Two classes of passive time-varying well-posed linear systems

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Abstract. We investigate two classes of time-varying well-posed linear systems. Starting from a time-invariant scattering-passive system, each of the time-varying systems is constructed by introducing a time-dependent inner product on the state space and modifying some of the generating operators. These classes of linear systems are motivated by physical examples such as the electromagnetic field around a moving object. To prove the well-posedness of these systems, we use the Lax-Phillips semigroup induced by a well-posed linear system, as in scattering theory.

Key words. Well-posed linear system, operator semigroup, linear time-varying system, Lax-Phillips semigroup, scattering passive system.

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1. Introduction

The concept of a time-varying well-posed linear systems has emerged over the years as researchers studied partial differential equations with time-dependent coefficients and then abstracted certain properties. In the absence of inputs and outputs, we have the theory of evolution families which is the natural generalization of the theory of strongly continuous semigroups. The relevant generation results have been developed by Kato [9, 10], but until today this theory is much less complete than the theory of strongly continuous semigroups (see, for example, Fattorini [6], Okazawa [13], Tanabe [21] and the survey by Schnaubelt [17]).

Various classes of time-varying linear systems with inputs and outputs have been introduced by Acquistapace and Terreni [1], Curtain and Pritchard [3], Hinrichsen and Pritchard [7], Jacob [8], and Schnaubelt [16]. The most general definition is the one in [16] which mimicks the concept of a (time-invariant) well-posed linear system from Weiss [24] (which is equivalent to the differently looking earlier definition in Salamon [15]). Unfortunately, for such systems, there is no complete representation theory available (unlike for time-invariant well-posed systems). It is difficult to verify that a given system of linear equations defines a time-varying well-posed system, and for this reason it is also difficult to construct non-trivial examples of such systems. The difficulties arise when we have unbounded control or observation operators and the evolution family of the system is not of parabolic type.

In this paper we want to introduce two classes of time-varying well-posed linear systems. Each such system is constructed using a scattering passive time-invariant system and a family of time-dependent inner products on the state space.

To state our main results, we need to recall some terminology and results concerning (time-invariant) well-posed linear systems from Staffans and Weiss [20] and from [24] (see also Staffans [19]). We assume that the reader is familiar with well-posed linear systems. In particular, we do not repeat their definition. The purpose of the following paragraphs is only to clarify the notation, and to remind which results on well-posed systems will be needed. In this paper we encounter both (time-invariant) well-posed linear systems and their time-varying counterparts. For the reader's convenience, we use the superscript i (for "invariant") for notation refering to a time-invariant system.

Let Σ^i be a well-posed linear system. It is known (see [20, Section 3]) that such a system Σ^i can be described locally in time by the equations

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= \overline{C}x(t) + Du(t). \end{aligned} \tag{1.1}$$

Here $u(\cdot)$ is the *input function*, $x(t)$ is the *state* at time t and $y(\cdot)$ is the *output function*. We have $u(t) \in U$, $x(t) \in X$ and $y(t) \in Y$, where the Hilbert spaces U, X and Y are called the *input space, the state space* and the *output space*, respectively. The operator $A : \mathcal{D}(A) \to X$ is the generator of a strongly continuous semigroup on X, and it determines two new Hilbert spaces as follows: X_1 is $\mathcal{D}(A)$ with the norm $\|z\|_1 = \|(\beta I - A)z\|$ where $\beta \in \rho(A)$. The *extrapolation* space X_{-1} is the completion of X with respect to the norm $\|(\beta I - A)^{-1}z\|$. These spaces are independent of the choice of $\beta \in \rho(A)$ (different choices lead to equivalent norms). The generator A has a unique extension to an operator in $\mathcal{L}(X, X_{-1})$ denoted by the same symbol. We have $B \in \mathcal{L}(U, X_{-1})$ and there exists $C \in \mathcal{L}(X_1, Y)$ such that if $u = 0$ and $x(0) \in \mathcal{D}(A)$, then $y(t) = Cx(t)$ for all $t \geq 0$.

The operators A, B and C are uniquely determined by Σ^i. We introduce the

Hilbert space $Z = \mathcal{D}(A) + (\beta I - A)^{-1}BU$ with the norm

$$\|z\|_Z = \inf\left\{ \left(\|w\|_1^2 + \|v\|^2\right)^{\frac{1}{2}} \mid w \in \mathcal{D}(A),\ v \in U,\ z = w + (\beta I - A)^{-1}Bv \right\}.$$

This space is also independent of the choice of $\beta \in \rho(A)$. Then the operator $\overline{C}$ appearing in (1.1) is a bounded extension of C to Z, which in general is not uniquely determined by the system. Finally, $D \in \mathcal{L}(U, Y)$. The operator D is uniquely determined if an extension $\overline{C}$ has been selected.

A function $x : [0, \infty) \to X$ is called a *classical solution* of (1.1) if

$$x \in C^1([0, \infty), X)$$

and the first equation in (1.1) holds for all $t \geq 0$. For inputs $u \in C([0, \infty), U)$, the first equation in (1.1) implies that any classical solution x belongs to $C([0, \infty), Z)$. Then the second equation in (1.1) makes sense and defines the output $y \in C([0, \infty), Y)$. For given $x(0)$ and u, the differential equation in (1.1) has a unique classical solution for $t \geq 0$ if

$$u \in \mathcal{H}^1((0, \infty), U) \quad \text{and} \quad Ax(0) + Bu(0) \in X. \tag{1.2}$$

(Note that the second condition above implies $x(0) \in Z$.) If we take all the pairs $(x(0), u)$ satisfying (1.2), then the corresponding classical solutions of (1.1) determine four families of operators $\mathbb{T}^i$, Φ^i, Ψ^i and $\mathbb{F}^i$, parametrized by $\tau \geq 0$, such that

$$\begin{bmatrix} x(\tau) \\ \mathbf{P}_\tau y \end{bmatrix} = \begin{bmatrix} \mathbb{T}^i_\tau & \Phi^i_\tau \\ \Psi^i_\tau & \mathbb{F}^i_\tau \end{bmatrix} \begin{bmatrix} x(0) \\ \mathbf{P}_\tau u \end{bmatrix}. \tag{1.3}$$

Here, $\mathbf{P}_\tau y$ denotes the restriction y to $[0, \tau]$. The 2×2 matrix appearing above is a bounded operator from $X \times L^2([0, \tau], U)$ to $X \times L^2([0, \tau], Y)$. (Here $L^2([0, \tau], U)$ is regarded as a subspace of $L^2([0, \infty), U)$, by extending any function in $L^2[0, \tau], U)$ to be zero for $t > \tau$.) In fact, in [20, 24] the system Σ^i is defined via the operator families $\mathbb{T}^i$, Φ^i, Ψ^i and $\mathbb{F}^i$. $\mathbb{T}^i$ is the strongly continuous semigroup generated by A.

Note that the bounded operators $\mathbb{T}^i_\tau$, Φ^i_τ, Ψ^i_τ and $\mathbb{F}^i_\tau$ are completely determined by their action on data pairs $(x(0), u)$ satisfying (1.2), because such pairs are dense in $X \times L^2([0, \infty), U)$. Moreover, for such $(x(0), u)$, we have

$$\mathbf{P}_\tau y \in \mathcal{H}^1((0, \tau), Y) \quad \text{and} \quad Ax(\tau) + Bu(\tau) \in X.$$

However, the formula (1.3) defines the state trajectory and the output function of Σ^i for every $x(0) \in X$ and for every $u \in L^2([0, \infty), U)$.

The well-posed system Σ^i is called *scattering passive* if for every classical solution of (1.1) corresponding to initial data satisfying (1.2) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|^2 \leq \|u(t)\|^2 - \|y(t)\|^2 \qquad \forall\, t \geq 0.$$

In particular, Σ^i is called *scattering energy preserving* if we always have equality in the above formula. Such systems occur often in mathematical physics, where $\frac{1}{2}\|x(t)\|^2$ is interpreted as the energy stored in the system at time t, $\frac{1}{2}\|u(t)\|^2$ is the incoming power and $\frac{1}{2}\|y(t)\|^2$ is the outgoing power. If we integrate the above inequality, we see that Σ^i is scattering passive if and only if

$$\left\| \begin{bmatrix} \mathbb{T}^i_\tau & \Phi^i_\tau \\ \Psi^i_\tau & \mathbb{F}^i_\tau \end{bmatrix} \right\| \leq 1$$

for some (hence, for every) $\tau > 0$. Similarly, Σ^i is scattering energy preserving if and only if the 2×2 matrix in (1.3) is isometric. Scattering passivity and scattering energy preservation can also be expressed in terms of $A, B, \overline{C}, D$, see Arov and Nudelman [2], Malinen *et al* [12], Staffans [18] and [20] for details (in [20], scattering passive systems were called dissipative).

Let Σ^i be a time-invariant scattering passive system described by (1.1). Let J be a closed interval of non-zero length (usually $J = [0, \infty)$). Let $P : J \to \mathcal{L}(X)$ be such that $P(t) = P(t)^* > 0$ and $P(t)^{-1}$ is bounded for every $t \in J$. Moreover, we assume that for every $z \in X$, both $P(\cdot)z$ and $P^{-1}(\cdot)z$ are continuously differentiable functions on J. We introduce two time-varying systems on the time interval J, informally defined by the equations

$$\begin{aligned} \dot{x}(t) &= P(t)^{-1}(Ax(t) + Bu(t)), \\ y(t) &= \overline{C}x(t) + Du(t), \end{aligned} \tag{1.4}$$

and

$$\begin{aligned} \dot{x}(t) &= AP(t)x(t) + Bu(t), \\ y(t) &= \overline{C}P(t)x(t) + Du(t). \end{aligned} \tag{1.5}$$

For the second system (described by (1.5)), in our main results we assume additionally that $P(\cdot)z$ is of class C^2 (instead of just C^1) for every $z \in X$.

At the first glance, it is not clear whether these equations make sense, even for smooth initial states and smooth u, because $P(t)^{-1}$ has not been defined on X_{-1} and it is not clear if $\overline{C}$ can be applied to $x(t)$ or to $P(t)x(t)$, respectively. It is even less clear if these equations have solutions for some subspace of initial conditions and inputs, and even if they have solutions, it is not clear if these depend continuously on the data (the initial state and the input function).

In this paper, we show that in fact both of the above systems of equations determine well-posed time-varying systems, a concept that we define in Section 3. Our definition is a slight generalization of the concept of a well-posed nonautonomous system introduced in [16]. We denote by Σ_l the system generated by (1.4) and by Σ_r the system generated by (1.5) (here, l and r stand for left and right). In particular, if $\tau \in J$ and the pair $(x(\tau), u)$ satisfies

$$u \in \mathcal{H}^1(J^0, U) \quad \text{and} \quad Ax(\tau) + Bu(\tau) \in X,$$

then (1.4) has a classical solution for $t \geq \tau$, $t \in J$, which satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle P(t)x(t), x(t)\rangle \leq \|u(t)\|^2 - \|y(t)\|^2 + \langle \dot{P}(t)x(t), x(t)\rangle \tag{1.6}$$

for every such t. (J^0 denotes the interior of J.) If the pair $(x(\tau), u)$ satisfies

$$u \in \mathcal{H}^1(J^0, U) \quad \text{and} \quad AP(\tau)x(\tau) + Bu(\tau) \in X\,,$$

then (1.5) has a classical solution for $t \geq \tau$, $t \in J$, which satisfies the same balance inequality (1.6). In fact, the well-posedness of the systems Σ_l and Σ_r is derived as a consequence of (1.6). If the original well-posed system Σ^i is energy preserving, then we have equality in (1.6) (for both time-varying systems). A time-varying well-posed system that satisfies (1.6) could be called *scattering P-passive*.

In the next section we prove generation results for a class of time-varying systems without inputs and outputs. In Section 3 we define linear time-varying well-posed systems and discuss their relationship with the Lax-Phillips evolution family which governs a certain time-varying system without inputs and outputs. The Lax-Phillips evolution family is the time-varying counterpart of the Lax-Phillips semigroup induced by a (time-invariant) well-posed linear system.

The results from Section 3 are used in Section 4 to establish our main well-posedness theorems for (1.4) and (1.5). In the last section (Section 5) we study a wave equation with time-dependent coefficients and with boundary control and observation. We transform the given system of equations into a system of the form (1.5) and then show the well-posedness of the latter system.

2. Time-varying multiplicative perturbations of m-dissipative operators

The standing assumptions of this section are the following: $A : \mathcal{D}(A) \to X$ is a maximally dissipative operator on the Hilbert space X. As in Section 1, J is a closed interval of non-zero length and the function $P : J \to \mathcal{L}(X)$ satisfies

- $P(t) = P(t)^* > 0$ (hence $P(t)^{-1}$ is bounded) for each $t \in J$,
- $P(\cdot)z \in C^1(J, X)$ for each $z \in X$, (2.1)
- $P(\cdot)^{-1}z \in C^1(J, X)$ for each $z \in X$.

We equip X with the following scalar products and the corresponding norms:

$$\langle w, z\rangle_t = \langle P(t)w, z\rangle \qquad \text{and} \qquad \|z\|_t = \|P(t)^{\frac{1}{2}}z\|, \qquad t \in J.$$

It is clear from (2.1) and the uniform boundedness principle that the functions $\|P(\cdot)\|$ and $\|P(\cdot)^{-1}\|$ are bounded on compact subintervals of J, which implies that the norms $\|\cdot\|_t$ are locally uniformly equivalent to the original norm on X.

Let $\tau \in J$ and set $J_\tau = J \cap [\tau, \infty)$. In this section we study the initial value problems

$$\dot{x}(t) = P(t)^{-1}Ax(t), \quad t \in J_\tau, \qquad x(\tau) = x_0, \tag{2.2}$$

and

$$\dot{x}(t) = AP(t)x(t), \quad t \in J_\tau, \qquad x(\tau) = x_0. \tag{2.3}$$

Here the operators $P(t)^{-1}A$ and $AP(t)$ are defined on their natural domains, namely,

$$\mathcal{D}(P(t)^{-1}A) = \mathcal{D}(A) \quad \text{and} \quad \mathcal{D}(AP(t)) = \{z \in X \mid P(t)z \in \mathcal{D}(A)\}.$$

From Engel and Nagel [5, Section VI.9], Kato [9], Schnaubelt [17] and Tanabe [21], we recall the following definition.

Definition 2.1. An *evolution family* $\mathbb{T}$ on X with time interval J is a family of operators $\mathbb{T}(t, \tau) \in \mathcal{L}(X)$ defined for $t, \tau \in J$ with $t \geq \tau$ which satisfies

(a) $\mathbb{T}(t, s)\mathbb{T}(s, \tau) = \mathbb{T}(t, \tau)$ for every $t, s, \tau \in J$ with $t \geq s \geq \tau$,

(b) $\mathbb{T}(t, t) = I$ for every $t \in J$,

(c) $(t, \tau) \mapsto \mathbb{T}(t, \tau)$ is strongly continuous for $t, \tau \in J$ with $t \geq \tau$.

The concept of an evolution family is the natural generalization of the concept of a strongly continuous semigroup to time-varying systems. When trying to generalize the concept of an infinitesimal generator, one runs into several difficulties, and as a result there is no standard notion of a time-varying infinitesimal generator. In this paper, we shall use the following definition:

Definition 2.2. Let $\{A(t) : \mathcal{D}(A(t)) \to X \mid t \in J\}$ be a family of densely defined linear operators on X. We say that $A(\cdot)$ *generates* the evolution family $\mathbb{T}$ if $\mathbb{T}(t, \tau)\mathcal{D}(A(\tau)) \subset \mathcal{D}(A(t))$ for all $t, \tau \in J$ with $t \geq \tau$ and, for every $\tau \in J$ with $\tau \neq \max J$ and every $x_0 \in \mathcal{D}(A(\tau))$, the function $x(\cdot) = \mathbb{T}(\cdot, \tau)x_0$ is continuously differentiable on J_τ and it is the unique solution of the Cauchy problem

$$\dot{x}(t) = A(t)x(t), \quad t \in J_\tau, \qquad x(\tau) = x_0. \tag{2.4}$$

It is known that the operators $A(t)$ generate at most one evolution family and that not every evolution family is generated in this way.

Proposition 2.3. *Under the standing assumptions stated at the beginning of this section, $P(\cdot)^{-1}A$ generates an evolution family $\mathbb{T}_l$ on X with time interval J. Let $x_0 \in \mathcal{D}(A)$. Then the map*

$$(t, \tau) \mapsto P(t)^{-1}A\mathbb{T}_l(t, \tau)x_0 \qquad (\textit{for } t, \tau \in J \ \textit{ with } \ \tau \leq t)$$

is continuous in X. Further, for each $t \in J$ with $t \neq \min J$, the map $\tau \mapsto \mathbb{T}_l(t, \tau)x_0$ (for $\tau \in J$ with $\tau \leq t$) is continuously differentiable and

$$\frac{\partial}{\partial \tau} \mathbb{T}_l(t, \tau)x_0 = -\mathbb{T}_l(t, \tau)P(\tau)^{-1}Ax_0. \tag{2.5}$$

Proof. Let $t \in J$. We have

$$\langle P(t)^{-1}Az, z\rangle_t = \langle Az, z\rangle \qquad \forall\, z \in \mathcal{D}(A),$$

so that $P(t)^{-1}A$ is dissipative on X with respect to the scalar product $\langle\cdot,\cdot\rangle_t$. Since this scalar product induces on X a norm that is equivalent to the original one, this implies that $I - P(t)^{-1}A$ has closed range.

It is clear that $P(t)^{-1}A$ is closed and that its adjoint is $A^*P(t)^{-1}$, with the domain $\mathcal{D}(A^*P(t)^{-1}) = \{x \in X \mid P(t)^{-1}x \in \mathcal{D}(A^*)\}$. It is also clear that $A^*P(t)^{-1}$ is dissipative with respect to the scalar product $\langle P(t)^{-1}w, z\rangle$. Therefore $I - A^*P(t)^{-1}$ is injective, so that $I - P(t)^{-1}A$ has dense range. Together with what we proved earlier, this implies that $\mathrm{Ran}\,(I - P(t)^{-1}A) = X$. As a result, $P(t)^{-1}A$ is maximally dissipative with respect to $\langle\cdot,\cdot\rangle_t$.

Take a compact interval $[a, b] \subset J$, $z \in X$, and set

$$L = \max_{t\in[a,b]} \|\dot{P}(t)\|, \qquad M = \max_{t\in[a,b]} \|P(t)^{-1}\|. \tag{2.6}$$

M and L are finite by the uniform boundedness principle. We estimate

$$\begin{aligned}\|z\|_t^2 &= \langle (P(t) - P(\tau) + P(\tau))z, z\rangle \le L\,|t-\tau|\,\|z\|^2 + \|z\|_\tau^2 \\ &\le LM\,|t-\tau|\,\|z\|_\tau^2 + \|z\|_\tau^2 \le e^{LM|t-\tau|}\,\|z\|_\tau^2\end{aligned}$$

(we have used the fact that $\|P(\tau)^{-\frac{1}{2}}\|^2 = \|P(\tau)^{-1}\|$). Due to this inequality and a result of Kato [9, Proposition 3.4] (see also [21, Proposition 4.3.2]), $P(\cdot)^{-1}A$ is a stable family in the sense of [9]. In addition, $P(\cdot)^{-1}A$ is strongly continuously differentiable on the (time-invariant) domain $\mathcal{D}(A)$. As a consequence, $P(\cdot)^{-1}A$ generates a unique evolution family $\mathbb{T}_l$ on the time interval $[a, b]$ by the corollary to Theorem 4.4.2 in [21]. Moreover, by the same corollary from [21], the mapping $(t, \tau) \mapsto P(t)^{-1}A\mathbb{T}_l(t, \tau)x_0$ is continuous in X and (2.5) holds for all τ, t such that $a \le \tau \le t \le b$ and $a < t$. The assertions then follow since $[a, b]$ was arbitrary. □

Proposition 2.4. *The functions $P(\cdot)$, $(I - P(\cdot)A)^{-1}$ and $(I - AP(\cdot))^{-1}$ are locally Lipschitz continuous in operator norm. These statements remain true if we replace $P(\cdot)$ with $P(\cdot)^{-1}$ and/or A with A^*.*

Proof. It follows easily from the first part of (2.6) that on any compact subiterval $[a, b] \subset J$, $P(\cdot)$ is Lipschitz continuous in operator norm, with Lipschitz constant L (we have already used this in the last proof).

We have seen in the proof of Proposition 2.3 that $P(t)^{-1}A$ is maximally dissipative with respect to the inner product $\langle\cdot,\cdot\rangle_t$, which implies that $I - P(t)^{-1}A$ has a bounded inverse that is a contraction with respect to the norm $\|\cdot\|_t$. Denoting $N = \max_{t\in[a,b]} \|P(t)\|$, we have that for $t \in [a, b]$,

$$\|(I - P(t)A)^{-1}z\| \le \|P(t)^{-\frac{1}{2}}\| \cdot \|(I - P(t)A)^{-1}z\|_t \le \|P(t)^{-\frac{1}{2}}\| \cdot \|z\|_t$$

$$\leq \|P(t)^{-\frac{1}{2}}\| \cdot \|P(t)^{\frac{1}{2}}\| \cdot \|z\| \leq (MN)^{\frac{1}{2}}\|z\|. \tag{2.7}$$

We have $A(I-P(t)A)^{-1} = P(t)^{-1}\left[-I + (I-P(t)A)^{-1}\right]$. From this combined with (2.7), it follows that for $t \in [a,b]$,

$$\|A(I-P(t)A)^{-1}z\| \leq M\left[1 + (MN)^{\frac{1}{2}}\right]. \tag{2.8}$$

For $t, \tau \in [a,b]$ we have, by a trivial computation,

$$(I-P(\tau)A)^{-1} - (I-P(t)A)^{-1} = (I-P(\tau)A)^{-1}[P(\tau)-P(t)]A(I-P(t)A)^{-1}.$$

From this, (2.7) and (2.8) we see that for $t, \tau \in [a,b]$,

$$\|(I-P(\tau)A)^{-1} - (I-P(t)A)^{-1}\| \leq (MN)^{\frac{1}{2}}L|t-\tau|M\left[1 + (MN)^{\frac{1}{2}}\right],$$

i.e., $(I-P(\cdot)A)^{-1}$ is locally Lipschitz continuous in operator norm. Taking adjoints, we see that also $(I-A^*P(\cdot))^{-1}$ is locally Lipschitz. Replacing A with A^*, we see that also $(I-AP(\cdot))^{-1}$ is locally Lipschitz. It is clear that we can also make the remaining replacements mentioned in the proposition. □

We shall prove below that under an additional smoothness assumption, $AP(\cdot)$ also generates an evolution family on X. This will be more difficult, since the domains $\mathcal{D}(AP(t))$ vary in time and the fundamental generation Theorem 6.1 from [9] does not apply. Under our standing assumptions (2.1), we construct an evolution family $\mathbb{T}_r$ on X such that for every $x_0 \in X$ the function $x(t) = \mathbb{T}_r(t,\tau)x_0$ is continuously differentiable in X_{-1} for $t \in J_\tau$ and it satisfies (2.4). Afterwards we show that $AP(\cdot)$ in fact generates $\mathbb{T}_r$ if, in addition, $P(\cdot)z \in C^2(J,X)$ for all $z \in X$.

We need several preparations. Remember that extrapolation spaces were defined after (1.1). For a detailed discussion of such spaces we refer to Tucsnak and Weiss [23, Chapter 2]. We take $\beta = 1$ in the definition of the norms on the domains and the extrapolation spaces of the operators A, A^*, $P(t)^{-1}A$, $A^*P(t)^{-1}$, $AP(t)$ and $P(t)A^*$ (this is possible according to the last proposition). The extrapolation space X_{-1} of A is isomorphic to $[\mathcal{D}(A^*)]'$ (duality with respect to the pivot space X) and hence they are identified. Similarly, for each $t \in J$, the spaces

$$X^t_{-1,l} = [D(A^*P(t)^{-1})]', \qquad X^t_{-1,r} = [D(P(t)A^*)]'$$

are identified with the extrapolation spaces of $P(t)^{-1}A$ and $AP(t)$, respectively. The norms of $\mathcal{D}(P(t)A^*)$ and $\mathcal{D}(A^*)$ are locally uniformly equivalent, so that the restrictions $I(t) : \mathcal{D}(P(t)A^*) \to \mathcal{D}(A^*)$ and $I(t)^{-1} : \mathcal{D}(A^*) \to \mathcal{D}(P(t)A^*)$ of the identity are locally uniformly bounded, for $t \in J$. Thus $I(t)^*$ is an isomorphism from X_{-1} to $X^t_{-1,r}$ and both $I(t)^*$ and its inverse $(I(t)^{-1})^*$ are locally uniformly bounded for $t \in J$. In the sequel we shall identify X_{-1} and $X^t_{-1,r}$ for all $t \in J$. It is then clear that the extension of $AP(t)$ to an operator in $\mathcal{L}(X, X_{-1})$ is simply the corresponding extension of A times $P(t)$.

Definition 2.5. A *backward evolution family* $\mathbb{S}$ on X with time interval J is a family of operators $\mathbb{S}(t,\tau) \in \mathcal{L}(X)$ defined for $t,\tau \in J$ with $t \geq \tau$ which satisfy properties (b) and (c) in Definition 2.1 and

$$\mathbb{S}(\sigma,\tau)\mathbb{S}(t,\sigma) = \mathbb{S}(t,\tau) \quad \text{for every} \quad t,\sigma,\tau \in J \quad \text{with} \quad t \geq \sigma \geq \tau.$$

With $A(\cdot)$ as in Definition 2.2, we say that $A(\cdot)$ *generates* the backward evolution family $\mathbb{S}$ if $\mathbb{S}(t,\tau)\mathcal{D}(A(t)) \subset \mathcal{D}(A(\tau))$ for all $t,\tau \in J$ with $t \geq \tau$ and, for each $t \in J$ with $t \neq \min J$ and every $x_0 \in \mathcal{D}(A(t))$, the function $v(\cdot) = \mathbb{S}(t,\cdot)x_0$ is continuously differentiable on $J \cap (-\infty, t]$ and it is the unique solution of the final value problem

$$\dot{v}(\tau) = \, - A(\tau)v(\tau), \quad \tau \leq t, \;\; \tau \in J, \qquad v(t) = x_0.$$

One can easily adapt the proofs of Kato's generation results for evolution familes to show the natural analogues of Theorem 4.4.2 and its corollary and of Proposition 4.3.2 in [21] for backward evolution families.

Remark 2.6. The function $t \mapsto P(t)^{-1}$ also satisfies the assumptions listed in (2.1). Proposition 2.3 thus shows that $P(\cdot)A$ generates an evolution family $\tilde{\mathbb{T}}_l$ on X with time interval J. We define a new evolution family $\mathbb{V}$ on X by

$$\mathbb{V}(t,\tau) = P(t)^{-1}\tilde{\mathbb{T}}_l(t,\tau)P(\tau) \qquad \text{for} \quad t,\tau \in J \quad \text{with} \quad \tau \leq t.$$

From $\frac{\mathrm{d}}{\mathrm{d}t}P(t)^{-1}z = -P(t)^{-1}\dot{P}(t)P(t)^{-1}z$ we see that $\mathbb{V}$ is generated by

$$A_1(t) \,=\, AP(t) - P^{-1}(t)\dot{P}(t)$$

with domains $\mathcal{D}(A_1(t)) = \mathcal{D}(AP(t))$, $t \in J$, and that

$$\frac{\partial}{\partial \tau}\mathbb{V}(t,\tau)x_0 = -\mathbb{V}(t,\tau)A_1(\tau)x_0 \tag{2.9}$$

for every $x_0 \in D(AP(\tau))$ and $t,\tau \in J$ with $\tau \leq t$ and $t \neq \min J$.

The main idea of the following proposition is to introduce the evolution family $\mathbb{T}_r$ as a bounded perturbation of $\mathbb{V}$, in the sense of Curtain and Pritchard [3, Ch. 9].

Proposition 2.7. *Under the standing assumptions stated at the beginning of this section, and using the notation* $\mathbb{V}$ *from Remark* 2.6*, there exists an evolution family* $\mathbb{T}_r$ *on* X *with time interval* J *having the following properties:*

(a) *For all* $x_0 \in X$ *and* $t,\tau \in J$ *with* $\tau \leq t$, $\mathbb{T}_r$ *satisfies the integral equations*

$$\mathbb{T}_r(t,\tau)x_0 = \mathbb{V}(t,\tau)x_0 + \int_\tau^t \mathbb{V}(t,s)P(s)^{-1}\dot{P}(s)\mathbb{T}_r(s,\tau)x_0 \,\mathrm{d}s, \tag{2.10}$$

$$\mathbb{T}_r(t,\tau)x_0 = \mathbb{V}(t,\tau)x_0 + \int_\tau^t \mathbb{T}_r(t,s)P(s)^{-1}\dot{P}(s)\mathbb{V}(s,\tau)x_0 \,\mathrm{d}s. \tag{2.11}$$

Moreover, $\mathbb{T}_r$ is the unique solution of each of these integral equations.

(b) *For all $t, \tau \in J$ with $t \geq \tau$, $t \neq \min J$, $\tau \neq \max J$, the derivatives*

$$\frac{\partial}{\partial \tau}\mathbb{T}_r(t,\tau)x_0 = -\,\mathbb{T}_r(t,\tau)AP(\tau)x_0 \qquad \text{and} \qquad \frac{\partial^+}{\partial t}\mathbb{T}_r(t,\tau)x_0\Big|_{t=\tau} = AP(\tau)x_0$$

exist in X for every $x_0 \in \mathcal{D}(AP(\tau))$.

(c) *For each $x_0 \in X$ and $\tau \in J$ with $\tau \neq \max J$, the function $t \mapsto x(t) = \mathbb{T}_r(t,\tau)x_0$ is continuously differentiable in X_{-1} and satisfies (2.3) in X_{-1} for all $t \in J_\tau$. The operators $\mathbb{T}_r(t,\tau)$ have locally uniformly bounded extensions to operators in $\mathcal{L}(X_{-1})$ denoted by the same symbols, for all $t, \tau \in J$ with $\tau \leq t$. Moreover, for each $x_0 \in X$ and $t \in J$ with $t \neq \min J$, the function $\tau \mapsto \mathbb{T}_r(t,\tau)x_0$ is continuously differentiable in X_{-1} and $\frac{\partial}{\partial \tau}\mathbb{T}_r(t,\tau)x_0 = -\mathbb{T}_r(t,\tau)AP(\tau)x_0$, for all $t, \tau \in J$ with $\tau \leq t$.*

(d) *The adjoint family $\{\mathbb{T}_r(t,\tau)^* \mid t,\tau \in J,\ t \geq \tau\}$ is a backward evolution family generated by $P(t)A^*$. Let $x_0 \in \mathcal{D}(A^*)$. Then the map*

$$(t,\tau) \mapsto P(\tau)A^*\mathbb{T}_r(t,\tau)^*x_0 \qquad (\text{for}\ \ (t,\tau) \in J^2 \ \ \text{with}\ \ \tau \leq t)$$

is continuous in X and, for $\tau \in J$ with $\tau \neq \max J$, the map $J_\tau \ni t \mapsto \mathbb{T}_r(t,\tau)^*x_0$ is continuously differentiable in X with derivative $\mathbb{T}_r(t,\tau)^*P(t)A^*x_0$.

In point (a) above, by the uniqueness of $\mathbb{T}_r$ we mean the following statement: Let $\mathbb{S}$ be a family of operators in $\mathcal{L}(X)$ indexed by pairs $(t,\tau) \in J^2$ with $\tau \leq t$ such that the map $(t,\tau) \mapsto \mathbb{S}(t,\tau)z$ is weakly continuous for every $z \in X$ and such that either (2.10) or (2.11) hold, with $\mathbb{S}$ in place of $\mathbb{T}_r$ and with weak integrals, for all $x_0 \in X$ and $t, \tau \in J$ with $\tau \leq t$. Then $\mathbb{S} = \mathbb{T}_r$.

Proof. (a) It is known that there is an evolution family $\mathbb{T}_r$ on X satisfying (2.10) and (2.11), see Theorem 9.2 and Corollary 9.4 of [3] (where the strong continuity of $\mathbb{T}_r$ follows from the proof of [3, Theorem 9.2]). The uniqueness assertion in (a) in the case of (2.10) is a direct consequence of Gronwall's inequality applied to the function $\phi(t) = \|\mathbb{S}(t,\tau)x_0 - \mathbb{T}_r(t,\tau)x_0\|$. In the case of the integral equation (2.11) we need the following observation: Let $T(t)$, $t \in [\tau, t]$, be bounded operators on X such $t \mapsto T(t)z$ is weakly continuous for every $z \in X$. Then the map $t \mapsto \|T(t)\|$ is lower semicontinuous and thus Lebesgue measurable. Using this for $T(t) = \mathbb{S}(t,\tau) - \mathbb{T}_r(t,\tau)$, we can derive from (2.11) that

$$\|T(t)x_0\| \leq \int_\tau^t \|T(s)\| \cdot \|P(s)^{-1}\dot{P}(s)\mathbb{V}(s,\tau)\| \cdot \|x_0\|\,\mathrm{d}s.$$

After taking here the supremum with respect to x_0 in the unit ball of X, we can again apply Gronwall's lemma to deduce that $T(t) = 0$.

(b) The first differential equation in (b) follows from (2.11) and (2.9), and the second one from (2.10) and the fact that $A_1(\cdot)$ (defined in Remark 2.6) generates $\mathbb{V}$.

(d) By the backward analogue of Proposition 2.3, the operators $P(t)A^*$ (with $t \in J$) generate a backward evolution family $\mathbb{S}$ on X with time interval J satisfying the continuity and differentiablity assertions in (d), with $\mathbb{T}_r(t,\tau)^*$ replaced by $\mathbb{S}(t,\tau)$. Fix $t,\tau \in J$ with $t > \tau$, and take $w \in \mathcal{D}(AP(\tau))$ and $z \in \mathcal{D}(A^*)$. We can differentiate

$$\begin{aligned}
\frac{\partial}{\partial s} \langle \mathbb{S}(t,s)^* \mathbb{V}(s,\tau) w, z \rangle &= \frac{\partial}{\partial s} \langle \mathbb{V}(s,\tau) w, \mathbb{S}(t,s) z \rangle \\
&= \langle [AP(s) - P^{-1}(s)\dot{P}(s)] \mathbb{V}(s,\tau) w, \mathbb{S}(t,s) z \rangle - \langle \mathbb{V}(s,\tau) w, P(s) A^* \mathbb{S}(t,s) z \rangle \\
&= -\langle \mathbb{S}(t,s)^* P^{-1}(s) \dot{P}(s) \mathbb{V}(s,\tau) w, z \rangle,
\end{aligned}$$

for all $s \in [\tau, t]$. Integrating this equation from τ to t, we deduce that the operators $\mathbb{S}(t,\tau)^*$ satisfy (2.11) with a weak integral for all $x_0 \in X$ and $t,\tau \in J$ with $\tau \le t$. Hence, $\mathbb{T}_r = \mathbb{S}^*$ due to the uniqueness part of statement (a).

(c) Let $x_0 \in X$, $z \in \mathcal{D}(A^*)$, $\tau \in J$ and $t \in J_\tau$. Then assertion (d) implies that

$$\begin{aligned}
\frac{\partial}{\partial t} \langle \mathbb{T}_r(t,\tau) x_0, z \rangle &= \frac{\partial}{\partial t} \langle x_0, \mathbb{T}_r(t,\tau)^* z \rangle = \langle x_0, \mathbb{T}_r(t,\tau)^* P(t) A^* z \rangle \\
&= \langle AP(t) \mathbb{T}_r(t,\tau) x_0, z \rangle .
\end{aligned}$$

This means that $x(t) = \mathbb{T}_r(t,\tau)x_0$ solves (2.3) in the space X_{-1}, as claimed. Let us denote by X_1^d the space $\mathcal{D}(A^*)$ with the norm $\|z\|_1^d = \|(I - A^*)z\|$. We know from (d) that $\mathbb{T}_r(t,\tau)^* \mathcal{D}(A^*) \subset \mathcal{D}(A^*)$. With the closed graph theorem we obtain that $\mathbb{T}_r(t,\tau)^* \in \mathcal{L}(X_1^d)$. According to (d), for every $x_0 \in X_1^d$, the function $(t,\tau) \mapsto \mathbb{T}_r(t,\tau)^* x_0$ is continuous in X_1^d and hence bounded on any compact set of pairs (t,τ) with $t \ge \tau$. Using the uniform boundedness principle, we conclude that the operators $\mathbb{T}_r(t,\tau)^*$ are locally uniformly bounded in $\mathcal{L}(X_1^d)$, and this implies that the operators $\mathbb{T}_r(t,\tau)$ are locally uniformly bounded in $\mathcal{L}(X_{-1})$. As above one further deduces from assertion (d) that

$$\begin{aligned}
\frac{\partial}{\partial \tau} \langle \mathbb{T}_r(t,\tau) x_0, z \rangle &= \frac{\partial}{\partial \tau} \langle x_0, \mathbb{T}_r(t,\tau)^* z \rangle = -\langle x_0, P(\tau) A^* \mathbb{T}_r(t,\tau)^* z \rangle \\
&= -\langle \mathbb{T}_r(t,\tau) AP(\tau) x_0, z \rangle .
\end{aligned}$$

We next show that the map $\tau \mapsto \mathbb{T}_r(t,\tau)AP(\tau)x_0$ is continuous in X_{-1}, for $\tau \in J$ with $\tau \le t$. Then the above weak derivative is in fact a strong derivative and assertion (c) will be established. To verify the asserted continuity, we fix $\tau \in J$ with $\tau \le t$. Take $x_n \in \mathcal{D}(AP(\tau))$ with $x_n \to x_0$ in X and $\sigma \in J$ with $\sigma \le t$ and $\sigma \to \tau$. Observe that the operators $T_r(t,\tau)AP(\tau)$ and $T_r(t,\sigma)AP(\tau)$ are uniformly bounded in $\mathcal{L}(X, X_{-1})$ by some constant $c > 0$. For each n, we have $AP(\tau)x_n \in X$ and thus

$$\begin{aligned}
&\limsup_{\sigma \to \tau} \|(T_r(t,\tau) - T_r(t,\sigma))AP(\tau)x_0\|_{-1} \\
&\le \|T_r(t,\tau)AP(\tau)(x_0 - x_n)\|_{-1} \\
&\quad + \limsup_{\sigma \to \tau} \Big(\|(T_r(t,\tau) - T_r(t,\sigma))AP(s)x_n\|_{-1} + \|T_r(t,\sigma)AP(\tau)(x_n - x_0)\|_{-1} \Big) \\
&\le 2c\|x_0 - x_n\| + \limsup_{\sigma \to \tau} \|(T_r(t,\tau) - T_r(t,\sigma))AP(\tau)x_n\|_{-1} = 2c\|x_0 - x_n\| .
\end{aligned}$$

Hence, $\|(T_r(t,\tau) - T_r(t,\sigma))AP(\tau)x_0\|_{-1} \to 0$ as $\sigma \to \tau$. We finally estimate

$$\begin{aligned}
&\|T_r(t,\tau)AP(\tau)x_0 - T_r(t,\sigma)AP(\sigma)x_0\|_{-1} \\
&\qquad \leq \|(T_r(t,\tau) - T_r(t,\sigma))AP(\tau)x_0\|_{-1} + \|T_r(t,\sigma)A(P(\tau)x_0 - P(\sigma)x_0)\|_{-1}.
\end{aligned}$$

Since the operators $T_r(t,\sigma)A : X \to X_{-1}$ are uniformly bounded, both terms on the right hand side tend to 0 as $\sigma \to \tau$. □

Proposition 2.8. *In addition to the standing assumptions of this section, we suppose that $P(\cdot)z \in C^2(J,X)$ for every $z \in X$. Let $\mathbb{T}_l$ and $\mathbb{T}_r$ be the evolution families from Propositions* 2.3 *and* 2.7*, respectively. Then the following statements hold:*

(a) *The operators $AP(t)$, with $t \in J$, generate the evolution family $\mathbb{T}_r$. The map*

$$(t,\tau) \mapsto AP(t)\mathbb{T}_r(t,\tau)[I - AP(\tau)]^{-1}z \qquad (\textit{for } t,\tau \in J \ \ \textit{with} \ \ \tau \leq t)$$

is continuous in X, for every $z \in X$.

(b) *The operators $A^*P(t)^{-1}$, with $t \in J$, generate the backward evolution family $\{\mathbb{T}_l(t,\tau)^* \mid t,\tau \in J,\ t \geq \tau\}$. For every $z \in X$, the map*

$$(t,\tau) \mapsto A^*P(\tau)^{-1}\mathbb{T}_l(t,\tau)^*[I - A^*P(t)^{-1}]^{-1}z \qquad (\textit{for } t,\tau \in J \ \ \textit{with} \ \ \tau \leq t)$$

is continuous in X. The operators $\mathbb{T}_l(t,\tau)$ have locally uniformly bounded extensions to operators in $\mathcal{L}(X^\tau_{-1,l}, X^t_{-1,l})$ denoted by the same symbols, for all $t,\tau \in J$ with $\tau \leq t$.

Proof. (a) It suffices to prove assertion (a) for every compact interval $[a,b] \subset J$. Fix such an interval $[a,b]$. As in the proof of Proposition 2.3, we see that the operators $P(t)A$, $t \in [a,b]$, are stable in the sense of Kato [9] and they generate an evolution family $\tilde{\mathbb{T}}_l$ satisfying the assertions of Proposition 2.3 with $P(t)^{-1}$ replaced by $P(t)$. We have already encountered $\tilde{\mathbb{T}}_l$ in Remark 2.6. Since $\|\dot{P}(t)P(t)^{-1}\|$ is bounded on $[a,b]$ (see (2.6)), the family of perturbed operators

$$A_2(t) = P(t)A + \dot{P}(t)P(t)^{-1}, \qquad t \in [a,b],$$

with domains $\mathcal{D}(A)$, is also stable in the sense of Kato, according to Proposition 4.3.3 of [21]. Since $A_2(\cdot)z$ is continuously differentiable for each $z \in \mathcal{D}(A)$, the corollary to Theorem 4.4.2 in [21] implies that $A_2(\cdot)$ generates an evolution family $\mathbb{S}$ such that the map $(t,\tau) \mapsto A\mathbb{S}(t,\tau)z$ is continuous for $a \leq \tau \leq t \leq b$ and for each $z \in \mathcal{D}(A)$. Therefore the function $s \mapsto \tilde{\mathbb{T}}_l(t,s)\mathbb{S}(s,\tau)z$ is continuously differentiable with derivative $\tilde{\mathbb{T}}_l(t,s)\dot{P}(s)P(s)^{-1}\mathbb{S}(s,\tau)z$ for $s \in [\tau,t]$. By integration and approximation (using that $\mathcal{D}(A)$ is dense in X), we derive that for every $x_0 \in X$,

$$\mathbb{S}(t,\tau)x_0 = \tilde{\mathbb{T}}_l(t,\tau)x_0 + \int_\tau^t \tilde{\mathbb{T}}_l(t,s)\dot{P}(s)P(s)^{-1}\mathbb{S}(s,\tau)x_0 \, \mathrm{d}s$$

if $a \leq \tau \leq t \leq b$. Consequently, the operators $P(t)^{-1}\mathbb{S}(t,\tau)P(\tau)$ satisfy (2.10), so that

$$\mathbb{T}_r(t,\tau) = P(t)^{-1}\mathbb{S}(t,\tau)P(\tau), \qquad a \leq \tau \leq t \leq b,$$

due to the uniqueness statement in Proposition 2.7(a). On the other hand, it is easy to see that the evolution family $P(t)^{-1}\mathbb{S}(t,\tau)P(\tau)$, with $a \le \tau \le t \le b$, is generated by $AP(t)$, so that the first part of (a) is proved.

To prove the continuity of $AP(t)\mathbb{T}_r(t,\tau)[I - AP(\tau)]^{-1}$, we compute

$$\begin{aligned} AP(t)\mathbb{T}_r(t,\tau)[I - AP(\tau)]^{-1} &= A\mathbb{S}(t,\tau)(I-A)^{-1}(I-A)P(\tau)[I - AP(\tau)]^{-1} \\ &= A\mathbb{S}(t,\tau)(I-A)^{-1}\left[I + (P(\tau)-I)[I - AP(\tau)]^{-1}\right]. \end{aligned} \tag{2.12}$$

From what we have shown earlier in this proof, we know that the map $(t,\tau) \mapsto A\mathbb{S}(t,\tau)(I-A)^{-1}x$ is continuous in X for $a \le \tau \le t \le b$ and for each $x \in X$. Moreover, by the uniform boundedness theorem, the operators $A\mathbb{S}(t,\tau)(I-A)^{-1}$, where $a \le \tau \le t \le b$, are uniformly bounded. The second factor in (2.12), $[I + (P(\tau)-I)[I - AP(\tau)]^{-1}]$, is Lipschitz continuous in norm, according to Proposition 2.4. Combining these facts, the desired continuity result follows.

(b) It is easy to see that $P(\cdot)^{-1}z \in C^2(J,X)$ for every $z \in X$. As in the proof of assertion (a) one can show that the operators $A^*P(t)^{-1}$, $t \in J$, generate a backward evolution family $\hat{\mathbb{T}}$ with time interval J satisfying the continuity assertion in (b) with $\mathbb{T}_l(t,\tau)^*$ replaced by $\hat{\mathbb{T}}(t,\tau)$. Fix $t,\tau \in J$ with $t > \tau$, and take $w \in \mathcal{D}(A)$ and $z \in \mathcal{D}(A^*P(t)^{-1})$. Then we obtain

$$\frac{\partial}{\partial s}\langle w, \mathbb{T}_l(s,\tau)^*\hat{\mathbb{T}}(t,s)z\rangle = \frac{\partial}{\partial s}\langle \mathbb{T}_l(s,\tau)w, \hat{\mathbb{T}}(t,s)z\rangle$$

$$= \langle P(s)^{-1}A\mathbb{T}_l(s,\tau)w, \hat{\mathbb{T}}(t,s)z\rangle - \langle \mathbb{T}_l(s,\tau)w, A^*P(s)^{-1}\hat{\mathbb{T}}(t,s)z\rangle = 0,$$

for all $s \in [\tau,t]$. This identity implies that $\mathbb{T}_l(t,\tau)^* = \hat{\mathbb{T}}(t,\tau)$. The last assertion is a consequence of the other results in (b). □

3. The Lax-Phillips semigroup induced by a well-posed system and its time-varying generalization

Starting from an arbitrary (time-invariant) well-posed linear system Σ^i, it is possible to define a strongly continuous semigroup which resembles those encountered in the scattering theory of Lax and Phillips [11], and which contains all the information about Σ^i. We recall this construction from Staffans and Weiss [20, Section 6]. Afterwards, we give the precise definition of a time-varying well-posed system and we construct a Lax-Phillips type evolution family induced by such a system.

Let H be a Hilbert space. For any interval J we regard $L^2(J,H)$ as a subspace of $L^2(\mathbb{R},H)$, by extending functions to be zero outside J. We denote by $\mathbf{P}_+$ the projection from $L^2(\mathbb{R},H)$ to $L^2(\mathbb{R}_+,H)$ (by truncation) and similarly, $\mathbf{P}_-$ is the projection from $L^2(\mathbb{R},H)$ to $L^2(\mathbb{R}_-,H)$. For every $t \in \mathbb{R}$, we designate by $\mathcal{S}_t$ the operator of left shift by t on $L^2(\mathbb{R},H)$, i.e., $\mathcal{S}_t f = f(\cdot + t)$. We further set $\mathcal{S}_t^{\pm} = \mathbf{P}_{\pm}\mathcal{S}_t$. For any open set $\Omega \subset \mathbb{R}^n$, we denote by $\mathcal{H}^1(\Omega,H)$ the space of all functions $f \in L^2(\Omega,H)$ whose distributional derivatives $\frac{\partial}{\partial x_k}f$ belong to $L^2(\Omega,H)$ $(1 \le k \le n)$.

Like in Section 1, we assume that Σ^i is a well-posed linear system with input space U, state space X and output space Y. The operators $A, B, \overline{C}$ and D are as in (1.1) and the families of operators $\mathbb{T}^i$, Φ^i, Ψ^i and $\mathbb{F}^i$ are as in (1.3).

Proposition 3.1. *Let $\mathcal{Y} = L^2((-\infty, 0], Y)$ and $\mathcal{U} = L^2([0,\infty), U)$. For all $t \geq 0$ we define on $\mathcal{Y} \times X \times \mathcal{U}$ the operator $\mathfrak{T}^i_t$ by*

$$\mathfrak{T}^i_t = \begin{bmatrix} \mathcal{S}^-_t & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \mathcal{S}^+_t \end{bmatrix} \begin{bmatrix} I & \Psi^i_t & \mathbb{F}^i_t \\ 0 & \mathbb{T}^i_t & \Phi^i_t \\ 0 & 0 & I \end{bmatrix} .$$

Then $\mathfrak{T}^i = (\mathfrak{T}^i_t)_{t\geq 0}$ is a strongly continuous semigroup. We denote the generator of $\mathfrak{T}^i$ by $\mathfrak{A}$. The domain of $\mathfrak{A}$ is given by

$$\mathcal{D}(\mathfrak{A}) = \left\{ \begin{bmatrix} y_0 \\ x_0 \\ u_0 \end{bmatrix} \in \mathcal{H}^1((-\infty,0), Y) \times X \times \mathcal{H}^1((0,\infty), U) \;\middle|\; \begin{matrix} Ax_0 + Bu_0(0) \in X\,, \\ y_0(0) = \overline{C}x_0 + Du_0(0) \end{matrix} \right\}$$

and on $\mathcal{D}(\mathfrak{A})$ the operator $\mathfrak{A}$ is given by

$$\mathfrak{A} \begin{bmatrix} y_0 \\ x_0 \\ u_0 \end{bmatrix} = \begin{bmatrix} y_0' \\ Ax_0 + Bu_0(0) \\ u_0' \end{bmatrix} . \tag{3.1}$$

For the proof we refer to [20]. $\mathfrak{T}^i$ is called the *Lax-Phillips semigroup* induced by Σ^i. The intuitive interpretation of the space $\mathcal{Y} \times X \times \mathcal{U}$ and of $\mathfrak{T}^i_t$ acting on it is that the first component is the past output, the second component is the current state, while the third component is the future input. Indeed, let $y_0 \in \mathcal{Y}$, $x_0 \in X$, $u_0 \in \mathcal{U}$, and define

$$\begin{bmatrix} y_t \\ x_t \\ u_t \end{bmatrix} = \mathfrak{T}_t \begin{bmatrix} y_0 \\ x_0 \\ u_0 \end{bmatrix} \qquad \text{for} \quad t \geq 0\,.$$

Let x and y be the state trajectory and the output function of Σ^i, i.e., the solutions of (1.1) corresponding to the initial state $x(0) = x_0$ and the input function $u = u_0$. We extend y to $\mathbb{R}$ by putting $y(t) = y_0(t)$ for $t \leq 0$. Then

$$y(t-\theta) = y_t(-\theta)\,, \quad x(t) = x_t\,, \quad u(t+\theta) = u_t(\theta)$$

for all $t \geq 0$ and for almost every $\theta \geq 0$.

Assume that x_0 and u_0 satisfy

$$u_0 \in \mathcal{H}^1((0,\infty), U) \quad \text{and} \quad Ax_0 + Bu_0(0) \in X\,.$$

Then for every $t \geq 0$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \left\| \begin{bmatrix} y_t \\ x_t \\ u_t \end{bmatrix} \right\|^2 = \|y(t)\|^2 + \frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|^2 - \|u(t)\|^2\,.$$

Thus, Σ^i is scattering passive if and only if $\mathfrak{T}^i$ is a contraction semigroup. Similarly, Σ^i is scattering energy preserving if and only if $\mathfrak{T}^i$ is an isometric semigroup.

In the sequel, we extend the concept of Lax-Phillips semigroup to the time-varying case. For this, first we give the definition of a time-varying well-posed system. Let J be a closed interval of non-zero length. For every $t, \tau \in \mathbb{R}$ with $\tau \leq t$, we denote by $\mathbf{P}(t,\tau)$ the truncation operator to the interval $[\tau, t]$, which can be applied to any function u defined on an interval containing $[\tau, t]$. (The function $\mathbf{P}(t,\tau)u$ is considered to be zero outside the interval $[\tau, t]$.)

Definition 3.2. A *time-varying well-posed system* Σ with input space U, state space X, output space Y and time interval J consists of four families of operators $\mathbb{T}$, Φ, Ψ and $\mathbb{F}$, parametrized by pairs $(t,\tau) \in J^2$ with $\tau \leq t$, with the following properties:

- $\mathbb{T}$ is an evolution family on X with time interval J (as defined in Section 2).
- The operators $\Phi(t,\tau) : L^2(J,U) \to X$, $\Psi(t,\tau) : X \to L^2(J,Y)$, $\mathbb{F}(t,\tau) : L^2(J,U) \to L^2(J,Y)$ are locally uniformly bounded.
- For all $t, \tau \in J$ with $\tau \leq t$, we have

$$\Phi(t,\tau) = \Phi(t,\tau)\mathbf{P}(t,\tau), \tag{3.2}$$
$$\Psi(t,\tau) = \mathbf{P}(t,\tau)\Psi(t,\tau), \tag{3.3}$$
$$\mathbb{F}(t,\tau) = \mathbf{P}(t,\tau)\mathbb{F}(t,\tau) = \mathbb{F}(t,\tau)\mathbf{P}(t,\tau). \tag{3.4}$$

- For all $t, s, \tau \in J$ with $\tau \leq s \leq t$, all $u \in L^2(J,U)$ and all $x_0 \in X$, we have

$$\Phi(t,\tau)u = \Phi(t,s)u + \mathbb{T}(t,s)\Phi(s,\tau)u, \tag{3.5}$$
$$\Psi(t,\tau)x_0 = \Psi(t,s)\mathbb{T}(s,\tau)x_0 + \Psi(s,\tau)x_0, \tag{3.6}$$
$$\mathbb{F}(t,\tau)u = \mathbb{F}(t,s)u + \mathbb{F}(s,\tau)u + \Psi(t,s)\Phi(s,\tau)u. \tag{3.7}$$

This definition is slightly more general than the one in [16] since we have weakened the uniform boundedness assumptions. Note that the operators $\mathbb{T}(t,\tau)$ are also locally uniformly bounded due to their strong continuity.

Remark 3.3. Suppose that the operator families $\mathbb{T}$, Φ, Ψ and $\mathbb{F}$ in the above definition are such that the operators

$$\mathbb{T}^i_\tau = \mathbb{T}(\theta+\tau,\theta), \qquad \Phi^i_\tau = \Phi(\theta+\tau,\theta)\mathcal{S}_{-\theta},$$
$$\Psi^i_\tau = \mathcal{S}_\theta\Psi(\theta+\tau,\theta), \qquad \mathbb{F}^i_\tau = \mathcal{S}_\theta\mathbb{F}(\theta+\tau,\theta)\mathcal{S}_{-\theta},$$

are independent of θ, for all pairs $(\theta+\tau,\theta) \in J^2$ with $\tau \geq 0$. Then the families $\mathbb{T}^i$, Φ^i, Ψ^i and $\mathbb{F}^i$ can be extended to all $\tau \geq 0$ in a natural way (which we do not detail), and these extended families form a (time-invariant) well-posed linear system.

If $\mathbb{T}$, Φ, Ψ and $\mathbb{F}$ form a well-posed time-varying system, then $\mathbb{T}$ is strongly continuous by definition. We next want to show that also the families Φ, Ψ and $\mathbb{F}$

are strongly continuous as functions of (t,τ). To this aim, we first observe that the identities (3.2), (3.3) and (3.4) imply that

$$\Psi(t,t) = 0, \qquad \Phi(t,t) = 0 \qquad \text{and} \qquad \mathbb{F}(t,t) = 0 \qquad \text{for} \ \ t \in J. \tag{3.8}$$

Second, let $t, s, \tau \in J$ with $\tau \le s \le t$. Then the formulas (3.3) and (3.6), respectively, (3.4) and (3.7) yield

$$\Psi(s,\tau) = \mathbf{P}(s,\tau)\Psi(t,\tau) \qquad \text{and} \qquad \mathbb{F}(s,\tau) = \mathbf{P}(s,\tau)\mathbb{F}(t,\tau). \tag{3.9}$$

In the next lemma we prove the continuity of $\Psi(t,\tau)$ at points of the form (r,r).

Lemma 3.4. *Let Σ be a time-varying well-posed system with time interval J. Let $x_0 \in X$ and $r \in J$. Then*

$$\lim_{\substack{(t,\tau)\to(r,r)\\ (t,\tau)\in J^2,\ t\ge\tau}} \Psi(t,\tau)x_0 = 0.$$

Proof. Fix $r \in J$. For any $t, \tau, \theta \in J$ with $\theta \le \tau \le t$, we rewrite (3.6) in the form

$$\Psi(t,\tau) = \Psi(t,\theta) - \Psi(\tau,\theta) - \Psi(t,\tau)[\mathbb{T}(\tau,\theta) - I]. \tag{3.10}$$

Due to the local uniform boundedness of Ψ, there is a constant $k > 0$ such that

$$\|\Psi(t,\tau)\| \le k \qquad \forall\, t,\tau \in [r-1, r+1] \cap J.$$

Take $\varepsilon > 0$. Since the map $(t,\tau) \mapsto \mathbb{T}(t,\tau)x_0$ is continuous and $\mathbb{T}(r,r)x_0 = x_0$, there exists a $\delta \in (0,1)$ such that

$$\|\mathbb{T}(\tau, r-\delta)x_0 - x_0\| \le \frac{\varepsilon}{2k} \qquad \forall\, \tau \in [r-\delta, r+\delta] \cap J.$$

Since $f = \Psi(r+\delta, r-\delta)x_0 \in L^2([r-\delta, r+\delta], Y)$, there exists a $\gamma > 0$ such that

$$t,\tau \in [r-\delta, r+\delta],\ \ t \ge \tau,\ \ t-\tau \le \gamma \implies \int_\tau^t |f(s)|^2 \,\mathrm{d}s \le \frac{\varepsilon^2}{4}. \tag{3.11}$$

For all (t,τ) as in (3.11), we can thus deduce from (3.9) that

$$\|\Psi(t, r-\delta)x_0 - \Psi(\tau, r-\delta)x_0\| \le \frac{\varepsilon}{2}.$$

Equation (3.10) with $\theta = r - \delta$ then implies that $\|\Psi(t,\tau)x_0\| \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ for all pairs (t,τ) as in (3.11). This fact shows the asserted convergence. $\square$

Proposition 3.5. *Suppose that the operator families $\mathbb{T}$, Φ, Ψ and $\mathbb{F}$ form a time-varying well-posed system Σ with time interval J. Then the operators $\Psi(t,\tau)$, $\Phi(t,\tau)$ and $\mathbb{F}(t,\tau)$ are strongly continuous with respect to $(t,\tau) \in J^2$ with $\tau \le t$.*

Proof. The strong continuity of the map $(t, \tau) \mapsto \Phi(t, \tau)$ was proved in [16, Proposition 3.5]. (The proof given there also works in the present, slightly more general situation.) In Lemma 3.4 we have shown that $\Psi(t, \tau)$ is strongly continuous as $(t, \tau) \to (r, r)$, where $r \in J$. Due to (3.9), the map $t \mapsto \Psi(t, \tau)$ is strongly continuous. Take $t, \tau, \theta \in J$ with $\theta \leq \tau \leq t$. Equation (3.10) implies that

$$\|[\Psi(t, \theta) - \Psi(t, \tau)]x_0\| \leq \|\Psi(t, \tau)\| \cdot \|[\mathbb{T}(\tau, \theta) - I]x_0\| + \|\Psi(\tau, \theta)x_0\|$$

for all $x_0 \in X$. We can think of the situations when τ is fixed and $\theta \to \tau$, or when θ is fixed and $\tau \to \theta$. In both cases, the above estimate shows that the strong continuity of the map $\tau \mapsto \Psi(t, \tau)$ because of the local uniform boundednes of Ψ, the strong continuity of $\mathbb{T}$ and Lemma 3.4.

Now take $(t_0, \tau_0) \in J^2$ with $\tau_0 < t_0$ and let (t_n, τ_n) be a sequence in J^2 such that $(t_n, \tau_n) \to (t_0, \tau_0)$. Denote $s = \frac{1}{2}(t_0 + \tau_0)$. For large n we have $\tau_n < s < t_n$. From (3.6) and the strong continuity of $t \mapsto \Psi(t, \tau)$ and $\tau \mapsto \Psi(t, \tau)$, we then deduce that $\Psi(t_n, \tau_n)x_0 \to \Psi(t_0, \tau_0)x_0$. Thus, the family Ψ is strongly continuous.

Let $u \in L^2(J, U)$ and let $a, b \in J$ with $a < b$. For any $t, \tau \in [a, b]$ with $\tau \leq t$, the local uniform boundedness of $\mathbb{F}$ yields that

$$\|\mathbb{F}(t, \tau)u\| \leq c\, \|\mathbf{P}(t, \tau)u\|, \tag{3.12}$$

where $c \geq 0$ is independent of t, τ and u. Arguing as in (3.11), we deduce from (3.12) that the map $(t, \tau) \mapsto \mathbb{F}(t, \tau)$, defined for $(t, \tau) \in [a, b]^2$ with $t \geq \tau$, is strongly continuous at points (r, r) with $r \in [a, b]$. Since a and b were arbitrary, it follows that $\mathbb{F}(t, \tau)$ is strongly continuous at (r, r) for every $r \in J$.

The map $t \mapsto \mathbb{F}(t, \tau)$ is strongly continuous by (3.9). We want to show the strong continuity of the map $\tau \mapsto \mathbb{F}(t, \tau)$. To this end, we take $\tau, s, t \in J$ with $\tau \leq s \leq t$. Then formula (3.7) implies that

$$\|\mathbb{F}(t, s)u - \mathbb{F}(t, \tau)u\| \leq \|\mathbb{F}(s, \tau)u\| + \|\Psi(t, s)\Phi(s, \tau)u\|.$$

This estimate combined with the strong continuity of Φ and $\mathbb{F}$ at (r, r) shows the strong continuity of $\tau \mapsto \mathbb{F}(t, \tau)$. Now the strong continuity of $(t, \tau) \mapsto \mathbb{F}(t, \tau)$ at points (t_0, τ_0) with $t_0 > \tau_0$ follows from (3.7) and the results established so far. □

Remark 3.6. Recall the notation $J_\tau = J \cap [\tau, \infty)$. If $\tau \in J$ with $\tau \neq \max J$, $x_0 \in X$ and $u \in L^2(J, U)$, then the funcion $x \in C(J_\tau, X)$ defined by $x(t) = \mathbb{T}(t, \tau)x_0 + \Phi(t, \tau)u$ is called the *state trajectory* of Σ corresponding to the initial time τ, the initial state x_0 and the input u. According to (3.9) there exists a unique $y \in L^2_{\text{loc}}(J_\tau, Y)$ such that

$$\mathbf{P}(t, \tau)y = \Psi(t, \tau)x_0 + \mathbb{F}(t, \tau)u$$

for every $t \in J_\tau$. This y is called the *output function* of Σ corresponding to the initial time τ, the initial state x_0 and the input u.

Proposition 3.7. *Let $\mathbb{T}$, Φ, Ψ and $\mathbb{F}$ be families of operators, indexed by pairs $(t,\tau) \in J^2$ with $\tau \le t$, such that*

$$\begin{aligned} &\mathbb{T}(t,\tau) : X \to X\,, && \Phi(t,\tau) : L^2(J,U) \to X\,, \\ &\Psi(t,\tau) : X \to L^2(J,Y)\,, && \mathbb{F}(t,\tau) : L^2(J,U) \to L^2(J,Y)\,, \end{aligned}$$

and (3.2)–(3.4) *hold. Then these families form a time-varying well-posed system with time interval J if and only if the family $\mathfrak{T}$ of the operators*

$$\mathfrak{T}(t,\tau) = \begin{bmatrix} \mathcal{S}^-_{t-\tau} & \mathcal{S}^-_t \Psi(t,\tau) & \mathcal{S}^-_t \mathbb{F}(t,\tau)\mathcal{S}_{-\tau} \\ 0 & \mathbb{T}(t,\tau) & \Phi(t,\tau)\mathcal{S}_{-\tau} \\ 0 & 0 & \mathcal{S}^+_{t-\tau} \end{bmatrix}, \qquad (t,\tau) \in J^2,\ \tau \le t, \tag{3.13}$$

is an evolution family on $\mathcal{Y} \times X \times \mathcal{U}$ with time interval J.

We call $\mathfrak{T}$ the *Lax-Phillips evolution family* induced by Σ. Note that if (3.3) and (3.4) hold, then in (3.13) the operators $\mathcal{S}^-_{t-\tau}$ and $\mathcal{S}^-_t$ could be replaced with $\mathcal{S}_{t-\tau}$ and $\mathcal{S}_t$ without changing the operator $\mathfrak{T}(t,\tau)$. (Recall that we identified a function defined on $\mathbb{R}_-$ with its extension by 0 to $\mathbb{R}$.)

Proof. Suppose that the four families form a time-varying well-posed system. Let $(t,\tau) \in J^2$ with $t \ge \tau$. Then the operator $\mathfrak{T}(t,\tau)$ defined in (3.13) is a bounded linear map on $\mathcal{Y} \times X \times \mathcal{U}$ satisfying $\mathfrak{T}(t,t) = I$. Proposition 3.5 further implies the strong continuity of $(t,\tau) \mapsto \mathfrak{T}(t,\tau)$. We show that property (a) from Definition 2.1 holds for the family $\mathfrak{T}$. Take $t,s,\tau \in J$ with $t \ge s \ge \tau$. Employing property (a) from Definition 2.1 for the evolution family $\mathbb{T}$, we obtain

$$\mathfrak{T}(t,s)\mathfrak{T}(s,\tau) = \begin{bmatrix} \mathcal{S}^-_{t-\tau} & T_{12} & T_{13} \\ 0 & \mathbb{T}(t,\tau) & T_{23} \\ 0 & 0 & \mathcal{S}^+_{t-\tau} \end{bmatrix}, \tag{3.14}$$

where, using (3.3) and (3.6) for T_{12}, (3.4) and (3.7) for T_{13}, (3.2) and (3.5) for T_{23},

$$\begin{aligned} T_{12} &= \mathcal{S}^-_{t-s}\mathcal{S}^-_s \Psi(s,\tau) + \mathcal{S}^-_t \Psi(t,s)\mathbb{T}(s,\tau) && (3.15)\\ &= \mathcal{S}^-_t [\Psi(s,\tau) + \Psi(t,s)\mathbb{T}(s,\tau)] \\ &= \mathcal{S}^-_t \Psi(t,\tau)\,, \\ T_{13} &= \mathcal{S}^-_{t-s}\mathcal{S}^-_s \mathbb{F}(s,\tau)\mathcal{S}_{-\tau} + \mathcal{S}^-_t \Psi(t,s)\Phi(s,\tau)\mathcal{S}_{-\tau} + \mathcal{S}^-_t \mathbb{F}(t,s)\mathcal{S}_{-s}\mathcal{S}^+_{s-\tau} && (3.16)\\ &= \mathcal{S}^-_t [\mathbb{F}(s,\tau) + \Psi(t,s)\Phi(s,\tau) + \mathbb{F}(t,s)]\mathcal{S}_{-\tau} \\ &= \mathcal{S}^-_t \mathbb{F}(t,\tau)\mathcal{S}_{-\tau}\,, \\ T_{23} &= \mathbb{T}(t,s)\Phi(s,\tau)\mathcal{S}_{-\tau} + \Phi(t,s)\mathcal{S}_{-s}\mathcal{S}^+_{s-\tau} && (3.17)\\ &= [\mathbb{T}(t,s)\Phi(s,\tau) + \Phi(t,s)]\mathcal{S}_{-\tau} \\ &= \Phi(t,\tau)\mathcal{S}_{-\tau}\,. \end{aligned}$$

Substituting these expressions for T_{12}, T_{13} and T_{23} into (3.14), we conclude that $\mathfrak{T}(t,s)\mathfrak{T}(s,\tau) = \mathfrak{T}(t,\tau)$, as expected.

Conversely, suppose that $\mathfrak{T}$ is an evolution family on $\mathcal{Y} \times X \times \mathcal{U}$. This fact easily implies that $\mathbb{T}(t,t) = I$ and $(t,\tau) \mapsto \mathbb{T}(t,\tau)$ is strongly continuous for $(t,\tau) \in J^2$ with $t \geq \tau$. Since $\mathfrak{T}(t,\tau)$ is locally uniformly bounded, we can deduce the locally uniform boundedness of the operators $\Phi(t,\tau)$, $\Psi(t,\tau)$ and $\mathbb{F}(t,\tau)$ using (3.3) and (3.4). Take $t, s, \tau \in J$ with $t \geq s \geq \tau$. By property (a) of Definition 2.1 for $\mathfrak{T}$, the expression in (3.13) is equal to

$$\mathfrak{T}(t,s)\mathfrak{T}(s,\tau) = \begin{bmatrix} \mathcal{S}^{-}_{t-\tau} & T_{12} & T_{13} \\ 0 & \mathbb{T}(t,s)\mathbb{T}(s,\tau) & T_{23} \\ 0 & 0 & \mathcal{S}^{+}_{t-\tau} \end{bmatrix},$$

where T_{12}, T_{13}, and T_{23} are given by (3.15), (3.16) and (3.17), respectively. Employing (3.2)–(3.4) and arguing as in the first part of the proof, we can then show that $\mathbb{T}$ is an evolution family and that (3.5)–(3.7) hold. □

4. Time-varying multiplicative perturbations of well-posed systems

Notation and standing assumptions. In this section, Σ^i is a time-invariant scattering passive system with input space U, state space X and output space Y. This system can be described in terms of the operators A, B, $\overline{C}$ and D as in (1.1). Recall from Section 1 that $Z = \mathcal{D}(A) + (\beta I - A)^{-1}BU$, which is a Hilbert space with a certain norm. (Thus, if $x_0 \in X$ and $v \in U$ are such that $Ax_0 + Bv \in X$, then $x_0 \in Z$.) We know from the theory of well-posed linear systems (see [20]) that $\overline{C} \in \mathcal{L}(Z, Y)$. J is a closed interval of non-zero length and J^0 is the interior of J. The function $P : J \to \mathcal{L}(X)$ satisfies the assumptions in (2.1). For $\tau \in J$, we set $J_\tau = J \cap [\tau, \infty)$. As in the previous section, $\mathbf{P}(t,\tau)$ is the truncation operator to the interval $[\tau, t]$. We denote by Π_k the projection onto the k-th factor of the product of n Hilbert spaces.

We shall construct the time-varying well-posed systems Σ_l and Σ_r corresponding to the equations (1.4) and (1.5), respectively. The families $\mathbb{T}_j$, Φ_j, Ψ_j and $\mathbb{F}_j$ forming the systems Σ_j $(j = l, r)$ provide the state trajectory and the output function of the system starting from any initial time $\tau \in J$ (if the initial state $x(\tau)$ and the input function u are given). Therefore, we expect to find solutions of (1.4) and (1.5) on intervals of the type J_τ. These solutions do not need to have an extension to J. Similarly to the time-invariant case, recalled in Section 1, we only expect to have classical solutions on J_τ for pairs $(x(\tau), u)$ in a dense subspace of $X \times L^2(J, U)$. These classical solutions will then determine the operator families of the systems Σ_l or Σ_r by continuous extension.

We start by looking at the first equation in (1.4), which is

$$P(t)\dot{x}(t) = Ax(t) + Bu(t). \tag{4.1}$$

A *classical solution* of (4.1) on an interval J_τ is a function

$$x \in C^1(J_\tau, X)$$

such that (4.1) holds for all $t \in J_\tau$. We claim that for such an x that corresponds to an input function $u \in C(J,U)$, we have $x \in C(J_\tau, Z)$. To see this, choose $\beta \in \rho(A)$, rewrite (4.1) as $(\beta I - A)x(t) = \beta x(t) - P(t)\dot{x}(t) + Bu(t)$ and apply $(\beta I - A)^{-1}$ to both sides, using that $(\beta I - A)^{-1}B \in \mathcal{L}(X,Z)$. Thus the right-hand side of the second equation in (1.4) is well defined and determines the output $y \in C(J_\tau, Y)$.

Next we introduce the similar concept for the system of equations (1.5). We repeat the first equation of (1.5) below:

$$\dot{x}(t) = AP(t)x(t) + Bu(t). \tag{4.2}$$

A *classical solution* of (4.2) on J_τ is a function

$$x \in C^1(J_\tau, X)$$

such that (4.2) holds for all $t \in J_\tau$. By a similar argument as above, if x is such a solution corresponding to an input $u \in C(J,U)$, then $P(\cdot)x(\cdot) \in C(J_\tau, Z)$ and hence the second equation in (1.5) determines $y \in C(J_\tau, Y)$.

In order to construct the systems Σ_l and Σ_r, we use the generator $\mathfrak{A}$ of the Lax-Phillips semigroup on $\mathcal{Y} \times X \times \mathcal{U}$, introduced in Proposition 3.1, and the Lax-Phillips evolution family, introduced in Proposition 3.7. We further define

$$\mathfrak{P}(t) = \begin{bmatrix} I & 0 & 0 \\ 0 & P(t) & 0 \\ 0 & 0 & I \end{bmatrix} \qquad \text{for every} \quad t \in J. \tag{4.3}$$

On $\mathcal{Y} \times X \times \mathcal{U}$ we consider for each $t \in J$ the operators

$$\mathfrak{P}(t)^{-1}\mathfrak{A} \begin{bmatrix} y_0 \\ x_0 \\ u_0 \end{bmatrix} = \begin{bmatrix} y_0' \\ P(t)^{-1}(Ax_0 + Bu_0(0)) \\ u_0' \end{bmatrix},$$

$$\mathfrak{A}\mathfrak{P}(t) \begin{bmatrix} y_0 \\ x_0 \\ u_0 \end{bmatrix} = \begin{bmatrix} y_0' \\ AP(t)x_0 + Bu_0(0) \\ u_0' \end{bmatrix},$$

defined on the domains $\mathcal{D}(\mathfrak{P}(t)^{-1}\mathfrak{A}(t)) = \mathcal{D}(\mathfrak{A})$ and

$$\mathcal{D}(\mathfrak{A}\mathfrak{P}(t)) = \{ \left[\begin{smallmatrix} y_0 \\ x_0 \\ u_0 \end{smallmatrix}\right] \in \mathcal{H}^1((-\infty,0),Y) \times X \times \mathcal{H}^1((0,\infty),U) \mid$$
$$AP(t)x_0 + Bu_0(0) \in X, \quad y_0(0) = \overline{C}P(t)x_0 + Du_0(0)\}.$$

For each $\tau \in J$ we set $V(\tau) = \{(x_0, u) \in X \times \mathcal{H}^1(J^0, U) \mid Ax_0 + Bu(\tau) \in X\}$.

Theorem 4.1. *(a) There exists a time-varying well-posed system Σ_l with time interval J consisting of operator families $\mathbb{T}_l$, Φ_l, Ψ_l and $\mathbb{F}_l$ such that the Lax-Phillips evolution family $\mathfrak{T}_l$ induced by Σ_l is generated by $\mathfrak{P}(\cdot)^{-1}\mathfrak{A}$. Let $\tau \in J$ with $\tau \neq \max J$ and $(x(\tau), u) \in V(\tau)$. Then (4.1) has a unique classical solution given by*

$$x(t) = \mathbb{T}_l(t,\tau)x(\tau) + \Phi_l(t,\tau)u \qquad \text{for all} \quad t \in J_\tau,$$

and we have $x \in C(J_\tau, Z)$*. Moreover, the function* y *from* (1.4) *satisfies*

$$\mathbf{P}(t,\tau)y = \Psi_l(t,\tau)x(\tau) + \mathbb{F}_l(t,\tau)u \quad \text{for all} \ \ t \in J_\tau,$$

so that it is the output function of Σ_l *corresponding to* τ*,* $x(\tau)$ *and* u *(see Remark* 3.6*). We have* $\mathbf{P}(t,\tau)y \in \mathcal{H}^1((\tau,t),Y)$ *for each* $t \in J_\tau$*. The balance inequality*

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle P(t)x(t), x(t)\rangle \le \|u(t)\|^2 - \|y(t)\|^2 + \langle \dot P(t)x(t), x(t)\rangle \tag{4.4}$$

holds for every $t \in J_\tau$*. If the original time-invariant system* Σ^i *is energy preserving, then we have equality in* (4.4)*.*

(b) Assume in addition that $P(\cdot)z \in C^2(J,X)$ *for every* $z \in X$*. Then there exists a time-varying well-posed system* Σ_r *with time interval* J *consisting of operator families* $\mathbb{T}_r$*,* Φ_r*,* Ψ_r *and* $\mathbb{F}_r$ *such that the Lax-Phillips evolution family* $\mathfrak{T}_r$ *induced by* Σ_r *is generated by* $\mathfrak{A}\mathfrak{P}(\cdot)$*. Let* $\tau \in J$ *with* $\tau \neq \max J$ *and* $(P(\tau)x(\tau), u) \in V(\tau)$*. Then* (4.2) *has a unique classical solution given by*

$$x(t) = \mathbb{T}_r(t,\tau)x(\tau) + \Phi_r(t,\tau)u \quad \text{for all} \ \ t \in J_\tau,$$

and we have $P(\cdot)x(\cdot) \in C(J_\tau, Z)$*. Moreover, the function* y *from* (1.5) *satisfies*

$$\mathbf{P}(t,\tau)y = \Psi_r(t,\tau)x(\tau) + \mathbb{F}_r(t,\tau)u \quad \text{for all} \ \ t \in J_\tau,$$

so that it is the output function of Σ_r *corresponding to* τ*,* $x(\tau)$ *and* u*. We have* $\mathbf{P}(t,\tau)y \in \mathcal{H}^1((\tau,t),Y)$ *for each* $t \in J_\tau$*. The balance inequality* (4.4) *holds for every* $t \in J_\tau$*. If* Σ^i *is energy preserving, then we have equality in* (4.4)*.*

Proof. We have seen in Section 3 that the operator $\mathfrak{A}$ is m-dissipative. Clearly, $\mathfrak{P}$ satisfies (2.1), and $\mathfrak{P}(\cdot)w_0 \in C^2(J, \mathcal{Y} \times X \times \mathcal{U})$ for every $w_0 \in \mathcal{Y} \times X \times \mathcal{U}$ in the case of assertion (b). Propositions 2.3 and 2.8 thus show that $\mathfrak{P}(\cdot)^{-1}\mathfrak{A}$ and $\mathfrak{A}\mathfrak{P}(\cdot)$ generate evolution families $\mathfrak{T}_l$ and $\mathfrak{T}_r$ on $\mathcal{Y} \times X \times \mathcal{U}$, respectively.

The second step is to show that for suitable $x(\tau)$ and u, the equations (4.1) and (4.2) have classical solutions, and the corresponding output functions y from (1.4) and (1.5) are indeed well defined and locally in $\mathcal{H}^1$. For this, we take $\tau \in J$ such that $\tau \neq \max J$ and we take $x(\tau) \in X$ and $u \in \mathcal{H}^1(J^0, U)$ such that they satisfy the conditions in the theorem: if $j = l$ then $(x(\tau), u) \in V(\tau)$, while if $j = r$ then $(P(\tau)x(\tau), u) \in V(\tau)$. Take $u_\tau \in \mathcal{H}^1((0,\infty), U)$ such that $u_\tau(t-\tau) = u(t)$ for all $t \in J_\tau$. Finally, take $y_\tau \in \mathcal{H}^1((-\infty,0), Y)$ such that $y_\tau(0) = \overline{C}x(\tau) + Du(\tau)$ if $j = l$, while $y_\tau(0) = \overline{C}P(\tau)x(\tau) + Du(\tau)$ if $j = r$. Then it is easy to see that $w(\tau) = \begin{bmatrix} y_\tau \\ x(\tau) \\ u_\tau \end{bmatrix} \in \mathcal{D}(\mathfrak{A})$ if $j = l$ and $w(\tau) \in \mathcal{D}(\mathfrak{A}\mathfrak{P}(\tau))$ if $j = r$. Define

$$w(t) = \begin{bmatrix} y_t \\ x(t) \\ u_t \end{bmatrix} = \mathfrak{T}_j(t,\tau)w(\tau) \tag{4.5}$$

for $t \in J_\tau$ and $j = l, r$. Temporarily, we set $\mathfrak{A}(t) = \mathfrak{P}(t)^{-1}\mathfrak{A}$ if $j = l$ and $\mathfrak{A}(t) = \mathfrak{A}\mathfrak{P}(t)$ if $j = r$. According to what we have shown in the first step, $\mathfrak{A}(\cdot)$ generates $\mathfrak{T}_j$. Observe that $w(\tau) \in \mathcal{D}(\mathfrak{A}(\tau))$ and this implies

$$w \in C^1(J_\tau, \mathcal{Y} \times X \times \mathcal{U}), \qquad w(t) \in \mathcal{D}(\mathfrak{A}(t)), \qquad \dot w(t) = \mathfrak{A}(t)w(t) \tag{4.6}$$

for $t \in J_\tau$. In particular, $x(\cdot) \in C^1(J_\tau, X)$ and the map $t \mapsto y_t$ is in $C^1(J_\tau, \mathcal{Y})$. We set $y(t) = y_t(0)$ and $\hat{y}(t,\theta) = y_t(\theta)$ for $t \in J_\tau$ and $\theta \le 0$. The relations (4.6) imply that $\hat{y} \in \mathcal{H}^1(J_\tau^0 \times (-\infty, 0), Y)$ (J_τ^0 is the interior of J_τ) and $\frac{\partial}{\partial t}\hat{y} = \frac{\partial}{\partial \theta}\hat{y}$, so that $\hat{y}$ is constant along the lines $\{(t+r, \theta - r) \mid r \in \mathbb{R}\} \cap (J_\tau \times \mathbb{R}_-)$. Therefore

$$y_t(\theta) = \hat{y}(t,\theta) = \begin{cases} \hat{y}(\tau, \theta + t - \tau) = y_\tau(\theta + t - \tau), & \theta \le \tau - t, \\ \hat{y}(t+\theta, 0) = y(t+\theta), & \tau - t \le \theta \le 0, \end{cases} \tag{4.7}$$

where $t \in J_\tau$. Similarly, we see that

$$u_t(\theta) = u_\tau(\theta + t - \tau) \qquad \text{for } \; t \in J_\tau \text{ and } \theta \ge 0. \tag{4.8}$$

which implies that $u_t(0) = u(t)$. So (4.6) implies that x, y and u satisfy (1.4) if $j = l$ and (1.5) if $j = r$. From the differential equations in (1.4) and (1.5) we conclude that $x(\cdot) \in C(J_\tau, Z)$ and $P(\cdot)x(\cdot) \in C(J_\tau, Z)$, respectively. The middle part of (4.6) with (4.7) imply that $\mathbf{P}(t,\tau)y \in \mathcal{H}^1((\tau,t), Y)$ for each $t \in J_\tau$.

The third step is to prove the inequality (4.4). First we consider $j = l$. Let τ, $x(\tau)$, u and y be as in the second step for $j = l$ and let $t \in J_\tau$. From (4.1) we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\langle P(t)x(t), x(t)\rangle &= 2\,\mathrm{Re}\,\langle P(t)x(t), \dot{x}(t)\rangle + \langle \dot{P}(t)x(t), x(t)\rangle \\ &= 2\,\mathrm{Re}\,\langle x(t), Ax(t) + Bu(t)\rangle + \langle \dot{P}(t)x(t), x(t)\rangle\,. \end{aligned}$$

Due to Proposition 5.2 of [12] and the second line in (1.4), we have

$$2\,\mathrm{Re}\,\langle x(t), Ax(t) + Bu(t)\rangle \le \|u(t)\|^2 - \|\overline{C}x(t) + Du(t)\|^2 = \|u(t)\|^2 - \|y(t)\|^2,$$

so that (4.4) follows for Σ_l. In the case that Σ^i is energy preserving, we obtain the equality in (4.4) in the same way using Theorem 3.2 of [12] (instead of Proposition 5.2 in [12]). Now consider $j = r$, let τ, $x(\tau)$, u and y be as in the second step for $j = r$ and let $t \in J_\tau$. From (4.2) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle P(t)x(t), x(t)\rangle = 2\,\mathrm{Re}\,\langle P(t)x(t), AP(t)x(t) + Bu(t)\rangle + \langle \dot{P}(t)x(t), x(t)\rangle\,.$$

Considering $\tilde{x}(t) = P(t)x(t)$ instead of $x(t)$, we can argue as above and deduce (4.4) as well as the corresponding equality also in this case.

The fourth step is to prove the uniqueness of the classical solutions of (4.1) and (4.2). Integrating (4.4), we derive

$$\|x(t)\|^2 + \int_\tau^t \|y(s)\|^2 \,\mathrm{d}s \le \|x(\tau)\|^2 + \int_\tau^t \|u(s)\|^2 \,\mathrm{d}s + \int_\tau^t \langle \dot{P}(s)x(s), x(s)\rangle \,\mathrm{d}s \tag{4.9}$$

for $(x(\tau), u) \in V(\tau)$ in the case of Σ_l and for $(P(\tau)x(\tau), u) \in V(\tau)$ in the case of Σ_r. Denoting $\phi(t) = \|x(t)\|^2 + \int_\tau^t \|y(s)\|^2 \,\mathrm{d}s$, the estimate (4.9) implies

$$\phi(t) \le \|x(\tau)\|^2 + \int_\tau^t \|u(s)\|^2 \,\mathrm{d}s + L(t)\int_\tau^t \phi(s)\,\mathrm{d}s,$$

where $L(t) = \max_{s\in[\tau,t]} \|\dot{P}(s)\|$. Applying here Gronwall's inequality thus yields

$$\|x(t)\|^2 + \int_\tau^t \|y(s)\|^2\, ds \le \omega(t-\tau)\Big(\|x(\tau)\|^2 + \int_\tau^t \|u(s)\|^2\, ds\Big) \tag{4.10}$$

for a nondecreasing function $\omega : [0,\infty) \to [1,\infty)$. This estimate shows the uniqueness of classical solutions to (1.4) and (1.5).

The fifth step is to introduce the time-varying well-posed systems Σ_l and Σ_r and to show that indeed the solutions of (1.4) and (1.5) are state trajectories and output functions of these systems. Let $\tau \in J$ with $\tau \ne \max J$ and let the functions u, x and y (defined on J_τ) be as in the second step. It is clear from the arguments in the second step that the functions x and y depend linearly on $(x(\tau), u)$. Thus, we can define the operators

$$\Sigma_j(t,\tau)\begin{bmatrix} x(\tau) \\ u \end{bmatrix} = \begin{bmatrix} x(t) \\ \mathbf{P}(t,\tau)y \end{bmatrix},$$

defined for $t,\tau \in J$ with $t \ge \tau$, $(x(\tau), u) \in V(\tau)$ if $j = l$ and $(P(\tau)x(\tau), u) \in V(\tau)$ if $j = r$. The inequality (4.10) allows us to extend the family $\Sigma_j(t,\tau)$ to locally uniformly bounded operators from $X \times L^2(J,U)$ to $X \times L^2(J,Y)$. We introduce the constituent blocks of Σ_j, which are the families of operators

$$\begin{aligned}
&\mathbb{T}_j(t,\tau) : X \to X; && \mathbb{T}_j(t,\tau)z = \Pi_1\Sigma_j(t,\tau)\left[\begin{smallmatrix} z \\ 0 \end{smallmatrix}\right],\\
&\Phi_j(t,\tau) : L^2(J,U) \to X; && \Phi_j(t,\tau)u = \Pi_1\Sigma_j(t,\tau)\left[\begin{smallmatrix} 0 \\ u \end{smallmatrix}\right],\\
&\Psi_j(t,\tau) : X \to L^2(J,Y); && \Psi_j(t,\tau)z = \Pi_2\Sigma_j(t,\tau)\left[\begin{smallmatrix} z \\ 0 \end{smallmatrix}\right],\\
&\mathbb{F}_j(t,\tau) : L^2(J,U) \to L^2(J,Y); && \mathbb{F}_j(t,\tau)u = \Pi_2\Sigma_j(t,\tau)\left[\begin{smallmatrix} 0 \\ u \end{smallmatrix}\right],
\end{aligned}$$

for $j = l, r$ and $t,\tau \in J$ with $t \ge \tau$. Consequently, the solutions x and y of (1.4) and (1.5) are given by

$$x(t) = \mathbb{T}_j(t,\tau)x(\tau) + \Phi_j(t,\tau)u, \qquad \mathbf{P}(t,\tau)y = \Psi_j(t,\tau)x(\tau) + \mathbb{F}_j(t,\tau)u$$

for $(x(\tau), u) \in V(\tau)$ and $j = l$, respectively, for $(P(\tau)x(\tau), u) \in V(\tau)$ and $j = r$. Combining these equalities with (4.5), (4.7), (4.8) and recalling that $u = \mathcal{S}_{-\tau}u_\tau$, we infer that

$$\mathfrak{T}_j(t,\tau) = \begin{bmatrix} \mathcal{S}^-_{t-\tau} & \mathcal{S}^-_t\Psi_j(t,\tau) & \mathcal{S}^-_t\mathbb{F}_j(t,\tau)\mathcal{S}_{-\tau} \\ 0 & \mathbb{T}_j(t,\tau) & \Phi_j(t,\tau)\mathcal{S}_{-\tau} \\ 0 & 0 & \mathcal{S}^+_{t-\tau} \end{bmatrix}$$

for $j = l, r$ and $t,\tau \in J$ with $t \ge \tau$. The definition of $\Sigma_j(t,\tau)$ implies that the families Φ_j, Ψ_j, and $\mathbb{F}_j$ satisfy the relations (3.2), (3.3) and (3.4), respectively. So Proposition 3.7 shows that the families $\mathbb{T}_j$, Φ_j, Ψ_j, and $\mathbb{F}_j$ ($j = l, r$) form time-varying well-posed systems Σ_j inducing the Lax-Phillips evolution families $\mathfrak{T}_j$. □

Now we derive representation formulas for the sytems Σ_l and Σ_r similar to those known for time invariant systems. Recall from Section 2 that the extrapolation spaces of $P(t)^{-1}A$ and $AP(t)$ are denoted by $X^t_{-1,l}$ and $X^t_{-1,r}$ respectively, and that $X^t_{-1,r}$ is isomorphic to (and identified with) X_{-1}, the extrapolation space of A.

Proposition 4.2. *In addition to the standing assumptions of this section, we assume that $P(\cdot)z \in C^2(J,X)$ for every $z \in X$. Then the systems Σ_l and Σ_r from Theorem 4.1 have the following properties, where $t, \tau \in J$ with $t \geq \tau$.*

(a) *The evolution family $\mathbb{T}_l$ is generated by $P(\cdot)^{-1}A$ and the evolution family $\mathbb{T}_r$ is generated by $AP(\cdot)$. These evolution families satisfy the assertions of Propositions 2.3, 2.7 and 2.8. Also, the operators $P(t)^{-1}$ extend to isomorphisms from X_{-1} to $X^t_{-1,l}$ which are locally uniformly bounded together with their inverses.*

(b) *For every $u \in L^2(J,U)$ we have*

$$\Phi_l(t,\tau)u = \int_\tau^t \mathbb{T}_l(t,s)P(s)^{-1}Bu(s)\,\mathrm{d}s, \quad \Phi_r(t,\tau)u = \int_\tau^t \mathbb{T}_r(t,s)Bu(s)\,\mathrm{d}s,$$

where the integrals are defined in $X^t_{-1,l}$ and $X^t_{-1,r}$, respectively.

(c) *If $z \in \mathcal{D}(A)$, then $[\Psi_l(t,\tau)z](s) = \overline{C}\mathbb{T}_l(s,\tau)z$, for all $s \in [\tau,t]$. If $z \in \mathcal{D}(AP(\tau))$, then $[\Psi_r(t,\tau)z](s) = \overline{C}P(s)\mathbb{T}_r(s,\tau)z$, for all $s \in [\tau,t]$.*

(d) *For every $u \in \mathcal{H}^1(J^0,U)$ with $u(\tau) = 0$ and $s \in [\tau,t]$, we have*

$$[\mathbb{F}_l(t,\tau)u](s) = \overline{C}\int_\tau^s \mathbb{T}_l(s,\sigma)P(\sigma)^{-1}Bu(\sigma)\,\mathrm{d}\sigma + Du(s),$$
$$[\mathbb{F}_r(t,\tau)u](s) = \overline{C}P(s)\int_\tau^s \mathbb{T}_r(s,\sigma)Bu(\sigma)\,\mathrm{d}\sigma + Du(s),$$

where the integrals take values in Z and $P(s)^{-1}Z$, respectively.

Proof. (a) The first part of assertion (a) follows from Propositions 2.3, 2.7 and 2.8 combined with Theorem 4.1 for the case $u = 0$. According to Proposition 2.4 the operator $P(t) - A$ has the locally uniformly bounded inverse $[I - P(t)^{-1}A]^{-1}P(t)^{-1}$ for $t \in J$. Using this fact, for all $z \in X$ we estimate

$$\begin{aligned}\|(I - P(t)^{-1}A)^{-1}P(t)^{-1}z\| &= \|(P(t) - A)^{-1}z\| \\ &= \|(P(t) - A)^{-1}[I - P(t) + P(t) - A](I - A)^{-1}z\| \\ &= \|[I + (P(t) - A)^{-1}(I - P(t))](I - A)^{-1}z\| \\ &\leq c(t)\,\|z\|_{X_{-1}},\end{aligned}$$

where $c(\cdot)$ is bounded on compact subsets of J. This means that $P(t)^{-1}$ has a locally uniformly bounded extension $P(t)^{-1} : X_{-1} \to X^t_{-1,l}$. A similar argument works to show that $P(t) : X^t_{-1,l} \to X_{-1}$ with a locally uniformly bounded norm.

(b) Fix $t, \tau \in J$ with $t > \tau$. We first consider the case $j = l$. Take $x_n \in \mathcal{D}(A)$ converging to x_0 in X as $n \to \infty$. Then for every $s \in [\tau,t]$ we have

$$\frac{\partial}{\partial s}\mathbb{T}_l(t,s)x_n = -\mathbb{T}_l(t,s)P(s)^{-1}Ax_n =: z_n,$$

due to Proposition 2.3. By the last part of (a), combined with the last sentence in Proposition 2.8, the vectors z_n converge to $-\mathbb{T}_l(t,s)P(s)^{-1}Ax_0$ in $X^t_{-1,l}$ as $n \to \infty$, uniformly with respect to $s \in [\tau, t]$. As a result, the map $[\tau, t] \ni s \mapsto \mathbb{T}_l(t,s)x_0$ is continuously differentiable in $X^t_{-1,l}$ with derivative $-\mathbb{T}_l(t,s)P(s)^{-1}Ax_0$. For $u \in \mathcal{H}^1(J^0, U)$ with $u(\tau) = 0$, Theorem 4.1 gives the solution $\Phi_l(\cdot, \tau)u = x(\cdot) \in C^1(J_\tau, X)$ of (4.1) with $x(\tau) = 0$. Then we can differentiate

$$\begin{aligned}\frac{\partial}{\partial s}\mathbb{T}_l(t,s)x(s) &= -\mathbb{T}_l(t,s)P(s)^{-1}Ax(s) + \mathbb{T}_l(t,s)[P(s)^{-1}Ax(s) + P(s)^{-1}Bu(s)] \\ &= \mathbb{T}_l(t,s)P(s)^{-1}Bu(s)\end{aligned}$$

in $X^t_{-1,l}$. By integration, we derive

$$\Phi_l(t,\tau)u = x(t) = \int_\tau^t \mathbb{T}_l(t,s)P(s)^{-1}Bu(s)\,\mathrm{d}s.$$

The case $u \in L^2(J, U)$ can now be treated by approximation using again part (a).

For the case $j = r$, take $u \in \mathcal{H}^1(J^0, U)$ with $u(\tau) = 0$. According to Theorem 4.1 the solution of (4.2) with $x(\tau) = 0$ is $\Phi_r(\cdot, \tau)u = x(\cdot) \in C^1(J_\tau, X)$. Using Proposition 2.7, we can differentiate

$$\begin{aligned}\frac{\partial}{\partial s}\mathbb{T}_r(t,s)x(s) &= -\mathbb{T}_r(t,s)AP(s)x(s) + \mathbb{T}_r(t,s)[AP(s)x(s) + Bu(s)] \\ &= \mathbb{T}_r(t,s)Bu(s)\end{aligned}$$

in X_{-1}. The asserted formula for $\Phi_r(t,\tau)u$ follows by integration. We then obtain the result for $u \in L^2(J, U)$ by approximation.

Assertions (c) and (d) follow from Theorem 4.1 and parts (a) and (b) above. □

If $P(\cdot)z \in C^1(J, X)$ for every $z \in X$, we can still construct a time-varying well-posed system Σ_r related to (1.5).

Proposition 4.3. *Under the standing assumptions of this section, there exists a time-varying well-posed system Σ_r with time interval J consisting of operator families $\mathbb{T}_r$, Φ_r, Ψ_r and $\mathbb{F}_r$ such that the following assertions hold. In these assertions, $\tau \in J$ with $\tau \neq \max J$.*

(a) *The evolution family $\mathbb{T}_r$ satisfies the assertions of Proposition 2.7. Let $x(\tau) \in X$ and $u \in L^2(J_\tau, U)$. Then the function*

$$t \mapsto x(t) = \mathbb{T}_r(t,\tau)x(\tau) + \Phi_r(t,\tau)u \qquad (\textit{for } t \in J)$$

belongs to $\mathcal{H}^1((\tau, b), X_{-1})$ for every $b \in J$ with $b > \tau$, and it satisfies (4.2) in X_{-1} for almost all $t \in J_\tau$.

(b) *For every $u \in L^2(J, U)$ we have (using integration in X_{-1})*

$$\Phi_r(t,\tau)u = \int_\tau^t \mathbb{T}_r(t,s)Bu(s)\,\mathrm{d}s.$$

(c) *For $x(\tau) \in X$ and $u \in L^2(J_\tau, U)$, we define the output $y : J_\tau \to Y$ of* (1.5) *by setting $\mathbf{P}(t,\tau)y = \Psi_r(t,\tau)x(\tau) + \mathbb{F}_r(t,\tau)u$ on $[\tau, t]$ for $t \in J_\tau$. Then we have the balance inequality*

$$\langle P(t)x(t), x(t)\rangle + \int_\tau^t \|y(s)\|^2 \, \mathrm{d}s \le \langle P(\tau)x(\tau), x(\tau)\rangle + \int_\tau^t \|u(s)\|^2 \, \mathrm{d}s + \int_\tau^t \langle \dot{P}(s)x(s), x(s)\rangle \, \mathrm{d}s \tag{4.11}$$

for every $t \in J_\tau$. If the system Σ^i is energy preserving, then the map $t \mapsto \langle P(t)x(t), x(t)\rangle$ is differentiable a.e. and (4.4) *holds with equality for a.e. $t \in J_\tau$.*

Proof. (a) We proceed similarly as in Remark 2.6 and Proposition 2.7. There exists a time-varying well-posed system $\tilde{\Sigma}_l$ consisting of operator families $\tilde{\mathbb{T}}_l$, $\tilde{\Phi}_l$, $\tilde{\Psi}_l$ and $\tilde{\mathbb{F}}_l$ which satisfy the assertions of Theorem 4.1(a) for the system Σ^i and the operators $P(t)^{-1}$. In particular, the corresponding Lax-Phillips evolution family $\tilde{\mathfrak{T}}_l$ is generated by $\mathfrak{P}(\cdot)\mathfrak{A}$, where we use the same notation as in the proof of Theorem 4.1. Let $w_0 \in \mathcal{Y} \times X \times \mathcal{U}$ and $t, s, \tau \in J_\tau$ with $t \ge s \ge \tau$. We set $\mathfrak{V}(t,\tau) = \mathfrak{P}(t)^{-1}\tilde{\mathfrak{T}}_l(t,\tau)\mathfrak{P}(\tau)$. By Proposition 2.7 there is an evolution family $\mathfrak{T}_r$ on $\mathcal{Y} \times X \times \mathcal{U}$ such that

$$\mathfrak{T}_r(t,\tau)w_0 = \mathfrak{V}(t,\tau)w_0 + \int_\tau^t \mathfrak{V}(t,s)\mathfrak{P}(s)^{-1}\dot{\mathfrak{P}}(s)\mathfrak{T}_r(s,\tau)w_0 \, \mathrm{d}s, \tag{4.12}$$

$$\mathfrak{T}_r(t,\tau)w_0 = \mathfrak{V}(t,\tau)w_0 + \int_\tau^t \mathfrak{T}_r(t,s)\mathfrak{P}(s)^{-1}\dot{\mathfrak{P}}(s)\mathfrak{V}(s,\tau)w_0 \, \mathrm{d}s. \tag{4.13}$$

Taking into account (3.13) and (4.3), we obtain

$$\mathfrak{V}(t,s) = \begin{bmatrix} \mathcal{S}_{t-s}^- & \mathcal{S}_t^-\tilde{\Psi}_l(t,s)P(s) & \mathcal{S}_t^-\tilde{\mathbb{F}}_l(t,s)\mathcal{S}_{-s} \\ 0 & P(t)^{-1}\tilde{\mathbb{T}}_l(t,s)P(s) & P(t)^{-1}\tilde{\Phi}_l(t,s)\mathcal{S}_{-s} \\ 0 & 0 & \mathcal{S}_{t-s}^+ \end{bmatrix}, \tag{4.14}$$

$$\mathfrak{V}(t,s)\mathfrak{P}(s)^{-1}\dot{\mathfrak{P}}(s) = \begin{bmatrix} 0 & \mathcal{S}_t^-\tilde{\Psi}_l(t,s)\dot{P}(s) & 0 \\ 0 & P(t)^{-1}\tilde{\mathbb{T}}_l(t,s)\dot{P}(s) & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{4.15}$$

Here we can replace $\mathcal{S}_t^-$ with $\mathcal{S}_t$ since (3.3) and (3.4) hold for $\tilde{\Sigma}_l$. Equation (4.12) now implies that the third line of $\mathfrak{T}_r(t,\tau)$ is given by $\begin{bmatrix} 0 & 0 & \mathcal{S}_{t-\tau}^+ \end{bmatrix}$ and that

$$[\mathfrak{T}_r(t,\tau)]_{21} = \int_\tau^t P(t)^{-1}\tilde{\mathbb{T}}_l(t,s)\dot{P}(s)[\mathfrak{T}_r(s,\tau)]_{21} \, \mathrm{d}s.$$

(Here and below the integrals are understood in a strong sense, and $[\mathfrak{T}_r(t,\tau)]_{jk}$ denotes the component of $\mathfrak{T}_r(t,\tau)$ in the jth line and kth row.) From Gronwall's inequality we deduce that $[\mathfrak{T}_r(t,\tau)]_{21} = 0$. Hence, $[\mathfrak{T}_r(t,\tau)]_{11} = \mathcal{S}_{t-\tau}^-$ by (4.12). On the other hand, there is a unique evolution family $\mathbb{T}_r$ on X satisfying the assertions of Proposition 2.7. Due to (4.12), (4.14) and (4.15), the components $[\mathfrak{T}_r(t,\tau)]_{22}$

satisfy (2.10) for all $x_0 \in X$ and $t, \tau \in J$ with $t \geq \tau$. The uniqueness part of Proposition 2.7(a) thus implies that $[\mathfrak{T}_r(t,\tau)]_{22} = \mathbb{T}_r(t,\tau)$. To determine the remaining components of $\mathfrak{T}_r(t,\tau)$, we set

$$\Psi_r(t,\tau) = \mathcal{S}_{-t}[\mathfrak{T}_r(t,\tau)]_{12}, \quad \mathbb{F}_r(t,\tau) = \mathcal{S}_{-t}[\mathfrak{T}_r(t,\tau)]_{13}\mathcal{S}_\tau, \quad \Phi_r(t,\tau) = [\mathfrak{T}_r(t,\tau)]_{23}\mathcal{S}_\tau.$$

We want to check that the operator families $\mathbb{T}_r$, Φ_r, Ψ_r and $\mathbb{F}_r$ form a time-varying well-posed system Σ_r. By Proposition 3.7 we have to verify the identities (3.2), (3.3) and (3.4) for these operators. At first, equations (4.12), (4.14) and (4.15) yield

$$\Psi_r(t,\tau) = \tilde{\Psi}_l(t,\tau)P(\tau) + \int_\tau^t \tilde{\Psi}_l(t,s)\dot{P}(s)\mathbb{T}_r(s,\tau)\,\mathrm{d}s. \tag{4.16}$$

Since $\tilde{\Psi}_l$ satisfies (3.3), we have

$$\mathbf{P}(t,\tau)\tilde{\Psi}_l(t,s) = \mathbf{P}(t,\tau)\mathbf{P}(t,s)\tilde{\Psi}_l(t,s) = \mathbf{P}(t,s)\tilde{\Psi}_l(t,s) = \tilde{\Psi}_l(t,s).$$

Equation (4.16) now implies the identity $\mathbf{P}(t,\tau)\Psi_r(t,\tau) = \Psi_r(t,\tau)$, i.e., (3.3) for Ψ_r holds. Second, we note that

$$\mathfrak{T}_r(t,s)\mathfrak{P}(s)^{-1}\dot{\mathfrak{P}}(s) = \begin{bmatrix} 0 & \mathcal{S}_t^-\Psi_r(t,s)P(s)^{-1}\dot{P}(s) & 0 \\ 0 & \mathbb{T}_r(t,s)P(s)^{-1}\dot{P}(s) & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

so that (4.13) and (4.14) yield

$$\Phi_r(t,\tau) = P(t)^{-1}\tilde{\Phi}_l(t,\tau) + \int_\tau^t \mathbb{T}_r(t,s)P(s)^{-1}\dot{P}(s)P(s)^{-1}\tilde{\Phi}_l(s,\tau)\,\mathrm{d}s. \tag{4.17}$$

As above, we first deduce the equality $\tilde{\Phi}_l(s,\tau)\mathbf{P}(t,\tau) = \tilde{\Phi}_l(s,\tau)$ from (3.2) for $\tilde{\Phi}_l$ and then (3.2) for Φ_r from (4.17). Finally, (4.12), (4.14) and (4.15) yield that

$$\mathbb{F}_r(t,\tau) = \tilde{\mathbb{F}}_l(t,\tau) + \int_\tau^t \tilde{\Psi}_l(t,s)\dot{P}(s)\Phi_r(s,\tau)\,\mathrm{d}s.$$

Arguing as above and using $\Phi_r(s,\tau)\mathbf{P}(t,\tau) = \Phi_r(s,\tau)$, we obtain (3.4) for $\mathbb{F}_r$. Consequently, $\mathbb{T}_r$, Φ_r, Ψ_r and $\mathbb{F}_r$ form a time-varying well-posed system Σ_r.

Let $\tau \in J$ with $\tau \neq \max J$, $x(\tau) \in X$ and $t \in J_\tau$. Due to Proposition 2.7, the function $t \mapsto \mathbb{T}_r(t,\tau)x(\tau)$ is continuously differentiable in X_{-1} with derivative $AP(t)\mathbb{T}_r(t,\tau)x_0$. For $(0,u) \in V(\tau)$, Theorem 4.1(a) shows that the function $t \mapsto P(t)^{-1}\tilde{\Phi}_l(t,\tau)u$ is continuously differentiable in X with the derivative

$$\frac{\partial}{\partial t}P(t)^{-1}\tilde{\Phi}_l(t,\tau)u = -P(t)^{-1}\dot{P}(t)P(t)^{-1}\tilde{\Phi}_l(t,\tau)u + A\tilde{\Phi}_l(t,\tau)u + Bu(t).$$

Equation (4.17) then implies that the map $t \mapsto \Phi_r(t,\tau)u$ is continuously differentiable in X_{-1} with the derivative $AP(\cdot)\Phi_r(\cdot,\tau)u + Bu$. Now, let $u \in L^2(J,U)$. By approximation, we see that the map $t \mapsto \Phi_r(t,\tau)u$ belongs to $\mathcal{H}^1((\tau,b),X_{-1})$ for every $b \in J$ with $b > \tau$ and that its derivative is equal to $AP(\cdot)\Phi_r(\cdot,\tau)u + Bu$ in $L^2((\tau,b),X_{-1})$. Summing up, we have established all assertions in (a).

(b) Assertion (b) can be proved as the corresponding identity in Proposition 4.2(b) using Proposition 2.7 and part (a) of the present proof.

(c) We show the balance inequality by an approximation argument. Fix an interval $[a,b] \subset J^0$. Let $\tau \in [a,b)$ and $(P(\tau)x(\tau),u) \in V(\tau)$. By a standard mollification procedure one constructs operators $P_n(t)$ for all $t \in [a,b]$ and $n \in \mathbb{N}$ such that (2.1) holds on $[a,b]$, $P_n(\cdot)x_0 \in C^2([a,b],X)$, and $P_n(\cdot)x_0 \to P(\cdot)x_0$ in $C^1([a,b],X)$ as $n \to \infty$ for every $x_0 \in X$. Moreover, the operators $P_n(t)$, $P_n(t)^{-1}$ and $\dot{P}_n(t)$ are uniformly bounded for $n \in \mathbb{N}$ and $t \in [a,b]$. We set $x_n(\tau) = P_n(\tau)^{-1}P(\tau)x(\tau)$. Then $x_n(\tau) \to x(\tau)$ in X as $n \to \infty$ and $(P_n(\tau)x_n(\tau),u) \in V(\tau)$. By Theorem 4.1(b) there are classical solutions x_n and y_n of (1.5) with P replaced by P_n and

$$\langle P_n(t)x_n(t),x_n(t)\rangle + \int_\tau^t \|y_n(s)\|^2\,\mathrm{d}s \le \langle P_n(\tau)x_n(\tau),x_n(\tau)\rangle + \int_\tau^t \|u(s)\|^2\,\mathrm{d}s + \int_\tau^t \langle \dot{P}_n(s)x_n(s),x_n(s)\rangle\,\mathrm{d}s \tag{4.18}$$

for every $t \in [\tau,b]$. We construct evolution families $\tilde{\mathfrak{T}}_l^n$, $\mathfrak{V}^n$ and $\mathfrak{T}_r^n$ as in part (a) of the proof, where we replace P with P_n and define $\mathfrak{P}_n$ corresponding to P_n as in (4.3). In view of the proof of Proposition 2.3 and the results cited there, the operators $\tilde{\mathfrak{T}}_l^n(t,\tau)$ are uniformly bounded for $n \in \mathbb{N}$ and $t,\tau \in [a,b]$ with $t \ge \tau$. Let $w_0 \in \mathcal{D}(\mathfrak{A})$. For $t,s \in [a,b]$ with $t \ge s$ we have

$$\frac{d}{ds}\tilde{\mathfrak{T}}_l^n(t,s)\tilde{\mathfrak{T}}_l(s,\tau)w_0 = \tilde{\mathfrak{T}}_l^n(t,s)(\mathfrak{P}(s) - \mathfrak{P}_n(s))\mathfrak{A}\tilde{\mathfrak{T}}_l(s,\tau)w_0.$$

Integrating this identity, we derive

$$\tilde{\mathfrak{T}}_l(t,\tau)w_0 - \tilde{\mathfrak{T}}_l^n(t,\tau)w_0 = \int_\tau^t \tilde{\mathfrak{T}}_l^n(t,s)(\mathfrak{P}(s) - \mathfrak{P}_n(s))\mathfrak{A}\tilde{\mathfrak{T}}_l(s,\tau)w_0\,\mathrm{d}s$$

for every $t \in [\tau,b]$. It is now straightforward to see that $\tilde{\mathfrak{T}}_l^n(t,\tau)$ converges strongly to $\tilde{\mathfrak{T}}_l(t,\tau)$ as $n \to \infty$ for all $\tau,t \in [a,b]$ with $t \ge \tau$, and so $\mathfrak{V}^n(t,\tau)$ converges strongly to $\mathfrak{V}(t,\tau)$. We observe that equation (4.12) holds with $\mathfrak{P}$, $\mathfrak{V}$ and $\mathfrak{T}_r$ replaced by $\mathfrak{P}_n$, $\mathfrak{V}^n$ and $\mathfrak{T}_r^n$, respectively. Combining this quation with Gronwall's inequality, we then deduce that the operators $\mathfrak{T}_r^n(t,\tau)$ converge strongly to $\mathfrak{T}_r(t,\tau)$ as $n \to \infty$ for all $\tau,t \in [a,b]$ with $t \ge \tau$. Hence, $x_n(t) \to x(t) = \mathbb{T}_r(t,\tau)x(\tau) + \Phi_r(t,\tau)u$ in X and $\mathbf{P}(t,\tau)y_n \to \mathbf{P}(t,\tau)y = \Psi_r(t,\tau)x(\tau) + \mathbb{F}_r(t,\tau)u$ in $L^2([\tau,t],Y)$ for all $t \in [\tau,b]$. So the balance inquality (4.11) for $(P(\tau)x(\tau),u) \in V(\tau)$ and $t \in [\tau,b]$ follows from (4.18). By approximation, one can extend (4.11) to the case of $x(\tau) \in X$ and $u \in L^2(J,U)$, still for $\tau,t \in [a,b]$ with $t \ge \tau$. Here $a,b \in J^0$ with $b > a$ are arbitrary, so that by continuity (4.11) holds for all $\tau,t \in J$ with $t \ge \tau$. If Σ^i is energy preserving, we have equality in (4.18) and we can then deduce also (4.11) with an equality sign for all $t \in J_\tau$, $x(\tau) \in X$ and $u \in L^2(J,U)$. This fact implies the final assertion in (b). □

5. An example with the wave equation

In this section we investigate a wave equation with control and observation at the boundary and with time-dependent coefficients in the differential equation. The corresponding time-invariant system was studied in Tucsnak and Weiss [22, Section 7] (see also [23, Section 7.5]). We refer to this work for further details and references. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary Γ. Moreover, let Γ_0 and Γ_1 be nonempty, disjoint, relatively open subsets of Γ such that $\overline{\Gamma_0 \cup \Gamma_1} = \Gamma$ and the boundaries of Γ_0 and Γ_1 with respect to Γ have surface measure 0. We note that, for almost all $\xi \in \Gamma$, there exists the outer unit normal $\nu(\xi)$ at ξ. We investigate the system described by the equations

$$\begin{aligned} \ddot{w}(t) &= \operatorname{div}(a(t)\nabla w(t)), && \text{on } [0,\infty)\times\Omega, \\ w(t) &= 0, && \text{on } [0,\infty)\times\Gamma_0 \\ \sqrt{2}\, bu(t) &= \nu\cdot a(t)\nabla w(t) + |b|^2\dot{w}(t), && \text{on } [0,\infty)\times\Gamma_1, \\ \sqrt{2}\, by(t) &= \nu\cdot a(t)\nabla w(t) - |b|^2\dot{w}(t), && \text{on } [0,\infty)\times\Gamma_1, \\ w(0) &= w_0, \qquad \dot{w}(0) = w_1, && \text{on } \Omega, \end{aligned} \tag{5.1}$$

where w and the output y are unknown and the input u and the initial data w_0, w_1 are given. In the above equations we have omitted the space variable ξ which belongs to Ω, Γ_0 or Γ_1, respectively, and we have denoted the scalar product in $\mathbb{R}^n$ by $a\cdot b$.

We assume that $b \in L^\infty(\Gamma_1)$ is real-valued and there exists $\delta > 0$ such that $b(\xi) \geq \delta$ for almost every $\xi \in \Gamma_1$. Moreover, $a(t,\xi)$ is a symmetric real $n\times n$–matrix with $a(t,\xi) \geq \delta I$ for all $t \in [0,\infty)$ and $\xi \in \overline{\Omega}$, and the components a_{ij} of a satisfy $a_{ij} \in C^2([0,\infty),\overline{\Omega})$ for all $i,j \in \{1,\dots,n\}$.

To put the equations (5.1) in the framework of the present paper, we set $H = L^2(\Omega)$ and $U = Y = L^2(\Gamma_1)$. We denote by γ the Dirichlet trace operator on Γ, so that $\gamma : \mathcal{H}^1(\Omega) \to \mathcal{H}^{\frac{1}{2}}(\Gamma)$. Let R be the usual restriction map from $L^2(\Gamma)$ to $L^2(\Gamma_1)$. Then the Dirichlet trace operator $\gamma_0 : \mathcal{H}^1(\Omega) \to \mathcal{H}^{\frac{1}{2}}(\Gamma_1)$ on Γ_1 is defined by $\gamma_0 = R\gamma$. We further introduce the Hilbert space

$$\mathcal{H}^1_{\Gamma_0}(\Omega) = \{f \in \mathcal{H}^1(\Omega) : (I-R)\gamma f = 0\}$$

which is a closed subspace of $\mathcal{H}^1(\Omega)$ endowed with the norm $\|f\|_{\mathcal{H}^1} = \|\nabla f\|_{H^n}$. The Neumann trace operator on Γ_1 is initially defined by $\gamma_1 f = R\frac{\partial}{\partial\nu} f$ for $f \in C^1(\overline{\Omega})$. One can extend γ_1 to all those $f \in \mathcal{H}^1_{\Gamma_0}(\Omega)$ such that $\Delta f \in H$, see e.g. [23, p. 107]. Here and later the derivatives are understood in the sense of distributions.

For vector-valued functions $v \in C^1(\overline{\Omega})^n$ we define a Dirichlet-type trace operator on Γ by $\overline{\gamma}_\nu v = \nu\cdot\gamma v$. Due to Theorem 1 and formula (1.10) in Section IX.1 of [4], we can extend $\overline{\gamma}_\nu$ to an operator from the space $H(\operatorname{div}) := \{v \in L^2(\Omega)^n \mid \operatorname{div} v \in L^2(\Omega)\}$ to $\mathcal{H}^{-\frac{1}{2}}(\Gamma)$ (still denoted by $\overline{\gamma}_\nu$) such that the equation

$$\langle \nabla\varphi, v\rangle + \langle \varphi, \operatorname{div} v\rangle = \langle \gamma\varphi, \overline{\gamma}_\nu v\rangle_{\mathcal{H}^{\frac{1}{2}}(\Gamma),\mathcal{H}^{-\frac{1}{2}}(\Gamma)}$$

holds for all $v \in H(\mathrm{div})$ and all $\varphi \in \mathcal{H}^1(\Omega)$, see equation (1.10) in Section IX.1 of [4]. Here, $H(\mathrm{div})$ is endowed with the norm $(\|v\|_{H^n}^2 + \|\operatorname{div} v\|_H^2)^{\frac{1}{2}}$, the brackets $\langle \cdot, \cdot \rangle$ denote the usual scalar product in $L^2(\Omega)^n$ or in $L^2(\Omega)$, and $\langle \cdot, \cdot \rangle_{\mathcal{H}^{\frac{1}{2}}(\Gamma), \mathcal{H}^{-\frac{1}{2}}(\Gamma)}$ denotes the duality pairing between $\mathcal{H}^{\frac{1}{2}}(\Gamma)$ and its dual space $\mathcal{H}^{-\frac{1}{2}}(\Gamma)$.

We also need a variant of $\overline{\gamma}_\nu$ on Γ_1. For this we introduce a subspace of $\mathcal{H}^{\frac{1}{2}}(\Gamma_1)$:

$$\tilde{\mathcal{H}}^{\frac{1}{2}}(\Gamma_1) = \left\{ \gamma_0 f \mid f \in \mathcal{H}^1_{\Gamma_0}(\Omega) \right\} = \left\{ \gamma f \mid f \in \mathcal{H}^1(\Omega),\ \operatorname{supp} \gamma f \subset (\Gamma \setminus \Gamma_0) \right\},$$

see [22, Section 5] and [23, p. 428]. The norm of $\psi \in \tilde{\mathcal{H}}^{\frac{1}{2}}(\Gamma_1)$ is given by the infimum of all norms $\|f\|_{\mathcal{H}^1}$ where $\psi = \gamma_0 f$ and $f \in \mathcal{H}^1_{\Gamma_0}(\Omega)$. We denote by $\tilde{\mathcal{H}}^{-\frac{1}{2}}(\Gamma_1)$ the dual space of $\tilde{\mathcal{H}}^{\frac{1}{2}}(\Gamma_1)$ with respect to the pivot space $L^2(\Gamma_1)$. The space $L^2(\Gamma_1)$ is densely embedded into $\tilde{\mathcal{H}}^{-\frac{1}{2}}(\Gamma_1)$ due to Remark 13.6.14 of [23]. For $v \in H(\mathrm{div})$ we can then define $\gamma_\nu v \in \tilde{\mathcal{H}}^{-\frac{1}{2}}(\Gamma_1)$ by

$$\langle \gamma_0 \varphi, \gamma_\nu v \rangle_{\tilde{\mathcal{H}}^{\frac{1}{2}}(\Gamma_1), \tilde{\mathcal{H}}^{-\frac{1}{2}}(\Gamma_1)} = \langle \nabla \varphi, v \rangle + \langle \varphi, \operatorname{div} v \rangle, \tag{5.2}$$

for all $\varphi \in \mathcal{H}^1_{\Gamma_0}(\Omega)$. Clearly, $\gamma_\nu : H(\mathrm{div}) \to \tilde{\mathcal{H}}^{-\frac{1}{2}}(\Gamma_1)$ is linear and continuous. In the sequel, the boundary conditions in (5.1) are understood in the sense of the trace operators $(I - R)\gamma$, γ_0 and γ_ν, respectively.

We now define the operator A_0 in H by setting

$$A_0 f = -\Delta f, \qquad \mathcal{D}(A_0) = \{ f \in \mathcal{H}^1_{\Gamma_0}(\Omega) \mid \Delta f \in H,\ \gamma_1 f = 0 \}.$$

It is known that A_0 is strictly positive, see [14, §1]. Moreover,

$$H_{\frac{1}{2}} = D(A_0^{\frac{1}{2}}) = \mathcal{H}^1_{\Gamma_0}(\Omega),$$

which is a Hilbert space with the norm $\|\varphi\|_{\frac{1}{2}} = \| \, |\nabla \varphi| \, \|_H$.

Theorem 5.1. *The equations* (5.1) *determine a time-varying well-posed system* Σ_r *with state* $(\nabla w, \dot{w})$*, state space*

$$X = H^n \times H = L^2(\Omega)^n \times L^2(\Omega)$$

and input and output space $U = L^2(\Gamma_1)$*. If* $w_0, w_1 \in H_{\frac{1}{2}} = \mathcal{H}^1_{\Gamma_0}(\Omega)$ *and* $u \in \mathcal{H}^1((0,\infty), U)$ *satisfy* $\operatorname{div}(a(0)\nabla w_0) \in L^2(\Omega)$ *and the compatibility condition*

$$\gamma_\nu a(0) \nabla w_0 + |b|^2 \gamma_0 w_1 = \sqrt{2}\, b u(0), \tag{5.3}$$

then there is unique solution w, y *of* (5.1) *satisfying*

$$w \in C^2([0,\infty), H) \cap C^1([0,\infty), H_{\frac{1}{2}}), \qquad y \in \mathcal{H}^1((0,b), U)$$

for each $b > 0$*. Moreover, in this case the functions* $t \mapsto \operatorname{div}(a(t)\nabla w(t))$ *and* $t \mapsto \gamma_\nu a(t) \nabla w(t)$ *belong to* $C([0,\infty), H)$ *and* $C([0,\infty), U)$*, respectively, and the following balance equality holds for all* $t \geq 0$*:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\langle a(t) \nabla w(t), \nabla w(t) \rangle + \|\dot{w}(t)\|^2 \Big) = \|u(t)\|^2 - \|y(t)\|^2 + \langle \dot{a}(t) \nabla w(t), \nabla w(t) \rangle .$$

Proof. 1) As in [22, p. 270] we define the Neumann map $N \in \mathcal{L}(U, H_{\frac{1}{2}})$ by setting $N\psi = v$, where $v \in H_{\frac{1}{2}}$ is the unique solution in $H_{\frac{1}{2}}$ of the Neumann problem $\Delta v = 0$ and $\gamma_1 v = \psi$ for a given $\psi \in U$. We further set $K = \sqrt{2}\, bN^* A_0 \in \mathcal{L}(H_{\frac{1}{2}}, U)$. It is known that $K = \sqrt{2}\, b\gamma_0$, see, e.g., [22, p. 271]. Moreover, we have $K^* \in \sqrt{2}\, A_0 N b \in \mathcal{L}(U, H_{-\frac{1}{2}})$, where $H_{-\frac{1}{2}}$ is the dual space of $H_{\frac{1}{2}}$ with pivot space H. We define

$$\tilde{A} = \begin{bmatrix} 0 & I \\ -A_0 & -\frac{1}{2}K^*K \end{bmatrix} \quad \text{with} \quad \mathcal{D}(\tilde{A}) = \left\{ \begin{bmatrix} f \\ g \end{bmatrix} \in H_{\frac{1}{2}} \times H_{\frac{1}{2}} \;\middle|\; A_0 f + \frac{1}{2}K^*Kg \in H \right\}$$
$$B = \begin{bmatrix} 0 \\ K^* \end{bmatrix}, \qquad \overline{C} = \begin{bmatrix} 0 & -K \end{bmatrix}, \qquad D = I.$$

In Proposition 7.1 of [22] it has been shown that $\tilde{A}, B, \overline{C}$, and D define via (1.1) a scattering energy-preserving time-invariant system $\tilde{\Sigma}^i$ with the state space $\tilde{X} = H_{\frac{1}{2}} \times H$ and the input and output space U.

2) In this step we transform the system $\tilde{\Sigma}$ to the new state space $X = H^n \times H$. To this aim, we observe that the subspace $E = \nabla H_{\frac{1}{2}}$ is closed in H^n and that E has the orthogonal complement

$$E_1 = \{ v \in H^n \mid \operatorname{div} v = 0,\, \gamma_\nu v = 0 \}.$$

These facts can be established like Proposition 1 in Section IX.1 of [4] using (5.2) above. We denote by L_0 the isometric map $\nabla : H_{\frac{1}{2}} \to E$. In particular, L_0 has the bounded inverse $L_0^{-1} : E \to H_{\frac{1}{2}}$. We then introduce the isometric isomorphism

$$J = \begin{bmatrix} L_0 & 0 \\ 0 & I \end{bmatrix} : H_{\frac{1}{2}} \times H \to E \times H.$$

In $E \times H$ we thus obtain the transformed operator

$$\hat{A} = J\tilde{A}J^{-1} = \begin{bmatrix} 0 & L_0 \\ -A_0 L_0^{-1} & -\frac{1}{2}K^*K \end{bmatrix} \quad \text{with} \quad \mathcal{D}(\hat{A}) = J\mathcal{D}(\tilde{A}).$$

The operator A_0 can be extended to an operator in $\mathcal{L}(H_{\frac{1}{2}}, H_{-\frac{1}{2}})$ again denoted by A_0, and we have $\langle \varphi, A_0\psi \rangle_{H_{\frac{1}{2}}} = \langle \nabla\varphi, \nabla\psi \rangle$ for every $\varphi, \psi \in H_{\frac{1}{2}}$. This fact yields

$$\langle \varphi, A_0 L_0^{-1} v \rangle_{H_{\frac{1}{2}}} = \langle \nabla\varphi, \nabla L_0^{-1} v \rangle = \langle L_0\varphi, v \rangle = \langle \varphi, L_0^* v \rangle_{H_{\frac{1}{2}}}$$

for every $v \in E$ and $\varphi \in H_{\frac{1}{2}}$. (The brackets $\langle \cdot, \cdot \rangle$ denote the standard scalar product on H^n, H, or U.) As a consequence, we have $A_0 = L_0^* L_0$ on $H_{\frac{1}{2}}$, and thus

$$\hat{A} = \begin{bmatrix} 0 & L_0 \\ -L_0^* & -\frac{1}{2}K^*K \end{bmatrix} \quad \text{with} \quad \mathcal{D}(\hat{A}) = \left\{ \begin{bmatrix} v \\ g \end{bmatrix} \in E \times H_{\frac{1}{2}} \;\middle|\; L_0^* v + \frac{1}{2}K^*Kg \in H \right\}.$$

We identify B and JB, thus considering B as a map from U to $E \times H_{-\frac{1}{2}}$, and we identify $\overline{C}$ and $\overline{C}J^{-1}$, considering $\overline{C}$ as map from $E \times H_{\frac{1}{2}}$ to U. It then easily follows that $\hat{A}$, B, $\overline{C}$, and D define a scattering energy-preserving time-invariant system $\hat{\Sigma}^i$ with the state space $\hat{X} = E \times H$ and the input and output space U.

3) To construct the time-varying system corresponding to (5.1), we want to consider the operators

$$P(t) = \begin{bmatrix} a(t) & 0 \\ 0 & I \end{bmatrix}$$

for each $t \geq 0$. However, these operators do not leave invariant $E \times H$, in general, though it is clear that they satisfy (2.1) on the space $X = H^n \times H = (E \oplus E_1) \times H$. In order to obtain a system on X, we have to extend $\hat{A}$ from $E \times H$ to X. To that purpose, we denote by L the bounded operator $\nabla : H_{\frac{1}{2}} \to H^n$. Equation (5.2) then implies

$$\langle \varphi, L^* v \rangle_{H_{\frac{1}{2}}} = \langle \nabla \varphi, v \rangle = -\langle \varphi, \operatorname{div} v \rangle + \langle \gamma_0 \varphi, \gamma_\nu v \rangle_{\tilde{\mathcal{H}}^{\frac{1}{2}}(\Gamma_1)} = 0$$

for every $v \in E_1$ and $\varphi \in H_{\frac{1}{2}}$. As a result, $L^* = L_0^* \oplus 0$. We now extend $\hat{A}$ to the operator

$$A = \begin{bmatrix} 0 & L \\ -L^* & -\frac{1}{2} K^* K \end{bmatrix} \quad \text{with} \quad \mathcal{D}(A) = \left\{ \begin{bmatrix} v \\ g \end{bmatrix} \in H^n \times H_{\frac{1}{2}} \;\middle|\; L^* v + \frac{1}{2} K^* K g \in H \right\}.$$

on X. Observe that $\mathcal{D}(A) = \mathcal{D}(\hat{A}) \oplus (E_1 \times \{0\})$ and that $A(E_1 \times \{0\}) = 0$. Therefore the operators A, B, $\overline{C}$, and D define a scattering energy-preserving time-invariant system Σ^i with the state space $X = H^n \times H$ and the input and output space U. Applying Theorem 4.1(b) to Σ^i and $P(\cdot)$, we thus obtain a time-varying well-posed system Σ_r with state space X and input and output space U which is determined by (1.5). In particular, for each initial state $x_0 \in X$ and each input function $u \in \mathcal{H}^1((0,\infty), U)$ satisfying $AP(0)x_0 + Bu(0) \in X$ there is a unique solution $x \in C^1([0,\infty), X)$ of

$$\dot{x}(t) = AP(t)x(t) + Bu(t). \tag{5.4}$$

Moreover, $P(\cdot)x(\cdot) \in C([0,\infty), Z)$, the output of the system is given by

$$y(t) = \overline{C}x(t) + u(t) \tag{5.5}$$

for all $t \geq 0$, and y belongs to $\mathcal{H}^1((0,b), U)$ for each $b > 0$. We recall that the space $Z = \mathcal{D}(A) + (I - A)^{-1} BU$ was defined in the Introduction. For the system $\tilde{\Sigma}^i$ defined by $\tilde{A}, B, \overline{C}$ and D, we have a corresponding space $\tilde{Z}$ which is equal to $Z_0 \times H_{\frac{1}{2}}$ where $Z_0 = \{ f \in H_{\frac{1}{2}} \mid \Delta f \in H,\ \gamma_1 f \in L^2(\Gamma_1) \}$, see p. 263 and p. 271 in [22]. Further, let $\hat{Z} = \mathcal{D}(\hat{A}) + (I - \hat{A})^{-1} BU$. Observe that $Z = \hat{Z} \oplus (E_1 \times \{0\})$. Moreover, equation (5.2) and the definition of γ_1 on p. 270 of [22] imply that $\gamma_\nu v = \gamma_1 f$ if $v = \nabla f$ and $f \in H_{\frac{1}{2}}$. It then follows that

$$\hat{Z} = J(Z_0 \times H_{\frac{1}{2}}) = \left\{ \begin{bmatrix} v \\ g \end{bmatrix} \in E \times H_{\frac{1}{2}} \;\middle|\; \gamma_\nu v \in L^2(\Gamma_1),\ \operatorname{div} v \in H \right\}. \tag{5.6}$$

Finally, the balance inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle P(t)x(t), x(t)\rangle = \|u(t)\|^2 - \|y(t)\|^2 + \langle \dot{P}(t)x(t), x(t)\rangle \tag{5.7}$$

holds for every $t \geq 0$.

4) We now have to relate the system (5.4) with the equations (5.1). First, the initial state x_0 of (5.4) is given by $x_0 = \left[\begin{smallmatrix}\nabla w_0 \\ w_1\end{smallmatrix}\right]$ for the functions w_0, w_1 from the statement of this proposition. Observe that $\nabla w_0 \in E \subset H^n$, $w_1 \in H_{\frac{1}{2}}$, and

$$AP(0)x_0 + Bu(0) = \begin{bmatrix} \nabla w_1 \\ -L^* a(0)\nabla w_0 + K^*(u(0) - \frac{1}{2}Kw_1) \end{bmatrix} =: \begin{bmatrix} \nabla w_1 \\ \psi \end{bmatrix}.$$

In view of step 3, we have to show that $AP(0)x_0 + Bu(0) \in H^n \times H$. This is clear for the first component. To treat the second component ψ, we first note that $a(0)\nabla w_0 \in H(\mathrm{div})$ by our assumptions. So we can apply γ_ν to $a(0)\nabla w_0$ and we can use formula (5.2). Let $\varphi \in H_{\frac{1}{2}}$. Recall that $K = \sqrt{2}\, b\gamma_0$. Using the compatibility condition (5.3) and equation (5.2), we deduce

$$\begin{aligned}
\langle \varphi, \psi\rangle_{H_{\frac{1}{2}}} &= \langle \varphi, -L^* a(0)\nabla w_0 + K^*(u(0) - \frac{1}{2}Kw_1)\rangle_{H_{\frac{1}{2}}} \\
&= \langle L\varphi, -a(0)\nabla w_0\rangle + \langle K\varphi, u(0) - \frac{1}{2}Kw_1\rangle \\
&= -\langle \nabla\varphi, a(0)\nabla w_0\rangle + \langle \gamma_0\varphi, \sqrt{2}\, bu(0) - |b|^2\gamma_0 w_1\rangle \\
&= -\langle \nabla\varphi, a(0)\nabla w_0\rangle + \langle \gamma_0\varphi, \gamma_\nu a(0)\nabla w_0\rangle \\
&= \langle \varphi, \mathrm{div}(a(0)\nabla w_0)\rangle.
\end{aligned}$$

It follows that $\psi \in H$, since $\mathrm{div}(a(0)\nabla w_0) \in H$ by assumption and $H_{\frac{1}{2}}$ is dense in H. Due to step 3), there is a solution $x = \left[\begin{smallmatrix}x_1 \\ x_2\end{smallmatrix}\right] \in C^1([0,\infty), X)$ of (5.4) with $P(\cdot)x(\cdot) \in C([0,\infty), Z)$. In particular, it holds that $x_2 \in C([0,\infty), H_{\frac{1}{2}})$ by (5.6). The first component of (5.4) is $\dot{x}_1(t) = Lx_2(t)$, so that

$$x_1(t) = \nabla w_0 + \nabla \int_0^t x_2(s)\,\mathrm{d}s$$

for every $t \geq 0$. We now define

$$w(t) = w_0 + \int_0^t x_2(s)\,\mathrm{d}s$$

for every $t \geq 0$, whence $\dot{w} = x_2$ and $x_1 = Lw = \nabla w$. The asserted balance equation thus follows from (5.7). We also obtain $w \in C^2([0,\infty), H) \cap C^1([0,\infty), H_{\frac{1}{2}})$. Hence, the second equation in (5.1) holds. Moreover, (5.4) and (5.5) yield the equations

$$\ddot{w}(t) = \dot{x}_2(t) = -L^* a(t)Lw(t) + K^*(u(t) - \frac{1}{2}K\dot{w}(t)) \tag{5.8}$$

$$y(t) = -K\dot{w}(t) + u(t) \tag{5.9}$$

for every $t \geq 0$. Applying a function $\varphi \in H_{\frac{1}{2}}$ to the equation (5.8), we derive

$$\langle \varphi, \ddot{w}(t) \rangle \;=\; -\,\langle \nabla\varphi, a(t)\nabla w(t) \rangle + \langle \gamma_0\varphi, \sqrt{2}\, bu - |b|^2 \gamma_0 \dot{w}(t) \rangle \tag{5.10}$$

for every $t \geq 0$. If φ is even a test function, we arrive at

$$\langle \varphi, \ddot{w}(t) \rangle \;=\; -\,\langle \nabla\varphi, a(t)\nabla w(t) \rangle .$$

Since $\ddot{w}(t) \in H$, the function $a(t)\nabla w(t)$ belongs to $H(\mathrm{div})$ and w satisfies the first line of (5.1). As a consequence, the map $t \mapsto \mathrm{div}(a(t)\nabla w(t))$ from $[0,\infty)$ to H is continuous. Going back to functions $\varphi \in H_{\frac{1}{2}}$, we deduce from equation (5.10) that

$$\begin{aligned} \langle \varphi, \ddot{w}(t) \rangle &= \langle \varphi, \mathrm{div}(a(t)\nabla w(t)) \rangle \\ &\qquad + \langle \gamma_0\varphi, \sqrt{2}\, bu(t) - |b|^2\gamma_0\dot{w}(t) - \gamma_\nu a(t)\nabla w(t) \rangle_{\tilde{\mathcal{H}}^{\frac{1}{2}}(\Gamma)}, \end{aligned} \tag{5.11}$$

using (5.2). We fix a function $\phi \in \tilde{\mathcal{H}}^{\frac{1}{2}}(\Gamma)$. Then there is a $g \in H_{\frac{1}{2}}$ with $\phi = \gamma_0 g$. There are Lipschitz functions χ_k with $0 \leq \chi_k \leq 1$ which are equal to one on $\overline{\Gamma}_1$ and vanish for those $\xi \in \mathbb{R}^n$ whose distance from $\overline{\Gamma}_1$ is larger than $\frac{1}{k}$, for each $k \in \mathbb{N}$. We set $\varphi_k(\xi) = \chi_k(\xi)g(\xi)$ for all $\xi \in \Omega$ and $k \in \mathbb{N}$ and note that $\varphi_k \in H_{\frac{1}{2}}$, $\gamma_0\varphi_k = \phi$ and $\varphi_k \to 0$ in H as $k \to \infty$. If we replace in (5.11) the function φ with φ_k, we conclude

$$\sqrt{2}\, bu(t) \;=\; |b|^2\gamma_0\dot{w}(t) + \gamma_\nu a(t)\nabla w(t)$$

for every $t \geq 0$. Thus the third equation in (5.1) has been shown and the function $t \mapsto \gamma_\nu a(t)\nabla w(t)$ belongs to $C([0,\infty), U)$. The fourth line in (5.1) is then a consequence of (5.9).

5) It remains to show the uniqueness assertion. Let w be a solution of (5.1) with the properties stated in the assertion. Then the function $x := \left[\begin{smallmatrix} \nabla w \\ \dot{w} \end{smallmatrix}\right]$ belongs to $C^1([0,\infty), X)$, its second component x_2 belongs to $C([0,\infty), H_{\frac{1}{2}})$, and it holds $\dot{x}_1 = Lx_2$. We have to verify that also the second component of (5.4) holds. Applying $\varphi \in H_{\frac{1}{2}}$ to the first line of (5.1), we obtain

$$\begin{aligned} \langle \varphi, \dot{x}_2(t) \rangle = \langle \varphi, \ddot{w}(t) \rangle \;&=\; \langle \varphi, \mathrm{div}(a(t)\nabla w(t)) \rangle \\ &= -\langle \nabla\varphi, a(t)\nabla w(t) \rangle + \langle \gamma_0\varphi, \gamma_\nu a(t)\nabla w(t) \rangle_{\tilde{\mathcal{H}}^{\frac{1}{2}}(\Gamma)} \\ &= -\langle L\varphi, a(t)x_1(t) \rangle + \langle \gamma_0\varphi, \sqrt{2}\, bu(t) - |b|^2\gamma_0\dot{w}(t) \rangle \\ &= -\langle \varphi, L^*a(t)x_1(t) \rangle_{H^{\frac{1}{2}}} + \langle K\varphi, u(t) - \frac{1}{2}Kx_2(t) \rangle \\ &= \langle \varphi, -L^*a(t)x_1(t) + K^*(u(t) - \frac{1}{2}Kx_2(t)) \rangle_{H^{\frac{1}{2}}}, \end{aligned}$$

where we also used the equation (5.2), the third line in (5.1), and $K = \sqrt{2}\, b\gamma_0$. As a result, x solves (5.4) which implies the uniqueness of the solutions to (5.1). $\square$

References

[1] P. Acquistapace and B. Terreni, Classical solutions of nonautonomous Riccati equations arising in parabolic boundary control problems, *Appl. Math. & Optim.* **39** (1999), pp. 361–409.

[2] D.Z. Arov and M.A. Nudelman, Passive linear stationary dynamical scattering systems with continuous time, *Integral Equations and Operator Theory* **24** (1996), pp. 1–45.

[3] R.F. Curtain and A.J. Pritchard, *Infinite Dimensional Linear Systems Theory*, Springer-Verlag, Berlin, 1978.

[4] R. Dautray and J.-L. Lions, *Mathematical Analysis and Numerical Methods for Science and Technology. Vol. 3: Spectral Theory and Applications,* Springer-Verlag, Berlin, 1990.

[5] K.-J. Engel and R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, Springer-Verlag, New York, 2000.

[6] H.O. Fattorini, *The Cauchy Problem*, Addison-Wesley, Reading, MA, 1983.

[7] D. Hinrichsen and A.J. Pritchard, Robust stability of linear evolution operators on Banach spaces, *SIAM J. Control & Optim.* **32** (1994), pp. 1503–1541.

[8] B. Jacob, *Time-Varying Infinite Dimensional State-Space Systems*, PhD thesis, Bremen, May 1995.

[9] T. Kato, Linear evolution equations of "hyperbolic" type, *J. Fac. Sci. Univ. Tokyo* **17** (1970), pp. 241–258.

[10] T. Kato, Linear evolution equations of "hyperbolic" type II, *J. Math. Soc. Japan* **25** (1973), pp. 648–666.

[11] P. Lax and R. Phillips, *Scattering Theory*, Academic Press, New York, London, 1967.

[12] J. Malinen, O.J. Staffans and G. Weiss, When is a linear system conservative? *Quarterly of Applied Math.* **64** (2006), pp. 61–91.

[13] N. Okazawa, Remarks on linear evolution equations of hyperbolic type in Hilbert space, *Advances in Math. Sciences and Appl., Tokyo* **8** (1998), pp. 399–423.

[14] A. Rodríguez-Bernal and E. Zuazua, Parabolic singular limit of a wave equation with localized boundary damping, *Discrete Contin. Dynam. Systems* **1** (1995), pp. 303–346.

[15] D. Salamon, Infinite dimensional systems with unbounded control and observation: A functional analytic approach, *Trans. Amer. Math. Soc.* **300** (1987), pp. 383–431.

[16] R. Schnaubelt, Feedbacks for nonautonomous regular linear systems, *SIAM J. Control & Optim.* **41** (2002), pp. 1141–1165.

[17] R. Schnaubelt, Well-posedness and asymptotic behaviour of non-autonomous linear evolution equations, *Evolution equations, semigroups and functional analysis*, A. Lorenzi and B. Ruf (eds.), Birkhäuser, Basel, 2002, pp. 311–338.

[18] O.J. Staffans, Passive and conservative continuous-time impedance and scattering systems. Part I: well-posed systems, *Math. of Control, Signals and Systems* **15** (2002), pp. 291–315.

[19] O.J. Staffans, *Well-Posed Linear Systems*, Cambridge University Press, UK, 2004.

[20] O.J. Staffans and G. Weiss, Transfer functions of regular linear systems, Part II: The system operator and the Lax-Phillips semigroup, *Trans. Amer. Math. Soc.* **354** (2002), pp. 3329–3262.

[21] H. Tanabe, *Equations of Evolution*, Pitman, London, 1979.

[22] M. Tucsnak and G. Weiss, How to get a conservative well-posed linear system out of thin air. Part I. Well-posedness and energy balance, *ESAIM Control Optim. Calc. Var.* **9** (2003), pp. 247–274.

[23] M. Tucsnak and G. Weiss, *Observation and Control for Operator Semigroups*, Birkhäuser Verlag, Basel, 2009.

[24] G. Weiss, Transfer functions of regular linear systems, Part I: Characterizations of regularity, *Trans. Amer. Math. Soc.* **342** (1994), pp. 827–854.

[25] G. Weiss, O.J. Staffans and M. Tucsnak, Well-posed linear systems -a survey with emphasis on conservative systems, *Appl. Math. Computer Sci.* **11** (2001), pp. 101–127.