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Interpolation of curves using variational subdivision surfaces

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Abstract. We introduce a variational method to interpolate or approximate given curves on a surface by subdivision within a given tolerance. In every subdivision step some new points are determined by the interpolation constraint and the other new points are varied by solving an optimization problem to minimize some surface energies such that the resulting surfaces have high quality. This method is simple to implement and can flexibly meet different quality requests.

Keywords: variational subdivision scheme, interpolation, fair surfaces, surface reconstruction

1 Introduction

Interpolation of space curves by fair surfaces often appears in practical applications for computer graphics and geometric modeling. For example, in product shape design we usually have some predefined curves such as *feature lines* from which an interpolating surface is to be reconstructed. In bio-medical modeling we often want to reconstruct an anatomical structure from *contour lines* obtained by e. g. CT scans. The general problem is how to reconstruct a surface that interpolates or approximates some arbitrary space curves which lie in principal on an arbitrary underlying surface. The reconstructed surfaces should have high quality, i. e., they are *fair surfaces* which minimize certain surface energies [20].

The interpolation problem is a fundamental problem in geometric design where smooth shapes under interpolation conditions are represented with mathematical models.

Since the given curves and the underlying surface could have arbitrary topology, subdivision schemes [5, 3, 17, 6, 25, 11] are appropriate to reconstruct the surfaces with arbitrary topology. Furthermore, subdivision schemes provide an efficient multiresolution representation [26, 12] of subdivision surfaces, i. e., during the proceeding of subdivision every subdivided polygonal mesh is an approximation of the limit surface and looks *smoother* than the previous one and thus, a subdivided polygonal mesh after only a few subdivision steps is mostly already appropriate for shape design in practice.

The usual interpolation subdivision schemes, e. g., the butterfly scheme [6, 25] and the Nasri's modified Doo-Sabin scheme [18], generate limit surfaces interpolating given points. These subdivision schemes are *stationary*, which means that the vertices of a subdivided mesh are computed by applying fixed affine combinations on the vertices of the old mesh. With this property, stationary schemes are inadequate to generate surfaces interpolating freeform curves.

It is well-known that the Lane-Riesenfeld subdivision algorithm [13] generates splines, which minimize some energies, e. g., an univariate spline function s of degree $2m+1$ with single knots $x_0(=a) < x_1 < \ldots < x_n(=b)$ has the minimum property

$$\int_a^b [f^{(m+1)}(x)]^2 dx \geq \int_a^b [s^{(m+1)}(x)]^2 dx$$

under the constraints

- $f \in C^m[a,\ b]$ and $f^{(m+1)}$ is quadratically integrable over $[a,\ b]$,
- $f(x_i) = s(x_i),\ i = 0, \ldots, n$,
- $s^{(i)}(a) = f^{(i)}(a),\ s^{(i)}(b) = f^{(i)}(b),\ i = 1, \ldots, m$, and
- $s^{(i)}(a) = s^{(i)}(b) = 0,\ i = m+1, \ldots, 2m$,

and the equality in the inequality only holds for $f = s$ over $[a,\ b]$, see [21] and for $m = 1$ also [2, P. 125]. Hence, Lane-Riesenfeld subdivision surfaces and most usual subdivision surfaces on regular part can be considered as the solution of some variational problems. Using variational technique we can extend the class of stationary subdivision schemes to a larger class of *variational subdivision schemes*, which are well-defined on both regular and irregular meshes, see an example of the *umbrella scheme* [9, Chp. 5.1] and the variational subdivision approach generating C^k interpolating curves [10].

In this paper we introduce a nonstationary subdivision scheme using variational technique in every subdivision step to interpolate given curves on a surface. To the best of our knowledge this is the first solution to the interpolation problem of arbitrary curves using a variational subdivision scheme.

2 Previous work

The *combined subdivision schemes* [14, 15] modify the Catmull-Clark scheme to interpolate some given curves by G^1 subdivision surfaces, where the subdivision rules are changed in the neighborhood of the interpolated curves in every subdivision step. The interpolated curves are given in a parametric representation and the changed rules depend on this parametrization, e. g., the second derivative of the curves, which leads to complicated calculations. Another limitation of the combined subdivision is that at most two curves could intersect at a point and, in addition to the interpolated curves, a control mesh should be given.

Based on the Doo-Sabin scheme, Nasri developed another subdivision scheme [19] to interpolate curves, which also needs to change the subdivision rules in the neighborhood of the interpolated curves.

Another modified Catmull-Clark scheme [24] can be used to interpolate cubic B-spline curves where the control mesh must have certain symmetric properties on both sides of the control polygon edges of the interpolated curves.

Besides the above subdivision approaches, Ju, Liu and others [8, 16] reconstructed interpolating surfaces from curves on parallel or non-parallel cross-section planes. Their works focused on segmenting the underlying object by medial axes that are determined by the given cross-section planes.

In this paper we consider a more general problem that the interpolated curves are freeform curves which lie in principal on an arbitrary underlying surface.

3 A variational subdivision scheme to interpolation of curves

3.1 Basic idea

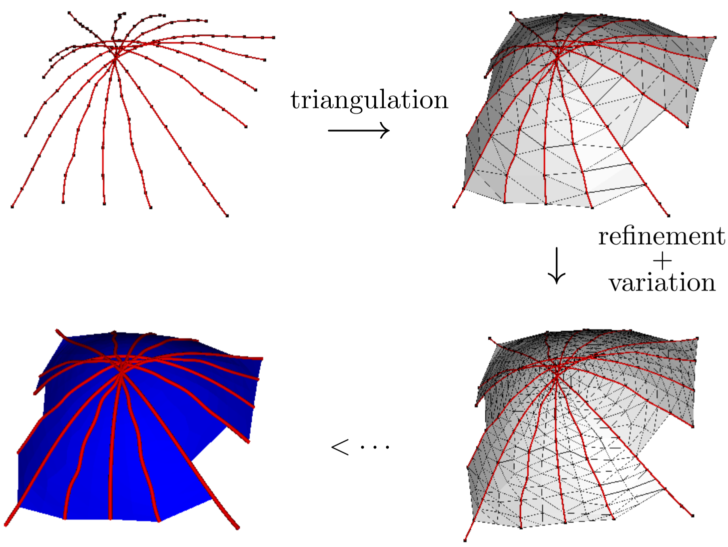

Fig. 1. From the given curves (top left) to the interpolating surface (bottom left).

The basic idea of the variational subdivision scheme is seen in Fig. 1. First, we choose some points from the given curves and construct an initial triangular mesh with these points as vertices. Second, in every subdivision step, the mesh is uniformly refined and new vertices are varied to meet the interpolating constraint and to minimize some surface energy by solving a variational problem. The solution of the variational problem is obtained by solving the associated Euler-Lagrange equation.

3.2 The umbrella energy

For a given triangular mesh, the *umbralla* of degree 1 of an inner vertex $\mathbf{p}$ is defined by

$$\Delta\mathbf{p} := \mathbf{p} - \frac{1}{n}\sum_{i=0}^{n-1}\mathbf{q}_i \ ,$$

see Fig. 2, where $\mathbf{q}_0, \ldots, \mathbf{q}_{n-1}$ are the neighbors of $\mathbf{p}$ which are connected with $\mathbf{p}$ by an edge.

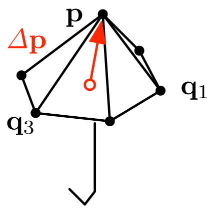

Fig. 2. The umbrella of degree 1 of $\mathbf{p}$.

The *umbrella* of degree 1 of a boundary vertex is defined as $\mathbf{0}$.

Iteratively, the *umbrella* of higher degree k is defined by the k-ring neighbors of an inner vertex

$$\Delta^k\mathbf{p} := \Delta^{k-1}\mathbf{p} - \frac{1}{n}\sum_{i=0}^{n-1}\Delta^{k-1}\mathbf{q}_i, \quad k \geq 2,$$

where the *k-ring neighbors* of $\mathbf{p}$ are the set of the vertices that are connected with $\mathbf{p}$ by a path consisting maximal k edges.

The *umbrella* of degree k of a boundary vertex is defined as $\mathbf{0}$.

Theorem 1 (Approximation property of the umbrella operator)

(i) For an arbitrary mesh, the umbrella operator of degree 1 *is a discrete approximation of the Laplace operator*

$$L\mathbf{f}(x,y) = \mathbf{f}_{xx}(x,y) + \mathbf{f}_{yy}(x,y) \ ,$$

where $\mathbf{f}$ *is a parametrization of the limit surface in the neighborhood of the vertex* $\mathbf{p}$.

(ii) For a regular triangular or quadrilateral mesh, the umbrella operator of higher degree k is a discrete approximation of the k-th Laplace operator

$$L^k\mathbf{f}(x,y) = L(L^{k-1}\mathbf{f}(x,y)), \quad k \geq 2 \ .$$

Proof: Let $\mathbf{f}$ be a local parametrization of the limit surface such that

$$\mathbf{p} = \mathbf{f}(0,0) \text{ and } \mathbf{q}_j = \mathbf{f}\left(h\cos\left(\frac{2\pi j}{n}\right), h\sin\left(\frac{2\pi j}{n}\right)\right),\ j = 0,\dots,n-1\ ,$$

where h is a sufficiently small number.

Suppose that $\mathbf{f}$ is sufficiently differentiable. Using the definition of $\Delta\mathbf{p}$ and the Taylor series with second degree Taylor polynomial

$$\mathbf{f}(x,y) = \mathbf{f}(0,0) + [x\ \ y]\begin{bmatrix}\mathbf{f}_x(0,0)\\ \mathbf{f}_y(0,0)\end{bmatrix} + \frac{1}{2}[x^2\ \ 2xy\ \ y^2]\begin{bmatrix}\mathbf{f}_{xx}(0,0)\\ \mathbf{f}_{xy}(0,0)\\ \mathbf{f}_{yy}(0,0)\end{bmatrix} + O(h^3) \tag{1}$$

we get

$$\Delta\mathbf{p} = -\frac{h^2}{4}L\mathbf{f}(0,0) + O(h^3)\ ,$$

by which we derive (i).

For a regular triangular or quadrilateral mesh, every inner vertex has valence $n = 6$ or $n = 4$, respectively. We first consider degree $k = 2$ and the 2-ring neighbors of $\mathbf{p}$. We can parametrize the limit surface by $\mathbf{f}$ such that

$$\mathbf{p} = \mathbf{f}(0,0), \quad \mathbf{q}_j = \mathbf{f}\left(h\cos\left(\frac{2\pi j}{n}\right), h\sin\left(\frac{2\pi j}{n}\right)\right),\ j = 0,\dots,n-1\ ,$$

and for $j,k = 0,\dots,n-1$

$$\mathbf{r}_{j,k} = \mathbf{f}\left(h\cos\left(\frac{2\pi j}{n}\right) + h\cos\left(\frac{2\pi k}{n}\right), h\sin\left(\frac{2\pi j}{n}\right) + h\sin\left(\frac{2\pi k}{n}\right)\right),$$

where h is a sufficiently small number and $\mathbf{r}_{j,0},\dots,\mathbf{r}_{j,n-1}$ are the neighbors of $\mathbf{q}_j$. Again, using the Taylor series with 4-th degree Taylor polynomial we get

$$\begin{aligned}\Delta\mathbf{p} = &-\frac{h^2}{4}(\mathbf{f}_{xx}(0,0) + \mathbf{f}_{yy}(0,0))\\ &-\frac{h^4}{64}(\mathbf{f}_{xxxx}(0,0) + 2\mathbf{f}_{xxyy}(0,0) + \mathbf{f}_{yyyy}(0,0)) + O(h^5)\end{aligned} \tag{2}$$

and

$$\begin{aligned}\Delta\mathbf{q}_j = &-\frac{h^2}{4}(\mathbf{f}_{xx}(c_j,s_j) + \mathbf{f}_{yy}(c_j,s_j))\\ &-\frac{h^4}{64}(\mathbf{f}_{xxxx}(c_j,s_j) + 2\mathbf{f}_{xxyy}(c_j,s_j) + \mathbf{f}_{yyyy}(c_j,s_j)) + O(h^5)\ ,\end{aligned} \tag{3}$$

where $c_j := h\cos(2\pi j/n), s_j := h\sin(2\pi j/n)$. From (2), (3) and (1) it follows

$$\begin{aligned}\Delta^2\mathbf{p} &= \frac{h^4}{16}(\mathbf{f}_{xxxx}(0,0) + 2\mathbf{f}_{xxyy}(0,0) + \mathbf{f}_{yyyy}(0,0)) + O(h^5)\\ &= \frac{h^4}{16}L^2\mathbf{f}(0,0) + O(h^5)\ .\end{aligned}$$

Similarly, we can show (ii) for $k \geq 3$. □

For a mesh, the *umbrella energy* of degree k is defined by the sum of the squared umbrellas of degree k of all vertices in the mesh, e. g., the umbrella energy of degree 1 and 2 are

$$E_1 := \sum_i \|\Delta \mathbf{p}_i\|_2^2 \quad \text{and} \quad E_2 := \sum_i \|\Delta^2 \mathbf{p}_i\|_2^2 \; .$$

Additionally, we define the *mixed umbrella energy*

$$E_{1.5} := (E_1 + E_2)/2 \; .$$

According to Theorem 1, the umbrella energy of a mesh is a discrete approximation for the Laplace energy of the continuous limit surface. By minimizing the umbrella energy in every subdivision step, Kobbelt developed the so-called *umbrella subdivision scheme* [9, Chp. 5.1], which is a variational nonstationary scheme.

Since the umbrella energy of each degree is a quadratic energy, one can solve this optimization problem by its Euler-Lagrange system, which is sparse and linear. For example, for the umbrella energy of degree 1, it holds

$$\begin{aligned} E_1 = \min &\Leftrightarrow \partial E_1(\mathbf{p}_1, \mathbf{p}_2, \ldots)/\partial \mathbf{p}_i = \mathbf{0} \quad \text{for all modifiable } \mathbf{p}_i \\ &\Leftrightarrow 2(\Delta \mathbf{p}_i - \textstyle\sum_{j=0}^{n_i-1} \frac{1}{n_j} \Delta \mathbf{q}_j) = \mathbf{0} \quad \text{for all modifiable } \mathbf{p}_i, \\ &\qquad \text{where } \mathbf{q}_0, \ldots, \mathbf{q}_{n_i-1} \text{ are the neighbors of } \mathbf{p}_i. \end{aligned} \tag{4}$$

This sparse linear systems can be efficiently solved, e. g. by the *Gauss-Seidel method*.

3.3 Algorithm

Algorithm 1: A variational subdivision algorithm

Input: Some arbitrary space curves which lie principally on a surface
Output: An interpolating surface

[Step 1] Construct an initial triangulation from the curves ;
repeat
  [Step 2] Uniformly refine the old triangular mesh ;
  [Step 3] Move the new *curve vertices* to the curves ;
  [Step 4] Fair the triangular mesh ;
until *the mesh is fine enough*;

To construct an initial triangulation in [Step 1], some equally spaced points on every curve are chosen, where the distance between two neighboring points is controlled by user through a parameter. All these points are marked as *curve vertices* and, for each curve vertex, the associated given curves are also recorded. With respect to these points we generate a triangulation using the well-known *Ball-Pivoting algorithm* [1] or the *modified Delaunay algorithm* [4, Chp. 9]. For the modified Delaunay algorithm, the points are projected onto a plane and the triangulation is first generated for the transformed points in the plane using Delaunay algorithm and then, the outputted triangulated topology is used for the original (non-transformed) points. Thus, the modified Delaunay algorithm works only for the curves that can be almost uniformly projected onto a plane.

In each subdivision step, the mesh is first uniformly refined by adding edge midpoints and thus, each old triangle is subdivided uniformly into four small triangles [Step 2]. If the both endpoints of an old edge are curve vertices and are associated to a same curve, then the new edge midpoint will be marked as a curve vertex too and the associated curve will be also recorded. Otherwise, the new edge midpoint will be marked as a *non-curve vertex*.

In [Step 3] the new curve vertices are modified to meet the interpolation constraint. Every new curve vertex $\mathbf{p}$ is moved to a nearest point $\mathbf{p}'$ on the associated curve and the neighboring new vertices $\mathbf{q}_i$ are correspondingly moved to $\mathbf{q}_i'$, see Fig. 3, where $\mathbf{d} = \mathbf{p}' - \mathbf{p}$ and f is a real value function with values between 0 and 1 depending on the distance between $\mathbf{p}$ and $\mathbf{q}_i$.

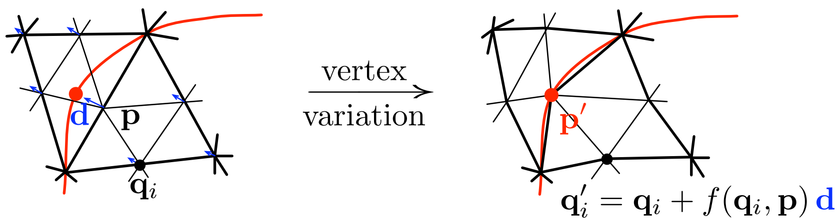

Fig. 3. Moving of a curve vertex $\mathbf{p}$ and its neighboring new vertices $\mathbf{q}_i$, such that the new position $\mathbf{p}'$ lies on the associated given curve.

The umbrella energy E_1 or $E_{1.5}$ is chosen to be the fairness energy of the mesh and this energy is minimized in [Step 4] by varying new non-curve vertices to the solution of a Euler-Lagrange system such as (4). Note that, if the new curve vertices are also allowed to be varied in the Euler-Lagrange system within a given tolerance to the associated curves, then we get an *approximation scheme*, which generates an approximation surface with better fairness. See examples in Section 4.

This algorithm is an interpolating scheme that the vertices of the initial mesh lie in the limit surface. Due to [Step 3] and [Step 4], the limit surface interpolates the given curves and each subdivided mesh is varied to obtain the minimum umbrella energy. If the new curve vertices are also allowed to be varied within a given tolerance in [Step 4], then the scheme is approximating.

To solve the variational problem in [Step 4], the Gauss-Seidel method is applied. The convergence of this iterative method is still an open problem. However, for the surfaces we used as test data, the method always converges.

4 Evaluation and examples

In this section we show some examples (Fig. 4 - Fig. 8) of the variational subdivision algorithm. To evaluate the reconstructed surfaces, we consider the widely used *thin plate energy*

$$\int_{\Omega} a(\kappa_1^2 + \kappa_2^2) + 2(1-b)\kappa_1\kappa_2 \, d\omega \ ,$$

where κ_1 and κ_2 are the principal curvatures of the surface and a and b are constants describing properties of the material. We calculate the principal curvatures at a vertex $\mathbf{p}$ according to Taubin's discrete approximation method [22], which approximates a 3×3 symmetric matrix $M_{\mathbf{p}}$ defined by the integral formula

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \kappa_n^{\mathbf{p}}(\mathbf{T}_\theta) \, \mathbf{T}_\theta \, \mathbf{T}_\theta^t \, d\theta \ ,$$

where $\mathbf{T}_\theta$ is the unit length tangent vector and $\kappa_n^{\mathbf{p}}(\mathbf{T}_\theta)$ is the normal curvature. The principal curvatures at $\mathbf{p}$ are derived by $\kappa_1 = 3\lambda_1 - \lambda_2$ and $\kappa_2 = 3\lambda_2 - \lambda_1$, where λ_1, λ_2 and 0 are the eigenvalues of $M_{\mathbf{p}}$.

The thin plate energy in a triangle is approximated by the linear interpolation of the values on vertices. In the test we simply calculate the thin plate energy for $a = b = 1$, which coincides with the *total curvature energy*

$$\int_{\Omega} \kappa_1^2 + \kappa_2^2 \, d\omega \ .$$

If the underlying surface $\mathcal{S}$ is known, where the given curves lie exactly on, then the similarity between a reconstructed surface $\mathcal{R}$ and $\mathcal{S}$ can be checked by the *Hausdorff distance*

$$d_H(\mathcal{S}, \mathcal{R}) = \max\{\sup_{\mathbf{s}\in\mathcal{S}} \inf_{\mathbf{r}\in\mathcal{R}} \operatorname{dist}(\mathbf{s}, \mathbf{r}), \sup_{\mathbf{r}\in\mathcal{R}} \inf_{\mathbf{s}\in\mathcal{S}} \operatorname{dist}(\mathbf{s}, \mathbf{r})\} \ .$$

Since the exact Hausdorff distance is complicated and expensive to be obtained, we simply approximate the Hausdorff distance by a discrete form, which is based on the distance between the vertices of $\mathcal{R}$ and the knots of the discrete representation of $\mathcal{S}$.

In Fig. 4, two subdivided meshes and the associated given curves are shown. Each given curve relates to a chain of edges whose vertices lie on the curve. By subdivision this chain converges to the curve and thus, the curve is interpolated by the subdivision surface.

Surfaces are reconstructed from the curves in Fig. 5 using different variants of the variational subdivision scheme. From total curvature energies in Table 1 follows that approximating surfaces usually have better fairness quality than interpolating surfaces and using the umbrella energy $E_{1.5}$ one usually get fairer surfaces than using E_1.

The quality of the initial triangulation is critical for the quality of the resulting surfaces. For a underlying surface that can be almost uniformly projected

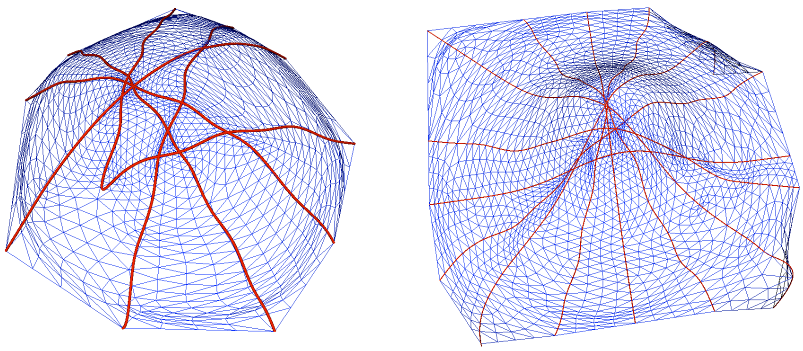

Fig. 4. Two subdivided meshes after 2 subdivision steps based on the umbrella energy $E_{1.5}$ with 5 (left) and 8 (right) given curves, respectively.

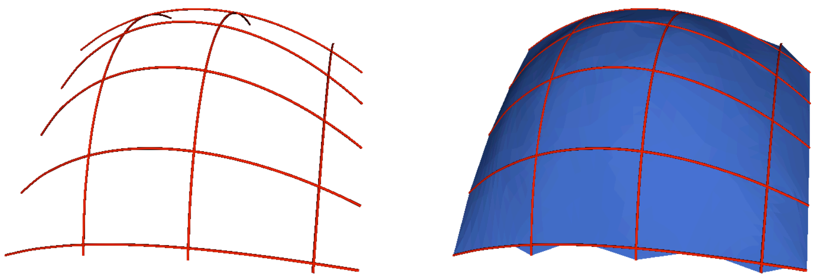

Fig. 5. 8 given curves (left) and the reconstructed surface (right) after 2 subdivision steps based on $E_{1.5}$.

Variant	Total curvature energy
interpolating surface using E_1	70.25
approximating surface with tolerance 0.1 using E_1	62.93
interpolating surface using $E_{1.5}$	68.85

Table 1. Total curvature energie of the reconstructed surfaces with the given curves in Fig. 5.

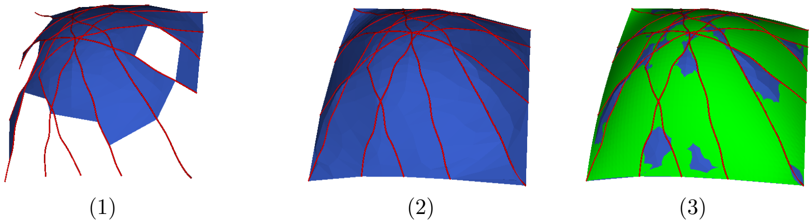

Fig. 6. A reconstructed surface using initial triangulation with the Ball-Pivoting algorithm (1), a reconstructed surface using initial triangulation with the modified Delaunay algorithm (2) and a comparison with the underlying surface in (3).

onto a plane, the triangulation by the modified Delaunay algorithm is usually better than that by the Ball-Pivoting algorithm, see Fig. 6. The Hausdorff distances of the reconstructed surfaces (1) and (2) in Fig. 6 are 0.75 and 0.23, respectively.

In Fig. 7 and 8, we show the reconstruction of a part torus and two branches from different given curves.

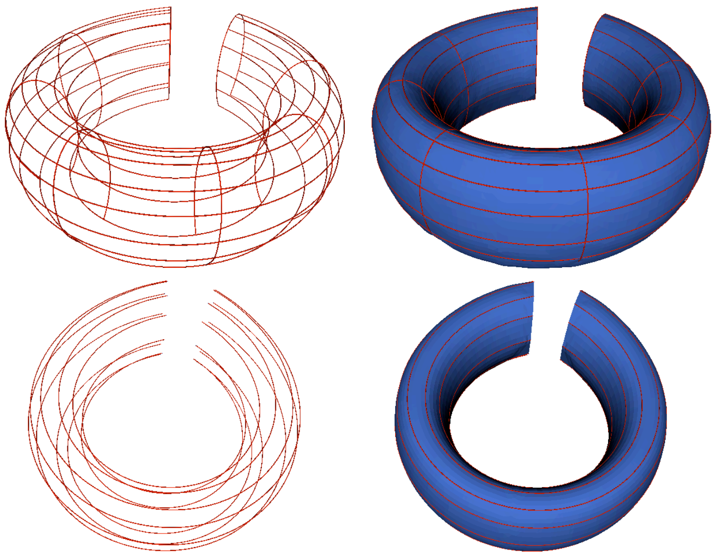

Fig. 7. Reconstructed surfaces of a part torus after 2 subdivision steps from different given curves.

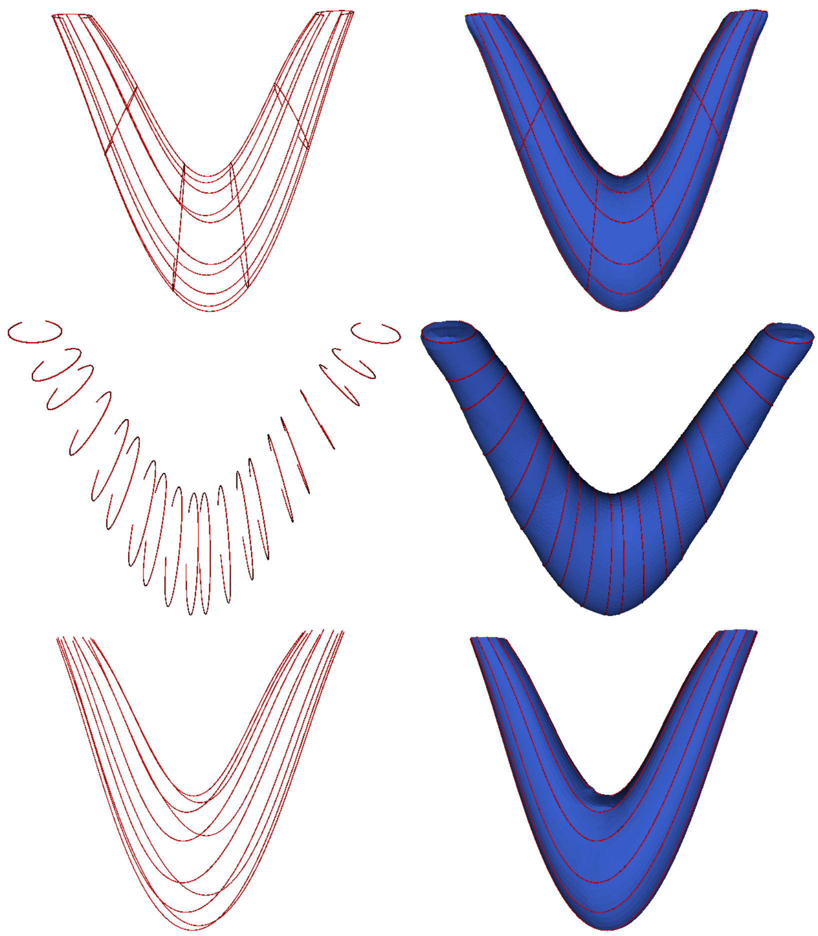

Fig. 8. Reconstructed surfaces of two branches after 2 subdivision steps from different given curves.

5 Conclusion and future work

We present a simple variational subdivision algorithm for surface reconstruction from some space curves which lie in principal on a surface with arbitrary shape and topology. The generated surfaces are guaranteed to interpolate or approximate given curves within a given tolerance. In addition, the reconstructed surfaces have high quality due to their fairness and this algorithm provides an efficient multiresolution representation of the reconstructed surfaces. To the best of our knowledge this is the first variational subdivision algorithm to solving this interpolation problem.

Some limitations and extensions of this algorithm are to be addressed in the future. First, a proper initial triangulation in [Step 1] is critical for the algorithm, see for example Fig. 6. For given curves on a complex underlying surface, the triangulation methods suggested in [Step 1] do not always work and we would like to develop convenient means for the user to select curve vertices and generate an initial triangulation automatically or half-automatically, such that the initial triangulation has the right topology and the desired initial shape. Second, in the variation phase [Step 4], other surface energies can be flexibly used as fairness energies to improve the quality of the reconstructed surfaces or to adapt for requirements in different applications [7]. For example, the Euler-Lagrange equation of the *mean-curvature energy* is a sixth-order flow and the surfaces with minimal mean-curvature energy under some constraints are visually C^2 [23]. However, some non-linear problems have to be dealt with to optimize these non-quadratic energies in the variation phase.

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