Solution Enclosures for Scalar Conservation Laws

Zur Erlangung des akademischen Grades eines

DOKTORS DER NATURWISSENSCHAFTEN

von der Fakultät für Mathematik des
Karlsruher Instituts für Technologie (KIT)
genehmigte

DISSERTATION

von
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Tag der mündlichen Prüfung: 30.6.2010
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Danksagung

An dieser Stelle möchte ich ganz herzlich Herrn Prof. Plum danken, der diese Arbeit betreut hat und mir die Möglichkeiten computergestützter Beweise in der Mathematik nahegebracht hat. Ferner möchte ich ihm für seine zahlreichen Verbesserungsvorschläge, diese Arbeit betreffend, danken.

Herrn Prof. Reichel danke ich herzlich für die freundliche Bereitschaft, das Korreferat zu übernehmen.

Maria Radosz möchte ich für zahlreiche Diskussionen danken, die zum Gelingen der Arbeit beigetragen haben, sowie für die genaue Durchsicht eines (schwierigen) Manuskripts auf Fehler.

Moralische Unterstützung ist mir von Giacomo K. zuteil geworden, weswegen ihm auch Dank gebührt.

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Introduction

1.1 General Introduction

1.1.1 Conservation laws in continuum physics

We typically find conservation laws in classical continuum physics, most notably in continuum mechanics, continuum thermodynamics and electrodynamics. The basic unifying principle in these disciplines consists in regarding physical systems as being continuously distributed in a region of space. Thus the physical quantities we use to describe the electromagnetic field or a deformable body of matter are represented by functions which assign values to *every* point in space. This point of view has up to now been very successful in modeling physical processes. It must be noted that even in the disciplines which take into account the quantified nature of matter (e.g. the quantum theory of a single electron or the kinetic theory of gases) the basic quantity is a field (the wave function or the molecular distribution function) in the sense mentioned before.

The basic physical laws are often expressed in the form of conservation laws. This is connected to the concept of *extensive quantities*. Extensive physical quantities (like mass, charge, energy, entropy) are distributed in space: to any region Ω in space at a certain time, a number $Q(t,\Omega)$ is assigned (the quantity of mass, charge etc. contained in Ω). A conservation law reflects the idea that the change of a certain physical extensive quantity $Q(t,\Omega)$, given by the integral over its density $\rho(x,t)$ over the region Ω, is balanced by its flux j through the boundary of Ω and the production $r(x,t)$:

$$\frac{d}{dt}Q(t,\Omega) = \frac{d}{dt}\int_{\Omega}\rho(x,t)dx = -\int_{\partial\Omega} j(x,t)\cdot\nu\, d\sigma + \int_{\Omega} r(x,t)dx$$

In general Q can be scalar- or vector-valued. Given sufficient smoothness of the fields ρ and j the above integral form of the conservation law is equivalent to its local form

$$\frac{\partial \rho}{\partial t}(x,t) + \text{div} j(x,t) = r(x,t), \quad (x,t) \in \mathbb{R}^d \times \mathbb{R}.$$

To arrive at the nonlinear hyperbolic PDE's which are studied in this work, one has to combine conservation laws and *constitutive relations*. In order to illustrate this, we take a look at the basic structure of mechanics. Here, the motion of a deformable body is described by a function

$$\chi : \mathbb{R}^3 \times [0,\infty) \to \mathbb{R}^3$$

(assumed sufficiently smooth) with the property $\chi(X,0) = X$, whose physical interpretation is as follows: $t \to \chi(X_0,t)$ is the trajectory of the material point whose position at time $t = 0$ is $X_0 \in \mathbb{R}^3$. This is the "Lagrangean" or "referential" description of the continuum, in which we think of $X \in \mathbb{R}^3$ indexing the individual material points making up the material. Assume for simplicity that $\chi(\cdot,t) : \mathbb{R}^3 \to \mathbb{R}^3$ is bijective for t fixed and that the inverse mapping $\psi(\cdot,t)$ is sufficiently smooth.

The basic quantities are the velocity field

$$v(X,t) := \frac{\partial \chi}{\partial t}(X,t),$$

the mass density $\rho(X,t)$ (a scalar field) and the *stress* $T(X,t)$ (a symmetric tensor field), thought as functions of the particle X. The Eulerian description of the continuum is constructed by considering the functions $\hat{v}, \hat{\rho}, \hat{T}$ defined by

$$\hat{v}(x,t) := v(\psi(x,t),t), \;\; \hat{\rho}(x,t) := \rho(\psi(x,t),t), \;\; \hat{T}(x,t) := T(\psi(x,t),t).$$

Henceforth we shall drop the hat from the notation (as is customary in mechanics) and write e.g. $v(X,t)$ for the velocity field in the Langrangean description and $v(x,t)$ for the velocity field in the Eulerian description. The basic laws of continuum mechanics, namely *balance of mass, balance of momentum and the balance of kinetic energy* [1] in Eulerian description are given follows:

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \text{div}(\rho v) &= 0 \\ \frac{\partial (\rho v)}{\partial t} + \text{div}(\rho v \otimes v) &= \text{div}\, T \end{aligned}$$

[1] For simplicity, we do not explain the balance of rotational momentum here.

$$\frac{1}{2}\frac{\partial}{\partial t}\rho|v|^2 + \frac{1}{2}\text{div}\rho|v|^2 v \quad = \quad \text{div}\,(Tv).$$

In the second line, the divergence of the tensor field T is given in cartesian components by $(\text{div}T)_i = \sum_{j=1}^{3} \partial_j T_{i,j}$. Using the balance of mass, the second line can be rewritten as

$$\rho\left(\frac{\partial v}{\partial t} + (v \cdot \nabla)v\right) = \text{div}\,T,$$

a form familiar from fluid dynamics.

The balance laws themselves, which are assumed to hold for all material bodies in the domain of classical mechanics, are not sufficient to determine the actual time evolution of the system. This is of course to be expected, since we want to describe a variety of bodies, e.g. elastic and fluid ones, gases, plasmas etc. within this framework. Therefore we supplement the above by so-called *constitutive relations*, which characterize concrete materials (or types of materials). We do not discuss the general principles on which the choice of the constitutive relations is based. It suffices to say that a very general form is given symbolically by

$$T(X,t) = \mathcal{F}(\{\chi(Y, t-s) : Y \in \mathbb{R}^d, 0 \leq s \leq t\}),$$

that is, the stress at some particle x at the present time depends through a particular functional $\mathcal{F}$ on the motion of the continuum at all earlier times.

Further specializing the constitutive relations we arrive at many special mechanical systems. Important examples are *elastic bodies* for which

$$T(X,t) = T(F(X,t)),$$

i.e. the stress at a particle is determined by the deformation gradient $F(X,t) := D_X\chi(X,t) \in \mathbb{R}^{3\times 3}$ at that particle at the present time. In particular,

$$T(X,t) = -p(\rho(X,t))I$$

for an elastic (compressible) fluid or gas, i.e. the stress is isotropic and depends only on the pressure p.

An interesting system of equations are the *magnetohydrodynamic fluid equations* (MHD), which describe the evolution of a gas composed of positively (ions) and negatively (electrons) charged particles, for which the total charge in every (macroscopic) region of space is zero:

$$\rho\left(\frac{\partial v}{\partial t} + (v \cdot \nabla)v\right) \quad = \quad j \times B - \text{grad}\,p$$

$$\begin{aligned}\eta j &= E + v \times B \\ \frac{\partial \rho}{\partial t} + \mathrm{div}(\rho v) &= 0 \\ \frac{\partial \sigma}{\partial t} + \mathrm{div} j &= 0.\end{aligned}$$

Here, σ and j stand for the electric charge density and electric current density, whereas E and B are the electric field and the magnetic induction. p is the hydrodynamic pressure and η the electric conductivity of the plasma.

For a deeper discussion of the concepts from continuum mechanics, we refer to [21]. [4] is an introduction to plasma physics and the equations of magnetohydrodynamics.

1.1.2 Hyperbolic conservation laws

Many balance laws in continuum physics, after combining them with constitutive relations, lead to *hyperbolic systems of conservation laws*. These are partial differential equations of the the form

$$U_t(x,t) + (\mathrm{div} F(U,t))(x,t) = \Pi(U(x,t),t)$$

where U is an unknown $\mathbb{R}^N$-valued function, whose values describes the state of a physical system at a point (x,t), e.g. $U(x,t) = (\rho(x,t), v(x,t))$ in a continuous medium. The (nonlinear) constitutive functions F and Π, coming from the constitutive relations mentioned above, determine the local flux and production of U. In this form, we may study the initial value problem for the above PDE, say on $\mathbb{R}^d \times (0,\infty)$, by first introducing a suitable concept of solution and then asking questions concerning existence and uniqueness. Since we will mainly deal with the scalar case ($N = 1$), we will not need the exact definition of hyperbolicity, but see [5] for more details.

We briefly sketch some basic results for the initial value problem for scalar conservation laws in one space dimension

$$u_t + f(u)_x = 0, \;\; u(\cdot,0) = u_0 \tag{1.1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a given nonlinear function with sufficient smoothness and $u_0 \in L^\infty(\mathbb{R})$. A function $u \in L^\infty(\mathbb{R} \times (0,T))$ is called a *weak solution* of the initial value problem if for all Lipschitz test functions $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ with compact support which satisfy $\phi(\cdot,T) = 0$, the following integral relation holds:

$$\int_{\mathbb{R}\times(0,T)} [u\phi_t + f(u)\phi_x]\, d(x,t) + \int_{\mathbb{R}} \phi(x,0)u_0(x)dx = 0.$$

The solution u is allowed to be discontinuous. The most notable feature of weak solutions is the non-uniqueness of solutions to the initial value problem, a phenomenon not seen when working with classical solutions.

This can be most easily illustrated by considering so-called *shock* solutions, i.e. solutions where $u(\cdot, t)$ contains one or more jump discontinuities as a function of the spatial variable x, which move with time. Consider e.g. Burgers' equation, where

$$f(u) = \frac{1}{2}u^2$$

and let

$$u_0(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}.$$

Then both of the two functions u, v defined by

$$u(x,t) := \begin{cases} 0 & x < \frac{1}{2}t \\ 1 & x > \frac{1}{2}t \end{cases}, \quad v(x,t) := \begin{cases} 0 & x < 0 \\ x/t & 0 < x < t \\ 1 & x > 0 \end{cases}$$

are weak solutions to the initial value problem (1.1). Usually criteria are now introduced to rule out the "inadmissible" or "unphysical" solutions. The first criteria were formulated for solutions with shocks. The *Lax admissibility condition* states that a shock is regarded as a physical one only if the characteristic curves enter the shock forward in time. Here, the characteristic curves are curves $(t, \kappa(t))$ which satisfy $\kappa'(t) = f'(u(x,t))$. According to that criterion, u is not admissible. On the other hand, the function

$$w(x,t) := \begin{cases} 1 & x < \frac{1}{2}t \\ 0 & x > \frac{1}{2}t \end{cases}$$

is also a shock solution to (1.1) with some other appropriate initial data u_0. Here, the characteristic curves enter the shock forward in time (see [5], [6]), and the shock is considered as admissible.

For the scalar conservation law an elegant existence and uniqueness theory, global in time, has been developed by Kruzhkov (see [5], [6]), even for an arbitrary number of space dimensions. It requires the introduction of the stronger notion of *entropy solution*; the initial value problem (1.1) then turns out to be well-posed in the class of entropy solutions, i.e. a unique entropy solution exists for given initial data $u_0 \in L^\infty(\mathbb{R})$ and depends in suitable sense continuously on the initial data. No comparable existence and uniqueness theorems for *systems of equations* in one space dimension exist. Generally speaking, weak solutions for systems can be constructed by several methods, most notably the Glimm scheme, the front tracking method and the vanishing viscosity method. This yields global-in-time existence theorems for initial data which is sufficiently small in a suitable norm and has sufficiently small total variation. For systems in several space dimensions, the situation appears to be unsettled (cf. the discussion in [5], chapter IV).

This work is about computer-assisted solution enclosures for scalar conservation laws, so in the next section, the general methodology of computer-assisted proofs will be described in more detail.

1.1.3 Computer-assisted proofs for partial differential equations

Computer assisted existence proofs and enclosure methods have mainly been developed for semi-linear elliptic boundary value problems, e.g.

$$\begin{cases} -\Delta u(x) + f(x, u(x)) = 0 & (x \in \Omega) \\ u(x) = 0 & x \in \partial\Omega \end{cases} \tag{1.2}$$

where Ω is some given domain. The problem of finding solutions to (1.2) can often be recast into a problem of the form

$$\mathcal{F}(u) = 0 \tag{1.3}$$

where $\mathcal{F} : X \to Y$ is a nonlinear, Frechet differentiable mapping between suitable Banach spaces. The method developed by M. Plum works schematically as follows: first a numerical approximate solution ω to (1.3) is computed. Linearizing at the approximate solution ω and introducing

$$L := \mathcal{F}'(\omega) : X \to Y,$$

(1.3) is equivalent to the fixed-point equation

$$v = -L^{-1}[\mathcal{F}(\omega) + \{\mathcal{F}(\omega + v) - \mathcal{F}(\omega) - L[v]\}] =: T(v)$$

for the difference $v = u - \omega \in X$ between the solution u and the approximate one ω. The goal is now to find a set $D \subset X$, which is mapped into itself under T. Application of a fixed-point theorem then ensures the existence of a solution to $v = T(v)$; hence a true solution u of (1.3) is contained in $\omega + D$. Since usually the set D has small diameter (see the conditions below), we obtain existence of a solution and a *tight* enclosure. We explicitly need quantities $\delta > 0, K > 0, g$ with the following properties:

$$\begin{aligned} \|\mathcal{F}(\omega)\|_Y &\leq \delta \\ \|w\|_X &\leq K\|L[w]\| \quad (w \in X) \end{aligned}$$

$$\|\mathcal{F}'(\omega+u)[w]-\mathcal{F}'(\omega)[w]\|_Y \quad \leq \quad g(\|u\|_X)\|w\|_X \quad (u\in X, w\in X).$$

Here, $g:[0,\infty)\to[0,\infty)$ is a non-decreasing function with $g(t)\to 0$ as $t\to 0^+$. The computation of these quantities depends on the concrete problem, but determining K is usually the most challenging task, since it requires to bound the *inverse of a linear operator*, acting on an *infinite-dimensional* space.

For the semilinear elliptic problems mentioned before, this can be achieved by computer-assisted enclosure of eigenvalues of the linearization L. For this and a more detailed discussion of the fixed-point theorems used, we refer to the survey article [17]. We just mention that one often tries to find a set D in the form of a ball with radius α; the condition

$$\delta \leq \frac{\alpha}{K} - G(\alpha),$$

where $G(t) = \int_0^t g(s)ds$ is then the main requirement for the ball with radius α to be mapped into itself under T. In particular, this condition is likely to be satisfied if δ is small, i.e. ω is a sufficiently good approximate solution.

Another approach was developed by M. Nakao, also based on the use of fixed-point theorems. Whereas in the previous method, the computation of the approximate solution ω was separated from the fixed point argument, Nakao works with a particular discretization (e.g. a Galerkin method with finite element or spectral basis functions) of the given boundary value problem. Using a projection operator corresponding to the discretization (e.g. the projection P_N onto space spanned by the first N basis functions) he splits the problem into a finite-dimensional one and a infinite-dimensional remainder problem. The finite-dimensional part can be treated by methods of verified numerical linear algebra and the infinite-dimensional part can be controlled by explicit remainder estimates for P_N (i.e. estimates for $(I-P_N)$). Nakao's splitting technique can also be used to treat eigenvalue problems. For further discussion, see [16, 18].

In order to correctly perform a computer-assisted proof, a number of computations, such as the evaluations of the defects of the numerical approximate solutions, have to be *verified*. This means that the results of all arithmetical operations performed by the computer must be known to approximate the *true* (i.e. mathematical) results of these operations within a *explicitly known* error bound. In practice, this can be very well done using *interval arithmetic* (see e.g. [8], [19]).

Finally, we would like to mention [14] and [13], where computer-assisted verifications for semilinear parabolic and second-order hyperbolic problems are studied.

1.1.4 Concluding remarks

There is a large body of literature treating the numerical analysis of conservation laws (see [12] and references therein). In particular, there are many good numerical methods available to compute approximate solutions to scalar conservation laws, systems of conservation laws in one space dimension or even systems in two space dimensions (see e.g. [11], chapter 10 and [12], chapter 18).

An important question is of course whether the numerically computed approximate solutions (e.g. given by a finite difference scheme) converge to a solution of the conservation law, e.g. when the grid is refined. The situation for scalar conservation laws seems to be satisfactory (e.g. the theory of monotone methods, see [12], chapter 15), but there are only few convergence results for methods computing solutions to systems in one space dimension (we refer to [12], chapter 15). A somewhat pessimistic statement has been given by P. Lax in [11], section 10.6, after the discussion of a method designed to compute flows of compressible fluids in two space dimensions:

> [...] How much credence can we give to these numerical calculations? There is no proof, and there never will be, that these results approximate the exact solutions of the Riemann problem within some acceptable error bound. Our confidence is based on the remarkable agreement of calculations carried out by Colella and Glaz, and others, using entirely different numerical methods.

Leaving the question aside if convergence proofs are altogether impossible or not, it is interesting to ask whether the numerical computation of solutions can be combined with analytical theorems to prove the existence of solutions in certain concrete cases. We then speak of a *computer-assisted existence proof* or *verified computation* of a solution. In this work, we do not study the convergence of numerical methods, but we make the attempt to verify solutions for conservation laws. It is the first contribution in this direction, and we only treat scalar conservation laws and only solutions which have a particular structure, namely those having a single shock.

1.2 Formulation of the mathematical problem

1.2.1 Single-shock solutions

We will consider weak solutions of the scalar conservation law

$$u_t + f(u)_x = 0, \quad u(\cdot, 0) = u_0 \tag{1.4}$$

(where $(x,t) \in \mathbb{R} \times (0,T)$) which however, have a rather special structure. We require that there exists a Lipschitz function $\gamma : (0,T) \to \mathbb{R}$ (which is unknown and a part of the solution) such that u is continuous to the left and to the right of the curve $(t, \gamma(t))$. $\gamma(t)$ is thus the position of the *shock*. Moreover we assume $\gamma(0) = 0$ and that the initial data is compatible with the structure of the solution, i.e. u_0 is continuous to the left and to the right of $x = 0$ and that $u_0(0^+) \neq u_0(0^-)$. In the following, solutions of that form are called *single-shock* solutions.

For future research, it would be desirable to develop a theory of computer-assisted existence proofs for conservation laws which enclose and prove the existence of discontinuous solutions of more general form, i.e. methods in which it is not needed to track the position of the shock(s). This does not seem to be easy, mainly due to difficulties in setting up a suitable fixed-point formulation. In view of this, we confine the discussion to single-shock solutions.

The special structure of the solution suggests to change independent variables by setting

$$y := x - \gamma(t)$$

so that in the (y,t)-coordinates, the discontinuity is always at $y = 0$. Assume that u is Lipschitz away from the shock. Introducing $w(y,t) := u(y + \gamma(t), t)$ and writing $a(u) = f'(u)$ we find that w solves

$$w_t(y,t) + (a(w(y,t)) - \gamma'(t))w_y(y,t) = 0$$

on $\{y < 0\} \times (0,T)$ and on $\{y > 0\} \times (0,T)$. For notational convenience, we now return to the notation x (instead of y) as the spatial variable.

Now since u was assumed to be a weak solution of (1.4), the **Rankine-Hugionot** condition

$$\gamma'(t) = \frac{f(u(\gamma(t)^+,t)) - f(u(\gamma(t)^-,t))}{u(\gamma(t)^+,t) - u(\gamma(t)^-,t)} = \frac{f(w(0^+,t)) - f(w(0^-,t))}{w(0^+,t) - w(0^-,t)} \tag{1.5}$$

necessarily holds, which is a well-known fact (see e.g. [6]).

Hence, writing $\chi(t) = \gamma'(t)$, we arrive at the following problem

$$\begin{cases} w_t^- + (a(w^-) - \chi)w_x^- = 0 & : \text{ on } \{x < 0\} \times (0,T) \\ w_t^+ + (a(w^+) - \chi)w_x^+ = 0 & : \text{ on } \{x > 0\} \times (0,T) \\ \chi(t) = \sigma(w^-(0,t), w^+(0,t)) & : \text{ on } (0,T) \end{cases} \tag{1.6}$$

where the *shock speed* function σ is given by

$$\sigma(w^-, w^+) := \frac{f(w^+) - f(w^-)}{w^+ - w^-} = \int_0^1 f'(\tau w^+ + (1-\tau)w^-)d\tau$$

$$= \int_0^1 a(\tau w^+ + (1-\tau)w^-)d\tau \tag{1.7}$$

It will be convenient to formulate the problems on bounded domains, since in the following chapters we will use compact imbeddings of Sobolev spaces. In practice, we always compute solutions on some "large" (but bounded) computational domain (say, with periodic boundary conditions in space direction) and then try to verify the existence of solutions on some smaller domain.

Thus, we consider domains Ω^-, Ω^+ defined by Lipschitz functions $\gamma_-, \gamma_+ : [0,T] \to \mathbb{R}$ through

$$\Omega^- := \{(x,t) : \gamma_-(t) < x < 0, t \in (0,T)\}$$
$$\Omega^+ := \{(x,t) : 0 < x < \gamma_+(t), t \in (0,T)\}.$$

and suppose, for technical reasons, that (A.3) holds. That is, that the right-sided derivatives of γ_- and γ_+ are supposed to exist for all $t \in [0,T)$. Motivated by this, we formulate the following problem:

Mathematical Problem: Let w_0^+ and w_0^- be Lipschitz continuous functions on the intervals $(0, \gamma_+(0))$, $(\gamma_-(0), 0)$ respectively. The task is to find (w^-, w^+, χ), where

$$w^\pm : \Omega^\pm \to \mathbb{R}, \quad \chi : (0,T) \to \mathbb{R}$$

$(w^\pm, \in W^{1,\infty}(\Omega^\pm), \chi \in L^\infty(0,T))$ such that

$$\begin{cases} w_t^- + (a(w^-) - \chi)w_x^- = 0 & : \text{ on } \Omega^- \\ w_t^+ + (a(w^+) - \chi)w_x^+ = 0 & : \text{ on } \Omega^+ \\ \chi(t) = \sigma(w^-(0,t), w^+(0,t)) & : \text{ on } (0,T) \\ w^-(\cdot,0) = w_0^-(\cdot), w^+(\cdot,0) = w_0^+(\cdot) & \end{cases} \tag{1.8}$$

and such that

$$a(w^-(0,t)) - \chi(t) > 0, \ \ a(w^+(0,t)) - \chi(t) < 0 \tag{1.9}$$

holds.

The conditions (1.9) correspond to the Lax shock admissibility conditions (see [6]). They characterize "admissible" shocks. We stress that it is not clear whether the above problem is well-posed or not. For given initial data, there may not exist a solution up to the given time T, since other discontinuities could develop in the initially smooth parts of the solution (besides the shock already present at time $t = 0$).

Note that a solution (w^-, w^+, χ) of (1.8) immediately gives a solution of (1.4) by setting

$$\gamma(t) := \int_0^t \chi(\tau)d\tau$$

$$u(x,t) \quad := \quad \begin{cases} w^-(x-\gamma(t),t) & : x < \gamma(t) \\ w^+(x-\gamma(t),t) & : x > \gamma(t) \end{cases}$$

which satisfies the initial condition

$$u(x,0) := \begin{cases} w_0^-(x) & : x < 0 \\ w_0^+(x) & : x > 0 \end{cases} .$$

If we choose $w_0^-(x) = u_0(x)$ for $x < 0$ and $w_0^+(x) = u_0(x)$ for $x > 0$, then we get a solution of the problem (1.4).

Henceforth we shall study problem (1.8) in view of proving the existence of solutions by computer-assisted means.

1.2.2 Linearization of the problem

Now suppose $(\omega^-, \omega^+, \mu)$ is a numerically computed approximate solution of (1.8) which satisfies the initial conditions exactly. We look for a true solution of the form $(\omega^- + v^-, \omega^+ + v^+, \mu + \xi)$, where $v^-(\cdot,0) = v^+(\cdot,0) = 0$. Inserting this into (1.8), we obtain

$$\begin{aligned} 0 &= (\omega^\pm + v^\pm)_t + (a(\omega^\pm + v^\pm) - \mu - \xi)(\omega^\pm + v^\pm)_x \\ &= \omega_t^\pm + (a(\omega^\pm) - \mu)\omega_x^\pm + v_t^\pm + (a(\omega^\pm + v^\pm) - \mu - \xi)(\omega_x^\pm + v_x^\pm) \\ &\quad -(a(\omega^\pm) - \mu)\omega_x^\pm \\ &= d^\pm + v_t^\pm + (a(\omega^\pm + v^\pm) - \mu - \xi)v_x^\pm + a(\omega^\pm)_x v^\pm \\ &\quad +[a(\omega^\pm + v^\pm) - a(\omega^\pm) - a'(\omega^\pm)v^\pm - \xi]\omega_x^\pm \end{aligned}$$

where $d^\pm = \omega_t^\pm + (a(\omega^\pm) - \mu)\omega_x^\pm$. We can rewrite this into an equation for $v^\pm$:

$$\begin{aligned} v_t^\pm + (a(\omega^\pm + v^\pm) - \mu - \xi)v_x^\pm + a(\omega^\pm)_x v^\pm &= \\ -d^\pm - [a(\omega^\pm + v^\pm) - a(\omega^\pm) - a'(\omega^\pm)v^\pm - \xi]\omega_x^\pm & \\ &= -d^\pm - (R^\pm[v^\pm] - \xi)\omega_x^\pm . \end{aligned}$$

Here,

$$R^\pm[v^\pm] := a(\omega^\pm + v^\pm) - a(\omega^\pm) - a'(\omega^\pm)v^\pm . \tag{1.10}$$

Furthermore, writing $\sigma^\pm(t) := \frac{\partial\sigma}{\partial w^\pm}(\omega^-(0,t), \omega^+(0,t))$,

$$\sigma(\omega^-(0,t) + v^-(0,t), \omega^+(0,t) + v^+(0,t)) = (\mu + \xi)(t)$$

holds if and only if

$$\mu(t) - \sigma(\omega^-(0,t), \omega^+(0,t)) + \xi(t) - \sum_{\pm} \sigma^\pm(t) v^\pm(0,\cdot) - R[v^-, v^+](t) = 0,$$

where

$$\begin{aligned} R[v^-, v^+](t) \quad &:= \quad \sigma((\omega^- + v^-)(0,t), (\omega^+ + v^+)(0,t)) \\ &\quad\; -\sigma(\omega^-(0,t), \omega^+(0,t)) - \sum_{\pm} \sigma^\pm(t) v^\pm(0,t). \end{aligned} \tag{1.11}$$

We summarize: given a numerically computed approximate solution $(\omega^-, \omega^+, \mu)$, we try to find (v^-, v^+, ξ) such that

$$\begin{cases} v_t^\pm + (a(\omega^\pm + v^\pm) - \mu - \xi) v_x^\pm + a(\omega^\pm)_x v^\pm = -d^\pm - (R^\pm[v^\pm] - \xi)\omega_x^\pm & : \text{ on } \Omega^\pm \\ \xi - \sum_\pm \sigma^\pm v^\pm(0,\cdot) = -d + R[v^-, v^+] & : \text{ on } (0,T) \\ v^-(\cdot, 0) = 0, v^+(\cdot, 0) = 0 \end{cases} . \tag{1.12}$$

Here, the *defects* are defined by

$$d^\pm \quad := \quad \omega_t^\pm + (a(\omega^\pm) - \mu)\omega_x^\pm \tag{1.13}$$

$$d \quad := \quad \mu - \sigma(\omega^-, \omega^+). \tag{1.14}$$

Later, we will reformulate (1.12) into a fixed-point equation for which existence of a solution will be shown, provided the defects are sufficiently small.

1.2.3 Some remarks on linear hyperbolic problems

As in all computer-assisted verifications, we need sufficient information on the corresponding linearized problems. In case of the hyperbolic equations considered here, the linearized problems pose additional, severe difficulties, apart from the problem of computing explicitly suitable constants bounding the norms of their inverse operators. The following remarks serve to illustrate this problem.

Consider a classical, smooth solution $v : \mathbb{R} \times [0,T] \to \mathbb{R}$ of

$$v_t + a v_x + c v = r, \quad v(\cdot, 0) = 0.$$

The standard L^2-estimate for hyperbolic problems (see e.g. [11], [15], also theorem A.4 in appendix A) is the one which is most easily proven. It gives

$$\|v(\cdot,t)\|_{L^2(\mathbb{R})} \le C \int_0^t \|r(\cdot,\tau)\|_{L^2(\mathbb{R})} d\tau \quad (t \in (0,T)), \tag{1.15}$$

where C depends on the L^∞-norm of a_x. It turns out that the choice of function spaces for a fixed-point argument is highly delicate. Because of the dependency of the constant C above on the L^∞-norm of a_x, the a-priori estimate (1.15) is not suitable for constructing a fixed-point argument. Note that, applying (1.15) to our problem (1.12), we obtain a constant C depending on the L^∞-norm of $v_x^\pm$. Instead we choose to work with L^∞-norms [2].

Also, it is not possible in general to estimate v_x by a norm involving the function values (and no derivatives) of r alone, i.e. it is not possible to have an estimate of the form

$$\|v_x(\cdot,t)\| \le C \int_0^t \|r(\cdot,\tau)\| d\tau \quad (t \in (0,T)),$$

where $\|\cdot\|$ stands e.g. for some L^p-norm on in the space variable. This is different from the situation that holds for linear parabolic and elliptic problems. In order to estimate v_x, for hyperbolic problems, we *also* need norms of the derivative r_x of r.

Consider again (1.12). The problem can be put into fixed-point form as follows: In chapter 2, we will introduce the operator $\mathbf{T}^\pm[u,\xi,r]$ which is the solution operator for the problem (up to modifications due to the presence of certain smoothing operators to be introduced later)

$$v_t^\pm + (a(\omega^\pm + u) - \mu - \xi)v_x^\pm + a(\omega^\pm)_x v^\pm = r, \quad v^\pm(\cdot,0) = 0,$$

the equation holding on $\Omega^\pm$. Then the first line of (1.12) can be written in the form:

$$v^\pm = \mathbf{T}^\pm[v^\pm, \xi, -d^\pm - (R^\pm[v^\pm] - \xi)\omega_x^\pm] \tag{1.16}$$

Now, at first glance, another way of writing a fixed point equation comes into mind, namely re-writing the first line of (1.12) as

$$\begin{aligned} v_t^\pm + (a(\omega^\pm) - \mu - \xi)v_x^\pm + a(\omega^\pm)_x v^\pm \\ = -d^\pm - (R^\pm[v^\pm] - \xi)\omega_x^\pm - (a(\omega^\pm + v^\pm) - a(\omega^\pm))v_x^\pm. \end{aligned}$$

Let us define $\hat{\mathbf{T}}^\pm$ to be the inverse of the linear hyperbolic operator on the left, i.e. define $\hat{\mathbf{T}}^\pm$ to be the solution operator for the problem

$$\begin{cases} v_t^\pm + (a(\omega^\pm) - \mu - \xi)v_x^\pm + a(\omega^\pm)_x v^\pm = r & \text{on } \Omega^\pm \\ v^\pm(\cdot,0) = 0 \end{cases}$$

[2] In general, L^∞-norms estimates for linear hyperbolic systems are only available when we have one space dimension.

with given r (and ξ) and then consider the fixed-point equation

$$v^{\pm} = \hat{\mathbf{T}}^{\pm}[\xi, -d^{\pm} - (R^{\pm}[v^{\pm}] - \xi)\omega_x^{\pm} - (a(\omega^{\pm} + v^{\pm}) - a(\omega^{\pm}))v_x^{\pm}].$$

This, however, leads to the following fundamental difficulty: in order to construct a set, which is mapped into itself, we need to control

$$-d^{\pm} - (R^{\pm}[v^{\pm}] - \xi)\omega_x^{\pm} - (a(\omega^{\pm} + v^{\pm}) - a(\omega^{\pm}))v_x^{\pm}. \tag{1.17}$$

in a suitable norm. That is, we have to control the norm of $v_x^{\pm}$. In order to satisfy the self-mapping property, we would then have to control

$$D_x\hat{\mathbf{T}}^{\pm}[\xi, -d^{\pm} - (R^{\pm}[v^{\pm}] - \xi)\omega_x^{\pm} - (a(\omega^{\pm} + v^{\pm}) - a(\omega^{\pm}))v_x^{\pm}];$$

since the inverses of hyperbolic operators have no smoothing properties, this requires control over the first derivative in x-direction of (1.17), hence we would need to control the *second derivative* of $v^{\pm}$. This "loss of derivatives" phenomenon represents a substantial difficulty in arriving at a suitable fixed-point formulation, which however we will be able to resolve using a modified form of the formulation (1.16), and L^{∞}-norms.

1.3 Mathematical preliminaries

In this section, we summarize certain facts from functional analysis which will be used in this work. In connection with linear hyperbolic problems, we often use the spaces ($\Omega \subset \mathbb{R}^2$ being a bounded domain with *Lipschitz boundary*, using the notation (x,t) for points in $\mathbb{R}^2$)

$$V^k(\Omega) := \{u \in L^{\infty}(\Omega) : D_x^j u \in L^{\infty}(\Omega) \text{ for } j = 0, \ldots, k\} \qquad (k \geq 1) \tag{1.18}$$

with norms

$$\|u\|_{V^k(\Omega)} := \sum_{j=0}^{k} \|D_x^j u\|_{L^{\infty}(\Omega)}.$$

In the definition above, we understand the derivatives appearing to be partial derivatives in the weak sense (see [6], [1]). We will sometimes use subscripts such as u_x, u_t to denote weak partial differentiation, but also

$$D_x u, D_t u, D_{xt} u,$$

etc. to indicate the corresponding derivatives, in particular if the symbol for the function to be differentiated already has too many subscripts. Consider now the space $W^{1,\infty}(\Omega)$ of bounded measurable functions with the weak partial derivatives of first order in $L^\infty(\Omega)$, where $\Omega \subset \mathbb{R}^2$ is supposed to have a Lipschitz boundary. It can be shown that $u \in W^{1,\infty}(\Omega)$ has a Lipschitz continuous representative $\tilde{u} : \bar{\Omega} \to \mathbb{R}$; more precisely there exists a $C > 0$ depending only on Ω such that each $u \in W^{1,\infty}(\Omega)$ has a representative $\tilde{u} \in C(\bar{\Omega})$ for which[3]

$$|\tilde{u}(x,t) - \tilde{u}(y,s)| \leq C\|\nabla u\|_{L^\infty(\Omega)}|(x,t) - (y,s)| \quad ((x,t),(y,s) \in \bar{\Omega}) \tag{1.19}$$

holds. That is, the Lipschitz constant of $\tilde{u}$ is less than $C\|\nabla u\|_{L^\infty(\Omega)}$. Clearly, we can define the trace of u on $\partial\Omega$ by

$$u|_{\partial\Omega} = \tilde{u}|_{\partial\Omega}$$

where $\tilde{u}$ is any Lipschitz representative of u on $\bar{\Omega}$. Note that the convergence of $u_j \in W^{1,\infty}(\Omega)$ in L^∞-norm to some $u \in W^{1,\infty}(\Omega)$ also implies the uniform convergence of $u_j|_{\partial\Omega}$ to $u|_{\partial\Omega}$.

Typically, we consider functions on domains of the form

$$\Omega^- := \{(x,t) : \gamma_-(t) < x < 0, t \in (0,T)\}$$
$$\Omega^+ := \{(x,t) : 0 < x < \gamma_+(t), t \in (0,T)\}$$

with given Lipschitz functions γ_-, γ_+. Then the discussion above leads to

Remark 1.1. *For any $v \in W^{1,\infty}(\Omega^\pm)$, the function $\varphi(t) := v(0,t)$ is Lipschitz on $(0,T)$ with Lipschitz constant less than $C\|\nabla v\|_{L^\infty(\Omega^\pm)}$, where C only depends on Ω.*

We will often have to use the following compactness theorems. Recall the notion of *weak-** convergence in L^∞ (see [7]); note that $L^\infty(\Omega)$ is the dual of $L^1(\Omega)$.

Definition 1.1. A sequence $\{u_j\} \subset L^\infty(\Omega)$ is said to converge in the weak-*-sense to $u \in L^\infty(\Omega)$ if and only if

$$\int_\Omega u_j \phi dx \to \int_\Omega u\phi dx \quad (j \to \infty)$$

for all $\phi \in L^1(\Omega)$.

[3] This can be proven as follows: using the mean-value theorem from calculus and and a standard approximation argument, we may show that any $v \in W^{1,\infty}(\mathbb{R}^2)$ has a representative which is Lipschitz on $\mathbb{R}^2$ and s.t.

$$|v(z_1) - v(z_2)| \leq |z_1 - z_2|\|\nabla v\|_{L^\infty(\mathbb{R}^2)} \quad (z_1, z_2 \in \mathbb{R}^2).$$

On the other hand, by theorem 3 in chapter 6, section 4 of [2], there exist a linear, bounded extension operator $T : W^{1,\infty}(\Omega) \to W^{1,\infty}(\mathbb{R}^2)$. Thus in particular, Tu for all $u \in W^{1,\infty}(\Omega)$ has a representative $\tilde{u} \in C(\bar{\Omega})$ such that

$$|\tilde{u}(z_1) - \tilde{u}(z_2)| \leq |z_1 - z_2|\|\nabla Tu\|_{L^\infty(\mathbb{R}^2)} \leq |z_1 - z_2|C\|\nabla u\|_{L^\infty(\Omega)} \quad (z_1, z_2 \in \bar{\Omega}).$$

Theorem 1.2. *A bounded sequence* $\{u_j\} \subset L^\infty(\Omega)$ *contains a weak-*-convergent subsequence.*

Keeping in mind that functions in $W^{k+1,\infty}(\Omega)$ have Lipschitz-continuous derivatives up to the k-th order, and in view of (1.19), the following theorem is a consequence of the Arzela-Ascoli theorem.

Theorem 1.3. *A sequence* $\{u_j\} \subset W^{k+1,\infty}(\Omega)$*, bounded in* $W^{k+1,\infty}(\Omega)$*, contains a subsequence which converges in* $W^{k,\infty}(\Omega)$.

The following standard subsequence argument will be found to be useful. Suppose we are given a sequence $\{u_j\}$ in some normed space and an element u with the property that *any* subsequence of $\{u_j\}$ contains a subsequence converging to u. Then an elementary argument shows that the whole sequence $\{u_j\}$ converges to u in norm.

1.3.1 Notation

Ω^-	typically denotes a domain bounded on the left by a Lipschitz function $\gamma_-(t)$ and such that $\{(0,t) : t \in (0,T)\}$ is contained in $\partial\Omega^-$. Moreover, the right-sided derivative $\gamma_-(t^+)$ is supposed to exist for all $t \in (0,T)$.
Ω^+	analogous to Ω^- above, bounded on the right by a Lipschitz function $\gamma_+(t)$.
$\Omega^\pm_\tau$	$\Omega^\pm_\tau = \{x : (x,\tau) \in \Omega^\pm\}$, $\tau \in (0,T)$
$\|v\|_\nu$	$\|v\|_\nu := \sup_{t\in(0,T)} e^{-\nu t}\|v(\cdot,t)\|_{L^\infty(\Omega_t)}$
$\|\|v\|\|_\nu$	$\|\|v\|\|_\nu := \int_{(0,T)} e^{-\nu t}\|v(\cdot,t)\|_{L^\infty(\Omega_t)}dt$
$V^k(\Omega)$	space of bounded, measurable functions on Ω such that all weak partial derivatives w.r.t. x up to order k exist and are bounded on Ω
$\|u\|_{V^k(\Omega)}$	$\|u\|_{V^k(\Omega)} := \sum_{j=0}^k \|D_x^j u\|_{L^\infty(\Omega)}$

Chapter 2

Fixed Point Formulation

In this chapter, we develop a suitable fixed-point formulation to overcome the aforementioned difficulties.

2.1 Some operators

2.1.1 Fixed point formulation; preliminary considerations

As in chapter 1, we assume that a numerical approximate solution $(\omega^-, \omega^+, \mu)$ of (1.8) is given. The precise regularity we require is as follows, understanding all derivatives in the weak sense:

$$\omega^{\pm}, \omega_x^{\pm}, \omega_t^{\pm}, \omega_{tx}^{\pm}, \omega_{xx}^{\pm} \in L^{\infty}(\Omega^{\pm}) \tag{2.1}$$

$$\mu \in BV(0,T), \tag{2.2}$$

Furthermore, we shall assume

$$a \in C^2(\mathbb{R}). \tag{2.3}$$

As a consequence of (2.1) and (2.2), the defects satisfy

$$\begin{aligned} d^{\pm} &= \omega_t^{\pm} + (a(\omega^{\pm}) - \mu)\omega_x^{\pm} \in L^{\infty}(\Omega^{\pm}), \\ d_x^{\pm} &\in L^{\infty}(\Omega^{\pm}) \\ d &= \mu - \sigma(\omega^-, \omega^+) \in BV(0,T). \end{aligned}$$

Recall that the function σ is given by

$$\sigma(w^-, w^+) = \int_0^1 a(\tau w^+ + (1-\tau)w^-)d\tau. \tag{2.4}$$

On account of our assumption (2.3), it is easy to show that $\sigma : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is a $C^2(\mathbb{R}\times\mathbb{R})$ function. We also repeat the definition

$$\sigma^{\pm}(t) := \frac{\partial \sigma}{\partial w^{\pm}}(\omega^-(0,t),\omega^+(0,t)) \quad (t \in (0,T)). \tag{2.5}$$

Recall that the task is to prove the existence of (v^-, v^+, ξ) satisfying (see 1.12)

$$\begin{cases} v_t^- + (a(\omega^- + v^-) - \mu - \xi)v_x^- + a(\omega^-)_x v^- = -d^- - (R^-[v^-] - \xi)\omega_x^- \\ v_t^+ + (a(\omega^+ + v^+) - \mu - \xi)v_x^+ + a(\omega^+)_x v^+ = -d^+ - (R^+[v^+] - \xi)\omega_x^+ \\ \xi(\cdot) - \sigma^- v^-(0,\cdot) - \sigma^+ v^+(0,\cdot) = -d + R[v^-, v^+] \\ v^{\pm}(\cdot, 0) = 0 \end{cases} \tag{2.6}$$

where

$$R^{\pm}[v^{\pm}] \;:=\; a(\omega^{\pm} + v^{\pm}) - a(\omega^{\pm}) - a'(\omega^{\pm})v^{\pm} \tag{2.7}$$

$$R[v^-, v^+] \;:=\; \sigma(\omega^- + v^-, \omega^+ + v^+) - \sigma(\omega^-, \omega^+) - \sum_{\pm} \sigma^{\pm} v^{\pm}. \tag{2.8}$$

Note again that $R^{\pm}[v^{\pm}]$ are elements of a space of functions on $\Omega^{\pm}$ and $R[v^-, v^+]$ is a function on $(0,T)$, i.e. in (2.8) we understand $v^{\pm}$ to mean $v^{\pm}(0,\cdot)$.

In the next subsection we will introduce the crucial fixed-point operator $\mathbf{\Phi}_\theta$. The discussion in the rest of this subsection is informal and serves as an outline for the following developments.

Recall the solution operator $\mathbf{T}^{\pm}[u^{\pm}, \xi, r]$ of[1]

$$\begin{cases} v_t^{\pm} + (a(\omega^{\pm} + u^{\pm}) - \mu - \xi)v_x^{\pm} + a(\omega^{\pm})_x v^{\pm} = r \\ v^{\pm}(\cdot, 0) = 0 \end{cases} \tag{2.9}$$

(i.e. $\mathbf{T}^{\pm}[u^{\pm}, \xi, r] = v^{\pm}$ if and only if $v^{\pm}$ solves the above linear hyperbolic problem with given r, ξ and $u^{\pm}$) and let $\mathbf{K}[u^-, u^+, \xi, r]$ denote the solution operator of the following problem for η:

$$\eta - \sigma^-(t)\mathbf{T}^-[u^-, \xi, \eta\omega_x^-](0,t) - \sigma^+(t)\mathbf{T}^+[u^+, \xi, \eta\omega_x^+](0,t) = r(t). \tag{2.10}$$

Here, note that the unknown η is a function on $(0,T)$ and r, ξ are given functions on $(0,T)$. For the moment, we leave aside the question whether (2.10) is well-posed or not. Later on, we use a modified version of $\mathbf{K}$ (denoted by $\mathbf{K}_\theta$) for which we prove well-posedness in Theorem 2.5.

[1] We leave out the precise definition of the domain of $\mathbf{T}^{\pm}$, since this will be the topic of the following subsections.

Lemma 2.1. *Any fixed point (v^-, v^+, ξ) of the following operator solves (2.6):*

$$\mathbf{\Phi}[v^-, v^+, \xi] := \begin{pmatrix} \mathbf{T}^-[v^-, \xi, \mathbf{K}[v^-, v^+, \xi, r(v^-, v^+, \xi)]\omega_x^- + r^-(v^-)] \\ \mathbf{T}^+[v^+, \xi, \mathbf{K}[v^-, v^+, \xi, r(v^-, v^+, \xi)]\omega_x^+ + r^+(v^+)] \\ \mathbf{K}[v^-, v^+, \xi, r(v^-, v^+, \xi)] \end{pmatrix}. \tag{2.11}$$

Here we write

$$\begin{aligned} r^\pm(v^\pm) &:= -d^\pm - R^\pm[v^\pm]\omega_x^\pm \\ r(v^-, v^+, \xi)(t) &:= -d(t) + R[v^-, v^+](t) + \sigma^-(t)\mathbf{T}^-[v^-, \xi, r^-(v^-)](0,t) \\ &\quad\; +\sigma^+(t)\mathbf{T}^+[v^+, \xi, r^+(v^+)](0,t). \end{aligned}$$

Proof. Let (v^-, v^+, ξ) be a fixed point of $\mathbf{\Phi}$. Thus ξ equals the third component of the triple $\mathbf{\Phi}[v^-, v^+, \xi]$, i.e.

$$\xi = \mathbf{K}[v^-, v^+, \xi, r(v^-, v^+, \xi)]. \tag{2.12}$$

From the definition of $r(v^-, v^+, \xi)$ and $\mathbf{K}$ we see that

$$\xi(t) - \sum_\pm \sigma^\pm(t)\mathbf{T}^\pm[v^\pm, \xi, \xi\omega_x^\pm + r^\pm(v^\pm)](0,t) = -d(t) + R[v^-, v^+](t) \tag{2.13}$$

holds for $t \in (0,T)$ (here we have used the fact that $r \mapsto \mathbf{T}^\pm[u, \xi, r]$ is linear for (u, ξ) fixed). Since on the other hand v^- equals the first and v^+ equals the second component of $\mathbf{\Phi}[v^-, v^+, \xi]$, it follows that

$$\begin{aligned} v^\pm &= \mathbf{T}^\pm[v^\pm, \xi, \mathbf{K}[v^-, v^+, \xi, r(v^-, v^+, \xi)]\omega_x^\pm + r^\pm(v^\pm)] = \\ &\quad \mathbf{T}^\pm[v^\pm, \xi, \xi\omega_x^\pm + r^\pm(v^\pm)] \end{aligned} \tag{2.14}$$

after recalling (2.12). Returning to (2.13), this implies

$$\xi(t) - \sum_\pm \sigma^\pm(t)v^\pm(0,t) = -d(t) + R[v^-, v^+](t) \quad (t \in (0,T)),$$

i.e. the third line of (2.6) holds true. From (2.14) and the definition of $\mathbf{T}^\pm$ we get

$$v_t^\pm + (a(\omega^\pm + v^\pm) - \mu - \xi)v_x^\pm + a(\omega^\pm)_x v^\pm = \xi\omega_x^\pm + r^\pm(v^\pm) = \xi\omega_x^\pm - d^\pm - R^\pm[v^\pm]\omega_x^\pm$$

and $v^\pm(\cdot, 0) = 0$, which shows that the the first two lines and the last line of (2.6) hold. □

As already mentioned in the first chapter, this particular fixed point formulation avoids the "loss-of-derivatives" phenomenon. Roughly speaking, this can be seen as follows.

One can prove a bound of the form (see Lemma 2.7)

$$\|r^{\pm}(v^{\pm})\|_{V^1(\Omega^{\pm})} \le b\left(\|v^{\pm}\|_{V^1(\Omega^{\pm})}\right),$$

where b is some nonnegative monotone nondecreasing function. By the a-priori estimates available for linear hyperbolic problems (see Appendix A, theorem A.16), the V^1-norm of

$$\mathbf{T}^{\pm}[v^{\pm}, \xi, \mathbf{K}[v^-, v^+, \xi, r(v^-, v^+, \xi)]\omega_x^{\pm} + r^{\pm}(v^{\pm})]$$

(first and second component of $\mathbf{\Phi}$) is controlled by the $L^{\infty}(0,T)$-norm of $\mathbf{K}[v^-, v^+, \xi, r(v^-, v^+, \xi)]$ and the V^1-norm of $r^{\pm}$. This works even though $v^{\pm}$ enters in the transport coefficients of the hyperbolic problem (2.9) defining $\mathbf{T}^{\pm}$, at least if (v^-, v^+, ξ) is "sufficiently" small in some sense. $\mathbf{K}$ in turn can be bounded by the V^1-norm of $v^{\pm}$ and the $L^{\infty}(0,T)$-norm of ξ.

It is hence natural to seek a set D, which is mapped into itself, in the space $V^1(\Omega^-) \times V^1(\Omega^+) \times L^{\infty}(0,T)$. Unlike in the discussion at the end of chapter 1, we see that no principal regularity trouble arises which would prevent from the outset the construction of D.

Unfortunately, the operator $\mathbf{\Phi}$ involves the operators $\mathbf{T}^{\pm}$ which are not compact considered as operators taking $r \in V^1$ into $\mathbf{T}^{\pm}[u^{\pm}, \xi, r] \in V^1$. Therefore, we cannot apply Schauder's fixed point theorem directly; other fixed point theorems (e.g. Banach's) are not easily applicable, due to the structure of the operator $\mathbf{\Phi}$.

The problem will be overcome in three steps:

(i) We regularize by introducing suitable smoothing operators (mollifiers). By using them, we will be able to build an operator $\mathbf{\Phi}_{\theta}$ depending on a parameter $\theta > 0$ which has the required compactness and continuity properties we need for Schauder's fixed point theorem.

(ii) We construct set a D, *independent* of θ, such that $\mathbf{\Phi}_{\theta}(D) \subset D$; The fixed point theorem will yield then for *each* sufficiently small $\theta > 0$ a fixed point of $\mathbf{\Phi}_{\theta}$.

(iii) By a weak compactness argument, these fixed points converge to some triple $(v^-, v^+, \xi) \in D$, a solution to (2.6).

Also, in checking compactness and continuity, it will not be necessary to keep track of the precise form of the estimations and values of constants. These kinds of considerations will become important only in chapter 3, where the set D is constructed by *concrete* inequalities.

2.1.2 Regularized operators.

In appendix A, we have introduced a smoothing operator $S_\theta^\pm : V^1(\Omega^\pm) \to V^1(\Omega^\pm)$ (with parameter θ) having the following properties (see Lemma A.8, recall that $\Omega_t^\pm = \{x \in \mathbb{R} : (x,t) \in \Omega^\pm\}$):

$$\left.\begin{array}{l} \|S_\theta^\pm v\|_{L^\infty(\Omega^\pm)} \le \|v\|_{L^\infty(\Omega^\pm)} \\ \|D_x S_\theta^\pm v\|_{L^\infty(\Omega^\pm)} \le \|D_x v\|_{L^\infty(\Omega^\pm)} \\ \|D_x^j (S_\theta^\pm v)\|_{L^\infty(\Omega^\pm)} \le C(j,\theta)\|D_x v\|_{L^\infty(\Omega^\pm)} \quad (j \ge 2) \\ S_\theta^\pm u \to u \ \text{ in } \ L^p(\Omega^\pm) \ \text{ as } \ \theta \to 0^+ \quad (p \in [1,\infty)) \end{array}\right\}. \tag{2.15}$$

We also have

$$\left.\begin{array}{l} \|S_\theta^\pm v\|_{L^\infty(\Omega_t^\pm)} \le \|v\|_{L^\infty(\Omega_t^\pm)} \quad (t \in (0,T)) \\ \|D_x S_\theta^\pm v\|_{L^\infty(\Omega_t^\pm)} \le \|D_x v\|_{L^\infty(\Omega_t^\pm)} \quad (t \in (0,T)) \end{array}\right\} \tag{2.16}$$

Here, $C(j,\theta) \to \infty$ as $\theta \to 0^+$. In addition to the smoothing operators $S_\theta^\pm$ above, we need a smoothing operator acting on functions in $L^\infty(0,T)$:

Definition 2.1. We set

$$(\hat{S}_\theta u)(t) := (\rho_\theta * Eu)(t) = \int_{\mathbb{R}} \rho_\theta(t-s)Eu(s)\, ds \quad (u \in L^\infty(0,T)) \tag{2.17}$$

where ρ_θ is the standard mollifier (see section A.2) and Eu is a trivial extension operator; $Eu = u$ on $(0,T)$ and zero outside.

Remark 2.2. *A well-known consequence (see [6]) of the preceding definition is*

$$\|\hat{S}_\theta u\|_{L^\infty(0,T)} \le \|u\|_{L^\infty(0,T)}. \tag{2.18}$$

Recall moreover that the approximate solutions $\omega^\pm$ are given on domains of the form

$$\Omega^- := \{(x,t) : \gamma_-(t) < x < 0, t \in (0,T)\}$$
$$\Omega^+ := \{(x,t) : 0 < x < \gamma_+(t), t \in (0,T)\}.$$

The sets

$$\Gamma^- := \{(x,t) : t \in [0,T], x = \gamma_-(t) \ \text{ or } \ x = 0\}$$
$$\Gamma^+ := \{(x,t) : t \in [0,T], x = \gamma_+(t) \ \text{ or } \ x = 0\}$$

play an important role in determining the well-posedness of the relevant linear hyperbolic problems. Namely, recall from appendix A that the hyperbolic problem defining the operator $\mathbf{T}^\pm$ above is well-posed if

$$a(\omega^\pm + u^\pm) - \mu - \xi$$

is outward-pointing on $\Gamma^{\pm}$.

We now introduce regularized versions of the operators $\mathbf{T}^{\pm}$. To clarify their precise domains of definition, the following sets are useful. Letting $\kappa > 0, \theta > 0$ be arbitrary positive numbers, we set

$$Z^{\pm}_{\theta,\kappa} := \{(u^{\pm},\xi) \in V^1(\Omega^{\pm}) \times L^{\infty}(0,T) : \|u^{\pm}\|_{V^1(\Omega^{\pm})} \le \kappa, \tag{2.19}$$

$$a(\omega^{\pm} + S^{\pm}_{\theta} u) - \mu - \xi \text{ is outward-pointing on } \Gamma^{\pm}\} \tag{2.20}$$

$$Z_{\theta,\kappa} := \{(u^-,u^+,\xi) : (u^-,\xi) \in Z^-_{\theta,\kappa}, (u^+,\xi) \in Z^+_{\theta,\kappa}\} \tag{2.21}$$

Definition 2.2. [2] $\mathbf{T}^{\pm}_{\theta} : Z^{\pm}_{\theta,\kappa} \times V^1(\Omega^{\pm}) \mapsto V^1(\Omega^{\pm})$ is defined as follows: $\mathbf{T}^{\pm}_{\theta}[u,\xi,r] = v^{\pm}$ if and only if $v^{\pm}$ satisfies[3]

$$\begin{cases} v^{\pm}_t + (a(\omega^{\pm} + S^{\pm}_{\theta} u) - \mu - \xi) v^{\pm}_x + S^{\pm}_{\theta} a(\omega^{\pm})_x v^{\pm} = r \\ v^{\pm}(\cdot,0) = 0. \end{cases} \tag{2.22}$$

Proposition 2.3. *There exists a $\nu_0 = \nu_0(\kappa) > 0$ (depending on a and $\omega^{\pm}$) such that*

$$\|D_x a(\omega^{\pm} + S^{\pm}_{\theta} u^{\pm})\|_{L^{\infty}(\Omega^{\pm})} + \|S^{\pm}_{\theta} a(\omega^{\pm})_x\|_{L^{\infty}(\Omega^{\pm})} \le \nu_0$$

for all $\theta > 0$ and all $(u^{\pm},\xi) \in Z^{\pm}_{\theta,\kappa}$.

Proof. Using (2.15) we see that

$$\begin{aligned} \|\omega^{\pm} + S^{\pm}_{\theta} u^{\pm}\|_{L^{\infty}(\Omega^{\pm})} &\le \|\omega^{\pm}\|_{L^{\infty}(\Omega^{\pm})} + \|S^{\pm}_{\theta} u^{\pm}\|_{L^{\infty}(\Omega^{\pm})} \\ &\le \|\omega^{\pm}\|_{L^{\infty}(\Omega^{\pm})} + \|u^{\pm}\|_{L^{\infty}(\Omega^{\pm})} \\ &\le \|\omega^{\pm}\|_{L^{\infty}(\Omega^{\pm})} + \kappa. \end{aligned}$$

We estimate, using (2.15) again:

$$\begin{aligned} \|D_x a(\omega^{\pm} + S^{\pm}_{\theta} u^{\pm})\|_{L^{\infty}(\Omega^{\pm})} &= \|a'(\omega^{\pm} + S^{\pm}_{\theta} u^{\pm})(\omega^{\pm}_x + D_x S_{\theta} u^{\pm})\|_{L^{\infty}(\Omega^{\pm})} \\ &\le \sup_{|y| \le \|\omega^{\pm}\|_{L^{\infty}(\Omega^{\pm})} + \kappa} |a'(y)| \left(\|\omega^{\pm}_x\|_{L^{\infty}(\Omega^{\pm})} + \|D_x S_{\theta} u^{\pm}\|_{L^{\infty}(\Omega^{\pm})} \right) \\ &\le \sup_{|y| \le \|\omega^{\pm}\|_{L^{\infty}(\Omega^{\pm})} + \kappa} |a'(y)| \left(\|\omega^{\pm}_x\|_{L^{\infty}(\Omega^{\pm})} + \|D_x u^{\pm}\|_{L^{\infty}(\Omega^{\pm})} \right) \\ &\le \sup_{|y| \le \|\omega^{\pm}\|_{L^{\infty}(\Omega^{\pm})} + \kappa} |a'(y)| \left(\|\omega^{\pm}_x\|_{L^{\infty}(\Omega^{\pm})} + \kappa \right), \end{aligned}$$

[2] Formally speaking, the mapping $\mathbf{T}^{\pm}_{\theta}$ depends on κ through its domain of definition. In order not to overburden the notation, we write $\mathbf{T}^{\pm}_{\theta}$ and not $\mathbf{T}^{\pm}_{\theta,\kappa}$.

[3] In writing $S^{\pm}_{\theta} a(\omega^{\pm})_x$ we mean $S^{\pm}_{\theta}(D_x a(\omega^{\pm}))$.

i.e. we have found an upper bound independent of θ. Moreover, we see that

$$\|S_\theta^\pm a(\omega^\pm)_x\|_{L^\infty(\Omega^\pm)} \leq \|a(\omega^\pm)_x\|_{L^\infty(\Omega^\pm)}$$

again using (2.15) to get rid of the smoothing. This yields the claim. □

Remark 2.4. *Note carefully that, because μ, ξ are functions of time only,*

$$a(\omega^\pm + S_\theta^\pm u^\pm) - \mu - \xi \in V^1(\Omega^\pm)$$

(only derivatives in x-direction are required in the definition of V^1). Moreover, $S_\theta^\pm a(\omega^\pm)_x \in V^1(\Omega^\pm)$ (see (2.1)*) and thus the problem* (2.22) *has a unique solution according to Theorem A.16. If we fix $\kappa > 0$ and choose $\nu > \nu_0(\kappa)$ in Theorem A.16, then* (A.32) *is fulfilled and we obtain (by taking the definition of the $\|\cdot\|_\nu$-norms into account, see e.g. subsection 1.3.1)*

$$\left.\begin{array}{l} \|\mathbf{T}_\theta^\pm[u,\xi,r](\cdot,t)\|_{L^\infty(\Omega_t^\pm)} \leq C_\kappa \int_0^t \|r\|_{L^\infty(\Omega_\tau^\pm)} d\tau \quad (t \in (0,T)) \\ \|D_x \mathbf{T}_\theta^\pm[u,\xi,r]\|_{L^\infty(\Omega^\pm)} \leq C_\kappa \|r\|_{V^1(\Omega^\pm)} \\ \|D_t \mathbf{T}_\theta^\pm[u,\xi,r]\|_{L^\infty(\Omega^\pm)} \leq C_\kappa \|r\|_{V^1(\Omega^\pm)} \end{array}\right\} \tag{2.23}$$

for all $(u,\xi) \in Z_{\theta,\kappa}^\pm$, $r \in V^1(\Omega^\pm)$ and all $\theta > 0$ (recall that $\Omega_\tau^\pm = \{x \in \mathbb{R} : (x,\tau) \in \Omega^\pm\}$).

Theorem 2.5. *Let $(u^-, u^+, \xi, r) \in Z_{\theta,\kappa} \times L^\infty(0,T)$; then the problem*

$$\eta - \sigma^- \mathbf{T}_\theta^-[u^-, \xi, \eta S_\theta^- \omega_x^-](0,\cdot) - \sigma^+ \mathbf{T}_\theta^+[u^+, \xi, \eta S_\theta^+ \omega_x^+](0,\cdot) = r \tag{2.24}$$

has a unique solution in $\eta \in L^\infty(0,T)$. Moreover, there exists a $C_{\theta,\kappa} > 0$ such that

$$\|\eta\|_{L^\infty(0,T)} \leq C_{\theta,\kappa} \|r\|_{L^\infty(0,T)}. \tag{2.25}$$

Definition 2.3. The problem given in (2.24) has a unique solution. Define the operator $\mathbf{K}_\theta : Z_{\theta,\kappa} \times L^\infty(0,T) \to L^\infty(0,T)$ by

$$\mathbf{K}_\theta[u^-, u^+, \xi, r] = \eta \Leftrightarrow \eta \text{ solves (2.24)}. \tag{2.26}$$

Proof. Let

$$H[\eta] = \sigma^-(\cdot)\mathbf{T}_\theta^-[u^-, \xi, \eta S_\theta^- \omega_x^-](0,\cdot) + \sigma^+(\cdot)\mathbf{T}_\theta^+[u^+, \xi, \eta S_\theta^+ \omega_x^+](0,\cdot); \tag{2.27}$$

the equation (2.24) now reads $\eta - H[\eta] = r$. First note that by (2.23),

$$\|\mathbf{T}_\theta^\pm[u^\pm, \xi, \eta S_\theta^\pm \omega_x^\pm](0,\cdot)\|_{L^\infty(0,t)} \quad \leq \quad C_\kappa \int_0^t \|\eta S_\theta^\pm \omega_x^\pm\|_{L^\infty(\Omega_\tau^\pm)} d\tau$$

$$\leq \; C_\kappa \int_0^t |\eta(\tau)| d\tau \tag{2.28}$$

for each $t \in (0,T)$, since $\|(S_\theta^\pm \omega_x^\pm)(\cdot,\tau)\|_{L^\infty(\Omega_\tau^\pm)} \leq \|\omega_x^\pm(\cdot,\tau)\|_{L^\infty(\Omega_\tau^\pm)}$ by (2.15). For any $t \in (0,T)$ we can now estimate $\|H[\eta]\|_{L^\infty(0,t)}$ as follows: by inequality (2.28)

$$\begin{aligned} \|H[\eta]\|_{L^\infty(0,t)} &\leq \sum_\pm \|\sigma^\pm\|_\infty \|\mathbf{T}_\theta^\pm[u^\pm, \xi, \eta S_\theta^\pm \omega_x^\pm](0,\cdot)\|_{L^\infty(0,t)} \\ &\leq C_\kappa \int_{(0,t)} |\eta(\tau)| d\tau \end{aligned} \tag{2.29}$$

for $t \in (0,T)$. Also note carefully that in the above estimation, only the norm of η on $(0,t)$ enters. Moreover we have

$$\begin{aligned} \|D_t H[\eta]\|_{L^\infty(0,t)} &\leq \sum_\pm \|\sigma^\pm\|_{L^\infty(0,T)} \|D_t \mathbf{T}_\theta^\pm[u^\pm, \xi, \eta S_\theta^\pm \omega_x^\pm]\|_{L^\infty(0,t)} \\ &\quad + \sum_\pm \|D_t \sigma^\pm\|_{L^\infty(0,T)} \|\mathbf{T}_\theta^\pm[u^\pm, \xi, \eta S_\theta^\pm \omega_x^\pm]\|_{L^\infty(0,t)}. \end{aligned} \tag{2.30}$$

In order to continue the above estimation, we use remark 1.1 in section 1.3. Namely, by (2.23), $\mathbf{T}_\theta^\pm[u^\pm, \xi, \eta S_\theta^\pm \omega_x^\pm]$ is a $W^{1,\infty}$ and hence a Lipschitz function on $\overline{\Omega^\pm}$, the derivative of

$$t \mapsto \mathbf{T}_\theta^\pm[u^\pm, \xi, \eta S_\theta^\pm \omega_x^\pm](0,\cdot)$$

can be estimated by the L^∞-norm of $\nabla \mathbf{T}_\theta^\pm[u^\pm, \xi, \eta S_\theta^\pm \omega_x^\pm]$ on $\Omega^\pm$, which by the two last lines of (2.23) can be estimated by the V^1-norm of $\eta S_\theta^\pm \omega_x^\pm$. Hence

$$\|D_t \mathbf{T}_\theta^\pm[u^\pm, \xi, \eta S_\theta^\pm \omega_x^\pm](0,\cdot)\|_{L^\infty(0,t)} \leq C_\kappa \|\eta S_\theta^\pm \omega_x^\pm\|_{V^1(\Omega^\pm)} \leq C_\kappa \|\eta\|_{L^\infty(0,T)}$$

by (2.23). In the last step, we used the properties (2.15) of the smoothing operator to write

$$\begin{aligned} \|\eta S_\theta^\pm \omega_x^\pm\|_{V^1(\Omega^\pm)} &\leq \|\eta\|_{L^\infty(0,T)} \{\|S_\theta^\pm \omega_x^\pm\|_{L^\infty(\Omega^\pm)} + \|D_x S_\theta^\pm \omega_x^\pm\|_{L^\infty(\Omega^\pm)}\} \\ &\leq \|\eta\|_{L^\infty(0,T)} \{\|\omega_x^\pm\|_{L^\infty(\Omega^\pm)} + \|\omega_{xx}^\pm\|_{L^\infty(\Omega^\pm)}\} \\ &\leq C\|\eta\|_{L^\infty(0,t)}. \end{aligned}$$

$\sigma^\pm, D_t\sigma^\pm$ can be easily estimated in L^∞-norm by constants depending on the L^∞-norms of ω and ω_t (recall that

$$\sigma^\pm(t) := \frac{\partial \sigma}{\partial w^\pm}(\omega^-(0,t), \omega^+(0,t))$$

and that $\sigma^\pm \in C^1(\mathbb{R})$). Thus from (2.30) we can conclude

$$\|D_t H[\eta]\|_{L^\infty(0,T)} \leq C_\kappa \|\eta\|_{L^\infty(0,T)} \tag{2.31}$$

which together with (2.29) implies that $H : L^\infty(0,T) \to L^\infty(0,T)$ is a linear compact operator, by the compact embedding of $W^{1,\infty}(0,T)$ into $L^\infty(0,T)$. This implies that (2.24) has a unique solution for every $r \in L^\infty(0,T)$ if the corresponding homogeneous equation only has the trivial solution. But for a solution η of the corresponding homogeneous problem we have $\eta = H\eta$ and hence for $t \in (0,T)$

$$|\eta(t)| \leq C_\kappa \int_{(0,t)} |\eta(\tau)| d\tau,$$

holds, by (2.29). But this implies $\eta = 0$ by a well-known integral form of Gronwall's inequality (see [6], [22]). Now Fredholm theory implies that $(I - H)^{-1} : L^\infty(0,T) \to L^\infty(0,T)$ is bounded, i.e. (2.24) is uniquely solvable and (2.25) holds. □

Now we can write down a regularized fixed-point operator by inserting smoothing operators in the right places. The fixed points of the regularized operator will give, in the limit as $\theta \to 0^+$, a solution of problem (2.6). The operator in question is given by (compare (2.11))

Definition 2.6.

$$\mathbf{\Phi}_\theta[v^-, v^+, \xi] := \begin{pmatrix} \mathbf{T}_\theta^-[v^-, \xi, \mathbf{K}_\theta[v^-, v^+, \xi, r_\theta(v^-, v^+, \xi)] S_\theta^-(\omega_x^-) + r_\theta^-(v^-)] \\ \mathbf{T}_\theta^+[v^+, \xi, \mathbf{K}_\theta[v^-, v^+, \xi, r_\theta(v^-, v^+, \xi)] S_\theta^+(\omega_x^+) + r_\theta^+(v^+)] \\ \mathbf{K}_\theta[v^-, v^+, \xi, r_\theta(v^-, v^+, \xi)] \end{pmatrix} \tag{2.32}$$

and is is well defined at least if $(v^-, v^+, \xi) \in Z_{\theta,\kappa}$. *Here we have used the abbreviations (compare the definitions of the quantities* $r^\pm(v^\pm), r(v^-, v^+, \xi)$ *directly following* (2.11)*)*

$$r_\theta^\pm(v^\pm) := S_\theta^\pm[-d^\pm - R^\pm[v^\pm]\omega_x^\pm] \tag{2.33}$$

$$r_\theta(v^-, v^+, \xi)(t) := p_\theta(v^-, v^+)(t) + \sum_\pm \sigma^\pm(t) \mathbf{T}_\theta^\pm[v^\pm, \xi, r_\theta^\pm(v^\pm)](0,t) \tag{2.34}$$

$$p_\theta(v^-, v^+)(t) := e^{\nu t} \hat{S}_\theta[e^{-\nu \cdot}(-d + R[v^-, v^+])] \tag{2.35}$$

(the somewhat strange form of p_θ *is more compatible with the exponentially weighted norms in the next chapter). Recall the definition of* $R^\pm$ *and* R *from* (2.7), (2.8). *We write*

$$\mathbf{\Phi}_\theta = (\mathbf{\Phi}_\theta^-, \mathbf{\Phi}_\theta^+, \mathbf{\Phi}_\theta^0).$$

For later reference, we will need the bounds

Lemma 2.7.

$$\begin{aligned}
\|r_\theta^\pm(v^\pm)\|_{V^1(\Omega^\pm)} &\leq b_1(\|v^\pm\|_{V^1(\Omega^\pm)}) && (2.36)\\
\|D_{xx}r_\theta^\pm(v^\pm)\|_{L^\infty(\Omega^\pm)} &\leq C_\theta b_1(\|v^\pm\|_{V^1(\Omega^\pm)}) && (2.37)\\
\|D_t p_\theta(v^-,v^+)\|_{L^1(0,T)} &\leq b_2(\|v^-\|_{W^{1,\infty}(\Omega^\pm)}, \|v^+\|_{W^{1,\infty}(\Omega^\pm)}) && (2.38)\\
\|D_t p_\theta(v^-,v^+)\|_{L^\infty(0,T)} &\leq C_\theta b_2(\|v^+\|_{L^\infty(\Omega^+)}, \|v^-\|_{L^\infty(\Omega^-)}) && (2.39)
\end{aligned}$$

where b_1, b_2 are nonnegative functions which are monotone nondecreasing in each of their arguments and which depend on d, d_x, $\omega, \omega_x, \omega_{xx}$ but are independent of θ, whereas C_θ depends on θ.

Proof. 1. (2.36) is a consequence of (B.12) in appendix B. Consider now (2.37). By (2.15), we can control the second derivative of $r_\theta^\pm(v^\pm)$:

$$\begin{aligned}
\|D_{xx}r_\theta^\pm(v^\pm)\|_{L^\infty(\Omega^\pm)} &\leq C_\theta\|D_x r_\theta^\pm(v^\pm)\|_{L^\infty(\Omega^\pm)}\\
&\leq C_\theta\|r_\theta^\pm(v^\pm)\|_{V^1(\Omega^\pm)}\\
&\leq C_\theta b_1(\|v^\pm\|_{V^1(\Omega^\pm)}).
\end{aligned}$$

2. To prove (2.38), first recall the following inequalities (see [23]). For any $f \in BV(0,T)$, the trivial extension Ef is in $BV(\mathbb{R})$. Then we have

$$\left.\begin{aligned}
&\|\rho_\theta * Ef\|_{L^1(\mathbb{R})} \leq \|Ef\|_{L^1(\mathbb{R})} = \|f\|_{L^1(0,T)}\\
&\|D_t(\rho_\theta * Ef)\|_{L^1(\mathbb{R})} \leq \|Ef\|_{BV(-1,T+1)}
\end{aligned}\right\} \qquad (2.40)$$

and all small $\theta > 0$. Note also that for any function $h \in L^\infty(0,T) \cap BV(0,T)$, we have

$$\|Eh\|_{BV(-1,T+1)} \leq \|h\|_{BV(0,T)} + 2\|h\|_{L^\infty(0,T)} \qquad (2.41)$$

(the last term accounting for the jump which is produced by extending h by zero outside of $[0,T]$). Now writing $h(t) := e^{-\nu t}(-d(t) + R[v^-,v^+](t))$, we have $Eh \in BV(\mathbb{R})$, since $d \in BV(0,T)$ and $R[v^-,v^+]$ is Lipschitz on $(0,T)$. Moreover

$$\begin{aligned}
D_t p_\theta(v^-,v^+)(t) &= \nu e^{\nu t}\hat{S}_\theta[h] + e^{\nu t}D_t\hat{S}_\theta[h]\\
&= \nu e^{\nu t}\rho_\theta * Eh + e^{\nu t}D_t(\rho_\theta * Eh)
\end{aligned}$$

on $(0,T)$. Thus, we obtain (using (2.40), (2.41))

$$\|D_t p_\theta(v^-,v^+)\|_{L^1(0,T)}$$

$$
\begin{aligned}
&\leq C\|h\|_{L^1(0,T)} + C\|Eh\|_{BV(-1,T+1)} \\
&\leq C\|h\|_{L^1(0,T)} + C\|h\|_{BV(0,T)} \\
&\leq C\|h\|_{L^\infty(0,T)} + C\|h\|_{BV(0,T)} \\
&\leq C\|e^{-\nu\cdot}(-d + R[v^-,v^+])\|_{L^\infty(0,T)} + C\|e^{-\nu\cdot}(-d + R[v^-,v^+])\|_{BV(0,T)} \\
&\leq C\|d\|_{L^\infty(0,T)} + C\|R[v^-,v^+]\|_{L^\infty(0,T)} + C\|d\|_{BV(0,T)} + C\|R[v^-,v^+]\|_{BV(0,T)} \\
&\leq C\|d\|_{L^\infty(0,T)} + C\|R[v^-,v^+]\|_{L^\infty(0,T)} + C\|d\|_{BV(0,T)} + \\
&\quad\ C\|D_tR[v^-,v^+]\|_{L^\infty(0,T)} \\
&\leq C(1 + \|R[v^-,v^+]\|_{L^\infty(0,T)} + \|D_tR[v^-,v^+]\|_{L^\infty(0,T)})
\end{aligned}
$$

where C depends on d, d_t, but not on θ. To obtain now the desired estimate, we only need to bound $R[v^-,v^+], D_tR[v^-,v^+]$ in terms of the norms $\|v^\pm\|_{W^{1,\infty}(\Omega^\pm)}$. This is easily done for $R[v^-,v^+]$. To bound $D_tR[v^-,v^+]$, we compute (using $\sigma \in C^2(\mathbb{R}\times\mathbb{R})$)

$$
\begin{aligned}
D_tR[v^-,v^+] \;=\; &\sum_\pm \frac{\partial\sigma}{\partial w^\pm}(\omega^-(0,\cdot)+v^-(0,\cdot),\omega^+(0,\cdot)+v^+(0,\cdot))(D_t\omega^\pm(0,\cdot)+D_tv^\pm(0,\cdot)) \\
&-\sum_\pm \frac{\partial\sigma}{\partial w^\pm}(\omega^\pm(0,\cdot))(D_t\omega^\pm(0,\cdot)) \\
&-\sum_\pm \{(D_t\sigma^\pm)v^\pm(0,\cdot) + \sigma^\pm D_tv^\pm(0,\cdot)\}.
\end{aligned}
$$

Hence $\|D_tR[v^-,v^+]\|_{L^\infty(0,T)} \leq \tilde{b}_2\left(\|v^\pm\|_{W^{1,\infty}(\Omega^\pm)}\right)$, if we set

$$
\begin{aligned}
\tilde{b}_2(l^\pm) \;:=\; &\sup_{|\lambda^\pm|\leq l^\pm, |\eta^\pm|\leq l^\pm, t\in(0,T)} \Big| \sum_\pm \frac{\partial\sigma}{\partial w^\pm}(\omega^+(0,t)+\lambda^-,\omega^+(0,t)+\lambda^+)(D_t\omega^\pm(0,t)+\eta^\pm) \\
&-\sum_\pm \frac{\partial\sigma}{\partial w^\pm}(\omega^\pm(0,t))(D_t\omega^\pm(0,t)) - \sum_\pm\{D_t\sigma^\pm(t)\lambda^\pm + \sigma^\pm(t)\eta^\pm\}\Big|
\end{aligned}
$$

the supremum being finite because of $\sigma \in C^2(\mathbb{R},\mathbb{R})$ and our assumptions (2.1) on $\omega^\pm$. This finishes the proof of (2.38).

3. We find

$$
D_tp_\theta(v^-,v^+) = D_t(\rho_\theta * Eh) = \int_{\mathbb{R}} (\rho_\theta)'(t-s)Eh(s)ds
$$

where E is the trivial extension operator and h from above. Thus

$$
\|D_tp_\theta(v^-,v^+)\|_{L^\infty(0,T)} \leq C_\theta\|Eh\|_{L^\infty(\mathbb{R})} \leq C_\theta(\|d\|_{L^\infty(0,T)} + \|R[v^-,v^+]\|_{L^\infty(0,T)}).
$$

$\|R[v^-,v^+]\|_{L^\infty(0,T)}$ can be estimated by the L^∞-norms of $v^\pm$, so we get (2.39). □

Up to now, we have constructed a family $\{\mathbf{\Phi}_\theta\}$ of operators depending on θ. In the following theorem, we check that if $\mathbf{\Phi}_\theta$ has fixed points in a fixed set D for all small θ, we can actually assert the existence of a solution to (2.6):

Theorem 2.8. *Suppose there is a nonempty, closed convex bounded set $D \subset V^1(\Omega^-) \times V^1(\Omega^+) \times L^\infty(0,T)$ and numbers $\theta_0 > 0, \kappa > 0$ such that for all $0 < \theta < \theta_0$,*

$$D \subset Z_{\theta,\kappa}$$

and suppose that for each sufficiently small θ a fixed point v_θ of $\mathbf{\Phi}_\theta$ in D exists; then there exists a $(v^-, v^+, \xi) \in V^1(\Omega^-) \times V^1(\Omega^+) \times L^\infty(0,T)$ satisfying (2.6).

Before we come to the proof, we need an auxiliary

Lemma 2.9. *If $\{\theta_j\}, \theta_j > 0$ is a sequence converging to zero and $\{v_j^\pm\}, v_j^\pm \in W^{1,\infty}(\Omega^\pm)$ are sequences converging to $v^\pm$ in $L^\infty(\Omega^\pm)$ such that $\|D_x v_j^\pm\|_{L^\infty(\Omega^\pm)}$ is bounded independent of j, then after choosing suitable subsequences the following convergence statements in $L^1(\Omega^\pm)$*

$$a(\omega^\pm + S_{\theta_j}^\pm v_j^\pm) \to a(\omega^\pm + v^\pm)$$

$$S_{\theta_j}^\pm a(\omega^\pm)_x \to a(\omega^\pm)_x$$

$$r_{\theta_j}^\pm(v_j^\pm) \to r^\pm(v^\pm) = -d^\pm - R^\pm[v]\omega_x^\pm$$

$$p_{\theta_j}(v_j^-, v_j^+) \to -d + R[v^-, v^+]$$

are true.

Proof. If $\{w_j\}$ is a sequence in $V^1(\Omega^-)$ or in $V^1(\Omega^+)$ which is uniformly convergent to w, then

$$\begin{aligned}
\|S_{\theta_j}^\pm w_j - w\|_{L^1(\Omega^\pm)} &\leq \|S_{\theta_j}^\pm (w_j - w)\|_{L^1(\Omega^\pm)} + \|S_{\theta_j}^\pm w - w\|_{L^1(\Omega^\pm)} \\
&\leq C\|S_{\theta_j}^\pm (w_j - w)\|_{L^\infty(\Omega^\pm)} + \|S_{\theta_j}^\pm w - w\|_{L^1(\Omega^\pm)} \\
&\leq C\|w_j - w\|_{L^\infty(\Omega^\pm)} + \|S_{\theta_j}^\pm w - w\|_{L^1(\Omega^\pm)}
\end{aligned}$$

(we have used the first line of (2.15)) and thus, in view of the fourth line of (2.15),

$$S_{\theta_j}^\pm w_j^\pm - w^\pm \to 0 \text{ in } L^1(\Omega^\pm). \tag{2.42}$$

Applying this to $v_j^\pm$, we see that $S_{\theta_j}^\pm v_j^\pm \to v^\pm$ in $L^1(\Omega^\pm)$. Upon selecting a suitable subsequence, we may assume $S_{\theta_j}^\pm v_j^\pm \to v^\pm$ a.e on $\Omega^\pm$ and thus $a(S_{\theta_j}^\pm v_j^\pm) \to a(v^\pm)$ a.e.. Since $\|S_{\theta_j}^\pm v_j^\pm\|_{L^\infty(\Omega^\pm)} \leq \|v_j^\pm\|_{L^\infty(\Omega^\pm)} \leq M$ for some $M > 0$ independent of j, we have

$$|a(\omega^\pm + S_{\theta_j}^\pm v_j^\pm)| \leq \sup\{|a(y)| : |y| \leq \|\omega^\pm\|_{L^\infty(\Omega^\pm)} + M\}$$

and thus the dominated convergence theorem shows that $a(\omega^\pm + S^\pm_{\theta_j} v^\pm_j) \to a(\omega^\pm + v^\pm)$ in $L^1(\Omega^\pm)$. The second statement about the convergence of $S^\pm_{\theta_j} a(\omega^\pm)_x$ follows from the fourth line of (2.15).

Now we proceed to prove the statement on $r^\pm_{\theta_j}(v^\pm_j)$. First note that from the continuity of a and the boundedness of a' on compact intervals, $R^\pm[v^\pm_j] \to R^\pm[v^\pm]$ (see (2.7)) uniformly on $L^\infty(\Omega^\pm)$. By (2.42), $S^\pm_{\theta_j}(R[v^\pm_j]\omega^\pm_x) \to R[v^\pm]\omega^\pm_x$ a.e. on $\Omega^\pm$. Moreover $S^\pm_{\theta_j} d^\pm \to d^\pm$ in $L^1(\Omega^\pm)$ by (2.15), so for a suitable subsequence, $r^\pm_{\theta_j}(v^\pm_j) \to r^\pm(v^\pm)$ a.e.. In view of the bound (2.36), we may apply the dominated convergence theorem again.

Finally, in view of the assumption that the Lipschitz functions $v^\pm_j$ converge uniformly on $\Omega^\pm$, $R[v^-_j, v^+_j]$ (see (2.8)) converges uniformly on $(0,T)$ to $R[v^-, v^+]$. Similarly as before, we see that $\hat{S}_{\theta_j}[e^{-\nu \cdot}(-d + R[v^-_j, v^+_j])] \to e^{-\nu \cdot}(-d + R[v^-, v^+])$ a.e.. Since by (2.18)

$$\|e^{\nu \cdot} \hat{S}_{\theta_j}[e^{-\nu \cdot}(-d + R[v^-_j, v^+_j])]\|_{L^\infty(0,T)} \leq C\|(-d + R[v^-_j, v^+_j])\|_{L^\infty(0,T)} \leq C$$

with some constant independent of j, we may apply the dominated convergence theorem once more to get the convergence of p_{θ_j}. □

Proof. (of Theorem 2.8.) Let $\{\theta_j\}, \theta_j > 0$ be any sequence converging to zero and and $(v^-_j, v^+_j, \xi_j) \in D$ be a corresponding sequence of fixed points of $\mathbf{\Phi}_{\theta_j}$. Since

$$v^\pm_j = \mathbf{\Phi}^\pm_{\theta_j}[v^-_j, v^+_j, \xi_j] = \mathbf{T}^\pm_{\theta_j}[v^\pm_j, \xi_j, \xi_j S^\pm_{\theta_j}\omega^\pm_x + r^\pm_\theta(v^\pm_j)]$$

we can use (2.23) to estimate

$$\begin{aligned}
\|D_t v^\pm_j\|_{L^\infty(\Omega^\pm)} &\leq C_\kappa \left[\|\xi_j S^\pm_{\theta_j}\omega^\pm_x\|_{V^1(\Omega^\pm)} + \|r^\pm_{\theta_j}(v^\pm_j)\|_{V^1(\Omega^\pm)}\right] \\
&\leq C_\kappa \left[\|\xi_j\|_{L^\infty(0,T)}\|S^\pm_{\theta_j}(\omega^\pm_x)\|_{V^1(\Omega^\pm)} + b_1(\|v^\pm_j\|_{V^1(\Omega^\pm)})\right] \\
&\leq C_\kappa \left[\|\xi_j\|_{L^\infty(0,T)}\|\omega^\pm_x\|_{V^1(\Omega^\pm)} + b_1\left(\|v^\pm_j\|_{V^1(\Omega^\pm)}\right)\right]
\end{aligned}$$

where in the second line we have used the fact that ξ is a function of time only as well as estimate (2.36) from Lemma 2.7; in the third line we have again used (2.15) to get rid of $S^\pm_\theta$. Note that C_κ above does not depend on θ. Thus the boundedness of D in $V^1 \times V^1 \times L^\infty$ implies that $\{D_t v^\pm_j\}_j$ is bounded in L^∞-norm. Hence $\{v^\pm_j\}_j$ is bounded in $W^{1,\infty}(\Omega^\pm)$ and upon passing to subsequences,

$$v^\pm_j \to v^\pm \text{ in } L^\infty(\Omega^\pm), \quad D_x v^\pm_j \rightharpoonup D_x v^\pm \text{ weak-* in } L^\infty(\Omega^\pm) \tag{2.43}$$

for some functions $v^\pm \in V^1(\Omega^\pm)$. Now the fact that (v^-_j, v^+_j, ξ_j) are fixed points of $\mathbf{\Phi}_{\theta_j}$ implies (by the definition of $\mathbf{\Phi}^\pm_{\theta_j}$, see (2.32)) that

$$\begin{cases} D_t v^\pm_j + (a(\omega^\pm + S^\pm_{\theta_j} v^\pm_j) - \mu - \xi_j) D_x v^\pm_j + S^\pm_{\theta_j} a(\omega^\pm)_x v^\pm_j = \xi_j S^\pm_{\theta_j}\omega^\pm_x + r^\pm_{\theta_j}(v^\pm_j) \\ v^\pm_j(\cdot, 0) = 0 \end{cases} \tag{2.44}$$

in the weak sense. Moreover $\xi_j = \mathbf{\Phi}^0_{\theta_j}[v_j^-, v_j^+, \xi_j] = \mathbf{K}_{\theta_j}[v_j^-, v_j^+, \xi_j, r_{\theta_j}(v_j^-, v_j^+, \xi_j)]$ implies by (2.10) the following equation

$$\xi_j - \sum_{\pm} \sigma^{\pm}\mathbf{T}^{\pm}_{\theta_j}[v_j^{\pm}, \xi_j, \xi_j S^{\pm}_{\theta_j}\omega^{\pm}_x + r^{\pm}_{\theta_j}(v^{\pm}_{\theta_j})](0,\cdot) \quad = \quad p_{\theta_j}(v^-, v^+) \tag{2.45}$$

on $(0,T)$, which can be rewritten as

$$\xi_j(\cdot) - \sigma^-(\cdot)v_j^-(0,\cdot) - \sigma^+(\cdot)v_j^+(0,\cdot) = p_{\theta_j}(v^-, v^+)(\cdot) \tag{2.46}$$

on $(0,T)$ again by the fact that (v_j^-, v_j^+, ξ_j) are fixed points of $\mathbf{\Phi}_{\theta_j}$. The boundedness of $\|D_t v_j^{\pm}\|_{L^\infty(\Omega^{\pm})}$, $\|D_x v_j^{\pm}\|_{L^\infty(\Omega^{\pm})}$ and (2.38) show that

$$\|D_t p_{\theta_j}(v^-, v^+)\|_{L^1(0,T)} \le b_2(\|v_j^-\|_{W^{1,\infty}(\Omega^{\pm})}, \|v_j^+\|_{W^{1,\infty}(\Omega^{\pm})}) \le C \tag{2.47}$$

where C is independent of j. Differentiating (2.46) we get

$$D_t\xi_j - \sum_{\pm}(D_t\sigma^{\pm})v_j^{\pm}(0,\cdot) - \sum_{\pm}\sigma^{\pm}(\cdot)(D_t v_j^{\pm})(0,\cdot) = D_t p_{\theta_j}(v^-, v^+)(\cdot).$$

Thus again the boundedness of $\|D_t v_j^{\pm}\|_{L^\infty(\Omega^{\pm})}, \|D_x v_j^{\pm}\|_{L^\infty(\Omega^{\pm})}$ and (2.47) imply that $\|D_t\xi_j\|_{L^1(0,T)}$ is uniformly bounded.

Since $\|\xi_j\|_{L^1(0,T)}$ is also uniformly bounded, we conclude that

$$\|\xi_j\|_{BV(0,T)} \le C,$$

hence upon selecting a further subsequence we can assume $\xi_j \to \xi$ in $L^1(0,T)$, by a compactness theorem for BV-functions [4]; see [23], theorem 4 in section 5.2. Note also that since $\xi_j \to \xi$ a.e. and the L^∞-norm of ξ_j is bounded independent of j, we have $\xi \in L^\infty(0,T)$.

Now we use Lemma 2.9 to send $j \to \infty$ in (2.44). Namely, since $a(\omega^{\pm} + S^{\pm}_{\theta_j}v_j^{\pm}) \to a(\omega^{\pm} + v^{\pm})$ in $L^1(\Omega^{\pm})$, $\xi_j \to \xi$ in $L^1(0,T)$ and $D_x v_j^{\pm} \rightharpoonup D_x v^{\pm}$ in the weak-* sense in $L^\infty(\Omega^{\pm})$, we have

$$\int_{\Omega^{\pm}} (a(\omega^{\pm} + S^{\pm}_{\theta_j}v_j^{\pm}) - \mu - \xi_j)D_x v_j^{\pm}\ \varphi d(x,t) \to \int_{\Omega^{\pm}} (a(\omega^{\pm} + v^{\pm}) - \mu - \xi)D_x v^{\pm}\varphi\ d(x,t)$$

[4]The theorem reads as follows: Let $U \subset \mathbb{R}^n$ be open and bounded, with ∂U Lipschitz. Assume $\{f_k\}_{k=1}^\infty$ is a sequence in $BV(U)$ satisfying

$$\sup_k \|f_k\|_{BV(U)} < \infty.$$

Then there exists a subsequence $\{f_{k_j}\}_{j=1}^\infty$ and a function $f \in BV(U)$ such that

$$f_{k_j} \to f \text{ in } L^1(U)$$

as $j \to \infty$.

$$\int_{\Omega^\pm} S^\pm_{\theta_j} a(\omega^\pm)_x v^\pm_j \varphi \, d(x,t) \to \int_{\Omega^\pm} a(\omega^\pm)_x v^\pm \varphi \, d(x,t)$$

$$\int_{\Omega^\pm} (\xi_j S^\pm_{\theta_j} \omega^\pm_x + r^\pm_{\theta_j}(v^\pm_j))\varphi \, d(x,t) \to \int_{\Omega^\pm} (\xi \omega^\pm_x + r^\pm(v^\pm))\varphi \, d(x,t)$$

for any $\varphi \in C^1(\overline{\Omega^\pm}), \varphi|_{\Gamma^\pm} = 0, \varphi(\cdot,T) = 0$. By the convergence of $v^\pm_j \to v^\pm$ in $L^\infty(\Omega^\pm)$,

$$-\int_{\Omega^\pm} v^\pm_j D_t \varphi dxdt \to -\int_{\Omega^\pm} v^\pm D_t \varphi dxdt,$$

and since $v^\pm_j$ is a weak solution of (2.44), we find, using the above convergence statements, that

$$\begin{cases} D_t v^\pm + (a(\omega^\pm + v^\pm) - \mu - \xi) D_x v^\pm + a(\omega^\pm)_x v^\pm = \xi \omega^\pm_x + r^\pm(v^\pm) \\ v^\pm(\cdot, 0) = 0 \end{cases} \tag{2.48}$$

holds. Now going back to (2.46), we finally conclude

$$\xi - \sigma^- v^- - \sigma^+ v^+ = -d + R[v^-, v^+],$$

since $p_{\theta_j}(v^-_j, v^+_j) \to -d + R[v^-, v^+]$ in $L^1(0,T)$. Thus, in total we see that (v^-, v^+, ξ) is a solution of the system

$$\begin{cases} v^-_t + (a(\omega^- + v^-) - \mu - \xi) v^-_x + a(\omega^-)_x v^- = -d^- - (R^-[v^-] - \xi)\omega^-_x \\ v^+_t + (a(\omega^+ + v^+) - \mu - \xi) v^+_x + a(\omega^+)_x v^+ = -d^+ - (R^+[v^+] - \xi)\omega^+_x \\ \xi(\cdot) - \sigma^- v^-(0,\cdot) - \sigma^+ v^+(0,\cdot) = -d - R[v^-, v^+] \end{cases} \tag{2.49}$$

i.e. (2.6). □

Remark 2.10. *The emphasis in the above theorem lies in the existence of* (v^-, v^+, ξ) *solving* (2.6). *Here, we are not interested in the question if the whole sequence of fixed points* $(v^-_\theta, v^+_\theta, \xi_\theta)$ *converges to* (v^-, v^+, ξ) *as* $\theta \to 0^+$. *If* D *is defined by norm inequalities, then* (v^-, v^+, ξ) *from above will also lie in* D *(see the next corollary).*

Corollary 2.11. *Suppose the set* D *in the previous theorem is defined by norm inequalities of the following form*

$$(v^-, v^+, \xi) \in D \Leftrightarrow \|v^\pm\|_\nu \le \alpha^\pm, \|v^\pm_x\|_\nu \le \beta^\pm, \|\xi\|_\nu \le \gamma,$$

where $\alpha^\pm, \beta^\pm, \gamma$ *are positive numbers and* $\|\cdot\|_\nu$ *denotes the exponentially weighted norm defined in Appendix A*[5]. *Then the solution* (v^-, v^+, ξ) *whose existence is asserted in Theorem 2.8 lies in* D.

[5]The norms are defined by $\|v^\pm\|_\nu := \sup_{t\in(0,T)} e^{-\nu t} \|v^\pm(\cdot,t)\|_{L^\infty(\Omega^\pm_t)}$, $\|\xi\|_\nu = \sup_{(0,T)} e^{-\nu t}|\xi(t)|$.

Proof. In the proof of Theorem 2.8, we constructed sequences $\{v_j^\pm\}, \{\xi_j\}$ such that $v_j^\pm \to v^\pm$ in $L^\infty(\Omega^\pm)$, $D_x v_j^\pm \overset{*}{\rightharpoonup} D_x v^\pm$ and such that $\xi_j \to \xi$ in $L^1(0,T)$. We then obviously have $\|v^\pm\|_\nu \leq \alpha^\pm$. Furthermore,

$$\|v_x^\pm\|_\nu \leq \liminf_{j\to\infty} \|D_x v_j^\pm\|_\nu \leq \beta^\pm$$

because of the weak-*-convergence. Finally, observe that we have $\xi_j \to \xi$ a.e. after choosing a suitable subsequence and thus also $e^{-\nu t}|\xi(t)| \leq \gamma$ a.e.. □

2.2 Compactness and Continuity Properties

In order to prepare for the application of Schauder's fixed point theorem to $\mathbf{\Phi}_\theta$, we need to show the continuity and compactness for $\mathbf{\Phi}_\theta$, $\theta > 0$ fixed. Consequently, the constants in the following estimations may depend on θ.

Several intermediate statements follow.

Theorem 2.12. *Let $(u_j, \xi_j) \in Z^\pm_{\theta,\kappa}$ be a sequence s.t. $(u_j, \xi_j) \to (u, \xi)$ in $V^1(\Omega^\pm) \times L^\infty(0,T)$ and $r_j \in V^1(\Omega^\pm)$ such that $r_j \rightharpoonup r$ weak-* in $L^\infty(\Omega^\pm)$ and such that $\|D_x r_j\|_{L^\infty(\Omega^\pm)}, \|D_{xx} r_j\|_{L^\infty(\Omega^\pm)}$ are uniformly bounded. Then*

$$\mathbf{T}^\pm_\theta[u_j, \xi_j, r_j] \to \mathbf{T}^\pm_\theta[u, \xi, r] \quad \textit{in } V^1(\Omega^\pm). \tag{2.50}$$

Proof. Write $v_j^\pm := \mathbf{T}^\pm_\theta[u_j, \xi_j, r_j]$. $v_j^\pm$ is a solution to the linear hyperbolic problem

$$\begin{cases} D_t v_j^\pm + (a(\omega^\pm + S^\pm_\theta u_j) - \mu - \xi_j) D_x v_j^\pm + S^\pm_\theta a(\omega^\pm)_x v_j^\pm = r_j \\ v_j^\pm(\cdot, 0) = 0. \end{cases}$$

(see (2.22)). We claim that the following bounds hold for the coefficient functions of the PDE above:

$$\left.\begin{array}{c} \|a(\omega^\pm + S^\pm_\theta u_j) - \mu - \xi_j\|_{V^2(\Omega^\pm)} \leq C_\theta \\ \|S^\pm_\theta a(\omega^\pm)_x\|_{V^2(\Omega^\pm)} \leq C_\theta \end{array}\right\} \tag{2.51}$$

with some C_θ independent of j. First note that $\|a(\omega^\pm + S^\pm_\theta u_j) - \mu - \xi_j\|_{L^\infty(\Omega^\pm)}$ is clearly uniformly bounded in j, since a is continuous and the L^∞-norms of u_j and ξ_j are uniformly bounded. To see this for the derivatives in x-direction, we estimate (note that μ, ξ_j only depend on t)

$$\begin{aligned} \|D_x[a(\omega^\pm + S^\pm_\theta u_j) - \mu - \xi_j]\|_{L^\infty(\Omega^\pm)} &= \|D_x a(\omega^\pm + S^\pm_\theta u_j)\|_{L^\infty(\Omega^\pm)} \\ &= \|a'(\omega^\pm + S^\pm_\theta u_j) D_x(\omega^\pm + S^\pm_\theta u_j)\|_{L^\infty(\Omega^\pm)} \end{aligned}$$

$$\leq \; C$$

since by (2.15), $\|D_x(\omega^\pm + S_\theta^\pm u_j)\|_{L^\infty(\Omega^\pm)} \leq \|\omega_x^\pm\|_{L^\infty(\Omega^\pm)} + \|D_x u_j\|_{L^\infty(\Omega^\pm)}$ and by the assumption that u_j is convergent in $V^1(\Omega^\pm)$, $\|D_x u_j\|_{L^\infty(\Omega^\pm)}$ is bounded in j. Moreover we have

$$\begin{aligned} \|D_{xx}[a(\omega^\pm + S_\theta^\pm u_j) - \mu - \xi_j]\|_{L^\infty(\Omega^\pm)} &= \|D_{xx} a(\omega^\pm + S_\theta^\pm u_j)\|_{L^\infty(\Omega^\pm)} \\ = \|a''(\omega^\pm + S_\theta^\pm u_j)[D_x(\omega^\pm + S_\theta^\pm u_j)]^2 + a'(\omega_x^\pm + S_\theta^\pm u_j) D_{xx}(\omega^\pm &+ S_\theta^\pm u_j)\|_{L^\infty(\Omega^\pm)} \\ &\leq C_\theta \end{aligned}$$

since $a \in C^2$, $\|D_x(\omega^\pm + S_\theta^\pm u_j)\|_{L^\infty(\Omega^\pm)}$ is bounded in j and

$$\|D_{xx} S_\theta^\pm u_j\|_{L^\infty(\Omega^\pm)} \leq C_\theta \|D_x u_j\|_{L^\infty(\Omega^\pm)}$$

according to (2.15). Note finally that

$$\|S_\theta^\pm a(\omega^\pm)_x\|_{V^2(\Omega^\pm)} \leq C_\theta \|a(\omega^\pm)_x\|_{V^1(\Omega^\pm)}$$

which gives the second line of (2.51).

Because of (2.51) and the fact that $a(\omega^\pm + S_\theta^\pm u_j) - \mu - \xi_j$ is outward-pointing, we may use Theorem A.18 to bound higher derivatives of $v_j^\pm$. Namely, by the uniform boundedness of $\|D_x r_j\|_{L^\infty(\Omega^\pm)}, \|D_{xx} r_j\|_{L^\infty(\Omega^\pm)}$, we obtain the following bounds

$$\|v_j^\pm\|_{L^\infty(\Omega^\pm)}, \|D_x v_j^\pm\|_{L^\infty(\Omega^\pm)}, \|D_t v_j^\pm\|_{L^\infty(\Omega^\pm)}, \|D_{xx} v_j^\pm\|_{L^\infty(\Omega^\pm)}, \|D_{xt} v_j^\pm\|_{L^\infty(\Omega^\pm)} \leq C; \qquad (2.52)$$

with C independent of j. As a consequence, any subsequence of the original sequence $\{v_j^\pm\}$ contains a subsequence with the property that there is a $v^\pm \in V^1(\Omega^\pm)$ s.t.

$$v_j^\pm \to v^\pm, \; D_x v_j^\pm \to D_x v^\pm \quad \text{in } L^\infty(\Omega^\pm).$$

Similarly as in the proof of Theorem 2.8, by writing out the integral relation satisfied by $v_j^\pm$ (see (A.8)), we conclude that $v^\pm$ satisfies (2.22) (note that the weak-* convergence of r_j to r suffices for this) and so $v^\pm = \mathbf{T}_\theta^\pm[u, \xi, r]$. But then the whole sequence $v_j^\pm$ converges to $\mathbf{T}_\theta^\pm[u, \xi, r]$ in $V^1(\Omega^\pm)$, by a standard subsequence argument. □

Theorem 2.13. $\mathbf{K}_\theta$ *is continuous on* $Z_{\theta,\kappa} \times L^\infty(0,T)$ *in the following sense: for any sequence* $(u_j^-, u_j^+, \xi_j, r_j)$ *converging to* (u^-, u^+, ξ, r) *in* $V^1(\Omega^-) \times V^1(\Omega^+) \times L^\infty(0,T) \times L^\infty(0,T)$ *such that* $r_j \in W^{1,\infty}(0,T)$*, and such that* $\|D_t r_j\|_{L^\infty(0,T)}$ *is uniformly bounded, we have*

$$\mathbf{K}_\theta[u_j^-, u_j^+, \xi_j, r_j] \to \mathbf{K}_\theta[u^-, u^+, \xi, r]$$

in $L^\infty(0,T)$.

Proof. Let $\eta_j := \mathbf{K}_\theta[u_j^-, u_j^+, \xi_j, r_j]$; then by (2.25), $\|\eta_j\|_{L^\infty(0,T)} \leq C$. Now note

$$D_t\eta_j - D_t H[\eta_j] = D_t r_j \quad \text{on } (0,T),$$

where $H[\eta]$ is defined as in (2.27). Also (2.30) remains valid, so the uniform boundedness of $\|D_t r_j\|_{L^\infty(0,T)}$ implies the uniform boundedness of $\|D_t\eta_j\|_{L^\infty(0,T)}$. This means that up to a subsequence, $\eta_j \rightharpoonup \eta$ in the weak-* sense for some $\eta \in L^\infty(0,T)$ in $L^\infty(0,T)$. Since

$$\eta_j S_\theta^\pm \omega_x^\pm \overset{*}{\rightharpoonup} \eta S_\theta^\pm \omega_x^\pm \text{ on } \Omega^\pm,$$
$$\|D_x(\eta_j S_\theta^\pm \omega_x^\pm)\|_{L^\infty(\Omega^\pm)}, \|D_{xx}(\eta_j S_\theta^\pm \omega_x^\pm)\|_{L^\infty(\Omega^\pm)} \leq C_\theta$$

by (2.15), we can apply Theorem 2.12 to obtain

$$\mathbf{T}_\theta^\pm[u_j^\pm, \xi_j, \eta_j S_\theta^\pm \omega_x^\pm] \to \mathbf{T}_\theta^\pm[u^\pm, \xi, \eta S_\theta^\pm \omega_x^\pm] \quad \text{in } V^1(\Omega^\pm)$$

and thus

$$\mathbf{T}_\theta^\pm[u_j^\pm, \xi_j, \eta_j S_\theta^\pm \omega_x^\pm](0,\cdot) \to \mathbf{T}_\theta^\pm[u^\pm, \xi, \eta S_\theta^\pm \omega_x^\pm](0,\cdot) \quad \text{in } L^\infty(0,T) \tag{2.53}$$

by remark 1.1 in section 1.3. Now we see that

$$\eta - \sum_\pm \sigma^\pm \mathbf{T}_\theta^\pm[u^\pm, \xi, \eta S_\theta^\pm \omega_x^\pm] = r, \tag{2.54}$$

that is, $\eta = \mathbf{K}_\theta[u^-, u^+, \xi, r]$ by Theorem 2.5. The usual subsequence argument now proves $\eta_j \to \mathbf{K}_\theta[u^-, u^+, \xi, r]$ in $L^\infty(0,T)$. □

The last two theorems now combine to yield a continuity assertion for $\mathbf{\Phi}_\theta$.

Theorem 2.14. *Suppose $D \subset Z_{\theta,\kappa}$. Then $\mathbf{\Phi}_\theta$ is continuous on D.*

Proof. Note first that the convergence of $f_j \to f$ in $L^\infty(\Omega^\pm)$ and $g_j \to g$ in $L^\infty(0,T)$ implies that

$$S_\theta^\pm f_j \to S_\theta^\pm f \text{ in } L^\infty(\Omega^\pm), \ \ \hat{S}_\theta g_j \to \hat{S}_\theta g \text{ in } L^\infty(0,T). \tag{2.55}$$

These to statements are implied by the first line of (2.15) and (2.18).

Let $(v_j^-, v_j^+, \xi_j) \in D$ be a sequence converging in $V^1(\Omega^-) \times V^1(\Omega^+) \times L^\infty(0,T)$. The uniform convergence of $v_j^\pm$ implies (see (2.7))

$$R^\pm[v_j^\pm] \to R^\pm[v^\pm]$$

in $L^\infty(\Omega^\pm)$. Thus using (2.55), we get $r^\pm_\theta(v^\pm_j) \to r^\pm_\theta(v^\pm)$ in $L^\infty(\Omega^\pm)$ (see (2.33)) and note moreover

$$\|D_x r^\pm_\theta(v^\pm_j)\|_{L^\infty(\Omega^\pm)}, \|D_{xx} r^\pm_\theta(v^\pm_j)\|_{L^\infty(\Omega^\pm)} \leq C_\theta, \tag{2.56}$$

due to (2.36) and (2.37). According to Theorem 2.12,

$$\mathbf{T}^\pm_\theta[v^\pm_j, \xi_j, r^\pm_\theta(v^\pm_j)] \to \mathbf{T}^\pm_\theta[v^\pm, \xi, r^\pm_\theta(v^\pm)] \tag{2.57}$$

in $V^1(\Omega^\pm)$. In view of

$$p_\theta(v^-_j, v^+_j)(t) = e^{\nu t}\hat{S}_\theta[e^{-\nu\cdot}(-d + R[v^-_j, v^+_j])]$$

and the fact that $R[v^-_j, v^+_j] \to R[v^-, v^+]$ in $L^\infty(0,T)$ (recall from (2.8) that $R[v^-, v^+] := \sigma(\omega^- + v^-, \omega^+ + v^+) - \sigma(\omega^-, \omega^+) - \sum_\pm \sigma^\pm v^\pm$), $r_\theta(v^-_j, v^+_j, \xi_j)$ converges to $r_\theta(v^-, v^+, \xi)$ in $L^\infty(0,T)$ because of (2.57).

By (2.39), $\|D_t p_\theta(v^-, v^+)\|_{L^\infty(0,T)} \leq C_\theta$. Furthermore, by (2.23),

$$\|\mathbf{T}^\pm_\theta[v^\pm_j, \xi_j, r^\pm_\theta(v^\pm_j)]\|_{W^{1,\infty}(\Omega^\pm)} \leq C_\kappa \|r^\pm_\theta(v^\pm_j)\|_{V^1(\Omega^\pm)} \leq C_\kappa b_1(\|v^\pm_j\|_{V^1(\Omega^\pm)}) \leq C_\kappa,$$

using (2.36). So we have (see (2.34))

$$\|D_t r_\theta(v^-_j, v^+_j, \xi_j)\|_{L^\infty(0,T)} \leq C_\theta.$$

Hence by theorem (2.13),

$$\mathbf{K}_\theta[v^-_j, v^+_j, \xi_j, r_\theta(v^-_j, v^+_j, \xi_j)] \to \mathbf{K}_\theta[v^-, v^+, \xi, r_\theta(v^-, v^+, \xi)],$$

in $L^\infty(0,T)$ so that $\mathbf{\Phi}^0_\theta$ (third component function of $\mathbf{\Phi}_\theta$) is continuous.

Next note that

$$\begin{aligned} &\mathbf{K}_\theta[v^-_j, v^+_j, \xi_j, r_\theta(v^-_j, v^+_j, \xi_j)]S^\pm_\theta \omega^\pm_x + r^\pm_\theta(v^\pm_j) \\ &\to \mathbf{K}_\theta[v^-, v^+\xi, r(v^-, v^+, \xi)]S^\pm_\theta \omega^\pm_x + r^\pm_\theta(v^\pm) \end{aligned} \tag{2.58}$$

in $L^\infty(\Omega^\pm)$, since $\mathbf{K}_\theta[v^-_j, v^+_j, \xi_j, r_\theta(v^-_j, v^+_j, \xi_j)]$ and $r^\pm_\theta(v^\pm_j)$ are uniformly convergent, as shown before. Now the D_x and D_{xx} derivatives of

$$\mathbf{K}_\theta[v^-_j, v^+_j, \xi_j, r_\theta(v^-_j, v^+_j, \xi_j)]S^\pm_\theta \omega^\pm_x + r^\pm_\theta(v^\pm_j)$$

are uniformly bounded in j, which can be seen by applying (2.36), (2.37) to estimate the D_x and D_{xx} derivatives of $r^\pm_\theta(v^\pm_j)$. Hence by applying Theorem 2.12 once more, we get the convergence of

$$\mathbf{T}^\pm_\theta[v^\pm_j, \xi_j, \mathbf{K}_\theta[v^-_j, v^+_j, \xi_j, r_\theta(v^-_j, v^+_j, \xi_j)]S^\pm_\theta \omega^\pm_x + r^\pm_\theta(v^\pm_j)]$$

to $\mathbf{T}^\pm_\theta[v^\pm, \xi, \mathbf{K}_\theta[v^-, v^+, \xi, r_\theta(v^-, v^+, \xi)]S^\pm_\theta \omega^\pm_x + r^\pm_\theta(v^\pm)]$ i.e. the continuity of $\mathbf{\Phi}^\pm_\theta$. □

Theorem 2.15. *Let $D \subset Z_{\theta,\kappa}$, $\theta > 0$ be fixed. Assume furthermore that*

$$\mathbf{\Phi}_\theta(D)$$

is a bounded subset of $V^1(\Omega^-) \times V^1(\Omega^+) \times L^\infty(0,T)$. Then $\mathbf{\Phi}_\theta$ is compact on D, i.e. $\mathbf{\Phi}_\theta$ maps bounded subsets of D into relatively compact subsets of $V^1(\Omega^-) \times V^1(\Omega^+) \times L^\infty(0,T)$.

Proof. Let $B \subset D$ be bounded. It will be sufficient of show that there is a constant $C(B)$, also dependent on θ, such that for all $(v^-, v^+, \xi) \in B$,

$$\begin{aligned}\|D_t\mathbf{\Phi}_\theta^\pm[v^+,v^-,\xi]\|_{L^\infty(\Omega^\pm)} \quad &+\|D_x\mathbf{\Phi}_\theta^\pm[v^+,v^-,\xi]\|_{L^\infty(\Omega^\pm)} + \|D_{xt}\mathbf{\Phi}_\theta^\pm[v^-,v^+,\xi]\|_{L^\infty(\Omega^\pm)} \\ &+\|D_{xx}\mathbf{\Phi}_\theta^\pm[v^-,v^+,\xi]\|_{L^\infty(\Omega^\pm)} \le C(B) \end{aligned} \tag{2.59}$$

$$\|D_t\mathbf{\Phi}_\theta^0[v^-,v^+,\xi]\|_{L^\infty(\Omega^\pm)} \le C(B). \tag{2.60}$$

Taking the above bounds for granted, the compactness assertion is immediate, since then $\mathbf{\Phi}_\theta^\pm(B)$ is relatively compact in $V^1(\Omega^\pm)$ and $\mathbf{\Phi}_\theta^0(B)$ is relatively compact in $L^\infty(0,T)$ (by the Arzéla-Ascoli Theorem, see Theorem 1.3).

Now we proceed to prove (2.59),(2.60). First by (2.36) and (2.37) we have the following crucial bound

$$\|r_\theta^\pm(v^\pm)\|_{V^2(\Omega^\pm)} \le C_\theta(B) \quad ((v^-,v^+,\xi) \in B). \tag{2.61}$$

Moreover we can estimate (see (2.34))

$$\begin{aligned}\|r_\theta(v^-,v^+,\xi)\|_{L^\infty(0,T)} &\le \|p_\theta(v^-,v^+)\|_{L^\infty(0,T)} + \sum_\pm \|\sigma^\pm\|_{L^\infty(0,T)}\|\mathbf{T}_\theta^\pm[v^\pm,\xi,r_\theta^\pm(v^\pm)]\|_{L^\infty(0,T)} \\ &\le C\|-d+R[v^-,v^+]\|_{L^\infty(0,T)} + C_\kappa \sum_\pm \|r_\theta^\pm(v^\pm)\|_{L^\infty(\Omega^\pm)} \\ &\le C(B) \end{aligned} \tag{2.62}$$

for any $(v^-,v^+,\xi) \in B$, by using (2.18) to get rid of the smoothing operator in $p_\theta(v^-,v^+)$, (2.23) to estimate the $T_\theta^\pm$ operators and (2.61) to estimate $\|r_\theta^\pm(v^\pm)\|_{L^\infty(\Omega^\pm)}$. We still need to estimate $\|D_t r_\theta(v^-,v^+,\xi)\|_{L^\infty(0,T)}$:

$$\begin{aligned} &\|D_t r_\theta(v^-,v^+,\xi)\|_{L^\infty(0,T)} \\ \le\ &\|D_t p_\theta(v^-,v^+)\|_{L^\infty(0,T)} + \sum_\pm \|D_t\sigma^\pm\|_{L^\infty(0,T)}\|T_\theta^\pm[v^\pm,\xi,r_\theta^\pm(v^\pm)]\|_{L^\infty(0,T)} \\ +\ &\sum_\pm \|\sigma^\pm\|_{L^\infty(0,T)}\|D_t T_\theta^\pm[v^\pm,\xi,r_\theta^\pm(v^\pm)]\|_{L^\infty(0,T)} \end{aligned}$$

$$\begin{aligned} &\leq C_\theta b_2(\|v^-\|_{L^\infty(\Omega^+)}, \|v^+\|_{L^\infty(\Omega^+)}) + C\sum_{\pm} \|r^\pm(v^\pm)\|_{V^1(\Omega^\pm)} \\ &\leq C_\theta(B), \end{aligned} \tag{2.63}$$

using (2.23) and (2.39).

Now recall that $\mathbf{\Phi}_\theta^\pm[v^-, v^+, \xi]$ solves a particular linear hyperbolic problem (see the definition of $\mathbf{T}_\theta^\pm$, (2.22)). We now can use theorem A.18 to control the second derivative of $\mathbf{\Phi}_\theta^\pm[v^-, v^+, \xi]$ to get

$$\begin{aligned} &\|D_{xx}\mathbf{\Phi}_\theta^\pm[v^-, v^+, \xi]\|_{L^\infty(\Omega^\pm)} \\ \leq\ & C\|\mathbf{K}_\theta[v^-, v^+, r_\theta(v^-, v^+, \xi)]S_\theta^\pm\omega_x^\pm + r_\theta^\pm(v^\pm)\|_{V^2(\Omega^\pm)} \\ \leq\ & C_\theta\left\{\|\mathbf{K}_\theta[v^-, v^+, r_\theta(v^-, v^+, \xi)]\|_{L^\infty(0,T)}\|S_\theta^\pm\omega_x^\pm\|_{V^2(\Omega^\pm)} + C_\theta(B)\right\} \\ \leq\ & C_\theta\left\{\|\mathbf{K}_\theta[v^-, v^+, r_\theta(v^-, v^+, \xi)]\|_{L^\infty(0,T)}\|\omega^\pm\|_{V^1(\Omega^\pm)} + C_\theta(B)\right\} \\ \leq\ & C_\theta(B). \end{aligned}$$

using (2.15) again to control the second derivatives of $S_\theta^\pm\omega_x^\pm$ and (2.62), (2.61) above.

The estimates for $D_x\mathbf{\Phi}_\theta^\pm$, $D_t\mathbf{\Phi}_\theta^\pm$ and $D_{xt}\mathbf{\Phi}_\theta^\pm$ follow in the same manner from (A.18).

To estimate $D_t\mathbf{\Phi}_\theta^0$, we note that $\eta = \mathbf{\Phi}_\theta^0[v^-, v^+, \xi]$ satisfies

$$\eta_t - D_t H[\eta] = D_t r_\theta(v^-, v^+, \xi)$$

where

$$H[\eta] = \sum_{\pm} \sigma^\pm(\cdot)\mathbf{T}_\theta^\pm[v^\pm, \xi, \eta S_\theta^\pm\omega_x^\pm](0, \cdot)$$

(recall (2.10)) and hence by (2.31),

$$\begin{aligned} \|\eta_t\|_{L^\infty(0,T)} &\leq C_{\kappa,\theta}\|\eta\|_{L^\infty(0,T)} + \|D_t r_\theta(v^-, v^+, \xi)\|_{L^\infty(0,T)} \\ &\leq C(B) \end{aligned}$$

by (2.63). Thus (2.60) follows. □

2.3 Main theorem on Φ_θ

Collecting the preceding results, we can state the following *main theorem*, thus overcoming the previously mentioned compactness difficulties.

Theorem 2.16. *Let $D \subset V^1(\Omega^-) \times V^1(\Omega^+) \times L^\infty(0,T)$ be nonempty, closed, convex and bounded. Assume furthermore that there is a θ_0 and a $\kappa > 0$ such that*

$$D \subset Z_{\theta,\kappa}, \boldsymbol{\Phi}_\theta(D) \subset D$$

for all $0 < \theta < \theta_0$. Then for all small θ the mapping $\boldsymbol{\Phi}_\theta$ has a fixed point and as a consequence, there exist a solution to (2.6)*. If moreover D is defined by norm inequalities, as in corollary 2.11, then the solution to* (2.6) *whose existence has been asserted is contained in D.*

Proof. By theorems 2.14 and 2.15, $\boldsymbol{\Phi}_\theta$ is continuous and compact. Under the assumptions of the theorem under consideration, D is mapped into itself by $\boldsymbol{\Phi}_\theta$ for all small $\theta > 0$. Hence by Schauder's fixed point theorem, there exist fixed points $(v_\theta^-, v_\theta^+, \xi_\theta) \in D$. Now apply theorem 2.8 and corollary 2.11. □

Chapter 3

Solution enclosure

3.1 The case of Burgers' Equation

In this chapter, we continue to study the operator $\mathbf{\Phi}_\theta$ from the last chapter. Our goal will be to construct a set D, (defined by concrete inequalities) independent of θ, which is mapped into itself under $\mathbf{\Phi}_\theta$. To simplify the exposition, it is convenient to consider first the case of Burgers' equation. More general enclosure conditions will be given below. Thus, in this section, $a(u) = u$; this gives

$$R^\pm[v^\pm] = a'(\omega^\pm + v^\pm) - a'(\omega^\pm) - a''(\omega^\pm)v^\pm = 0, \tag{3.1}$$

moreover, $\sigma(w^-, w^+) = \int_0^1 a(\tau w^+ + (1-\tau)w^-)d\tau = \frac{1}{2}(w^- + w^+)$ and so

$$\begin{aligned} \sigma^\pm(t) &= \frac{1}{2}, \\ R[v^-, v^+] = \sigma(\omega^- + v^-, \omega^+ + v^+) - \sigma(\omega^-, \omega^+) &- \frac{1}{2}\sum_\pm v^\pm = 0. \end{aligned} \tag{3.2}$$

Note also that

$$r_\theta^\pm(v^\pm) = S_\theta^\pm[-d^\pm]. \tag{3.3}$$

The fixed point operator in question now reads

$$\mathbf{\Phi}_\theta[v^-, v^+, \xi] := \begin{pmatrix} \mathbf{T}_\theta^-[v^-, \xi, \mathbf{K}_\theta[v^-, v^+, \xi, r_\theta(v^-, v^+, \xi)]S_\theta^-(\omega_x^-) - S_\theta^-[d^-]] \\ \mathbf{T}_\theta^+[v^+, \xi, \mathbf{K}_\theta[v^-, v^+, \xi, r_\theta(v^-, v^+, \xi)]S_\theta^+(\omega_x^+) - S_\theta^+[d^+]] \\ \mathbf{K}_\theta[v^-, v^+, \xi, r_\theta(v^-, v^+, \xi)] \end{pmatrix} \tag{3.4}$$

where

$$r_\theta(v^-, v^+, \xi)(t) \;:=\; p_\theta(v^-, v^+)(t) + \sum_\pm \sigma^\pm(t)\mathbf{T}_\theta^\pm[v^\pm, \xi, -S_\theta^\pm[d^\pm]]$$

$$= \quad -e^{\nu t}\hat{S}_\theta[e^{-\nu \cdot}d] + \frac{1}{2}\sum_{\pm} \mathbf{T}^{\pm}_\theta[v^\pm, \xi, -S^\pm_\theta[d^\pm]]$$

3.1.1 Estimation of $\mathbf{T}^{\pm}_\theta$ and $\mathbf{K}_\theta$

Recall that we use the following exponentially weighted norms:

$$\|v\|_\nu := \sup_{t\in(0,T)} e^{-\nu t}\|v(\cdot,t)\|_{L^\infty(\Omega^\pm_t)}$$
$$\|\!|v|\!\|_\nu := \int_{(0,T)} e^{-\nu t}\|v(\cdot,t)\|_{L^\infty(\Omega^\pm_t)}dt.$$

for functions v either in $L^\infty(\Omega^-)$ or $L^\infty(\Omega^+)$. We also use

$$\|f\|_\nu := \sup_{(0,T)} e^{-\nu t}|f(t)|$$

for functions $f \in L^\infty(0,T)$.

In order to construct D explicitly, we will rely on the following explicit estimates for $\mathbf{T}^\pm_\theta$:

$$\|\mathbf{T}^\pm_\theta[u,\xi,r]\|_\nu \quad \le \quad \|r\|_\nu \tag{3.5}$$

$$\begin{aligned}\|D_x\mathbf{T}^\pm_\theta[u,\xi,r]\|_\nu \quad &\le \quad \|r_x\|_\nu + \left(\int_0^T \|D_xS^\pm_\theta\omega^\pm_x\|_{L^\infty(\Omega^\pm_t)}dt\right)\|\mathbf{T}^\pm_\theta[u,\xi,r]\|_\nu \\ &\le \quad \|r_x\|_\nu + \left(\int_0^T \|\omega^\pm_{xx}\|_{L^\infty(\Omega^\pm_\tau)}d\tau\right)\|\mathbf{T}^\pm_\theta[u,\xi,r]\|_\nu,\end{aligned} \tag{3.6}$$

To obtain them, recall that $\mathbf{T}^\pm_\theta[u,\xi,r]$ solves a linear hyperbolic problem with transport coefficient $\omega^\pm + S^\pm_\theta u - \mu - \xi$ and production coefficient (denoted by c in Appendix A) given by $S^\pm_\theta a(\omega^\pm)_x$. Thus, (3.5) and (3.6) are just the bounds (A.31) from appendix A (using, as usual, (2.15) to estimate the smoothing operator). However, for these bounds to be valid, we need the condition (A.32) to hold, which now reads

$$\nu \ge \max\{\sup_{\Omega^\pm}(-S^\pm_\theta(\omega^\pm_x)), \sup_{\Omega^\pm}(-S^\pm_\theta\omega^\pm_x - \omega^\pm_x - D_xS^\pm_\theta u)\}. \tag{3.7}$$

Lemma 3.1. *(see theorem A.16). Suppose* (3.7) *holds. Then we have the following estimate for* $\mathbf{K}_\theta$*:*

$$\|\mathbf{K}_\theta[u^-,u^+,\xi,r]\|_\nu \le C_1\|r\|_\nu \quad (r\in C[0,T]) \tag{3.8}$$

with

$$C_1 = \exp\left(\int_0^T h(\tau)d\tau\right), \tag{3.9}$$
$$h(\tau) = \sum_{\pm} \frac{1}{2}\|\omega_x^{\pm}(\cdot,\tau)\|_{L^\infty(\Omega_\tau^{\pm})}.$$

Proof. Let $\eta = \mathbf{K}_\theta[u^-, u^+, \xi, r]$. Then, by (2.10),

$$\eta - \frac{1}{2}\mathbf{T}_\theta^-[u^-, \xi, \eta S_\theta^- \omega_x^-](0,\cdot) - \frac{1}{2}\mathbf{T}_\theta^+[u^+, \xi, \eta S_\theta^+ \omega_x^+](0,\cdot) = r.$$

and hence by using (3.5), we obtain for $t \in (0,T)$ a.e.,

$$\begin{aligned} e^{-\nu t}|\eta(t)| &\leq e^{-\nu t}|r(t)| + e^{-\nu t}\frac{1}{2}\sum_{\pm}|\mathbf{T}_\theta^{\pm}[u^{\pm}, \xi, \eta S_\theta^{\pm}(\omega_x^{\pm})](0,t)| \\ &\leq e^{-\nu t}|r(t)| + \frac{1}{2}\sum_{\pm}\int_0^t |\eta(\tau)|\|\omega_x^{\pm}\|_{L^\infty(\Omega_\tau^{\pm})}e^{-\nu\tau}d\tau \\ &\leq e^{-\nu t}|r(t)| + \int_0^t h(\tau)|\eta(\tau)|e^{-\nu\tau}d\tau \end{aligned}$$

where $h(\tau) = \frac{1}{2}\sum_{\pm}\|\omega_x^{\pm}(\cdot,\tau)\|_{L^\infty(\Omega_\tau^{\pm})}$. So by [22], Chapter 1.III (Gronwall's inequality), we have the estimate

$$e^{-\nu t}|\eta(t)| \leq e^{-\nu t}|r(t)| + \int_0^t h(\tau)e^{H(t,\tau)}|r(\tau)|e^{-\nu\tau}d\tau,$$

with $H(t,\tau) = \int_\tau^t h(s)ds$. This gives

$$\begin{aligned} \|\eta\|_\nu &\leq \left(1 + \int_0^t h(\tau)e^{H(t,\tau)}d\tau\right)\|r\|_\nu \\ &\leq \left(1 - e^{H(T,T)} + e^{H(T,0)}\right)\|r\|_\nu \\ &\leq e^{H(T,0)}\|r\|_\nu \end{aligned}$$

after estimating $e^{-\nu\tau}|r(t)|$ by $\|r\|_\nu$ in the first line. □

For practical purposes and in case of Burgers' equation ($|\sigma^{\pm}| = \frac{1}{2}$) however, we use the following less precise estimate for C_1:

$$C_1 := \exp\left(\frac{1}{2}\sum_{\pm}\|\omega_x^{\pm}\|_{L^\infty(\Omega^{\pm})}T\right). \tag{3.10}$$

This follows immediately by noting $h(\tau) \leq \frac{1}{2}\sum_{\pm}\|\omega_x^{\pm}\|_{L^\infty(\Omega^{\pm})}$.

3.1.2 Construction of D

Now consider a set D defined by the inequalities

$$\begin{cases} \|v^\pm\|_\nu & \leq & \alpha^\pm \\ \|v_x^\pm\|_\nu & \leq & \beta^\pm \\ \|\xi\|_\nu & \leq & \gamma \end{cases} \tag{3.11}$$

Our goal is to give sufficient conditions on the parameters $\nu, \alpha^\pm, \beta^\pm, \gamma$ such that $\mathbf{\Phi}_\theta(D) \subset D$ for all $\theta > 0$.

Theorem 3.2. *Suppose D, defined by (3.11), is contained in $Z_{\theta,\kappa}$ for some $\kappa > 0$ and for all sufficiently small $\theta > 0$. Define the following numbers $C_2^\pm, C_3^\pm$ explicitly computable from the approximate solution $\omega^\pm$:*

$$C_2^\pm := C_1 \int_0^T \|\omega_x^\pm\|_{L^\infty(\Omega_\tau^\pm)} d\tau \tag{3.12}$$

$$C_3^\pm := \int_0^T \|\omega_{xx}^\pm\|_{L^\infty(\Omega_\tau^\pm)} d\tau \tag{3.13}$$

(and C_1 given by (3.10)*).*

Then the following conditions are sufficient for $\mathbf{\Phi}_\theta(D) \subset D$ (for all sufficiently small θ):

$$2\|\omega_x^\pm\|_{L^\infty(\Omega^\pm)} + \beta^\pm e^{\nu T} \leq \nu \tag{3.14}$$

$$C_1 \left[\|d\|_\nu + \frac{1}{2} \sum_\pm \|d^\pm\|_\nu \right] \leq \gamma \tag{3.15}$$

$$C_2^\pm \left[\|d\|_\nu + \frac{1}{2} \sum_\pm \|d^\pm\|_\nu \right] + \|d^\pm\|_\nu \leq \alpha^\pm \tag{3.16}$$

$$C_3^\pm (C_1 + C_2^\pm) \left[\|d\|_\nu + \frac{1}{2} \sum_\pm \|d^\pm\|_\nu \right] + \|d_x^\pm\|_\nu + C_3^\pm \|d^\pm\|_\nu \leq \beta^\pm \tag{3.17}$$

Proof. We have for $(u^-, u^+, \xi) \in D$ (i.e. (3.11) holds for $u^\pm, \xi$), by use of (2.15),

$$\begin{aligned} \max\{\sup_{\Omega^\pm}(-S_\theta^\pm \omega_x^\pm), \sup_{\Omega^\pm}(-S_\theta^\pm \omega_x^\pm - \omega_x^\pm - D_x S_\theta^\pm u^\pm)\} \\ \leq 2\|\omega_x^\pm\|_{L^\infty(\Omega^\pm)} + \|u_x^\pm\|_{L^\infty(\Omega^\pm)} \\ \leq 2\|\omega_x^\pm\|_{L^\infty(\Omega^\pm)} + \beta^\pm e^{\nu T}, \end{aligned}$$

hence the condition posed in (3.14) is sufficient for (3.7) to hold.

From (3.5) we find for $(v^-, v^+, \xi) \in D$ the following inequality

$$\|\mathbf{T}_\theta^\pm[v^\pm, \xi, -S_\theta^\pm d^\pm]\|_\nu \leq \|S_\theta^\pm d^\pm\|_\nu \tag{3.18}$$

since (3.14) and therefore (3.7) holds. Note moreover the following two inequalities (use (2.16) and (2.35)):

$$\begin{aligned}
\|p_\theta(v^-, v^+)\|_\nu &= \|e^{\nu\cdot}\hat{S}_\theta[de^{-\nu\cdot}]\|_\nu \\
&= \sup_{\tau\in(0,T)} |e^{-\nu\tau}e^{\nu\tau}\hat{S}_\theta[de^{-\nu\cdot}](\tau)| \\
&\leq \|\hat{S}_\theta[de^{-\nu\cdot}]\|_{L^\infty(0,T)} \leq \|de^{-\nu\cdot}\|_{L^\infty(0,T)} = \|d\|_\nu, \qquad (3.19)\\
\|S_\theta^\pm d^\pm\|_\nu &= \int_0^T e^{-\nu\tau}\|S_\theta^\pm d^\pm\|_{L^\infty(\Omega_\tau^\pm)}d\tau \\
&\leq \int_0^T e^{-\nu\tau}\|d^\pm\|_{L^\infty(\Omega_\tau^\pm)}d\tau \leq \|d^\pm\|_\nu. \qquad (3.20)
\end{aligned}$$

We are now prepared to estimate $\mathbf{\Phi}_\theta^0$. Using the definition of $r_\theta(v^-, v^+, \xi)$ (see (2.34)), (3.18), (3.5) and (3.19) we have

$$\begin{aligned}
\|r_\theta(v^-, v^+, \xi)\|_\nu &\leq \|p_\theta(v^-, v^+)\|_\nu + \frac{1}{2}\sum_\pm \|\mathbf{T}_\theta^\pm[v^\pm, \xi, -S_\theta^\pm[d^\pm]]\|_\nu \\
&\leq \|p_\theta(v^-, v^+)\|_\nu + \frac{1}{2}\sum_\pm \|S_\theta^\pm d^\pm\|_\nu \\
&\leq \|d\|_\nu + \frac{1}{2}\sum_\pm \|S_\theta^\pm d^\pm\|_\nu \\
&\leq \|d\|_\nu + \frac{1}{2}\sum_\pm \|d^\pm\|_\nu \qquad (3.21)
\end{aligned}$$

Note moreover that by (2.15)

$$\begin{aligned}
\|D_x(S_\theta^\pm d^\pm)\|_\nu &= \int_0^T \|D_x(S_\theta^\pm d^\pm)(\cdot,\tau)\|_{L^\infty(\Omega_\tau^\pm)}e^{-\nu\tau}d\tau \\
&\leq \int_0^T \|d_x^\pm(\cdot,\tau)\|_{L^\infty(\Omega_\tau^\pm)}e^{-\nu\tau}d\tau = \|d_x^\pm\|_\nu \qquad (3.22)
\end{aligned}$$

Thus, using (3.8) and (3.21), we arrive at (note that $r_\theta(v^-, v^+, \xi)$ is continuous due to the smoothing operator, see (2.34)).

$$\|\mathbf{\Phi}_\theta^0[v^-, v^+, \xi]\|_\nu = \|\mathbf{K}_\theta[v^-, v^+, \xi, r_\theta(v^-, v^+, \xi)]\|_\nu$$

$$\leq C_1 \left[\|d\|_\nu + \frac{1}{2} \sum_\pm \|d^\pm\|_\nu \right]. \tag{3.23}$$

Hence we find using (3.18), (3.20), (3.23) and (2.32)

$$\begin{aligned}
\|\mathbf{\Phi}_\theta^\pm[v^-, v^+, \xi]\|_\nu &\leq \|\mathbf{K}_\theta[v^-, v^+, \xi, r_\theta(v^-, v^+\xi)] S_\theta^\pm(\omega_x^\pm)\|_\nu + \|S_\theta^\pm d^\pm\|_\nu \\
&\leq \left(\int_0^T \|\omega_x^\pm\|_{L^\infty(\Omega_\tau^\pm)} d\tau \right) \|\mathbf{K}_\theta[v^-, v^+, \xi, r_\theta(v^-, v^+\xi)]\|_\nu + \|d^\pm\|_\nu \\
&\leq C_1 \left(\int_0^T \|\omega_x^\pm\|_{L^\infty(\Omega_\tau^\pm)} d\tau \right) \left[\|d\|_\nu + \frac{1}{2} \sum_\pm \|d^\pm\|_\nu \right] + \|d^\pm\|_\nu \\
&\leq C_2^\pm \left[\|d\|_\nu + \frac{1}{2} \sum_\pm \|d^\pm\|_\nu \right] + \|d^\pm\|_\nu. \qquad (3.24)
\end{aligned}$$

Next we estimate $D_x \mathbf{\Phi}_\theta^\pm[v^-, v^+, \xi]$ using (3.6), (3.23) and (3.24):

$$\begin{aligned}
\|D_x \mathbf{\Phi}_{\theta^\pm}[v^-, v^+, \xi]\|_\nu &\leq \|\mathbf{K}_\theta[v^-, v^+, \xi, r_\theta(v^-, v^+, \xi)] D_x S_\theta^\pm \omega_x\|_\nu \qquad (3.25) \\
&\quad + \|D_x S_\theta^\pm d^\pm\|_\nu + \left(\int_0^T \|\omega_{xx}^\pm\|_{L^\infty(\Omega_\tau^\pm)} d\tau \right) \|\mathbf{\Phi}_\theta^\pm\|_\nu \\
&\leq \left(\int_0^T \|\omega_{xx}^\pm\|_{L^\infty(\Omega_\tau^\pm)} d\tau \right) \|\mathbf{K}_\theta[v^-, v^+, \xi, r_\theta(v^-, v^+, \xi)]\|_\nu \\
&\quad + \|d_x^\pm\|_\nu + \left(\int_0^T \|\omega_{xx}^\pm\|_{L^\infty(\Omega_\tau^\pm)} d\tau \right) \|\mathbf{\Phi}_\theta^\pm\|_\nu \\
&\leq C_3^\pm C_1 \left[\|d\|_\nu + \frac{1}{2} \sum_\pm \|d^\pm\|_\nu \right] + \|d_x^\pm\|_\nu \\
&\quad + C_3^\pm \left[C_2^\pm \left(\|d\|_\nu + \frac{1}{2} \sum_\pm \|d^\pm\|_\nu \right) + \|d^\pm\|_\nu \right] \\
&= C_3^\pm (C_1 + C_2^\pm) \left[\|d\|_\nu + \frac{1}{2} \sum_\pm \|d^\pm\|_\nu \right] + \|d_x^\pm\|_\nu + \\
&+ \quad C_3^\pm \|d^\pm\|_\nu. \qquad (3.26)
\end{aligned}$$

The statement of the theorem is a direct consequence of (3.23), (3.24) and (3.26). □

In the above theorem, the only non-trivial condition is the condition on ν, since we may define the numbers $\alpha^\pm, \beta^\pm, \gamma$ by the left-hand sides of the inequalities in the theorem. This is a consequence of the linearity of the function a. The corresponding conditions for general a are discussed in the next section.

If $\alpha^\pm, \beta^\pm$ and γ can be chosen sufficiently small, i.e. if the defects

$$\|d^\pm\|_\nu, \|d_x^\pm\|_\nu, \|d\|_\nu$$

of the numerical approximate solution are sufficiently small (note that the norms depend on ν) then there is a good chance of satisfying (3.14) in a concrete situation (with some $\nu > 0$); a large value of T, however, makes the condition more difficult to satisfy.

Once we have $\mathbf{\Phi}_\theta(D) \subset D, D \subset Z_{\theta,\kappa}$, the fixed point argument from chapter 2 will yield the existence of a solution in D, i.e. we get enclosures of the form

$$\|w^\pm(\cdot,\tau) - \omega^\pm(\cdot,\tau)\|_{L^\infty(\Omega_\tau^\pm)} \le \alpha^\pm e^{\nu\tau}.$$

with $w^\pm$ denoting the true solution.

If we assume that the numerical approximate solution is strictly monotone increasing, then we can get rid of the exponentially growing factors, i.e. we may set $\nu = 0$.

Theorem 3.3. *Assume that there is a constant $k > 0$ such that*

$$\omega_x^\pm \ge k > 0 \quad on \;\; \Omega^\pm. \tag{3.27}$$

Let D be defined by inequalities (3.11)*, where now $\nu = 0$. Then the conditions*

$$-2k + \beta^\pm \le 0 \tag{3.28}$$

$$C_1 \left[\|d\|_0 + \frac{1}{2} \sum_\pm \|d^\pm\|_0 \right] \le \gamma \tag{3.29}$$

$$C_2^\pm \left[\|d\|_0 + \frac{1}{2} \sum_\pm \|d^\pm\|_0 \right] + \|d^\pm\|_0 \le \alpha^\pm \tag{3.30}$$

$$C_3^\pm(C_1 + C_2^\pm) \left[\|d\|_0 + \frac{1}{2} \sum_\pm \|d^\pm\|_0 \right] + \|d_x^\pm\|_0 + C_3^\pm \|d^\pm\|_0 \le \beta^\pm \tag{3.31}$$

are sufficient for $\mathbf{\Phi}_\theta(D) \subset D$. The constants $C_2^\pm, C_3^\pm$ are the same as in the previous theorem.

Proof. We note that now under the conditions of the theorem, $(u^-, u^+, \xi) \in D$ implies

$$\max\{\sup_{\Omega^\pm}(-S_\theta^\pm \omega_x^\pm), \sup_{\Omega^\pm}(-S_\theta^\pm \omega_x^\pm - \omega_x^\pm - D_x S_\theta^\pm u^\pm)\}$$
$$\le \max\{-k, -2k + \beta^\pm\} \le 0,$$

hence (3.7) holds. Here we have used (A.15). The other estimates go as before. □

3.2 The general case

We return now to the case of general $a \in C^2$, and give the analogous conditions which are sufficient for D of the form (3.11) to be mapped into itself. Recall that the fixed point operator is given by

$$\mathbf{\Phi}_\theta[v^-, v^+, \xi] := \begin{pmatrix} \mathbf{T}_\theta^-[v^-, \xi, \mathbf{K}_\theta[v^-, v^+, \xi, r_\theta(v^-, v^+, \xi)]S_\theta^-(\omega_x^-) + r_\theta^-(v^-)] \\ \mathbf{T}_\theta^+[v^+, \xi, \mathbf{K}_\theta[v^-, v^+, \xi, r_\theta(v^-, v^+, \xi)]S_\theta^+(\omega_x^+) + r_\theta^+(v^+)] \\ \mathbf{K}_\theta[v^-, v^+, \xi, r_\theta(v^-, v^+, \xi)] \end{pmatrix}. \tag{3.32}$$

with $r_\theta, r_\theta^\pm$ from equations (2.33), (2.34).

The estimate (3.8) continues to hold, but in the definition (3.9) of C_1, h has to be replaced by

$$h(\tau) := \sum_\pm |\sigma^\pm(\tau)| \|\omega_x^\pm\|_{L^\infty(\Omega_\tau^\pm)}$$

We need some estimates from appendix B. Assume that nonnegative monotone nondecreasing functions $g, m, l : [0, \infty) \to [0, \infty)$ and a number $M > 0$ are known such that

$$|a(\omega + v) - a(\omega) - a'(\omega)v| \leq g(|v|) \tag{3.33}$$

$$|a'(\omega + v) - a'(\omega) - a''(\omega)| \leq m(|v|) \tag{3.34}$$

$$|a'(v)| \leq l(v) \tag{3.35}$$

for all $v, \omega \in \mathbb{R}, |\omega| \leq M$ and such that $g(v) = o(|v|), m(v) = o(|v|)$ as $|v| \to 0^+$. Assume moreover that a function $q : [0, \infty) \times [0, \infty) \to [0, \infty)$ with $q(v^-, v^+) = o(|v^-| + |v^+|)$ as $|v^-|, |v^+| \to 0^+$ is known such that (recall the definition of R from (2.8)):

$$|R[v^-, v^+]| \leq q(|v^-|, |v^+|). \tag{3.36}$$

Define the quantities M_1, M_2 as in appendix B and set furthermore

$$M_3(\alpha^-, \alpha^+) := \sup_{(0,T)} e^{-\nu t} q(e^{\nu t}\alpha^-, e^{\nu t}\alpha^+) \quad (\alpha^-, \alpha^+ \in [0, \infty)) \tag{3.37}$$

Assume furthermore $\|\omega^\pm\|_{L^\infty(\Omega^\pm)} \leq M$.

Then we have the following estimates (see appendix B):

$$\left.\begin{array}{rcl} \|r_\theta^\pm(v^\pm)\|_\nu & \leq & \|d^\pm\|_\nu + M_1(\|v^\pm\|_\nu) \\ \|r_\theta^\pm(v^\pm)_x\|_\nu & \leq & \|d_x^\pm\|_\nu + M_2(\|v^\pm\|_\nu, \|v_x^\pm\|_\nu) \end{array}\right\}. \tag{3.38}$$

Theorem 3.4. *The following conditions are sufficient for $\Phi_\theta(D) \subset D$:*

$$\begin{aligned}
l(\|\omega^\pm\|_{L^\infty(\Omega^\pm)})\|\omega^\pm_x\|_{L^\infty(\Omega^\pm)} + l\left(\|\omega^\pm\|_{L^\infty(\Omega^\pm)} + \alpha^\pm e^{\nu T}\right)\left(\|\omega^\pm_x\|_{L^\infty(\Omega^\pm)} + \beta^\pm e^{\nu T}\right) &\leq \nu \\
C_1\left[\|d\|_\nu + M_3(\alpha^-,\alpha^+) + \sum_\pm \left(\|d^\pm\|_\nu + M_1^\pm(\alpha^\pm)\right)\right] &\leq \gamma \\
C_2^\pm\left[\|d\|_\nu + M_3(\alpha^-,\alpha^+) + \sum_\pm \|\sigma^\pm\|_\infty\left[\|d^\pm\|_\nu + M_1^\pm(\alpha^\pm)\right]\right] + \|d^\pm\|_\nu + M_1^\pm(\alpha^\pm) &\leq \alpha^\pm \\
C_3^\pm(C_1 + C_2^\pm)\left[\|d\|_\nu + M_3(\alpha^-,\alpha^+) + \sum_\pm \|\sigma^\pm\|_\infty\left[\|d^\pm\|_\nu + M_1^\pm(\alpha^\pm)\right]\right] & \\
+\|d^\pm_x\|_\nu + C_3^\pm\|d^\pm\|_\nu + M_2(\alpha^\pm,\beta^\pm) &\leq \beta^\pm
\end{aligned}$$

where $C_1, C_2^\pm, C_3^\pm$ are as in (3.10)*,* (3.12)*,* (3.13)*.*

Proof. To obtain a sufficient condition for (3.7), we note that

$$\begin{aligned}
\sup_{\Omega^\pm} -S_\theta^\pm a(\omega^\pm)_x &\leq \|a(\omega^\pm)_x\|_{L^\infty(\Omega^\pm)} \\
&= \|a'(\omega^\pm)\omega^\pm_x\|_{L^\infty(\Omega^\pm)} \\
&\leq l\left(\|\omega^\pm\|_{L^\infty(\Omega^\pm)}\right)\|\omega^\pm_x\|_{L^\infty(\Omega^\pm)}
\end{aligned}$$

using (3.35) and (2.15). Moreover we have

$$\begin{aligned}
&|-S_\theta^\pm a(\omega^\pm)_x - a'(\omega^\pm + S_\theta^\pm v^\pm)(\omega^\pm_x + D_x S_\theta^\pm v^\pm)| \\
\leq\ & l\left(\|\omega^\pm\|_{L^\infty(\Omega^\pm)}\right)\|\omega^\pm_x\|_{L^\infty(\Omega^\pm)} + l\left(\|\omega^\pm\|_{L^\infty(\Omega^\pm)} + \|v^\pm\|_{L^\infty(\Omega^\pm)}\right)\left(\|\omega^\pm_x\|_{L^\infty(\Omega^\pm)} + \|v^\pm_x\|_{L^\infty(\Omega^\pm)}\right).
\end{aligned}$$

again by (3.35) and (2.15)), and thus

$$\begin{aligned}
&\max\{\sup_{\Omega^\pm} -a(S_\theta\omega^\pm)_x, \sup_{\Omega^\pm}\left(-a(S_\theta\omega^\pm)_x - a(\omega^\pm + S_\theta v^\pm)_x\right)\} \\
&\leq l(\|\omega^\pm\|_{L^\infty(\Omega^\pm)})\|\omega^\pm_x\|_{L^\infty(\Omega^\pm)} + l\left(\|\omega^\pm\|_{L^\infty(\Omega^\pm)} + \alpha^\pm e^{\nu T}\right)\left(\|\omega^\pm_x\|_{L^\infty(\Omega^\pm)} + \beta^\pm e^{\nu T}\right).
\end{aligned}$$

This is just the first line of the set of conditions in the theorem.

Next we need to estimate $\|p_\theta(v^-, v^+)\|_\nu$:

$$\begin{aligned}
\|p_\theta(v^-,v^+)\|_\nu &\leq \|d\|_\nu + \sup_{(0,T)} |\hat{S}_\theta[e^{-\nu\cdot}R[v^-,v^+]](t)| \\
&\leq \|d\|_\nu + \sup_{(0,T)} e^{-\nu t}|R[v^-,v^+]| \\
&\leq \|d\|_\nu + \sup_{(0,T)} e^{-\nu t} q(e^{\nu t}\alpha^-, e^{\nu t}\alpha^+)
\end{aligned}$$

$$=: \quad \|d\|_\nu + M_3(\alpha^-, \alpha^+) \tag{3.39}$$

for $(v^-, v^+, \xi) \in D$.

1. *Estimation of* $\mathbf{\Phi}^0_\theta$. We find for $(v^-, v^+, \xi) \in D$ the following inequality

$$\begin{aligned}
\|r_\theta(v^-, v^+, \xi)\|_\nu &\leq \|p_\theta(v^-, v^+)\|_\nu + \sum_\pm \|\sigma^\pm\|_\infty \|\mathbf{T}^\pm_\theta[v^\pm, \xi, r^\pm_\theta(v^\pm)]\|_\nu \\
&\leq \|d\|_\nu + M_3(\alpha^-, \alpha^+) + \sum_\pm \|\sigma^\pm\|_\infty \|r^\pm_\theta(v^\pm)\|_\nu \\
&\leq \|d\|_\nu + M_3(\alpha^-, \alpha^+) \\
&+ \sum_\pm \|\sigma^\pm\|_\infty \left[|\!|\!| d^\pm |\!|\!|_\nu + M^\pm_1(\alpha^\pm)\right]
\end{aligned} \tag{3.40}$$

where we have used (3.39), (3.19) and (3.38). Hence we obtain

$$\begin{aligned}
&\|\mathbf{\Phi}^0_\theta(v^-, v^+, \xi)\|_\nu = \|\mathbf{K}_\theta[v^-, v^+, \xi, r_\theta(v^-, v^+\xi)]\|_\nu \\
&\leq C_1 \left[\|d\|_\nu + M_3(\alpha^-, \alpha^+) + \sum_\pm \|\sigma^\pm\|_\infty \left[|\!|\!| d^\pm |\!|\!|_\nu + M^\pm_1(\alpha^\pm)\right]\right].
\end{aligned} \tag{3.41}$$

2. *Estimation of* $\mathbf{\Phi}^\pm_\theta$. From (3.38), (3.41) and (B.10) we get

$$\begin{aligned}
&\|\mathbf{\Phi}^\pm_\theta\|_\nu \leq \|\mathbf{K}_\theta[v^-, v^+, \xi, r_\theta(v^-, v^+\xi)]\|_\nu \left(\int_0^T \|\omega^\pm_x\|_{L^\infty(\Omega^\pm_\tau)} d\tau\right) + \|r^\pm_\theta(v^\pm)\|_\nu \\
&\leq \quad C^\pm_2 \left[\|d\|_\nu + M_3(\alpha^-, \alpha^+) + \sum_\pm \|\sigma^\pm\|_\infty \left[|\!|\!| d^\pm |\!|\!|_\nu + M^\pm_1(\alpha^\pm)\right]\right] + |\!|\!| d^\pm |\!|\!|_\nu + M^\pm_1(\alpha^\pm)
\end{aligned}$$

with $C^\pm_2$ as before. By analogous estimations as before, we find

$$\begin{aligned}
&\|D_x \mathbf{\Phi}^\pm_\theta\|_\nu \leq C^\pm_3(C_1 + C^\pm_2) \left[\|d\|_\nu + M_3(\alpha^-, \alpha^+) + \sum_\pm \|\sigma^\pm\|_\infty \left[|\!|\!| d^\pm |\!|\!|_\nu + M^\pm_1(\alpha^\pm)\right]\right] \\
&+ \quad |\!|\!| d^\pm_x |\!|\!|_\nu + C^\pm_3 |\!|\!| d^\pm |\!|\!|_\nu + M_2(\alpha^\pm, \beta^\pm) + C^\pm_3 M_1(\alpha^\pm)
\end{aligned}$$

where $C^\pm_3$ is the same as before. Now it is obvious that $\mathbf{\Phi}_\theta(D) \subset D$ is a consequence of the conditions in the theorem. □

Note that M_1, M_2, M_3 are quadratic in $\alpha^\pm, \beta^\pm$ so that we have a good chance of satisfying the above conditions if the defects are small and T is not too large.

Chapter 4

Numerical treatment

In this chapter, we explain, for the case of Burger's equation, how to compute and verify solutions. The considerations on the numerical scheme are slightly more general and hold for arbitrary a, but later on we will set $a(u) = u$.

4.1 Computation of solutions

Recall that we intend first to find numerical approximate solutions to (see (1.8))

$$\begin{cases} w_t^- + (a(w^-) - \chi)w_x^- = 0 & : \text{ on } \Omega^- \\ w_t^+ + (a(w^+) - \chi)w_x^+ = 0 & : \text{ on } \Omega^+ \\ \chi(t) = \sigma(w^-(0,t), w^+(0,t)) & : \text{ on } (0,T) \\ w^-(\cdot,0) = w_0^-(\cdot), w^+(\cdot,0) = w_0^+(\cdot) & \end{cases}, \tag{4.1}$$

together with the Lax admissibility condition (see (1.9))

$$a(w^-(0,t)) - \chi(t) > 0, \;\; a(w^+(0,t)) - \chi(t) < 0. \tag{4.2}$$

If we want to verify the existence of true solutions close to the approximations $(\omega^-, \omega^+, \mu)$, the theory from the previous chapters requires the following smoothness:

$$\omega^\pm, \omega_x^\pm, \omega_t^\pm, \omega_{xx}^\pm, \omega_{xt}^\pm \in L^\infty(\Omega^\pm), \mu \in BV(0,T). \tag{4.3}$$

The process consists of two steps: first we compute an approximate solution defined on a grid, then we interpolate to construct the approximate solution $(\omega^-, \omega^+, \mu)$ with the required smoothness properties.

4.1.1 A finite difference scheme

Since we are looking for *smooth* solutions w^+, w^- to the left and to the right of the shock, we use a Lax-Wendroff scheme for discretizing the PDE's in (4.1). The Lax-Wendroff scheme should allow us to compute the smooth parts of the solution with accuracy, whereas the motion of the shock is characterized here only by its shock velocity χ. The idea of separating the computation of the smooth parts of the solution and the computation of the shock velocity or position is called *shock-tracking* (see e.g. [3]). In contrast, a large part of the literature on the numerical analysis of conservation laws is devoted to the methodology of *shock-capturing*. In a shock-capturing numerical scheme, we do not have extra variables (like χ) characterizing the position and/or the velocity of the discontinuites in the solution. Rather, the finite-difference scheme by itself produces the steep changes in the solution (see [12]).

Return now to our problem (4.1), writing $f(u,t) = \mathcal{A}(u) - \chi(t)u$, where $\mathcal{A}$ is a primitive of a, we can write the relevant PDE's in (4.1) in the conservative form

$$w_t^\pm + f(w^\pm, t)_x = 0 \quad (\text{on } \Omega^\pm). \tag{4.4}$$

This is useful for writing down a *conservative* (see [12]) Lax-Wendroff scheme. We introduce a computational domain of the form $(-A, A) \times (0, T)$ (which is larger than the domain $\Omega^- \cup \Omega^+$ on which we verify the existence of a solution) and two grids

$$G^- := \{(x_n, t_m) : x_n = nh,\ t_m = mk,\ n \leq q,\ m \in \mathbb{N}_0,\ x_n \geq -A,\ t_m \leq T\} \tag{4.5}$$

$$G^+ := \{(x_n, t_m) : x_n = nh,\ t_m = mk,\ n \geq -q,\ m \in \mathbb{N}_0, x_n \leq A,\ t_m \leq T\} \tag{4.6}$$

with spatial stepsize $h > 0$ and temporal stepsize $k > 0$. The number $q \in \mathbb{N}$ is positive, hence for a number of grid points, the left grid continues into $\{x > 0\}$ and the right grid into $\{x < 0\}$ (see figure 4.1.1). The use of overlapping grids empirically turned out to be helpful in increasing the precision of the numerical approximate solutions (i.e. to reduce the defect bounds). We choose initial conditions

$$w_{n,0}^- = \hat{w}_n^-,\ \ w_{n,0}^+ = \hat{w}_n^+,\ \ \chi_0 = \sigma(\hat{w}_0^-, \hat{w}_0^+)$$

for the discretized quantities $w_{n,m}^+, w_{n,m}^-, \chi_m$ to be computed (see section 4.2 for discussion how to choose the $\hat{w}^\pm$).

Suppose χ_m has already been computed. If $x_n \neq qh$, we compute $w_{n,m}^- = w^-(x_n, t_m)$ using the following (rather standard) Lax-Wendroff scheme

$$w_{n,m+1}^- = w_{n,m}^- - \frac{1}{2}\lambda[f(w_{n+1,m}^-, t_m) - f(w_{n-1,m}^-, t_m)]$$

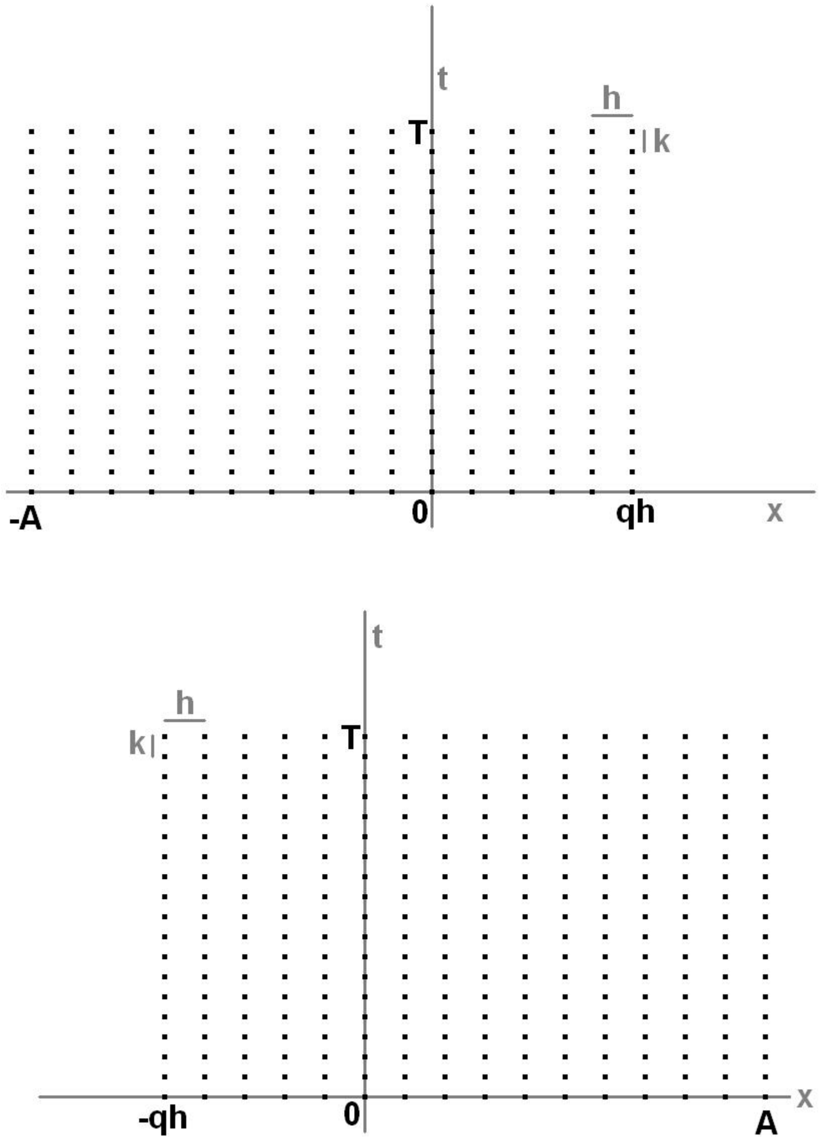

Figure 4.1: The computational domain

$$+\frac{1}{2}\lambda^2[a_1(f(w^-_{n+1,m},t_m)-f(w^-_{n,m},t_m))$$
$$-a_2(f(w^-_{n,m},t_m)-f(w^-_{n-1,m},t_m))]$$

where

$$\begin{aligned} f(w^-_{n,m},t_m) &= \mathcal{A}(w^-_{n,m})-\chi_m w^-_{n,m}, \\ a_1 &= a\left(\frac{1}{2}(w^-_{n,m}+w^-_{n+1,m})\right)-\chi_m, \\ a_2 &= a\left(\frac{1}{2}(w^-_{n-1,m}+w^-_{n,m})\right)-\chi_m. \end{aligned}$$

See [12] for a discussion of the Lax-Wendroff-scheme, in particular for the motivation of the terms $\frac{1}{2}\lambda^2[\ldots]$.

In the actual computation, $\lambda = \frac{k}{h}$ is chosen to satisfy the Courant-Friedrichs-Lewy (CFL) condition. At the right boundary of the grid G^-, i.e. if $n = q$, we use

$$w^-_{n,m+1} = w^-_{n,m} + \lambda[f(w^-_{n-1,m},t_m) - f(w^-_{n,m},t_m)]. \tag{4.7}$$

The *order* of the numerical scheme (in the sense of truncation error) is hence reduced around the grid points $x_n = qh$, but this does not affect the overall accuracy since in our numerical examples the characteristic direction is always pointing into the shock. The errors due to (4.7) will therefore not propagate into $(-A, 0)$.

Analogous formulas are used for the computation of $w^+_{n,m+1}$. After computing $w^+_{n,m+1}, w^-_{n,m+1}$, we set

$$\chi_{m+1} = \sigma(w^-_{0,m}, w^+_{0,m}).$$

Finally, at $x = -A$ and $x = A$, we impose the periodic boundary conditions

$$w^-(-A,t_m) = w^+(A,t_m) \tag{4.8}$$

for convenience. Later on, we choose Ω^-, Ω^+ such that $\Omega^- \cup \Omega^+$ is "sufficiently separated" from the boundary of the computational domain (i.e. the characteristic curves leave $\Omega^- \cup \Omega^+$) and so (4.8) has no influence on the verified solution.

4.1.2 Construction of the smooth approximate solution ω

To construct ω^- on $(-A, 0) \times (0, T)$ (ω^+ on $(0, A) \times (0, T)$ respectively), we first use spline interpolation in x-direction at fixed time t_m. Recall the following theorem on cubic spline interpolation ([20]):

Theorem 4.1. *Let* $z_0 = a < z_1 < \ldots < z_p = b$ *be a partition of the interval* $[a, b]$ *and* $Y = (y_i)_{i=0,\ldots,p}$ *be real numbers. There exists a unique cubic spline* S_Y*, i.e. a function in* $C^2[a, b]$ *which is a cubic polynomial on each* $[z_i, z_{i+1}]$ *such that*

$$S_Y(z_i) = y_i \quad (i = 0, \ldots, p)$$

and such that

$$S_Y''(z_0) = S_Y''(z_p) = 0.$$

In fact, we *define* the approximate solution ω^- as follows: for each t_m, let $\omega^-(x, t_m)$ be the spline interpolation as above corresponding to the equidistant partition $\{x_n\}$ of the interval $[-A, qh]$ and interpolating the values $(w^-_{n,m})_{n \le q}$.

For $t \in [t_m, t_{m+1}]$ we now *define*, by linear interpolation,

$$\omega^-(x, t) := \frac{t_{m+1} - t}{k}\omega^-(x, t_m) + \frac{t - t_m}{k}\omega^-(x, t_{m+1}).$$

ω^- thus satisfies the regularity requirements (4.3). ω^+ is constructed in the same way, by spline interpolation on the interval $[-qh, A]$. On each interval $[x_n, x_{n+1}]$ the coefficients of the spline $\omega^\pm(\cdot, t_m)$ can be calculated using well-known formulas (see [20], 2.4.2). To take rounding errors into account, we implemented a interpolation routine with interval output.

To define μ, the approximate solution for the shock velocity, we put

$$\mu(t) := \chi_m \quad (t \in [t_m, t_{m+1}]). \tag{4.9}$$

Evidently, this yields a μ in $BV(0, T)$.

4.1.3 Evaluation of the defects

To evaluate the defects $\|d^\pm\|_\nu$, $\|d^\pm_x\|_\nu$ (see (1.13), (1.14)) we have to estimate the suprema of $|d^\pm|$ and $|d^\pm_x|$ on each rectangle $B_{n,m} := [x_n, x_{n+1}] \times [t_m, t_{m+1}]$. This is easily possible if a is e.g. a polynomial. Since in our numerical examples we work with Burgers' Equation, we will describe the evaluation of the defects only in that case and thus put $a(u) = u$.

On the aforementioned rectangles, we have $d^\pm = \omega^\pm_t + (\omega^\pm - \chi(t))\omega^\pm_x$, which is a polynomial of at most fifth order in x and quadratic in t (with *interval* valued coefficients). Evidently, $d^\pm$ and $d^\pm_x$ can be explicitly expressed in terms of the data from the spline interpolation. Lagrange interpolation is useful in evaluating the supremum over $B_{n,m}$.

Therefore we write by third-order Langrange interpolation[1] ($\eta_\nu = x_n + \frac{\nu h}{3}, \nu = 0, \ldots, 3$) inside the interval $[x_n, x_{n+1}]$:

$$\begin{aligned} & d^\pm(x_n + y, t_m + \tau) \\ = & \sum_{\nu=0}^{3} d^\pm(\eta_\nu, t_m + \tau) l_\nu(y) + \frac{1}{4!} d^\pm_{xxxx}(\xi^\tau_x, t_m + \tau) y(y - \frac{1}{3}h)(y - \frac{2}{3}h)(y - h) \\ = & \sum_{\nu=0}^{3} d^\pm(\eta_\nu, t_m + \tau) l_\nu(y) + R_2 \end{aligned}$$

for $(y, \tau) \in [0, h] \times [0, k]$ and $\xi^\tau_x \in [x_m, x_{m+1}]$. Now using linear interpolation on the terms in the first sum of the last line (w.r.t the variable τ), we write

$$\begin{aligned} d^\pm(x_n + y, t_m + \tau) &= \sum_{\nu=0}^{3} \{(1 - \frac{\tau}{k}) d^\pm(\eta_\nu, t_m) + \frac{\tau}{k} d^\pm(\eta_\nu, t_{m+1})\} l_\nu(y) \\ & \quad + \frac{1}{2} \sum_{\nu=0}^{3} d^\pm_{tt}(\eta_\nu, \sigma^\nu_\tau) l_\nu(y) \tau(k - \tau) + R_2 \\ &= d^\sharp(x_n + y, t_m + \tau) + R_1 + R_2, \end{aligned}$$

where $\sigma^\nu_\tau \in [t_m, t_{m+1}]$ and $d^\sharp$ denoting the first sum in the above equation. Using simple bounds for l_ν we can now estimate

$$\sup_{B_{n,m}} |d^\sharp(x, t)| \leq \frac{3}{2} \sum_{\nu=0}^{3} \max\{|d^\pm(\eta_\nu, t_m)|, |d^\pm(\eta_\nu, t_{m+1})|\}$$

which allows us to calculate an upper bound on the supremum by evaluating $d^\sharp$ at a finite number of points in $B_{n,m}$. To bound R_1 we note that $d^\pm_{tt}$ is a polynomial of order zero in t on $[t_m, t_{m+1}]$, and since $\tau(k - \tau) \leq \frac{k^2}{4}$,

$$\sup_{B_{n,m}} |R_1| \leq \frac{3k^2}{16} \sum_{\nu=0}^{3} |d_{tt}(\eta_\nu, t_m)|.$$

Now write $d^\pm(x_n+y, t_m+\tau) = \sum_{i=1}^{6} \alpha_i(\tau) y^{6-i}$, i.e. as a polynomial in $y \in [0, h]$ with coefficients which are at most second-order polynomials in τ (actually, only α_1 and α_2 are needed in the following). Some lengthy calculations and estimations show that $\sup_{B_{n,m}} |R_2| \leq \frac{h^4}{324} \sup_{B_{n,m}} |d^\pm_{xxxx}|$, where

$$\sup_{B_{n,m}} |d^\pm_{xxxx}| \quad \leq \quad 5! \max\{|\alpha_1(0)|, |\alpha_1(k)|\} h$$

[1] Recall that $l_\nu(y) = \prod_{j=0, j\neq\nu}^{3} \frac{y - \eta_j}{\eta_\nu - \eta_j}$.

$$+4!\max\{|\alpha_2(0)|,|\alpha_2(k)|\}+\frac{5!}{16}k^2h|\alpha_1''(0)|+\frac{4!}{16}k^2|\alpha_2''(0)|.$$

Thus, we have reduced the estimation of $\sup_{B_{n,m}}|d^\pm|$ to the evaluation of known polynomials at a finite number of points, which can be done by the computer in interval arithmetic (here the `polyval` routine of INTLAB is used). Entirely similar considerations serve to find bounds for $\|d_x^\pm\|_{L^\infty(\Omega_\tau)}$ and the quantities $\|\omega_x^\pm\|_{L^\infty(\Omega_\tau^\pm)}$, $\|\omega_{xx}^\pm\|_{L^\infty(\Omega_\tau^\pm)}$.

To calculate an upper bound for $\|d\|_\nu$, we note that, by (4.9) and (1.7),

$$\begin{aligned} d(t) &= \mu(t)-\sigma(\omega^-(0,t),\omega^+(0,t)) \\ &= \chi_m-\sigma(\omega^-(0,t),\omega^+(0,t)) \\ &= \chi_m-\frac{1}{2}(\omega^-(0,t)+\omega^+(0,t)), \quad t\in[t_m,t_{m+1}] \end{aligned}$$

where $\frac{1}{2}(\omega^-(0,t)+\omega^+(0,t))$ is linear in t on $[t_m,t_{m+1}]$. Hence

$$\sup_{t\in[t_m,t_{m+1}]}|d(t)|\le\max\{|\chi_m-\frac{1}{2}(\omega^-(0,t_m)+\omega^+(0,t_m))|,|\chi_m-\frac{1}{2}(\omega^-(0,t_{m+1})+\omega^+(0,t_{m+1}))|\}.$$

This finally gives an upper bound on $\|d\|_\nu$.

4.1.4 Verification procedure

To accomplish the actual verification, we use the theorem below (recall $a(u)=u$).

Theorem 4.2. *Suppose two pairs of domains $\tilde\Omega^\pm,\Omega^\pm$ are given such that $\Omega^\pm\subset\tilde\Omega^\pm$ and such that*

$$\begin{aligned} \Omega^- &= \{(x,t):\gamma_-(t)<x<0,t\in(0,T)\} \\ \Omega^+ &= \{(x,t):0<x<\gamma_+(t),t\in(0,T)\}. \end{aligned}$$

with Lipschitz functions γ_-,γ_+. Suppose moreover that ω^-,ω^+ are given as functions on the larger domains $\tilde\Omega^\pm$. Define the quantities $\tilde C_1,\tilde C_2^\pm,\tilde C_3^\pm$ (i.e. in the same way as in chapter 3, equations (3.10), (3.12), (3.13)*, but now referring to the domains $\tilde\Omega^\pm$) by*

$$\begin{aligned} \tilde C_1 &:= \exp\left(\frac{1}{2}\sum_\pm\|\omega_x^\pm\|_{L^\infty(\tilde\Omega^\pm)}T\right) \\ \tilde C_2^\pm &:= \tilde C_1\int_0^T\|\omega_x^\pm\|_{L^\infty(\tilde\Omega_\tau^\pm)}d\tau \\ \tilde C_3^\pm &:= \int_0^T\|\omega_{xx}^\pm\|_{L^\infty(\tilde\Omega_\tau^\pm)}d\tau \end{aligned}$$

and let $\delta, \delta^{\pm}, \eta^{\pm}$ *(defect bounds) be nonnegative numbers such that*

$$\|d^{\pm}\|_{\nu,\tilde{\Omega}^{\pm}} \leq \delta^{\pm}, \ \ \|d_x^{\pm}\|_{\nu,\tilde{\Omega}^{\pm}} \leq \eta^{\pm}, \ \ \|d\|_{\nu} \leq \delta$$

(the norms $\|\cdot\|_{\nu,\tilde{\Omega}^{\pm}}$ *are defined analogously to* $\|\cdot\|_{\nu}$ *by replacing* $\Omega^{\pm}$ *by* $\tilde{\Omega}^{\pm}$*). Moreover let* $\alpha^{\pm}, \beta^{\pm}, \gamma$ *be numbers such that*

$$\left.\begin{array}{rcl} \tilde{C}_1\left[\delta + \frac{1}{2}\sum_{\pm}\delta^{\pm}\right] & \leq & \gamma \\ \tilde{C}_2^{\pm}\left[\delta + \frac{1}{2}\sum_{\pm}\delta^{\pm}\right] + \delta^{\pm} & \leq & \alpha^{\pm} \\ \tilde{C}_3^{\pm}(\tilde{C}_1 + \tilde{C}_2^{\pm})\left[\delta + \frac{1}{2}\sum_{\pm}\delta^{\pm}\right] + \eta^{\pm} + \tilde{C}_3^{\pm}\delta^{\pm} & \leq & \beta^{\pm}. \end{array}\right\} \tag{4.10}$$

Assume furthermore that the following inequalities with the numbers $\alpha^{\pm}, \gamma$ *from above are satisfied:*

$$\left.\begin{array}{lcl} \omega^-(0,t) - \mu(t) - e^{\nu T}(\alpha^- + \gamma) & \geq & c > 0 \\ \omega^+(0,t) - \mu(t) + e^{\nu T}(\alpha^+ + \gamma) & \leq & -c < 0 \\ \omega^-(\gamma_-(t),t) - \mu(t) + (\alpha^- + \gamma)e^{\nu T} - \gamma_-'(t) & \leq & -c < 0 \\ \omega^+(\gamma_+(t),t) - \mu(t) - (\alpha^+ + \gamma)e^{\nu T} - \gamma_+'(t) & \geq & c > 0 \end{array}\right\} \tag{4.11}$$

for some $c > 0$*, and suppose finally that*

$$s^{\pm} := 2\|\omega_x^{\pm}\|_{L^{\infty}(\tilde{\Omega}^{\pm})} + \beta^{\pm}e^{\nu T} \leq \nu. \tag{4.12}$$

holds. Then there exists a solution (w^-, w^+, χ) *of* (1.8)*, such that*

$$\|\omega^{\pm} - w^{\pm}\|_{\nu} \leq \alpha^{\pm}, \|D_x(\omega^{\pm} - w^{\pm})\|_{\nu} \leq \beta^{\pm}, \|\chi - \mu\|_{\nu} \leq \gamma \tag{4.13}$$

(the norms are taken on $\Omega^{\pm}$*) holds, which in addition satisfies the Lax admissibility condition* (1.9)*.*

Proof. We claim that inequalities (4.11), imply that D defined by (3.11) is contained in $Z_{\theta,\kappa}$ (see (2.21)) with $\kappa := e^{\nu T}\max\{(\alpha^- + \beta^-), (\alpha^+ + \beta^+)\}$.

It is clear that $\|v^{\pm}\|_{V^1(\Omega^{\pm})} \leq \kappa$ if $(v^-, v^+, \xi) \in D$, since $\|v^{\pm}\|_{V^1(\Omega^{\pm})} \leq e^{\nu T}(\alpha^{\pm} + \beta^{\pm})$. Consider first Ω^-, whose left boundary is described by the graph of a Lipschitz function $\gamma_- : [0,T] \to \mathbb{R}$. We want to show that for $(v^-, v^+, \xi) \in D$, $(v^-, \xi) \in Z_{\theta,\kappa}^-$ (see (2.20)). Thus it suffices to check that

$$(\omega^- + S_\theta^- v^- - \mu - \xi, 1) \cdot n \geq c$$

for some $c > 0$ on the two parts $\Gamma_1^- := \{(0,t) \in t \in (0,T)\}$ and $\Gamma_2^- := \{(\gamma_-(t), t) : t \in (0,T)\}$ of the boundary of Ω^- (where n denotes the outward-pointing unit-normal given by (A.4)) for *all* $(v^-, v^+, \xi) \in D$. Since on Γ_1^- we have $n = (1, 0)$ and

$$\begin{aligned} \omega^-(0,t) + (S_\theta v^-)(0,t) - \mu(t) - \xi(t) &\geq \omega^-(0,t) + \inf_{(x,t)\in\Omega^-} v^-(x,t) - \mu(t) - \xi(t) \\ &\geq \omega^-(0,t) - \mu(t) - e^{\nu t}(\alpha^- + \gamma) \\ &\geq c > 0 \end{aligned}$$

by (4.11). Here we have used that, by (3.11),

$$|v^-(x,t)| \leq e^{\nu T}\alpha^{\pm}, |\xi(t)| \leq e^{\nu T}\gamma \quad (t \in (0,T)).$$

Now consider Γ_2^-. The argument for Γ_2^- is similar, taking into account that

$$n(x,t) = \frac{1}{\sqrt{1 + |\gamma_-'(t^+)|^2}}(-1, -\gamma_-'(t^+))$$

on Γ_2^-.

Now note that the constants $C_1, C_2^{\pm}, C_3^{\pm}$ from theorem 3.2 are dominated by the constants $\tilde{C}_1, \tilde{C}_2^{\pm}, \tilde{C}_3^{\pm}$ and also the defects $\|d^{\pm}\|_\nu$, $\|d_x^{\pm}\|_\nu$, $\|d\|_\nu$ are not greater than $\delta^{\pm}, \eta^{\pm}$ and δ. So, in view of the conditions in the above theorem, theorem 3.2 implies $\mathbf{\Phi}_\theta(D) \subset D$. Hence we may apply our main theorem 2.16 to get the conclusion. □

Remark 4.3. *The first two lines of conditions* (4.11) *are reminiscent of the* Lax shock *admissibility conditions. This means that we require the numerical approximate solution* $(\omega^-, \omega^+, \mu)$ *to satisfy a somewhat stronger form of the Lax shock condition, in order that the linearized hyperbolic problems (see chapter 1) are sufficiently well-behaved (see appendix A). This is reasonable, since we do not expect to be able to verify (and, from a physical viewpoint, are not interested in) inadmissible shocks.*

In practice, we proceed as follows: Choose a number $\nu > 2\|\omega_x^{\pm}\|_{L^\infty(\tilde{\Omega}^{\pm})}$ and evaluate all the defects $\|d^{\pm}\|_{\nu,\tilde{\Omega}^{\pm}}$, $\|d^{\pm}\|_{\nu,\tilde{\Omega}^{\pm}}$ on rectangular domains $\tilde{\Omega}^- = (-B, 0) \times (0,T)$, $\tilde{\Omega}^+ = (0, B) \times (0,T)$; here these domains are chosen smaller than the computational domain ($B \leq A$). Also we calculate the quantities $\tilde{C}_1, \tilde{C}_2^{\pm}, \tilde{C}_3^{\pm}$. We then define $\alpha^{\pm}, \beta^{\pm}, \gamma$ by the left-hand sides of the corresponding inequalities (4.10). $\alpha^{\pm}, \beta^{\pm}, \gamma$ are small if the defects are sufficiently small. Then we check the crucial condition

$$s^{\pm} := 2\|\omega_x^{\pm}\|_{L^\infty(\tilde{\Omega}^{\pm})} + \beta^{\pm} e^{\nu T} \leq \nu. \tag{4.14}$$

To verify the outward-pointing conditions (4.11) at $x = 0$, we verify that some $c > 0$ exists with

$$\min\{\omega^-(0, t_m), \omega^-(0, t_{m+1})\} - \mu(t_m) - e^{\nu T}(\alpha^- + \gamma) \geq c \tag{4.15}$$

for all time steps t_m under consideration. This gives the first inequality in (4.11), the second is treated analogously. We expect that this condition should be rather easy to fulfill, since otherwise the numerical approximate solution would not qualify as an admissible shock (see the remark 4.3).

Finally, we determine domains $\Omega^\pm$ such that the remaining conditions of (4.11) referring to γ_-, γ_+ are satisfied. Consider e.g. the third line of (4.11), referring to the left boundary of Ω^-. The chance of satisfying this condition is good if $\gamma_-'(t)$ is sufficiently positive. For simplicity, we only consider left boundaries of the form

$$\gamma_-(t) = \frac{h}{k}t + mh$$

with some $m \in \mathbb{N}$. Thus Ω^- has in general a trapezoidal shape (see e.g. figure 4.13 which shows the Ω^- and Ω^+ used in numerical example 3 below) and the boundary goes through the grid points. At each rectangle $B_{n,m} = [x_n, x_{n+1}] \times [t_m, t_{m+1}]$ the boundary passes through, we verify the existence of some c such that

$$\sup_{(x,t)\in B_{n,m}} \omega^-(x, t) - \mu_m + e^{\nu T}(\alpha^- + \gamma) - \frac{h}{k} \leq -c < 0 \tag{4.16}$$

The supremum can be easily evaluated using the `polyval` routine of INTLAB (for interval evaluation of polynomials). (4.16) ensures the third condition in (4.11); the fourth is obtained analogously.

4.2 Numerical examples

The computations were done in MATLAB, using the INTLAB interval-arithmetic package developed by S. Rump (see [19]). We used a workstation with Intel Core 2 Quad Q9659 processor (3 GHz).

In the numerical examples below, we choose initial data consisting of spline functions. This is done for simplicity[2], since the numerical approximate solutions are constructed by spline interpolation in x-direction and thus will satisfy the spline initial conditions exactly.

[2]More general initial conditions can be treated by altering the initial condition in the hyperbolic problems (2.22). The homogeneous initial condition $v^\pm(\cdot, 0) = 0$ has to be replaced by the inhomogeneous one

$$v^\pm(\cdot, 0) = w_0^\pm(\cdot) - \omega^\pm(\cdot, 0),$$

Example 1. In the first numerical example we choose as initial data the spline interpolation of $(A = 5)$

$$w(x,0) := \begin{cases} 0.04\,(x+5)^2 & : x \in (-5,0) \\ -0.1x + 0.5 & : x \in (0,5) \end{cases}, \tag{4.17}$$

i.e. we choose some $h > 0$ and then take as initial data the spline interpolation of $w|_{[-5,qh]}$ and $w|_{[-qh,5]}$ with respect to the equidistant partition $\{x_n\}$ of the intervals $[-A, qh]$ and $[-qh, A]$ (see the definition (4.5), (4.6) of the grids). The initial data for the finite difference scheme are chosen to be

$$\begin{aligned} \hat{w}_n^- &:= 0.04\,(x_n+5)^2, \text{ for } x_n = nh, \quad -A \le x_n, \ n \le q \\ \hat{w}_n^+ &:= -0.1x_n + 0.5, \text{ for } x_n = nh, \ x_n \le A, \quad -q \le n. \end{aligned}$$

The initial conditions for the other numerical examples are constructed in the same way.

We also choose $T = 0.5, \tilde{\Omega}^- = (-2,0) \times (0,0.5), \tilde{\Omega}^+ = (0,2) \times (0,0.5)$. The table below gives the results of calculations made with different grid sizes ($N = \frac{1}{h}$); the column "successful" indicates whether the condition (4.14) was satisfied and thus whether the verification was successful or not. Figures 4.6 and 4.7 show plots of the approximate solution at different times. Figure 4.8 shows the domains $\Omega^\pm$ on which the existence of a solution was actually verified, i.e. the domains for which we successfully checked the third and fourth of the conditions in (4.11).

Example 2. For the second example, we choose $T = 0.7$ and as initial data the piecewise spline interpolation (analogously to Example 1) of

$$w(x,0) := \begin{cases} 0 & : \ x \in (-5,-4) \\ 0.15(x+4)^2 & : \ x \in (-4,0) \\ 0.9\exp(1/2.2)\exp\left(-\frac{1}{\sqrt{2.2^2+0.01}-\sqrt{(x-2.3)^2+0.01}}\right) & : \ x \in (0.1, 4.5) \\ 0 & : \ \text{else} \end{cases} \tag{4.18}$$

With $N = 1000, \nu = 3$ we were not able to enclose the solution. The reason was that β^+ was still too large. Using $N = 2000$, the defects were sufficiently reduced and the enclosure was successful (see also figures 4.9, 4.10, 4.8 for $\Omega^\pm$ in the case of successful enclosure).

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where $w_0^\pm$ is the *given* (non-spline) initial condition for (1.8). The "small" quantities $\|w_0^\pm(\cdot) - \omega^\pm(\cdot,0)\|_{L^\infty(\Omega^\pm)}, \|D_x(w_0^\pm(\cdot) - \omega^\pm(\cdot,0))\|_{L^\infty(\Omega^\pm)}$ then also enter the a-priori-estimates (3.5), (3.6) and thus also the enclosure inequalities (4.10).

N	ν	δ^-	η^-	δ^+	η^+	δ
500	2	0.00077644	0.093041	0.0041946	0.023859	0.00056253
	α^-	α^+	β^-	β^+	γ	s^-
	0.0013906	0.0043627	0.09321	0.02386	0.0034442	1.0454
	s^+	successful				
	0.26548	yes				
N	ν	δ^-	η^-	δ^+	η^+	δ
1000	2	0.00027194	0.048025	0.002125	0.012173	0.00028267
	α^-	α^+	β^-	β^+	γ	s^-
	0.00057374	0.0022097	0.048105	0.012174	0.0016778	0.9283
	s^+	successful				
	0.239	yes				

Figure 4.2: Defects and enclosure parameters $\alpha^{\pm}, \beta^{\pm}, \gamma$ for Example 1.

N	ν	δ^-	η^-	δ^+	η^+	δ
1000	3	0.0036968	0.0049065	0.0051336	0.060435	0.0027418
	α^-	α^+	β^-	β^+	γ	s^-
	0.017867	0.015917	0.010285	0.10893	0.019041	2.6036
	s^+	successful.				
	3.6895	no				
N	ν	δ^-	η^-	δ^+	η^+	δ
2000	3	0.0018464	0.0024458	0.0025521	0.029909	0.0013777
	α^-	α^+	β^-	β^+	γ	s^-
	0.0089491	0.0079586	0.0051384	0.054171	0.0095392	2.5025
	s^+	successful				
	2.6128	yes				

Figure 4.3: Defects and enclosure parameters $\alpha^{\pm}, \beta^{\pm}, \gamma$ for Example 2.

Example 3. We choose $[-A, A] := [-3, 3]$ as computational domain and

$$w(x,0) := \begin{cases} 1.5 + 1.7896x + 1.2753x^2 - 0.1623x^3 - 0.1480x^4 & : \ x < 0 \\ -0.3(x-3) & : \ x > 0 \end{cases}. \tag{4.19}$$

This example presents a difficulty for the verification procedure here. As seen from the plots (figures 4.11, 4.12), a discontinuity forms in the smooth left-hand part of the solution, causing $\|\omega_x^-\|_{L^\infty(\Omega^-)}$ to become large. First we take $T = 0.25$ and were able to verify a computed solution, as seen from the table below:

N	ν	δ^-	η^-	δ^+	η^+	δ
1500	10	0.00065959	0.14656	0.002062	0.024415	0.00056341
	α^-	α^+	β^-	β^+	γ	s^-
	0.0016015	0.0022539	0.15004	0.024419	0.0024979	5.3537
	s^+	successful				
	0.92975	yes				

Figure 4.4: Defects and enclosure parameters $\alpha^\pm, \beta^\pm, \gamma$ for Example 3, $T = 0.25$.

Figure 4.13 shows the domains $\Omega^\pm$ on which the existence of the solution was actually verified.

We tried to increase T and took, say, $T = 0.45, \nu = 9$. Here the verification procedure was not successful (s^- was a little bit too large). The reason is of course the large value of $\|\omega_x^-\|_{\Omega_\tau}$ for τ close to 0.45.

N	ν	δ^-	η^-	δ^+	η^+	δ
3000	9	0.00035836	0.080298	0.001243	0.012524	0.00028345
	α^-	α^+	β^-	β^+	γ	s^-
	0.0018504	0.0015269	0.087547	0.012536	0.0019675	9.5893
	s^+	sucessful				
	1.4006	no				

Figure 4.5: Defects and enclosure parameters $\alpha^\pm, \beta^\pm, \gamma$ for Example 3, $T = 0.45$.

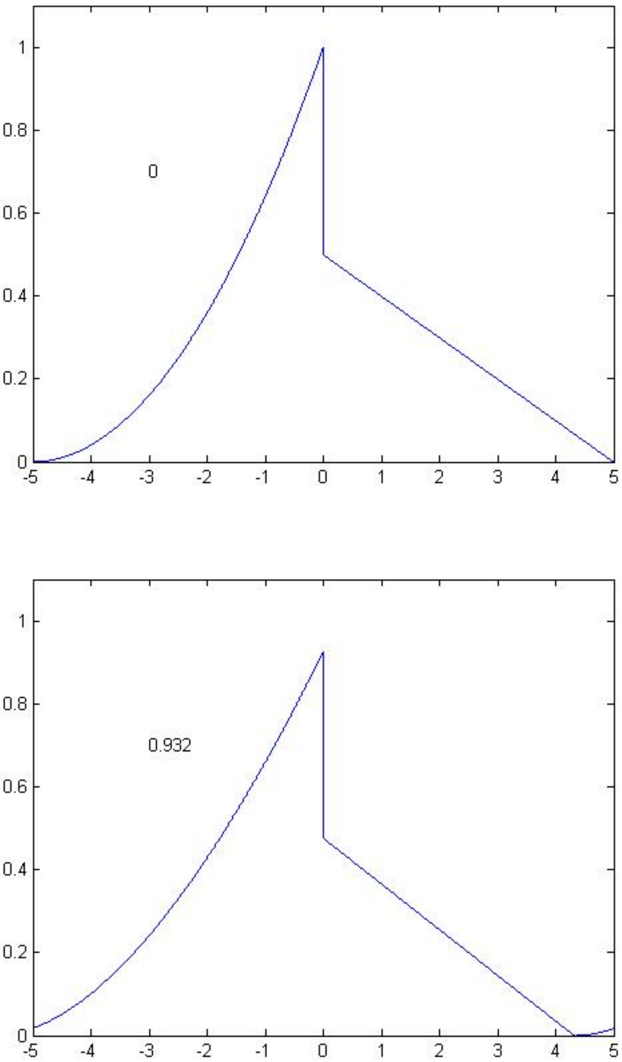


Figure 4.6: Example 1, $t = 0$ and $t = 0.932$

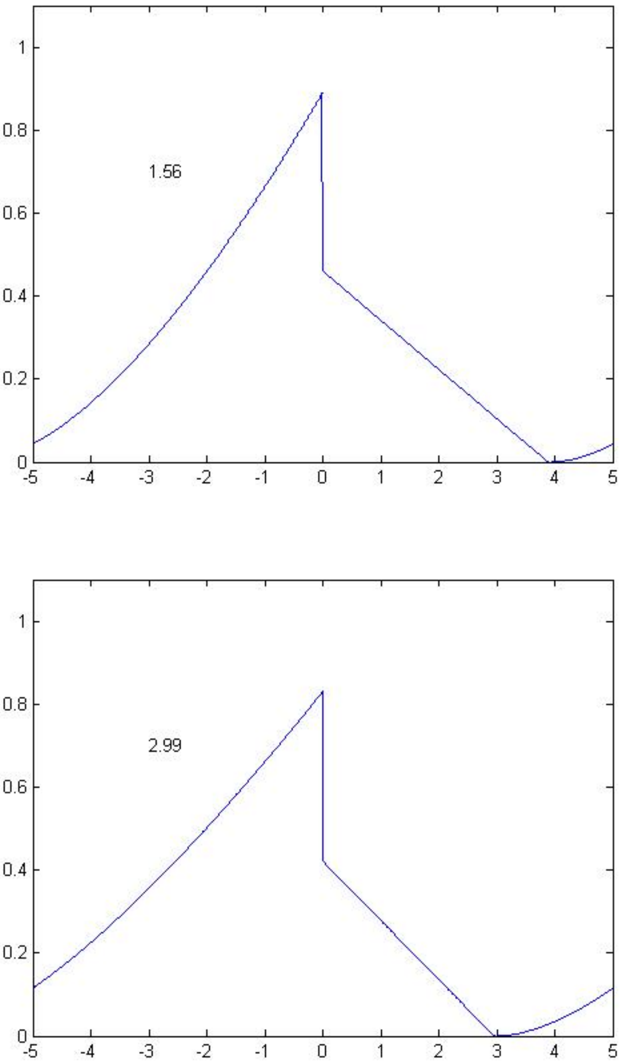


Figure 4.7: Example 1, $t = 1.56$ and $t = 2.99$

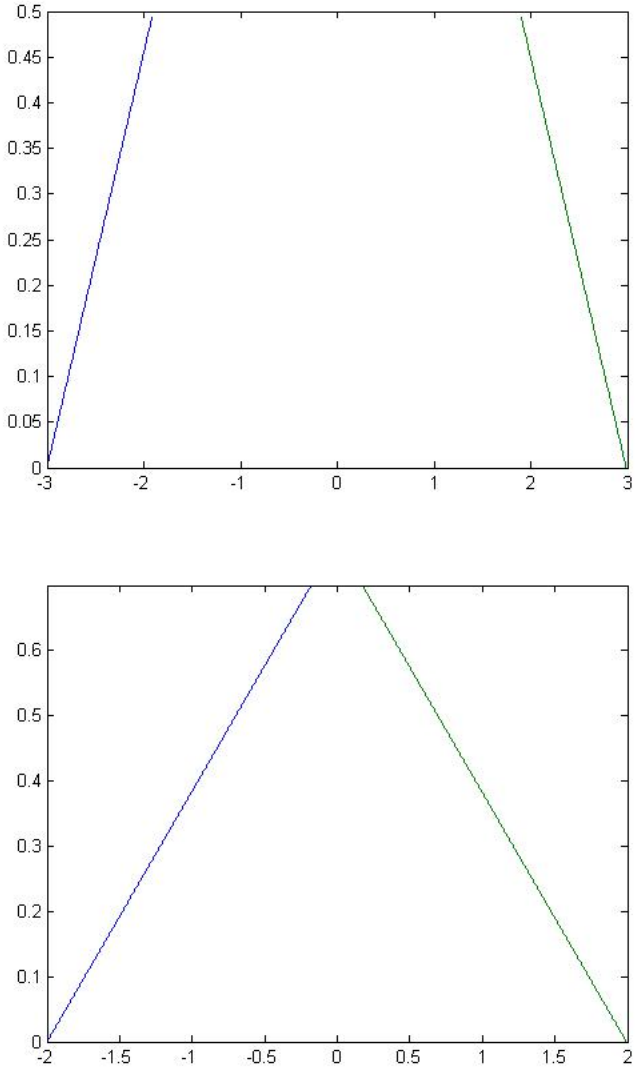
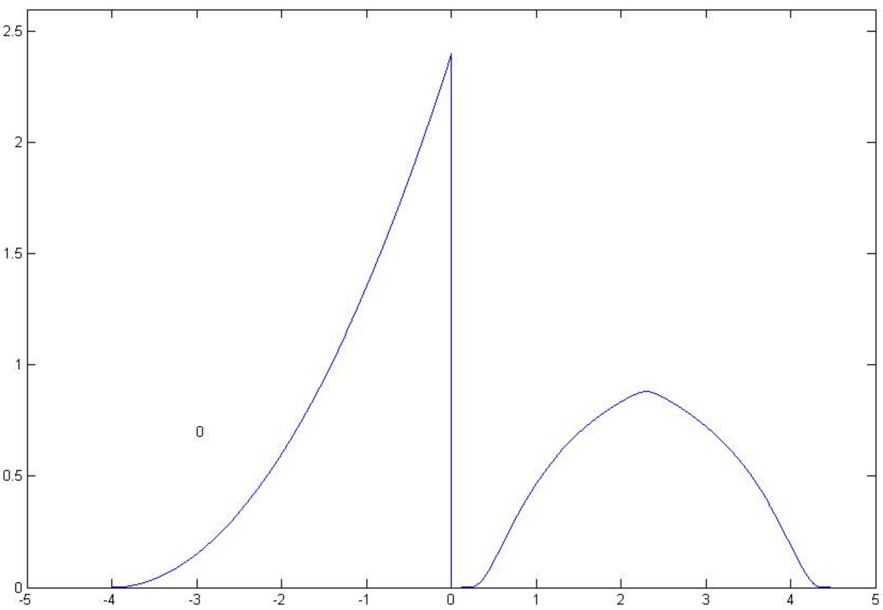


Figure 4.8: The domains $\Omega^{\pm}$ for examples 1 (upper figure) and 2 (lower figure)



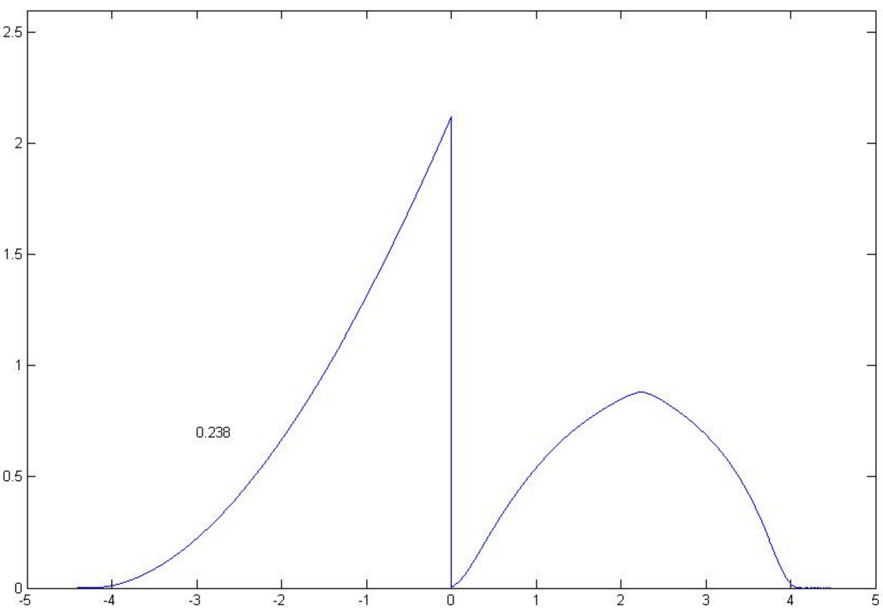


Figure 4.9: Example 2, $t = 0$ and $t = 0.238$

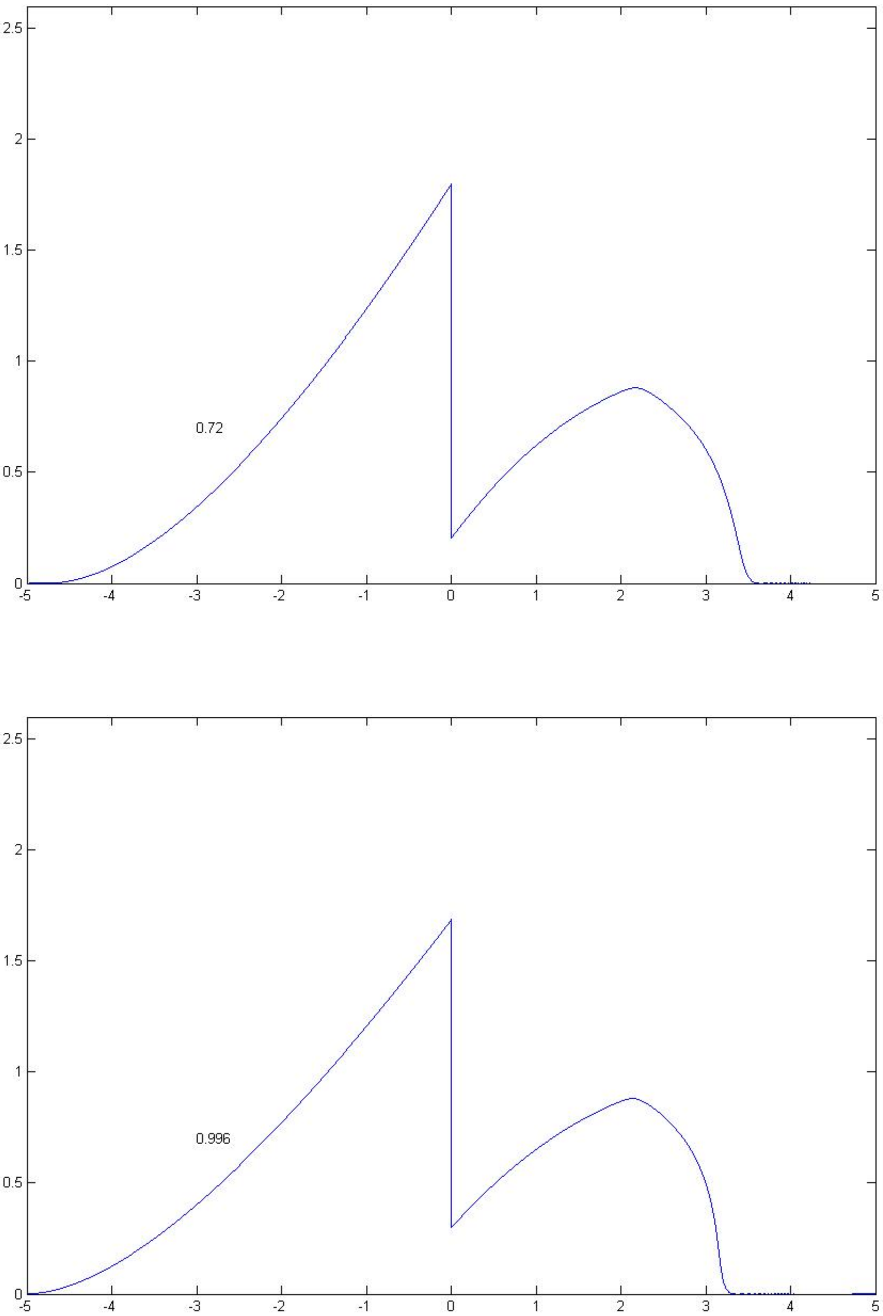


Figure 4.10: Example 2, $t = 0.72$ and $t = 0.996$

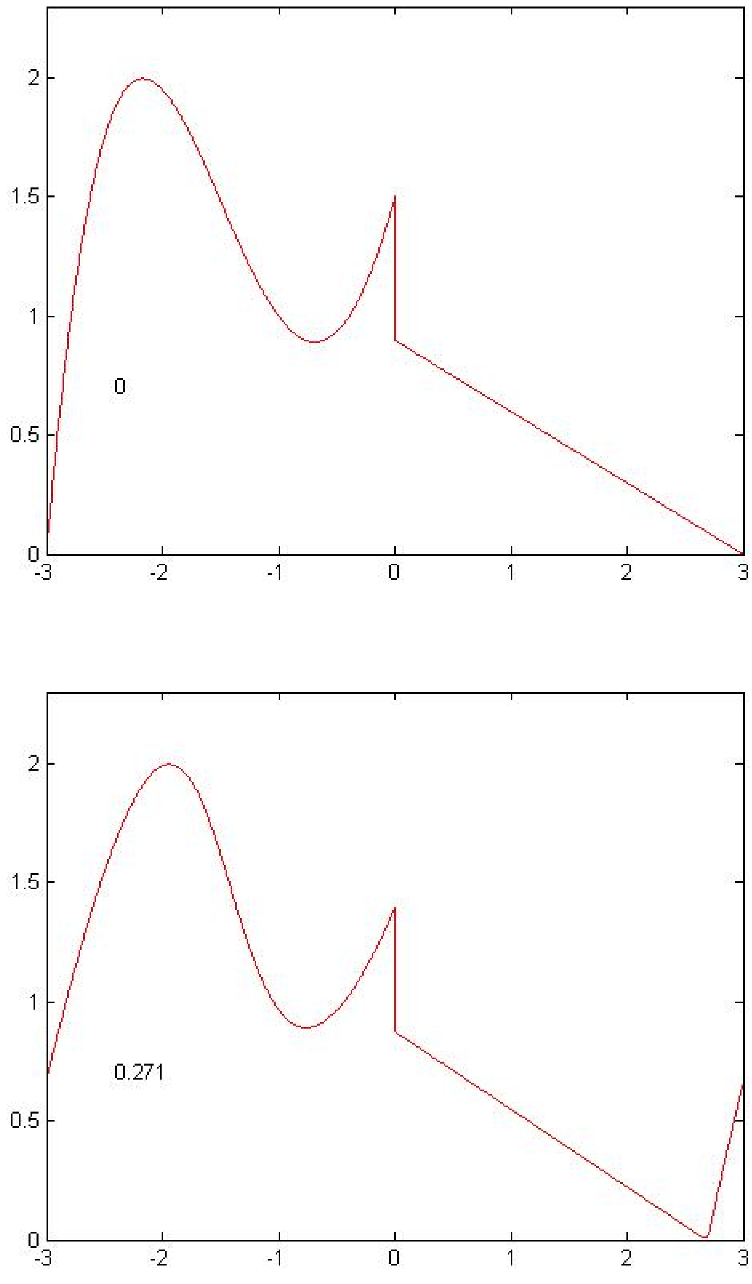


Figure 4.11: Example 3, $t = 0$ and $t = 0.271$

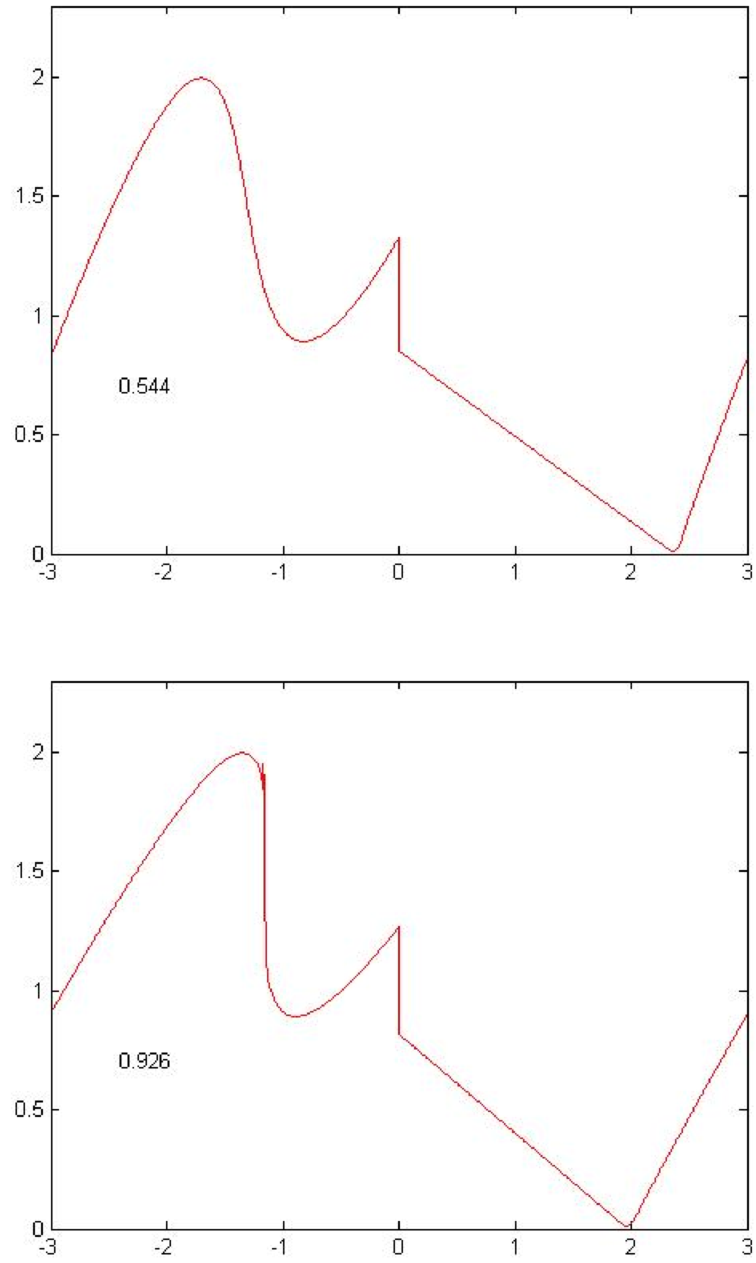
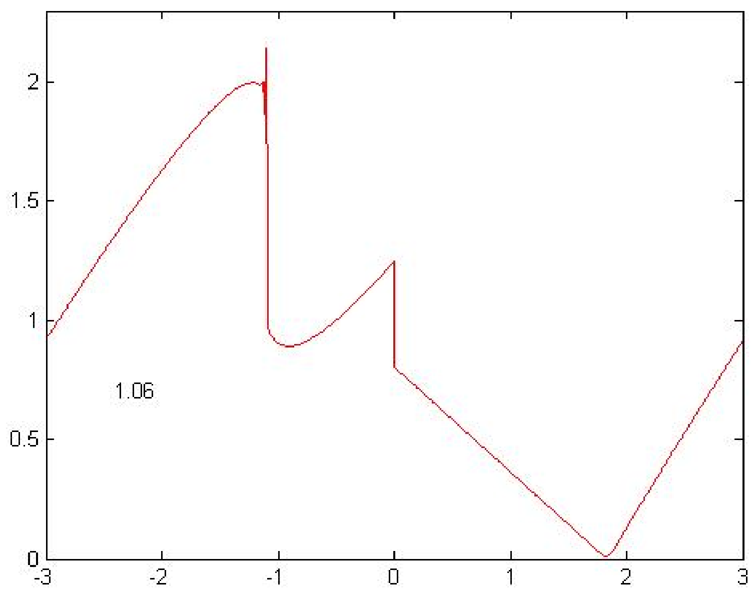


Figure 4.12: Example 3, $t = 0.544$ and $t = 0.926$



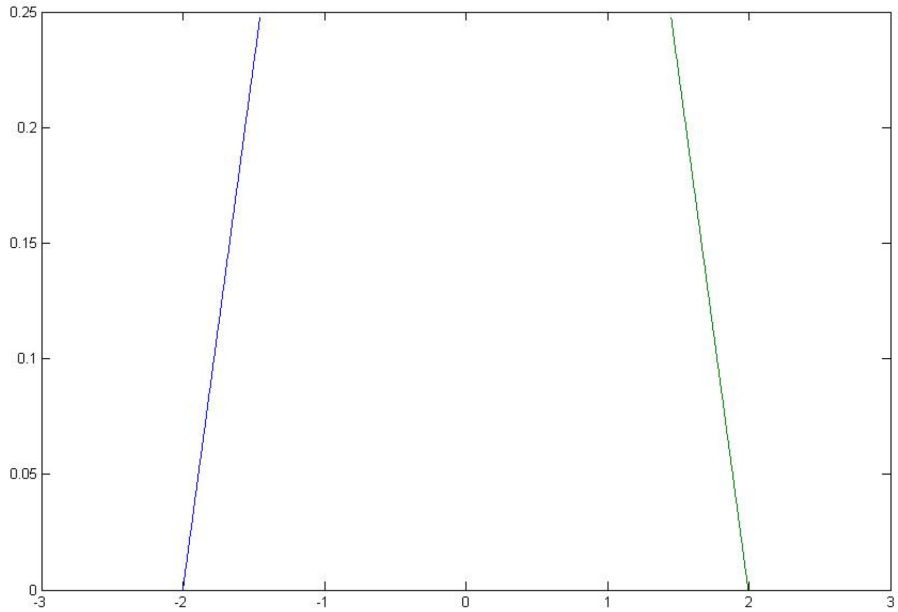


Figure 4.13: Example 3: approximate solution at $t = 1.06$ and the domains $\Omega^{\pm}$ on which the existence of the solution was verified in case of $T = 0.25, \nu = 10$.

Chapter 5

Systems of Conservation Laws

In this chapter, we make some tentative steps to extend the foregoing theory to systems of conservation laws. There remain however still a lot of technicalities to be worked out in detail.

5.1 Preliminaries

Consider single-shock solutions of the following N by N system of conservation laws in one space dimension:

$$u_t + F(u)_x = 0 \quad \text{on } \mathbb{R} \times (0,T) \tag{5.1}$$

Here, $F : \Sigma \subset \mathbb{R}^N \to \mathbb{R}^N$ is the given nonlinearity defined on an open subset Σ of $\mathbb{R}^N$ (e.g. the "state space" of the physical system which is modeled by (5.1)) and $u : \mathbb{R} \times (0,T) \to \Sigma$ is the unknown. Writing $A(u) = DF(u) \in \mathbb{R}^{N,N}$ (DF denoting the Jacobian matrix of F) considerations entirely analogous to those of chapter 1 lead to (compare (1.8)):

$$\left\{ \begin{array}{ccl} w_t^{\pm} + (A(w^{\pm}) - \chi I) w_x^{\pm} = 0 & : & \text{on } \Omega^{\pm} \\ \sigma(w^-(0,t), w^+(0,t), \chi(t)) = 0 & : & \text{on } (0,T) \\ w^-(\cdot,0) = w_0^-(\cdot), w^+(\cdot,0) = w_0^+(\cdot) & & \end{array} \right. \tag{5.2}$$

where I is the $N \times N$ identity matrix and

$$\sigma(w^-, w^+, \chi) := F(w^+) - F(w^-) - \chi(w^+ - w^-) \in \mathbb{R}^N \tag{5.3}$$

(i.e. $\sigma(w^-, w^+, \chi) = 0$ is just the Rankine-Hugionot condition). We will always assume that *strict hyperbolicity* holds:

Definition 5.1. If for all $u \in \Sigma$, $A(u)$ has N distinct real eigenvalues

$$\lambda_1(u) < \lambda_2(u) < \ldots < \lambda_N(u),$$

then the equation (5.1) is called strictly hyperbolic.

The Rankine-Hugionot condition is necessary for a given discontinuity to be a shock. We will need however to restrict further the classes of shocks considered.

Definition 5.2. Let (w^+, w^-, χ) be a solution of (5.2). If

$$\lambda_1(w^+(0,t)) < \lambda_2(w^+(0,t)) < \ldots < \lambda_j(w^+(0,t)) < \chi(t) < \lambda_j(w^-(0,t)) \tag{5.4}$$

and

$$\lambda_{j-1}(w^-(0,t)) < \chi(t) \quad (\text{if } j > 1) \tag{5.5}$$

hold on $(0,T)$ for some $j \in \{1,\ldots,N\}$, then the shock considered is called an admissible j-shock (if $j = 1$ then (5.5) is omitted).

Our definition of j-shock is stronger than the definition commonly given in the literature, which consists only of the requirement

$$\lambda_j(w^+(0,t)) < \chi(t) < \lambda_j(w^-(0,t))$$

(see [6], [5]). The condition informally means that on the right, the characteristics corresponding to $\lambda_1, \lambda_2, \ldots, \lambda_j$ enter and those corresponding to $\lambda_{j+1}, \ldots, \lambda_N$ leave the shock, whereas on the left, $\lambda_j, \ldots, \lambda_N$ enter and $\lambda_1, \ldots, \lambda_{j-1}$ leave. We will see in a moment that (5.4) is crucial to ensure the well-posedness of the following linear system, arising from linearization of (5.2) (compare (1.12)):

$$\begin{cases} v_t^\pm + (A(\omega^\pm + v^\pm) - \mu I - \xi I)v_x^\pm + A(\omega^\pm)_x v^\pm = -d^\pm - (R^\pm[v^\pm] - \xi)\omega_x^\pm & : \text{ on } \Omega^\pm \\ (\omega^+ - \omega^-)\xi + \frac{\partial\sigma}{\partial w^+}v^+ + \frac{\partial\sigma}{\partial w^-}v^- = -d + R[v^-, v^+] & : \text{ on } (0,T) \\ v^-(\cdot, 0) = 0, v^+(\cdot, 0) = 0 \end{cases} \tag{5.6}$$

5.2 Linearization

Recall that $(\omega^-, \omega^+, \mu)$ is an approximate solution of (5.4); assume furthermore (5.4) holds for $(\omega^-, \omega^+, \mu)$, i.e. $(\omega^-, \omega^+, \mu)$ is an *approximate* j-shock. To prepare for a suitable fixed-point formulation, we inquire whether the linear system arising from (5.6) (analogous to the considerations in chapter 2, section 2.1.1) is well-posed. Thus consider, say, the problem for v^-:

$$v_t^- + (A(\omega^- + u^-) - \mu I - \xi I)v_x^- + A(\omega^-)_x v^- = r \tag{5.7}$$

where u^-, r, ξ are given and u^-, ξ are "small". From the theory of linear hyperbolic systems, we know that in addition to the initial condition $v^-(0,\cdot) = 0$, q additional boundary conditions have to be supplied, where q is the number of characteristics leaving the boundary at $x = 0$. Disregarding the "small" perturbations u and ξ for the moment, q is given by the number of negative eigenvalues of

$$A(\omega^-(0,t)) - \mu I \quad (t \in (0,T)).$$

Linear hyperbolic systems. To make the specification of boundary conditions more precise, consider the general linear hyperbolic system

$$v_t + B(x,t)v_x + C(x,t)v = r(x,t) \tag{5.8}$$

on $(-\infty, 0) \times (0,T)$ with initial condition $v(\cdot, 0) = 0$. The matrix $B(x,t) \in \mathbb{R}^{N\times N}$ (sufficiently smooth in (x,t)) is assumed to have N real eigenvalues

$$\mu_1(x,t) < \mu_2(x,t) < \ldots < \mu_N(x,t)$$

such that $\mu_q(0,t) < 0 < \mu_{q+1}(0,t)$ on $(0,T)$. Use $\{u_i(x,t)\}_{i=1,\ldots,N}$ to denote the (x,t) dependent eigenvectors of $B(x,t)$, assumed to depend sufficiently smoothly on (x,t). Writing

$$v(x,t) = \sum_{i=1}^{N} \alpha_i(x,t)u_i(x,t)$$

and inserting this into (5.8), we obtain

$$\begin{aligned} r &= \sum_i \left\{ \frac{\partial \alpha_i}{\partial t} u_i + \alpha_i \frac{\partial u_i}{\partial t} \right\} + \sum_i B \frac{\partial \alpha_i}{\partial x} u_i + \sum_i \alpha_i B \frac{\partial u_i}{\partial x} + \sum_i \alpha_i C u_i \\ &= \sum_i \left\{ \frac{\partial \alpha_i}{\partial t} + \mu_i \frac{\partial \alpha_i}{\partial x} \right\} u_i + \tilde{C} \cdot (\alpha_1, \ldots, \alpha_N)^T \end{aligned}$$

with some new (x,t)-dependent matrix $\tilde{C}$. Evidently, since the u_i are linearly independent, this can be understood as a linear hyperbolic system for the α_i

$$\frac{\partial \alpha_i}{\partial t} + \mu_i \frac{\partial \alpha_i}{\partial x} + \hat{C}(x,t) \cdot (\alpha_1, \ldots, \alpha_N)^T = \hat{r} \quad (i = 1, \ldots, N)$$

with *uncoupled* principal part. Since $\mu_1, \ldots, \mu_q < 0, \mu_{q+1}, \ldots, \mu_N > 0$ at $x = 0$ it is now clear from the linear hyperbolic theory (see e.g. [11]) that we have to impose q conditions of the form

$$\alpha_i(0,t) = \rho_i(t) \quad (i = 1, \ldots, q) \tag{5.9}$$

to make (5.8) well-posed.

Boundary conditions for (5.6). Let domains $\Omega^\pm$ be given as in chapter 2. We now discuss the prescription of boundary values at $x = 0$, ignoring the other parts of the boundary for now. Let now $\{\theta_i(w)\}_{i=1,\dots,N}$ be eigenvectors for $A(w)$, depending sufficiently smoothly on w. For functions $v^\pm$ taking their values in $\mathbb{R}^N$ and defined on either Ω^- or Ω^+ we define $[v^\pm]_{i,u^\pm}(t)$ to be the expansion coefficients of $v(0,t)$ in the basis $\{\hat\theta^\pm_{i,u^\pm}(t)\}, \hat\theta^\pm_{i,u^\pm}(t) = \theta^\pm_i(\omega^\pm(0,t) + u^\pm(0,t))$, with $u^\pm$ given:

$$v^\pm(0,t) = \sum_{i=1}^{N} [v^\pm]_{i,u^\pm}(t)\hat\theta^\pm_{i,u^\pm}(t).$$

For sufficiently "small" $u^\pm, \xi$, we expect from the foregoing that

$$\begin{aligned} [v^-]_{i,u^-}(t) &= \rho_i(t) \quad (t \in (0,T), i = 1,\dots,j-1) \\ [v^+]_{i,u^+}(t) &= \rho_i(t) \quad (t \in (0,T), i = j+1,\dots,N) \end{aligned}$$

(with given functions $\rho_1,\dots,\rho_{j-1},\rho_{j+1},\dots,\rho_N$) is the correct boundary condition to impose on (5.6). Indeed, we may regard (5.6) as a linear system of hyperbolic equations with transport coefficient

$$B^\pm(x,t) := A(\omega^\pm(x,t) + u^\pm(x,t)) - \mu(t)I - \xi(t)I.$$

Now consider $B^-(0,t)$. On account of the definition of a j-shock and we have the following inequalities

$$\lambda_1(\omega^-(0,t)) < \lambda_2(\omega^-(0,t)) < \ldots < \lambda_{j-1}(\omega^-(0,t)) < \mu(t) < \lambda_j(\omega^-(0,t))$$

and since for sufficiently "small" perturbations u, ξ, the (ordered) eigenvalues $\zeta_j(t)$ of $B^-(0,t)$ should be approximately be given by $\lambda_j(\omega^-(0,t)) - \mu(t)$, we can expect to have

$$\zeta_1(t) < \zeta_2(t) < \zeta_{j-1}(t) < 0 < \zeta_j(t)$$

so that the first $j-1$ characteristics are leaving the boundary. Thus we define the operator $\mathbf{T}^-$ as the solution operator to

$$\begin{cases} v^-_t + (A(\omega^- + u^-) - \mu I - \xi I)v^-_x + A(\omega^-)_x v^- = r \;\; : \;\; \text{on } \Omega^\pm \\ v^-(\cdot,0) = 0 \\ (v^-)_i(t) = \rho_i(t) \quad (i = 1,\dots,j-1) \end{cases} \tag{5.10}$$

with given $u, r, \xi, \rho_i (i = 1,\dots,j)$. We write $\mathbf{T}^-[u,\xi,\rho,r]$ and define $\mathbf{T}^+[u,\xi,\rho,r]$ analogously (here $N-j$ functions $\rho_{j+1},\dots,\rho_N$ are given). ρ stands for the collection $(\rho_1,\dots,\rho_{j-1},\rho_{j+1},\rho_N)$ of $L^\infty(0,T)$-functions.

5.3 Fixed-point formulation

Now we formally define a suitable fixed-point operator for (5.6) in analogy to the developments in the previous chapters, without making any effort to introduce smoothing operators at the right places in order to obtain compactness properties.

The operator K. First consider the analogue of the operator **K** (see (2.10)) from chapter 2; as just explained, the hyperbolic equations for $v^\pm$ in (5.6) have to be supplied with boundary conditions given by functions $\rho_p \in L^\infty(0,T)$; motivated by this, we consider (compare (1.12))

$$\begin{cases} v_t^\pm + (A(\omega^\pm + v^\pm) - \mu - \xi)v_x^\pm + A(\omega^\pm)_x v^\pm = -d^\pm + (R^\pm[v^\pm] - \xi)\omega_x^\pm & : \text{ on } \Omega^\pm \\ (\omega^+ - \omega^-)\xi + \frac{\partial\sigma}{\partial w^+}v^+ + \frac{\partial\sigma}{\partial w^-}v^- = -d^\pm + R[v^-, v^+] & : \text{ on } (0,T) \\ v^-(\cdot,0) = 0, v^+(\cdot,0) = 0 \\ [v^-(0,t)]_i = \rho_i(t) \quad i = 1,\dots,j-1 \\ [v^+(0,t)]_i = \rho_i(t) \quad i = j+1,\dots,N \end{cases} \tag{5.11}$$

where the unknowns are $v^\pm, \xi, \rho_1, \dots, \rho_{j-1}, \rho_{j+1} \dots, \rho_N$. We let ρ stand for the collection $\rho_1, \dots, \rho_{j-1}, \rho_{j+1} \dots, \rho_N$.

Suppose in the following that (v^-, v^+, ξ, ρ) solves (5.11). Now write

$$\lambda_i^\pm = \lambda_i^\pm(t) := \lambda_i(\omega^\pm(0,t))$$

and

$$\left.\begin{array}{rcl} \alpha_i^-(t) & := & [\mathbf{T}^-[\xi, v^-, \rho, -d^- + (R^-[v^-] - \xi)\omega_x^-]]_{i,0} \\ \alpha_i^+(t) & := & [\mathbf{T}^+[\xi, v^+, \rho, -d^+ + (R^+[v^+] - \xi)\omega_x^+]]_{i,0} \end{array}\right\}. \tag{5.12}$$

Since

$$\frac{\partial\sigma}{\partial w^\pm}(w^-, w^+, \chi) = A(w^\pm) - \chi I.$$

the left-hand side of the second line of (5.11) can be written as (all functions evaluated at $x = 0, t \in (0,T)$),

$$\begin{aligned} (\omega^+ - \omega^-)\xi + \sum_\pm (A(\omega^\pm) - \mu I)v^\pm &= \\ (\omega^+ - \omega^-)\xi + \sum_{i=1}^N (\lambda_i^- - \mu)\alpha_i^-(t)\hat\theta_{i,0}^- + \sum_{i=1}^N (\lambda_i^+ - \mu)\alpha_i^+(t)\hat\theta_{i,0}^+ &= \\ (\omega^+ - \omega^-)\xi + \sum_{i=1}^{j-1} (\lambda_i^- - \mu)\rho_i(t)\hat\theta_{i,0}^- + \sum_{i=j}^N (\lambda_i^- - \mu)\alpha_i^-(t)\hat\theta_{i,0}^- &+ \end{aligned}$$

$$\sum_{i=1}^{j}(\lambda_i^+ - \mu)\alpha_i^+(t)\hat{\theta}_{i,0}^- + \sum_{i=j+1}^{N}(\lambda_i^+ - \mu)\rho_i(t)\hat{\theta}_{i,0} \tag{5.13}$$

where we used $(A(\omega^\pm) - \mu I)\hat{\theta}_{i,0}^\pm = (\lambda_i^\pm(\omega^\pm(0,t)) - \mu(t))\hat{\theta}_{i,0}^\pm$. Consider the linear mapping

$$\begin{aligned} \mathbf{G}(t) : \mathbb{R} \times \mathbb{R}^{N-1} \to \mathbb{R}^N \\ \mathbf{G}(t)[\xi, \rho] := (\omega^+(0,t) - \omega^-(0,t))\xi + \\ \sum_{i=1}^{j-1}(\lambda_i^-(t) - \mu(t))\rho_i(t)\hat{\theta}_{i,0}^-(t) + \sum_{i=j+1}^{N}(\lambda_i^+(t) - \mu(t))\rho_i(t)\hat{\theta}_{i,0}^+(t) \in \mathbb{R}^N \end{aligned}$$

where t is a parameter. We *assume* that $\mathbf{G}$ is invertible; this assumption is reasonable, since for sufficiently weak j-shocks (i.e. shocks for which $|\omega^+(0,t) - \omega^-(0,t)|$ is sufficiently small for $t \in (0,T)$), $(\omega^+(0,t) - \omega^-(0,t))$ approximately has the direction $\theta_j(\omega^-(0,t))$ [1], so that

$$(\omega^+(0,t) - \omega^-(0,t)), \theta_1, \ldots, \theta_{j-1}, \theta_{j+1}, \ldots, \theta_N$$

are linearly independent. Hence, the expression (5.13) can be rewritten as

$$\begin{aligned} (\omega^+ - \omega^-)\xi + \mathbf{G}(t)[\xi, \rho] + \\ \sum_{i=j+1}^{N}(\lambda_i^- - \mu)\alpha_i^-(t)\hat{\theta}_{i,0}^-(t) + \sum_{i=1}^{j-1}(\lambda_i^+ - \mu)\alpha_i^+(t)\hat{\theta}_{i,0}^+(t) \end{aligned}$$

with $\alpha_i^\pm$ given by (5.12). Inverting the second line of (5.11) for ξ, ρ, we therefore arrive at the following problem

$$\begin{aligned} & (\xi, \rho_1, \ldots, \rho_{j-1}, \rho_j, \ldots, \rho_N) + \\ & \mathbf{G}(t)^{-1}\left\{\sum_{i=j+1}^{N}(\lambda_i^-(t) - \mu(t))\alpha_i^-(t)\hat{\theta}_{i,0}^-(t) + \sum_{i=1}^{j-1}(\lambda_i^+(t) - \mu(t))\alpha_i^+(t)\hat{\theta}_{i,0}^+(t)\right\} \\ = \; & \mathbf{G}(t)^{-1}r \end{aligned} \tag{5.14}$$

with given $u^\pm, r$ and $\xi, \rho_1, \ldots, \rho_{j-1}, \rho_j, \ldots, \rho_N$ as unknowns (the $\alpha_i^\pm$ are given by (5.12) and are regarded as functions of $\xi, \rho_1, \ldots, \rho_{j-1}, \rho_{j+1}, \ldots, \rho_N$ and u^-, u^+). Now (5.12) implies

$$\alpha_i^-(t) \;\; := \;\; -\left[\mathbf{T}^-[\xi, u^-, \rho, \xi\omega_x^-]\right]_{i,0} + \left[\mathbf{T}[\xi, u^-, \rho, -d^- + R^-[v^-]\omega_x^-]\right]_{i,0}$$

[1] See [6]

So in analogy to the definition of **K** we now formulate the following problem for unknowns η, ρ:

$$(\eta,\rho) + \mathbf{G}(t)^{-1}\Big\{ \sum_{i=j+1}^{N} (\lambda_i^-(t) - \mu(t)) \left[\mathbf{T}^-[\xi, u^-, \rho, \eta\omega_x^-]\right]_{i,0} \hat{\theta}_{i,0}^-(t) \\ + \sum_{i=1}^{j-1} (\lambda_i^+(t) - \mu(t)) \left[\mathbf{T}^+[\xi, u^+, \rho, \eta\omega_x^+]\right]_{i,0} \hat{\theta}_{i,0}^+(t) \Big\} = \mathbf{G}(t)^{-1} r$$

The solution of (5.14) defines the operator **K** by $(\xi, \rho_1, \dots, \rho_{j-1}, \rho_{j+1}, \dots, \rho_N) = \mathbf{K}[u^-, u^+, \xi, r]$ in analogy to the operator **K** in chapter 2.

Fixed point operator. The first line of (5.11) can then be written as

$$v^{\pm} = \mathbf{T}^{\pm}[\xi, v^{\pm}, \mathbf{K}[v^-, v^+, \xi, r(v^-, v^+, \xi, \rho)])\omega_x^{\pm} + r^{\pm}(v^{\pm})]$$

where

$$\begin{aligned} r^{\pm}(v^{\pm}) &:= -d^{\pm} - R^{\pm}[v^{\pm}]\omega_x^{\pm} \\ r(v^-, v^+, \xi, \rho) &:= -d + R[v^-, v^+] \\ &\quad + \sum_{i=j+1}^{N} (\lambda_i^-(t) - \mu(t)) \left[\mathbf{T}^-[v^-, \xi, \rho, r^{\pm}(v^{\pm})]\right]_{i,0} \hat{\theta}_{i,0}^+(t) \\ &\quad + \sum_{i=1}^{j-1} (\lambda_i^+(t) - \mu(t)) \left[\mathbf{T}^+[v^+, \xi, \rho, r^{\pm}(v^{\pm})]\right]_{i,0} \hat{\theta}_{i,0}^-(t). \end{aligned}$$

The second line of (5.11) is then written as

$$(\xi, \rho) = \mathbf{K}[v^-, v^+, \xi, \rho, r(v^-, v^+, \xi)].$$

The presence of the additional quantities ρ complicates the fixed-point argument. The question whether suitable spaces of functions for these additional quantities can be found such that the problem admits a treatment analogous to the scalar case, remains open at the present time.

Appendix A

Linear Hyperbolic Problems

We collect some facts about linear hyperbolic equations of the form

$$v_t + av_x + cv = r.$$

Standard literature references are [9], [11], [6]; however none of those have existence theorems immediately applicable to our setting. Therefore, existence proofs will be given below.

A.1 Weak solutions

We consider domains Ω of the following form:

$$\Omega := \{(x,t) : \gamma_1(t) < x < \gamma_2(t), t \in [0,T]\} \subset \mathbb{R}^2 \tag{A.1}$$

with Lipschitz functions γ_1, γ_2. The set Ω_t is defined by

$$\Omega_t := \{x \in \mathbb{R} : (x,t) \in \Omega\}.$$

Define furthermore

$$\Gamma := \{(\gamma_1(t), t), (\gamma_2(t), t) : t \in [0,T]\}. \tag{A.2}$$

Ω has a Lipschitz boundary, but we require a bit more. Assume that the right-sided derivative

$$\lim_{h\to 0^+} \frac{\gamma_i(t+h) - \gamma_i(t)}{h} \text{ exists for each } t \in [0,T) \text{ and } i = 1,2. \tag{A.3}$$

We use the symbol $\gamma_i'(t^+)$ to denote the above limit and define the vector field $n(x,t)$ for each $(x,t) \in \Gamma$ by

$$\left.\begin{array}{l} n(\gamma_1(t),t) := \frac{1}{\sqrt{1+|\gamma_1'(t^+)|^2}}(-1,\gamma_1'(t^+)) \\ n(\gamma_2(t),t) := \frac{1}{\sqrt{1+|\gamma_2'(t^+)|^2}}(1,-\gamma_2'(t^+)) \end{array}\right\} \tag{A.4}$$

(A.3) is satisfied e.g. if γ_1, γ_2 are in addition piecewise C^1. If γ_1 is differentiable in t_0, then $n(\gamma_1(t_0), t_0)$ is the usual unit outer normal on $\partial\Omega$ at $(\gamma_1(t_0), t_0) \in \Gamma$.

Recall the definition of the spaces V^k:

$$V^k(\Omega) := \{u \in L^\infty(\Omega) : D_x^j u \in L^\infty(\Omega) \text{ for } j = 0, \dots, k\}. \tag{A.5}$$

Definition A.1. $a \in V^1(\Omega)$ is outward-pointing *on* Γ *if there is a constant* $C > 0$ *such that*

$$(a(x,t),1) \cdot n(x,t) \geq C > 0 \quad ((x,t) \in \Gamma) \tag{A.6}$$

with $n(x,t)$ *defined above.*

We shall work exclusively with the following notion of weak solution for linear hyperbolic problems.

Definition A.2. *Let* $a \in V^1(\Omega)$ *be outward pointing on* Γ *and* $c, r \in V^1(\Omega)$. $v \in V^1(\Omega)$ *is called a weak solution of the initial value problem*

$$v_t + av_x + cv = r, \quad v(\cdot, 0) = 0 \tag{A.7}$$

if the following relation holds for all $\phi \in C^1(\overline{\Omega})$ *such that* $\phi(\cdot, T) = 0, \phi|_\Gamma = 0$*:*

$$-\int_\Omega v\phi_t \, dx \, dt + \int_\Omega (av_x + cv)\phi \, dx \, dt = \int_\Omega r\phi \, dx \, dt. \tag{A.8}$$

As an immediate consequence of the above definition, let us note that $v_t \in L^\infty(\Omega)$ for a weak solution v. Indeed, we have

$$\left| \int_\Omega v\phi_t dxdt \right| \leq \|av_x + cv - r\|_{L^\infty(\Omega)} \int_\Omega |\phi| dxdt$$

for any $\phi \in C_0^1(\Omega)$, whence it follows that $v_t \in L^\infty(\Omega)$.

Remark A.3. v *is a weak solution of* (A.7) *if and only if* $v \in W^{1,\infty}(\Omega), v(\cdot, 0) = 0$ *and*

$$v_t + av_x + cv = r$$

almost everywhere on Ω.

We have the important

Theorem A.4 (L^2-apriori estimate)**.** *Let $\nu \geq \sup_\Omega(-c + \frac{1}{2}a_x)$ and v be a weak solution of (A.7). Then there exists a constant $C > 0$ such that*

$$e^{-\nu\tau}\|v(\cdot,\tau)\|_{L^2(\Omega_\tau)} \leq C\sqrt{\int_0^\tau e^{-2\nu t}\|r(\cdot,t)\|^2_{L^2(\Omega_t)}dt} \tag{A.9}$$

holds true for any $\tau \in (0,T)$.

Proof. According to the last remark, $v_t \in L^\infty(\Omega)$. Set $w := e^{-\nu t}v$; note that

$$w_t + aw_x + (c+\nu)w = e^{-\nu t}r$$

a.e. on Ω. Multiplying with w and integrating over $\Omega \cap (\mathbb{R} \times (0,\tau))$, the expression

$$\int_0^\tau \int_{\Omega_t} aww_x \, dx \, dt = \int_0^\tau \int_{\Omega_t} aD_x\left(\frac{1}{2}w^2\right) dx \, dt$$

arises, on which we may use integration by parts (which is valid for $W^{1,\infty}(\Omega)$-functions) to arrive at

$$\frac{1}{2}\|w(\cdot,\tau)\|^2_{L^2(\Omega_\tau)} - \frac{1}{2}\|w(\cdot,0)\|^2_{L^2(\Omega_0)} +$$
$$\frac{1}{2}\int_{\Gamma_\tau}(n_2+n_1a)w^2 d\sigma + \int_0^\tau\int_{\Omega_t}\left(c - \frac{1}{2}a_x + \nu\right)w^2 \, dx \, dt = \int_0^\tau\int_{\Omega_t} e^{-\nu t}rw \, dx \, dt.$$

with $\Gamma_\tau := \Gamma \cap (\mathbb{R} \times (0,\tau))$ and $n = (n_1, n_2)$ denoting the outer-unit normal. Note that since γ_1, γ_2 are differentiable a.e. on $(0,T)$, the unit outer normal is given by (A.4).

Since a is outward-pointing, $n_2 + n_1 a \geq 0$. Taking the choice of ν into account and noting that $\|w(\cdot,0)\|_{L^2(\Omega_0)} = 0$, we obtain the inequality

$$\begin{aligned}\frac{1}{2}\|w(\cdot,\tau)\|^2_{L^2(\Omega_\tau)} &\leq \int_0^\tau e^{-\nu\tau}\|r\|_{L^2(\Omega_t)}\|w\|_{L^2(\Omega_t)}dt \\ &\leq \frac{1}{2}\int_0^\tau e^{-2\nu t}\|r\|^2_{L^2(\Omega_t)}dt + \frac{1}{2}\int_0^\tau \|w(\cdot,t)\|^2_{L^2(\Omega_t)}dt.\end{aligned}$$

Gronwall's inequality (see [22]) now implies

$$\|w(\cdot,\tau)\|^2_{L^2(\Omega_\tau)} \leq C\int_0^\tau e^{-2\nu t}\|r\|^2_{L^2(\Omega_t)}dt.$$

Hence we conclude

$$e^{-2\nu\tau}\|v(\cdot,\tau)\|^2_{L^2(\Omega_\tau)} \leq C\int_0^\tau e^{-2\nu t}\|r\|^2_{L^2(\Omega_t)}dt.$$

□

A.2 Smoothing operators

We introduce an important smoothing operation for functions $v \in V^1(\Omega)$.

Definition A.5. *For $v \in V^1(\Omega)$, we set*

$$\tilde{v}(y,t) := \begin{cases} v(y,t) & : \gamma_1(t) \le y \le \gamma_2(t) \\ v(\gamma_1(t),t) & : y < \gamma_1(t) \\ v(\gamma_2(t),t) & : y > \gamma_2(t) \end{cases} \qquad (y \in \mathbb{R}, t \in (0,T)) \tag{A.10}$$

Here we identify $v(\cdot,t) \in W^{1,\infty}(\Omega_t)$ for almost all $t \in (0,T)$ with its Lipschitz-continuous representative.

Note that $\tilde{v} = v$ on Ω.

Remark A.6. *For almost all t, $\tilde{v}(\cdot,t)$, is a Lipschitz function and we clearly have*

$$\|\tilde{v}\|_{L^\infty(\mathbb{R}\times(0,T))} = \|v\|_{L^\infty(\Omega)}, \quad \|\tilde{v}_x\|_{L^\infty(\mathbb{R}\times(0,T))} = \|v_x\|_{L^\infty(\Omega)}, \tag{A.11}$$

since $v_x = 0$ outside of Ω.

Choose now a nonnegative function $\rho \in C^\infty(\mathbb{R})$ with compact support in $(-1,1)$ and having the property

$$\int_{\mathbb{R}} \rho(y)\, dy = 1.$$

Let $\rho_\theta(y) := \theta^{-1}\rho(y/\theta)$, which defines the well-known *standard mollifier*.

Definition A.7. *We define the smoothing operator $S_\theta : V^1(\Omega) \to V^1(\Omega)$ by*

$$(S_\theta v)(x,t) := \int_{\mathbb{R}} \tilde{v}(y,t)\rho_\theta(x-y)dy. \tag{A.12}$$

Note that $S_\theta v(\cdot,t) \in C^\infty(\bar{\Omega}_t)$ for a.e. $t \in (0,T)$. An important relation between mollification and weak differentiation is given by

$$(D_x S_\theta v)(x,t) = \int_{\mathbb{R}} (D_x\tilde{v})(y,t)\rho_\theta(x-y)dy. \tag{A.13}$$

Lemma A.8. *We have the following properties of S_θ ($v \in V^1(\Omega)$):*

(i) $\|S_\theta v\|_{L^\infty(\Omega)} \le \|v\|_{L^\infty(\Omega)}$

(ii) $\|D_x(S_\theta v)\|_{L^\infty(\Omega)} \leq \|v_x\|_{L^\infty(\Omega)}$

(iii) $\|D_x^j(S_\theta v)\|_{L^\infty(\Omega)} \leq C(j,\theta)\|v_x\|_{L^\infty(\Omega)} \quad (j \geq 2)$

(iv) $S_\theta v \to v$ *in* $L^p(\Omega)$ *as* $\theta \to 0^+$, *for any* $p \in [1,\infty)$.

Here, $C(j,\theta) \to \infty$ *as* $\theta \to 0^+$.

Proof. To prove (i), we note that by the nonnegativity of ρ_θ,

$$|S_\theta v(x,t)| \leq \int_{\mathbb{R}} \rho_\theta(x-y)|\tilde{v}(y,t)|dy \leq \|\tilde{v}\|_{L^\infty(\mathbb{R}\times(0,T))} \int_{\mathbb{R}} \rho_\theta(x-y)dy = \|v\|_{L^\infty(\Omega)}.$$

Furthermore, (A.13) and (A.11), imply (ii).

We have for $j \geq 1$ (see [6])

$$(D_x^j S_\theta v)(x,t) = \int_{\mathbb{R}} \theta^{-j} \rho^{(j-1)} \left(\frac{x-y}{\theta}\right)(D_x\tilde{v})(y,t)dy$$

and thus

$$\|D_x^j S_\theta v\|_{L^\infty(\Omega)} \leq C(j,\theta)\|D_x v\|_{L^\infty(\Omega)},$$

proving (iii).

In order to prove (iv) we note that

$$\int_\Omega |(S_\theta v)(x,t) - v(x,t)|^p d(x,t) \quad = \quad \int_0^T \int_{\Omega_t} |(S_\theta v)(x,t) - v(x,t)|^p dxdt$$

$$= \int_0^T \int_{\Omega_t} |(\rho_\theta * \tilde{v})(x,t) - v(x,t)|^p dxdt \tag{A.14}$$

For each $t \in (0,T)$,

$$\int_{\Omega_t} |(\rho_\theta * \tilde{v})(x,t) - v(x,t)|^p \, dx = \int_{\Omega_t} |(\rho_\theta * \tilde{v})(x,t) - \tilde{v}(x,t)|^p \, dx \to 0,$$

as $\theta \to 0^+$, which is a well-known property of the mollifier (see [6]). Moreover for all $\theta < 1$,

$$|(\rho_\theta * \tilde{v})(x,t) - v(x,t)|^p \leq C|(\rho_\theta * \tilde{v})(x,t)|^p + C|v(x,t)|^p \leq C$$

holds with some $C > 0$ independent of θ, since $|(\rho_\theta * \tilde{v})(x,t)| \leq \|v\|_{L^\infty(\Omega)}$ The dominated convergence theorem applied to the integral over $(0,T)$ in (A.14) now finishes the proof of (iv). □

Remark A.9. *Sometimes it is useful to note the inequality*

$$\inf_\Omega v \leq (S_\theta v)(x,t) \leq \sup_\Omega v \tag{A.15}$$

which follows from (A.12) *and the fact that* ρ_θ *is nonnegative.*

A.3 Existence Theorem

A.3.1 A-priori estimates in L^∞-norm

Consider the hyperbolic initial value problem

$$v_t + av_x + cv = r, \quad v(\cdot, 0) = 0 \tag{A.16}$$

with *smooth* coefficients $a, c, r \in C^\infty(\bar\Omega)$ and a satisfying the condition (n denoting the unit outer normal on Γ from (A.4))

$$(a(x,t), 1) \cdot n(x,t) > 0 \quad \text{for all } \; t \in (0,T), x \in \{\gamma_1(t), \gamma_2(t)\}, \tag{A.17}$$

which is somewhat weaker than (A.6). Recall the following norms ($\nu > 0$)

$$\|v\|_\nu := \sup_{t\in(0,T)} e^{-\nu t}\|v(\cdot,t)\|_{L^\infty(\Omega_t)}$$
$$|\!|\!|v|\!|\!|_\nu := \int_{(0,T)} e^{-\nu t}\|v(\cdot,t)\|_{L^\infty(\Omega_t)}dt$$

Theorem A.10 (Basic a-priori bound)**.** *Let* $\nu \geq \sup_{(x,t)\in\Omega}(-c(x,t))$ *and* $v \in C^\infty(\bar\Omega)$ *be a classical solution of (A.16); then*

$$\|v\|_\nu \leq |\!|\!|r|\!|\!|_\nu \tag{A.18}$$

holds.

Proof. Let $\psi_\varepsilon \in C^1(\mathbb{R})$ be a family of nonnegative functions such that $\psi_\varepsilon(x) \to |x|, \psi'_\varepsilon(x) \to \operatorname{sign} x$ as $\varepsilon \to 0^+$ and such that $\psi_\varepsilon(0) = 0$, $|\psi'_\varepsilon(x)| \leq 1$ for $\varepsilon \in (0,1)$. Multiply the hyperbolic PDE (A.16) with $e^{-\nu t}\psi'_\varepsilon(v)$ to obtain

$$u^\varepsilon_t + au^\varepsilon_x + (c+\nu)u^\varepsilon = re^{-\nu t}\psi'_\varepsilon(v) + c(u^\varepsilon - v\psi'_\varepsilon(v)e^{-\nu t}) \tag{A.19}$$

on Ω, where $u^\varepsilon(x,t) := e^{-\nu t}\psi_\varepsilon(v(x,t))$. Now fix $(x_0, t_0) \in \Omega$. $\vartheta : (s, t_0] \to \mathbb{R}$ defines a characteristic curve passing through (x_0, t_0) if

$$\vartheta(t) \in \Omega_t \quad (t \in (s, t_0])$$

and

$$\vartheta'(t) = a(\vartheta(t), t) \quad (t \in (s, t_0]), \;\; \vartheta(t_0) = x_0. \tag{A.20}$$

Since solutions ϑ to the above ODE problem with some $s < t_0$ close to t_0 exist, characteristic curves through (x_0, t_0) exist.

Now let $\vartheta : J \to \mathbb{R}$ (where $J \subset (0, t_0]$ is a left open interval) be a maximal solution of (A.20), that is, a solution with the property that for any other characteristic curve $\tilde{\vartheta} : I \to \mathbb{R}, \tilde{\vartheta}(t) \in \Omega_t, \tilde{\vartheta}(t_0) = x_0$ defined on an interval $I = (s, t_0]$ we have $I \subset J$ and $\tilde{\vartheta} = \vartheta$ on I. Such a maximal solution exists (see [10], chapter 4 (III)).

We claim that ϑ is necessarily defined on the interval $J = (0, t_0]$. A theorem on maximal solutions of ODE's ([10], chapter 4 (III)) states that the curve $(\vartheta(t), t)$ must leave any compact subset $\mathcal{C} \subset \Omega$. Note furthermore that ϑ is uniformly continuous on its interval of definition (this follows from (A.20) and the fact that a is bounded) so that the right-hand limit of ϑ at $t = s$ exists. We write

$$\vartheta(s) := \lim_{t \to s^+} \vartheta(t).$$

Thus $(\vartheta(s), s)$ lies on the boundary of Ω. If $s > 0$ then this implies $\vartheta(s) = \gamma_1(s)$ or $\vartheta(s) = \gamma_2(s)$. Suppose that $\gamma_1(s) = \vartheta(s)$. Since $\vartheta(s + h) > \gamma_1(s + h)$, we have for small $h > 0$:

$$\frac{1}{h}[-\vartheta(s + h) + \vartheta(s) + \gamma_1(s + h) - \gamma_1(s)] \leq 0,$$

but as $h \to 0^+$, the above expression converges to (note assumption (A.3))

$$\begin{aligned} -\vartheta'(s) + \gamma_1'(s^+) &= -a(\gamma_1(s), s) + \gamma_1'(s^+) \\ &= \sqrt{1 + |\gamma_1'(s^+)|^2}(a(\gamma_1(s), s), 1) \cdot n(\gamma_1(s), s) > 0 \end{aligned}$$

because of (A.17), which gives a contradiction. We can argue similarly if $\gamma_2(s) = \vartheta(s)$, so that $s > 0$ is impossible. Thus, $s = 0$ and ϑ is defined up to $t = 0$ and $\gamma_1(0) \leq \vartheta(0) \leq \gamma_2(0)$.

Consider now (A.19), evaluated at $(\vartheta(\tau), \tau)$. Integrating w.r.t. τ and using $v(\vartheta(0), 0) = 0$, we obtain

$$\begin{aligned} & u^\varepsilon(x_0, t_0) + \int_0^{t_0} (c(\vartheta(\tau), \tau) + \nu) u^\varepsilon(\vartheta(\tau), \tau) d\tau \\ = & \int_0^{t_0} [r(\vartheta(\tau), \tau) e^{-\nu\tau} \psi_\varepsilon'(v(\vartheta(\tau), \tau)) \\ & + c(\vartheta(t), t))(u^\varepsilon(\vartheta(\tau), \tau) - \psi_\varepsilon'(v(\vartheta(\tau), \tau)) v(\vartheta(\tau), \tau) e^{-\nu\tau})] d\tau \end{aligned}$$

which gives, since $c(\vartheta(\tau), \tau) + \nu \geq 0$ and $u^\varepsilon \geq 0$,

$$u^\varepsilon(x_0, t_0) \leq \int_0^{t_0} \|r(\cdot, \tau)\|_{L^\infty(\Omega_\tau)} e^{-\nu\tau} d\tau + \int_0^{t_0} \|c\|_{L^\infty(\Omega_\tau)} \|u^\varepsilon - v\psi_\varepsilon'(v) e^{-\nu\tau}\|_{L^\infty(\Omega_\tau)} d\tau.$$

Since the second integral converges to zero as $\varepsilon \to 0^+$ (note that $u^\varepsilon \to |v|$, $v\psi'_\varepsilon(v) \to v\text{sign}v = |v|$ and that the second integrand in the last formula is bounded by a constant independent of ε) we get the desired a-priori bound, after sending $\varepsilon \to 0^+$. □

Theorem A.11. *Let $v \in C^\infty(\bar{\Omega})$ be a solution of (A.16), and $\nu \geq \sup_\Omega(-a_x - c)$. Then*

$$\|v_x\|_\nu \leq \|r_x\|_\nu + \|c_x v\|_\nu. \tag{A.21}$$

and moreover

$$\|v_t\|_{L^\infty(\Omega)} \leq C\|r\|_{V^1(\Omega)} \tag{A.22}$$

for some constant C depending only on $\|a\|_{L^\infty(\Omega)}$, $\|a_x\|_{L^\infty(\Omega)}$,$\|c\|_{L^\infty(\Omega)}$ and ν.

Proof. We differentiate (A.16) with respect to x to obtain

$$v_{tx} + av_{xx} + (a_x + c)v_x = -c_x v + r_x, \quad v_x(\cdot, 0) = 0$$

thus v_x satisfies a hyperbolic problem of the type considered before. Now applying theorem A.10, we get

$$\|v_x\|_\nu \leq \|r_x - c_x v\|_\nu \leq \|r_x\|_\nu + \|c_x v\|_\nu,$$

provided ν is chosen as in the statement of the theorem. Note that (A.21) also implies

$$\|v_x\|_{L^\infty(\Omega)} \leq C\|r\|_{V^1(\Omega)}, \tag{A.23}$$

where C depends on $\|a_x\|_{L^\infty(\Omega)}$, $\|c_x\|_{L^\infty(\Omega)}$.

To prove the estimate for v_t, we write

$$v_t = -av_x - cv + r$$

and thus

$$\begin{aligned} \|v_t\|_{L^\infty(\Omega^\pm)} &\leq \|a\|_{L^\infty(\Omega)}\|v_x\|_{L^\infty(\Omega)} + \|c\|_{L^\infty(\Omega)}\|v\|_{L^\infty(\Omega)} + \|r\|_{L^\infty(\Omega)} \\ &\leq C\|r\|_{V^1(\Omega)} \end{aligned}$$

by (A.23). □

The following theorem gives important estimates (without explicit constants) on higher derivatives of a smooth solution.

Theorem A.12. *Let v be as in the previous theorem. Then for $j \geq 2$,*

$$\|D_x^j v\|_{L^\infty(\Omega)} \leq C(\|a\|_{V^j}, \|c\|_{V^j})\|r\|_{V^j(\Omega)}$$
$$\|D_t D_x^{j-1} v\|_{L^\infty(\Omega)} \leq C(\|a\|_{V^j}, \|c\|_{V^j})\|r\|_{V^j(\Omega)}$$

Proof. We differentiate (A.16) repeatedly and apply the estimate from theorem A.10, choosing ν appropriately. □

A.3.2 Existence of weak solutions.

There are a variety of techniques available to prove the existence of solutions to hyperbolic initial-value problems, for example the method of characteristics (see [9], [11]), parabolic regularization (see e.g. [6]) and a method based on energy estimates. The method of characteristics is especially straightforward for one space dimension and can be used to prove the existence of C^1-solutions. However, we need an existence theorem for weak solutions, where the data of the problem is given in the space V^1. The method of characteristics presumably can also be applied here, but we choose to rely on the following existence theorem, which can be found in [11]:

Theorem A.13. *Let $S := \mathbb{R} \times (0, T)$ and $a, c, r \in C^\infty(\bar{S})$. Then there exists a $u \in C^\infty(\bar{S})$ solving the hyperbolic initial-value problem*

$$u_t + au_x + cu = r \quad \text{on } S, \quad u(\cdot, 0) = 0. \tag{A.24}$$

The strategy will be to smooth out the data of the given hyperbolic problem, to apply theorem A.13 and then to use the a-priori-estimates in theorems A.10 and A.11 to get the existence of a weak solution. In building the approximate problems by smoothing out the data, we have to take care not to destroy the important outward-pointing property. We need several auxiliary lemmata.

Lemma A.14. *Define the mapping $\Lambda : S \to S$ by*

$$\Lambda(x,t) := \left(\frac{x - \gamma_1(t)}{\gamma_2(t) - \gamma_1(t)}, t \right) = (\Lambda_1(x,t), t). \tag{A.25}$$

Then $\Lambda(\Omega) = (0,1) \times (0,T) =: \hat{\Omega}$. Then, if v is a weak solution of

$$v_t + av_x + cv = r \tag{A.26}$$

on Ω, then the function $\hat{v} : (0,1) \times (0,T) \to \mathbb{R}$ defined by $\hat{v} = v \circ \Lambda^{-1}$ is a weak solution of

$$\hat{v}_t + \hat{a}\hat{v}_x + \hat{c}\hat{v} = \hat{r} \tag{A.27}$$

on $\hat{\Omega}$, with

$$\left.\begin{array}{l}\hat{a}(y,t) := \frac{-\gamma_1'(t^+)-y(\gamma_2'(t^+)-\gamma_1'(t^+))+a(\Lambda^{-1}(y,t))}{\gamma_2(t)-\gamma_1(t)} \\ \hat{c}(y,t) := c(\Lambda^{-1}(y,t)), \quad \hat{r}(y,t) := r(\Lambda^{-1}(y,t))\end{array}\right\}. \tag{A.28}$$

Conversely, if $\hat{v}$ solves (A.27) *with some coefficient functions $\hat{a}, \hat{c}$ and right-hand side $\hat{r}$ on $\hat{\Omega}$, then the function $v(x,t) := \hat{v}(\Lambda(x,t))$ solves* (A.26) *with*

$$\left.\begin{array}{l}a(x,t) := \gamma_1'(t^+) + \frac{\gamma_2'(t^+)-\gamma_1'(t^+)}{\gamma_2(t)-\gamma_1(t)}(x-\gamma_1(t)) + (\gamma_2(t)-\gamma_1(t))\hat{a}(\Lambda(x,t)) \\ c(x,t) := \hat{c}(\Lambda(x,t)), \quad r(x,t) := \hat{r}(\Lambda(x,t))\end{array}\right\} \tag{A.29}$$

Proof. $\hat{v}$ defined by $\hat{v} = v \circ \Lambda^{-1}$ is clearly Lipschitz on $\hat{\Omega}$, since any weak solution of (A.26) is Lipschitz on Ω. Hence, using the almost everywhere differentiability of Lipschitz functions together with the chain rule, we get

$$\begin{aligned} v_t(x,t) &= \hat{v}_y(\Lambda(x,t))\left(\frac{-\gamma_1'(t)-\Lambda_1(x,t)(\gamma_2'(t)-\gamma_1'(t))}{\gamma_2(t)-\gamma_1(t)}\right) + \hat{v}_t(\Lambda(x,t)) \\ v_x(x,t) &= \hat{v}_y(\Lambda(x,t))\frac{1}{\gamma_2(t)-\gamma_1(t)} \end{aligned}$$

a.e. on Ω. Since $\gamma_i'(t)$ exist for a.e. $t \in (0,T)$ we have in particular $\gamma_i'(t) = \gamma_i'(t^+)$ a.e.. Thus we obtain

$$\begin{aligned} \hat{v}_t(\Lambda(x,t)) + \left(\frac{-\gamma_1'(t)-\Lambda_1(x,t)(\gamma_2'(t^+)-\gamma_1'(t^+))+a(x,t)}{\gamma_2(t)-\gamma_1(t)}\right)\hat{v}_y(\Lambda(x,t)) \\ +c(x,t)\hat{v}(\Lambda(x,t)) = r(x,t). \end{aligned}$$

The other direction is proved analogously. □

Lemma A.15. *If $a \in V^1(\Omega)$ is outward-pointing, then $\hat{a}$ defined by* (A.28) *is also outward-pointing w.r.t. $\hat{\Omega}$, i.e.*

$$\left.\begin{array}{ll}\hat{a}(0,t) \le -C < 0 & t \in (0,T) \\ \hat{a}(1,t) \ge C > 0 & t \in (0,T)\end{array}\right\} \tag{A.30}$$

for some $C > 0$. If conversely some function $\hat{a} \in V^1(\hat{\Omega})$ satisfies (A.30)*, then a defined in the first line of* (A.29) *is outward-pointing on Ω.*

Proof. Let $\hat{a}$ be defined as in (A.28) and a be outward-pointing. We have

$$\hat{a}(0,t) = \frac{a(\gamma_1(t),t)-\gamma_1'(t^+)}{\gamma_2(t)-\gamma_1(t)}$$

$$= \quad -\frac{\sqrt{1+|\gamma_1'(t^+)|^2}(a(\gamma_1(t),t),1)\cdot n(x,t)}{\gamma_2(t)-\gamma_1(t)} \le -C < 0.$$

On the other hand, let $\hat{a}$ satisfy (A.30) and define a by (A.29). Then

$$a(\gamma_1(t),t) = \gamma_1'(t^+) + (\gamma_2(t)-\gamma_1(t))\hat{a}(0,t)$$

and thus for a.e. t,

$$\begin{aligned} & (a(\gamma_1(t),t),1)\cdot n(\gamma_1(t),t) \\ = \quad & \frac{1}{\sqrt{1+|\gamma_1'(t^+)|^2}}(a(\gamma_1(t),t),1)\cdot(-1,\gamma_1'(t^+)) \\ = \quad & \frac{1}{\sqrt{1+|\gamma_1'(t^+)|^2}}(-\gamma_1'(t^+) - (\gamma_2(t)-\gamma_1(t))\hat{a}(0,t) + \gamma_1'(t^+)) \\ = \quad & -\frac{1}{\sqrt{1+|\gamma_1'(t^+)|^2}}(\gamma_2(t)-\gamma_1(t))\hat{a}(0,t) \ge C > 0. \end{aligned}$$

On the other part of Γ, we argue similarly. □

Theorem A.16. *The problem (A.7) has a unique solution $v \in V^1(\Omega)$ and moreover we have the estimates*

$$\left.\begin{aligned} \|v\|_\nu \le \|r\|_\nu \\ \|v_x\|_\nu \le \|r_x\|_\nu + \left(\int_0^T \|c_x\|_{L^\infty(\Omega_t)}dt\right)\|v\|_\nu \end{aligned}\right\} \tag{A.31}$$

provided ν satisfies

$$\nu > \max\{\sup_\Omega(-c), \sup_\Omega(-c-a_x)\}. \tag{A.32}$$

Proof. 1. Since $a(\cdot,t)$ is Lipschitz, we first extend a to S by setting $a(z,t) = a(\gamma_2(t),t)$ if $z > \gamma_2(t)$ and $a(z,t) = a(\gamma_1(t),t)$ for $z < \gamma_1(t)$ (see section A.2), and do the same for $c, r \in V^1(\Omega)$. In order to reduce the number of symbols, we continue to denote these extended functions by the same letters a, c and r. This implies

$$(-c-a_x)(y,t) \le \max\{\sup_\Omega(-c), \sup_\Omega(-c-D_x a)\}, \quad -c(y,t) \le \sup_\Omega(-c) \tag{A.33}$$

for $(y,t) \in S$. Moreover we set

$$a(x,t) = c(x,t) = r(x,t) = 0 \quad ((x,t) \notin S).$$

Let $\hat{a}, \hat{c}, \hat{r}$ be defined from a, c, r as in (A.28).

Now define functions $\hat{a}_\delta, \hat{c}_\delta, \hat{r}_\delta \in C^\infty(S)$ by smoothing out in x and t directions in the following way, with ρ_δ denoting the standard mollifier:

$$\begin{aligned}
\hat{a}_\delta(y,t) &:= \int_{\mathbb{R}}\int_{\mathbb{R}} \rho_\delta(y-z)\rho_\delta(t-s)\hat{a}(z,s)\;ds\,dz \\
\hat{c}_\delta(y,t) &:= \int_{\mathbb{R}}\int_{\mathbb{R}} \rho_\delta(y-z)\rho_\delta(t-s)\hat{c}(z,s)\;ds\,dz \\
\hat{r}_\delta(y,t) &:= \int_{\mathbb{R}}\int_{\mathbb{R}} \rho_\delta(y-z)\rho_\delta(t-s)\hat{r}(z,s)\;ds\,dz.
\end{aligned}$$

$\hat{a}_\delta, \hat{c}_\delta, \hat{r}_\delta$ converge to $\hat{a}, \hat{c}, \hat{r}$ in $L^1_{\text{loc}}(\hat{\Omega})$ as $\delta \to 0$. Note that

$$-D_x\hat{a}_\delta(y,z) = \int_{\mathbb{R}}\int_{\mathbb{R}} \rho_\delta(y-z)\rho_\delta(t-s)(-D_y\hat{a})(z,s)\;ds\,dz. \tag{A.34}$$

According to theorem A.13, there exists a $\hat{v}_\delta \in C^\infty(\bar{S})$ such that

$$D_t\hat{v}_\delta + \hat{a}_\delta D_x\hat{v}_\delta + \hat{c}_\delta\hat{v}_\delta = \hat{r}_\delta, \quad \hat{v}_\delta(\cdot,0) = 0.$$

We now study the behaviour of $\hat{a}_\delta$ at $y = 1$ and $y = 0$, respectively. Write $L := \|\hat{a}_x\|_{L^\infty(\Omega)}$, $\chi_{(0,T)}$ for the characteristic function of the interval $(0,T)$ and note that for all $t \in (0,T)$,

$$\begin{aligned}
\hat{a}_\delta(0,t) &= \int_{\mathbb{R}} \rho_\delta(-z) \int_{\mathbb{R}} \rho_\delta(t-s)\hat{a}(z,s)\chi_{(0,T)}(s)\,ds\,dz \\
&\le \int_{\mathbb{R}} \rho_\delta(-z) \int_{\mathbb{R}} \rho_\delta(t-s)\hat{a}(0,s)\chi_{(0,T)}(s)\,ds\,dz \\
&+ L\int_{\mathbb{R}} \rho_\delta(-z) \int_{\mathbb{R}} \rho_\delta(t-s)|z|\chi_{(0,T)}(s)\,ds\,dz \\
&\le \left\{-C + L\left(\int_{\mathbb{R}} \rho_\delta(z)|z|\,dz\right)\right\}\left(\int_{\mathbb{R}} \rho_\delta(t-s)\chi_{(0,T)}(s)\,ds\right) \\
&= \left\{-C + L\delta\int_{\mathbb{R}} \rho(z)|z|\,dz\right\}\left(\int_{\mathbb{R}} \rho_\delta(t-s)\chi_{(0,T)}(s)\,ds\right)
\end{aligned}$$

with C from (A.30); moreover, we have $\int_{\mathbb{R}} \rho_\delta(t-s)\chi_{(0,T)}(s)ds > 0$ for all $t \in (0,T)$. The above implies, that for sufficiently small δ,

$$\hat{a}_\delta(0,t) < 0 \quad (t \in (0,T)). \tag{A.35}$$

Analogously we check $\hat{a}_\delta(1,t) > 0$. Now define functions $a_\delta, c_\delta, r_\delta \in V^1(\Omega)$ on the original domain by

$$a_\delta(x,t) \;:=\; \gamma_1'(t) + \frac{\gamma_2'(t) - \gamma_1'(t)}{\gamma_2(t) - \gamma_1(t)}(x - \gamma_1(t)) + (\gamma_2(t) - \gamma_1(t))\hat{a}_\delta(\Lambda(x,t))$$

$$c_\delta(x,t) := \hat{c}_\delta(\Lambda(x,t)), \quad r_\delta(x,t) := \hat{r}_\delta(\Lambda(x,t)).$$

Note that the L^∞-norms of $c_\delta, r_\delta, D_x c_\delta, D_x r_\delta$ can be bounded independently of $\delta > 0$, since this can be done for the corresponding quantities $\hat{c}_\delta, \hat{r}_\delta, D_x\hat{c}_\delta, D_x\hat{r}_\delta$, using the fact that Λ is a Lipschitz transformation. More precisely, the following proposition holds.

Proposition A.17. *There exist $K_\delta > 0, F_\delta > 0$ with $K_\delta \to 1$, $F_\delta \to 0$ such that*

$$\begin{aligned} \|r_\delta\|_\nu &\leq K_\delta \|r\| && \text{(A.36)} \\ \|D_x r_\delta\|_\nu &\leq K_\delta \|r_x\| + F_\delta && \text{(A.37)} \\ \int_0^T \|D_x c_\delta\|_{L^\infty(\Omega_t)} dt &\leq \int_0^T \|D_x c\|_{L^\infty(\Omega_t)} dt + F_\delta. && \text{(A.38)} \end{aligned}$$

Proof. We have, by definition of r_δ and Λ,

$$\begin{aligned} \|r_\delta\|_{L^\infty(\Omega_\tau)} &= \sup_{x\in\Omega_\tau} |\hat{r}_\delta(\Lambda(x,\tau))| \\ &= \sup_{y\in(0,1)} |\hat{r}_\delta(y,\tau)| \\ &\leq \sup_{y\in(0,1)} \int_{\mathbb{R}^2} \rho_\delta(y-z)\rho_\delta(\tau-s)|\hat{r}(z,s)|\, ds\, dz \\ &= \sup_{y\in(0,1)} \int_{\mathbb{R}^2} \rho_\delta(y-z)\rho_\delta(\tau-s)|r(\Lambda^{-1}(z,s))|\, ds\, dz \\ &\leq \sup_{y\in(0,1)} \int_{\mathbb{R}^2} \rho_\delta(y-z)\rho_\delta(\tau-s)\|r(\cdot,s)\|_{L^\infty(\Omega_s)}\, ds\, dz \\ &= \sup_{y\in(0,1)} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \rho_\delta(y-z)\, dz\right) \rho_\delta(\tau-s)\|r(\cdot,s)\|_{L^\infty(\Omega_s)}\, ds \\ &= \int_{\mathbb{R}} \rho_\delta(\tau-s)\|r(\cdot,s)\|_{L^\infty(\Omega_s)}\, ds \end{aligned}$$

and thus

$$\begin{aligned} \|r_\delta\|_\nu &= \int_0^T e^{-\nu\tau}\|r_\delta\|_{L^\infty(\Omega_\tau)} d\tau \\ &\leq \int_0^T e^{-\nu\tau} \int_{\mathbb{R}} \rho_\delta(\tau-s)\|r(\cdot,s)\|_{L^\infty(\Omega_s)}\, ds\, d\tau \\ &\leq \int_0^T e^{-\nu\tau} \int_{\mathbb{R}} \rho_\delta(\xi)\|r(\cdot,\tau-\xi)\|_{L^\infty(\Omega_{\tau-\xi})}\, d\xi\, d\tau \\ &\leq \int_{\mathbb{R}} e^{-\nu\xi}\rho_\delta(\xi) \left(\int_0^T e^{-\nu(\tau-\xi)}\|r(\cdot,\tau-\xi)\|_{L^\infty(\Omega_{\tau-\xi})} d\tau\right) d\xi \end{aligned}$$

$$
\begin{aligned}
&\leq \int_{\mathbb{R}} e^{-\nu\xi}\rho_\delta(\xi)\left(\int_{-\xi}^{T-\xi} e^{-\nu\eta}\|r(\cdot,\eta)\|_{L^\infty(\Omega_\eta)}d\eta\right)d\xi \\
&\leq \int_{\mathbb{R}} e^{-\nu\xi}\rho_\delta(\xi)\|r\|_\nu d\xi \\
&=: K_\delta\|r\|_\nu
\end{aligned}
$$

where

$$
K_\delta = \int_{\mathbb{R}} e^{-\nu\xi}\rho_\delta(\xi)d\xi \to e^{-\nu 0} = 1 \quad (\delta \to 0^+),
$$

hence we have (A.36). A calculation gives

$$
\begin{aligned}
&(D_x c_\delta)(x,t) \\
=& \int_{\mathbb{R}^2} \rho_\delta(z)\rho_\delta(t-s)(D_x c)\left(\gamma_1(s) + \left(\frac{x-\gamma_1(t)}{\gamma_2(t)-\gamma_1(t)} - z\right)(\gamma_2-\gamma_1)(s), s\right)\frac{(\gamma_2-\gamma_1)(s)}{(\gamma_2-\gamma_1)(t)}\, ds\, dz.
\end{aligned}
$$

Now we can estimate

$$
\begin{aligned}
\int_0^T \|D_x c_\delta\|_{L^\infty(\Omega_t)}dt &\leq \int_0^T \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \rho_\delta(z)dz\right)\rho_\delta(t-s)\|D_x c\|_{L^\infty(\Omega_s)}\frac{(\gamma_2-\gamma_1)(s)}{(\gamma_2-\gamma_1)(t)}\, ds\, dt \\
&= \int_0^T \int_{\mathbb{R}} \rho_\delta(t-s)\|D_x c\|_{L^\infty(\Omega_s)}\left(1 + \frac{(\gamma_2-\gamma_1)(s)-(\gamma_2-\gamma_1)(t)}{(\gamma_2-\gamma_1)(t)}\right)\, ds\, dt \\
&\leq \int_0^T \int_{\mathbb{R}} \rho_\delta(t-s)\|D_x c\|_{L^\infty(\Omega_s)}\left(1 + C_0|s-t|\right)\, ds\, dt \\
&\leq \int_{\mathbb{R}} \int_{\mathbb{R}} \rho_\delta(t-s)\|D_x c\|_{L^\infty(\Omega_s)}\left(1 + C_0|s-t|\right)\, ds\, dt \\
&\leq \int_0^T \|D_x c\|_{L^\infty(\Omega_t)}dt + F_\delta
\end{aligned}
$$

with $F_\delta := C_0 \int_{\mathbb{R}}\int_{\mathbb{R}} \rho_\delta(t-s)\|D_x c\|_{L^\infty(\Omega_s)}|s-t|\, ds\, dt \to 0$ as $\delta \to 0^+$. Here we have used that $s \mapsto (\gamma_2-\gamma_1)(s)$ is Lipschitz continuous and bounded from below by some constant. The proof for (A.37) is similar. □

Continuation of the proof of Theorem A.16. The same calculations as in lemma A.15 and (A.35) imply that (A.17) holds for a_δ. On account of lemma A.14, $v_\delta := \hat{v}_\delta \circ \Lambda$ solves

$$
D_t v_\delta + a_\delta D_x v_\delta + c_\delta v_\delta = r_\delta, \quad v_\delta(\cdot,0) = 0. \tag{A.39}
$$

on Ω.

2. Now we will send $\delta \to 0^+$ to prove the existence of a weak solution, simultaneously getting (A.31). We would like to apply theorems A.10 and A.11 to the solutions v_δ with ν satisfying (A.32). Therefore we must check that

$$\nu > \max\{\sup_\Omega(-c_\delta), \sup_\Omega(-c_\delta - D_x a_\delta)\} \tag{A.40}$$

holds for all sufficiently small $\delta > 0$. From the definition of a_δ we compute, using (A.34),

$$\begin{aligned}(D_x a_\delta)(x,t) &= \frac{\gamma_2'(t) - \gamma_1'(t)}{\gamma_2(t) - \gamma_1(t)} + (D_y \hat{a}_\delta)(\Lambda(x,t)) \\ &= \frac{\gamma_2'(t) - \gamma_1'(t)}{\gamma_2(t) - \gamma_1(t)} + \int_{\mathbb{R}} \int_{\mathbb{R}} \rho_\delta \left(\frac{x - \gamma_1(t)}{\gamma_2(t) - \gamma_1(t)} - z \right) \rho_\delta(t-s)(D_y \hat{a})(z,s) \; ds \, dz.\end{aligned}$$

Combining this with

$$(D_y \hat{a})(y,t) = -\frac{\gamma_2'(t) - \gamma_1'(t)}{\gamma_2(t) - \gamma_1(t)} + (D_x a)(\Lambda^{-1}(y,t))$$

we obtain

$$(D_x a_\delta)(x,t) = \int_{\mathbb{R}} \int_{\mathbb{R}} \rho_\delta \left(\frac{x - \gamma_1(t)}{\gamma_2(t) - \gamma_1(t)} - z \right) \rho_\delta(t-s)(D_x a)(\Lambda^{-1}(z,s)) \; ds \, dz.$$

A similar calculation for c_δ finally yields

$$\begin{aligned}&(-c_\delta - D_x a_\delta)(x,t) \\ =& \int_{\mathbb{R}} \int_{\mathbb{R}} \rho_\delta \left(\frac{x - \gamma_1(t)}{\gamma_2(t) - \gamma_1(t)} - z \right) \rho_\delta(t-s)(-c(\Lambda^{-1}(z,s)) - (D_x a)(\Lambda^{-1}(z,s))) \; ds \, dz\end{aligned}$$

and in view of the nonnegativity of ρ_δ we have thus by (A.33):

$$(-c_\delta - D_x a_\delta)(x,t) \le \max\{\sup_\Omega(-c), \sup_\Omega(-c - (D_x a))\} < \nu \tag{A.41}$$

for all $(x,t) \in \Omega$. Analogously, we check that $\nu > -c_\delta(x,t)$. Thus we may apply theorems A.10, A.11 to get the estimates

$$\begin{aligned} e^{-\nu t} \|v_\delta(\cdot,t)\|_{L^\infty(\Omega_t)} &\le \|\!|r_\delta|\!\|_\nu \qquad &(A.42)\\ e^{-\nu t} \|D_x v_\delta(\cdot,t)\|_{L^\infty(\Omega_t)} &\le \|\!|D_x r_\delta|\!\|_\nu + \|\!|(D_x c_\delta) v_\delta|\!\|. \qquad &(A.43)\end{aligned}$$

From these bounds we obviously obtain, using (A.37), (A.38),

$$\|v_\delta\|_{L^\infty(\Omega)}, \|D_x v_\delta\|_{L^\infty(\Omega)} \le C$$

with C independent of $\delta > 0$. From the last bounds we also get

$$\|D_t v_\delta\|_{L^\infty(\Omega)} = \|r_\delta - a_\delta D_x v_\delta - c_\delta v_\delta\|_{L^\infty(\Omega)} \leq C;$$

thus, upon passing to a suitable subsequence, we may assume

$$v_\delta \to v \quad \text{in } \ L^\infty(\Omega), v_\delta \overset{*}{\rightharpoonup} v \quad \text{in } \ W^{1,\infty}(\Omega) \tag{A.44}$$

for some $v \in V^1(\Omega)$. Since we have $r_\delta \to r, a_\delta \to a, c_\delta \to c$ in $L^1(\Omega)$, v is easily shown to be a weak solution of the hyperbolic equation. Moreover by (A.36) and (A.18) we get

$$\|v_\delta\|_\nu \leq \|r_\delta\|_\nu \leq K_\delta \|r\|.$$

Thus, from (A.42), we obtain the first inequality of (A.31) by sending $\delta \to 0^+$. The second inequality of (A.31) is proven similarly, using (A.37), (A.38).

3. It remains to show that $v(\cdot, 0) = 0$. But since $v_\delta(\cdot, 0) = 0$, this follows directly from the convergence statement (A.44). Thus, v is a weak solution to the problem (A.7). That the solution is unique follows by applying theorem A.4. □

Theorem A.18. *Let $j \geq 1$ and $a, c, r \in V^j(\Omega)$. Then for the unique weak solution of the problem* (A.7) *we have the estimates*

$$\|D_x^j v\|_{L^\infty(\Omega)} \leq C(\|a\|_{V^j}, \|c\|_{V^j}) \|r\|_{V^j(\Omega)}$$
$$\|D_t D_x^{j-1} v\|_{L^\infty(\Omega)} \leq C(\|a\|_{V^j}, \|c\|_{V^j}) \|r\|_{V^j(\Omega)}.$$

Proof. We build a sequence of approximative solutions v_δ as in the proof of theorem A.16. From [2], corollary 2 in 6.1 we take the following lemma: for any real interval (x_1, x_2) there exists a linear and bounded extension operator

$$T_{x_1,x_2} : W^{j,\infty}(x_1, x_2) \to W^{j,\infty}(\mathbb{R}), \ \ T_{x_1,x_2} w|_{(x_1,x_2)} = w \quad (w \in W^{j,\infty}(x_1, x_2))$$

where the norm of T_{x_1,x_2} is bounded by a constant independent of x_1, x_2. We define an extension of a (denoted by the same letter) by

$$a(\cdot, t) := T_{\gamma_1(t),\gamma_2(t)} a(\cdot, t) \in W^{j,\infty}(\Omega_t) \quad (t \in (0, T)).$$

It follows that the extended a is in $V^j(S)$ and $\|D_x^k a\|_{L^\infty(\mathbb{R}\times(0,T))} \leq C\|a\|_{V^k(\Omega)} (0 \leq k \leq j)$. Doing the same with c and r, we define $\hat{a}, \hat{c}, \hat{r}$ from the extended functions a, c, r exactly as before. We produce $\hat{a}_\delta, \hat{c}_\delta, \hat{r}_\delta$ by mollifying $\hat{a}, \hat{c}, \hat{r}$. The V^j-norms of $\hat{a}_\delta, \hat{c}_\delta, \hat{r}_\delta$ are controlled in a standard way by the corresponding norms of the extended a, c and r on S and therefore by the V^j norms of a, c, r on Ω. The fact that $\hat{a}_\delta$ is outward-pointing for sufficiently small δ can be checked as before. v_δ is defined as before. The inequalities of the theorem follows by application of theorem A.12 to the functions v_δ and then sending $\delta \to 0$. □

Appendix B

Auxiliary estimations

B.1 Estimation of $r_\theta^\pm(v^\pm)$

Recall the definition of $r_\theta^\pm$ from (2.33):

$$r_\theta^\pm(v^\pm) := S_\theta[-d^\pm - R^\pm[v^\pm]\omega_x^\pm].$$

We want to estimate $r_\theta^\pm(v^\pm)$ and $D_x r_\theta^\pm(v^\pm)$ in the $L^\infty(\Omega_t^\pm)$-norm, for a fixed time t. These estimates are needed in chapter 2, where they are used in proving compactness and continuity properties of the fixed point operator $\mathbf{\Phi}_\theta$ and moreover in chapter 3, where they enter in the inequalities used to define the set D, which is mapped into itself by $\mathbf{\Phi}_\theta$.

Let $M > 0$ be a number such that we have $|\omega^\pm(x,t)| \leq M$ for all $(x,t) \in \Omega^\pm$. Let $g, m : [0,\infty) \to [0,\infty)$ be two monotone nondecreasing functions s.t.

$$\left.\begin{array}{l} |a(\omega+v) - a(\omega) - a'(\omega)v| \leq g(|v|) \\ |a'(\omega+v) - a'(\omega) - a''(\omega)v| \leq m(|v|) \end{array}\right\} \tag{B.1}$$

for all $v \in \mathbb{R}, \omega \in \mathbb{R}, |\omega| \leq M$ and such that $g(v) = o(|v|), m(v) = o(|v|)$ as $|v| \to 0^+$.

For the purposes in chapter 2 it suffices to know that such g, m exist:

Lemma B.1. *Functions g, m (depending on M) with the above property exist.*

Proof. The existence of g is a easy consequence of Taylor's formula applied to $a \in C^2$. Moreover,

$$|a'(\omega+v) - a'(\omega) - a''(\omega)v| \quad = \quad \left| \int_0^1 [a''(\tau(\omega+v) + (1-\tau)\omega) - a''(\omega)]\, v \, d\tau \right|$$

$$\leq \quad |v| \int_0^1 |a''(\tau v + \omega) - a''(\omega)| d\tau$$

Hence m has the required properties if we set

$$m(y) := y \sup\{|a''(\tau v + \omega) - a''(\omega)| : \tau \in [0,1], |v| \leq y, |\omega| \leq M\}.$$

□

Lemma B.2. *Let* $M > 0$ *and* g, m *be as in* (B.1)*. Then we have*

$$\left.\begin{array}{rcl} \|r_\theta^\pm(v^\pm)\|_\nu & \leq & \|d^\pm\|_\nu + M_1\left(\|v^\pm\|_\nu\right) \\ \|r_\theta^\pm(v^\pm)_x\|_\nu & \leq & \|d_x^\pm\|_\nu + M_2\left(\|v^\pm\|_\nu, \|v_x^\pm\|_\nu\right) \end{array}\right\}. \tag{B.2}$$

Here,

$$M_2\left(\|v^\pm\|_\nu, \|v_x^\pm\|_\nu\right) := \int_0^T \|\omega_x^\pm\|^2_{L^\infty(\Omega_\tau^\pm)} m(e^{\tau\nu}\|v^\pm\|_\nu) e^{-\nu\tau} d\tau$$
$$+\|v_x^\pm\|_\nu \int_0^T \|\omega_x^\pm\|^2_{L^\infty(\Omega_\tau^\pm)} \left(m(e^{\nu\tau}\|v^\pm\|_\nu) + e^{\nu\tau} \sup_{|y|\leq M} |a''(y)| \|v^\pm\|_\nu \right) d\tau$$
$$+ \int_0^T \|\omega_{xx}\|_{L^\infty(\Omega_\tau^\pm)} g(e^{\nu\tau}\|v^\pm\|_\nu) e^{-\nu\tau} d\tau$$

as well as

$$M_1\left(\|v^\pm\|_\nu\right) \quad := \quad \int_0^T \|\omega_x^\pm\|_{L^\infty(\Omega_\tau^\pm)} g(e^{\nu\tau}\|v^\pm\|_\nu) e^{-\nu\tau} d\tau.$$

Proof. First note that by (B.1)

$$|a'(\omega^\pm + v^\pm) - a'(\omega^\pm)| \leq m(|v^\pm|) + |v^\pm| \sup_{|y|\leq M} |a''(y)|. \tag{B.3}$$

We have

$$|R^\pm[v^\pm]| \quad = \quad |a(\omega^\pm + v^\pm) - a(\omega^\pm) - a'(\omega^\pm)v^\pm| \leq g(|v^\pm|) \tag{B.4}$$

and moreover

$$\begin{aligned} |D_x R^\pm[v^\pm]| &= |[a'(\omega^\pm + v^\pm) - a'(\omega^\pm) - a''(\omega^\pm)v^\pm]\omega_x^\pm| \\ &+ |[a'(\omega^\pm + v^\pm) - a'(\omega^\pm)]v_x^\pm| \\ &\leq +m\left(\|v^\pm\|_{L^\infty(\Omega_t^\pm)}\right) \|\omega_x^\pm\|_{L^\infty(\Omega_t^\pm)} + \end{aligned}$$

$$\left(m(\|v^{\pm}\|_{L^\infty(\Omega_t^{\pm})}) + \|v^{\pm}\|_{L^\infty(\Omega_t^{\pm})} \sup_{|y|\le M} |a''(y)|\right) \|v_x^{\pm}\|_{L^\infty(\Omega_t^{\pm})}, \tag{B.5}$$

where in the last line we applied (B.3) to estimate $a'(\omega^{\pm}+v^{\pm}) - a'(\omega^{\pm})$. It follows by (2.15) that

$$\begin{aligned} \|r_\theta^{\pm}(v^{\pm})_x\|_{L^\infty(\Omega_\tau^{\pm})} &\le \|-d_x^{\pm} - (R^{\pm}[v^{\pm}]\omega_x^{\pm})_x\|_{L^\infty(\Omega_\tau^{\pm})} \\ &\le \|d_x^{\pm}\|_{L^\infty(\Omega_\tau^{\pm})} + \|R^{\pm}[v^{\pm}]_x \omega_x^{\pm}\|_{L^\infty(\Omega_\tau^{\pm})} + \|R^{\pm}[v^{\pm}]\omega_{xx}^{\pm}\|_{L^\infty(\Omega_\tau^{\pm})} \end{aligned} \tag{B.6}$$

and thus by (B.4), (B.5)

$$\begin{aligned} \|r_\theta^{\pm}(v^{\pm})_x\|_{L^\infty(\Omega_\tau^{\pm})} &\le \|d_x^{\pm}\|_{L^\infty(\Omega_\tau^{\pm})} + \|\omega_x^{\pm}\|_{L^\infty(\Omega_\tau^{\pm})} \Big[m\left(\|v^{\pm}\|_{L^\infty(\Omega_\tau^{\pm})}\right) \|\omega_x^{\pm}\|_{L^\infty(\Omega_\tau^{\pm})} + \Big(m(\|v^{\pm}\|_{L^\infty(\Omega_\tau^{\pm})}) \\ &+ \sup_{|y|\le M} |a''(y)| \|v^{\pm}\|_{L^\infty(\Omega_\tau^{\pm})} \Big) \|v_x^{\pm}\|_{L^\infty(\Omega_\tau^{\pm})} \Big] + \|\omega_{xx}^{\pm}\|_{L^\infty(\Omega_\tau^{\pm})} g\left(\|v^{\pm}\|_{L^\infty(\Omega_\tau^{\pm})}\right). \end{aligned} \tag{B.7}$$

Moreover by (2.15), (B.4),

$$\|r_\theta^{\pm}(v^{\pm})\|_{L^\infty(\Omega_\tau^{\pm})} \le \|-d^{\pm} - R^{\pm}[v^{\pm}]\omega_x^{\pm}\|_{L^\infty(\Omega_\tau^{\pm})} \tag{B.8}$$

$$\le \|d^{\pm}\|_{L^\infty(\Omega_\tau^{\pm})} + g\left(\|v^{\pm}\|_{L^\infty(\Omega_\tau^{\pm})}\right) \|\omega_x^{\pm}\|_{L^\infty(\Omega_\tau^{\pm})} \tag{B.9}$$

Multiplying with $e^{-\nu\tau}$ and integrating the above expressions over $(0,T)$, we arrive at

$$\|r_\theta^{\pm}(v^{\pm})_x\|_\nu \le \|d_x^{\pm}\|_\nu + M_2\left(\|v^{\pm}\|_\nu, \|v_x^{\pm}\|_\nu\right) \tag{B.10}$$

$$\|r_\theta^{\pm}(v^{\pm})\|_\nu \le \|d^{\pm}\|_\nu + M_1\left(\|v^{\pm}\|_\nu\right) \tag{B.11}$$

where M_1, M_2 are as in the statement of the lemma.

We note here that also the following estimate can be derived. Not by integrating over $(0,T)$, but by taking the supremum of (B.9) and (B.7) over $(0,T)$, we get

$$\left.\begin{aligned} \|r_\theta^{\pm}(v^{\pm})_x\|_{L^\infty(\Omega^{\pm})} &\le \|d_x^{\pm}\|_\nu + G_2\left(\|v^{\pm}\|_{L^\infty(\Omega^{\pm})}, \|v_x^{\pm}\|_{L^\infty(\Omega^{\pm})}\right) \\ \|r_\theta^{\pm}(v^{\pm})\|_{L^\infty(\Omega^{\pm})} &\le \|d^{\pm}\|_\nu + G_1\left(\|v^{\pm}\|_{L^\infty(\Omega^{\pm})}\right) \end{aligned}\right\} \tag{B.12}$$

where G_1, G_2 are easily calculated (the explicit form is not needed in this work). □

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