_2} \sum_{i_3=i_2+2}^{n-2r+4} a_{i_3} \cdots \sum_{i_r=i_{r-1}+2}^{n-2} a_{i_r}, \tag{6}$$

and

$$A(r, n) = \sum_{i_1=0}^{n} a_{i_1} \sum_{i_2=0}^{i_1+1} a_{i_2} \sum_{i_3=0}^{i_2+1} a_{i_3} \cdots \sum_{i_r=0}^{i_{r-1}+1} a_{i_r},\ \forall\ n \in \mathbb{N},\ A(0, n) = 1. \tag{7}$$

From (5), we observe that the transition function $P_{mn}(t)$ satisfies

$$\pi_m P_{mn}(t) = \pi_n P_{nm}(t), \quad m, n \in \mathcal{N},\ t \geq 0, \tag{8}$$

where

$$\pi_0 \equiv 1,\ \pi_n \equiv \frac{a_0 a_2 \cdots a_{2n-2}}{a_1 a_3 \cdots a_{2n-1}},\ n = 1, 2, 3, \ldots,$$

are the potential coefficients and (8) shows that $P_{mn}(t)$ is symmetric. Therefore $\{X(t), t \geq 0\}$ is reversible [1].

THEOREM 2. The transition probability $P_{00}(t)$ has an addition formula with parameters $\left\{\left(\prod_{i=0}^{2m-1} a_i, -\sum_{i=0}^{2m} a_i\right)\middle| m \geq 0\right\}$.

PROOF. From (5),

$$P_{0m}(t) = L_{m-1} f_m(t), \quad P_{m0}(t) = M_m f_m(t),$$

where

$$f_m(t) = \frac{t^m}{m!} - A(1,2m)\frac{t^{m+1}}{(m+1)!} + A(2,2m)\frac{t^{m+2}}{(m+2)!} - A(3,2m)\frac{t^{m+3}}{(m+3)!} + \cdots. \tag{9}$$

Using the Kolmogorov equation,

$$P_{00}(t+u) = \sum_{m=0}^{\infty} P_{0m}(t) P_{m0}(u) = \sum_{m=0}^{\infty} p_m f_m(t) f_m(u),$$

where

$$p_m = \prod_{i=0}^{2m-1} a_i.$$

From (9),

$$q_{m+1} = -A(1,2m) = -\sum_{i=0}^{2m} a_i.$$

Therefore, for $m = 0, 1, 2, \ldots$,

$$q_{m+1} - q_m = -(a_{2m-1} + a_{2m}),\ a_{-1} = 0 \text{ and } p_{m+1} p_m^{-1} = a_{2m} a_{2m+1}.$$

Hence, $P_{00}(t) = f_0(t)$ has an addition formula with parameters

$$\left\{\left(\prod_{i=0}^{2m-1} a_i, -\sum_{i=0}^{2m} a_i\right) i|\ m \geq 0\right\}$$

and the corresponding J-fraction coincides with (4).

The following two examples present the addition formula representation of the transition probability $P_{00}(t)$.

EXAMPLE 1 (Linear rates). Consider a BDP with birth and death rates

$$a_{2n} = a_{2n+1} = n+1, \quad n = 0, 1, 2, \ldots.$$

so that

$$\pi_n = 1, \quad n = 0, 1, 2, \ldots.$$

The corresponding J-fraction for $\hat{P}_{00}(s)$ is

$$\hat{P}_{00}(s) = \frac{1}{s+1-}\ \frac{1^2}{s+3-}\ \frac{2^2}{s+5-}\ \cdots = \int_0^{\infty} \frac{e^{-t}}{s+t}\, dt = \sum_{r=0}^{\infty} (-1)^r \frac{r!}{s^{r+1}},$$

[3, p. 582, (4.6.1)].

This leads to the transition probability,

$$P_{00}(t) = \frac{1}{1+t}.$$

From Theorem 1,

$$P_{0m}(t) = \frac{t^m}{(1+t)^{m+1}},\ m \geq 0.$$

Using (8),

$$P_{m0}(t) = \frac{\pi_0}{\pi_m} P_{0m}(t) = \frac{t^m}{(1+t)^{m+1}}.$$

Therefore,

$$P_{00}(t+u) = \frac{1}{1+t+u} = \sum_{m=0}^{\infty} P_{0m}(t) P_{m0}(u) = \sum_{m=0}^{\infty} (m!)^2 f_m(t) f_m(u),$$

where

$$f_m(t) = \frac{1}{m!} \frac{t^m}{(1+t)^{m+1}} = \frac{t^m}{m!} - 1! \binom{m+1}{1}^2 \frac{t^{m+1}}{(m+1)!} + \mathcal{O}(t^{m+2}).$$

Hence $P_{00}(t)$ has an addition formula with parameters $\{(\ (m!)^2, -(m+1)^2\ )|m \geq 0\}$.

EXAMPLE 2 (Infinite sever queue) Consider a BDP with birth and death rates

$$a_{2n} = 1,\ n = 0, 1, \ldots, \text{ and } \quad a_{2n-1} = n,\ n = 1, 2, \ldots,$$

so that

$$\pi_n = \frac{1}{n!}, \quad n = 0, 1, 2, \ldots.$$

This rates corresponds to the $M/M/\infty$ (self service) model. It is well known that,

$$P_{0m}(t) = \frac{(1-e^{-t})^m \exp(e^{-t}-1)}{m!}, \quad m \geq 0.$$

Using (8),

$$P_{m0}(t) = (1-e^{-t})^m \exp(e^{-t}-1),\ m \geq 0.$$

Therefore,

$$P_{00}(t+u) = \exp[e^{-(t+u)} - 1] = \sum_{m=0}^{\infty} P_{0m}(t) P_{m0}(u) = \sum_{m=0}^{\infty} m! f_m(t) f_m(u),$$

where

$$f_m(t) = \frac{(1-e^{-t})^n \exp(e^{-t}-1)}{m!} = \frac{t^m}{m!} - \frac{(m+1)(m+2)}{2} \frac{t^{m+1}}{(m+1)!} + \mathcal{O}(t^{m+2}).$$

Hence $P_{00}(t)$ has an addition formula with parameters $\{(m!, -(m+1)(m+2)/2)|\ m \geq 0\}$.

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