October 18, 2010, 12:00
Draft, don't distribute

DEPENDENCE PROPERTIES OF EXIT TIMES WITH APPLICATIONS TO RISK MANAGEMENT

NICOLE BÄUERLE* AND ALEXA MANGER‡

Abstract. We investigate the dependence structure of d-dimensional Itô processes which are not necessarily time-homogeneous. Sufficient conditions are given which imply that the processes are associated, i.e. show a certain kind of positive dependence. We also prove that associated processes have associated hitting times. Some applications in risk management are given.

1. Introduction

We investigate the dependence structure of d-dimensional Itô processes which are not necessarily time-homogenous. Sufficient conditions are given which imply that the processes are associated, i.e. show a certain kind of positive dependence. We also consider hitting times of the processes and prove that association of the processes implies in particular that the hitting times are associated. Hitting times are used in stochastic models to define the failure time of a system. This could be the time of the breakdown of a system, the time an unacceptable quality is achieved or the default time point of a firm. The paper is motivated by Ebrahimi (2002) who investigates questions of this type. However we use a different approach for our proofs by investigating the infinitesimal generator of the process along the lines in Liggett (1985). We specialize these results to Itô processes and prove similar statements for time-inhomogeneous processes. The results in Ebrahimi (2002) are extended to more general Itô processes where the coefficient functions may depend on time and on some of the states.

For further investigations of associated Markov processes we refer the reader to Szekli (1995). Rüschendorf (2008) uses the infinitesimal generator to work out comparison results of Markov processes on Polish spaces. For further comparison results of Lévy processes see Bäuerle et al. (2008) or Bergenthum and Rüschendorf (2007).

The paper is organized as follows: In the next section we provide the basic definitions and concepts like association and stochastic monotonicity. In Section 3 we investigate association of hitting times and in Section 4 we consider association and stochastic monotonicity of Itô processes. In the last section an application to risk management is given.

2. Dependence Concepts and Monotonicity

In this section we summarize definitions and facts about dependence concepts and monotonicity of stochastic processes. Let us start with the concept of association of

Key words and phrases. association, exit time, diffusion process, risk management.

random vectors which has been introduced by Esary et al. (1967). The association property reflects positive dependence within a random vector. It is widely used in applications and weaker than other well-known dependence concepts (see e.g. Szekli (1995), Müller and Stoyan (2002), Joe (1997)).

Definition 1. *An $\mathbb{R}^d$-valued random vector X is said to be* associated*, if*

$$\mathrm{Cov}(f(X), g(X)) \geq 0$$

for all measurable, increasing functions f, g: $\mathbb{R}^d \to \mathbb{R}$ for which the covariance exists.

From the definition it follows immediately that if X is associated we have $Cov(X_i, X_j) \geq 0$. If $X = (X_1, \dots, X_d) \sim \mathcal{N}_d\big(\mu, \Sigma = (\sigma_{ij})\big)$ has a multivariate normal distribution, then it is well-known (see Tong (1990), Theorem 5.1.1) that X is associated if and only if $\sigma_{ij} \geq 0$ for all i, j. For a random vector $X = (X_1, \dots, X_d)$ we denote by $X^\perp = (X_1^\perp, \dots, X_d^\perp)$ a random vector with same margins as X but independent components. The following properties of association will be crucial. For a proof of (a)-(d) see Esary et al. (1967) and for (e) the reader is referred to Denuit et al. (2001). Recall that for two real-valued random variables we have $X \leq_{icx} Y$ if and only if $\mathbb{E}\, f(X) \leq \mathbb{E}\, f(Y)$ for all increasing, convex $f : \mathbb{R} \to \mathbb{R}$ for which the expectations exist.

Lemma 2.
(a) *If $X = (X_1, \dots, X_d)$ is associated and $f_1, \dots, f_k$: $\mathbb{R}^d \to \mathbb{R}$ are increasing (or decreasing) functions with arbitrary $k \in \mathbb{N}$, then the random vector $(f_1(X), \dots, f_k(X))$ is associated.*
(b) *If $X_1, \dots, X_d$ are independent, then $X = (X_1, \dots, X_d)$ is associated.*
(c) *If $X = (X_1, \dots, X_d)$ and $Y = (Y_1, \dots, Y_k)$ are associated and stochastically independent, then $(X_1, \dots, X_d, Y_1, \dots, Y_k)$ is associated.*
(d) *If $\{X_n\}_{n\in\mathbb{N}}$ is a sequence of associated, $\mathbb{R}^d$-valued random vectors converging to X in distribution, then X is again associated.*
(e) *If X is associated then*

$$\sum_{i=1}^d X_i^\perp \leq_{icx} \sum_{i=1}^d X_i.$$

Association of random vectors can be extended to stochastic processes in a natural way. In what follows we assume that (X_t) is a stochastic process on a given probability space $(\Omega, \mathcal{P}, \mathbb{P})$.

Definition 3. *We say that an $\mathbb{R}^d$-valued stochastic process $(X_t)_{t\geq 0}$ is* associated *if for all $k \in \mathbb{N}$ and all times $0 \leq t_1 < \dots < t_k$ the $\mathbb{R}^{dk}$-valued random vector $\big(X(t_1), \dots, X(t_k)\big)$ is associated.*

Note that this definition implies that the process is associated in time and space. Alternatively one could only require that X_t is associated for all $t \geq 0$ (association in space) or that $(X_1(t_1), \dots, X_d(t_d))$ is associated for all time points $0 \leq t_1, \dots, t_d$. This last definition has been used in Ebrahimi (2002). Obviously if (X_t) is associated according to Definition 3, this implies that (X_t) is associated according to the other two definitions where association in space is the weakest one.

Now suppose that $(X_t)_{t\geq 0}$ is a homogeneous Markov process with values in $\mathbb{R}^d$ and with transition semigroup $(P_t)_{t\geq 0}$, i.e. for $f \in \mathcal{C}_b(\mathbb{R}^d)$, the set of bounded and continuous functions on $\mathbb{R}^d$ we have

$$P_t f(x) = \mathbb{E}_x\, f(X_t)$$

where $\mathbb{E}_x$ is the conditional expectation given $X_0 = x$. By $\mathbb{E}_{t,x}$ we denote the conditional expectation given $X_t = x$. In addition we assume that $(P_t)_{t\geq 0}$ is *Feller-continuous* which means that the operators P_t map $\mathcal{C}_b$ into itself. Let us denote by $\mathcal{F}_\uparrow^+$ the set of bounded, non-decreasing, non-negative functions $f : \mathbb{R}^d \to \mathbb{R}$. The following definition can be applied to general, not necessarily time-homogeneous processes.

Definition 4. *We say that an $\mathbb{R}^d$-valued stochastic process $(X_t)_{t\geq 0}$ is* stochastically monotone *if for all $f \in \mathcal{F}_\uparrow^+$ we have $x \mapsto \mathbb{E}_{t,x}\, f(X_{t+h}) \in \mathcal{F}_\uparrow^+$ for all $t, h \geq 0$.*

3. Association of Hitting Times

In this section we will first discuss the implications that arise for the dependence properties of hitting times when the process X is associated.

In what follows we assume that $(X_t)_{t\geq 0} = \big(X_1(t), \dots, X_d(t)\big)_{t\geq 0}$ is a stochastic process on a probability space $(\Omega, \mathcal{P}, \mathbb{P})$ with values in $\mathbb{R}^d$ and $\mathbb{P}$-a.s. continuous sample paths. For arbitrary values $a_1, \dots, a_d \in \mathbb{R}$ we define the hitting times

$$H_i := \inf\{t \geq 0 : X_i(t) < a_i\}, \quad i = 1, \dots, d$$

where $\inf \emptyset := \infty$. For the Itô processes we consider later, H_i is most often in distribution equal to $\inf\{t \geq 0 : X_i(t) \leq a_i\}$. But we avoid some technical consideration when we define H_i with a strict inequality. We obtain the following result:

Theorem 3.1. *Let $(X_t)_{t\geq 0}$ be an associated, $\mathbb{R}^d$-valued stochastic process with $\mathbb{P}$-a.s. continuous sample paths and suppose that $T > 0$ is an arbitrary time point. It holds:*

(a) *The random indicators $\big(1_{[X_1(T)<a_1]}, \dots, 1_{[X_d(T)<a_d]}\big)$ are associated.*
(b) *The random indicators $\big(1_{[\min_{0\leq t\leq T} X_1(t)<a_1]}, \dots, 1_{[\min_{0\leq t\leq T} X_d(t)<a_d]}\big)$ are associated.*
(c) *The hitting times $(H_1, \dots, H_d)$ are associated.*

Proof. (a) The assumption implies that in particular $\big(X_1(T), \dots, X_d(T)\big)$ is associated. Obviously $f_i : \mathbb{R} \to \mathbb{R}$ given by $f_i(x) := 1_{[x<a_i]}$ are decreasing functions for all i, thus according to Lemma 2 part a) the random vector $\big(1_{[X_1(T)<a_1]}, \dots, 1_{[X_d(T)<a_d]}\big)$ is associated.

(b) Let $n \in \mathbb{N}$. By assumption $\big(X(0), X(\frac{T}{2^n}), \dots, X(\frac{T(2^n-1)}{2^n}), X(T)\big)$ is associated. Moreover the functions $f_i : \mathbb{R}^{2^n+1} \to \mathbb{R}$ defined by

$$f_i\big(x(0), \dots, x(2^n)\big) := 1_{[\min_{0\leq k\leq 2^n} x(\frac{kT}{2^n})<a_i]}$$

are decreasing functions for all i, thus according to Lemma 2 part a) the random vector

$$\big(1_{[\min_{0\leq k\leq 2^n} X_1(\frac{kT}{2^n})<a_1]}, \dots, 1_{[\min_{0\leq k\leq 2^n} X_d(\frac{kT}{2^n})<a_d]}\big)$$

is associated. Now we obviously have due to the $\mathbb{P}$-a.s. continuity of the paths of X that

$$1_{[\min_{0\leq k\leq 2^n} X_i(\frac{kT}{2^n})<a_i]} \uparrow 1_{[\min_{0\leq t\leq T} X_i(t)<a_i]}, \quad \mathbb{P}-a.s. \text{ for } n \to \infty$$

With Lemma 2 part d) the statement follows.

(c) For $n \in \mathbb{N}$ and all i define

$$H_i^n := \Big(\frac{1}{2^n}\inf\{1 \le k \le n2^n : X_i(\frac{k}{2^n}) < a_i\}\Big) \wedge n.$$

Obviously H_i^n is an increasing function of

$$\Big(X_i(0), X_i(\frac{1}{2^n}), \dots, X_i(n)\Big)$$

and our assumption implies that $(H_1^n, \dots, H_d^n)$ are associated. Since

$$(H_1^n, \dots, H_d^n) \to (H_1, \dots, H_d)$$

$\mathbb{P}$-a.s. for $n \to \infty$ the statement follows.

□

4. Association of Itô Processes

Now we turn to a special class of Markov processes, the so-called Itô processes. In order to introduce them, let $B = (B_t)_{t \ge 0} = (B_1(t), \dots, B_m(t))$ be an m-dimensional Brownian motion and $\mu_i : \mathbb{R}^{d+1} \to \mathbb{R}, \sigma_{ij} : \mathbb{R}^{d+1} \to \mathbb{R}$ suitable coefficient functions. We define $\big(X_i(t)\big),\ i = 1, \dots, d$ as the solution of the stochastic differential equation

$$dX_i(t) = \mu_i(t, X_t)dt + \sum_{j=1}^m \sigma_{ij}(t, X_t)dB_j(t) \tag{4.1}$$

with $X_i(0) = x_i$. For later considerations it is reasonable to introduce $\Sigma(t,x) := \sigma(t,x)\sigma(t,x)^\top$ where $\sigma(t,x) = \big(\sigma_{ij}(t,x)\big) \in \mathbb{R}^{d \times m}$. Conditions on the coefficient functions μ_i and σ_{ij} are available in the literature such that a unique strong solution to (4.1) exists. This is for example guaranteed if there exists a constant $K \ge 0$, such that

- for all $t \ge 0$ and $x, y \in \mathbb{R}^d$:
$$|\mu(t,x) - \mu(t,y)| + |\sigma(t,x) - \sigma(t,y)| \le K\,|x - y|\,,$$
where $|\sigma|^2 = \sum |\sigma_{ij}|^2$.
- for all $t \ge 0$ and $x \in \mathbb{R}^d$:
$$|\mu(t,x)| + |\sigma(t,x)| \le K\big(1 + |x|\big).$$
- μ and σ have bounded derivatives of order one and two.

Let us denote by $\mathcal{A}$ the generator of the process (X_t) and by $D_{\mathcal{A}}$ its domain. If we denote by $\mathcal{C}^{1,2}$ the set of all functions $f : \mathbb{R}^{d+1} \to \mathbb{R}$ such that f is once continuously differentiable in the time component and twice continuously differentiable in the other components then $f \in \mathcal{C}^{1,2}$ implies that $f \in D_{\mathcal{A}}$. Note that the corresponding transition semigroup is Feller-continuous. For $f \in \mathcal{C}^{1,2}$ the infinitesimal generator of the process is given by:

$$\mathcal{A}f(t,x) = \frac{\partial}{\partial t}f(t,x) + \sum_{i=1}^d \mu_i(t,x)\frac{\partial}{\partial x_i}f(t,x) + \frac{1}{2}\sum_{i,j=1}^d \Sigma_{ij}(t,x)\frac{\partial^2}{\partial x_i \partial x_j}f(t,x). \tag{4.2}$$

When the coefficient functions μ_i and σ_{ij} depend only on x and not on t, i.e.

$$dX_i(t) = \mu_i(X_t)dt + \sum_{j=1}^m \sigma_{ij}(X_t)dB_j(t) \tag{4.3}$$

then the Itô process is homogeneous in time. In this case the generator is given by

$$\mathcal{A}f(x) = \sum_{i=1}^{d} \mu_i(x) \frac{\partial}{\partial x_i} f(x) + \frac{1}{2} \sum_{i,j=1}^{d} \Sigma_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} f(x). \tag{4.4}$$

We obtain from Theorem 2.14 in Liggett (1985) or as a corollary of Theorem 4.3 below (indeed a slightly weaker statement can be found in Herbst and Pitt (1991) where only association in space is considered):

Theorem 4.1. *Let (X_t) be a homogeneous Itô process as given in* (4.3). *Suppose (X_t) is stochastically monotone. If $\Sigma_{ij}(x) \geq 0$ for all $x \in \mathbb{R}^d$ and i, j, then (X_t) is associated.*

Note that the stochastic monotonicity is in general not necessary to obtain an associated process. The next theorem can be found in Herbst and Pitt (1991) (Theorem 1.1) and characterizes stochastic monotonicity of the homogeneous process (X_t).

Theorem 4.2 (Herbst and Pitt (1991))**.** *Let (X_t) be a homogeneous Itô process as given in* (4.3). *(X_t) is stochastically monotone if and only if $\mu_j(x)$ is increasing in $x_k, k \neq j$ and $\Sigma_{ij}(x)$ depends only on x_i, x_j.*

These conditions are quite restrictive. In particular when we turn to the time-inhomogeneous Itô process given by (4.1). This process can be interpreted as a homogeneous process $X = (X_0(t), X_1(t), \dots, X_d(t))$ by setting $X_0(t) = t$. Stochastic monotonicity would here require that $\mu_j(t, x)$ is increasing in t for all $j \neq 0$. This is however too restrictive for the association property. We obtain the following theorem:

Theorem 4.3. *Let (X_t) be an Itô process as given in* (4.1). *Suppose (X_t) is stochastically monotone. If $\Sigma_{ij}(t, x) \geq 0$ for all $x \in \mathbb{R}^d$ i, j and $t \geq 0$ then (X_t) is associated.*

The proof follows the ideas of Liggett (1985) Theorem 2.14 who proved association of general homogeneous Markov processes.

Proof. Recall that for $f \in D_{\mathcal{A}}$ we denote $P_h f(t, x) := \mathbb{E}_{t,x} f(t + h, X_{t+h})$. We suppose now that $f, g : \mathbb{R}^{d+1} \to \mathbb{R}$ are functions which do not depend on t, but only on $(x_1, \dots, x_d)$, thus we can interpret f, g as functions from $\mathbb{R}^d$ to $\mathbb{R}$. Further assume that $f, g \in D_{\mathcal{A}} \cap \mathcal{F}_{\uparrow}^{+}$. We show for $h \geq 0$ that

$$F(h) := P_h fg - \big(P_h f\big)\big(P_h g\big) \geq 0$$

which implies that X_{t+h} is associated. In order to do this consider

$$F'(h) = \mathcal{A}P_h fg - \big(\mathcal{A}P_h f\big)\big(P_h g\big) - \big(\mathcal{A}P_h g\big)\big(P_h f\big).$$

We know by assumption that $P_h f, P_h g$ are again in $D_{\mathcal{A}}$ and increasing in x. We obtain that $\mathcal{A}fg \geq f\mathcal{A}g + g\mathcal{A}f$ since

$$\begin{aligned}
& \mathcal{A}fg(t, x) \\
= \;& \sum_{i=1}^{d} \mu_i(t, x) \frac{\partial}{\partial x_i} (fg)(t, x) + \frac{1}{2} \sum_{i,j=1}^{d} \Sigma_{ij}(t, x) \frac{\partial^2}{\partial x_i \partial x_j} (fg)(t, x) \\
= \;& f(t, x)\mathcal{A}g(t, x) + g(t, x)\mathcal{A}f(t, x) + \sum_{i,j=1}^{d} \Sigma_{ij}(t, x) \frac{\partial}{\partial x_i} f(t, x) \frac{\partial}{\partial x_j} g(t, x).
\end{aligned}$$

In view of our assumption and because f and g are increasing, it holds that $\mathcal{A}fg \geq f\mathcal{A}g + g\mathcal{A}f$. This implies that

$$F'(h) \geq \mathcal{A}P_h fg - \mathcal{A}\big[\big(P_h f\big)\big(P_h g\big)\big] = \mathcal{A}F(h).$$

Hence $F'(h) = \mathcal{A}F(h) + G(h)$ for a function $G \geq 0$. A solution of this equation is given by (see Liggett (1985) Theorem 2.14):

$$F(h) = P_h F(0) + \int_0^h P_{h-s}G(s)ds$$

which implies that $F \geq 0$ since $G \geq 0$. Thus X_{t+h} is associated. By induction it follows from the Markov property that $(X(t_1), \dots, X(t_n))$ is associated for $0 \leq t_1 < \ldots < t_n$ (cf. Liggett (1985)). □

Theorem 4.4. *Let (X_t) be an Itô process as given in* (4.1). *(X_t) is stochastically monotone if and only if $\mu_j(t,x)$ is increasing in $x_k, k \neq j$ and $\Sigma_{ij}(t,x)$ depends only on x_i, x_j and t.*

Proof. We first show that stochastic monotonicity implies the stated conditions. We know that for $f \in D_{\mathcal{A}}$ with f depending only on x the Dynkin formula reads (see e.g. Øksendal and Sulem (2005), Theorem 1.23)

$$\mathbb{E}_{t,x}\, f(X_{t+h}) = f(x) + \mathbb{E}_{t,x}\Big[\int_t^{t+h} \mathcal{A}f(X_u)du\Big] \tag{4.5}$$

and for $x \leq x'$ and $f \in \mathcal{F}_{\uparrow}^{+}$ we have

$$\mathbb{E}_{t,x}\, f(X_{t+h}) \leq \mathbb{E}_{t,x'}\, f(X_{t+h}) \tag{4.6}$$

for all $t, h \geq 0$. Now choose the test function $f(y) = e^{\alpha y_i}$ with $\alpha > 0$ in an environment around the fixed x_i and extend it to a function in $D_{\mathcal{A}} \cap \mathcal{F}_{\uparrow}^{+}$. When we suppose that $x \leq x'$ and $x_i = x_i'$, then dividing (4.6) by h and letting h tend to zero we obtain $\mathcal{A}f(x) \leq \mathcal{A}f(x')$, i.e.

$$\mu_i(t,x) + \frac{1}{2}\alpha\Sigma_{ii}(t,x) \leq \mu_i(t,x') + \frac{1}{2}\alpha\Sigma_{ii}(t,x'). \tag{4.7}$$

Letting $\alpha \downarrow 0$ we obtain $\mu_i(t,x) \leq \mu_i(t,x')$. Since $x_i = x_i'$ this implies that $\mu_i(t,x)$ is increasing in x_j for $j \neq i$. Diving equation (4.7) by α and letting $\alpha \to \infty$ we obtain $\Sigma_{ii}(t,x) \leq \Sigma_{ii}(t,x')$. Using in the same way a test function which locally behaves as $f(y) = 1 - e^{-\alpha y_i}$ with $\alpha > 0$ we obtain $\Sigma_{ii}(t,x) \geq \Sigma_{ii}(t,x')$ which implies that $\Sigma_{ii}(t,x)$ can only depend on t and x_i. Finally when we suppose that $x \leq x'$and $x_i = x_i', x_j = x_j'$ where $i \neq j$ and use a test function which locally behaves as $f(x) = e^{\alpha_i x_i + \alpha_j x_j}$ with $\alpha_i, \alpha_j > 0$ we obtain $\Sigma_{ij}(t,x) \leq \Sigma_{ij}(t,x')$. Doing the same with $f(x) = 1 - e^{-\alpha_i x_i - \alpha_j x_j}$ we obtain $\Sigma_{ij}(t,x) \geq \Sigma_{ij}(t,x')$ which implies that $\Sigma_{ij}(t,x)$ can only depend on x_i, x_j and t.

Now suppose that μ_i and Σ_{ij} have the stated properties. Suppose that $x \leq x'$ and let (X_t) and (X_t') be the Itô processes given by (4.1) with $X_0 = x$ and $X_0' = x'$ respectively. We have to show that for $f : \mathbb{R}^d \to \mathbb{R}$ increasing

$$\mathbb{E}_{t,x}\, f(X_{t+h}) \leq \mathbb{E}_{t,x'}\, f(X_{t+h}'), \quad \forall t, h \geq 0. \tag{4.8}$$

It is well-known (see Müller and Stoyan (2002) Theorem 3.3.5) that this is equivalent to the fact that X_{t+h} and X_{t+h}' can be constructed on a common probability space such that $X_{t+h}(\omega) \leq X_{t+h}'(\omega)$ for all ω. Moreover, it is enough to show (4.8) for twice continuously differentiable functions f (see Müller and Stoyan (2002) Theorem

2.5.5). Thus it is sufficient to show for $x \le x'$ with $f(x) = f(x')$ that $\mathcal{A}f(x) \le \mathcal{A}f(x')$. Now let $I := \{i \in \{1,\dots,d\} : x_i = x_i'\}$. For $i \notin I$ we have $\frac{\partial}{\partial x_i} f(x) = 0$. Moreover, the fact that f is increasing implies that $\frac{\partial}{\partial x_i} f(x) = \frac{\partial}{\partial x_i} f(x')$ for $i \in I$. This can be seen since (e_i denotes the i-th unit vector)

$$\begin{aligned} \frac{f(x'+he_i)-f(x')}{h} &\ge \frac{f(x+he_i)-f(x)}{h} \\ \frac{f(x')-f(x'-he_i)}{h} &\le \frac{f(x)-f(x-he_i)}{h} \end{aligned}$$

and for $h \downarrow 0$ the statement follows. Similarly we also obtain using Taylor's formula that $\frac{\partial^2}{\partial x_i \partial x_j} f(x) = \frac{\partial^2}{\partial x_i \partial x_j} f(x')$ for $i, j \in I$. Thus, $\mathcal{A}f(x) \le \mathcal{A}f(x')$ holds if

$$\sum_{i\in I} \mu_i(t,x)\frac{\partial}{\partial x_i} f(x) + \frac{1}{2}\sum_{i,j\in I} \Sigma_{ij}(t,x)\frac{\partial^2}{\partial x_i \partial x_j} f(x) \quad \le$$

$$\le \sum_{i\in I} \mu_i(t,x')\frac{\partial}{\partial x_i} f(x') + \frac{1}{2}\sum_{i,j\in I} \Sigma_{ij}(t,x')\frac{\partial^2}{\partial x_i \partial x_j} f(x')$$

which is true due to our assumption. Note also that since f is increasing we have $\frac{\partial}{\partial x_i} f(x) \ge 0$. □

5. Applications to Risk Management

Theorem 4.3 and 4.4 immediately imply that the Itô processes given by

$$dS_i(t) = S_i(t)\Big(\mu_i(t)dt + \sum_{j=1}^{d} \sigma_{ij}(t)dB_j(t)\Big)$$

for $i = 1,\dots,d$ with $S_i(0) = s_i(0)$ and deterministic coefficients $\mu_i(t), \sigma_{ij}(t)$ are associated if $\Sigma(t) = \sigma(t)\sigma(t)^\top \ge 0$. The processes $(S_1,\dots,S_d)$ give the price evolution of d risky assets in the Black-Scholes model. In the classical firm value model of Merton it is assumed that S_i gives the evolution of the total market value of firm i. The default of firm i occurs when its market value falls below the value of its debt or a given (deterministic) threshold a_i. Thus, default can be described by

$$D_i = \{S_i(T) < a_i\}$$

where $T > 0$ is the maturity of the debts. Alternatively default may also be described by

$$D_i = \{\min_{0\le t\le T} S_i(t) < a_i\}.$$

The exposure of firm i is a random variable $L_i, i = 1,\dots,d$ which is supposed to be independent of the evolution of the asset values and independent of $L_j, j \ne i$. Hence the total loss of the credit portfolio is given by

$$L = \sum_{i=1}^{d} L_i 1_{D_i}.$$

From Theorem 3.1 it follows that $(1_{D_1},\dots,1_{D_d})$ are associated and since L_i are independent also $(L_1 1_{D_1},\dots,L_d 1_{D_d})$ is associated. Now suppose $L^\perp = \sum_{i=1}^{d} L_i 1_{D_i^\perp}$ where $D_i^\perp$ indicates that the asset prices evolve independent, i.e. we consider the same model with independent asset processes $S_1,\dots S_d$. Using Lemma 2 part (e) we obtain $L^{\perp} \leq_{icx} L$. If the probability measure of the underlying probability space is non-atomic, this implies for all monotone, convex and law-invariant risk measures ρ that $\rho(L^{\perp}) \leq \rho(L)$ (see Theorem 4.4. in Bäuerle and Müller (2006)). In particular the *Average-value-at-risk* of X at level $\gamma \in (0,1)$ given by

$$AVaR_{\gamma}(X) = \frac{1}{1-\gamma}\int_{\gamma}^{1} VaR_u(X)du$$

where the Value-at-Risk VaR_{γ} is the smallest γ-quantile of $-X$, satisfies these condition. Thus, the risk measured by a monotone, convex and law-invariant risk measure increases when the firms show some kind of dependence in terms of association compared to the independent case.

Remark 5.1. (a) This application has also been considered in Bäuerle (2002). The result in the present paper extends the findings in Bäuerle (2002) since the underlying processes and the risk measures may be more general.

(b) In Ebrahimi (2002) two-dimensional Itô processes of the type

$$dX_i(t) = \mu_i(t)dt + \sum_{j=1}^{m} \sigma_{ij}(t)dB_j(t) \tag{5.1}$$

where the coefficients are only time-dependent are considered. Under further restrictive conditions it is shown that the exit times are PQD (positively quadrant dependent) which is implied by association (see Barlow and Proschan (1981) Theorem 4.2). Thus Theorem 4.3 together with Theorem 3.1 part (c) implies the results in Ebrahimi (2002).

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(N. Bäuerle) Institute for Stochastics, Karlsruhe Institute of Technology, D-76128 Karlsruhe, Germany

E-mail address: nicole.baeuerle@kit.edu

(A. Manger) Department of Mathematics, University of Duisburg-Essen, D-45141 Essen, Germany

E-mail address: alexa.manger@uni-due.de