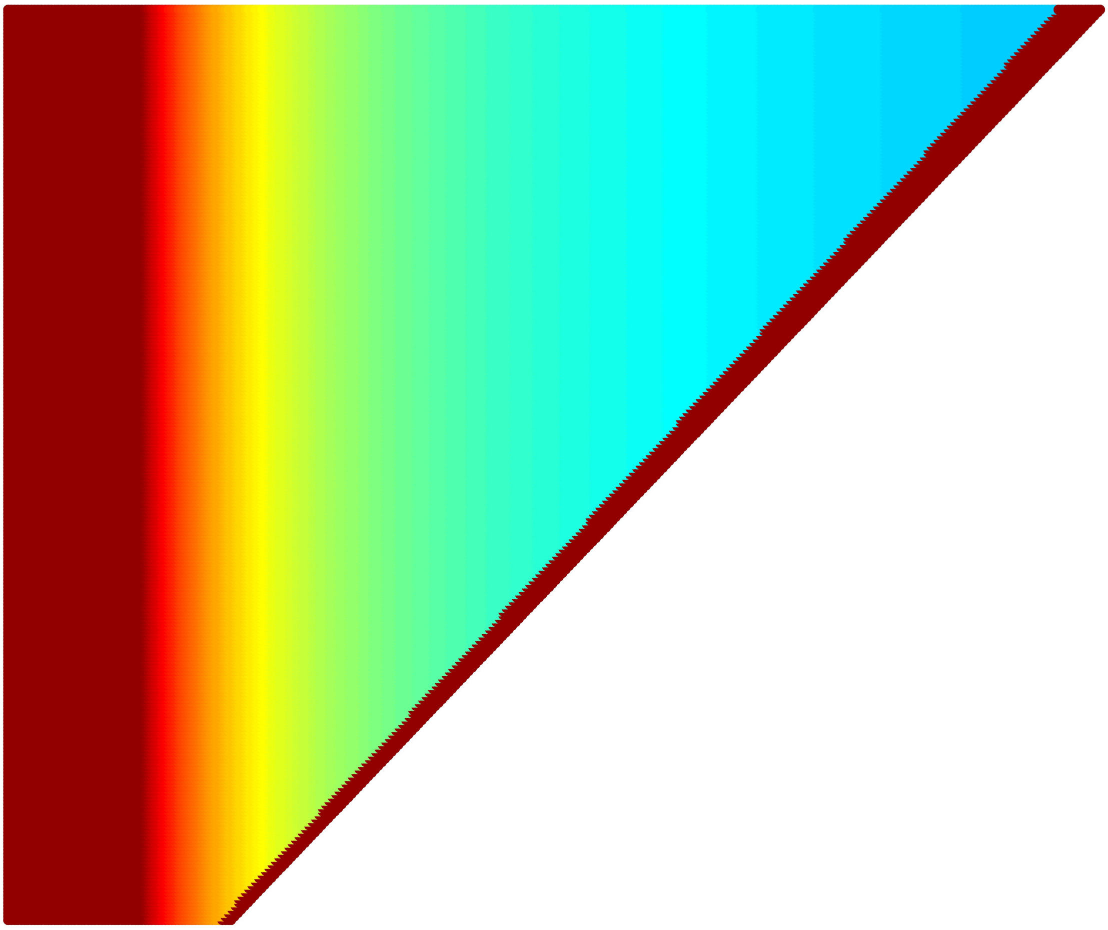
Sebastian P. Urban

Compensation, Incentives and Risk-Taking in Principal-Agent Models



Compensation, Incentives and Risk-Taking in Principal-Agent Models

Zur Erlangung des akademischen Grades eines

DOKTORS DER NATURWISSENSCHAFTEN

von der Fakultät für Mathematik des
Karlsruher Instituts für Technologie (KIT)
genehmigte

DISSERTATION

von

Dipl.-Math. Sebastian Philipp Urban
aus Ludwigshafen am Rhein

Tag der mündlichen Prüfung: 19. Dezember 2012

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To my family

Acknowledgements

My most sincere and heartfelt gratitude belongs to Dr. Luitgard Veraart, who offered me the opportunity to work on this project and took me under her caring and thorough supervision. In these years she has never stopped encouraging me and safely guided me to the successful conclusion.

I am particularly indebted to Prof. Dr. Nicole Bäuerle for taking me under her wings when the continuation of my work was at stake, and for later offering me a position in her work group at the KIT when the project funding expired. Additionally, she provided the second official evaluation of this thesis and stood by with good advice.

The major part of my work was carried out within the research project lead by Dr. L. Veraart on "Risikomanagement in Finanzmärkten und optimale Anreizsysteme" which was funded by the Ministry of Science, Research and Arts, Baden-Württemberg, Germany as part of the "Juniorprofessoren-Programms". This support is gratefully acknowledged.

I will miss my friends and colleagues from the Institute for Stochastics. Thank you for almost 4 very enjoyable years and good luck to you all!

Karlsruhe, February 2013
Sebastian Urban

Contents

1 Introduction

The idea of a principal-agent setup is to let two parties with distinct goals interact in a financial market. The principal provides some initial capital and employs the agent to invest it in the market and to subsequently control the portfolio in his name.

A wage schedule, that is a stream of payments or a terminal salary or possibly a combination of both, is agreed to compensate the agent for her actions and reward or punish the development of the managed process.

Given a specific contract and equipped with some personal utility the agent chooses her investment strategy to maximize her own expected aggregated utility of wages without considering the principal. The principal on the other hand can influence the agent's expected wage by selecting the wage schedule. In this interaction the principal now tries to induce the agent to behave in a specific way by picking a suitable compensation scheme and setting the appropriate incentives.

For example, the principal may be concerned with the risk of the agent's portfolio strategy and might prefer investment policies that leave at least half of the capital in riskless positions. Rather than issuing and enforcing that constraint he will search for a wage schedule such that the agent's optimal strategy automatically respects it.

It is of particular interest how the choice of a specific wage schedule incites the agent in her actions and how the principal can transfer his own attitude towards risk to the agent.

1.1 Problem Description

We consider a financial market with one riskless and one risky asset and finite time horizon $T > 0$.

The principal provides some capital $x_0 > 0$ which is invested in the market and then controlled by the agent with a portfolio strategy π, yielding the portfolio wealth process X. As additional quantity the running maximum of portfolio wealth is recorded in the so-called high-water mark process X^*.

A contract between both parties specifies intermediate wages ψ_t^π and a terminal wage Ψ_T^π, payable by the principal, to compensate the agent for her actions. All wages may depend on the evolution of wealth and high-water mark up to the time of payment, and the time-point itself. As wealth and high-water mark are influenced by the agent's control, the wages indirectly depend on π, too.

The agent's personal risk preference is expressed via the utility U, which she applies to all wage payments she collects.

In a time-continuous market the agent aims to maximize the expected utility of wages

$$\mathbb{E}\left[U\left(\Psi_T^\pi\right) + \int_0^T U\left(\psi_t^\pi\right) dt\right]$$

by choosing an optimal strategy $\pi^* = (\pi^*(X(t), X^*(t), t))_{0\le t\le T}$. If $[0, T]$ is discretized to n periods with trading times $\{0 = t_0, t_1, \ldots, t_{n-1}, t_n = T\}$ the target reads as

$$\mathbb{E}\left[U\left(\Psi_{t_n}^{\pi}\right) + \sum_{i=1}^{n} U\left(\psi_{t_i}^{\pi}\right)\right]$$

and a discrete optimal strategy $\pi^* = (\pi^*(X(t_i), X^*(t_i), t_i))_{i=1,\ldots,n}$ is sought.

We investigate what kind of contract is suitable to transfer the principal's attitude towards risk to the agent: Given a compensation scheme, what strategy will the agent consider optimal? And what influence does the agent's personal utility have on that strategy? Further, what kind of contract sets the right incentives and motivates the agent to take only those risks the principal complies with?

1.2 Literature Review

Before discussing optimal contracts, we initially have to clarify what risk means in mathematical terms. One will consider a venture risky, if its outcome is subject to some randomness. In the language of stochastic this corresponds to random variables which are evaluated by some real-valued function, referred to as the risk measure (c.f. Artzner, Delbaen, Eber & Heath (1999) and Föllmer & Schied (2002) for the notion of coherent and convex risk measures or Föllmer & Schied (2011) for an excellent overview).

In the portfolio optimization branch of financial mathematics one is interested in deriving optimal strategies for portfolio management problems. Offered the opportunity to continuously consume parts of the managed portfolio and given some utility preference, the consumption problem is to find a strategy maximizing the expected utility of the aggregated consumption. If the market admits a finite horizon, the terminal utility problem is to maximize the expected utility of terminal wealth, instead. Both may be combined and there are lots of variations, of course. Bingham & Kiesel (2004) provide a detailed introduction to the latter as well as to financial market modeling.

Portfolio Optimization with Prescribed Risk Constraints

In a first step one may investigate what optimal strategies arise in terminal utility or consumption problems when the agent is additionally constrained to respect bounds on the risk measure of the assets she manages.

A lot of research has been done on this topic. Probably one of the most cited papers is by Basak & Shapiro (2001) in which they solve the terminal utility problem with bounded Value at Risk as well as the problem with Limited Expected Loss in a Black-Scholes market. A dynamically applied Value at Risk constrained is considered by Yiu (2004). A variation of Limited Expected Losses is to account for all losses excessing the Value-at-Risk. For this as well as for a bounded shortfall probability Gabih, Grecksch & Wunderlich (2005) offer an analytical solution along with a numerical study. Emmer, Klüppelberg & Korn (2001) give a solution for the Bounded Capital at Risk case and in the same work as above, Föllmer & Schied (2002) discuss that problem for the convex Bounded Shortfall risk measure.

More generally, Rogers (2009) demonstrates how terminal utility problems incorporating a fixed risk constraint can be related to solution-equivalent problems without that constraint by modifying the utility function, as long as the initial constraint is given as bound on a law-invariant and coherent risk measure.

The Average Value at Risk case is treated by Bäuerle & Mundt (2009) in a binomial market. De Giorgi (2002) states a linear program to solve for the latter case in a time-continuous market when the distribution of terminal wealth is known. A demonstration on real market data is included, but no closed-form solution is provided. In an infinite time setting Bayraktar & Young (2008) discuss optimal behavior when the probability of a life-time ruin is bounded.

Gandy (2005) reconsiders the Value-at-Risk, Expected Shortfall, and Conditional Value-at-Risk problems in a Black-Scholes market under the risk-neutral measure. Extending the focus to insurance mathematics, Artzner (1999) applies risk control with coherent measures to control problems in insurance.

In all the above setups the agent is, by definition, virtually forced to respect a bound on the risk implied by her actions. No risk management is applied, yet.

Inciting by Bonus Contracts

For the second step consider problem formulations that do not prescribe a fixed risk constraint but instead try to influence the agent by providing incentives.

Leaving the established ground of Inada[1] utilities, the reasonable asymptotic elasticity criterion of Kramkov & Schachermayer (1999) provides a sufficient condition to use a dual approach to solve the optimization problem. Their results are applied by Bichuch & Sturm (2012) to discuss existence and uniqueness of solutions to non-concave terminal utility problems. They impose mild conditions on the agent's utility and the principal's compensation scheme and do not require completeness of the market. He & Zhou (2011) suggest another technique to attack terminal utility and some variations of goal-reaching problems in complete markets. They propose to change the point of view from the usual dynamic problems to a static approach of maximizing over quantiles, similar to well-known martingale techniques, and solve three classes of goal-reaching problems.

Browne (1999) discusses the simplest case in which the agent is paid one monetary unit if and only if the terminal wealth under his control attains a predefined level. The agent's utility is replaced by a compensation or wage schedule, in this case the discontinuous indicator function. Refining his initial suggestion, Browne (2000) offers an infinite-time problem setup, in which the agent is continuously rewarded until she beats a reference portfolio or falls short of it. Strategies may be chosen to maximize to probability of goal-reaching before shortfall or to minimize the waiting time until the goal is reached the first time.

In another line of research, Carpenter (2000) investigates the agent's attitude towards risk when her reward consists of Call options on the managed assets. Against previously common folklore and intuition (see e.g. DeFusco, Johnson & Zorn (1990)) it turns out that offering Call options does not necessarily lower the agent's risk aversion. More abstractly, Ross (2004) gives a characterization of the change in risk aversion implied when moving the agent from plainly collecting her private utility to collecting the utility of a compensation scheme. He names the three independently adjustable effects of translation, scaling and convexity of the schedule which together sign responsible for the complete change in risk aversion and behavior.

Various works have been carried out in settings with continuously instead of just terminally consuming agents. Very notable is the setup of Heinricher & Stockbridge (1991) in which the continuous reward process may depend on the running maximum process of the managed asset. The authors give conditions of optimality in a very general

[1] Compare definition 2.1.

diffusion model. Panageas & Westerfield (2009) exploit these results for a high-water mark compensation scheme which, aside of a continuous wage payment proportional to the value of the managed asset, rewards the agent whenever the asset value exceeds the recorded running maximum.

Risk Transfer in Principal-Agent Models

Step three is to add a principal to the considerations, who employs the agent by contracting her to a wage schedule or compensation scheme. The agent's assignment is to trade the principal's capital in the market while getting paid for her observable actions, efforts and results or, depending on the setup, to receive a terminal compensation for her aggregated efforts. All previously discussed (and more) wage schedules may be used to compensate the agent, and the principal may additionally be subject to such (or other) risk constraints as introduced above. We now deal with a two-fold optimization problem: the agent's utility-of-expected-wage optimization problem is nested into the principal's choose-the-best-wage problem.

A general and quantitative idea of incentive structures implemented in major US companies give Aggarwal & Samwick (2003). Given detailed data on the salary and compensation of all United States S&P 500, MidCap and SmallCap high-ranking managers for one year, they range the bonus paid to the CEO per net increase in the firm value around 3 per cent. This is about half of the total incentives offered to the complete management team.

Stole (1999) gives a complete overview over so-called static models that incorporate one discrete period only. In these setups, both parties bring along their personal preference and the agent must accept whatever contract the principal issues. Every choice of a contract implies a predictable optimal behavior of the agent, by which the principal tries to delegate his own risk constraints. The static model is extendable to multiple periods or continuous-time. In one of the latter, Chen & Pelger (2012) let the principal offer one of four stock options and punish the observable portfolio variance. In each case they discuss the optimal strategies for both parties.

In a finite Black-Scholes like setup incorporating an effort-optimizing agent with exponential utility Ou-Yang (2003) are able to name the unique solution. Ju & Wan (2012) allow path dependency in their non-negative and continuous compensation schemes and use backward stochastic differential equations to characterize solutions for the bounded and unbounded case. They restrict themselves to a risk-neutral principal and CRRA type agents in a finite-horizon market.

Adopting the high-water mark scheme to the principal-agent setup, Guasoni & Obłój (2011) additionally continuously compensate the agent with a fraction of the value of the managed assets and discuss which fraction the principal should choose. Finally, adding a new type of contract to the field, Rogers (2009) introduces his so-called robust contracts that are implicitly defined by perfectly aligning the principal's and agent's utility.

Principal-Agent Models with Hidden Information

As fourth and final step incorporate partial unobservability. Whether or not all parameters are known can have a severe effect on the optimal strategy. For instance Bäuerle, Urban & Veraart (2012) show that in a discrete-time approximation of a continuous-time market the optimal strategies no longer converge if partial information is incorporated.

In so-called asymmetric information setups the principal cannot monitor the agent's course of action and efforts or is unaware of the agent's private preference. A brief survey on several variations of partial information models can be found in Stole (1999). Starks (1987) considers bonus contracts similar to those the before-mentioned Browne (2000) analyzes, but includes unobservability of the agent's actions, and compares to linear compensation schemes rewarding the over-performance of a reference portfolio. Regarding the mean-variance problem, she concludes the latter should be preferred.

Holmstrom & Milgrom (1987) ask for the functional form of optimal continuous compensation contracts in single- and multi-period asymmetric information setups. For both cases they find the optimal contracts to be linear functions of all aggregated observable information variables. Related work is performed by Williams (2004), naming deterministic partial differential equation characterizations of optimal contracts in a broadly general setup additionally offering a private and unobservable bank account to the agent's disposal. A recursive algorithm to approximate solutions is suggested. Edmans & Gabaix (2011) demonstrate how to explicitly derive closed-form optimal contracts in time-discrete as well as in time-continuous markets, when restricting the interaction model to agents reporting the sum of their action and some noise which needs to be independent of action.

Focusing on a Black-Scholes type market with infinite time horizon, Sannikov (2008) lets the agent continuously receive a mixture of long- and short-term incentives. The agent will resign if his participation constraint is no longer satisfied, and may be fired as well as promoted. Assuming exponential preference in both parties, the author discusses properties of optimal compensation strategies.

Stressing the links to game theory, Zhang & Zenois (2008) suggest a multi-period expansion of the static setup wherein only the agent is aware of the state of a Markov decision process and is paid for her observable actions. The authors show that, if offered a properly designed set of compensation options at each step, the agent can be incited to gradually reveal the true state.

1.3 Main Contributions & Scope of this Thesis

In this thesis we first evaluate the implications of contracting the agent to bonus wage schedules in continuous time. These compensation schemes are typically discontinuous, non-differentiable, or partially concave and partially convex. We discuss how solution candidates for the associated non-standard portfolio problems can be verified. We also line out how to approximate these wages by functions with more convenient properties.

To examine portfolio problems with arbitrary wage schedules numerically we expand the well-known Markov chain approximation technique in two directions: It shall incorporate one-step transitions to freely configurable neighborhoods and it shall work on arbitrarily spaced grids.

Then we extend the market model by incorporating the high-water mark as contractable quantity. Throughout the high-water mark literature the principal-agent interaction is embedded into a continuous-time framework with infinite or finite, but random, time horizon.

The incentives of intermediately compensating the agent with a share in the increase of the high-water mark in these setups induce a finite and time-independent optimal strategy that is closely related to the optimal choice for an investor with power utility. In that case the high-water mark does not promote unbounded risk-taking (c.f. the discussion of known results in section 5.1).

However, in practice no contract is entered for an infinite time. On the contrary, typical businesses which offer this kind of compensation explicitly limit the period of agreement to a few years.

When a finite time horizon is stochastically approximated by a random termination time with finite expectation and vanishing variance, the agent's optimal strategy diverges when termination is approached. But no results are available for a deterministically finite time horizon.

With this thesis we contribute to the existing theory by, for the first time, considering high-water mark portfolio problems in a time-discrete model with finite time horizon. We analytically and numerically examine several portfolio problems in this context and derive the following main results:

In one-period models we identify the presence of a share in terminal wealth as sufficient property for the agent's optimal strategy to be bounded (c.f. theorem 5.6, conclusion 5.7, theorem 5.10, and conclusion 5.11).

Even when the agent is not intrinsically motivated to respect bounds on her strategy, the principal's and agent's goals can, regardless of their preferences, be aligned in an economical balance, if we assume an additional participation constraint for the agent and a non-conflicting risk-constraint for the principal (c.f. theorem 5.8 and conclusion 5.9).

Getting back to the conjecture of the agent taking unbounded risks when termination is approached:

We can confirm this behavior for an agent with shifted log preference that is paid an intermediate share of the high-water mark (c.f. (5.56) and the subsequent discussion).

But we also find that one can in the same situation arrive at an, in the limit of approximation, bounded optimal strategy by adding another share of terminal wealth to the agent's wage schedule (c.f. conclusion 5.14).

1.4 Outline

This thesis is organized as follows:

The general diffusion market model, some common notation, and the prototype portfolio problem are introduced in chapter 2.

In chapter 3 bonus wages and the arising non-standard portfolio problems are introduced. It is discussed how solution candidates for portfolio problems with discontinuous boundary conditions can be verified. Also the approximation of discontinuous wages by a sequence of continuous yet non-concave and non-convex functions is suggested and a modified martingale approach to the associated portfolio problems is sketched. Finally an adaption of SAHARA type utility functions to incorporate non-zero thresholds is elaborated and the terminal SAHARA utility problem is solved.

Chapter 4 employs a Markov chain approximation of arbitrary wage schedules. Locally consistent schemes that incorporate transitions of up to 2 steps up or down are developed and specifically applied to Black-Scholes markets. Extensions of the latter which cover even larger neighborhoods and arbitrarily spaced grids are suggested.

High-water marks are introduced in chapter 5 as a new quantity agents may be contracted to. A time-continuous setup with finite horizon and an approximation of the latter in discrete time are presented. Analytical solutions for one-period setups and special cases of multi-period problems are derived and analyzed for their incitations on the agent and their implications on risk-behavior. A computational method for solving high-water mark portfolio problems is constructed and applied to various multi-period cases. Finally the incentives set by intermediately rewarding an agent are extracted and compared to known results in continuous time without a finite horizon.

Chapter 6 summarizes all results and concludes.

2 The Market

This chapter provides the common market model for the analysis of bonus type wage schedules in chapter 3 and the construction of Markov chain approximation schemes in chapter 4.

2.1 A General Diffusion Model

Let $(\Omega, \mathfrak{F}, \mathbb{P})$ a probability space and $(B(t))_{t\geq 0}$ a $\mathbb{P}$-Brownian motion with filtration $\mathcal{F} := (\mathcal{F}(t))_{t\geq 0}$ that already contains all $\mathbb{P}$-zero sets and respects $\bigvee_{t\geq 0} \mathcal{F}(t) \subseteq \mathfrak{F}$. We consider in general a time-continuous financial market with finite horizon $T > 0$ that includes at least one risk-free and one risky asset. Using some strategy $u = (u(x,t))_{t\geq 0}$ that is admissible in that market (in a yet to define sense), we assume the wealth evolves as the Itô process

$$dX^u(t) = f(X^u(t), t, u(X^u(t), t))dt + g(X^u(t), t, u(X^u(t), t))dB(t), \;\; 0 \leq t \leq T \tag{2.1}$$

with suitable functions $f, g : \mathbb{R}^+ \times [0, T) \times U \to \mathbb{R}^+$. Wealth starts with $X^u(0) \equiv X(0) = x_0 > 0$ where the initial endowment is positive and independent of the investor's control.

To facilitate readability we will in the following refrain from explicitly referring to the dependence of X on the control u and simply write $X(t)$ whenever there is no danger of ambiguity.

2.2 The Time- and State-Continuous Portfolio Problem

When it comes to their attitude towards risk, there are many types of investors. One way to model this attitude is to equip the investor with a preference, expressed via the utility function U. As the term utility function is understood slightly different by individual authors, let us agree on the following:

Definition 2.1 *A* ***utility function*** *is a strictly increasing, strictly concave, and twice continuously differentiable mapping* $U : \mathbb{R} \to \mathbb{R}$ *or* $U : (0, \infty) \to \mathbb{R}$.
$U : (0, \infty) \to \mathbb{R}$ *is said to exhibit the* ***Inada*** *property, if*

$$\lim_{y \to 0+} U'(y) = \infty \quad \textit{and} \quad \lim_{y \to \infty} U'(y) = 0.$$

The Inada property means that a gain in the good under evaluation is (in the limit) valued infinitely high when its reserves deplete and does not improve the value of an already infinitely large supply.

Let us consider general stochastic optimization problems of type

$$\begin{aligned} V(x,t) &:= \sup_{u \in \mathcal{U}(x,t)} \{J(x,t;u)\}, \qquad (2.2) \\ J(x,t;u) &:= \mathbb{E}\left[\int_t^T \psi(X(s), s, u(X(s), s))ds + \Psi(X(T)) \,\middle|\, X_t = x \right], \end{aligned}$$

where ψ serves as a running utility or wage and Ψ denotes the terminal utility or wage[1] achieved at $t = T$. Given capital x at time t, $J(x, t; u)$ is the reward earned when further applying the strategy u and $V(x, t)$ is the value of the situation. As we allow for a state-, time- and control-dependent running utility, one may easily include a discounting function or factor, if desired.

The investor's aim is to find some optimal strategy u^* yielding the maximum expected utility, i.e. to solve

$$\mathbb{E}\left[\int_0^T \psi(X(s), s, u(X(s), s))ds + \Psi(X(T))\right] \xrightarrow{u^*} \max . \tag{2.3}$$

A solution u^* of (2.3) translates to the problem (2.2) as it satisfies $J(x_0, 0; u^*) = V(x_0, 0)$.

We can transfer the portfolio problem to a deterministic setting using the HJB decomposition. Standard techniques (e.g. (Kushner & Dupuis, 2001, Ch. 12.1)) yield the problem formulation

$$\begin{cases} \sup_{u\in\mathcal{U}} \{\mathcal{A}^u v(x,t) + \psi(x,t,u)\} = 0, & (x,t) \in Q, \\ v(x,t) = \Psi(x), & (x,t) \in \partial^* Q. \end{cases} \tag{2.4}$$

Here the set $Q := \mathbb{R}^+ \times [0, T)$ denotes the control region, composed of the open set $\mathbb{R}^+$ for wealth states and the half-open time-domain $[0, T)$ over which control is executed. Depending on the problem it may be appropriate[2] to consider only a subset of Q. The terminal condition needs to hold at the boundary with respect of time, i.e. at $\partial^* Q := [0, \infty] \times \{T\}$. Note that $\partial^* Q \subset \partial Q$ where $\partial Q = \overline{Q} \setminus Q^o$ refers to the usual topological boundary.

$\mathcal{A}^u$ is the stochastic counterpart of the Laplace operator:

$$\mathcal{A}^u := v(x,t) \mapsto \frac{\partial}{\partial t}v(x,t) + f(x,t,u)\frac{\partial}{\partial x}v(x,t) + \frac{1}{2}\sigma(x,t,u)^2\frac{\partial^2}{(\partial x)^2}v(x,t). \tag{2.5}$$

The deterministic $\mathcal{U} \subseteq \mathbb{R}$ is the (in terms of intersection) smallest set with

$$\forall\ (x,t) \in Q,\ \forall\ (u(x,s))_{0\le s\le t} \in \mathcal{U}(x,t),\ \text{and}\ \forall\ 0 \le s \le t : u(x,s)(\Omega) \subseteq \mathcal{U}.$$

2.3 The Black-Scholes Case

We will in particular investigate portfolio problems in a classical Black-Scholes model, i.e. in the special case of a one-dimensional Black-Scholes market with constant interest rate, drift, and volatility parameters. Available for trading in this market is one riskless bond S^0 and one risky asset S exhibiting

$$\begin{aligned} dS^0(t) &= rS^0(t)dt,\ 0 \le t \le T,\ S^0(0) > 0, \\ dS(t) &= S(t)\left(\mu dt + \sigma dB(t)\right),\ 0 \le t \le T,\ S(0) > 0. \end{aligned}$$

As usual choose the time horizon $T > 0$, a rate $r > 0$, some drift $\mu \in \mathbb{R}$, and a volatility $\sigma > 0$. This market is arbitrage-free and complete, compare for instance (Bingham &

[1] At this point no specific properties of Ψ and ψ are assumed. We will later in more detail discriminate between the utility (in the sense of definition 2.1) and the wage schedule and replace Ψ with $U \circ \Psi$ and ψ with $U \circ \psi$.

[2] In section 3.2 we have examples for bounded rectangular sets Q and time-dependent wealth-domains.

Kiesel, 2004, Chapter 6.2.1 and particularly Theorem 6.2.2), and the unique risk-neutral measure $\mathbb{Q}$ satisfies

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}(T)} = L(T) := \exp\left(-\vartheta B(T) - \frac{1}{2}\vartheta^2 T\right) \tag{2.6}$$

in the Radon-Nikodym sense. $\vartheta := \frac{\mu - r}{\sigma}$ denotes the so-called market price of risk. Replacing T with t in (2.6) yields the exponential change-of-measure martingale $(L(t))_{t\geq 0}$. And discounting the latter gives

$$\zeta(t) := \frac{L(t)}{S^0(t)} = \exp\left(-rt - \vartheta B(t) - \frac{1}{2}\vartheta^2 t\right), \tag{2.7}$$

the state-price-density. There are (at least) two common setups how an investor may control the evolution of wealth.

Controlling the Fraction of Wealth in the Risky Asset

Given some $(\mathcal{F}_t)$-adapted and progressively measurable investment strategy $(\pi(t))$ denoting the fraction of wealth to invest in the risky asset, an initial capital $x_0 > 0$ evolves according to

$$dX^\pi(t) = X^\pi(t)\left(\left(\pi(t)\left(\mu - r\right) + r\right)dt + \pi(t)\sigma dB(t)\right), \; X(0) = x_0. \tag{2.8}$$

Here the control is $u = \pi$, $f = (x,t,\pi) \mapsto x(\pi(\mu - r) + r)$, and $g = (x,t,\pi) \mapsto x\pi\sigma$. Applying the formula of Itô and Doeblin immediately yields the solution

$$X^\pi(t) = x_0 \exp\left(rt + \int_0^t \left(\pi(s)\left(\mu - r\right) - \frac{1}{2}\sigma^2\pi(s)^2\right)ds + \int_0^t \sigma\pi(s)dB(s)\right) \tag{2.9}$$

for the wealth at time t. We additionally require $\int_0^t \pi(s)^2 ds < \infty$ $\mathbb{P}$-a.s. $\forall t$ to ensure the stochastic integral in (2.9) is a local martingale.

As we have a finite time horizon and the wealth is by definition strictly positive, control is required over $Q := (0,\infty) \times [0,T)$ with boundary region $\partial^* Q := (0,\infty) \times \{T\}$. The set $\mathcal{U}(x,t)$ of all control processes which are feasible at time t when the intermediate capital $X(t) = x$ is available for trading, is

$$\mathcal{U}(x,t) := \left\{ (\pi(X^\pi(s),s))_{t\leq s<T} \;\middle|\; \begin{array}{l} \mathcal{F}\text{-adapted, progressively measurable,} \\ \text{self-financing, and } \mathbb{P}\text{-a.s. } \forall t < t' \leq T \\ \int_t^{t'} |\varphi_0(X^\pi(t),t)|\, ds < \infty \text{ as well as} \\ \int_t^{t'} \left(\varphi(X^\pi(t),t)S(t)\right)^2 ds < \infty \end{array} \right\}, \text{ where} \tag{2.10}$$

$$\begin{aligned} \varphi_0(X^\pi(t),t) &:= (1 - \pi(X^\pi(t),t))\frac{X^\pi(t)}{S^{(0)}(t)} \quad \text{and} \\ \varphi(X^\pi(t),t) &:= \pi(X^\pi(t),t)\frac{X^\pi(t)}{S(t)} \end{aligned}$$

denote the number of bonds and stock, respectively, held at time t.

Controlling the Amount of Wealth in the Risky Asset

If alternatively the investor chooses to control her wealth by specifying the amounts $(\theta(X^\theta(t),t))$ of money in the risky position, it holds

$$\begin{aligned} dX^\theta(t) &= [\theta(X^\theta(t),t)(\mu-r)+X(t)r]dt+[\theta(X^\theta(t),t)\sigma]dB(t), \\ X^\theta(t) &= e^{rt}\left(x_0+(\mu-r)\int_0^t \theta(X^\theta(s),s)e^{-rs}ds+\sigma\int_0^t \theta(X^\theta(s),s)e^{-rs}dB(s)\right). \end{aligned} \tag{2.11}$$

Now $u=\theta$, $f=(x,t,\theta)\mapsto\theta(\mu-r)+xr$, and $g=(x,t,\theta)\mapsto\theta\sigma$. This type of control allows negative wealth, thus require $\theta(X^\theta(t),t)^2\leq K^2\left(1+\left(X^\theta(t)\right)^2\right)$ for some $K>0$ and all $t\in[0,T]$. Hence, one may invest $|\theta(X^\theta(t),t)|\leq K$ even when the wealth drops to zero, but is (in the limit) bound to invest no more than the proportion K of his wealth.

We have to ensure that no more capital is initially invested in stock than the investor has been equipped with. Thus we require $\theta(x_0,0)\leq x_0$. As any strategy must globally fulfill this property, we define the set of initially admissible strategies as

$$\begin{aligned} \mathcal{U}(x,t) := \Bigg\{ (\theta(X^\theta(t),t))_{0\leq t<T} \;\Bigg|\; & \mathcal{F}\text{-adapted, progressively measurable,} \\ & \text{self-financing, } \theta(x_0,0)\leq x_0, \text{ and } \mathbb{P}\text{-a.s.} \\ & \int_t^{t'} |\varphi_0(X^\pi(t),t)|\,ds<\infty \text{ as well as } \exists K>0 \text{ s.t.} \\ & \theta(X^\theta(t),t)^2\leq K^2(1+(X^\theta(t))^2)\ \forall t\in[0,T)\Bigg\}, \end{aligned} \tag{2.12}$$

where in this case the number of bonds held at t is

$$\varphi_0(X^\theta(t),t)=\frac{X^\theta(t)-\theta(X^\theta(t),t)}{S^{(0)}(t)}.$$

3 Analysis of Bonus Type Wage Schedules

In systems of Principal-Agent interaction it is desirable to consider a broad class of wages and compensation schedules. Such schemes in practice often set performance goals and reward reaching these with bonus payouts of cash, company stocks or option packages. Translated to a mathematical model one has to consider discontinuities in the wage schedule.

Additionally, incentive schemes are often convex in rewarding the agent in order to encourage her to perform better. Hence deriving the utility of these in general yields non-concave, possibly not everywhere differentiable or even discontinuous functions. The associated optimization problems do not fit in the standard context required for martingale arguments or techniques such as the HJB approach.

This chapter evaluates extensions of the latter and introduces some new ideas how to examine these non-standard optimization problems. The theoretical results are accompanied and illustrated by example applications.

3.1 HJB Decomposition with Discontinuous Boundaries

The HJB methodology provides a well-understood translation of portfolio problems to problems of solving deterministic partial differential equations. However, considering discontinuous utilities or wages yields discontinuous boundaries of the control region in the decomposition. For instance Bäuerle & Rieder (2004) and Browne (1999) demonstrate the solution of a basic case, but no general technique to solve or at least verify solution candidates is known. Both articles rely on results from van Mellaert & Dorato (1972) and Fleming & Rishel (1975) who also provide the basis for the following discussion.

Let us reconsider the portfolio problem (2.2) in a Black-Scholes market with portfolios controlled by fractions of wealth and for the moment abstain from a running utility. In this context and when writing partial derivatives as subindices (e.g. v_{xx} is the second partial derivative with respect to x), the HJB setup translates to

$$\begin{cases} \sup_{\pi \in \mathcal{U}} \big\{ v_t(x,t) + x(\pi(\mu - r) + r) v_x(x,t) + \\ \qquad\qquad \frac{1}{2} x^2 \pi^2 \sigma^2 v_{xx}(x,t) \big\} = 0, & (x,t) \in Q, \\ v(x,t) = \Psi(x,t), & (x,t) \in \partial^* Q, \end{cases} \tag{3.1}$$

where $Q = (0,\infty) \times [0,T)$, $\partial^* Q = (0,\infty) \times \{T\}$ reflect the finite time horizon and strictly positive, finite wealth.

If we assume v is convex in π, we can determine the unique global maximizer by differentiating v with respect to π and solving for zero. In this case one finds the abstract optimal

$$\pi^* = -\frac{\mu - r}{\sigma^2} \cdot \frac{v_x(x,t)}{x v_{xx}(x,t)}. \tag{3.2}$$

Plugging it into (3.1), we find the necessary condition

$$v_t(x,t) + rxv_x(x,t) - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 \frac{v_x(x,t)^2}{v_{xx}(x,t)} = 0,\ (x,t) \in Q. \tag{3.3}$$

How to approach the latter problem heavily depends on what terminal condition Ψ is required. Educated guessing a separation ansatz for $v(x,t)$ can often simplify (3.3) to (a set of) ordinary differential equations:

Logarithmic boundary conditions for instance motivate the additive $v(x,t) = \log(x) + f(t)$ which yields

$$f_t(t) + r + \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 = 0.$$

It is easily solved (recall the boundary condition $f_t(T) = 0$) and provides the value function

$$v(x,t) = \log(x) + \left(r + \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2\right)(T - t),\ (x,t) \in Q.$$

From v we can with (3.2) deduce the constant optimal strategy

$$\pi^* = \pi^*(x,t) \equiv \frac{\mu - r}{\sigma^2},$$

which is well-known as Merton ratio. We will later frequently rediscover the discretized version of the Merton ratio as part of the solution to various portfolio problems.

Another popular try (working e.g. for power utility) is the multiplicative $v(x,t) = f(t)g(x)$, producing

$$\begin{aligned} f_t(t) &= cf(t),\ c \in \mathbb{R}, \\ g_{xx}(x) &= \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 \frac{(g_x(x))^2}{dg(x) + xrg_x(x)},\ d \in \mathbb{R}. \end{aligned}$$

The first ODE means $f(t) = \exp(ct)$ for some $c \in \mathbb{R}$, while there is no obvious solution to the second equation. Note that in many cases the actual solution V may not separate at all (an example for the latter is discussed by the end of the next section). In that case candidates cannot be derived at all by schematically varying the ansatz.

Whenever (and howsoever) an approach yields a candidate value function v, that candidate still needs to be verified. In particular it needs to comply with the boundary condition. Here a serious issue arises: Any candidate v derived by an exploitation of (3.3) is necessarily continuous on Q. Hence it cannot be continued (in a continuous manner) to match a discontinuous boundary condition.

In the next section we will discuss why and in what sense continuous candidates for v can indeed solve the HJB problem problem (3.1) and, interpreted as a stochastic functional, also solve the original portfolio problem (2.2).

3.2 Verification with Discontinuous Boundaries

When some candidate value function has been named, it is still subject to verification. The arguments used for verification are often quite subtly adapted to the specific portfolio problem.

A Standard Verification Theorem for Continuous Problems

Let us start with the basic type of verification theorem. For portfolio problems with continuous boundaries it suffices to provide some regularity properties:

Theorem 3.1 (Fleming & Rishel, 1975, chapter VI, theorem 4.1)
Let $v(x,t)$ be a candidate solution of (3.1). If $v \in C^{2,1}(Q)$ satisfies the polynomial growth condition

$$\exists C > 0, \exists l > 0 : \; v(x,t) \leq C(1 + |x|^l) \quad \forall (x,t) \in Q$$

and further is continuous on $\overline{Q}$, then

a) $v(x,t) \geq J(x,t;u)$ for any admissible control process $u \in \mathcal{U}(x,t)$ and for any intermediate state $(x,t) \in Q$, and

b) if some $u^* \in \mathcal{U}(x,t)\ \forall (x,t) \in Q$ and further

$$\mathcal{A}^{u^*} v(x,t) = \sup_{u \in \mathcal{U}} \{\mathcal{A}^u v(x,t)\} \quad \forall (x,t) \in Q,$$

then $v(x,t) = J(x,t;u^*)$ for all $(x,t) \in Q$, thus u^* is an optimal solution. $\diamond$

To derive at the statement of theorem 3.1, the stochastic control problem considered in (Fleming & Rishel, 1975, chapter VI, theorem 4.1) was reformulated as maximization problem. The original is recovered by replacing $\geq$ with $\leq$ in part a) and sup with inf in part b).

A First Example with Discontinuous Boundary

As very first example for a stochastic optimization problem that yields discontinuous boundaries consider van Mellaert & Dorato (1972). They study some (in their context: physical) system with a continuously evolving state process $\xi = (\xi(t))$,

$$d\xi(t) = f(\xi(t), u(t))dt + Gdw(t), \; t \leq T,$$

where f is a deterministic function of the current state and the control $u(t)$ applied at t, G is constant and the random noise $(w(t))$ is a Wiener process. Evolution is terminated at $T > 0$. The original setup considers a multi-dimensional evolution with matrix G. For our intentions, however, it suffices to examine the one-dimensional case.

ξ can be considered as a special case of the general diffusion model (2.1) and fits in our context with the exception that van Mellaert & Dorato (1972) do not specify an initial time-point and initial state. This shortcoming is of no further consequence[1] as for any $s \leq T$ and $x \in \mathbb{R}$ they study the probability

$$v(x,t) := \mathbb{P}(\xi(s) \in D, \; t \leq s \leq T \,|\, \xi(t) = x)$$

of ξ not leaving some open real interval D in the remaining time to termination.

Note that ξ is controlled by some strategy $u = (u(\xi(t), t))$, thus the values $v(x,t) = v(x,t;u)$ also depend on u. Accordingly the goal of van Mellaert & Dorato (1972) is to find an optimal control u^* that maximizes $v(x,t;u)$.

Figure 3.1 sketches the states (x,t) of ξ and the implied values of $v(x,t)$ for some fixed strategy u:

[1] Although not specified it seems van Mellaert & Dorato (1972) mean to start at $t = 0$. Assuming that, the Wiener process can be understood in the usual way with $\mathbb{P}(w(0) = 0) = 1$.

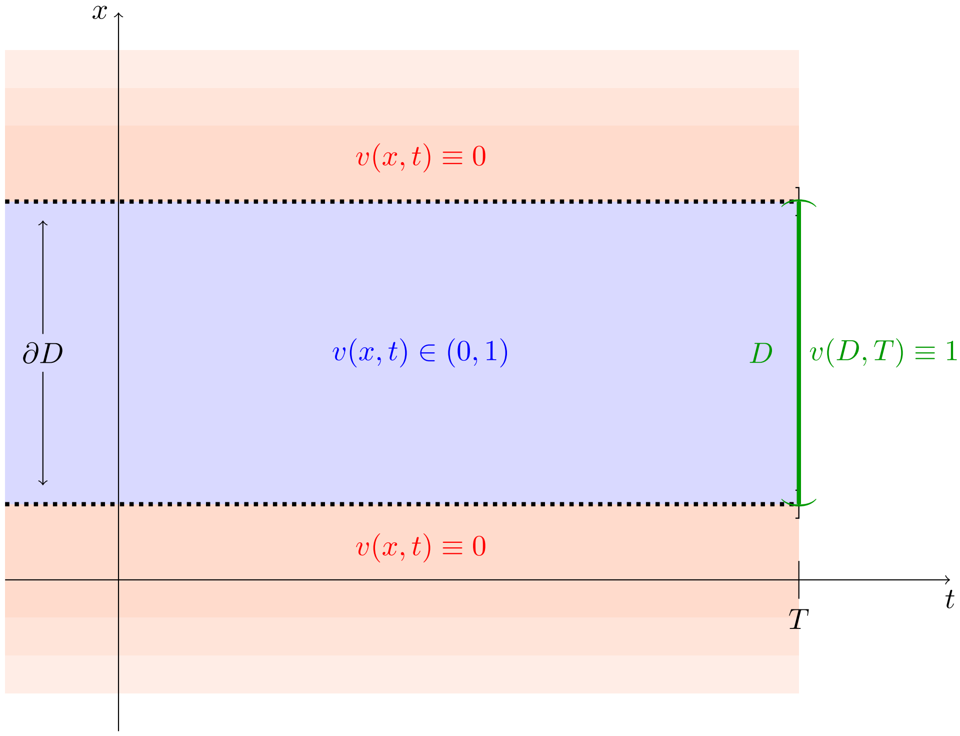


Figure 3.1: States and implied values for the problem studied by van Mellaert & Dorato (1972).

For any fixed $t < T$ the mapping $x \mapsto v(x,t)$ is continuous but at $t = T$ it holds

$$v(x,T) = \mathbb{P}(\xi(s) \in D,\ T \leq s \leq T \,|\, \xi(T) = x) = \mathbb{1}_{[D]}(x),$$

so there are discontinuities at the so-called critical set $C := \partial D \times \{T\}$.

In their article van Mellaert & Dorato (1972) provide a candidate for the optimal control u^* along with the values $v(x,t) = v(x,t;u^*)$ and discuss both numerically. However, they do not verify their conjecture in the strict sense of stochastic optimization.

An Adaption to Portfolio Optimization

To fit in the context of portfolio optimization the setup of van Mellaert & Dorato (1972) needs to be equipped with a left-hand boundary, corresponding to an initial time t_0. (Fleming & Rishel, 1975, chapter IV, sections 3 and 4) provide that addition. They also reconsider it as a minimization problem by putting

$$J(x,t;u) := 1 - \mathbb{P}(\xi^u(s) \in D,\ t \leq s \leq T \,|\, \xi^u(t) = x) \tag{3.4}$$

and aiming for u^* minimizing $J(x,t;u)$. In their notation ξ^u explicitly carries the control applied. Figure 3.2 sketches the modified situation.

The condition $\xi^u(s) \in D,\ t \leq s \leq T$, incorporates the complete remaining path $(\xi^u(s))_{t \leq s \leq T}$ and thus cannot be formulated in terms of a terminal Ψ. But observing

$$\begin{aligned} J(x,t;u) &= 1 - \mathbb{P}(\xi^u(s) \in D,\ t \leq s \leq T \,|\, \xi^u(t) = x) \\ &= \mathbb{P}(\exists s \in [t,T] \text{ such that } \xi^u(s) \notin D \,|\, \xi^u(t) = x) \end{aligned}$$

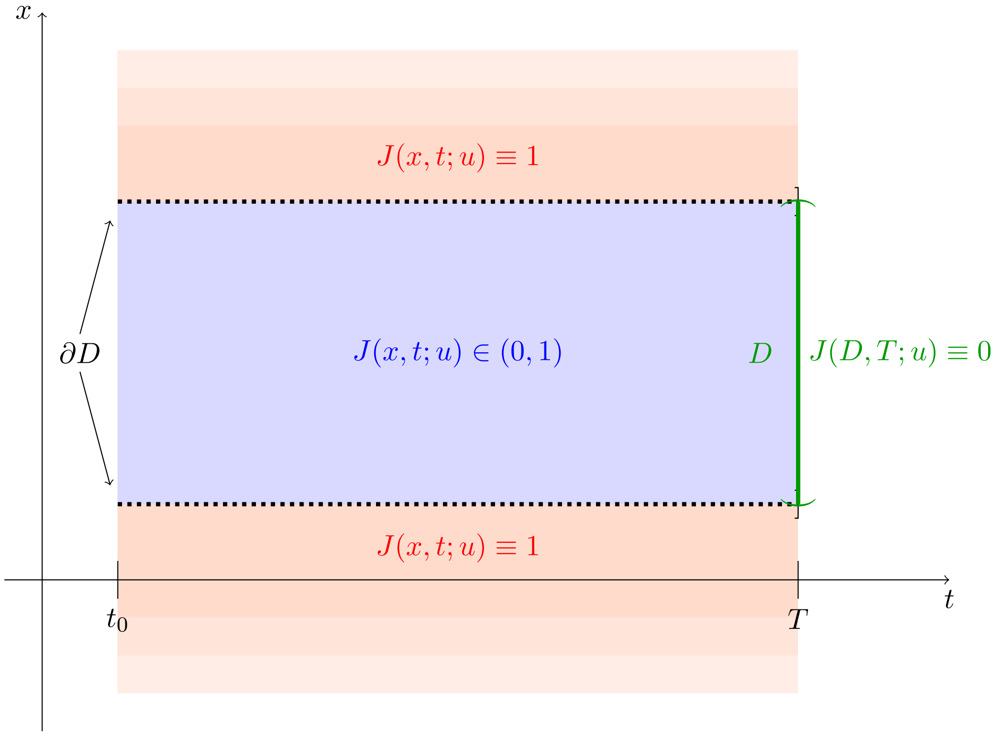


Figure 3.2: States and values for the minimization problem (3.4) as considered by Fleming & Rishel (1975).

and assuming $\xi(0) \in D$ we can define the exit time $\tau(\omega) := \min\{s : \ \xi^u(s) \notin D\} \wedge T$ and rewrite

$$J(x,t;u) = \mathbb{E}\left[\Psi(\xi^u(\tau),\tau) \,|\, \xi^u(t) = x\right] \tag{3.5}$$

with the discontinuous

$$\Psi(x,\tau) := \begin{cases} 1, & \tau <= T \text{ and } x \in \partial D \\ 0, & \tau = T \text{ and } x \in D \end{cases}.$$

The stopping time τ is by construction bounded to $\tau(\omega) \in (t_0, T]$ for all $\omega \in \Omega$. Let $Q := D \times [t_0, T)$ and $\partial^* Q := \partial D \times \{t_0, T\}$. Then $J(x,t;u)$ is defined for all $(x,t) \in \overline{Q}$. Figure 3.3 sketches the states and values for (3.5):

In this variation control is terminated when the steered process reaches ∂D, which can very well occur before T. Hence the setup does not yet fit into the fixed time horizon framework.

One way to align both is to allow random termination times in the framework. That approach is discussed for instance in (Fleming & Rishel, 1975, chapter VI, section 2). But the verification of solution candidates then requires some technical adjustments (compare e.g. Rishel (1970)).

Another approach is to continue the control on $(\tau, T]$ in a consistent way such that it suffices to consider a terminal $\Psi = \Psi(T)$. This is what we will examine in the next example.

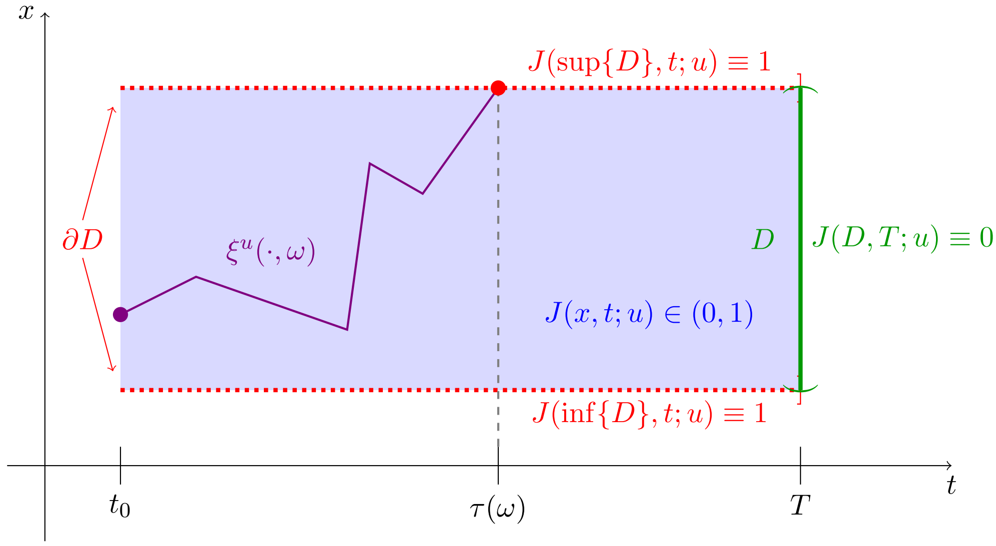


Figure 3.3: States and values for (3.5) along with an exemplary path $(\xi^u(t,\omega))_{t_0 \le t \le \tau(\omega)}$ for some $\omega \in \Omega$.

A More Elaborate Example

In so-called goal-reaching models the agent is rewarded with a premium if and only if a previously fixed target wealth is terminally achieved. This setup assumes control takes place during the complete time interval $[0, T]$; stopping at some premature $\tau < T$ is not designated.

The terminal wage for that type of contract is $\Psi(T, x) = \mathbb{1}_{[x \ge b]}$, $x > 0$, where $b > x_0 > 0$ is the target wealth level and without loss of generality[2] the bonus is one monetary unit.

We follow Browne (1999) for this example, but restrict ourselves to a classical Black-Scholes market with one risky asset and constant market parameters. The problem then reads as

$$(P_{\text{Browne}}) \qquad \begin{cases} \sup_\theta \left\{ \mathbb{E}\left[\Psi(T, X^\theta(T))\right] \right\} \\ X(0) = x_0, \ \theta \in \mathcal{U}(x_0) \cap \{\theta : \ X^\theta(t) \ge 0, \ 0 < t \le T\} \end{cases}$$

and the agent's objective is to maximize

$$J(x, t; \theta) = \mathbb{E}^{x,t}\left[\Psi(T, X^\theta(T))\right] = \mathbb{P}^{x,t}(X^\theta(T) \ge b), \ 0 \le t \le T, \ 0 < x < \infty.$$

over θ.

Note that the agent is set up to control the amount $\theta(t)$ of wealth in stock at times $0 \le t \le T$. Given an initial endowment $x_0 > 0$ the proper set of admissible strategies $\mathcal{U}(x_0)$ from (2.12) is reduced by further requiring θ to provide non-negative wealth at all times.

Let $R(t, T) := \exp(r(T - t))$ the amount of money which, when put in the bond at time t, yields exactly 1 monetary unit at T. If at some $t < T$ a wealth $x \ge bR(t, T)$ is attained, the goal can be safely achieved by investing in the riskless bond.

[2] An arbitrary bonus c yields $\Psi(T, x) = c\mathbb{1}_{[x \ge b]}$. But due to the linearity of the target (recall the agent being risk-neutral) the optimal strategy does not depend on c and the value is simply scaled by c.

This is an important observation: An agent who has already attained the necessary wealth level $bR(t,T)$ at t is not obliged to put $\theta(s) \equiv 0$ for $t \le s \le T$. But she has nothing to win if not choosing that strategy. On the contrary – if she places some money in the stock she risks to fall below $bR(s,T)$ at some later time $t < s \le T$.

On the other hand, due to the multiplicative nature of the Black-Scholes stock price dynamics, an intermediate wealth of $X_t^\theta = x = 0$ leads to $X_T^\theta = 0$ $\mathbb{P}$-a.s., regardless of θ.

We can now collect the boundary conditions

$$J(x,t;\theta) = \begin{cases} 0, & t < T,\ x = 0, \\ 0, & t = T,\ x < b, \\ 1, & t = T,\ x \ge b, \end{cases}$$

for any admissible strategy θ along with the necessary additional property

$$J(x,t;\theta^*) = V(x,t) = 1,\ t < T,\ x \ge bR(t,T)$$

for any strategy θ^* that is optimal.

Before going into further detail let us in figure 3.4 line out the states, values and boundaries as discussed above:

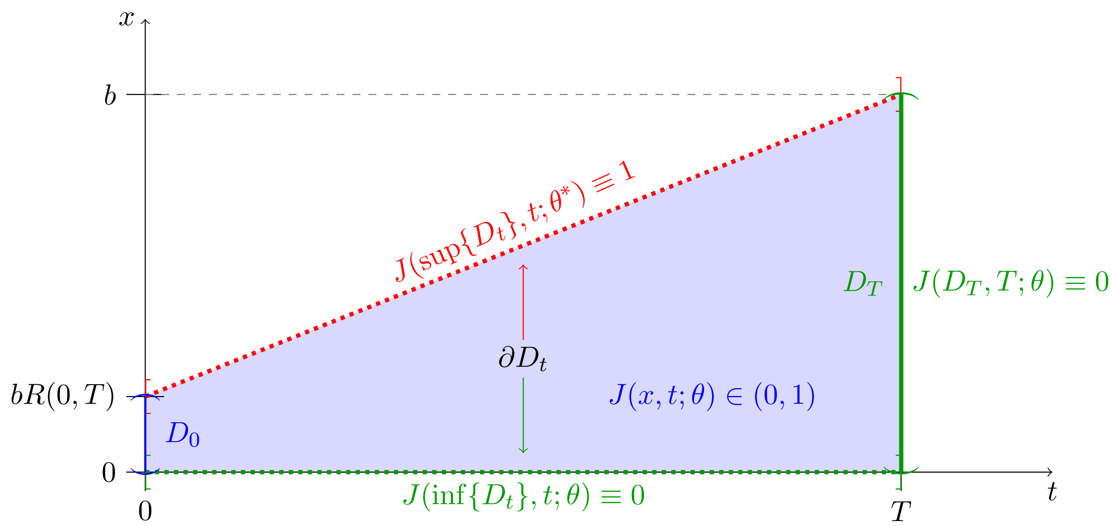


Figure 3.4: States and values for (P_{Browne}), interpreted as a problem with a bounded (open) control region in the sense of Fleming & Rishel (1975).

The sets $D_t := (0, bR(t,T))$, $0 \le t \le T$, depicted in figure 3.4 correspond to the open intervals of wealth values at time t for which it is not yet clear what terminal wage will be achieved at T. Wealth starts in $x_0 \in D_0$ (as nothing is to be done otherwise), is then controlled by some strategy θ and evolves in $\{D_t,\ 0 \le t \le T\}$ until it hits ∂D_τ at the exit time

$$\tau(\omega) := \min\{t \in (0,T] :\ X^\theta(t) \notin D\} \wedge T$$

or arrives at D_T if no exit occurs.

In terms of Fleming & Rishel (1975) we have the control region $Q := \{(x,t) :\ x \in D_t,\ 0 \le t < T\}$ with relative boundary $\partial^* Q := \overline{D_T} \times \{T\} = [0,b] \times \{T\}$.

As sketched, $x \mapsto J(x,t;\theta)$ is continuous for $0 \le t < T$ and any[3] admissible θ. But at $t = T$ it holds

$$J(x,T;\theta) = \Psi(T,x) = \mathbb{1}_{[x\ge b]} = \begin{cases} 1, & x = \sup\{D_T\} \\ 0, & x \in D_T \\ 0, & x = \inf\{D_T\} \end{cases}$$

and this function is discontinuous at $x = b$. Hence it suffices[4] to consider

$$C := \{(b,T)\}$$

as critical set.

Browne (1999) provides the following solution:

Theorem 3.2 (Browne, 1999, theorem 3.1)
Let ϕ the density, Φ the cumulative density and Φ^{-1} the quantile of the standard normal distribution. Further set

$$\phi(\pm\infty) := 0,\ \Phi(x) := \begin{cases} 0, & x = -\infty \\ 1, & x = +\infty \end{cases},\ \text{and}\ \Phi^{-1}(x) := \begin{cases} -\infty, & x \le 0 \\ +\infty, & x \ge 1 \end{cases}.$$

Then the solution of the portfolio problem (P_{Browne}) is

$$\begin{aligned} v(t,x) &= \Phi\left(\Phi^{-1}\left(\frac{x}{bR(t,T)}\right) + |\vartheta|\sqrt{T-t}\right), \\ \theta^*(t,x) &= \operatorname{sign}(\vartheta)\frac{bR(t,T)}{\sigma\sqrt{T-t}}\phi\left(\Phi^{-1}\left(\frac{x}{bR(t,T)}\right)\right), \end{aligned} \tag{3.6}$$

for all $0 \le t \le T$ and $x > 0$, where the optimal strategy $\theta^*(t)$ denotes the amount of wealth in stock at time t. $\diamond$

Note that wealth is controlled by amounts θ of wealth in stock. In fractional terms the optimal strategy of theorem 3.2 reads as $\pi^*(t,x) = \frac{1}{x}\theta^*(t,x)$ for all $0 \le t \le T$ and all wealths $x > 0$. If $x = 0$ the value of π^* is of no consequence as long as it is finite, w.l.o.g. set $\pi^*(t,0) := 1$.

Let us also observe and record that $0 \le v(x,t) \le 1$ by construction, thus the growth condition

$$v(x,t) \le C(1+|x|^l) \quad \forall (x,t) \in Q$$

of theorem 3.1 is satisfied e.g. for $C = l = 1$.

The prominent idea of Browne (1999) is to construct a value function v that is continuous and strictly increasing on Q but cuts its values when reaching the upper or lower boundary by employing an inverse distribution function: The (standard normal) distribution function Φ maps the real line to $[0,1]$ where $\Phi(x) \in (0,1)$ but $\lim_{x\to-\infty}\Phi(x) = 0$ and $\lim_{x\to+\infty}\Phi(x) = 1$. Hence the inverse distribution function Φ^{-1} reversely maps the open interval $(0,1)$ to $\mathbb{R}$ and the boundaries 0 and 1 to $-\infty$ and $+\infty$. If now some value larger than 1 is to be mapped by Ψ^{-1}, the result can only be $+\infty$ as defined in theorem

[3] If $\theta = \theta^*$ is an optimal strategy, then additionally $J(\sup\{D_t\}, t; \theta) = 1$ for all $0 \le t < T$, but continuity does not depend on whether θ is optimal or not.

[4] Following Fleming & Rishel (1975) one would expect $C := \{(T,0),(T,b)\}$. But as v is continuous in $(T,0)$ we do not have to explicitly exclude that point.

3.2. Analogously, values less than 0 are mapped to $-\infty$ by Ψ^{-1}. As adding finite values to infinities has no effect, the cut of v can then be realized by applying the distribution function on the value of its inverse (plus some time-depending term).

If at any time $t \in [0, T]$ the wealth $x \geq bR(t, T)$ is achieved, the goal can be reached without risk. In that case $\frac{x}{bR(t,T)} \geq 1$ and $v(t, x) = \Phi(\infty) = 1$. The optimal strategy is $\theta^*(t, x) = 0$ as $\phi(\infty) = 0$, i.e. from that point on all capital is invested in the bond.

Should wealth drop to zero, it holds $\Phi^{-1}(0) = -\infty$ and $v(t, x) = \Phi(-\infty) = 0$. With $\phi(-\infty) = 0$ the optimal strategy is $\theta^*(t, x) = 0$, too, i.e. no capital is invested in the stock (and also none is invested in the bond as $x = 0$).

Let us discuss the optimal strategy when termination approaches. Fixing $t < T$ (imagine it close to T), three cases need to be distinguished:

- If $x = bR(t, T)$, we know $\theta^*(x, t) = 0$ and thus $X^{\theta^*}(s) = bR(s, T)$ for all $t \leq s \leq T$.
- If $x \in (0, bR(t, T))$, it holds $\Phi^{-1}\left(\frac{x}{bR(t,T)}\right) \in \mathbb{R}$ and thus $\phi\left(\Phi^{-1}\left(\frac{x}{bR(t,T)}\right)\right) \in \mathbb{R}$. As $\sqrt{T-t} \to 0$ it follows $\lim_{t\to T} \theta^*(x, t) = \infty$.
- If $x = 0$, we know $\theta^*(x, t) = 0$ and $X^{\theta^*}(s) = 0$ for all $t \leq s \leq T$.

An agent who has not already attained the necessary capital to reach her goal (and has some money left) will by all means try to reach it in the remaining time. Indeed the optimal strategy in that case steps over any finite bound when $t \to T$, implying the agent is willing to take any risk. The result of that plan of action is either $x = bR(s, T)$ (success) or $x = 0$ (complete loss of all capital) for some $s \in (t, T]$, there is nothing in between.

We can now understand the effects of prematurely hitting one of the boundaries at τ when the agent employs the strategy θ^*, as illustrated in figure 3.5:

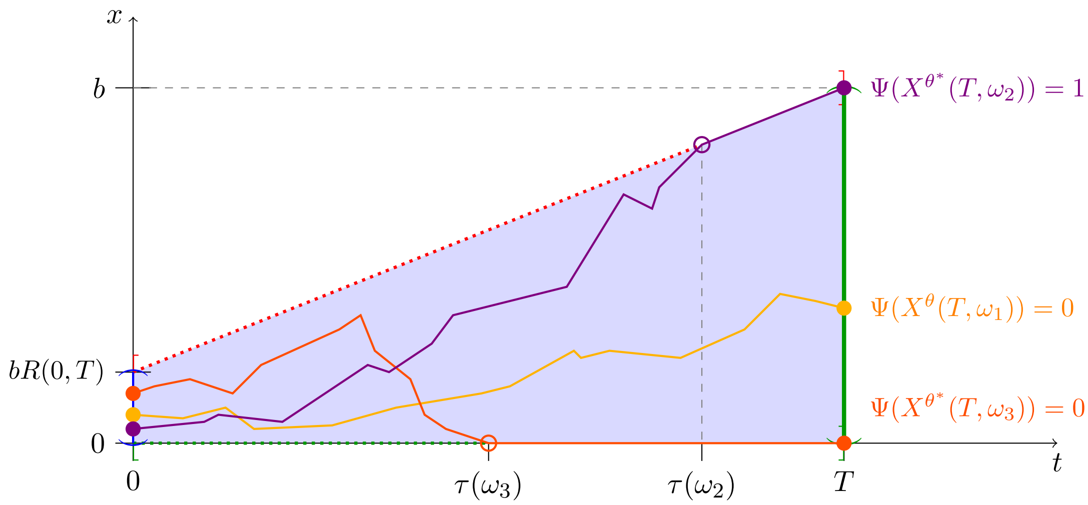


Figure 3.5: States of (P_{Browne}) as illustrated in figure 3.4. Additionally, one exemplary path $(X^{\theta}(t, \omega_1))_{0\leq t\leq T}$ (yellow) of wealth, controlled by some strategy θ, is plotted along with two different paths $(X^{\theta^*}(t, \omega_2))_{0\leq t\leq T}$ (violet) and $(X^{\theta^*}(t, \omega_3))_{0\leq t\leq T}$ (orange) of optimally controlled wealth.

In figure 3.5 three paths are sketched:

The yellow path corresponds to the situation that at no time any of the boundaries is reached. As we have discussed above, this cannot occur when the strategy θ^* is applied (but it may happen for some other admissible choice θ).

The violet path successfully attains the value $bR(t,T)$ at $\tau(\omega_2) \in (0,T)$. From that moment on the optimal strategy is zero and wealth develops along $\{bR(s,T) : \tau(\omega_2) \leq s \leq T\}$.

Analogously when the orange path hits zero at $\tau(\omega_3)$, both $X^{\theta^*}(s)$ and $\theta^*(X^{\theta^*}(s), s)$ are zero for $\tau(\omega_3) \leq s \leq T$.

Concluding we find θ^* provides $X^{\theta^*}(T) \in \{0, b\} = \partial D_T$ $\mathbb{P}$-a.s..

To verify θ^* and v we need a variation of theorem 3.1 that incorporates discontinuous boundaries. (Fleming & Rishel, 1975, chapter VI, example 2) provide that the conclusions of that theorem still hold under the following modified conditions:

- The value function $v \in C^{2,1}(Q)$ respects polynomial bounds on its growth and solves the HJB condition (3.1) on Q.
- v is continuous (at least) on $\overline{Q} \setminus C$.
- The optimally controlled process X^{θ^*} does $\mathbb{P}$-a.s. not terminate in D_T, i.e.

$$\mathbb{P}(X^{\theta^*}(T) \in D_T \,|\, X(0) = x_0 \in D_0) = 0.$$

  θ^* ensures that once ∂D_t is hit at $0 < t \leq T$, the process walks along that boundary, and further provides that one of the extremal terminal values in ∂D_T is attained. Thus one can equivalently require

$$\mathbb{P}(X^{\theta^*}(T) \in \partial D_T \,|\, X(0) = x_0 \in D_0) = 1.$$

We have discussed and made plausible all of the above conditions. While continuous differentiability on Q and continuity on $\overline{Q} \setminus C$ are evident, a rigorous proof of $\mathbb{P}$-a.s. termination in ∂D_T requires some technical accuracy.

To this end Browne (1999) considers the normalized future optimal wealth process $H_t := \frac{1}{bR(t,T)} X_t^{\theta^*}$ for $0 \leq t < T$, which he analytically continues for $t = T$. That $\mathbb{P}(H_T \in \{0,1\} \,|\, H_0 = \frac{x}{b} e^{-rT}) = 1$ is then covered by Heath (1993).

Summarizing the findings of this section, we have seen that it is possible to verify solution candidates for portfolio optimization problems with discontinuous boundary conditions. The technique introduced by Fleming & Rishel (1975) provides a good start but needs adjustment to the particular problem under consideration. Unlike in the continuous case there is no generic set of requirements that one can check to obtain verification.

Besides, one needs to find a solution candidate before commencing verification – and that can be a demanding task. In the followings sections of this chapter we will therefor consider two different approaches to non-standard portfolio optimization which do not require the derivation of solution candidates.

3.3 Martingale Approximation of Bonus Wages

We have seen in section 3.2 that it is possible to verify conjectured solutions of portfolio optimization problems with discontinuous wage schedules. But it is still unclear how to initially derive or deduce such candidates.

In this section we approach the problem from another direction: We will examine if and how discontinuous wages can be approximated by a sequence of continuous wage functions and what can be asserted about the associated sequence of optimization problems.

Continuously Approximating a Discontinuous Terminal Wage

Let us again focus on the Black-Scholes case of (2.3) with $\psi \equiv 0$. The terminal wage Ψ is assumed to be strictly increasing in wealth. It is supposed to exhibit at least one but no more than a finite number of discontinuities but shall be continuously differentiable everywhere else. We do not require concavity or the Inada properties.

The overall idea is to use a sequence (Ψ_n) of continuously differentiable functions that point-wise approximate Ψ. As the limit needs to be discontinuous, point-wise convergence is the best one can achieve. In the following we will even allow a finite number of exceptional points at which the approximating sequence does not convergence[5] at all.

Let $C \subset \mathbb{R}^+$ with $|C| < \infty$ the finite set of exceptional points. Now fix a sequence $(\Psi_n)_{n\in\mathbb{N}}$ with

- $\Psi_n : (0,\infty) \to \mathbb{R}$ continuously differentiable and strictly increasing, $n \in \mathbb{N}$,
- $\lim_{n\to\infty} \Psi_n(x) = \Psi(x)$ for all $x \in \mathbb{R}^+ \setminus C$, and
- $\Psi_n(x) \leq \Psi(x)$ for all $n \in \mathbb{N}$ and all $x \in (0,\infty)$.

We then consider the portfolio problems

$$(P_n) \qquad \begin{cases} \sup_{\pi_n} \left\{\mathbb{E}\left[\Psi_n(X^{\pi_n}(T))\right]\right\} \\ X(0) = x_0,\ \pi_n = (\pi_n(t))_{0\leq t\leq T} \in \mathcal{U}(x_0, 0) \end{cases}$$

for all $n \in \mathbb{N}$.

The requirement of an approximation from below provides

$$\lim_{n\to\infty} \sup_{\pi_n} \left\{\mathbb{E}\left[\Psi_n(X^{\pi_n}(T))\right]\right\} \leq \sup_{\pi} \left\{\mathbb{E}\left[\Psi(X^{\pi}(T))\right]\right\},$$

thus also the sequence of problem values approximates the true value from below.

[5] This is uncritical as all finite subsets of $(0,\infty)$ are zero-sets in the Borel sense.

A Modified Martingale Approach

For some (not necessarily optimal) feasible strategy π we can consider $X^\pi(T)$ itself a variable and search for the optimal terminal wealth X_n^* with the Lagrangian

$$L_n(X^\pi(T), \lambda_n) := \mathbb{E}\left[\Psi_n(X^\pi(T)) - \lambda_n(\zeta(T)X^\pi(T) - x_0)\right], \tag{3.7}$$

where $(\zeta(t))_{0\le t\le T}$ is the state-price density and the Lagrangian constraint coupled with the multiplier λ_n ensures that $X^\pi(T)$ is attainable with endowment x_0. (3.7) resembles the usual martingale methodology in complete markets and yields the necessary first-order condition

$$\mathbb{E}\left[\Psi_n'(X^\pi(T)) - \lambda_n\zeta(T)\right] = 0, \tag{3.8}$$

where λ_n is chosen to match the above budget constraint. Note that $\Psi_n'(x) > 0$ for all $x \in (0,\infty)$ by assumption and $\zeta(T,\omega) > 0$ for all $\omega \in \Omega$ by (2.7), thus necessarily $\lambda_n > 0$, too.

If Ψ_n was strictly convex or strictly concave, Ψ_n' was one-to-one on its domain and a candidate solution was found by simply inverting it:

$$X_n^* = (\Psi_n')^{-1}(\lambda_n\zeta(T)). \tag{3.9}$$

But as we are employing Ψ_n to approximate a target function with sharp bends or upward jumps, Ψ_n will have to increase its slope before a discontinuity or bend and to decrease it again afterwards. Hence no global inverse of Ψ_n' exists.

By assumption each Ψ_n is continuous and strictly increasing but not globally concave or globally convex. Let $C = \{x_n^{(1)}, \dots, x_n^{(N)}\}$ collect all roots of Ψ_n'', i.e. all stationary points of Ψ_n, and call that set the exceptional points of Ψ_n. We assume that C is the same for all indices n, meaning that the approximating functions are chosen to share the stationary points. As one would conveniently aim to keep C as small as possible, it is natural to construct the sequence in such a way that stationary points occur only where jumps or bends in Ψ make them necessary; so that is a mild constraint.

We may now split the domain $(0,\infty)$ in the open intervals

$$(0, x^{(1)}), (x^{(1)}, x^{(2)}), \dots, (x^{(N)}, \infty)$$

such that on each of these either all Ψ_n are strictly convex or all Ψ_n are strictly concave. On each interval we can then locally invert Ψ_n' with the purpose to rewrite the first-order condition (3.9) as

$$\begin{aligned} 0 \quad \overset{!}{=} \quad & \mathbb{E}\left[\mathbb{1}_{[x^{(N)}<X_T^\pi]}\left(\Psi_n'(X_T^\pi) - \lambda\zeta(T)\right)\right] + \\ & \sum_{i=1}^{N} \mathbb{E}\left[\mathbb{1}_{[x^{(i-1)}<X_T^\pi<x^{(i)}]}\left(\Psi_n'(X_T^\pi) - \lambda\zeta(T)\right)\right], \end{aligned} \tag{3.10}$$

where $x^{(0)} := 0$.

(3.10) is particularly true, if the random variables in all expectations vanish. Thus, we may consider the problem individually on each interval $(x^{(i-1)}, x^{(i)})$ and use the proper local inverse $I_n^{(i)} := [\frac{\partial}{\partial x}\Psi_n]^{-1}_{x^{(i-1)}<x<x^{(i)}}$. With I_n^∞ denoting the local inverse over $(x^{(N)}, \infty)$ one may informally suggest

$$X_n^* := \begin{cases} I_n^{(1)}(\lambda_n^*\zeta(T)), & \text{if } (\lambda_n^*\zeta(T)) \text{ corresponds to } X_n^* \text{ in } (x^{(0)}, x^{(1)}), \\ \dots, & \dots \\ I_n^{(N)}(\lambda_n^*\zeta(T)), & \text{if } (\lambda_n^*\zeta(T)) \text{ corresponds to } X_n^* \text{ in } (x^{(N-1)}, x^{(N)}), \\ I_n^{(\infty)}(\lambda_n^*\zeta(T)), & \text{if } (\lambda_n^*\zeta(T)) \text{ corresponds to } X_n^* \text{ in } (x^{(N)}, \infty) \end{cases} \tag{3.11}$$

as a combined solution. The tricky part is to find the right branch, i.e. given the value $\lambda_n^*\zeta_T$ to decide which interval the corresponding value of X_n^* lies in.

We know that the state-price density $\zeta(T) = \zeta(T, y)$ strictly decreases in the realization $y = W(T, \omega)$ of the value of the Brownian motion at terminal time T. But as the paths of Brownian motions are almost surely continuous, changing its value at time T implies a change in the path leading there, too. Thus for any $\delta \neq 0$ we can almost surely find some $\varepsilon > 0$ such that if $W(T, \omega_1) = y$ and $W(T, \omega_2) = y + \delta$, then $W(t, \omega_1) \neq W(t, \omega_2)$ for all $t \in (T - \varepsilon, T)$. Hence changing the value of $W(T)$ affects the path and – via the stochastic integral $\int_0^T \pi(s)dW(s)$ in (2.9) – also the value of $X(T)^\pi$ in an unpredictable way. As monotonicity seems the only sensible criterion to decide (3.11) there is no way to achieve a monotone relation between $\zeta(T)$ and X_n^* in the generality of this setup.

Nevertheless, a strictly monotone relation exists, if we can drop the path dependence in $\int_0^T \pi(s)dW(s)$ by confining ourselves to constant strategies $\pi(s) \equiv \pi \in \mathbb{R}$. In this case we can explicitly derive the value of $X(T)^\pi = f(W(T))$ by extracting $y = W(T, \omega)$ from $\zeta(T, \omega)$.

Although narrowing the search domain means lowering the bound, it still proves useful in judging the strategies proposed by numerical approximations to the strategy. In particular the Markov Chain Monte Carlo method (that we will discuss in detail in chapter 4) requires initial fine-tuning for each individual problem to produce sensible solutions in finite time.

The next section provides an example application of the above martingale technique to the discontinuous and neither convex nor concave problem of Browne (1999).

3.4 Continuously Approximating an Example

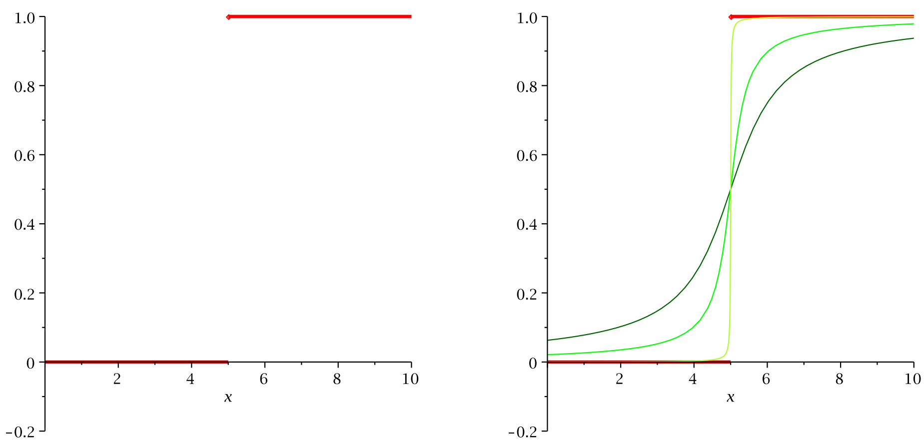
Let us now apply the martingale technique to a discontinuous example and look into further details. The simple bonus contract $\Psi(x) := \mathbb{1}_{[x \geq b]}(x)$, $x > 0$, of Browne (1999) provides a simple yet interesting example. Here the agent is paid one monetary unit if and only if she reaches the target wealth level b at terminal time T.

An Arcus Tangens Approximation

We will approximate this discontinuous and non-differentiable function by a sequence of continuously differentiable functions Ψ_n, $n \in \mathbb{N}$, that are strictly convex on $(0, b)$ and strictly concave on (b, ∞). To this end define

$$\Psi_n(x) := \frac{1}{\pi} \arctan\left(n(x - b)\right) + \frac{1}{2}, \ n \in \mathbb{N}, \ 0 < x_0 < b. \tag{3.12}$$

Figure 3.6 relates the original terminal wage Ψ with the approximating sequence (Ψ_n):



(a) Ψ only

(b) Ψ and Ψ_1, Ψ_3, Ψ_{100}

Figure 3.6: First step in constructing a lower-bound approximation to the target function. The red curve is Ψ with barrier $b = 5$, the green curves are Ψ_1 (dark green), Ψ_3 (medium green), and Ψ_{100} (light green).

Note that Ψ_n as defined in (3.12) does not satisfy $\Psi_n(x) \leq \Psi(x)$ for $0 < x < b$ and thus will not yield a lower bound as promoted above. This deliberate imprecision is to facilitate a clear and easily comprehensible presentation of all (somewhat technical) steps to come. A modification $\hat{\Psi}_n$ of Ψ_n that feasibly establishes a lower-bound approximation is provided later in (3.21)..

$(\Psi_n)_{n \in \mathbb{N}}$ approximates both $\Psi(x) = \mathbb{1}_{[x \geq b]}$ and the slight variation $\mathbb{1}_{[x > b]}$ in the sense

$$\Psi_n(x) \to \begin{cases} 0, & x < b \\ 1, & x > b \end{cases}, \ (n \to \infty),$$

while the limit does not exist at $x = b$.

As Ψ_n is not globally concave (and neither globally convex), $x \mapsto \Psi_n'(x)$ is not injective and we have to split the domain into $(0, x^{(1)}) = (0, x^{(N)}) = (x, b)$ and $(x^{(N)}, \infty) = (b, \infty)$. We follow (3.10) and separately consider the local inverses $I_n^- := I_n^{(1)}$ and $I_n^+ := I_n^{(\infty)}$ of the two branches of the derivative:

$$\begin{aligned}
\psi_n^+ &: \quad [b, \infty) \to (0, \tfrac{n}{\pi}], \\
\psi_n^- &: \quad (0, b] \to \big(\tfrac{n}{\pi(1+b^2n^2)}, \tfrac{n}{\pi}\big], \\
\psi_n^\pm(x) &:= \quad \frac{n}{\pi(1 + n^2(x-b)^2)}, \\
I_n^+ &: \quad (0, \tfrac{n}{\pi}] \to [b, \infty), \\
I_n^- &: \quad \big(\tfrac{n}{\pi(1+b^2n^2)}, \tfrac{n}{\pi}\big] \to (0, b], \\
I_n^\pm(y) &:= \quad b \pm \frac{1}{n}\sqrt{\frac{n}{\pi y} - 1}
\end{aligned}$$

Here the (globally existing but not globally injective) derivative Ψ_n' is split at $x^{(N)} = b$ in the two locally injective branches ψ_n^+ and ψ_n^-. It holds $I_n^\pm \circ \psi_n^\pm = \psi^\pm \circ I_n^\pm = \mathrm{id}$ on the appropriate domains. Compare figure 3.7 for a sketch:

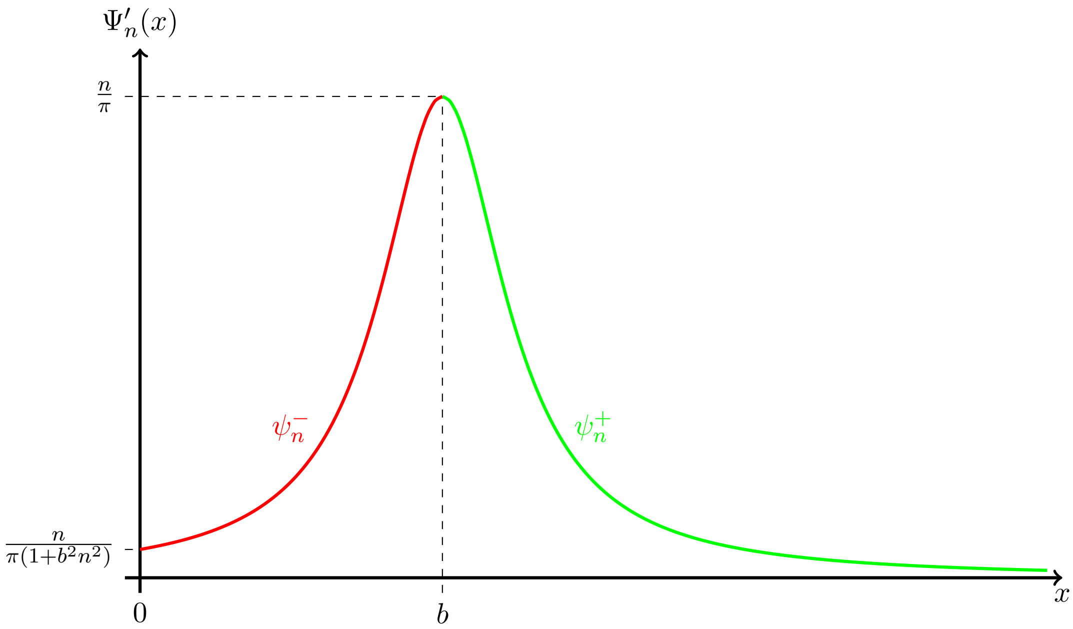


Figure 3.7: Splitting the domain of the derivative in the arcus tangens approximation (3.12) of $\Psi(x) = \mathbb{1}_{[x \geq b]}(x)$. The left branch (red) is ψ_n^- while the (green) right branch represents ψ_n^+.

Deriving a Split Criterion

Let us consider a classical Black-Scholes market. For constant market parameters the state-price density (2.7) is

$$\zeta(t) = \exp\left(-rt - \vartheta B(t) - \frac{1}{2}\vartheta^2 t\right), \ 0 \le t \le T.$$

As discussed in (3.11), in order to derive a split criterion in the Black-Scholes market we need to restrict the agent to choose among constant strategies. For any $\pi \neq 0$ the terminal portfolio wealth[6]

$$X^\pi(T) = x_0 \exp\left(\left(r + \pi\left(\mu - r\right) - \frac{1}{2}\sigma^2\pi^2\right) T + \sigma\pi B(T)\right) \tag{3.13}$$

is strictly monotone in $y = B(T, \omega)$. Thus, if $\pi > 0$ then $X^\pi(T)$ increases in y and there exists a unique y_b (depending on the barrier b) such that

$$X^\pi(T,\omega) \ge b \iff B(T,\omega) \ge y_b \iff \zeta(T,\omega) \le Z(y_b), \tag{3.14}$$

where $Z : \mathbb{R} \to (0, \infty)$, $y \mapsto \zeta(T,\omega)|_{B(T,\omega)=y}$ maps the terminal value of the Brownian motion to the value of the state-price density on the same path. If otherwise $\pi < 0$, $X^\pi(T)$ decreases in y and we can equivalently find a unique y_b such that

$$X^\pi(T,\omega) \ge b \iff B(T,\omega) \le y_b \iff \zeta(T,\omega) \ge Z(y_b). \tag{3.15}$$

For constant π we can explicitly derive y_b from (3.13) and find the split parameter

$$z_b := Z(y_b) = \left(\frac{x_0}{b}\right)^{\frac{\vartheta}{\sigma\pi}} \exp\left(-rT + \frac{1}{2}\left(\vartheta^2 + \left(\frac{2r}{\pi\sigma} - \pi\sigma\right)\vartheta\right)T\right).$$

The optimal terminal wealths associated with positive or negative strategies then are

$$\begin{aligned} X_n^{*,+} &:= \begin{cases} I_n^+(\lambda_n^*\zeta(T)), & \zeta(T) \le z_b \\ I_n^-(\lambda_n^*\zeta(T)), & \zeta(T) > z_b \end{cases}, \\ X_n^{*,-} &:= \begin{cases} I_n^+(\lambda_n^*\zeta(T)), & \zeta(T) \ge z_b \\ I_n^-(\lambda_n^*\zeta(T)), & \zeta(T) < z_b \end{cases}, \end{aligned} \tag{3.16}$$

respectively, where here (and further on) an upper index $^+$ denotes the positive strategy case and $^-$ refers to the negative strategy case.

[6] Choosing $\pi = 0$ yields the deterministic (bond) wealth $X(T) = x_0 e^{rT}$. We can easily check later, if the terminal wealth with non-zero strategies beats this benchmark, and disregard this case for now.

Determining the Optimal Lagrange Parameter

We cannot yet calculate the solutions $X_n^{*,\pm}$ as the value of the Lagrange parameter λ_n^* has not yet been derived. Note that the same multiplier covers both cases, as it is necessarily unique in (3.8).

Exploiting the budget constraint $\mathbb{E}\left[\zeta(T)X_n^{*,\pm}\right] \overset{!}{=} x_0$ and assuming positive strategies, it holds

$$\begin{aligned}
&\mathbb{E}\left[\zeta(T)X_n^{*,+}\right] - x_0 && (3.17)\\
=\ & \int_{-\infty}^{y_b} \phi_{0,T}(y)Z(y)I_n^+(\lambda_n^* Z(y))dy + \int_{y_b}^{\infty} \phi_{0,T}(y)Z(y)I_n^-(\lambda_n^* Z(y))dy - x_0\\
\approx\ & \int_{\underline{y}}^{y_b} \phi_{0,T}(y)Z(y)I_n^+(\lambda_n^* Z(y))dy + \int_{y_b}^{\bar{y}} \phi_{0,T}(y)Z(y)I_n^-(\lambda_n^* Z(y))dy - x_0 =: \chi_n^+(\lambda_n^*).
\end{aligned}$$

Here $\phi_{0,T}$ is the probability density function of the $\mathcal{N}(0,T)$-distributed random variable $B(T)$. The approximation in χ_n^+ with finite boundaries $\underline{y}$ and $\bar{y}$ arises from the necessity of computability as well as from some conditions on the domain of I_n^+ to be discussed shortly. For decreasing wealth $X_n^{*,-}$ we have the analogue

$$\begin{aligned}
&\mathbb{E}\left[\zeta(T)X_n^{*,-}\right] - x_0 && (3.18)\\
=\ & \int_{-\infty}^{y_b} \phi_{0,T}(y)Z(y)I_n^-(\lambda_n^* Z(y))dy + \int_{y_b}^{\infty} \phi_{0,T}(y)Z(y)I_n^+(\lambda_n^* Z(y))dy - x_0\\
\approx\ & \int_{\underline{y}}^{y_b} \phi_{0,T}(y)Z(y)I_n^-(\lambda_n^* Z(y))dy + \int_{y_b}^{\bar{y}} \phi_{0,T}(y)Z(y)I_n^+(\lambda_n^* Z(y))dy - x_0 =: \chi_n^-(\lambda_n^*).
\end{aligned}$$

numerical approximation of the budget constraint.

In (3.17) and (3.18) we have to ensure that the argument $Z(y)\lambda_n^*$ to $I_n^\pm$ is member of the appropriate domains

$$(0, \tfrac{n}{\pi}] \text{ and } \big(\tfrac{n}{\pi(1+b^2n^2)}, \tfrac{n}{\pi}\big].$$

With (3.8), $Z(y)\lambda > 0$ holds true for all $y \in \mathbb{R}$ and all proper Lagrange parameters λ, but as

$$\begin{aligned}
Z(y)\lambda > \frac{n}{\pi(1+b^2n^2)} \quad &\Longleftrightarrow \quad \begin{cases} y < \alpha_n(\lambda), & \vartheta > 0 \\ y > \alpha_n(\lambda), & \vartheta < 0 \end{cases} \quad \text{and}\\
Z(y)\lambda \le \frac{n}{\pi} \quad &\Longleftrightarrow \quad \begin{cases} y \ge \beta_n(\lambda), & \vartheta > 0 \\ y \le \beta_n(\lambda), & \vartheta < 0 \end{cases}, \qquad (3.19)
\end{aligned}$$

where

$$\begin{aligned}
\alpha_n(\lambda) \ &:= \ -\frac{1}{\vartheta}\log\left(\frac{n}{\lambda\pi(1+b^2n^2)}\right) - \frac{rT}{\vartheta} - \frac{\vartheta T}{2},\\
\beta_n(\lambda) \ &:= \ -\frac{1}{\vartheta}\log\left(\frac{n}{\lambda\pi}\right) - \frac{rT}{\vartheta} - \frac{\vartheta T}{2},
\end{aligned}$$

we have to pose necessary conditions on the integration boundaries $\underline{y}$ and $\bar{y}$.

Note that all conditions in (3.19) simplify to $y \in \mathbb{R}$, thus vanish, if $n \to \infty$. For computational purposes we nevertheless have to enforce finite boundaries for all degrees n of approximation.

Table 3.1: Intervals $(\underline{y}, \bar{y})$ for y when in χ_n^+ from (3.17) integrating the factors $I_n^+(Z(y)\lambda)$ and $I_n^-(Z(y)\lambda)$, all parameter cases.

	$I_n^+(\cdot)$	$I_n^-(\cdot)$
$\vartheta > 0$	$(\max\{\beta_n(\lambda), -K\}, \min\{y_b, K\})$	$(\max\{y_b, \beta_n(\lambda), -K\}, \min\{\alpha_n(\lambda), K\})$
$\vartheta < 0$	$(-K, \min\{y_b, \beta_n(\lambda), K\})$	$(\max\{y_b, \alpha_n(\lambda), -K\}, \min\{\beta_n(\lambda), K\})$

Table 3.2: Intervals $(\underline{y}, \bar{y})$ for y when in χ_n^- from (3.18) integrating the factors $I_n^+(Z(y)\lambda)$ and $I_n^-(Z(y)\lambda)$, all parameter cases.

	$I_n^+(\cdot)$	$I_n^-(\cdot)$
$\vartheta > 0$	$(\max\{y_b, \beta_n(\lambda), -K\}, K)$	$(\max\{\beta_n(\lambda), -K\}, \min\{y_b, \alpha_n(\lambda), K\})$
$\vartheta < 0$	$(\max\{y_b, -K\}, \min\{\beta_n(\lambda), K\})$	$(\max\{\alpha_n(\lambda), -K\}, \min\{y_b, \beta_n(\lambda), K\})$

As $\phi_{0,T}$ is peaked at zero and vanishes quickly with increasing distance to zero, a convenient choice may be the symmetric interval $[-K, K]$, with $K > 0$. Reasonable performance is e.g. achieved for $K := 10\mathrm{Var}(B_T) = 10T$. Table 3.1 explicitly collects all integration boundaries for χ_n^+ and table 3.2 provides the same for χ_n^-.

Note that plugging in the intervals specified in tables 3.1 and 3.2 may yield a lower integration boundary that is larger than the upper when for small n a rather coarse approximation is chosen. In that case the referred intervals are supposed to[7] just vanish.

Depending on the sign of the strategy π we then solve

$$\chi_n^+(\lambda) = 0 \quad \text{or} \quad \chi_n^-(\lambda) = 0$$

numerically for λ to find the (approximately) optimal Lagrange parameter λ_n^*.

The Arcus Tangens Approximation Refined

Finally, let us get back to the previously advertised adjustment of the approximating sequence to a feasible choice.

To establish the lower-bound property, we have to shift each Ψ_n below Ψ. As $\Psi_n(b) = \frac{1}{2}$ for all $n \in \mathbb{N}$ and as all Ψ_n are continuous, we can for any $\delta > 0$ and for all $n \in \mathbb{N}$ still find an interval $\mathcal{I}_n = (b - \varepsilon_n, b)$ such that $\Psi_n(x) \geq \frac{1}{2} - \delta$ for all $x \in \mathcal{I}_n$.

Thus, just shifting Ψ_n downwards by some δ_n such that as desired $\Psi_n(x) - \delta_n \leq 0 = \Psi(x)$ for all $x < b$ implies $\delta_n \equiv \frac{1}{2}$ and no convergence $\Psi_n \to \Psi$ $(n \to \infty)$ can arise from that.

The resolution is to simultaneously shift to the right and downwards. This means to choose a strictly decreasing sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ with $\varepsilon_n \to 0$ $(n \to \infty)$ and shift each Ψ_n to the right by ε_n, i.e. to set

$$\tilde{\Psi}_n(x) := \Psi_n(x - \varepsilon_n), \ x \in \mathbb{R}, \ n \in \mathbb{N}. \tag{3.20}$$

[7] Technically, in that case one would flip the integration boundaries along with the sign of the integral. To prevent that we would have to explicitly ensure the proper alignment. But as this means wrapping each condition in another $\min\{\dots, y_b\}$ or $\max\{\dots, y_b\}$ it is omitted to facilitate understanding.

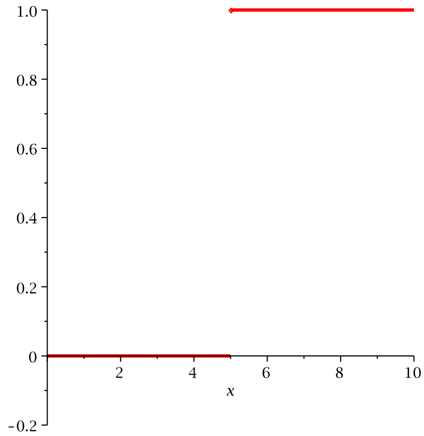
This yields $\tilde{\Psi}_n(b) < \frac{1}{2}$ for all $n \in \mathbb{N}$ and we can shift $\tilde{\Psi}_n$ below the discontinuous target $\mathbb{1}_{[x \geq b]}$ by setting

$$\hat{\Psi}_n(x) := \tilde{\Psi}_n(x) - \tilde{\Psi}_n(b) = \frac{1}{\pi}\left(\arctan\left(n(x - b - \varepsilon_n)\right) + \arctan\left(n\varepsilon_n\right)\right). \tag{3.21}$$

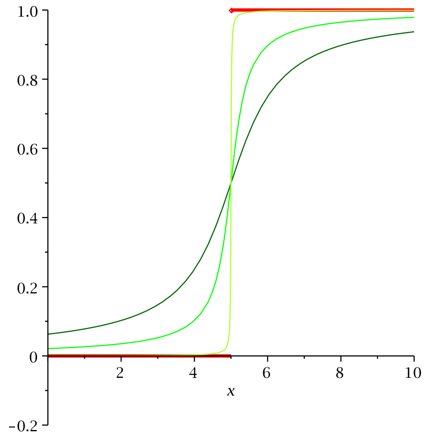
In order to preserve the desired convergence property

$$\hat{\Psi}_n \overset{!}{\to} \begin{cases} 0, & x < b \\ 1, & x > b \end{cases} \quad (n \to \infty)$$

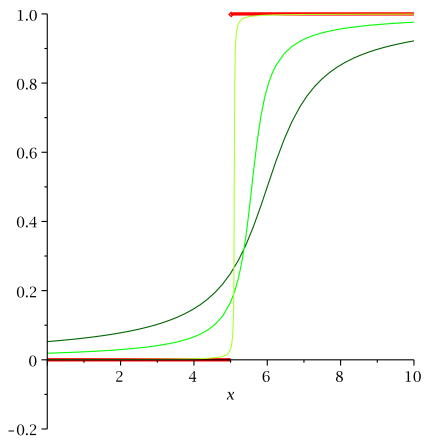
in (3.21), we have to restrict our choice of $(\varepsilon_n)_{n \in \mathbb{N}}$ to sequences with convergence order less than 1. Figure 3.8 sketches the stepwise construction of the lower-bound approximation for $\varepsilon_n := \sqrt{\frac{1}{n}}$:



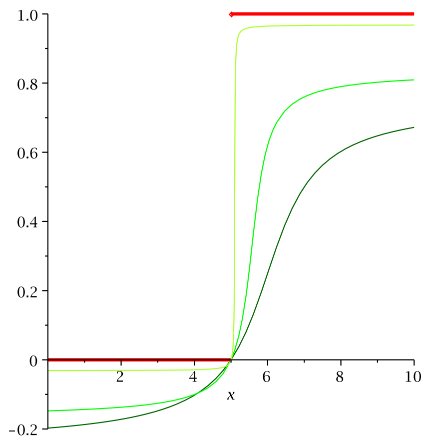
(a) Ψ only



(b) Ψ and Ψ_1, Ψ_3, Ψ_{100}



(c) Ψ and $\tilde{\Psi}_1$, $\tilde{\Psi}_3$, $\tilde{\Psi}_{100}$



(d) Ψ and $\hat{\Psi}_1$, $\hat{\Psi}_3$, $\hat{\Psi}_{100}$

Figure 3.8: Construction of a lower-bound approximation to the target function. The red curve is Ψ with barrier $b = 5$, the green curves are Ψ_n, $\tilde{\Psi}_n$, or $\hat{\Psi}_n$, where in each subfigure only one of these is plotted for $n \in \{1, 3, 100\}$ (dark to light green).

Parts a) and b) are the same as in figure 3.6 and illustrate the target and the first step in approximation by picking Ψ_n as described in (3.12). The green curves in part c) correspond to the right-shifted $\tilde{\Psi}_n$'s from (3.20) and part d) depicts the right- and down-shifted $\hat{\Psi}_n$ for $n = 1, 3, 100$.

Discussion

We have in general discussed a variation of the martingale methodology that splits the domain of a non-concave target in a finite number of open intervals and individually operates on each of those. For the example of Browne (1999) we then have in various steps constructed a suitable point-wise approximation of the discontinuous bonus wage Ψ by a sequence of such continuous but non-concave functions.

A key assumption to that approach is the existence of a mapping that allows to allocate the proper open interval to any randomly observed value of the state-price density. To satisfy that assumption in a Black-Scholes market we had to restrict the choice of admissible strategies to constant ones. Hence only a lower boundary for the problem value can be achieved.

There are of course many ways to approach the original problem of non-concave and non-convex utilities of wages. The next section suggests a less direct and more elegant approach to the problem.

3.5 SAHARA Utility

Wage schedules often come in the form of bonus contracts that reward the agent significantly more generous (or even: only) if a previously fixed bonus level is achieved. The basic example is the all-or-nothing contract of Browne (1999) but many variations can be thought of.

The bonus level reflects the principal's goals for the portfolio. From his point of view the agent is supposed to take some risk if portfolio wealth is (still) below the targeted level and shall reduce the riskiness of her actions when the target can already be reached.

In mathematical terms this translates to an increasing absolute risk aversion for wealths lower than the target level and a decreasing absolute risk aversion for wealths above that level. But that property cannot be realized by standard tools as power, logarithmic, or exponential utility.

Chen, Pelsser & Vellekoop (2011) introduce the SAHARA (Symmetric Asymptotic Hyperbolic Absolute Risk Aversion) class of utility functions which exhibits the property asked for. They call the wealth, at which risk aversion changes its slope from increasing to decreasing, the threshold level and assume it is located at zero wealth. But in practice target wealths and bonus goals are set to positive levels, yielding thresholds larger than zero.

In this section the SAHARA class of utility functions is discussed for non-zero threshold levels. Along the lines of Chen, Pelsser & Vellekoop (2011) we discuss the necessary modifications and solve the SAHARA portfolio problem.

SAHARA Utility with Non-Zero Threshold Level

Any utility $U : \mathbb{R} \to \mathbb{R}$ of SAHARA type is characterized and defined by its absolute risk aversion

$$\mathrm{ARA}_U(x) = -\frac{U''(x)}{U'(x)} := \frac{\alpha}{\sqrt{\beta^2 + (x-d)^2}}, \; x \in \mathbb{R}. \tag{3.22}$$

The parameter $\alpha > 0$ can, in the limit, be interpreted as the relative risk aversion:

$$\lim_{x\to\infty} \mathrm{RRA}_U(x) = \lim_{x\to\infty} x\mathrm{ARA}_U(x) = \alpha.$$

$\beta > 0$ scales the effect of increasing and decreasing absolute risk aversion around the threshold.

And d finally is the threshold wealth and determines the location where absolute risk aversion ceases to increase and starts to decrease.

We want to derive U from (3.22). Set $V(x) := U'(x)$. Then one has to solve

$$V'(x) = -\frac{\alpha}{\sqrt{\beta^2 + (x-d)^2}} V(x).$$

The solution is

$$V(x) = c_2 \left((x-d) + \sqrt{\beta^2 + (x-d)^2} \right)^{-\alpha} \tag{3.23}$$

where the free parameter c_2 needs to be positive to ensure U is strictly increasing. Using $\operatorname{arsinh}(x) = \log(x + \sqrt{x^2+1})$ one may rewrite (3.23) to

$$V(x) = c_2 \beta^{-\alpha} \exp\left(\operatorname{arsinh}\left(\frac{x-d}{\beta} \right) \right)^{-\alpha}.$$

Abbreviate $\gamma(x) := \sqrt{\beta^2 + (x-d)^2}$. To find $U(x) = \int V(x)dx$, substitute $y := x - d$, $x = y + d$, $dy = dx$. Then, following Chen, Pelsser & Vellekoop (2011, Prop. 2.2),

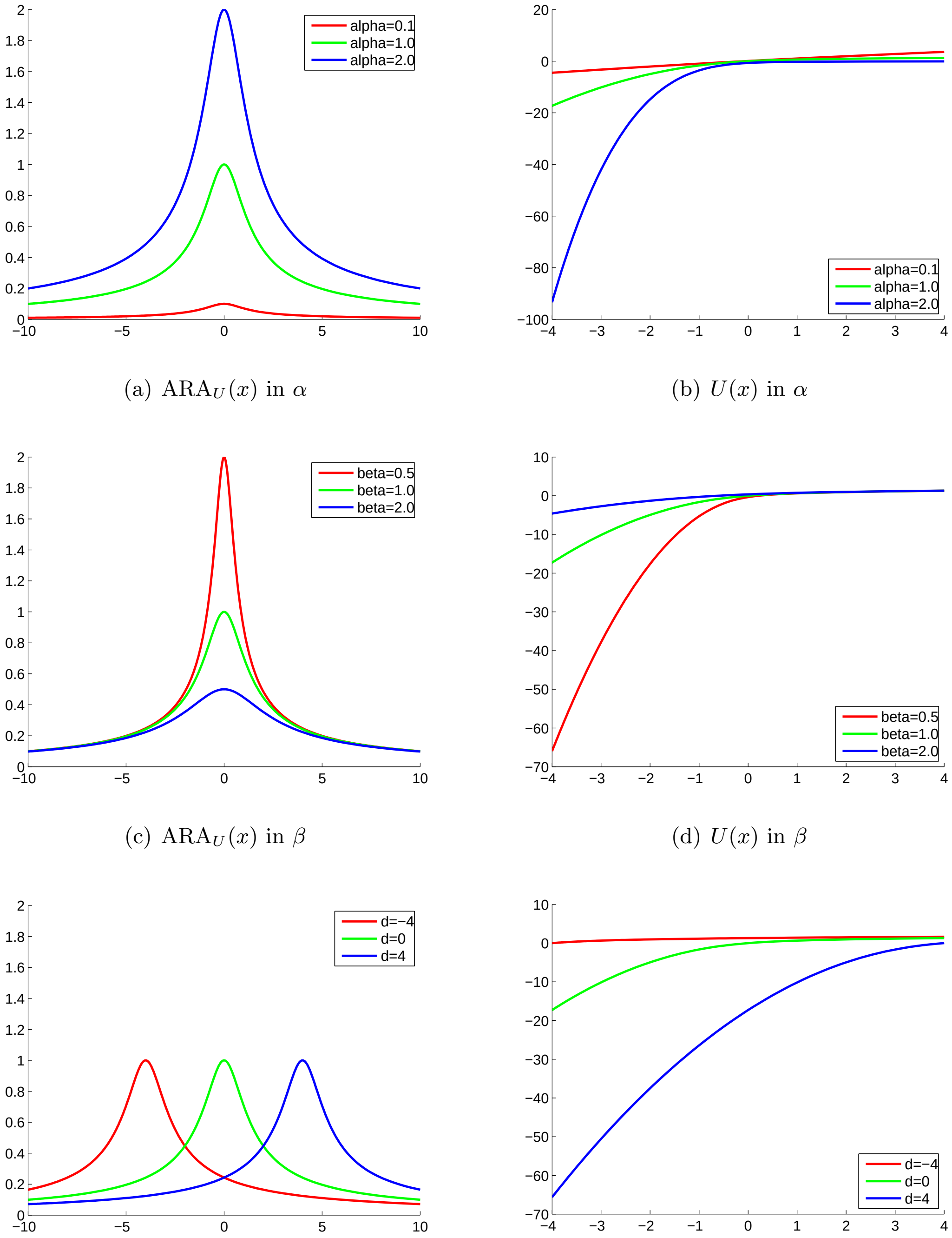
$$U(x) = c_1 + c_2 \cdot \begin{cases} -\frac{1}{\alpha^2-1} \left((x-d) + \gamma(x)\right)^{-\alpha} \left((x-d) + \alpha\gamma(x)\right) & \alpha \neq 1 \\ \frac{1}{2} \log\left((x-d) + \gamma(x)\right) - \frac{1}{2}\beta^{-2}(x-d)\left((x-d) - \gamma(x)\right) & \alpha = 1 \end{cases} \tag{3.24}$$

with $c_1 \in \mathbb{R}$, $c_2 > 0$ describes the family of all SAHARA utility functions.

Figure 3.9 gives an intuition of the role of all three parameters in terms of absolute risk aversion and the implied SAHARA utility (with $c_1 = 0$ and $c_2 = 1$):

In parts a) and b) the asymptotic relative risk aversion α is varied. The maximum of the absolute risk aversion is located at d and has the value $\mathrm{ARA}_U(d) = \frac{\alpha}{\beta}$. Thus increasing α also increases the maximal risk aversion exhibited by the agent while the effect is reversed by increasing β, as parts c) and d) illustrate.

Shifting the threshold level in parts e) and f) translates ARA_U and U by d.



(a) $\mathrm{ARA}_U(x)$ in α

(b) $U(x)$ in α

(c) $\mathrm{ARA}_U(x)$ in β

(d) $U(x)$ in β

(e) $\mathrm{ARA}_U(x)$ in d

(f) $U(x)$ in d

Figure 3.9: Sensitivity of $\mathrm{ARA}_U(x)$ (left column) and $U(x)$ (right column) in α (first row), β (second row), and d (third row). All green curves coincide and represent $(\alpha, \beta, d) = (1, 1, 0)$. This setting is varied in each row by decreasing (the red curve) or increasing (the blue curve) one parameter.

Note that U is concave for any parameter choice. The threshold level is not so easy to deduce from the graph of U, but of course obvious when examining ARA_U.

Finally note that similar to the steps in Chen, Pelsser & Vellekoop (2011) one finds the asymptotic elasticity

$$\lim_{x\to\pm\infty} \frac{xU'(x)}{U(x)} = 1 \mp \alpha,\ \alpha > 0. \tag{3.25}$$

As $\alpha > 0$ it holds

$$\limsup_{x\to\infty} \frac{xU'(x)}{U(x)} < 1,$$

thus U exhibits reasonable asymptotic elasticity in the sense of (Kramkov & Schachermayer, 1999, definition 2.2). We will need that property when later determining the dual SAHARA utility and approaching the portfolio problem.

The Dual SAHARA Utility

To determine the maximizer of a function it is sometimes beneficial to consider the so-called convex dual of that function:

Definition 3.1 *Let $f \in C^1(\mathbb{R})$. Then*

$$\tilde{f}(y) := \sup_{x\in\mathbb{R}}\{f(x) - xy\}$$

*is denoted the **convex dual** of f.*

As the name suggests, the convex dual is always convex: For all $y, z \in \mathbb{R}$ and for all $\lambda \in [0,1]$ it holds

$$\begin{aligned}
& \tilde{f}(\lambda y + (1-\lambda)z) \\
= \ & \sup_{x\in\mathbb{R}}\{(\lambda + 1 - \lambda)f(x) - x(\lambda y + (1-\lambda)z)\} \\
\leq \ & \lambda \sup_{x\in\mathbb{R}}\{f(x) - xy\} + (1-\lambda)\sup_{x\in\mathbb{R}}\{f(x) - xz\} \\
= \ & \lambda\tilde{f}(y) + (1-\lambda)\tilde{f}(z).
\end{aligned}$$

That maximum is attained at $x = x(y) = (f')^{-1}(y)$. One may rewrite

$$\tilde{f}(y) = f\left((f')^{-1}(y)\right) - y(f')^{-1}(y)$$

and conclude that if $f'(0) = y_0$ then $(f')^{-1}(y_0) = 0$ and $\tilde{f}(y_0) = f(0)$. Further note

$$\tilde{f}'(y) = \underbrace{f'\left((f')^{-1}(y)\right)}_{=y}\left((f')^{-1}\right)'(y) - y\left((f')^{-1}\right)'(y) - (f')^{-1}(y) = -(f')^{-1}(y),$$

thus

$$\tilde{f}(y) = -\int (f')^{-1}(y)dy. \tag{3.26}$$

Let us now examine the dual SAHARA utility $\tilde{U}$. Employing $y = \operatorname{arsinh}(x)$ being the inverse of $x = \sinh(y) = \frac{1}{2}(e^y - e^{-y})$, we can conclude

$$\begin{aligned}
& y &=& \ U'(x) = V(x) = c_2\left((x-d) + \gamma(x)\right)^{-\alpha} \\
\iff & x &=& \ d + \frac{1}{2}\left(\left(\frac{y}{c_2}\right)^{-\frac{1}{\alpha}} - \beta^2\left(\frac{y}{c_2}\right)^{\frac{1}{\alpha}}\right) \\
& &=& \ d + \beta\sinh\left(-\frac{1}{\alpha}\log\left(\frac{y}{c_2}\right) - \log(\beta)\right) =: I(y).
\end{aligned} \tag{3.27}$$

The inverse first derivative I from (3.27) is called the inverse marginal utility.
We can now use (3.26) to determine $\tilde{U}$:
For SAHARA utilities with risk aversion $\alpha \neq 1$ we have

$$\begin{aligned}\tilde{U}(y) &= -\int I(y)dy \\ &= -\frac{1}{2}\int\left(\left(\frac{y}{c_2}\right)^{-\frac{1}{\alpha}} - \beta^2\left(\frac{y}{c_2}\right)^{\frac{1}{\alpha}} + 2d\right)dy \\ &= \frac{1}{2}\left(\frac{\alpha}{\alpha+1}\beta^2\left(\frac{y}{c_2}\right)^{\frac{1}{\alpha}} - \frac{\alpha}{\alpha-1}\left(\frac{y}{c_2}\right)^{-\frac{1}{\alpha}} - 2d\right)y + D,\ D \in \mathbb{R}.\end{aligned}$$

As $V(0) = c_2(\gamma(0) - d)^{-\alpha} =: y_0$, necessarily $\tilde{U}(y_0) = U(0)$. Hence,

$$\begin{aligned}\tilde{U}(y) = \frac{1}{2}\Bigg(&\frac{\alpha}{\alpha+1}\beta^2 c_2^{-\frac{1}{\alpha}}\left(y^{\frac{\alpha+1}{\alpha}} - y_0^{\frac{\alpha+1}{\alpha}}\right) - \\ &\frac{\alpha}{\alpha-1}c_2^{\frac{1}{\alpha}}\left(y^{\frac{\alpha-1}{\alpha}} - y_0^{\frac{\alpha-1}{\alpha}}\right) - 2d(y-y_0)\Bigg) + U(0).\end{aligned} \tag{3.28}$$

Let $\alpha = 1$. Then it holds

$$\begin{aligned}\tilde{U}(y) &= -\int I(y)dy \\ &= -\frac{1}{2}\int\left(\frac{c_2}{y} - \beta^2\frac{y}{c_2} + 2d\right)dy \\ &= \frac{1}{2}\left(\frac{\beta^2}{2c_2}y^2 - c_2\log(y) - 2dy\right) + E,\ E \in \mathbb{R}.\end{aligned}$$

With $y_0 = \frac{c_2}{\gamma(0)-d}$ this yields

$$\tilde{U}(y) = \frac{1}{2}\left(\frac{\beta^2}{2c_2}\left(y^2 - y_0^2\right) - c_2\log\left(\frac{y}{y_0}\right) - 2d(y-y_0)\right) + U(0). \tag{3.29}$$

The SAHARA Portfolio Problem

SAHARA type utilities in general operate on the complete real domain and can also map negative wealths to their associated utilities. Complementing that fact it seems sensible to choose a market model that admits negative wealths.

Let us consider a classical Black-Scholes market as introduced in section 2.3 and advise the agent to control the portfolio by choosing amounts θ of wealth. The wealth X^θ then develops according to the stochastic differential equation (2.11) and may become negative.

The SAHARA agent aims to achieve

$$\max_{\theta \in \mathcal{U}(x_0)} \mathbb{E}\left[U\left(X_T^\theta\right)\right] \tag{3.30}$$

where $X(0) = x_0 > 0$ and she can choose θ among the set $\mathcal{U}(x_0)$ of admissible strategies as defined in (2.12).

As discussed in section 3.3, decompose (3.30) in a static and a dynamic subproblem.

The Static SAHARA Problem

The static problem is to find the optimal terminal wealth X_T^* as that random variable solving

$$\max_{\theta \in \mathcal{U}(x_0)} \mathbb{E}\left[U\left(X^\theta(T)\right)\right] = \mathbb{E}\left[U\left(X^*(T)\right)\right]$$

under the condition that $X^*(T)$ is attainable by initial wealth x_0.

It is beneficial to reconsider the state-price density ζ before approaching the static problem: The definition $\zeta(t) = \frac{L(t)}{S^0(t)}$ yields the stochastic differential equation

$$d\zeta(t) = -r\zeta(t)dt - \vartheta\zeta(t)dB(t),\ 0 \leq t \leq T,$$

with initial value $\zeta(0) = \frac{L(0)}{S^0(0)} = \frac{1}{S^0(0)}$. Let us slightly generalize this to $\zeta(0) = \frac{\zeta_0}{S^0(0)}$ for some $\zeta_0 > 0$. The new parameter ζ_0 can be interpreted as Lagrange multiplier when translating the static problem in a Lagrangian setup.

With that modification $\zeta(T)$ can be rewritten as

$$\begin{aligned} \zeta(T) &= \frac{1}{S^0(T)}\left(\frac{S(T)}{\hat{C}}\right)^{-\frac{\vartheta}{\sigma}} \text{ with} \\ \hat{C} &:= S(0)\zeta_0^{\frac{\sigma}{\vartheta}} \exp\left(\frac{1}{2}\left(\mu + r - \sigma^2\right)T\right), \end{aligned} \tag{3.31}$$

i.e. it can be represented in terms of the stock price process.

Following (Cox & Huang, 1989, theorem 2.1), the solution to the static problem is found by calculating

$$\begin{aligned} X^*(T) &= I(\zeta(T)) \\ &= d + \beta \sinh\left(-\frac{1}{\alpha}\log\left(\frac{\zeta(T)}{c_2}\right) - \log(\beta)\right) \\ &\overset{(3.31)}{=} d + \beta \sinh\left(\frac{\lambda}{\alpha\sigma}\log\left(\frac{S(T)}{c_2^{-\frac{\sigma}{\lambda}}\hat{C}\exp\left(-\frac{\sigma r T}{\lambda}\right)}\right) - \log(\beta)\right) \\ &= d + \beta \sinh\left(p\log\left(\frac{S(T)}{C}\right) - \log(\beta)\right) \\ &= d + \frac{1}{2}\left(\left(\frac{S(T)}{C}\right)^p - \beta^2\left(\frac{S(T)}{C}\right)^{-p}\right). \end{aligned} \tag{3.32}$$

Here $p := \frac{\vartheta}{\alpha\sigma}$ and $C := c_2^{-\frac{\sigma}{\lambda}}\hat{C}\exp\left(-\frac{\sigma r T}{\vartheta}\right)$.

Note that the optimal terminal wealth is still parameterized with the Lagrangian ζ_0 via C and $\hat{C}$.

The Dynamic SAHARA Problem

As the market is complete, there exists a replicating portfolio for the claim $X^*(T)$ as acquired in (3.32). We can solve this dynamic problem by delta hedging. To that purpose we have to determine the complete optimal wealth process $(X^*(t))_{0 \leq t \leq T}$. As this market is arbitrage-free, the latter can be established by calculating

$$X^*(t) = S^0(t)\mathbb{E}^{\mathbb{Q}}\left[\left.\frac{X^*(T)}{S^0(T)}\right|\mathcal{F}(t)\right]. \tag{3.33}$$

To that end the following identity (obtained by an application of Bayes' rule and the moment generating function of the normal distribution) will prove useful:

$$\mathbb{E}^{\mathbb{Q}}\left[S(T)^{\nu} \,|\, \mathcal{F}(t)\right] = S(t)^{\nu} \exp\left(\nu(T-t)(r+\frac{1}{2}(\nu-1)\sigma^2)\right), \ \nu \in \mathbb{R}. \tag{3.34}$$

We can now directly assess (3.33):

$$\begin{aligned} X^*(t) &= S^0(t)\mathbb{E}^{\mathbb{Q}}\left[\frac{X^*(T)}{S^0(T)} \,\middle|\, \mathcal{F}(t)\right] \\ &= de^{-r(T-t)} + \frac{1}{2}e^{-r(T-t)} \times \\ &\quad \left(\left(\frac{1}{C}\right)^p \mathbb{E}^{\mathbb{Q}}\left[S(T)^p \,|\, \mathcal{F}(t)\right] - \beta^2 \left(\frac{1}{C}\right)^{-p} \mathbb{E}^{\mathbb{Q}}\left[S(T)^{-p} \,|\, \mathcal{F}(t)\right]\right) \\ &\overset{(3.34)}{=} de^{-r(T-t)} + \frac{1}{2}e^{(\frac{1}{2}\sigma^2 p^2 - r)(T-t)} \times \\ &\quad \left(\left(\frac{S_t e^{(T-t)(r-\frac{1}{2}\sigma^2)}}{C}\right)^p - \beta^2 \left(\frac{S_t e^{(T-t)(r-\frac{1}{2}\sigma^2)}}{C}\right)^{-p}\right) \\ &= de^{-r(T-t)} + b(t)\sinh\left(p\log\left(\frac{S_t e^{(T-t)(r-\frac{1}{2}\sigma^2)}}{C}\right) - \log(\beta)\right), \end{aligned} \tag{3.35}$$

where

$$b(t) := \beta \exp\left((T-t)\left(\frac{\vartheta^2}{2\alpha^2} - r\right)\right), \ 0 \le t \le T.$$

Note that $b(t) \le b(0) < \infty$, thus b is bounded.

We can now solve for the Lagrange parameter ζ_0 using the attainability constraint

$$X^*(0) \overset{!}{=} x_0.$$

Exploiting it yields

$$C = S(0)e^{(r-\frac{1}{2}\sigma^2)T}\left(\beta \exp\left(\operatorname{arsinh}\left(\frac{x_0 - de^{-rT}}{b(0)}\right)\right)\right)^{-\frac{1}{p}},$$

which can be re-translated in terms of $\hat{C}$ and ζ_0. As a result $X^*(t)$ can now be completely determined from the known market and portfolio parameters.

Considering $X^*(t) = X^*(t, S(t))$ a (feedback) function of stock prices, the associated optimal strategy replicating X^* is found by building the delta hedge portfolio:

$$\begin{aligned} \theta^*(t) &= S(t)\frac{\partial X^*(t)}{\partial S(t)} \\ &= b(t)\cosh\left(p\log\left(\frac{S(t)e^{(T-t)(r-\frac{1}{2}\sigma^2)}}{C}\right) - \log(\beta)\right)p \\ &= p\sqrt{b(t)^2 + \left(X^*(t) - de^{-r(T-t)}\right)^2}. \end{aligned} \tag{3.36}$$

As p is constant and $b(t)$ is bounded, this is an admissible solution.

The SAHARA Value Function

Of particular interest are the values $V(x,t) = \mathbb{E}\left[U(X^*(T))\,|\,X^{\theta^*}(t)=x\right]$ of expected SAHARA utility from terminal wealth given state (x,t).

As $V(x,t) = c_1 + c_2\hat{V}(x,t)$ with

$$\hat{V}(x,t) := \mathbb{E}\left[U(X^*(T))\,|\,X^{\theta^*}(t)=x,\ c_1=0,\ c_2=1\right],$$

we can set $c_1 = 0$ and $c_2 = 1$ in what follows. It holds

$$\begin{aligned}
\mathbb{E}\left[\zeta(T)\,|\,\mathcal{F}(t)\right] &= \mathbb{E}\Bigg[\zeta_0 \exp\left(-rt - \vartheta B(t) - \frac{1}{2}\vartheta^2 t\right) \times \\
&\qquad \exp\left(\left(-r - \frac{1}{2}\vartheta^2\right)(T-t) - \vartheta(B(T)-B(t))\right)\Bigg] \\
&= \zeta(t)e^{-r(T-t)}, \\
\mathbb{E}\left[\zeta(T)^{1+\frac{1}{\alpha}}\,\middle|\,\mathcal{F}(t)\right] &= \mathbb{E}\Bigg[\left(\zeta_0 \exp\left(-rt - \vartheta B(t) - \frac{1}{2}\vartheta^2 t\right)\right)^{1+\frac{1}{\alpha}} \times \\
&\qquad \exp\left(\frac{\alpha+1}{\alpha}\left(-r - \frac{1}{2}\vartheta^2\right)(T-t)\right) \times \\
&\qquad \exp\left(-\vartheta\frac{\alpha+1}{\alpha}(B(T)-B(t))\right)\,\Bigg|\,\mathcal{F}(t)\Bigg] \\
&= \zeta(t)^{1+\frac{1}{\alpha}} \exp\left(\frac{\alpha+1}{\alpha}\left(\frac{\vartheta^2}{2\alpha} - r\right)(T-t)\right),
\end{aligned}$$

and, with similar steps,

$$\mathbb{E}\left[\zeta(T)^{1-\frac{1}{\alpha}}\,\middle|\,\mathcal{F}(t)\right] = \zeta(t)^{1-\frac{1}{\alpha}} \exp\left(\frac{\alpha-1}{\alpha}\left(-\frac{\vartheta^2}{2\alpha} - r\right)(T-t)\right).$$

We can rewrite $X^*(t)$ from (3.35) in terms of $\zeta(t)$:

$$\begin{aligned}
& X^*(t) \\
= {} & \mathbb{E}\left[\frac{\zeta(T)}{\zeta(t)}X^*(T)\,\middle|\,\mathcal{F}(t)\right] \\
= {} & \mathbb{E}\left[\frac{\zeta(T)}{\zeta(t)}\left(d + \frac{1}{2}\left(\left(\frac{S(T)}{C}\right)^p - \beta^2\left(\frac{S(T)}{C}\right)^{-p}\right)\right)\,\middle|\,\mathcal{F}(t)\right] \\
= {} & \mathbb{E}\left[\frac{\zeta(T)}{\zeta(t)}\left(d + \frac{1}{2}\left(\zeta(T)^{-\frac{1}{\alpha}} - \beta^2\zeta(T)^{\frac{1}{\alpha}}\right)\right)\,\middle|\,\mathcal{F}(t)\right] \\
= {} & d\zeta(t)^{-1}\mathbb{E}\left[\zeta(T)\,|\,\mathcal{F}(t)\right] + \frac{1}{2}\zeta(t)^{-1}\mathbb{E}\left[\zeta(T)^{1-\frac{1}{\alpha}}\,\middle|\,\mathcal{F}(t)\right] - \frac{1}{2\beta^2}\zeta(t)^{-1}\mathbb{E}\left[\zeta(T)^{1+\frac{1}{\alpha}}\,\middle|\,\mathcal{F}(t)\right] \\
= {} & de^{-r(T-t)} + b(t)\sinh\left(-\frac{1}{\alpha}\log(\zeta(t)) + \frac{1}{\alpha}\left(r - \frac{\vartheta^2}{2}\right)(T-t) - \log(\beta)\right). \qquad (3.37)
\end{aligned}$$

The representation (3.37) can be solved for $\zeta(t)$:

$$\begin{aligned}
\zeta(t) &= e^{(r-\alpha r - \frac{\vartheta^2}{2} + \frac{\vartheta^2}{2\alpha})(T-t)} \times \\
&\qquad \left(X^*(t) - de^{-r(T-t)} + \sqrt{b(t)^2 + \left(X^*(t) - de^{-r(T-t)}\right)^2}\right)^{-\alpha}. \qquad (3.38)
\end{aligned}$$

Let

$$i(x,t) := e^{(r-\alpha r-\frac{\vartheta^2}{2}+\frac{\vartheta^2}{2\alpha})(T-t)}\left(x-de^{-r(T-t)}+\sqrt{b(t)^2+\left(x-de^{-r(T-t)}\right)^2}\right)^{-\alpha}$$

denote the right-hand side of (3.38) given $X^*(t)=x$. Then by (Cox & Huang, 1989, equation (2.53)) we can rewrite

$$\frac{\partial}{\partial x}\hat{V}(x,t) = i(x,t),$$

thus we find the value function by integrating $i(x,t)$ with respect to x.

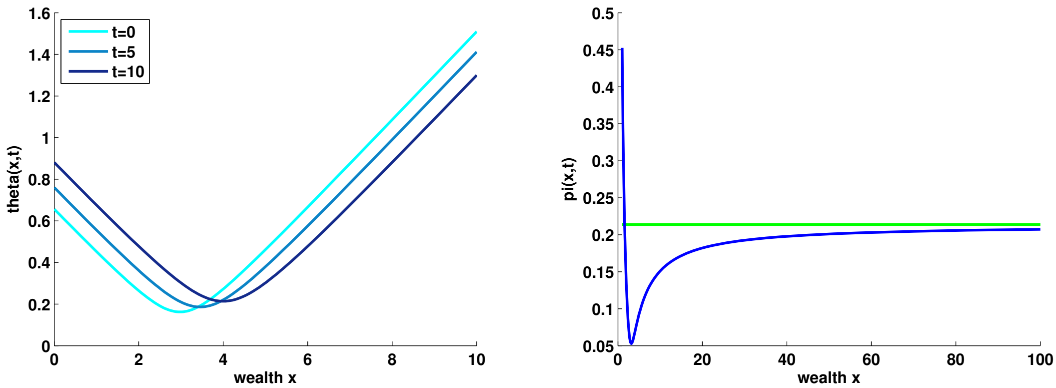
The substitution $y = x - de^{-r(T-t)}$ reduces this task to the case already discussed in Chen, Pelsser & Vellekoop (2011, proof of theorem 3.3). Resubstituting and taking into account the boundary condition $V(x,T) \overset{!}{=} U(x)$ for all $x \in \mathbb{R}$ and all choices of c_1 and c_2 finally yields

$$V(x,t) = c_1 + c_2\hat{V}(x,t), \tag{3.39}$$
$$\hat{V}(x,t) = \begin{cases} E(t)\left(x-D(t)+\gamma(x-D(t),t)\right)^{-\alpha}\left(x-D(t)+\alpha\gamma(x-D(t),t)\right) & \alpha \neq 1 \\ \frac{1}{2}\log\left(x-D(t)+\gamma(x-D(t),t)\right)+\frac{x-D(t)}{2b(t)^2}\left(\gamma(x-D(t),t)-(x-D(t))\right) & \alpha = 1 \end{cases},$$

where $D(t) := de^{-r(T-t)}$ and $E(t) := \frac{1}{1-\alpha^2}e^{(T-t)(1-\alpha)\left(r+\frac{\vartheta^2}{2\alpha}\right)}$, $t \in [0,T]$.

Discussion

The optimal SAHARA strategy exhibits several interesting properties. Let us discuss them with the aid of figure 3.10:



(a) $\theta(x,t)$ in x for various fixed t.

(b) $\pi(x,t)$ in x for $t=0$.

Figure 3.10: Aspects of the optimal SAHARA strategy $\theta = \theta(x,t)$.

For both parts of figure 3.10 a classical Black-Scholes market with riskless rate $r = \log(1.03)$, stock drift $\mu = \log(1.05)$, and volatility $\sigma = 30\%$ on a time horizon of $T = 10$ is the basis. The agent applies SAHARA utility with parameters $\alpha = 1$, $\beta = 1$, and $d = 4$. She aims to maximize her expected utility of terminal wealth as described in (3.30).

In part a) the agent's optimal strategy $\theta = \theta(x,t)$ is plotted as function of wealth x. The three curves correspond to the situations at $t = 0$ (light blue), $t = 5$ (medium blue), and $t = T = 10$ (dark blue).

Let $\underline{x}(t) := de^{-r(T-t)}$ the amount of wealth required at time t to reach terminal wealth d when from now on putting all capital in the bond. Then the minimum of each curve is located at $x = \underline{x}(t)$, i.e. the agent invests the smallest amount of money in the stock when her current portfolio wealth only just suffices to reach terminal wealth d at T.

The strategy value at the minimum is $\theta(\underline{x}(t), t) = pb(t) \geq 0$. The factor $b(t)$ linearly depends on the scaling factor β which in the limit reflects the agent's relative risk aversion. Hence a very cautions agent with β very close to zero (recall $\beta > 0$ by assumption) in that situation chooses to put (almost) all capital in the bond. Accordingly her wealth from then on evolves along the pure bond curve, yielding $\theta(X^\theta(s), s) \approx 0$ for all $t \leq s \leq T$. If the same agent arrives at some wealth $x > \underline{x}(t)$, she reserves $\underline{x}(t)$ for the bond and invests the surplus $x - \underline{x}(t)$ in the risky investment. Analogously, she invests no more than the missing amount $(\underline{x}(t) - x)$ in the stock when at some time she arrives at $x < \underline{x}(t)$.

Consider a risk-neutral agent who is contracted to a terminal wage that coincides with the SAHARA utility of terminal wealth, i.e. let $\Psi(x) = U(x)$ with U the SAHARA utility function on $\alpha > 0$, $\beta > 0$, and $d > 0$. We can then compare the incentives with the bonus contract $\Psi(x) = \mathbb{1}_{[x \geq d]}$ investigated by Browne (1999). The main difference is that the SAHARA agent is rewarded not only when beating d but also for smaller terminal wealths, though of course with less wage. Also it pays off to achieve terminal wealths larger than d. The latter effects in the agent still investing in the risky stock when it is no longer necessary to achieve d (scalable by β). Both agents are willing to risk the more the farther they are below of $\underline{x}(t)$, but the SAHARA agent will never choose to place arbitrarily large amounts of money in the stock, even when $t \to T$.

The relative risk attitude of the above SAHARA type agent is illustrated in part b) of figure 3.10. Here the fraction $\pi(x,t) := \frac{\theta(x,t)}{x}$ of wealth to be placed in the stock is plotted as function of wealth for fixed $t = 0$. As discussed for part a) the minimum is not zero since $\beta = 1 > 0$.

For x less than $\underline{x}(0)$ and decreasing, $\theta(x,t)$ increases, hence $\pi(x,t)$ grows, too. This reflects the decreasing absolute risk aversion as modeled in the first place. Analogously, the agent is willing to bet more capital on the stock when she has accumulated more than $\underline{x}(0)$, but in that direction the relative amount of wealth is bounded:

$$\lim_{x\to\infty} \frac{\theta(x,t)}{x} = p \lim_{x\to\infty} \sqrt{\frac{b(t)^2 + \left(X^*(t) - de^{-r(T-t)}\right)^2}{x^2}} = p = \frac{1}{\alpha} \cdot \frac{\mu - r}{\sigma^2}.$$

The limiting strategy coincides with the optimal behavior for an investor with power utility and relative risk aversion α. Since here $\alpha = 1$, the Merton ratio (in our market approximately 21%, sketched as the green line) is yielded.

Concluding, contracting a risk-neutral agent to a SAHARA type terminal wage sets similar incentives as a plain bonus contract, but avoids the unfavorable overreaction one observes for closing termination. Also, the SAHARA agent is motivated to obtain terminal wealths larger than the target level d while her bonus-type counterpart never exceeds that barrier.

4 Markov Chain Approximation of Bonus Wages

This chapter explains how time- and space-discrete Markov chains can be used to approximate the wealth diffusion process of an investor in a time- and space-continuous market setting and how this leads to approximately optimal strategies and value functions.

We will in the following consider portfolio problems of the type described in (2.3) for some wages ψ and Ψ.

The original thought is adapted from Munk (1997), its application to Merton's problem originates from Munk (2003). Both articles discuss chains with tridiagonal transition matrix, that is with non-zero probabilities for moves to the direct adjacent states only. Using their transition weights in setups with discontinuous initial (or bounding, for finite time) value function however resulted in slow convergence and turned out to be numerically unstable.

The main intention of the following is thus to generalize the setup to produce approximating chains with strictly positive transition probabilities for multiple states up and down. Additionally, approximation schemes using arbitrary grids are discussed. All results are derived for a broad class of markets with Itô stock dynamics and then specifically applied to the Black-Scholes market.

4.1 Discretization with Regular Grids

In order to compute solutions we first have to discretize the (originally continuous) state- and time domains.

To that purpose fix $h > 0$ and construct the lattice $\mathcal{R}_h := \{zh : z \in \mathbb{Z}\}$ of all possible[1] discretized wealth values. Then for the number N of time-steps choose a fixed interval[2] τ such that $N := \frac{T}{\tau} \in \mathbb{N}$ and write $\mathcal{T}_\tau := \{0, \tau, \ldots, N\tau\}$.

The general idea is to approximate the diffusion $(X_t)_{0\leq t\leq T}$ of (2.1) by a Markov chain $(\xi_n^h)_{n=0,\ldots,N}$ with state space $\mathcal{R}_h$ and transition probabilities $p^h(x, y|u, t)$ from state x to y at time $t \in \mathcal{T}_\tau \setminus \{T\}$ when the discrete strategy value is chosen as $u \in U$. Write $u^h := (u_0^h, \ldots, u_{N-1}^h) \in \mathcal{U}^h(x, t)$ if it is admissible at time $t \in \mathcal{T}_\tau$ with intermediate wealth x. For admissibility we require the resulting chain to be Markov, that is to satisfy

$$\begin{aligned} & \mathbb{P}(\xi_{n+1} = y \mid (u_0^h, \ldots, u_n^h), \xi_0 = x_0, \ldots, \xi_{n-1} = x_{n-1}, \xi_n = x) \\ = \; & \mathbb{P}(\xi_{n+1} = y \mid u_n^h, \xi_n = x) \end{aligned} \tag{4.1}$$

for all $n \in \{0, \ldots, N-1\}$ and whenever both conditional probabilities are well-defined. In particular we want to construct an approximation with transition probabilities

$$p(x, y|u_n^h, \tau n) := \mathbb{P}(\xi_{n+1} = y \mid u_n^h, \xi_n = x)$$

[1] One may also choose a subset, e.g. non-negative wealths only, or introduce an upper bound. We will get back to that later.

[2] Though not absolutely necessary, having a fixed time-interval is very convenient. Munk (1997) discusses a more general framework with value- and action-dependent intervals.

that do not depend on the time-step. Hence we additionally require the Markov chain to be stationary.

The discretization of (2.2) at $t = n\tau$ with $n = 0, \dots, N$ and $x \in \mathcal{R}_h$ then reads as

$$\begin{aligned} V^h(x,t) &:= \sup_{u \in \mathcal{U}^h(x,t)} \left\{ J^h(x,t;u) \right\}, \\ J^h(x, n\tau; u^h) &:= \mathbb{E}\left[\sum_{m=n}^{N-1} \psi(\xi_m^h, n, u_m^h)\tau + \Psi(\xi_N^h) \,\middle|\, \xi_n^h = x \right]. \end{aligned} \tag{4.2}$$

4.2 Local Consistency Conditions

While any choice of transition probabilities $p^h(x, y|u, t)$ defines some Markov chain (ξ_n^h), we are only interested in those which approximate the continuous-time diffusion.

With $\Delta\xi_n^h := \xi_{n+1}^h - \xi_n^h$ abbreviate the random increment after step $n < N$ and by $\mathbb{E}_{x,n}^{h,u}[\dots]$ and $\mathrm{Var}_{x,n}^{h,u}(\dots)$ denote the conditional expectation and variance given the history $\{\xi_i^h, u_i^h : i = 0, \dots, n-1\}$ along with the current state $(\xi_n^h, u_n^h) = (x, u)$.

Definition 4.1 (Local consistency)
The chain (ξ_n^h) is called ***locally consistent*** *with (X_t), if*

1. *their drifts coincide, that is for some $\alpha > 0$*

$$\mathbb{E}_{x,n}^{h,u}\left[\Delta\xi_n^h\right] = \tau f(x, n\tau, u) + o(\tau h^\alpha), \ (x, n\tau, u) \in \mathcal{R}_h \times \mathcal{T}_\tau \times U,$$

2. *further for some $\alpha > 0$ their variances satisfy*

$$Var_{x,n}^{h,u}\left(\Delta\xi_n^h\right) = \tau g(x, n\tau, u)^2 + o(\tau h^\alpha), \ (x, n\tau, u) \in \mathcal{R}_h \times \mathcal{T}_\tau \times U,$$

3. *and the one-step increments vanish in the sense*

$$\sup_{n,\omega} |\xi_{n+1}^h(\omega) - \xi_n^h(\omega)| \longrightarrow 0 \ (h \to 0).$$

We can translate a discrete-time Markov chain to continuous time by embedding it. Let $(\xi_t^h)_{0 \le t \le T}$ the time-continuous Markov chain with embedded chain $(\xi_n^h)_{n=0,\dots,N}$. Local consistency is then a necessary (and with some additional assumptions also sufficient) condition to provide weak convergence of (ξ_t^h) to (X_t) when $h \to 0$. This topic is discussed in detail in (Kushner & Dupuis, 2001, chapters 10 & 11), (Munk, 1997, section 8) and (Kushner, 1990, section 8).

4.3 A Scheme Including 2-Step Jumps

In order to solve the optimization problem (4.2) numerically, let us approximate the partial derivatives with finite differences.

There are many ways to specify a scheme of differences that converge to the partial derivatives, and in the way described below each choice leads to a weighted recursive problem formulation that can be evaluated numerically. By this choice one may that way e.g. influence to which (neighboring) states the chain can jump in one step.

If the scheme is selected with care and foresight, the resulting weights may be interpreted as (generalized) transition probabilities. But despite all degrees of freedom, it is crucial to pick a scheme which produces non-negative weights to ensure the convergence of those recursively found approximations to the correct value function.

Munk (2003) describes a scheme with jumps to the next neighbors only. We will now discuss a scheme that also allows for jumps of 2 states up or down and sketch how this can be generalized to arbitrary jumps.

At points $(x,t) \in \mathcal{R}_h \times \mathcal{T}_\tau$ and for arbitrary control values u one may approximate the derivatives in $\mathcal{A}^u J(x,t;u)$ with the following finite-differences scheme:

$$\begin{aligned}
J_t(x,t;u) &\approx D_t^{h,-} J^h(x,t;u) \\
&:= \frac{1}{\tau}\Big(J^h(x,t;u) - J^h(x,t-\tau;u)\Big), \\
J_x(x,t;u) &\approx \begin{cases} D_x^{h,+} J^h(x,t;u), & f(x,t,u) \geq 0 \\ D_x^{h,-} J^h(x,t;u), & f(x,t,u) < 0 \end{cases}, \\
&:= \begin{cases} \frac{1}{2h}\left(J^h(x+2h,t;u) - J^h(x,t;u)\right), & f(x,t,u) \geq 0 \\ \frac{1}{2h}\left(J^h(x,t;u) - J^h(x-2h,t;u)\right), & f(x,t,u) < 0 \end{cases}, \\
J_{xx}(x,t;u) &\approx D_x^{h,2} J^h(x,t;u) \\
&:= \frac{1}{h^2}\Big(J^h(x+h,t;u) - 2J^h(x,t;u) + J^h(x-h,t;u)\Big). \qquad (4.3)
\end{aligned}$$

Here D^+ denotes a forward difference and D^- denotes a backward difference. A positive drift f provides that an increase in wealth is more probable than a decrease. In order to reproduce the direction of wealth evolution, the forward difference is chosen whenever $f > 0$ and accordingly the backward difference is chosen whenever $f < 0$. It does not matter which one is assigned in the case $f = 0$.

For D^2 a second-order difference is employed to approximate second derivatives. We will in the following consider various variations of difference operators, but the basic principle is always the same.

Exploiting (4.3) in (2.4) we have

$$\begin{aligned}
0 &= \sup_{u\in U}\{\mathcal{A}^u V(x,t) + \psi(x,t,u)\} \\
&= \sup_{u\in U}\{\mathcal{A}^u J(x,t;u) + \psi(x,t,u)\} \\
&\approx \sup_{u\in U}\big\{[D_t^{h,-}J^h(x,t;u)] + [D_x^{h,+}J^h(x,t;u)]f^+(x,t,u) - \\
&\quad [D_x^{h,-}J^h(x,t;u)]f^-(x,t,u) + [\tfrac{1}{2}D_x^{h,2}J^h(x,t;u)]S(x,t,u) + \psi(x,t,u)\big\}. \qquad (4.4)
\end{aligned}$$

We reformulate the optimization problem in (4.4) as

$$J^h(x,t-\tau;u^*) \approx \sup_{u\in U} \sum_{y\in Y(x)} p^{h,\tau}(x,y|t,u) J^h(y,t;u) + \tau\psi(x,t,u). \qquad (4.5)$$

where u^* is the optimal strategy we are looking for. The set

$$Y(x) = Y(x,\mathcal{R}_h) := \{x-2h, x-h, x, x+h, x+2h\} \cap \mathcal{R}_h \qquad (4.6)$$

denotes the neighbors of $x \in \mathcal{R}_h$, relative to the chosen lattice $\mathcal{R}_h$, and for any $u \in U$ the weights $p^{h,\tau}(\cdot,\cdot|t,u)$,

$$\begin{aligned}
p^{h,\tau}(x, x \pm h|t,u) &:= \frac{\tau}{2h^2} S(x,t,u), \\
p^{h,\tau}(x, x \pm 2h|t,u) &:= \frac{\tau}{2h} f^{\pm}(x,t,u), \\
p^{h,\tau}(x, x|t,u) &:= 1 - \frac{\tau}{2h}|f(x,t,u)| - \frac{\tau}{h^2} S(x,t,u), \\
p^{h,\tau}(x, y|t,u) &:= 0, \ y \notin Y(x),
\end{aligned} \tag{4.7}$$

shall be interpreted as transition probabilities. Here $S(x,t,u) := g(x,t,u)^2$ is the dispersion of wealth.

The idea is to consider the right-hand side of (4.5) as target function in u and employ it to find an optimal strategy value at time $t-\tau$, where $t = T, \dots, \tau$ backwards iterates through $\mathcal{T}_\tau$.

In order to ensure strictly non-negative transition probabilities, we have to require

$$p^{h,\tau}(x,x|t,u) > 0 \iff \tau < \left(\frac{1}{2h}|f(x,t,u)| + \frac{1}{h^2} S(x,t,u)\right)^{-1} \ \forall (x,t,u) \in \mathcal{R}_h \times \mathcal{T}_\tau \times U.$$

Although it is desirable to keep the number of iterations small, it is convenient and sufficient to choose

$$\mathring{\tau} := \sup \left\{ \tau : \ \tfrac{T}{\tau} \in \mathbb{N}, \ \tau < \inf \left\{ \left(\frac{1}{2h}|f(x,t,u)| + \frac{1}{h^2} S(x,t,u) \right)^{-1} \right\} \right\} \tag{4.8}$$

where the infimum is taken among all $(x,t,u) \in \mathcal{R}_h \times \mathcal{T}_\tau \times U$.

Note that the transition probabilities $p^{h,\tau}(\cdot,\cdot|\cdot,\cdot)$ from (4.7) will only yield a stationary Markov chain if the maps f and g are time-independent. In section 4.4 we will see that this is in particular true for a classical Black-Scholes market.

Before moving on let us check the consistency of the scheme lined out by (4.5), (4.6), (4.7), and (4.8).

Local Consistency of the 2-Step Scheme

As the 2-step scheme does not allow jumps farther than 2 h-steps up or down, clearly

$$\sup_{n,\omega} |\xi^h_{n+1}(\omega) - \xi^h_n(\omega)| \le 2h \to 0 \ (h \to 0).$$

For $\xi^h_n = x \in \mathcal{R}_h$, arbitrary step numbers $n < N$ and strategy values $u \in U^h(x, n\tau)$ it holds

$$\begin{aligned}
\mathbb{E}^{h,u}_{x,n}\left[\Delta \xi^h_n\right] &= \left(\sum_{y \in Y(x)} y p^{h,\tau}(x,y|t,u) \right) - x \\
&= \frac{\tau}{2h}\big((x+2h) f^+(x,n\tau,u) + (x-2h) f^-(x,n\tau,u)\big) + \\
&\quad \frac{\tau}{2h^2}\left((x+h+x-h) S(x,n\tau,u)\right) + \\
&\quad x\left(1 - \frac{\tau}{2h}|f(x,n\tau,u)| - \frac{\tau}{h^2} S(x,n\tau,u)\right) - x \\
&= \tau f(x,n\tau,u)
\end{aligned}$$

and, taking (4.8) into account,

$$\begin{aligned}
&\mathrm{Var}^{h,u}_{x,n}\left(\Delta\xi^h_n\right)\\
=\ &\mathbb{E}^{h,u}_{x,n}\left[\left(\xi^h_n\right)^2\right] - \mathbb{E}^{h,u}_{x,n}\left[\xi^h_n\right]^2\\
=\ &\sum_{y\in Y(x)}\Big((y-x)^2 p^{h,\tau}(x,y|t,u)\Big) - (\tau f(x,t,u))^2\\
=\ &(2h)^2\frac{\tau}{2h}\left(f^+(x,n\tau,u)+f^-(x,n\tau,u)\right) + h^2\frac{\tau}{h^2}g(x,n\tau,u)^2 - (\tau f(x,n\tau,u))^2\\
=\ &\tau g(x,n\tau,u)^2 + o(\tau h^{1+\epsilon})\ \forall\epsilon>0.
\end{aligned}$$

Thus, the 2-step scheme yields a locally consistent Markov chain approximation.

4.4 Black-Scholes Approximation with Regular Grids

Let us now focus on the Black-Scholes market and discuss both control frameworks introduced in Section 2.3.

Controlling the Fraction of Wealth in the Risky Asset

In this market the wealth process is a geometric Brownian motion and therefore strictly positive. For all practical purposes we need a finite state space, so we introduce the artificial upper bound $L \in \mathbb{N}$ and redefine the state-space as $\mathcal{R}_h := \{0, h, \ldots, Lh\}$ with neighborhood $Y(\cdot, \mathcal{R}_h)$ constructed as in (4.6).

The control u coincides with the fraction π of wealth held in the risky asset. If we symmetrically constrain short-selling bond and stock to $\pi_t \in [-M, M]$ for all $t \in \mathcal{T}_\tau$, we can derive an explicit term for a small enough time delay τ^* respecting (4.8):

$$\begin{aligned}
&\inf_{(x,t,u)\in\mathcal{R}_h\times\mathcal{T}_\tau\times U}\left\{\left(\frac{1}{2h}|f(x,t,u)| + \frac{1}{h^2}S(x,t,u)\right)^{-1}\right\}\\
=\ &\inf_{(x,\pi)\in\mathcal{R}_h\times U}\left\{\left(\frac{1}{2h}|x(\pi(\mu-r)+r)| + \frac{1}{h^2}x^2\pi^2\sigma^2\right)^{-1}\right\}\\
=\ &\left(\sup_{(x,\pi)\in\mathcal{R}_h\times U}\left\{\frac{1}{2h}|x(\pi(\mu-r)+r)| + \frac{1}{h^2}x^2\pi^2\sigma^2\right\}\right)^{-1}\\
\geq\ &\left(\frac{1}{2h}\sup_{x\in\mathcal{R}_h}\{x\}\left(\sup_{\pi\in U}\{\pi\}|\mu-r|+r\right) + \frac{\sigma^2}{h^2}\sup_{x\in\mathcal{R}_h}\{x^2\}\sup_{\pi\in U}\{\pi^2\}\right)^{-1}\\
=\ &\left(\tfrac{1}{2}L\left(M|\mu-r|+r\right)+\sigma^2L^2M^2\right)^{-1} \ =:\ \tau_{\max},
\end{aligned}$$

hence $\tau^* := \sup\{\tau :\ \frac{T}{\tau}\in\mathbb{N},\ \tau\leq\tau_{\max}\}$ is a feasible choice.

Considering the modified state space we can now determine the matching transition probabilities: Assuming no external investors appear, insolvency terminates all business. Hence, $x = 0$ shall be an absorbing state. As even large amounts of wealth do not guarantee an enduring fortune, it seems natural to define the artificial upper bound $x = Lh$ as a reflecting state. Figure 4.1 lines out the scheme.

Before putting down all transition probabilities note that f and g describing the Black-Scholes wealth are time-dependent only in feedback form, that is $f(x,t;u) = f(x(t);u(t))$

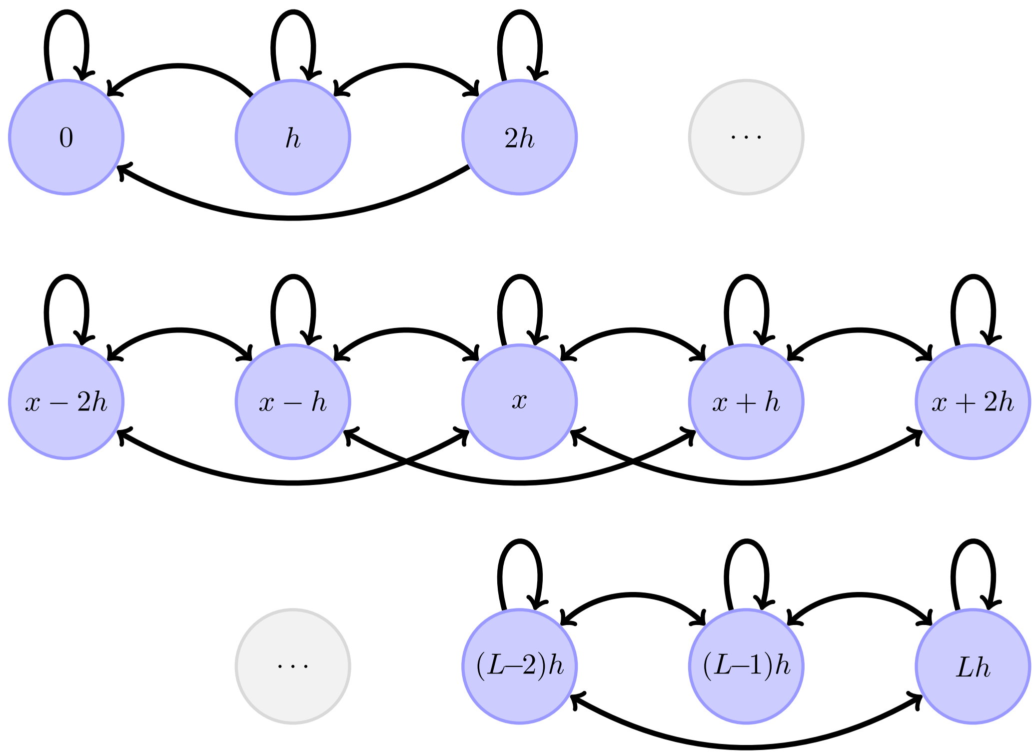


Figure 4.1: 2-step transition scheme with absorbing left and reflecting right boundary.

and $g(x,t;u) = g(x(t);u(t))$. Hence we can drop the time-condition in $\bar{p}^{h,\tau^*}(\cdot,\cdot|t,\cdot)$. Altogether, using (4.7), this yields the probabilities

$$
\begin{aligned}
p^{h,\tau^*}(x,y|\pi) &= 0,\ y \notin Y(x),\\
p^{h,\tau^*}(x,x\pm h|\pi) &= \frac{\tau^*}{2h^2}x^2\pi^2\sigma^2,\ x \in \mathcal{R}_h \setminus \{0, Lh\},\\
p^{h,\tau}(Lh,(L-1)h|\pi) &= \frac{\tau^*}{2h^2}x^2\pi^2\sigma^2,\\
p^{h,\tau^*}(x,x+2h|\pi) &= \begin{cases} \frac{\tau^*}{2h}x(\pi(\mu-r)+r), & \pi(\mu-r) > -r \text{ or } \mu = r\\ 0, & \text{otherwise}\end{cases}\\
&\quad \text{for } x \in \mathcal{R}_h \setminus \{0,(L-1)h, Lh\},\\
p^{h,\tau^*}(x,x-2h|\pi) &= \begin{cases} -\frac{\tau^*}{2h}x(\pi(\mu-r)+r), & \pi(\mu-r) < -r\\ 0, & \text{otherwise}\end{cases},\\
&\quad \text{for } x \in \mathcal{R}_h \setminus \{0,h\},\\
p^{h,\tau^*}(x,x|\pi) &= 1 - \sum_{y\in Y(x)\setminus\{x\}} p^{h,\tau^*}(x,y|\pi),\ x \in \mathcal{R}_h.
\end{aligned}
\tag{4.9}
$$

With the last line of (4.9) we ensure that for any fixed state x and any strategy π all transition probabilities $p^{h,\tau}(x,\cdot|\pi)$ sum up to 1, as one would expect.

Having truncated the state-space we have to reconsider the local consistency conditions:

For $\xi_n^h = x \in \mathcal{R}_h \setminus \{0, h, (L-1)h, Lh\}$, arbitrary step numbers $n < N$ and strategy values $u \in U^h(x, n\tau)$ everything works out as in section 4.2.

Similar calculations for $x = h$ and $x = (L-1)h$ yield the desired first and seconds moments one state below the upper boundary.

As (Munk, 1997, remark 8.1) explains, the transition probabilities at the reflecting boundary $x = Lh$ itself are of no concern for consistency.

The absorbing boundary $x = 0$ causes both moments to vanish, so to ensure local consistency here, we have to demand

$$f(0, n\tau, u) = 0 = g(0, n\tau, u),\ n = 0, \dots, N-1,\ u \in U^h(0, n\tau).$$

This is in particular true for the Black-Scholes market.

Controlling the Amount of Wealth in the Risky Asset

In this setup, wealth is no longer strictly positive. We therefore model

$$\mathcal{R}_h := \{-Lh, -(L-1)h, \dots, Lh\}$$

for some $L \in \mathbb{N}$ and again construct the appropriate relative neighborhood $Y(\cdot, \mathcal{R}_h)$ as in (4.6).

As explained in (2.12), to exclude riskless doubling we can only consider strategies that respect

$$\theta(t)^2 \leq K^2 \left(1 + \left(X^\theta(t)\right)^2\right)$$

for some fixed $K > 0$. As $|X^\theta(t)| \leq Lh$ at all times t, it holds

$$\theta(t)^2 \leq K^2(1 + (Lh)^2) \leq (K\tilde{L}h)^2$$

for all t, where

$$\tilde{L} := \min\left\{n \in \mathbb{N} : n^2 \geq L^2 + \frac{1}{h^2}\right\} > L$$

is chosen to simplify the following considerations.

We can now pick a safe τ^* by exploiting (4.8) in

$$\begin{aligned}
&\inf_{(x,t,u) \in \mathcal{R}_h \times \mathcal{T}_\tau \times U} \left\{\left(\frac{1}{2h}|f(x,t,u)| + \frac{1}{h^2}S(x,t,u)\right)^{-1}\right\} \\
=\ & \left(\sup_{(x,\theta) \in \mathcal{R}_h \times U} \left\{\frac{1}{2h}|\theta(\mu - r) + xr| + \frac{1}{h^2}\theta^2\sigma^2\right\}\right)^{-1} \\
\geq\ & \left(\frac{1}{2h}\left(\sup_{\theta \in U}\{|\theta|\}|\mu - r| + \sup_{x \in \mathcal{R}_h}\{|x|\}r\right) + \frac{1}{h^2}\sup_{\theta \in U}\{\theta^2\}\sigma^2\right)^{-1} \\
\geq\ & \left(\frac{1}{2}\left(K\tilde{L}|\mu - r| + Lr\right) + (K\tilde{L}\sigma)^2\right)^{-1} \quad =: \ \tau_{\max},
\end{aligned}$$

where again $\tau^* := \sup\{\tau : \frac{T}{\tau} \in \mathbb{N},\ \tau \leq \tau_{\max}\}$. For computational purposes, nevertheless, a stricter bound[3] is used.

[3]Using $\tilde{L}$ is convenient, but yields additional iterations in the implementation. Going back to L, it holds $\tau_{\max} < \tau_L$,

$$\tau_L := \left(\frac{1}{2h}\left(K\sqrt{1 + (Lh)^2}|\mu - r| + Lhr\right) + \frac{1}{h^2}K^2(1 + (Lh)^2)\sigma^2\right)^{-1}$$

Though falling to or below zero wealth no longer means bankruptcy, we still have to incorporate the artificial minimum and maximum wealth levels in the transition probabilities. We therefore choose a reflecting upper boundary at $x = Lh$ and also a reflecting lower boundary at $x = -Lh$. Figure 4.2 illustrates the proper scheme.

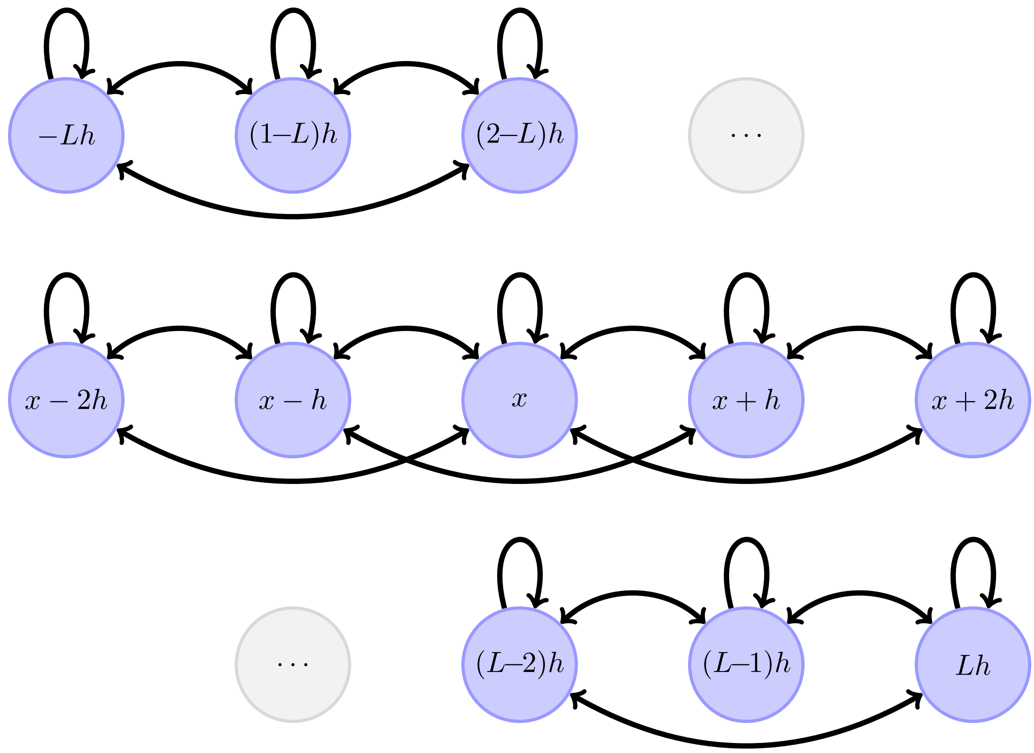


Figure 4.2: 2-step transition scheme with reflecting boundaries.

As already discovered for the fractionally controlled case, the wealth diffusion is time-dependent in feedback form only and we drop its explicit notation. With (4.7) we arrive at

$$\begin{aligned}
p^{h,\tau^*}(x,y|\theta) &= 0,\ y \notin Y(x),\\
p^{h,\tau^*}(x,x\pm h|\theta) &= \frac{\tau^*}{2h^2}\theta^2\sigma^2,\ x\in\mathcal{R}_h\setminus\{-Lh,Lh\},\\
p^{h,\tau}(\pm Lh,\pm(L-1)h|\theta) &= \frac{\tau^*}{2h^2}\theta^2\sigma^2,\\
p^{h,\tau^*}(x,x+2h|\theta) &= \begin{cases}\frac{\tau^*}{2h}\left(\theta(\mu-r)+xr\right), & \theta(\mu-r)>-xr\\ 0, & \text{otherwise}\end{cases},\\
&\quad\ \text{for } x\in\mathcal{R}_h\setminus\{(L-1)h,Lh\},\\
p^{h,\tau^*}(x,x-2h|\theta) &= \begin{cases}-\frac{\tau^*}{2h}\left(\theta(\mu-r)+xr\right), & \theta(\mu-r)<-xr\\ 0, & \text{otherwise}\end{cases},\\
&\quad\ \text{for } x\in\mathcal{R}_h\setminus\{-Lh,-(L-1)h\},\\
p^{h,\tau^*}(x,x|\theta) &= 1-\sum_{y\in Y(x)}p^{h,\tau^*}(x,y|\theta),\ x\in\mathcal{R}_h.
\end{aligned}\tag{4.10}$$

As a direct consequence of the consistency considerations in section 4.2 we have local consistency on $\mathcal{R}_h \setminus \{\pm Lh, \pm(L-1)h\}$, which may be extended to the states $\pm(L-1)h$ with similar calculations.

The choice of transition probabilities at the reflecting boundaries does, again, not matter for local consistency, as (Munk, 1997, remark 8.1) argues. Hence, (4.10) defines a locally consistent transition scheme.

An Example Application of the Markov Chain Method

Recall the SAHARA agent from section 3.5. We will now apply the Markov chain method to her optimization problem and compare the numerically approximated optimal strategy with the analytical solution.

Consider the classical Black-Scholes model with riskless rate $r = \log(1.03)$, stock drift $\mu = \log(1.05)$, and volatility $\sigma = 30\%$ on a time horizon of $T = 2$. The agent has relative limit risk aversion $\alpha = 2$, scales with $\beta = 1$ and is contracted to the threshold $d = 2$.

As Chen, Pelsser & Vellekoop (2011) suggest, let the agent control the amounts of wealth. With an initial endowment of one monetary unit, considering a terminal wealth in $[-10, 10]$ should suffice for most realizations. A finer discretization of states yields better results but at the same time increases the number of time steps and thereby the computation time. So let us agree on the symmetric state space

$$\mathcal{R}_h := \{-10, -9.75, -9.5, \ldots, 0, \ldots, 9.5, 9.75, 10\},$$

i.e. choose $h = \frac{1}{4}$ and $L = 40$. Constraining investments[4] with $K = \frac{1}{2}$, we arrive at $N = 38$ steps of $\tau \approx 0.05$ units of time. Figure 4.3 illustrates the computational results:

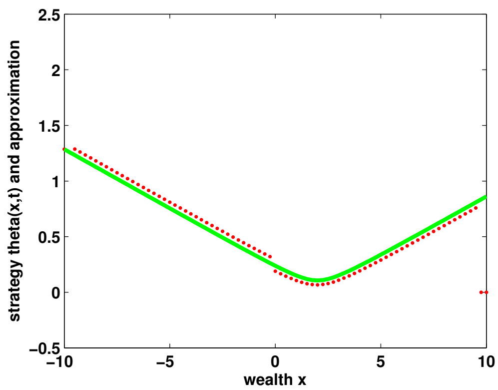
The approximated strategy values in part a) quite well approximate the analytical solution at most wealth states x. For $x > 0$ the value $\theta(x, t)$ is slightly underestimated while for $x < 0$ the approximation overestimates. The outliers at the boundaries are due to the artificially introduced reflecting states.

Due to the recursive nature of the algorithm, the approximated strategy values at step $t = T - \tau$ determine the value approximation at that step and these, in turn, influences the strategy at the next step, and so on. Accordingly, the errors arising at the boundaries propagate to the center of $\mathcal{R}_h$. The approximated strategy after $N = 38$ steps is depicted in part c). Here values below $x \approx -8$ and above $x \approx 8$ are unreliable while the remaining central values offer a good approximation to the analytical solution. One has to keep that effect in mind and, depending on the individual wealth and step number, reserve a sufficiently large buffer to the boundaries.

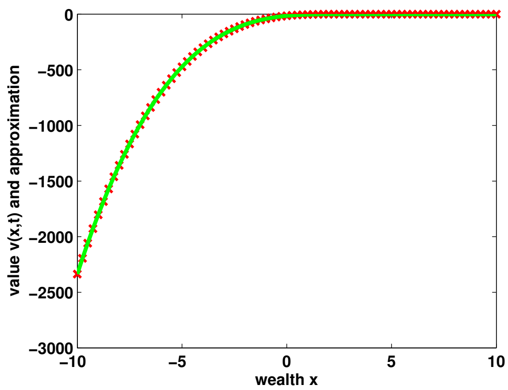
Accordingly the quality of approximation of the implied value function is good for central states and decreases from step to step at the boundaries. As we can learn from parts b) and d) the impact is not so severe for SAHARA utility as the value function covers a large domain and errors are relatively small. In fact, they are hard to discover at that scale.

Part e) provides a zoomed view on the right boundary of $v(x, t)$ at $t = 0$. Here the propagated errors yield dislocated values for large x.

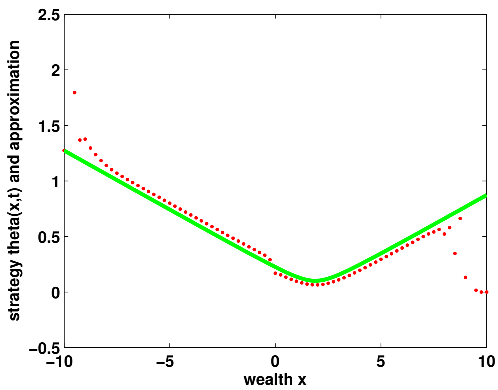
[4] The choice of K is important as the larger K the smaller τ^* and accordingly the more time-steps are necessary. But in order to find accurate solutions one should not begin the search with such small values as $K = \frac{1}{2}$. However, that choice is feasible here, and provides a conveniently small N.



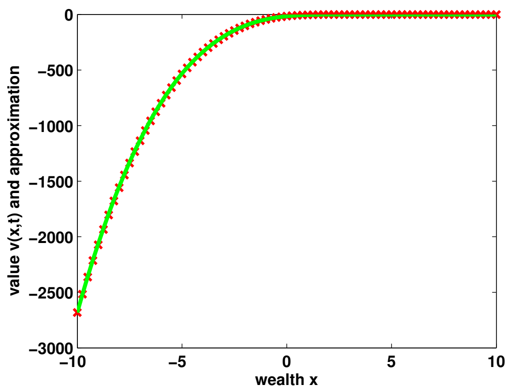
(a) $\theta(x, t)$ at $t = T - \tau$.



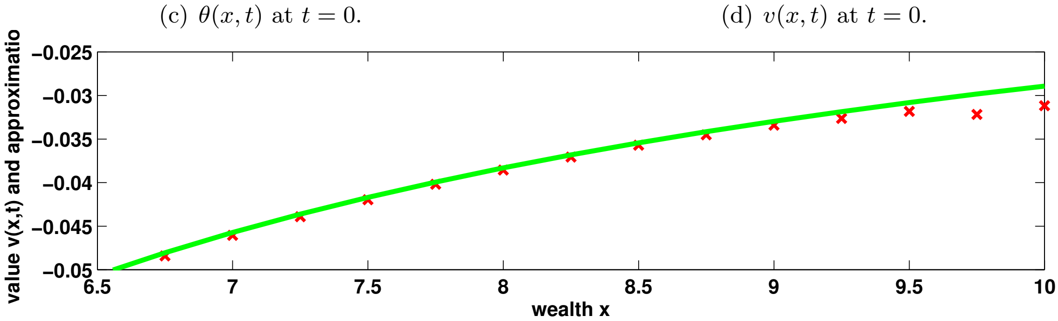
(b) $v(x, t)$ at $t = T - \tau$.



(c) $\theta(x, t)$ at $t = 0$.



(d) $v(x, t)$ at $t = 0$.



(e) $v(x, t)$ at $t = 0$, clipped and zoomed in.

Figure 4.3: Optimal strategies (left column) and value functions (right column) for the example SAHARA agent. The first row shows $t = T - \tau$ and the second row illustrates $t = 0$. In each part the straight green curve is the analytical solution and the red markings denote numerically computed values over $\mathcal{R}_h$.

4.5 Schemes with Farther Jumps

Any scheme we specify in this section will in our context get applied to the expected discounted utility function $J^h(x,t;u)$. As we have to deal with derivatives with respect to x or t only, we simplify the notation by informally omitting the discretization parameter h and also the strategy value u, writing just $J(x,t)$.

In the same spirit as when initially suggesting the 2-step scheme let us now proceed: The goal of this section is to derive non-vanishing transition probabilities to farther neighbors, e.g. to $Y(x) = \{x \pm ih : \ i = 0, \dots, K\} \cap \mathcal{R}_h$ for some $K \in \mathbb{N}$.

Indeed each choice of a finite-difference scheme for J_x, J_{xx}, and J_t implies a selection of weights. And we will achieve weights for exactly those transitions $x + zh$, $z \in \mathbb{Z}$, that appear in the finite-difference schemes.

The challenge is to pick a scheme that produces non-negative weights, allowing the interpretation as transition probabilities. We present 2 suggestions how to systematically approach the problem:

Augmenting the Spread

In the first or second derivative choose the augmented difference

$$\begin{aligned} D_x^{h,+} J(x,t) &= \frac{1}{nh}\left(J(x+nh,t) - J(x,t)\right), \\ D_x^{h,-} J(x,t) &= \frac{1}{nh}\left(J(x,t) - J(x-nh,t)\right), \\ D_x^{h,2} J(x,t) &= \frac{1}{mh^2}\left(J(x+mh) - 2J(x) + J(x-mh)\right). \end{aligned} \tag{4.11}$$

If one aims for an explicit approximation, the finite difference $D_t^{h,-}$ approximation to the time-derivative should stay unchanged: An augmentation to

$$D_t^{h,-} J(x,t) = \frac{1}{k\tau}(J(x,t) - J(x,t-k\tau))$$

yields a reverse recursive algorithm that skips $k-1$ time-steps, which does not make[5] much sense.

The scheme (4.11) always produces non-negative weights, but allows transitions only to the relative neighbors $\{x, x \pm nh, x \pm mh\}$. To include the full set $Y(x)$, one has to employ a different technique.

[5] at least the choice of τ should then be adjusted such that $N = \frac{T}{\tau} \equiv 0 \mod k$

Taylor Expansion

Assume the map $(x,t) \mapsto J(x,t)$ is continuous and sufficiently often differentiable with respect to x. Then the Taylor expansion yields

$$J(x+h) \approx \sum_{k=0}^{K} \frac{J^{(k)}(x)}{k!} h^k, \; J(x) \approx \sum_{k=0}^{K} \frac{J^{(k)}(x-h)}{k!} h^k, \; K \in \mathbb{N}.$$

Exploiting that, one can introduce additional evaluation points $x \pm kh$, $k = \{1, \dots, K\}$. The idea is applicable in various ways – and depending on the chosen depth levels for the approximation of the first and second derivative, each ansatz yields a different scheme.

A systematical approach is first to apply Taylor to the second derivative in (2.5) for some depth level $K \in \mathbb{N}$ and then to recursively approximate all appearing derivatives with finite differences.

To keep the finite differences as symmetric as possible, we alternately use the forward and backward version in the recursion. Putting

$$J^{(i,j)}(x,t) := \left(\frac{\partial}{\partial x}\right)^i \left(\frac{\partial}{\partial t}\right)^j J(x,t), \; i,j \in \mathbb{N},$$

this yields the finite difference approximations

$$\begin{aligned} J_{xx}(x,t) &\approx \sum_{k=1}^{K} \frac{h^k}{k!} J^{(k+2,0)}(x-h,t), \\ J^{(k,0)}(x,t) &\approx \begin{cases} D_x^{h,+} J^{(k-1,0)}(x,t), & k \text{ odd} \\ D_x^{h,-} J^{(k-1,0)}(x,t), & k \text{ even} \end{cases}, \; k = 1, \dots, K+2. \end{aligned} \tag{4.12}$$

In this context, $D_x^{h,+}$ and $D_x^{h,-}$ are meant as in (4.11) without augmentation (the $n = 1$ case).

With $K \in \mathbb{N}$ the resulting scheme will in general offer transition weights for

$$Y(x) = \{x \pm ih : \; i = 0, \dots, K+2\} \cap \mathcal{R}_h.$$

As an example let us consider the next obvious step in the chain: A scheme for up to 3 states up or down. Indeed, setting $K = 1$ in (4.12) we find

$$J_{xx}(x,t) \approx J^{(2,0)}(x-h,t) + hJ^{(3,0)}(x-h,t),$$

coinciding with Newton's rule. Now

$$\begin{aligned} J^{(1,0)}(x,t) &\approx D_x^{h,+} J(x,t) \\ &= \frac{1}{h}\left(J(x+h,t) - J(x,t)\right), \\ J^{(2,0)}(x,t) &\approx D_x^{h,-} D_x^{h,+} J(x,t) \\ &= \frac{1}{h^2}\left(J(x+h,t) - 2J(x,t) + J(x-h,t)\right), \\ J^{(3,0)}(x,t) &\approx D_x^{h,+} D_x^{h,-} D_x^{h,+} J(x,t) \\ &= \frac{1}{h^3}\left(J(x+2h,t) - 3J(x+h,t) + 3J(x,t) - J(x-h,t)\right), \end{aligned}$$

which by plugging it into the above yields the symmetric scheme

$$J_{xx}(x,t) \approx \frac{1}{h^2}\left(J(x+2h,t) - 3J(x+h,t) + 4J(x,t) - 3J(x-h,t) + J(x-2h,t)\right).$$

The weights for moving from x to $x \pm h$ are negative. This effect is intrinsic to the Taylor approach and cannot always be corrected. In the 3-step scheme we can compensate it by choosing the finite-difference scheme for the first derivative as in (4.3), that is

$$J_x(x,t) \approx \begin{cases} D_x^{h,+} J(x,t), & f(x,t,u) \geq 0 \\ D_x^{h,-} J(x,t), & f(x,t,u) < 0 \end{cases}.$$

By that we found another (non-negative) summand for the critical transition weights.

Note that though the construction of this scheme followed the Taylor idea, the above definition of $D_x^{h,\pm}$ slightly differs from what is formally used in the Taylor scheme (4.12). But as we are free to choose any scheme, as long as the local consistency conditions hold, it does not matter how we derive it.

4.6 Discretization with Arbitrary Grids

Until now we have chosen a fixed-width lattice to discretize the wealth space. The obvious benefit is the clarity and simplicity of the approach.

But when deploying such schemes in numerical calculations some drawbacks arise: As computations require finite minimal and maximal wealth levels which are not natural to the portfolio problem, artificial reflections need to be introduced at the boundaries. Those reflecting states cause numerical disturbances that originate at the boundaries and, due to the recursive calculation, propagate from there.

One way to counteract the issue is picking reflecting boundaries very far away from the region under examination. But that significantly adds to the computation time.

Observing that errors can in one step never propagate farther than the maximal distance to a neighbor, as determined by the transition scheme, another idea is to either diminish the number of time-steps or to choose a finer discretization of states. Unfortunately both approaches counteract in the sense that a finer discretization immediately yields a smaller time-step τ and that for a given discretization the largest τ to still ensure local consistency is fixed and one may not step over it.

Instead of trying any of the above suggestions let us weaken the assumption of equally spaced state lattices and allow arbitrary grid meshes. The intuition is to refine the grid at the boundaries and thereby introduce a certain buffer of additional states to hamper and slow down the error propagation.

Let

$$\mathcal{R}_H := \{x_1, \ldots, x_H\} \subset \mathbb{R},\ H \in \mathbb{N},$$

denote the pairwise distinct but apart from that arbitrary wealth levels. Without loss of generality assume $x_1 < \ldots < x_H$ (otherwise rearrange the order of indices). We keep the time-discretization $\mathcal{T}_\tau = \{0, \tau, \ldots, N\tau\}$ and by $(\xi_n^H)_{n=0,\ldots,N}$ denote the time- and state-discrete Markov chain running on $\mathcal{R}_H$.

For $t = n\tau$ with $n = 0, \ldots, N$ and $x \in \mathcal{R}_H$ the corresponding reformulation of the optimization problem (4.2) then reads as

$$\begin{aligned} J^H(x, n\tau; u^H) &:= \mathbb{E}\left[\sum_{m=n}^{N-1} \psi(\xi_m^H, n, u_m^H)\tau + \Psi(\xi_N^H) \,\middle|\, \xi_n^H = x\right], \\ V^H(x,t) &:= \sup_{u \in \mathcal{U}^H(x,t)} \left\{J^H(x,t;u)\right\}, \end{aligned} \tag{4.13}$$

where $\mathcal{U}^H$ is the straightforward modification of $\mathcal{U}^h$ to match the new state-space.

The next section discusses how to generalize the 2-step scheme (4.5) and derives a modified set of transition probabilities.

4.7 A Generalized 2-Step Scheme

To achieve a 2-step scheme similar to (4.5), fix $(x_i, t) \in \mathcal{R}_H \times \mathcal{T}_\tau$ and $u \in \mathcal{U}^H(x,t)$. One may approximate

$$\begin{aligned} J_t(x_i,t;u) &\approx \frac{1}{\tau}\left(J^H(x_i,t;u) - J^H(x_i,t-\tau;u)\right), \\ J_x(x_i,t;u) &\approx \begin{cases} \frac{1}{x_{i+2}-x_i}\left(J^H(x_{i+2},t;u) - J^H(x_i,t;u)\right), & f(x_i,t,u) \geq 0,\ 1 \leq i \leq H\text{–}2 \\ \frac{1}{x_i - x_{i-2}}\left(J^H(x_i,t;u) - J^H(x_{i-2},t;u)\right), & f(x_i,t,u) < 0,\ 3 \leq i \leq H \end{cases}, \\ J_{xx}(x_i,t;u) &\approx \frac{1}{(x_{i+1}-x_i)(x_i-x_{i-1})}\left(J^H(x_{i+1},t;u) - 2J^H(x_i,t;u) + J^H(x_{i-1},t;u)\right). \end{aligned} \tag{4.14}$$

The above scheme is well-defined for $H \geq 5$ only (what for any practical purpose may be assumed) and does, for obvious reasons, not (yet) specify finite-difference approximations at the boundaries. Exploit (4.14) in (2.4):

$$J^H(x_i, t-\tau; u) \approx \sum_{y \in Y(x_i)} p^{H,\tau}(x_i, y|t,u) J^H(y,t;u) + \tau\psi(x_i,t,u), \tag{4.15}$$

where now

$$Y(x_i) = Y(x_i, \mathcal{R}_H) := \{x_k : 1 \leq k \leq H, |i-k| \leq 2\} \subset \mathcal{R}_H,\ x_i \in \mathcal{R}_h,$$

is the relative neighborhood of x_i in $\mathcal{R}_H$ and the transition probabilities $p^{H,\tau}(\cdot,\cdot|t,u)$ are found as

$$\begin{aligned} p^{H,\tau}(x_i, x_{i+2}|t,u) &= \frac{\tau}{x_{i+2}-x_i} f^+(x_i,t,u),\ 2 \leq i \leq H-2, \\ p^{H,\tau}(x_i, x_{i-2}|t,u) &= \frac{\tau}{x_i - x_{i-2}} f^-(x_i,t,u),\ 3 \leq i \leq H, \\ p^{H,\tau}(x_i, x_{i\pm 1}|t,u) &= \frac{\tau}{2(x_{i+1}-x_i)(x_i-x_{i-1})} S(x_i,t,u),\ 2 \leq i \leq H-1, \\ p^{H,\tau}(x_i, x_i|t,u) &= 1 - \sum_{y \in Y(x_i)\setminus\{x_i\}} p^{H,\tau}(x_i,y|t,u),\ 3 \leq i \leq H-2, \\ p^{H,\tau}(x_i, y|t,u) &= 0,\ y \notin Y(x_i). \end{aligned} \tag{4.16}$$

Note that (4.15) formally holds for $x_i \in \mathcal{R}_H \setminus \{x_1, x_2, x_{H-1}, x_H\}$ only, while some of the transition probabilities in (4.16) can naturally be extended to further elements of $\mathcal{R}_H$. The choice of the remaining transition probabilities depends on what behavior at the boundaries is desired:

- For a reflecting upper boundary (as appropriate for all applications discussed here) choose

$$\begin{aligned}
p^{H,\tau}(x_H, x_{H-1}|t,u) &:= 0, \\
p^{H,\tau}(x_H, x_H|t,u) &:= 1 - \sum_{y \in Y(x_H)\setminus\{x_H\}} p^{H,\tau}(x_H, y|t,u), \\
p^{H,\tau}(x_{H-1}, x_{H-1}|t,u) &:= 1 - \sum_{y \in Y(x_{H-1})\setminus\{x_{H-1}\}} p^{H,\tau}(x_{H-1}, y|t,u).
\end{aligned}$$

- Analogously, a reflecting lower boundary (needed e.g. when using the amount of wealth as a control) exhibits

$$\begin{aligned}
p^{H,\tau}(x_1, x_2|t,u) &:= 0, \\
p^{H,\tau}(x_1, x_3|t,u) &:= \frac{\tau}{x_3 - x_1} f^+(x_1, t, u), \\
p^{H,\tau}(x_1, x_1|t,u) &:= 1 - \sum_{y \in Y(x_1)\setminus\{x_1\}} p^{H,\tau}(x_1, y|t,u), \\
p^{H,\tau}(x_2, x_2|t,u) &:= 1 - \sum_{y \in Y(x_2)\setminus\{x_2\}} p^{H,\tau}(x_2, y|t,u).
\end{aligned}$$

- An absorbing lower boundary (for control-by-fraction problems) finally needs

$$p^{H,\tau}(x_1, x_3|t,u) := 0,$$

  while all other probabilities can be chosen as in the reflecting lower boundary case.

Figure 4.4 gives sketches of all above situations.

In contrast to the situation with fixed-width grids, transition probabilities from x_i to $x_j \in Y(x_i)$ now depend on the particular position of the nearest neighbors. To ensure all these probabilities lie in $[0,1]$, we have to demand

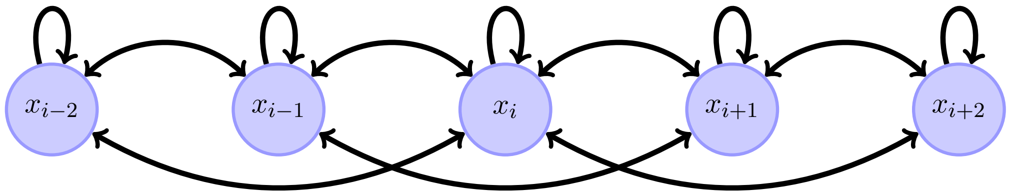
$$p^{H,\tau}(x_i, x_i|t,u) > 0, \ i \in \{1, \dots, H\}. \tag{4.17}$$

There will in general be no other way to satisfy (4.17) than to introduce state-dependent $\tau = \tau(x_i)$, potentially varying along the entire grid. One may be tempted, to individually choose the locally largest $\tau(x_i)$ at each grid position, but this will in general not yield computable schemes.

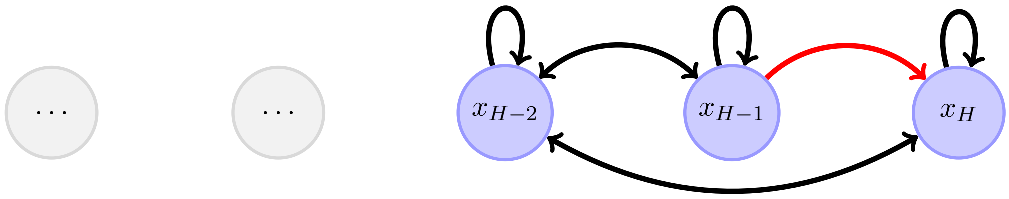
One idea to provide a well-defined and computable scheme is to select admissible time-steps such that each next-larger value is a natural multiple of the next-smaller one.

Consider, for example, the grid $\{0, 1, \dots, 9\} \cup \{10, 20, \dots, 90\} \cup \{91, 92, \dots, 100\}$. This is a coarse cast of $[0, 100]$ with refined tails. Having picked an appropriate τ_1 with $T = N_1\tau_1$ and $N_1 \in \mathbb{N}$ for the central region, a smaller $\tau_2 < \tau_1$ is required at the boundaries. Choosing $\tau_1 = N_2\tau_2$ with $N_2 \in \mathbb{N}$ such that τ_2 is admissible for the finer grid, one can follow algorithm 4.1:

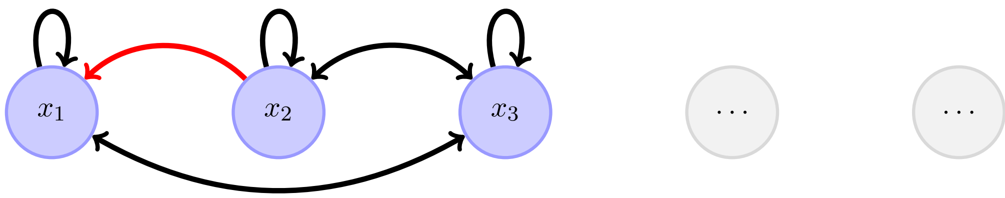
It remains the question how to deal with transitions from states on the finer grid to states on the coarser grid. While working the inner loop, these cannot take place, so one has to artificially adjust the affected transition probabilities. No modification is needed for the execution of the outer loop, as then every transition is feasible.



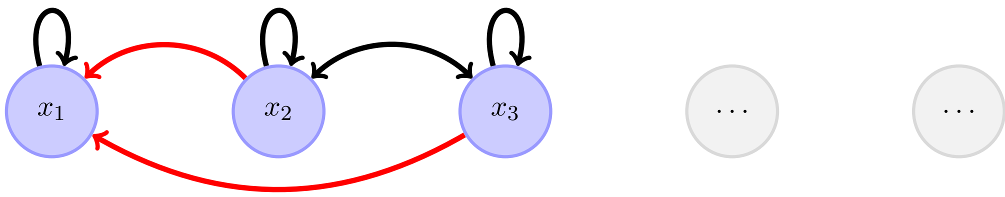
(a) The common interior for $H \geq 5$, $i \in \{3, \dots, H-2\}$.



(b) The reflecting upper boundary for $H \geq 5$ – note there is no transition from x_H to x_{H-1}.



(c) The reflecting lower boundary case for $H \geq 5$ – note there is no transition from x_1 to x_2.



(d) the absorbing lower boundary case for $H \geq 5$ – there are no transition leaving x_1

Figure 4.4: Sketch of allowed transition for all model variations of the 2-step scheme on arbitrary grids.

Algorithm 4.1: ITERATEGRIDREFINEMENT

input: tau1, tau2, T
for t *from* T *down to* tau1 **do**
  for s *from* (t − tau2) *down to* (t − tau1 + tau2) **do**
    calculate the MC approximation on the finer grid parts at time s
  end
  calculate the MC approximation on the entire grid at time (t − tau1)
end

The local consistency property given in Definition 4.1 is not applicable to non-equidistant grids, so the desired convergence of the Markov chain approximation still needs to be established. (Kushner & Dupuis, 2001, chapters 5.5 and 6.6) discuss the convergence issue and find justification[6] for a two-dimensional variation of the example given above.

Another approach is to use the smallest grid inter-space $\Delta x := \min\{x_{i+1} - x_i : i = 1, \ldots, H-1\}$ to resemble an equidistant grid. This yields a common τ applicable to the whole arbitrary grid. Following the latter suggestion, a sufficient rule for choosing τ in a Black-Scholes market is discussed in the next section.

4.8 Black-Scholes Approximation with Arbitrary Grids

Let us now apply the transition scheme for arbitrary grids, as suggested in section 4.7, to the Black-Scholes market and specify all missing details.

Drift and volatility are constant over time in the classical Black-Scholes market, so as above drop the time argument in notation. The portfolio is controlled either by the fraction $u = \pi$ of wealth in stock, or by the amount $u = \theta$ of wealth in the risky asset. We discuss both variations:

Controlling the Fraction of Wealth in the Risky Asset

In a sensible model for this case one will set $x_1 = 0$ to cover bankruptcy. As the sequence $x_1, \ldots, x_H$ strictly increases, we have $x_i \geq 0$ for $i = 1, \ldots, H$ and thus

$$f^{\pm}(x_i, t, u) \equiv f^{\pm}(x_i, \pi) = x(\pi(\mu - r) + r)^{\pm}$$

and

$$S(x_i, t, u) \equiv S(x_i, \pi) = x_i^2 \pi^2 \sigma^2.$$

Initially, fix an arbitrary $\tau > 0$. Exploiting (4.16) with absorbing lower and reflecting upper boundary, the transition probabilities read as

$$\begin{aligned}
p^{H,\tau}(x_i, x_j|\pi) &= 0,\ x_j \notin Y(x_i), \\
p^{H,\tau}(x_1, x_3|\pi) &= 0, \\
p^{H,\tau}(x_i, x_{i+2}|\pi) &= \frac{\tau x_i}{x_{i+2} - x_i}(\pi(\mu - r) + r)^+,\ i = 2, \ldots, H-2, \\
p^{H,\tau}(x_i, x_{i-2}|\pi) &= \frac{\tau x_i}{x_i - x_{i-2}}(\pi(\mu - r) + r)^-,\ i = 3, \ldots, H, \\
p^{H,\tau}(x_1, x_2|\pi) &= 0, \\
p^{H,\tau}(x_H, x_{H-1}|\pi) &= 0, \\
p^{H,\tau}(x_i, x_{i+1}|\pi) &= p^{H,\tau}(x_i, x_{i-1}|\pi) \\
&= \frac{\tau \sigma^2 \pi^2 x_i^2}{2(x_{i+1} - x_i)(x_i - x_{i-1})},\ i = 2, \ldots, H-1, \\
p^{H,\tau}(x_i, x_i|\pi) &= 1 - \sum_{y \in Y(x_i) \setminus \{x_i\}} p^{H,\tau}(x_i, y|\pi),\ i = 1, \ldots, H.
\end{aligned}$$

[6] Note that the overall setup of their book does not include the control-dependent volatility case which is admitted here. In a research article, Kushner (1990) argues that virtually all properties of the control-independent case still hold under mild assumptions, though he does not explicitly address variable grids.

As in the equidistant case, bind π to $[-M, M]$. We now derive a $\tau_{\max}$ sufficiently small such that the T-fractional $\tau^* := \sup\{\tau : \frac{T}{\tau} \in \mathbb{N}, \tau \leq \tau_{\max}\}$ ensures local consistency.

To this purpose note that the probability to stay in the same state is minimal at inner states, that is for $i \in \{3, \ldots, H-2\}$. Hence we pick an inner state x_i and sufficiently require $p^{H,\tau}(x_i, x_i|\pi) \geq 0$, which is equivalent to

$$\tau \leq \left(\sum_{y \in Y(x_i) \setminus \{x_i\}} \frac{p^{H,\tau}(x_i, y|\pi)}{\tau} \right)^{-1}.$$

A safe and simpler (particularly state-independent and faster to calculate) choice is found by instead using

$$\begin{aligned}
&\left(\sum_{y \in Y(x_i) \setminus \{x_i\}} \frac{p^{H,\tau}(x_i, y|\pi)}{\tau} \right)^{-1} \\
= &\left(\frac{x_i}{x_{i+2} - x_i} (\pi(\mu - r) + r)^+ + \frac{x_i}{x_i - x_{i-2}} (\pi(\mu - r) + r)^- + \frac{\sigma^2 \pi^2 x_i^2}{(x_{i+1} - x_i)(x_i - x_{i-1})} \right)^{-1} \\
\geq &\left(\frac{x_i}{\Delta x} |\pi(\mu - r) + r| + \left(\frac{\sigma \pi x_i}{\Delta x} \right)^2 \right)^{-1} \\
\geq &\left(\frac{x_H}{\Delta x} (M|\mu - r| + r)| + \left(\frac{x_H}{\Delta x} \right)^2 (M\sigma)^2 \right)^{-1} \quad =: \tau_{\max}
\end{aligned}$$

to define τ^* as above.

Controlling the Amount of Wealth in the Risky Asset

As portfolio values in an amount-controlled Black-Scholes market may fall below zero, assume $x_1 < 0$ and $x_H > 0$. Both boundaries shall be reflecting, but the grid does not need to be symmetric in zero.

Particularly, the maximal and absolute minimal wealth may differ. Denote the larger value $\hat{x} := \max\{-x_1, x_H\}$ and recall that admissible strategies comply with

$$\theta(t)^2 \leq K^2 \left(1 + \left(X^\theta(t) \right)^2 \right), \ t \in \mathcal{T}_\tau.$$

We have

$$f^\pm(x_i, t, u) \equiv f^\pm(x_i, \theta) = (\theta(\mu - r) + x_i r)^\pm$$

and

$$S(x_i, t, u) \equiv S(x_i, \theta) = \sigma^2 \theta^2$$

and therefore, according to (4.16), find

$$\begin{aligned}
p^{H,\tau}(x_i, x_j|\theta) &= 0,\ x_j \notin Y(x_i), \\
p^{H,\tau}(x_i, x_{i+2}|\theta) &= \frac{\tau}{x_{i+2} - x_i}(\theta(\mu - r) + x_i r)^+,\ i = 1, \dots, H-2, \\
p^{H,\tau}(x_i, x_{i-2}|\theta) &= \frac{\tau}{x_i - x_{i-2}}(\theta(\mu - r) + x_i r)^-,\ i = 3, \dots, H, \\
p^{H,\tau}(x_1, x_2|\theta) &= 0, \\
p^{H,\tau}(x_H, x_{H-1}|\theta) &= 0, \\
p^{H,\tau}(x_i, x_{i+1}|\theta) &= p^{H,\tau}(x_i, x_{i-1}|\theta) \\
&= \frac{\tau\sigma^2\theta^2}{2(x_{i+1} - x_i)(x_i - x_{i-1})},\ i = 2, \dots, H-1, \\
p^{H,\tau}(x_i, x_i|\theta) &= 1 - \sum_{y \in Y(x_i) \setminus \{x_i\}} p^{H,\tau}(x_i, y|\theta),\ i = 1, \dots, H.
\end{aligned}$$

With similar steps as in the fractional control case one finds

$$\tau_{\max} := \left(\frac{1}{\Delta x} \left(K\sqrt{1 + \hat{x}^2}|\mu - r| + \hat{x}r \right) + \left(\frac{\sigma K}{\Delta x} \right)^2 (1 + \hat{x}^2) \right)^{-1}$$

to induce a grid-wide sufficiently small time-discretization.

5 Portfolio Optimization in High-Water Mark Market Models

In this chapter we take a different approach to modeling risk-transfer in principal-agent systems: We introduce the so-called high-water mark as the running maximum of portfolio wealths. The high-water mark process is indirectly controlled by the agent's actions. Contracts based on, or at least incorporating, the high-water mark thus provide feedback to the agent. That kind of feedback structure is suitable to set incentives and influence the agent's risk behavior.

Mathematically, incorporating the high-water mark translates to an enlargement of the state space. At each time the state of wealth and high-water mark can now be represented as two-dimensional tuple $(x, x^*) \in \mathbb{R}^+ \times [x_0, \infty)$.

In the first section of this chapter a continuous-time market model with high-water marks is presented. The second section suggests an adequate extension of the Cox-Ross-Rubinstein model to incorporate high-water marks in discrete time such that it approximates the continuous-time market. In section 5.3 we explain how solutions can be computed in the discrete model. Section 5.4 collects two terminal wealth problems that we will frequently refer to as benchmarks. In the remaining sections 5.5 to 5.9 several high-water mark portfolio problems in discrete time are discussed.

Main Results

In the following we show that offering the agent a terminal share of the high-water mark in one-period models tempts her to take deliberately high risks while adding another share of terminal wealth immediately guarantees the agent's choice of strategy is bounded (c.f. theorems 5.6 and 5.10).

Even with pure high-water mark contracts an economic balance can be achieved in some situations when considering participation constraints for both parties (c.f. theorem 5.8 and conclusion 5.9).

We discuss and numerically analyze optimal strategies in multi-period markets for pure high-water mark wages (c.f. section 5.6) as well as for compensations combining wealth and high-water mark (c.f. section 5.8).

We can establish the convergence of optimal strategy and value function for combined compensations in one period: If the share of wealth dominates the high-water mark share, an agent with logarithmic preference will in the limit behave as if rewarded by terminal wealth only (c.f. theorems 5.12 and 5.13).

In section 5.9 the effects of intermediate wages are discussed. If offering a share of the high-water mark, the agent tends to accept arbitrarily high risks when termination approaches, which is in line with previously known results for a continuous-time market with random termination time. If additionally providing a share of terminal wealth, the optimal behavior at the last step becomes bounded in limit. Hence by properly combining the individual instruments the principal can set the incentives he desires.

5.1 A High-Water Mark Model in Continuous Time

5.1.1 A Continuous-Time Framework with High-Water Marks

Recall the classical Black-Scholes market from section 2.3. Classical means that only one stock is considered and riskless rate, stock drift and volatility are constant over time. This market is arbitrage-free and complete. It features the major financial instruments and is elaborate enough to cover the setups and situations relevant here. Finally, everything shall be observable by both parties.

We now introduce the high-water mark process $X^* = (X^*(t))_{0 \leq t \leq T}$ by setting

$$X^*(t) := \max\{X(s) : \; 0 \leq s \leq t\}, \; 0 \leq t \leq T. \tag{5.1}$$

X^* is $\mathcal{F}$-adapted and exhibits non-decreasing and $\mathbb{P}$-a.s. continuous paths that start at $X(0)^* = X(0) = x_0$. It further holds $dX^*(t) = 0$ whenever the wealth X does not increase[1] at time t.

Assume the agent controls the wealth evolution by investing the fractions $\pi(t)$ of wealth in the stock at each time $t \in [0, T)$. She is rewarded for her efforts by the continuously paid wages $\psi(X(t), X^*(t), t)$, $t \in (0, T]$, and a terminal wage $\Psi(X(T), X(T)^*)$. All wages may now – in contrast to the wages paid in section 2.2 – incorporate the high-water mark and omit the actual strategy. The latter omission can be viewed as contracting to results rather than to effort and models the common practice.

In the generality of this setup a large number of contracts and associated portfolio problems is conceivable. Let us particularly consider the following basic types to facilitate the understanding of the effects the high-water mark has on both parties:

1. The simplest contract on the high-water mark is the agreement to leave the agent with a share of her best performance. It is accomplished by setting $\psi \equiv 0$ and $\Psi(X(T), X(T)^*) := \beta X(T)^*$. The agent will require $\beta > 0$ while the principal has an interest to keep the share small. Note that $X(T)^* \geq X(T)$ so the principal will never offer contracts with $\beta \geq 1$.

   It is a common belief that this kind of contract incites agents to take great risks as they do not participate in the actual wealth remaining at termination. We will get back to that.

2. Elaborating on that thought let us also consider contracts with terminal-only rewards that incorporate both the wealth and the high-water mark at termination. The straightforward modification of the first case is $\psi \equiv 0$ and $\Psi(X(T), X(T)^*) := \alpha X(T) + \beta X(T)^*$ where a linear combination of both is offered to the agent. To avoid trivial cases let $\alpha, \beta > 0$. Again the principal will not be willing to offer $\alpha \geq 1$ or $\beta \geq 1$.

   Can the principal already control the agent's risk behavior by setting both weights? We will get back to that question, too.

3. The intuition is that more control can be exercised when more instruments are present, so finally let us add intermediate wages to the previous setup. With $\Psi(X(T), X(T)^*) := \alpha X(T) + \beta X(T)^*$ as before and $d\psi(X(t), X^*(t), t) := \gamma X(t)dt +$

[1] Note that on any path of wealth X one can with probability 1 find some open region $(t-\varepsilon, t+\varepsilon) \subset [0, T]$ over which wealth does not increase. In contrast to X, the high-water mark X^* is indeed (pathwise!) differentiable in each of those regions.

$\delta dX^*(t)$ that is established in a consistent manner. The factors γ and δ are of course subject to the same requirements as α and β.

Note that γ is indeed a share in the running wealth while δ only makes a share of the increase of the high-water mark available to the agent. The idea is not to reward stagnation but only improvement of performance.

In the first and second setup suggested above no intermediate wages are paid, hence wealth starts with $X(0) = x_0$ and evolves with

$$dX^\pi(t) = X^\pi(t) \left((\pi(t) (\mu - r) + r) \, dt + \pi(t)\sigma dB(t) \right), \ 0 \leq t \leq T, \qquad (5.2)$$

compare (2.8). The agent collects the random terminal wage $\alpha X_T + \beta X_T^*$ (where the first setup is just the special case $\alpha = 0$) which she possibly assesses with some private utility.

Additionally paying intermediate wages of the type suggested in setup number three translates to deducing $d\psi(X(t), X^*(t), t)$ from the wealth process and yields the evolution $X(0) = x_0$,

$$dX^\pi(t) = X^\pi(t) \left((\pi(t) (\mu - r) + r - \gamma) \, dt + \pi(t)\sigma dB(t) \right) - \delta dX^*(t), \ 0 \leq t \leq T. \quad (5.3)$$

The accrued intermediate wages $\int_0^T \gamma X(t)dt + \int_0^T \delta dX^*(t)$ are added to the agent's balance.

The crucial point here is the finite time-horizon. The portfolio problems are much simpler when instead of $T > 0$ an infinite horizon is considered as then the optimal strategy must be time-independent. But for any practical purpose an infinite time horizon (and even a possibly infinite time horizon) is impossible to contract to.

5.1.2 Known Results

Panageas & Westerfield (2009) consider the infinite version of the market described here by continuing bond and stock evolutions to price processes $(S^0(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ that do not terminate at any finite time. Accordingly the agent is employed for an infinite time and produces the wealth $(X(t))_{t\geq 0}$. Her high-water mark is recorded in $(X^*(t))_{t\geq 0}$ where (5.1) is straightforwardly continued for $t > T$. By the lack of a finite time horizon no terminal wage can be paid and the infinite time domain requires a discounting of intermediate wages (which can be implemented directly into $\psi = \psi(\cdot, \cdot, t)$ as time-dependence is explicitly scheduled).

They approach the problem in finite time by introducing a random termination time: Consider a Poisson process $P = (P(t))_{t\geq 0}$ with intensity $\lambda > 0$. The contract is now terminated at the random time τ of the first Poisson jump, i.e. at

$$\tau = \min\{t \,|\, P(t) \geq 1\} \sim \mathrm{Exp}(\lambda).$$

Hence the expected termination time is $\mathbb{E}[\tau] = \frac{1}{\lambda}$ and can be adjusted to – in expectation – match any finite horizon desired. The drawback is that the time horizon is still random and there are uncountable many paths with infinite termination. The benefit is that a Hamilton-Jacobi-Bellman approach that omits the time derivate of the value is valid.

For a discount rate $\eta > 0$ their agent collects wages

$$\int_0^\tau \delta e^{-(\eta+\lambda)t} dX^*(t)$$

which of course depend on the strategy chosen to manage the wealth process X that underlies the integrator X^*. Their main result, translated in our notation, is the following:

Theorem 5.1 (Proposition 1 of Panageas & Westerfield (2009))
Assume $\mu > r$ and $\nu(1+\delta) > 1$, where ν is the smaller root of

$$\nu^2 r - \nu(r + \eta + \lambda + \omega) + (\eta + \lambda) = 0$$

and $\omega = \frac{1}{2}\frac{(\mu-r)^2}{\sigma^2}$. Then the portfolio problem

$$(P_{\mathrm{P\&W}}) \quad \sup_{\pi} \left\{ \mathbb{E}\left[\int_0^{\tau} \delta e^{-(\eta+\lambda)t} dX^*(t) \right] \right\}$$

admits the unique optimal strategy $\pi^* = (\pi(t)^*)_{t\geq 0}$ where

$$\pi(t)^* \equiv \frac{1}{1-\nu} \cdot \frac{\mu - r}{\sigma^2}.$$

$\diamond$

The agent's optimal behavior in this case is stunningly simple and renders the optimal strategy of an investor of power type with relative risk aversion $(1-\nu)$. The principal can predict his agent's strategy but as market parameters and discounting are quantities out of his reach, he cannot influence it.

To better approach a deterministic finite horizon Panageas & Westerfield (2009) now consider a sequence of Poisson processes $(P_n)_{n\in\mathbb{N}}$ where $P_n = (P_n(t))_{t\geq 0}$ has intensity $\lambda_n > \frac{n}{T}$. Termination in the n-th market model is triggered when P_n jumps for the n-th time, i.e. at

$$\tau_n = \min\{t \,|\, P_n(t) \geq n\}.$$

P_n can be considered as the sum of n independent Poisson processes with intensity $\frac{1}{T}$ each, thus τ_n is Erlang distributed with expectation T and its variance vanishes for $n \to \infty$. The authors then establish the following result:

Theorem 5.2 (Lemma 1 of Panageas & Westerfield (2009))
Let all assumptions of theorem 5.1 hold and consider the portfolio problem

$$(P^n_{\mathrm{P\&W}}) \quad \sup_{\pi} \left\{ \mathbb{E}\left[\int_0^{\tau_n} \delta e^{-(\eta+\lambda)t} dX^*(t) \right] \right\}.$$

Then the agent's optimal strategy at the time of the $(n-1)$-th jump approaches infinity when $n \to \infty$. $\diamond$

Interpretation is as follows: Shortly before (expected) termination the agent mainly aims to improve the high-water mark as this is what her wage depends on. The consequences of wealth dropping due to risky strategies is negligible as there is not much time left. With n increasing the expected time remaining between the $(n-1)$-th jump and termination decreases and induces arbitrarily risky[2] investments.

[2]The less time remains the less potential future earnings are affected by changes in the strategy. The agent effectively gets more and more myopic. In order to earn something now, she tends to place arbitrarily high positions in the stock.

It remains to investigate if the same conclusion also holds in a truly finite market and if the same arguments lead to unbounded strategies when the intermediate shares of the running maximum are replaced by a terminal share of the high-water mark.

The question what incentives are set by continuously rewarding the agent with shares of both wealth and high-water mark is addressed by Guasoni & Obłój (2011) in a market with infinite time-horizon. In their setup the agent is continuously rewarded with a rate γ of wealth and a rate $\delta = \frac{\tilde{\delta}}{1-\tilde{\delta}}$ of the high-water mark, where $\tilde{\delta} \in (0,1)$. She draws the power utility with some relative risk aversion κ of her accrued wages $F(T)$ where $F(0) = 0$ and

$$dF(t) = rF(t)dt + \gamma X(t)dt + \delta dX^*(t).$$

Note that in particular the agent is in this setup not allowed to bet on her own portfolio, thus her earnings are invested in the bond and yield the interests $rF(t)dt$.

Now there is again a finite horizon T. The authors immediately send $T \to \infty$ and approach the agent's portfolio problem by optimizing the so-called certainty equivalence rate instead of the utility of aggregated wages F:

$$(P_{\mathrm{G\&O}}) \quad \begin{cases} \sup_\pi \{\mathrm{CER}_\kappa(F^\pi)\} \\ \pi = (\pi(t))_{t\geq 0},\ \pi(t) \in \mathbb{R}\ \forall t \geq 0 \end{cases}.$$

Here $F^\pi = (F(t)^\pi)_{t\geq 0}$ denotes the process of collected wages when strategy π is applied and the certainty equivalence rate is defined as

$$\mathrm{CER}_\kappa(F^\pi) := \lim_{T\to\infty} \frac{1}{(1-\kappa)T} \log\left(\mathbb{E}\left[(F(T)^\pi)^{1-\kappa}\right]\right).$$

The certainty equivalence rate has the unique property that the agent (equipped with her power utility) would immediately choose to avoid any risks and put everything in the money market if its riskless rate exceeded the certainty equivalence rate. Among all those rates with the before-mentioned property it is the smallest, i.e. if the money market admits a lower rate than the certainty equivalence rate, the agent will immediately prefer trading all financial instruments again.

The key result is:

Theorem 5.3 (Theorem 1 of Guasoni & Obłój (2011))
Let $\kappa^* := (1-\delta)\kappa + \delta$ and assume the parametric restriction

$$\gamma - r < \frac{(\mu - r)^2}{2\sigma^2} \cdot \frac{\min(\kappa^*, 1)}{(\kappa^*)^2}.$$

Then $(P_{\mathrm{G\&O}})$ admits the unique optimal strategy $\pi^* = (\pi(t)^*)_{t\geq 0}$ where

$$\pi(t)^* \equiv \frac{1}{\kappa^*} \cdot \frac{\mu - r}{\sigma^2}.$$

◇

So again the optimal strategy is that of a power-investor, this time with a relative risk aversion of $(1-\kappa^*)$. Surprisingly it does not directly depend on the rate γ the agent is offered to consume from the running wealth (though the omitted parametric restrictions pose an upper boundary on the excess rate $(\gamma - r)$ above which any strategy is trivially optimal).

The implications for the agent's risk behavior are ambiguous: If the agent brings along a relative risk aversion $\kappa < 1$, i.e. if she is more risk-loving than the log-investor, including the high-water mark into the contract makes the agent more risk-averse. If on the other hand the agent has a relative risk aversion $\kappa > 1$, i.e. if she is more cautious than the log-investor, the high-water mark will incite her to be less risk-averse.

5.1.3 Guiding Research Questions

Both Panageas & Westerfield (2009) and Guasoni & Obłój (2011) do not consider a finite time horizon, but obviously this is an essential property for realistic contracts.

The intuition of theorem 5.2 suggests that no solution can exist for $T < \infty$.

We will investigate this question and choose another approach: Fix the time horizon at a finite time T and then approximate the continuous-time market itself by a sequence of market models in discrete time. The upcoming section introduces the proper extension of the well-known Cox-Ross-Rubinstein market and revisits the portfolio problems introduced here in a time-discrete context.

5.2 A High-Water Mark Model in Discrete Time

The famous continuous-time market model of Black and Scholes can be considered as the limit of a sequence of discrete-time Cox-Ross-Rubinstein models. This section explains how the original setup of Cox, Ross and Rubinstein can be extended to incorporate the high-water mark and determines the spaces of admissible states for each discrete time-step.

5.2.1 Market Dynamics

Cox, Ross and Rubinstein's binomial model features the same financial instruments as the Black-Scholes market and shares the finite horizon, but assumes time- and value-discrete price dynamics: For a fixed time horizon $T > 0$ and a number $n \in \mathbb{N}$ of discretization steps consider the time-grid $\{t_0, \dots, t_n\}$ with $t_i := \frac{i}{n}T$, $i = 0, \dots, n$. The money market account S^0 and the[3] stock S^1 evolve with

$$\begin{aligned} S^0_{t_i} &:= (1 + r_n)S_{t_{i-1}} = (1 + r_n)^i, \; i = 1, \dots, n, \quad \text{as } S^0_{t_0} := 1, \\ S^1_{t_i} &:= Y_{t_i} S_{t_{i-1}}, \; i = 1, \dots, n, \quad \text{with } S^1_{t_0} := s^1_0 \in \mathbb{R}^+. \end{aligned}$$

The stock price factors $Y_{t_1}, \dots, Y_{t_n}$ are assumed i.i.d. random variables on a probability space $(\{\text{up}, \text{down}\}, \mathcal{P}(\{\text{up}, \text{down}\}), \tilde{\mathbb{P}})$ with $\tilde{\mathbb{P}}(Y_{t_1} = u_n) = p_n = 1 - \tilde{\mathbb{P}}(Y_{t_1} = d_n)$. This market is arbitrage-free and complete if and only if $0 < d_n < 1 + r_n < u_n < \infty$, compare e.g. (Shreve, 2004, Theorem 1.2.2) or (Bingham & Kiesel, 2004, Corollary 4.5.1 and Proposition 4.5.2). In this case the unique equivalent martingale measure $\mathbb{Q}$ is induced by the risk-neutral one-step probabilities $q_n := \tilde{\mathbb{Q}}(Y_1 = \text{up}) = \frac{1+r_n-d_n}{u_n-d_n}$.

In the literature the time horizon usually coincides with the number of time steps, thus $T = n$ and the latter parameter is omitted. The reason for additionally taking it into account here, is that we want to use the Cox-Ross-Rubinstein model as a discrete approximation to the Black-Scholes model. By splitting the trading interval $[0, T]$ in a

[3]We consider one stock only to match the time-continuous reference market. The CRR model itself can easily be extended to a larger number of stocks.

growing number $n \in \mathbb{N}$ of equidistant periods and accordingly adjusting the market parameters, one can achieve the (typically weak) convergence of stock and option prices in the Cox-Ross-Rubinstein market to their counterparts in a Black-Scholes market (compare e.g. (Föllmer & Schied, 2011, Chapter 5.7 and particularly Theorem 5.52) for fairly complete discussion; Willinger & Taqqu (1991) offer an overview of weak-convergence results and Cutland, Kopp & Willinger (1993) discuss a stronger type of convergence and applications to option-pricing). The respective market parameter sets translate as follows.

- Approximate a Black-Scholes market with horizon $T > 0$, riskless rate $r > 0$, stock drift $\mu \in \mathbb{R}$, and volatility $\sigma > 0$ to the degree $n \in \mathbb{N}$:

Δ_n	$:=$	$\frac{T}{n}$	length of one time-step
r_n	$:=$	$e^{r\Delta_n} - 1$	discrete interest rate
u_n	$:=$	$e^{\sigma\sqrt{\Delta_n}}$	upwards factor
d_n	$:=$	$e^{-\sigma\sqrt{\Delta_n}}$	downwards factor
p_n	$:=$	$\frac{e^{\mu\Delta_n} - d_n}{u_n - d_n}$	upwards probability under $\tilde{\mathbb{P}}$

- Find the Black-Scholes market approximated by a Cox-Ross-Rubinstein market with horizon $T > 0$, $n \in \mathbb{N}$ periods, discrete interest rate r_n, stock factors u_n and d_n s.t. $0 < d_n < 1 + r < u_n < \infty$, and physical one-step upwards probability $p_n \in (0, 1)$:

Δ_n	$:=$	$\frac{T}{n}$	length of one time-step
r	$:=$	$\frac{1}{\Delta_n} \log(1 + r_n)$	continuous interest rate
μ	$:=$	$\frac{1}{\Delta_n} \log(p_n u_n + (1 - p_n) d_n)$	stock drift
σ	$:=$	$\frac{1}{\sqrt{\Delta_n}} \log(u_n)$	volatility

Cox-Ross-Rubinstein markets can only converge to a Black-Scholes market with well-defined (symmetric) volatility $\sigma > 0$ if the up- and down-factors satisfy $u_n d_n = 1$. On the other hand, the discretization of a Black-Scholes market to a Cox-Ross-Rubinstein market will yield an arbitrage free model with well-defined physical measure if and only if n is large enough:

$$\begin{aligned}
\text{(NA)} \quad &\iff \quad d_n < 1 + r_n < u_n \\
&\iff \quad e^{\sigma\sqrt{\Delta_n}} < e^{r\Delta_n} < e^{\sigma\sqrt{\Delta_n}} \\
&\iff \quad n > T\left(\frac{r}{\sigma}\right)^2, \\
p_n \in (0,1) \quad &\iff \quad 0 < p_n < 1 \\
&\iff \quad 0 < \frac{e^{\mu\Delta_n} - d_n}{u_n - d_n} < 1 \\
&\iff \quad n > T\left(\frac{\mu}{\sigma}\right)^2,
\end{aligned}$$

so necessarily and sufficiently require

$$n > T\left(\frac{\max\{r, \mu\}}{\sigma}\right)^2. \tag{5.4}$$

From now on let us assume (5.4) holds and consider an arbitrage-free Cox-Ross-Rubinstein model that is feasible to approximate a Black-Scholes market. Together with (NA) and the standing assumptions, the necessary condition $d_n u_n = 1$ implies

$$0 < d_n < 1 < 1 + r_n < u_n = d_n^{-1} < \infty.$$

To assist readability in what follows the subscripts $_n$ of r_n, u_n, d_n, p_n and q_n shall be dropped. To prevent confusing the discrete interest rate r_n with the continuous interest rate r, from now on let us write r_{BS} whenever the interest rate of the Black-Scholes market is referred to.

Evolution of Wealth

An investor with initial endowment $x_0 > 0$ and strategy $\pi = (\pi_{t_i})_{i=0}^{n-1}$ denoting the fraction of wealth invested in the stock achieves wealth

$$X_{t_1-} = [(1+r) + \pi_{t_0}(Y_{t_1} - (1+r))]x_0$$

after step 1. As we additionally want to record the best wealth so far in the high-water mark, we have to agree if that mark is updated prior to deducing a potential intermediate wage or afterwards. Let us first update[4] the record. To formalize the concept the in-between time t_1-, as already used above, is employed. More generally collect all time-points in $\mathcal{T} := \{t_0, t_{1-}, t_1, \dots, t_{n-}, t_n\}$. $\mathcal{T}$ shall be ordered by $t_i < t_j$ if and only if $i < j$ and $t_{j-} < t_j$ for all $i = 0, \dots, n-1$ and $j = 1, \dots, n$. Back at step 1 we can now perform the first high-water mark update and set

$$X^*_{t_1} := \max\{x_0, X_{t_1-}\}.$$

Now the investor (think of her as the agent) is compensated with intermediate wage $\psi = \psi(X_{t_1-}, X^*_{t_1}, t_1)$ which, as it takes time into account, may also introduce some discounting of the payoffs. The wealth then decreases by ψ and what remains is

$$X_{t_1} = X_{t_1-} - \psi(X_{t_1-}, X^*_{t_1}, t_1).$$

This amount is reinvested in the market with strategy π and the scheme above is applied at each further time-step. Finally, we additionally allow for a terminal wage Ψ at time T. We have to make sure that any wage paid may depend on the complete trading history up to the time of payment but may not incorporate information not yet available. To that end consider the stopped processes $(X^t_s)_{s\in\mathcal{T}}$ and $(X^{t,*}_s)_{s\in\mathcal{T}}$ with

$$\begin{aligned} X^t_s &:= \begin{cases} X_s, & t_0 \le s \le t \\ X_t, & t < s \le t_n \end{cases} \quad \text{and} \\ X^{t,*}_s &:= \begin{cases} X^*_s, & t_0 \le s \le t \\ X^*_t, & t < s \le t_n \end{cases}. \end{aligned}$$

For $t = t_1, \dots, t_n$ intermediate wages $\psi = \psi(X^{t-}, X^{t,*}, t)$ now incorporate no more information than is available at t. Equivalently require $t \mapsto \psi(\cdot, \cdot, t)$ to be $\mathcal{F}_{t-}$-measurable for $t = t_1, \dots, t_n$ where $\mathcal{F} = (\mathcal{F}_t)_{t\in\mathcal{T}}$ denotes the natural filtration of X (and X^* as $X^*_{t_i}$

[4] That way the momentary compensation is higher but subsequently beating the record is harder.

is already $\mathcal{F}_{t_i-}$-measurable by construction). The terminal wealth $\Psi = \Psi(X^{t_n-}, X^{t_n,*})$ shall be $\mathcal{F}_{t_n-}$-measurable. Altogether this yields the wealth evolution

$$\begin{aligned} X_0 &= x_0, \quad X_0^* = X_0, \\ X_{t_i-} &= [(1+r) + \pi_{t_{i-1}}(Y_{t_i} - (1+r))]X_{t_{i-1}}, \\ X_{t_i}^* &= \max\{X_{t_{i-1}}^*, X_{t_i-}\}, \\ X_{t_i} &= X_{t_i-} - \psi(X^{t_i-}, X^{t_i,*}, t_i) - \mathbb{1}_{[i=n]}\Psi(X^{t_n-}, X^{t_n,*}), \end{aligned} \tag{5.5}$$

$i = 1, \dots, n$.

In any sensible market setting, intermediate compensations will be non-negative and no larger than the actual wealth, so we require $0 \le \psi = \psi(X^{t_i-}, X^{t_i,*}, t_i) \le X_{t_i}$ for $i = 1, \dots, n$ and $0 \le \Psi = \Psi(X^{t_n-}, X^{t_n,*})$. The terminal wage is not required to underbid the terminal wealth as there is no imperative obligation to do so from the agent's point of view. We will, however, in the following only consider contracts that leave the principal with non-negative wealth after wages or, at least, with non-negative expected wealth after wages.

Figure 5.1 sketches the order of pay-off, rebalancing, and portfolio update.

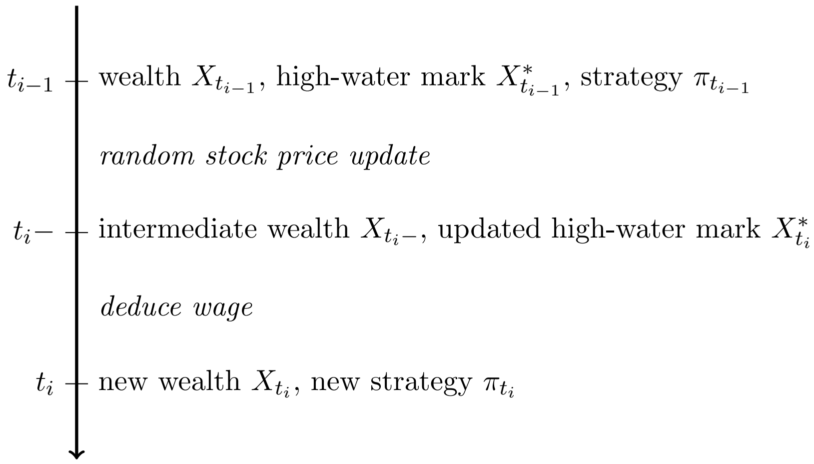


Figure 5.1: update of wealth, high-water mark and strategy in the Cox-Ross-Rubinstein market

5.2.2 The Portfolio Problem

The investor aims to optimize her expected aggregated utility of wages by choosing an optimal strategy π^* that maximizes

$$\mathbb{E}\left[\sum_{i=1}^{n} U\left(\psi(X^{t_i-}, X^{t_i,*}, t_i)\right) + U\left(\Psi(X^{t_n-}, X^{t_n,*})\right) \,\middle|\, X_{t_0} = X_{t_0}^* = x_0\right] \tag{5.6}$$

when controlling the wealth $(X_{t_i})_{i=0,\dots,n}$ with the $\mathcal{F}$-adapted process $\pi^* = (\pi_{t_k}^*)_{k=0,\dots,n-1}$. Note that the expectation is taken under the physical measure $\mathbb{P}$, which is composed as the product of the one-step measures $\tilde{\mathbb{P}}$.

Discrete optimization problems with a prescribed terminal condition are most naturally solved backwards by applying the principle of dynamic programming. To this purpose define the k-th step value function for our extended problem with recorded high-water marks as

$$\begin{aligned} J_k(x,x^*) &:= \sup_\pi \{J_k(x,x^*;\pi)\}, \text{ where} \qquad (5.7)\\ J_k(x,x^*;\pi) &:= \mathbb{E}^{k,x,x^*}\left[\sum_{i=k+1}^{n} U\left(\psi(X^{t_i-},X^{t_i,*},t_i)\right) + U\left(\Psi(X^{t_n-},X^{t_n,*})\right)\right] \end{aligned}$$

and $\mathbb{E}^{k,x,x^*}[\cdot]$ refers to the conditional expectation $\mathbb{E}[\,\cdot\,|\,X_{t_k}=x, X^*_{t_k}=x^*]$ for time indices $k=0,\dots,n$, pairs (x,x^*) of intermediate wealth and intermediate high-water mark that are admissible at that time index, and strategies π that are admissible for the remaining time-steps up to t_n. Notice that although it is not explicitly denoted on the right-hand side of (5.7), the processes X and X^* are controlled by choosing a control process π. In what follows

$$g_k(x,x^*;\pi_k) := [1+r+\pi_k(x,x^*)(Y_{t_{k+1}}-(1+r))]x \qquad (5.8)$$

abbreviates the (random) new wealth at time t_{k+1} produced from intermediate wealth x and intermediate high-water mark x^* when applying the strategy $\pi_k=\pi_k(x,x^*)$ at time t_k. The latter transition functions g_k, $k=0,\dots,n-1$ are, of course, specific to the Cox-Ross-Rubinstein market.

Note that $J_0(x_0,x_0)$ is the target given in (5.6) and that $J_n(x,x^*)=U(\Psi(x,x^*))$ yields an initial condition for the backwards recursion. The essential ingredient to that recursion is Bellman's equation (compare e.g. (Bertsekas, 2000, Chapter 1.3 for an introduction and particularly Proposition 1.3.1)), which in this context reads

$$\begin{aligned} J_k(x,x^*) = \sup_{\pi_k}\Big\{&\mathbb{E}^{k,x,x^*}\big[U(\psi(X^{t_{k+1}-},X^{t^*_{k+1}},t_{k+1}))+\\ &J_{k+1}\left(g_k(x,x^*;\pi_k),\max\{g_k(x,x^*;\pi_k),x^*\}\right)\big]\Big\}, \qquad (5.9)\end{aligned}$$

for $k=0,\dots,n-1$.

(5.9) covers a very large set of wage schedules. A notable special case is a contract without intermediate compensations, yielding a pure terminal utility problem. For $k=0,\dots,n-1$ Bellman's equation then simplifies to

$$J_k(x,x^*) = \sup_{\pi_k}\left\{\mathbb{E}^{k,x,x^*}\left[J_{k+1}\left(g_k(x,x^*;\pi_k),\max\{g_k(x,x^*;\pi_k),x^*\}\right)\right]\right\}. \qquad (5.10)$$

Tools for the Analysis of High-Water Marks

There are some quantities intrinsic to the analysis of high-water mark problem. We will frequently use the following terms:

Definition 5.1 *For $x\in\mathbb{R}$ and $\pi\in\mathbb{R}$ let*

$$\begin{aligned} \hat{x}(x,\pi) &:= x\,(1+r+\pi\,(u-(1+r))) \quad \textit{and}\\ \mathring{x}(x,\pi) &:= x\,(1+r+\pi\,(d-(1+r))) \end{aligned}$$

denote the new wealth produced in one step from x when trading with strategy π when stock prices go up or down, respectively.

Note that $\hat{x}$ and $\mathring{x}$ are special cases of the wealth update function (5.8) in which the strategy value π is plugged in for $\pi(x, x^*)$ (thus dropping the dependence on x^*) and additionally substituting Y_{k+1} with the deterministic and time-independent values u or d (thus dropping the dependence on k).

Definition 5.2 *By $z := \frac{x^*}{x}$ denote the* ***performance ratio****. If by intermediate wages or large enough admissible strategy spaces the wealth $x = 0$ is produced, set $z := +\infty$.*

The quantity z measures how far the current high-water mark is away from being improved. Its inverse z^{-1} can be interpreted as the percentage of the running record currently held in wealth.

Definition 5.3 *The* ***update margins*** *$\hat{\pi}(x, x^*)$ and $\mathring{\pi}(x, x^*)$ denote those strategy values above which the current high-water mark x^* is at least matched by the current wealth x when stock prices go up or, respectively below which x^* is at least matched by x when stock prices go down:*

$$
\begin{aligned}
\hat{\pi}(x, x^*) &:= \min\{y \in \mathbb{R} : \ \hat{x}(x, \pi) \geq x^* \ \forall \pi \geq y\} \\
&= (z - 1 - r)(u - 1 - r)^{-1}, \\
\mathring{\pi}(x, x^*) &:= \max\{y \in \mathbb{R} : \ \mathring{x}(x, \pi) \geq x^* \ \forall \pi \leq y\} \\
&= (z - 1 - r)(d - 1 - r)^{-1}.
\end{aligned}
$$

In one-period models the abbreviations

$$
\begin{aligned}
\hat{\pi} &:= \hat{\pi}(x_0, x_0) = -\frac{r}{u - 1 - r} \quad \text{and} \\
\mathring{\pi} &:= \mathring{\pi}(x_0, x_0) = \frac{r}{1 + r - d}
\end{aligned}
$$

are employed and in multiple periods one can call upon the performance ratio to shorten

$$
\begin{aligned}
\hat{\pi}(z) &:= \hat{\pi}(x, x^*) \quad \text{and} \\
\mathring{\pi}(z) &:= \mathring{\pi}(x, x^*)
\end{aligned}
$$

while $z = \frac{x^*}{x}$ is defined as above.

Linear Portfolio Problems

In what follows we will examine the following general classes of linear problems in detail:

Definition 5.4 *The* ***agent's terminal optimization problem*** *is*

$$
\left(P^{\mathfrak{A},n}_{\alpha,\beta,U}\right) \quad \begin{cases} \sup_\pi \left\{\mathbb{E}\left[U\left(\alpha X_{t_n} + \beta X^*_{t_n}\right)\right]\right\} \\ \pi = (\pi_{t_0}, \ldots, \pi_{t_{n-1}}), \ \pi_{t_k} = \pi_{t_k}(x, x^*) \in \mathfrak{A}, \ k = 0, \ldots, n-1 \end{cases}
$$

where $\alpha \geq 0$ and $\beta \geq 0$ weight wealth and high-water mark and at least one is non-zero. U is the agent's utility function, $\mathfrak{A} \subseteq \mathbb{R}$ the set of admissible strategy values, and n the number of periods.

*For $\gamma \geq 0$ and $\delta \geq 0$ the **agent's general optimization problem** is*

$$\left(P^{\mathfrak{A},n}_{\alpha,\beta,\gamma,\delta,U}\right) \quad \begin{cases} \sup_\pi \Big\{ \mathbb{E}\Big[\sum_{i=1}^n U\left(\gamma X_{t_k}(t_k - t_{k-1}) + \delta(X^*_{t_k} - X^*_{t_{k-1}})\right) + \\ \qquad\qquad U\left(\alpha X_{t_n} + \beta X^*_{t_n}\right) \Big] \Big\} \\ \pi = (\pi_{t_0}, \dots, \pi_{t_{n-1}}),\ \pi_{t_k} = \pi_{t_k}(x, x^*) \in \mathfrak{A},\ k = 0, \dots, n-1 \end{cases}$$

where U, $\mathfrak{A}$, and n are as introduced above.
For both problems we further require $X(0) = X^*(0) = x_0 > 0$.

The step-wise value functions associated with the terminal problem are

$$\begin{aligned} J_k(x, x^*) &= \sup_\pi \{J_k(x, x^*; \pi)\}, \text{ where} \\ J_k(x, x^*; \pi) &= \mathbb{E}\left[U(\alpha X_{t_n} + \beta X^*_{t_n}) \,\middle|\, X_{t_k} = x,\ X^*_{t_k} = x^* \right], \end{aligned} \tag{5.11}$$

and the agent's general optimization problem has step-wise values

$$\begin{aligned} J_k(x, x^*) &= \sup_\pi \{J_k(x, x^*; \pi)\}, \text{ where} \\ J_k(x, x^*; \pi) &= \mathbb{E}\Bigg[\sum_{i=1}^n U\left(\gamma X_{t_k}(t_k - t_{k-1}) + \delta(X^*_{t_k} - X^*_{t_{k-1}})\right) + \\ &\qquad\qquad U\left(\alpha X_{t_n} + \beta X^*_{t_n}\right) \,\Bigg|\, X_{t_k} = x,\ X^*_{t_k} = x^* \Bigg], \end{aligned} \tag{5.12}$$

each for $k = 0, \dots, n$. Both (5.11) and (5.12) stand in line with the more general (5.7).

We will make good use of that notation later, but have to clarify something else first: Which intermediate wealth and high-water mark combinations (x, x^*) can occur when trading with the scheme sketched in figure 5.1?

5.2.3 Spaces of Admissible States

The No-Shortselling Case

Let us initially assume the strategy π_k is bound to $[0, 1]$ for all step k to exclude short-selling. As we can learn from (5.8), wealth at step k then may reach any value within $\left[x_0 d^k, x_0 d^{-k}\right]$. Things are slightly more complicated with the high-water mark. Naturally, x_0 is the sole admissible value for X^*_0. Let us draw a picture of the new wealth x_1 and the new high-water mark x^*_1 in dependence of π_0 and the market change $Y_1 \in \{d, \frac{1}{d}\}$:

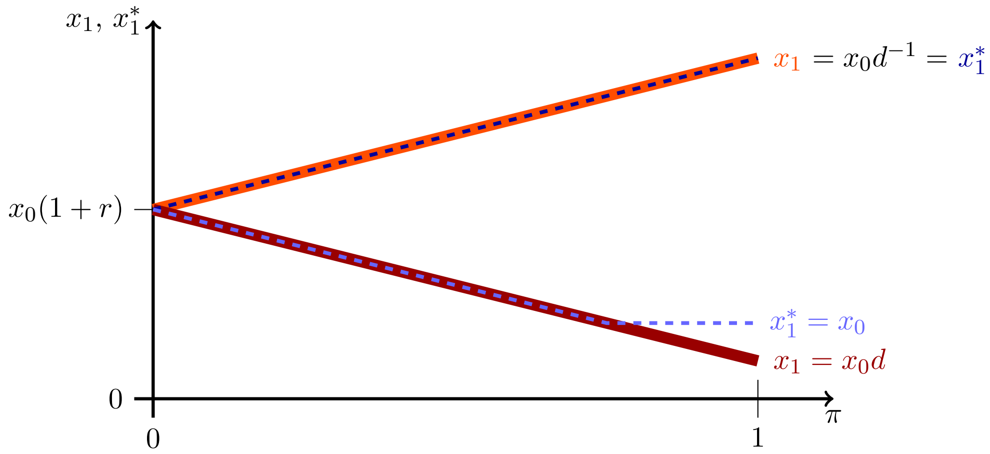


Figure 5.2: Wealth x_1 (in red) and high-water mark x_1^* (in blue) at step $k = 1$ as function of strategy π_0, plotted for upwards and downwards market movement Y_1.

If stock prices go up, x_1 linearly increases with π_0 starting at the all-bond $x_0(1+r)$ and ending at all-in $x_0 d^{-1}$. This is the lighter red graph in figure 5.2. Falling stock prices yield a wealth x_1 between $x_0(1+r)$ and the worse $x_0 d$. Compare the darker red graph. The high-water mark x_1^* increases with x_1 when prices go up (the darker blue line) and decreases with x_1 along the lighter blue line, when prices go down, but is capped at $x_0^* = x_0$. Translating this to the wealth-high-water-mark plane yields the set $\mathcal{X}_1$ depicted in figure 5.3.

$\mathcal{X}_1$ is a one-dimensional subset of $\mathbb{R}^+ \times \mathbb{R}^+$. But from now on path-dependence kicks in: As we may have commanded any wealth $x_1 \in \left[x_0 d, x_0^{-1}\right]$, the translation described in figure 5.2 may start at each such level x_1 (replace x_0 with x_1 in all labels of figure 5.2 and do not forget that $x_1^* = \max\{x_0, x_1\}$ is no longer fixed). Consequentially the admissible region $\mathcal{X}_2$ is the union of all one-dimensional regions that arise from different values of x_1.

In figure 5.3 $\mathcal{X}_1$ is constructed from (x_0, x_0) using $\hat{x}$ (the lighter green curve) and $\mathring{x}$ (the darker green curve). Note that though we have assumed $\pi \in [0,1]$ for the moment, $\hat{x}(x,\pi)$ and $\mathring{x}(x,\pi)$ are the proper wealth update functions for arbitrary $\pi \in \mathbb{R}$, too. We will later elaborate on the fact that for any real x both $\pi \mapsto \hat{x}(x,\pi)$ and $\pi \mapsto \mathring{x}(x,\pi)$ are linear and continuously differentiable with respect to π.

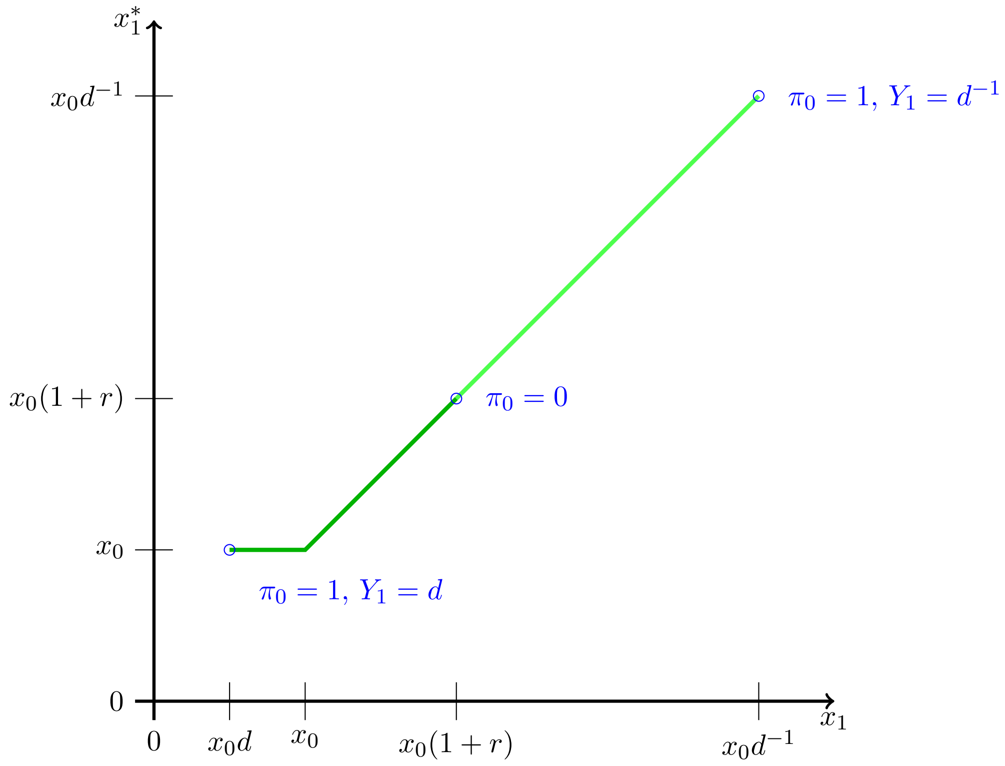


Figure 5.3: The set $\mathcal{X}_1$ of admissible states when π is restricted to $[0,1]$ and no intermediate wage is paid.

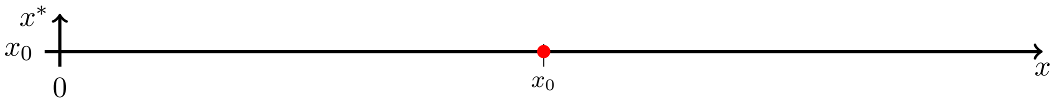
To further derive the spaces $\mathcal{X}_3, \mathcal{X}_4, \ldots$ let us formalize the principle: With the preliminary considerations we can now put

$$\begin{aligned} \hat{X}(Z) &:= \{(\hat{x}(x,\pi), \max\{\hat{x}(x,\pi), x^*\}) : (x,x^*) \in Z,\ \pi \in [0,1]\} \quad \text{and} \\ \mathring{X}(Z) &:= \{(\mathring{x}(x,\pi), \max\{\mathring{x}(x,\pi), x^*\}) : (x,x^*) \in Z,\ \pi \in [0,1]\} \end{aligned} \tag{5.13}$$

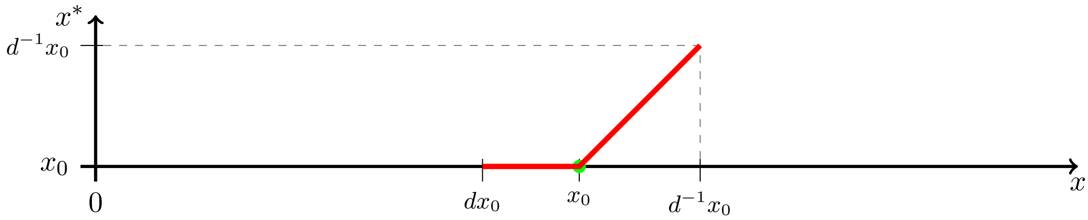
for sets $Z \subset \mathbb{R}^2$ and recursively describe

$$\begin{aligned} \mathcal{X}_0 &:= \{(x_0, x_0)\}, \\ \mathcal{X}_k &:= \hat{X}(\mathcal{X}_{k-1}) \cup \mathring{X}(\mathcal{X}_{k-1}),\ k = 1, \ldots, n. \end{aligned} \tag{5.14}$$

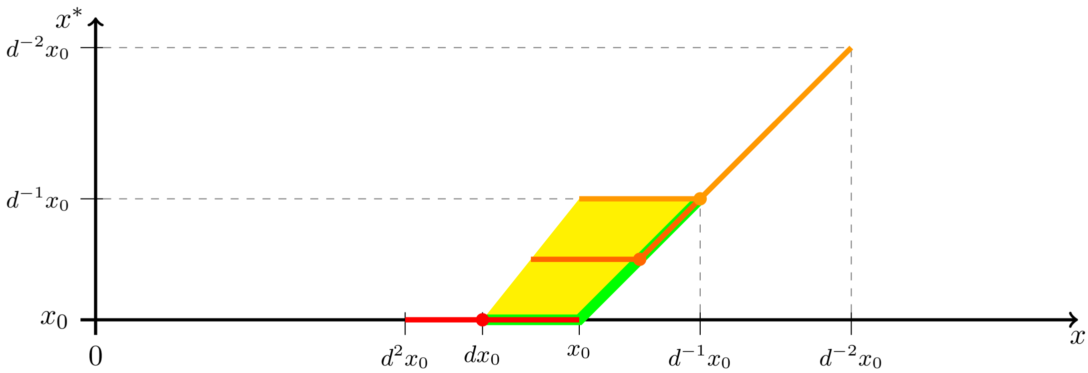
Figure 5.4 visualizes the construction:



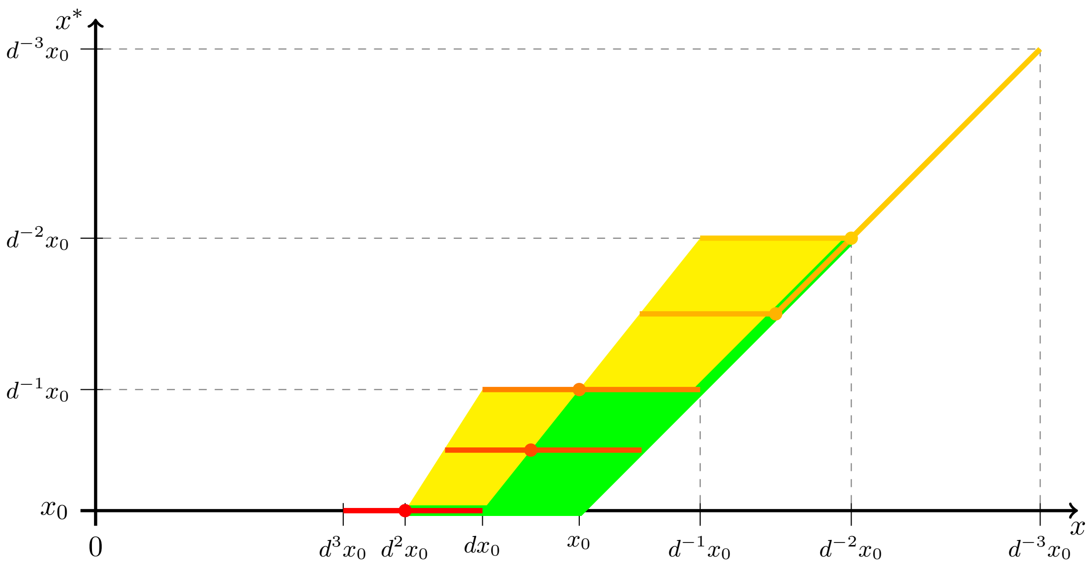
(a) $\mathcal{X}_0$



(b) $\mathcal{X}_1$



(c) $\mathcal{X}_2$



(d) $\mathcal{X}_3$

Figure 5.4: $\mathcal{X}_0$ and the sets $\mathcal{X}_1$, $\mathcal{X}_2$, and $\mathcal{X}_3$, successively constructed from their predecessors. In step $k \geq 1$, $\mathcal{X}_{k-1}$ is sketched green. The red to orange dots are prominent points in $\mathcal{X}_{k-1}$, where each point (x, x^*) spawns a likewise colored line denoting the points $\hat{X}(\{(x, x^*)\}) \cup \mathring{X}(\{(x, x^*)\})$ that are reachable from there. These together with the yellow shape in the background form the resulting union $\mathcal{X}_k$.

Though (5.14) easily characterizes the sets $\mathcal{X}_k$ in recursive terms, a closed description is more elaborate. Apart from that we will need to systematically discretize all $\mathcal{X}_k$ in order to compute approximate solutions to the various portfolio problems. To that end the true shape is somewhat unhandy and the question of a suitable simplification arises. As in particular the sharp one-dimensional peaks in the upper right and lower left corners cause numerical trouble, the idea here is to pick two-dimensional sets $\tilde{\mathcal{X}}_k$ which include $\mathcal{X}_k$ but are more easily defined in a closed form.

Note that in $\mathcal{X}_k$ the line segment spawning from the point $(d^j x_0, x_0)$, $j = 0, \ldots, k-1$, has slope d^j. Add another one at the lower left tip $(d^k x_0, x_0)$ with the proper slope d^k, extend it to the point $(x_0, d^{-k} x_0)$, and then connect it with the upper right tip at $(d^{-k} x_0, d^{-k} x_0)$. This yields the set $\tilde{\mathcal{X}}_k$ sketched in figure 5.5 and formally defined as

$$\tilde{\mathcal{X}}_k := \left\{ (x, x^*) \in \mathbb{R}_+^2 : \; x_0 d^k \le x \le \frac{x_0}{d^k}, \; x_0 \le x^* \le \frac{x_0}{d^k}, \; 1 \le \frac{x^*}{x} \le \frac{1}{d^k} \right\}, \; k = 1, \ldots, n. \tag{5.15}$$

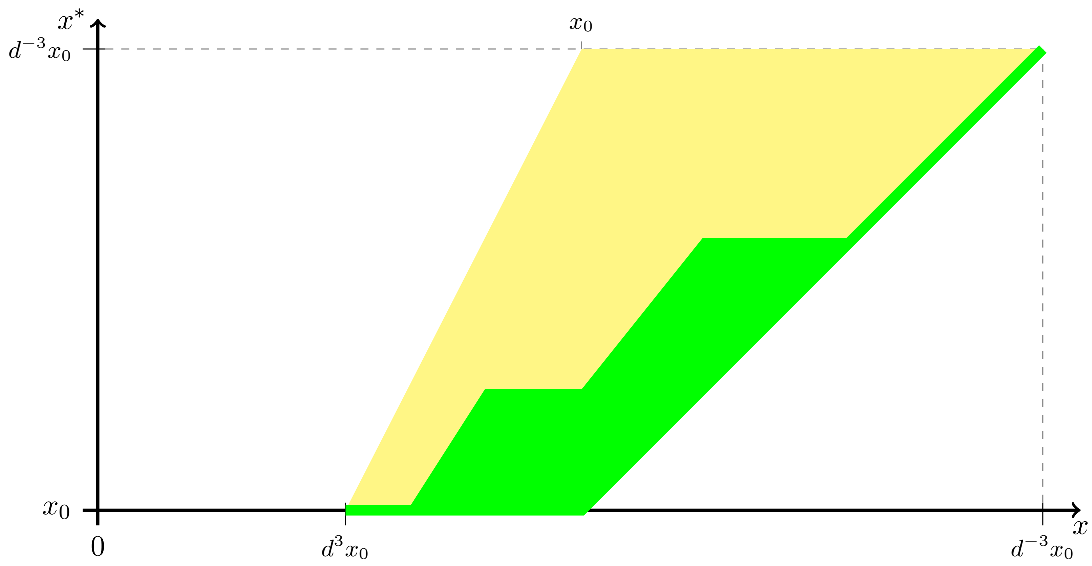
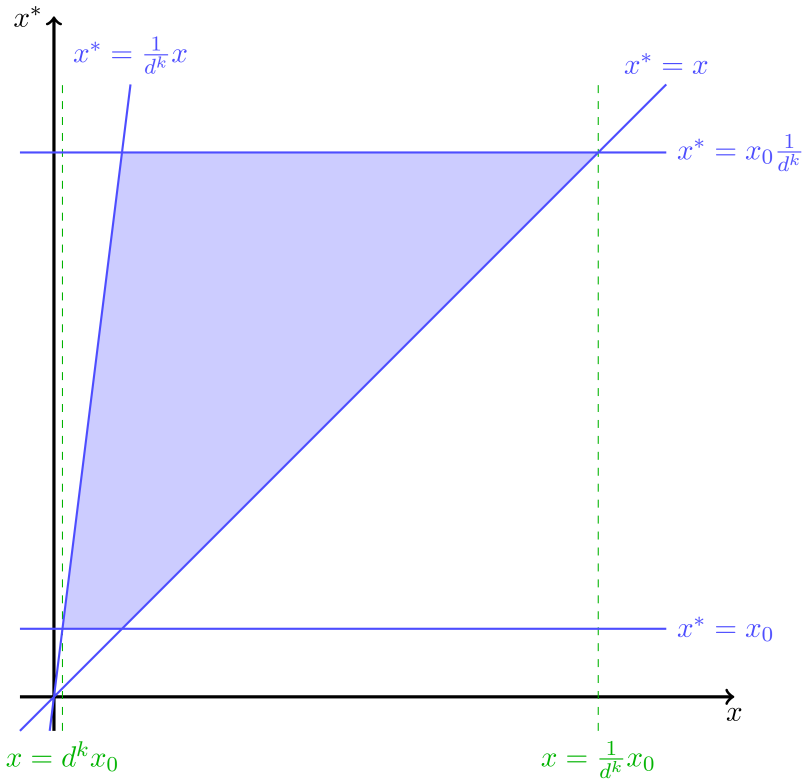
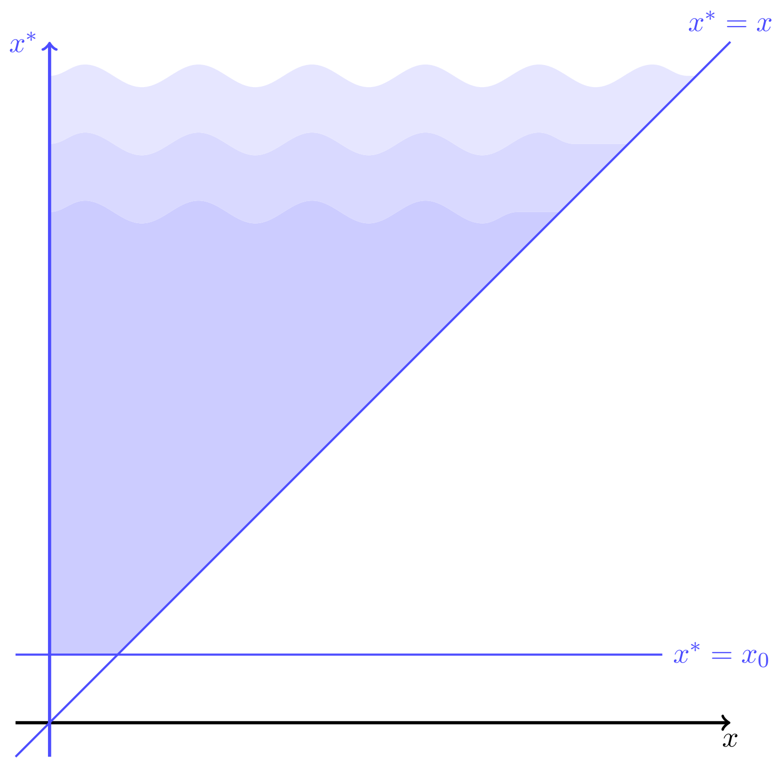


Figure 5.5: Construction of $\tilde{\mathcal{X}}_k$ (yellow) from $\mathcal{X}_k$ (green), exemplary for $k = 3$.

$\tilde{\mathcal{X}}_k$ is a convex polygon in the $x - x^*$ plane with the property $\mathcal{X}_k \subset \tilde{\mathcal{X}}_k$. It is more suitable for computational approaches and can be discretized with a grid as described in (5.21). For increasing k (and n, of course), $\tilde{\mathcal{X}}_k$ approaches a 45° cone, capped at x_0, and coincides with the limit of $\mathcal{X}_k$. Figure 5.6 sketches the general construction of $\tilde{\mathcal{X}}_k$ and the limit situation.



(a) $\tilde{\mathcal{X}}_k$ forms a convex polygon for fixed k.



(b) In the limit case $k, n \to \infty$, $\tilde{\mathcal{X}}_k$ is an unbounded (capped) convex cone and coincides with the limit of $\mathcal{X}_k$.

Figure 5.6: Sketch of $\tilde{\mathcal{X}}_k$ in the $x - x^*$ plane.

Admitting Short-Selling and Intermediate Wage Deduction

Until now we have considered the special case of terminal utility problems with strategies constrained to no-shortselling. If with $\psi \neq 0$ intermediate wages are allowed or if the agent is offered a range larger than $[0,1]$ to choose her strategies from, different state spaces arise.

Negative intermediate wealths shall be excluded at any time. We have already required the intermediate wages to ensure a non-negative capital remains. To provide the same property when enlarging the space of admissible strategies we need to pose boundaries for the choice of π: $\mathring{x}(x,\pi)$ linearly increases in π, so there is a minimal value $\hat{\pi}_0$ such that the wealth produced from x and π is non-negative for all strategies at least as large as $\hat{\pi}_0$. As $\mathring{x}(x,\pi)$ linearly decreases with π the analogue bound is a maximal value $\mathring{\pi}_0$. These are

$$\begin{aligned} \hat{\pi}_0 &:= \min\{y \in \mathbb{R} : \hat{x}(x,\pi) \geq 0 \ \forall \pi \geq y\} = -\frac{1+r}{u-1-r}, \\ \mathring{\pi}_0 &:= \max\{y \in \mathbb{R} : \mathring{x}(x,\pi) \geq 0 \ \forall \pi \leq y\} = \frac{1+r}{1+r-d}. \end{aligned} \tag{5.16}$$

Note that both $\hat{\pi}_0$ and $\mathring{\pi}_0$ are independent of x.

Let us now incorporate intermediate wages and admit $\pi \in [\pi_{\min}, \pi_{\max}] \subseteq [\hat{\pi}, \mathring{\pi}]$ for some $\pi_{\min} < \pi_{\max}$ and construct the associated state spaces $\mathcal{Y}_0^{[\pi_{\min},\pi_{\max}]}, \ldots, \mathcal{Y}_n^{[\pi_{\min},\pi_{\max}]}$. As before wealth and high-water mark start at (x_0, x_0), so $\mathcal{Y}_0^{[\pi_{\min},\pi_{\max}]} = \mathcal{X}_0 = \{(x_0, x_0)\}$.

From then everything works in almost the same way as in the construction of the sets $\mathcal{X}_k$. The differences are:

- Given intermediate wealth $X_{t_{k-1}} = x$ we have to consider new wealths X_{t_k-} from $\{\hat{x}(x,\pi) : \pi_{\min} \leq \pi \leq \pi_{\max}\}$ and $\{\mathring{x}(x,\pi) : \pi_{\min} \leq \pi \leq \pi_{\max}\}$ in the upwards or downwards case. Recapitulate figure 5.3: If now $\pi_{\max} > 1$ the lighter green line stretches farther out to the upper right and the darker green line extends farther to the left as borrowing from the bond induces a leverage effect. If $\pi_{\min} < 0$ the green lines do no longer start at the same point (corresponding to $\pi = 0$), but begin to cross that point and to extend in the reverse direction due to short-selling. The domain for the updated wealth X_{t_k-} after one stock price change (and before wages are deduced) now is

$$\left[\min\left\{\hat{x}(x,\pi_{\min}), \mathring{x}(x,\pi_{\max})\right\}, \max\left\{\hat{x}(x,\pi_{\max}), \mathring{x}(x,\pi_{\min})\right\}\right]. \tag{5.17}$$

- Next the high-water mark $X^*_{t_{k-1}} = x^*$ is updated. As the running maximum can never decrease only the new upper boundary for the wealth effects $X^*_{t_k}$ which now lies somewhere within $[x^*, \max\{\hat{x}(x,\pi_{\max}), \mathring{x}(x,\pi_{\min})\}]$, depending on what new wealth X_{t_k-} was achieved.

- To arrive at X_{t_k} wages have to be deduced. As no further restrictions were put on ψ, an intermediate wealth $X_{t_k-} = x$ can be reduced to any value within $[0, x]$.

We are now ready to recursively construct the appropriate state spaces by slightly modifying the set operators (5.13) to

$$\begin{aligned} \hat{Y}_{\pi_{\min}}^{\pi_{\max}}(Z) &:= \{(y, \max\{\hat{x}(x,\pi), x^*\}) : y \in [0, \hat{x}(x,\pi)], (x,x^*) \in Z, \pi \in [\pi_{\min}, \pi_{\max}]\}, \\ \mathring{Y}_{\pi_{\min}}^{\pi_{\max}}(Z) &:= \{(y, \max\{\mathring{x}(x,\pi), x^*\}) : y \in [0, \mathring{x}(x,\pi)], (x,x^*) \in Z, \pi \in [\pi_{\min}, \pi_{\max}]\} \end{aligned} \tag{5.18}$$

for sets $Z \subset \mathbb{R}^2$ and restating (5.14) as

$$\begin{aligned} \mathcal{Y}_0^{\pi_{\min},\pi_{\max}} &:= \{(x_0, x_0)\}, \\ \mathcal{Y}_k^{\pi_{\min},\pi_{\max}} &:= \hat{Y}_{\pi_{\min}}^{\pi_{\max}}(\mathcal{Y}_{k-1}) \cup \mathring{Y}_{\pi_{\min}}^{\pi_{\max}}(\mathcal{Y}_{k-1}),\ k = 1, \dots, n. \end{aligned} \tag{5.19}$$

Finally let $\mathcal{Y}_k := \mathcal{Y}_k^{0,1}$ abbreviate the special case of an agent with intermediate wages paid but still being bound to no-shortselling. In figure 5.7 the recursive construction of the latter is sketched.

The visual impression is true: We can easily represent $\mathcal{Y}_k$ in closed form as

$$\mathcal{Y}_k = \left\{(x, x^*) \in \mathbb{R}_+^2 :\ 0 \le x \le x^*,\ x_0 \le x^* \le \frac{x_0}{d^k}\right\},\ k = 0, \dots, n. \tag{5.20}$$

We can now formally explain what admissibility of states means:

Definition 5.5 *At step $k \in \{0, \dots, n\}$ the state (x, x^*) is called*

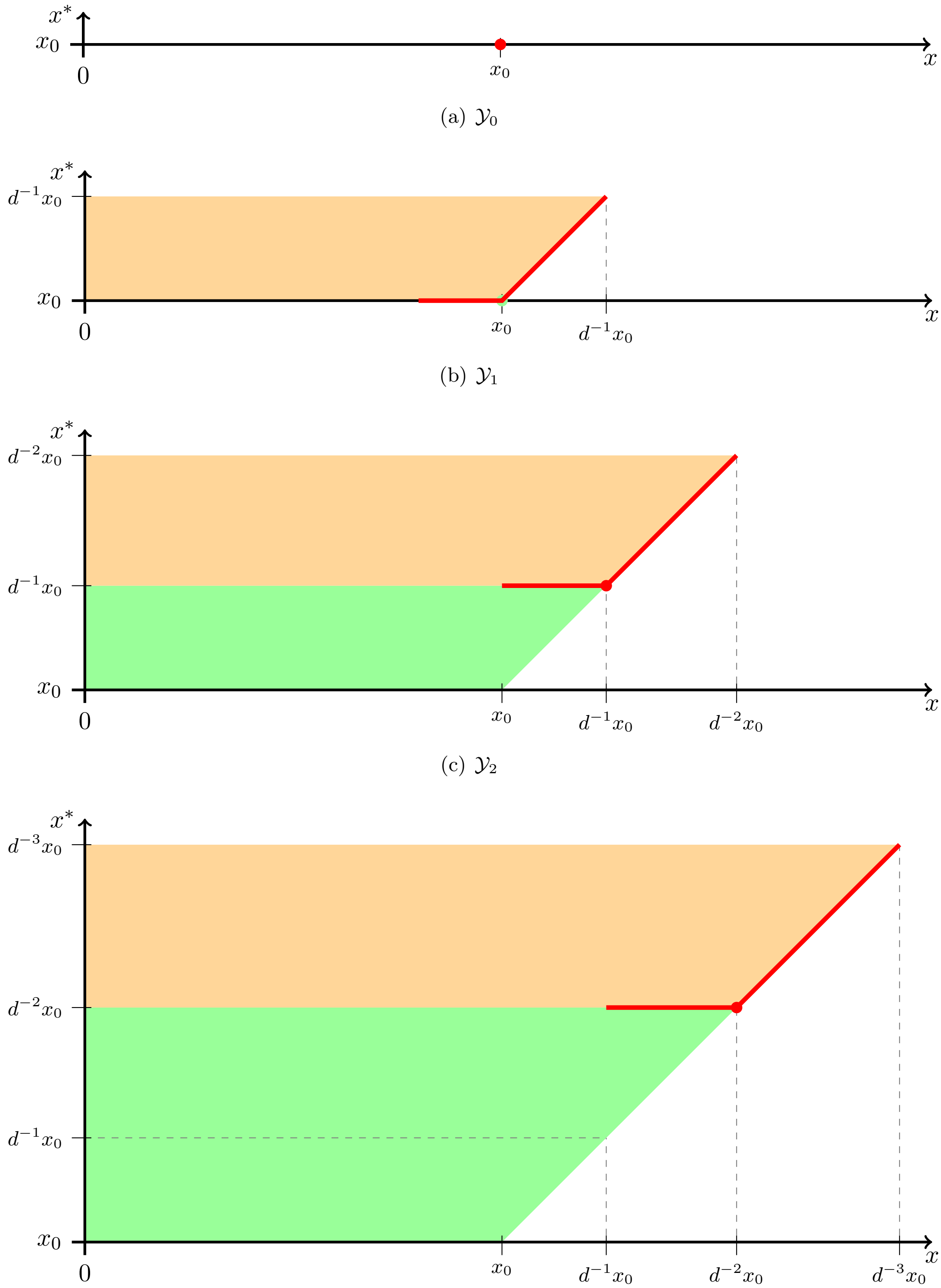
a) ***strictly admissible for the terminal portfolio problem with no-shortselling and without intermediate wages*** *if $\omega \in \Omega$ and an admissible control exists such that $(X_{t_k}(\omega), X^*_{t_k}(\omega)) = (x, x^*)$, where the control of X is subject to no-shortselling constraints and no intermediate wages are paid. This true if and only if $(x, x^*) \in \mathcal{X}_k$.*

b) ***admissible for the terminal portfolio problem with no-shortselling and without intermediate wages*** *if $(x, x^*) \in \tilde{\mathcal{X}}_k$.*

c) ***admissible for the portfolio problem with intermediate wages and strategy bounds*** *$\pi_{\min}$* ***and*** *$\pi_{\max}$ such that $[\pi_{\min}, \pi_{\max}] \subseteq [\hat{\pi}, \mathring{\pi}]$ if $\omega \in \Omega$, an admissible control, and an intermediate wage schedule ψ exist such that $(X_{t_k}(\omega), X^*_{t_k}) = (x, x^*)$, where the control of X is restricted to $\pi \in [\pi_{\min}, \pi_{\max}]$ and intermediate wages ψ are deduced. This is true if and only if $(x, x^*) \in \mathcal{Y}_k^{\pi_{\min},\pi_{\max}}$.*

Whenever the context already clarifies whether or not intermediate wages are present and what the strategy bounds are, (x, x^*) is called an ***admissible state*** *if it is included in the proper set as listed above.*

The spaces $\tilde{\mathcal{X}}$ and $\mathcal{Y}$ in all variations have now been constructed to be suitable for computations. A proper discretization and a matching triangulation is what the next section is about.



(a) $\mathcal{Y}_0$

(b) $\mathcal{Y}_1$

(c) $\mathcal{Y}_2$

(d) $\mathcal{Y}_3$

Figure 5.7: $\mathcal{Y}_0$ and the sets $\mathcal{Y}_1$, $\mathcal{Y}_2$, and $\mathcal{Y}_3$, successively constructed from their predecessors. In step $k \geq 1$, $\mathcal{Y}_{k-1}$ is sketched green. The red dot is the prominent point $(x, x^*) = (d^{1-k}x_0, d^{1-k}x_0) \in \mathcal{Y}_{k-1}$, the red line starting from there denotes the points $\hat{Y}(\{(x, x^*)\}) \cup \mathring{Y}(\{(x, x^*)\})$ that are admissible for X_{t_k-}, and the lighter red shaded region is what can be reached with X_{t_k} after wages deduced. The union of everything colored is $\mathcal{Y}_k$.

5.3 Computing Solutions in the Discrete Model

In order to compute solutions to this kind of problems numerically, one has to discretize the uncountable (enlarged) state spaces $\tilde{\mathcal{X}}_k$ and $\mathcal{Y}_k$ for $k \geq 1$ with a finite subset. In this section a grid-like discretization is proposed and a matching interpolation scheme over the grid is introduced.

5.3.1 Discretizing the Admissible State Spaces

To discretize $\tilde{\mathcal{X}}_k$ or $\mathcal{Y}_k$ we need to specify a two-dimensional grid. As we want to use that grid to construct an interpolation, we want it to provide the following properties:

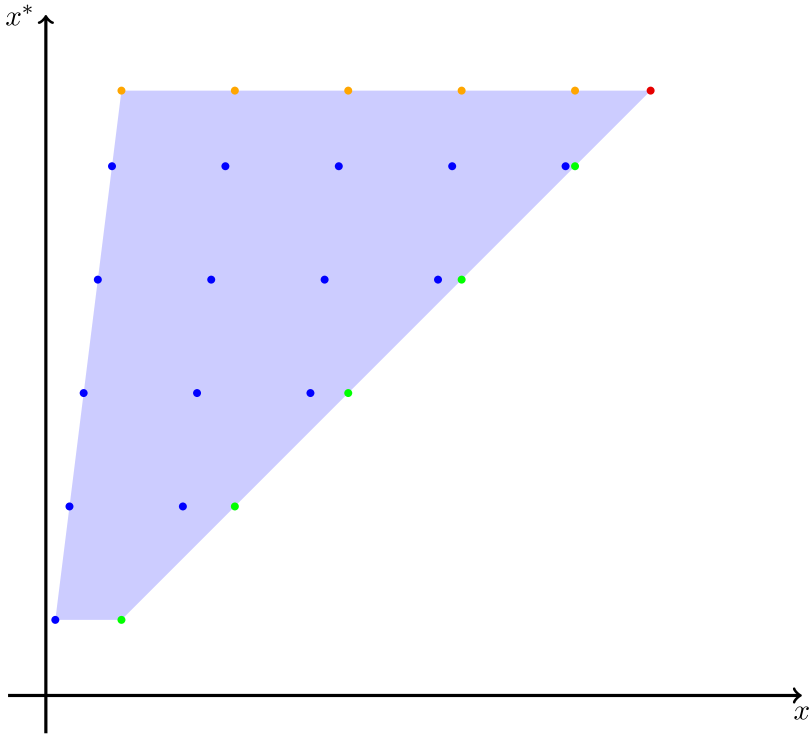
- Every vertex of the grid must be admissible.
- There must exist a triangulation of the grid.
- Every admissible state must lie in exactly one triangle of that triangulation.
- The grid shall be simple to construct and as regularly laid out as possible.

The first 3 properties are essential to finding a well-defined interpolation on that grid, while the fourth is for convenience reasons. Let us start with the latter and choose a discretization step-width $h > 0$. The principal idea is to individually walk along the domains of wealth and high-water mark in h-steps. As x^* attributes the simpler bounds, let us first define gradually increasing levels of x^*. Starting with the lower bound x_0 and working upwards in h-steps yields $\{x_0 + jh : \; j \in \mathbb{N}_0, \; x_0 + jh < x_0 d^{-k}\}$. We have to stop before stepping over the upper bound $x_0 d^{-k}$, but to ensure the third grid property we have to add the additional level $\{x_0 d^{-k}\}$, which we will call upper-irregular.

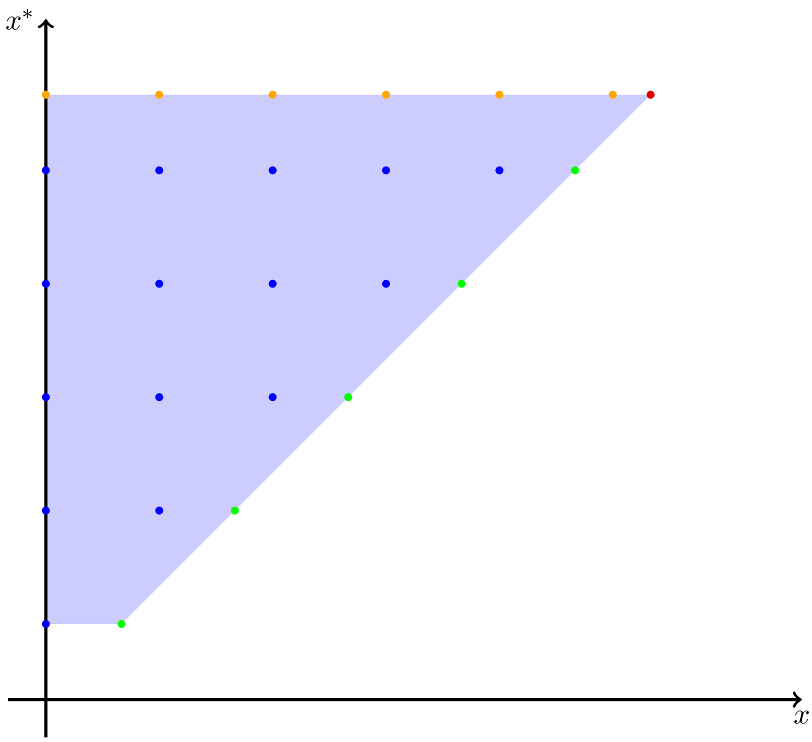
For each of those levels we pick the first grid point on that level on the left-hand boundary, i.e. at $(x^* d^k, x^*)$ for $\tilde{\mathcal{X}}_k$ or at $(0, x^*)$ for $\mathcal{Y}_k$, and then walk to the right in h-steps, collecting the wealth values x in $\{x^* d^k + jh : \; j \in \mathbb{N}_0, \; x^* d^k + jh < x^*\}$ for $\tilde{\mathcal{X}}_k$ or the simpler x in $\{jh : \; j \in \mathbb{N}_0, \; jh < x^*\}$ for $\mathcal{Y}_k$. For the third property, again add the point (x^*, x^*) on the right-hand boundary. For general h that last point will have a distance to its left neighbor that is shorter than h, so we call that one right-irregular, too. Altogether we have the discretized state spaces

$$
\begin{aligned}
\Delta\mathcal{X}_k &:= \{(x, x^*) : \; x^* \in \Delta_k^*, \; x \in \Delta_k(x^*, x^* d^k)\} \quad \text{and} && (5.21)\\
\Delta\mathcal{Y}_k &:= \{(x, x^*) : \; x^* \in \Delta_k^*, \; x \in \Delta_k(x^*, 0)\}, \quad \text{where} && (5.22)\\
\Delta_k^* &:= \{x_0 + jh : \; j \in \mathbb{N}_0, \; x_0 + jh < x_0 d^{-k}\} \cup \{x_0 d^{-k}\}, &&\\
\Delta_k(x^*, \lambda) &:= \{\lambda + jh : \; j \in \mathbb{N}_0, \; \lambda + jh < x^*\} \cup \{x^*\}, \; x^* \in \Delta_k^*, \; \lambda \in \mathbb{R}. &&
\end{aligned}
$$

Note that $\Delta\mathcal{X}_k \subset \tilde{\mathcal{X}}_k = \mathrm{conv}(\Delta\mathcal{X}_k)$ and $\Delta\mathcal{Y}_k \subset \mathcal{Y}_k = \mathrm{conv}(\Delta\mathcal{Y}_k)$, so though the grids are very small subsets of the spaces they discretize they are still large enough to span these spaces in terms of the convex hull. Figure 5.8 sketches how the grids are formed.



(a) $\mathcal{X}_k$ for some $k \geq 1$.



(b) $\mathcal{Y}_k$ for some $k \geq 1$.

Figure 5.8: Discretizations $\Delta\mathcal{X}_k$ and $\Delta\mathcal{Y}_k$. Blue vertices are regular, orange vertices are upper-irregular, green vertices right-irregular, and the red vertex in the upper right corner attributes both irregularities.

The grids $\Delta\mathcal{Y}_k$ are simpler than their counterparts $\Delta\mathcal{X}_k$ and, as can be seen in the construction (5.22), a special case of $\Delta\mathcal{X}_k$ for $\lambda = 0$. Let us now focus on the more complex grids $\Delta\mathcal{X}_k$ and discuss their properties as well as introduce a triangulation and interpolation on them.

Obviously, the grid is regular on the left and lower boundaries but irregular when approaching the right and upper boundaries, so why do we not smooth out the effects by deviantly choosing an $h = \frac{x_0 d^{-k} - x_0}{N}$ for some $N \in \mathbb{N}$ that yields an equidistant discretization in x^*-direction? Firstly for general d there is no h which satisfies the desired equidistance property for all steps $k = 2, \ldots, n-1$. This would imply that we choose a different h for every time-step. And secondly even when taking into consideration only one certain step k, there is in general no h that would provide equidistance in x^*- as well as in x-direction, not even on just one level of Δ_k^*. Additionally allowing the choice of two discretization widths h for the x^*-direction and $\tilde{h}$ for the x-direction also will not solve the problem, as in general there does not exist a value of $\tilde{h}$ that equidistantly discretizes more than one level of Δ_k^*. Finally, even individually choosing equidistant grid-widths for each time-step, each Δ_k^*, and for each level in that set will not produce a simple grid, as there may now exist various triangulations (mind that between two grid points on level x^* there now might be one or none or more than one grid points on the level above and below) and it is not trivial to pick a good one.

Accounting for these difficulties, $\Delta\mathcal{X}_k$ provides a fairly simple grid. We will now have a closer look on the implied triangulation scheme.

5.3.2 Triangulation and Interpolation over the Grid

Any triangle can be represented by one vertex and the two side vectors pointing from that vertex to the remaining two vertices. Also recall that in two-dimensional vector spaces any point can be uniquely characterized by a linear combination of 2 base vectors.

If now a triangulation of that vector space is available, one can interpolate the value of an arbitrary function (which is at least defined for all grid points) at a non-grid point (x, y) by identifying that triangle which contains (x, y), picking one of its vertices, and constructing a vector space base from the two side vectors.

Figure 5.10 explains the construction in detail. The essential point here is that the three vertices of a triangle just suffice while any polygon with more vertices yields more base vectors than necessary.

We now proceed with constructing a triangulation for the admissible state spaces of high-water mark problems.

A Scheme for Finding Enclosing Triangles

First, consider the boundaries as lines in the triangulation scheme that connect all those vertices that lie on the edge. Then, of course, add the horizontal levels $\Delta_k(x^*)$ by connecting the left-most vertex $x^* d^k$ with the right-most vertex x^* of that level. Let us call the inter-space

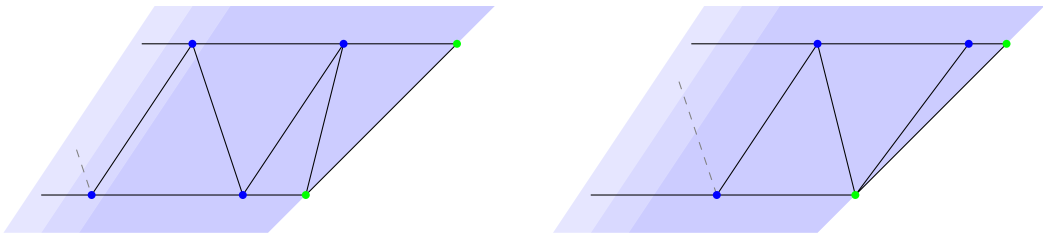
$$R(x^*) := \left\{ (x, y) : \ (x, y) \in \tilde{\mathcal{X}}_k, \ x^* \leq y < \min\{x^* + h, x_0 d^{-k}\} \right\}, x^* \in \Delta_k^* \setminus \{x_0 d^{-k}\}$$

the row with lower boundary x^*. In this definition we have not yet taken care of the top boundary points with $x^* = x_0 d^{-k}$. No harm is done by adding these to the topmost row $R(\max\{x_0 + jh : \ j \in \mathbb{N}_0, \ x_0 + jh < x_0 d^{-k}\})$.

Now let us triangulate each row $R(x^*)$ by starting at the lower left edge $(x^* d^k, x^*)$ and walking along the left-hand boundary up to the first vertex $(\min\{x^*+h, x_0 d^{-k}\}d^k, \min\{x^*+h, x_0 d^{-k}\})$ on the level above, then downwards again to the second vertex $(x^* d^k + h, x^*)$ on the lower level, then upwards again, and so on. We can without difficulty continue the scheme until we reach the last (right-irregular) vertex on the lower level. Then we need to consider another property of the row:

As we can easily verify in figure 5.8, the number of grid points within the level $\Delta_k(x^*)$ may coincide with the number of grid points on the level $\Delta_k(x^* - h)$ below or be 1 larger. Analogously, the level $\Delta_k(\min\{x^* + h, x_0 d^{-k}\})$ above may contain the same number of points or one more. This is a trivial consequence of the slope of the left-hand boundary of $\tilde{\mathcal{X}}_k$ being larger than the slope of the right-hand boundary (thus the number of points may increase), but the angle between both boundaries being less than 45° (so no more than one point may be added per level).

Let us call a row $R(x^*)$ regular if the number of grid points on the lower level coincides with the number of grid points on the upper level, and irregular otherwise. How to continue the triangulation to the right edge of $\tilde{\mathcal{X}}_k$ depends on the regularity of the row: For regular rows there is nothing to be done, as the right-irregular vertices of both levels are already connected, but in an irregular row there is one regular vertex on the upper level left, which we connect next and then terminate. In the latter case we encounter two adjacent triangles which both point downwards while in regular rows the orientation of triangles alternates. C.f. figure 5.9.



(a) A regular row. (b) An irregular row.

Figure 5.9: Triangulation of rows when approaching the irregularities on the right edge.

We will now formally state an algorithm to determine the unique triangle $\triangle(A, B, C)$ that contains the state $(x, x^*) \in \tilde{\mathcal{X}}_k$. In order to keep the presentation simple, let us introduce some notation:

For some $p = (p_x, p_{x^*}) \in \Delta\mathcal{X}_k$ the functions `leftNeighbor`(p) and `rightNeighbor`(p) return that grid point in $\Delta_k(p_{x^*})$ which is located on the same x^* level, does not coincide with p and is closest to the left or the right, respectively and if existent.

By the definition of $\Delta\mathcal{X}_k$ there always exists a point $l = (l_x, l_{x^*}) \in \Delta\mathcal{X}_k$ that is located lower left of (x, x^*) in the sense that

- its h-wm coordinate l_{x^*} is maximal among those in Δ_k^* that are less or equal to x^* and

- its wealth coordinate l_x is maximal among those in $\Delta_k(l_{x^*})$ which are less than or equal to x.

The function `lowerLeft`(x, x^*) will locate and return that point.

Analogously, there may (but needs not) exist a point $r = (r_x, r_{x^*}) \in \Delta\mathcal{X}_k$ that is located lower right of (x, x^*) in the sense that

- its h-wm coordinate r_{x^*} coincides with l_{x^*} of the lower left point l and
- its wealth coordinate r_x is minimal among those in $\Delta_k(l_{x^*})$ which are larger than x.

If existent, the call to `rightNeighbor`(l) will find and return it.

As long as (x, x^*) is not located on the top boundary of $\tilde{\mathcal{X}}_k$ (that case will be handled separately), we can also find a point $u = (u_x, u_{x^*}) \in \Delta\mathcal{X}_k$ that is located above it in the sense that

- the h-wm coordinate u_{x^*} of u is minimal among those in Δ_k^* that are larger than x^* and
- its wealth coordinate u_x is minimal among those in $\Delta_k(u_{x^*})$ which are larger than x.

Delegate that task to `upper`(x, x^*) and note that the resulting point u will mostly lie between l and r (i.e. $l_x < u_x \le r_x$), but as we approach the irregular triangles on the right edge, it may also be located even further on the right than r (given r exists).

The triangulation on the top-line of $\tilde{XX}_k$ is performed by algorithm 5.1:

Algorithm 5.1: TRIANGULATETOP

---

input : interpolation point (x, x^*) in the top row, grid $\Delta\mathcal{X}_k$
output: points A and B defining a line that contains (x, x^*) and a point $C =$ `Null`

```
l ← lowerLeft((x, x*));
if l is in the top right corner of ΔX_k then
    A ← leftNeighbor(l);
    B ← l;
else
    A ← l;
    B ← rightNeighbor(l);
end
C ← Null;
```

Algorithm 5.1 and all above considerations culminate in the main interpolation algorithm 5.2.

Algorithm 5.2: TRIANGULATE

```
input  : interpolation point (x, x*), grid ΔX_k
output: points A, B, and C defining the triangle that contains (x, x*)
if (x, x*) is located on the top edge of ΔX_k then
    return the results of TriangulateTop((x, x*), ΔX_k) and exit;
end
l ← lowerLeft((x, x*));
if lowerRight((x, x*)) exists then
    r ← rightNeighbor (l);
    u ← upper((x, x*));
    if u_x ≤ r_x then
        if (x, x*) lies above the straight through l and u then
            located in the regular triangle pointing downwards to l
            A ← l, B ← u, C ← leftNeighbor(u);
        else if (x, x*) lies above the straight through r and u then
            located in the regular triangle pointing downwards to r
            A ← r, B ← rightNeighbor (u), C ← u;
        else
            located in the regular triangle with base l and r and pointing upwards to u
            A ← l, B ← r, C ← u;
        end
    else
        if (x, x*) lies above the straight through l and u then
            located in the last regular triangle of that row, pointing downwards to r
            A ← l, B ← u, C ← leftNeighbor (u);
        else
            located in the first irregular triangle of that row, pointing upwards to u
            A ← l, B ← r, C ← u;
        end
    end
else
    if this is a regular row then
        if (x, x*) lies above the straight through leftNeighbor (l) and u then
            located in the last regular triangle pointing downwards to leftNeighbor (l)
            A ← leftNeighbor (l), B ← u, C ← rightNeighbor (u);
        else if (x, x*) lies above the straight through l and u then
            located in the first irregular triangle pointing upwards to u
            A ← leftNeighbor (l), B ← l, C ← u;
        else
            located in the second irregular triangle pointing downwards to l
            if rightNeighbor(u) exists then
                A ← l, B ← rightNeighbor (u), C ← u;
            else
                this can occur in the top-most row
                A ← l, B ← u, C ← leftNeighbor (u);
            end
        end
    else
        if (x, x*) lies above the straight through l and u then
            located in the first irregular triangle pointing downwards to l
            A ← l, B ← u, C ← leftNeighbor (u);
        else
            located in the second irregular triangle also pointing downwards to l
            A ← l, B ← rightNeighbor (u), C ← u;
        end
    end
end
```

Interpolating Non-Grid Values

Now that we have established the triangulation for each point $(x, x^*) \in \tilde{\mathcal{X}}_k$, we can linearly interpolate the value of $J_k(x, x^*)$ using the known values of J_k at the triangulation points.

If (x, x^*) is located in the top row, the triangulation returned those two points A and B whose connecting line contains (x, x^*) (and an empty point C). As the high-water mark coordinates of A and B match x^*, we can write

$$\begin{pmatrix} x \\ x^* \end{pmatrix} = A + \lambda(B - A) = (1 - \lambda)A + \lambda B$$

and interpolate

$$J_k(x, x^*) \approx (1 - \lambda)J_k(A_x, A_{x^*}) + \lambda J_k(B_x, B_{x^*}), \tag{5.23}$$

where the factor λ is simply the percentage of the distance between A and B that falls upon the line segment from A to (x, x^*):

$$\lambda = \frac{x - A_x}{B_x - A_x}.$$

In the more general case of (x, x^*) being located somewhere else within $\tilde{\mathcal{X}}_k$, the triangulation returns 3 pairwise distinct points A, B, and C that span a triangle containing (x, x^*). Similarly to above write

$$\begin{pmatrix} x \\ x^* \end{pmatrix} = A + \lambda_1(B - A) + \lambda_2(C - A) = (1 - \lambda_1 - \lambda_2)A + \lambda_1 B + \lambda_2 C$$

and interpolate

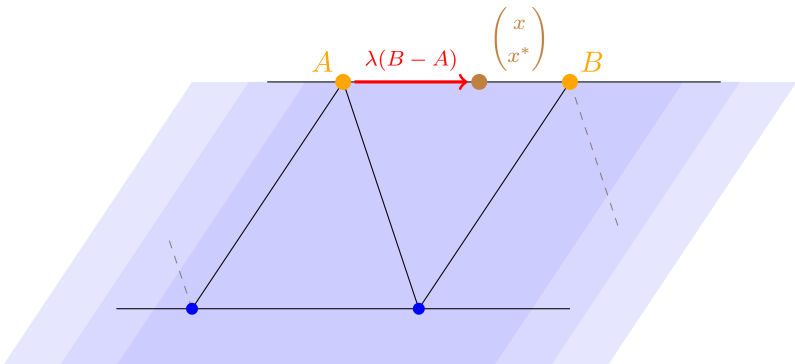
$$J_k(x, x^*) \approx (1 - \lambda_1 - \lambda_2)J_k(A_x, A_{x^*}) + \lambda_1 J_k(B_x, B_{x^*}) + \lambda_2 J_k(C_x, C_{x^*}). \tag{5.24}$$

The vector $\lambda = (\lambda_1, \lambda_2)^\top$ is easily found by rewriting

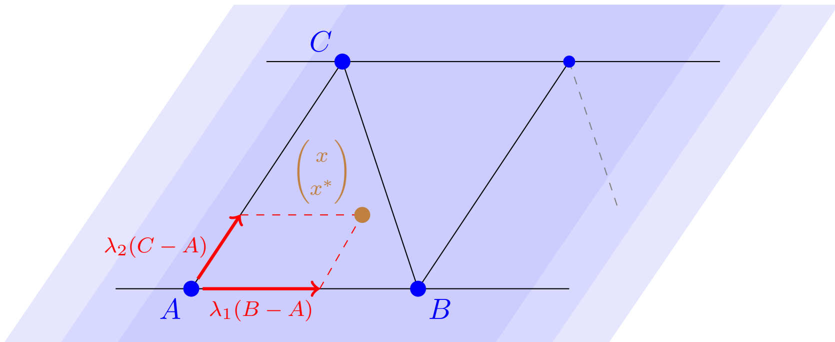
$$\begin{aligned}
\begin{pmatrix} x \\ x^* \end{pmatrix} &= A + \lambda_1(B - A) + \lambda_2(C - A) \quad \Longleftrightarrow \\
\begin{pmatrix} x \\ x^* \end{pmatrix} - A &= (B - A \mid C - A)\,\lambda \quad \Longleftrightarrow \\
\lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} &= \begin{pmatrix} B_x - A_x & C_x - A_x \\ B_{x^*} - A_{x^*} & C_{x^*} - A_{x^*} \end{pmatrix}^{-1} \cdot \begin{pmatrix} x - A_x \\ x^* - A_{x^*} \end{pmatrix}.
\end{aligned}$$

Figure 5.10 depicts the interpolation in both cases.

We are now ready to apply the discretization and interpolation to portfolio problems.



(a) (x, x^*) at the top edge of $\tilde{\mathcal{X}}_k$.



(b) (x, x^*) somewhere else within $\tilde{\mathcal{X}}_k$.

Figure 5.10: Linear interpolation of $J_k(x, x^*)$.

5.4 Pure Wealth Benchmarks

This section paves the way for all discrete-time portfolio problems that we are going to examine. To provide the grounds for a comparative analysis the important and well-known log portfolio problem is recited. It further serves to discuss numerical issues of the computation and provides a benchmark. As yet another benchmark the less common shifted-log-of-wealth portfolio problem is introduced. It will be of great benefit when discussing the incentives of intermediate wage schedules.

5.4.1 The Log Investor

In discrete time it is necessary to restrict the admissible strategies to a finite interval to avoid the possibility of non-positive wealth as the logarithmic utility is not defined for such values. It holds

$$\begin{aligned} \hat{x}(x_0,\pi) &= [1+r+\pi(u-1-r)]\,x_0 > 0 \quad &\Longleftrightarrow \quad \pi > -\frac{1+r}{u-1-r} = \hat{\pi}(x_0,0) \text{ and} \\ \mathring{x}(x_0,\pi) &= [1+r+\pi(d-1-r)]\,x_0 > 0 \quad &\Longleftrightarrow \quad \pi < \frac{1+r}{1+r-d} = \mathring{\pi}(x_0,0). \end{aligned}$$

One way to achieve a well-posed problem is restricting strategies to an arbitrary closed subinterval of $(\hat{\pi}(x_0,0),\mathring{\pi}(x_0,0))$. We will later see that in that particular case the whole (open) interval is a feasible choice, too, as the optimal strategy will be included in it for any parameter choice.

For an economic interpretation the no-shortselling interval $\mathfrak{A} := [0,1]$ is a sensible choice. It is feasible as $\hat{\pi}(x_0,0) < 0$ and $\mathring{\pi}(x_0,0) > 1$ and we will consider the log portfolio problem with no-shortselling constraints as first benchmark:

$$\left(P_{1,0,\log}^{[0,1],n}\right) \quad \begin{cases} \sup_\pi \{\mathbb{E}[\log(X_{t_n})]\} \\ \pi = (\pi_{t_0},\ldots,\pi_{t_{n-1}}),\ \pi_{t_k} = \pi_{t_k}(x) \in [0,1],\ k=0,\ldots,n-1 \end{cases}. \tag{5.25}$$

We can immediately state the solution:

Proposition 5.4
The portfolio problem $\left(P_{1,0,\log}^{[0,1],n}\right)$ admits the unique optimal strategy

$$\pi_{t_k}^* \equiv \pi^* := \max\{0,\min\{1,\pi_{\text{Merton}}^*\}\},\ k=0,\ldots,n-1, \tag{5.26}$$

where

$$\pi_{\text{Merton}}^* := (1+r)\left(\frac{p}{1+r-d} + \frac{1-p}{1+r-d^{-1}}\right). \tag{5.27}$$

The value implied is

$$J_k(x,x^*) = \log(x) + (n-k)\left(\log(1+r) + \mathbb{E}[\log(1+\pi^* R)]\right),\ k=0,\ldots,n, \tag{5.28}$$

where

$$\begin{aligned} &\mathbb{E}[\log(1+\pi^* R)] \\ = \;& p\log\left(1+\pi^*\left(\frac{u}{1+r}-1\right)\right) + (1-p)\log\left(1+\pi^*\left(\frac{d}{1+r}-1\right)\right) \end{aligned}$$

and R is the discounted and normalized random return at step[5] one. □

[5] Note that the returns $R_1,\ldots,R_n$ are i.i.d. random variables.

Proof. By definition it holds $J_n(x, x^*) = J_n(x) = \log(x)$ where we can safely drop the high-water mark x^* as it is not considered in this setup. The Bellman principle (5.10) can accordingly be simplified to the version

$$J_k(x) = \sup_{\pi_k} \left\{ \mathbb{E}^{k,x} \left[J_{k+1} \left(g_k(x; \pi_k) \right) \right] \right\} \tag{5.29}$$

without intermediate wages and without high-water marks. Here $\mathbb{E}^{k,x}[\cdot] := \mathbb{E}[\cdot \mid X_{t_k} = x]$ and

$$g_k(x; \pi_k) := [1 + r + \pi_k(x)(Y_{t_{k+1}} - (1+r))]x$$

is the naturally simplified version of the wealth update function $g_k(x, x^*; \pi_k)$ from (5.8).

An application of (5.29) yields

$$\begin{aligned}
J_{n-1}(x) &= \sup_{\pi_{n-1}} \left\{ \mathbb{E}^{n-1,x} \left[\log(X_{t_n-}) \right] \right\} \\
&= \sup_{\pi_{n-1}} \left\{ p \log(\hat{x}(x,\pi)) + (1-p) \log(\mathring{x}(x,\pi)) \right\}, \ x > 0; \\
\frac{\partial}{\partial \pi} J_{n-1}(x;\pi) &= p \frac{x(u-1-r)}{\hat{x}(x,\pi)} + (1-p) \frac{x(d-1-r)}{\mathring{x}(x,\pi)} \\
&= 0 \iff \\
\pi &= \frac{(1+r)\left(p(u-1-r) + (1-p)(d-1-r)\right)}{(u-1-r)(1+r-d)} = \pi^*_{\text{Merton}}.
\end{aligned}$$

As

$$\left(\frac{\partial}{\partial \pi} \right)^2 J_{n-1}(x;\pi) = -p \frac{x^2(u-1-r)^2}{\hat{x}(x,\pi)^2} - (1-p) \frac{x^2(d-1-r)^2}{\mathring{x}(x,\pi)^2} < 0,$$

$J_{n-1}(x;\pi)$ is strictly concave in π. If $\pi^*_{\text{Merton}} \in [0,1]$, it is the unique maximizer; if not the nearest boundary is. This yields

$$\pi^*_{t_{n-1}} = \max\{0, \min\{1, \pi^*_{\text{Merton}}\}\} = \pi^*.$$

The value at step $k = n-1$ arises to

$$\begin{aligned}
J_{n-1}(x) &= p \log(\hat{x}(x,\pi^*)) + (1-p) \log(\mathring{x}(x,\pi^*)) \\
&= p \log(x) + p \log(1+r) + p \log \left(1 + \pi^* \cdot \frac{u-1-r}{1+r} \right) + \\
&\quad (1-p) \log(x) + (1-p) \log(1+r) + (1-p) \log \left(1 + \pi^* \cdot \frac{d-1-r}{1+r} \right) \\
&= \log(x) + \log(1+r) + \\
&\quad p \log \left(1 + \pi^* \left(\frac{u}{1+r} - 1 \right) \right) + (1-p) \log \left(1 + \pi^* \left(\frac{d}{1+r} - 1 \right) \right).
\end{aligned}$$

This establishes the basis for an induction.

Now assume that for some $k \in \{1, \ldots, n-1\}$ it holds $\pi^*_{t_{n-1}} = \ldots = \pi^*_{t_{k+1}} = \pi^*$ and

$$J_l(x) = \log(x) + (n-l) \left(\log(1+r) + \mathbb{E} \left[\log \left(1 + \pi^* R \right) \right] \right)$$

for $l = n, n-1, \ldots, k+1$. It follows

$$\begin{aligned}
J_k(x) &= \sup_{\pi_k} \left\{ \mathbb{E}^{k,x} \left[J_{k+1}(g_k(x;\pi_k)) \right] \right\} \\
&= \sup_{\pi_k} \left\{ p \log(\hat{x}(x,\pi_k)) + (1-p) \log(\mathring{x}(x,\pi_k)) \right\} + \\
&\quad (n-k-1) \left(\log(1+r) + \mathbb{E} \left[\log \left(1 + \pi^* R \right) \right] \right).
\end{aligned}$$

The solution to the supremal problem above is again π^* as the target function coincides with what we discussed for step $k = n-1$. Plugging it in we find

$$\begin{aligned} J_k(x) &= \log(x) + \log(1+r) + \mathbb{E}\left[\log\left(1+\pi^* R\right)\right] + \\ &\quad (n-k-1)\left(\log(1+r) + \mathbb{E}\left[\log\left(1+\pi^* R\right)\right]\right) \\ &= \log(x) + (n-k)\left(\log(1+r) + \mathbb{E}\left[\log\left(1+\pi^* R\right)\right]\right) \end{aligned}$$

as asserted. ■

For later comparison we recall another property: The log investor is of hyperbolic absolute risk aversion (HARA) type as

$$\mathrm{ARA}_U(y) = -\frac{U''(y)}{U'(y)} = \frac{1}{y}$$

and attributes the constant relative risk aversion (CRRA) of

$$\mathrm{RRA}_U(y) = -\frac{yU''(y)}{U'(y)} = 1.$$

Note that the optimal strategy, among many other properties, in particular is time-independent. This is true for CRRA utilities but, as we are going to see, no longer holds in multi-period models when the high-water mark is incorporated.

(Bäuerle & Rieder, 2011, Theorem 4.2.11 and following remarks) discuss a more general version of (5.25) that covers HARA type utility functions. The statement of proposition 5.4 can be considered a special case of the latter.

A Reference Market

Let us compare the analytical solution to the results of the approximation calculated over the grid. For the numerical approach we need to fix market parameters for the targeted Black-Scholes market and a degree of approximation. In all examples below a Black-Scholes model with time horizon $T = 1$, riskless rate $r_{\mathrm{BS}} = \log(1.03)$ and one stock with drift $\mu = \log(1.05)$ and volatility $\sigma = 30\%$ is assumed. It is discretized to an $n = 5$ step Cox-Ross-Rubinstein market yielding

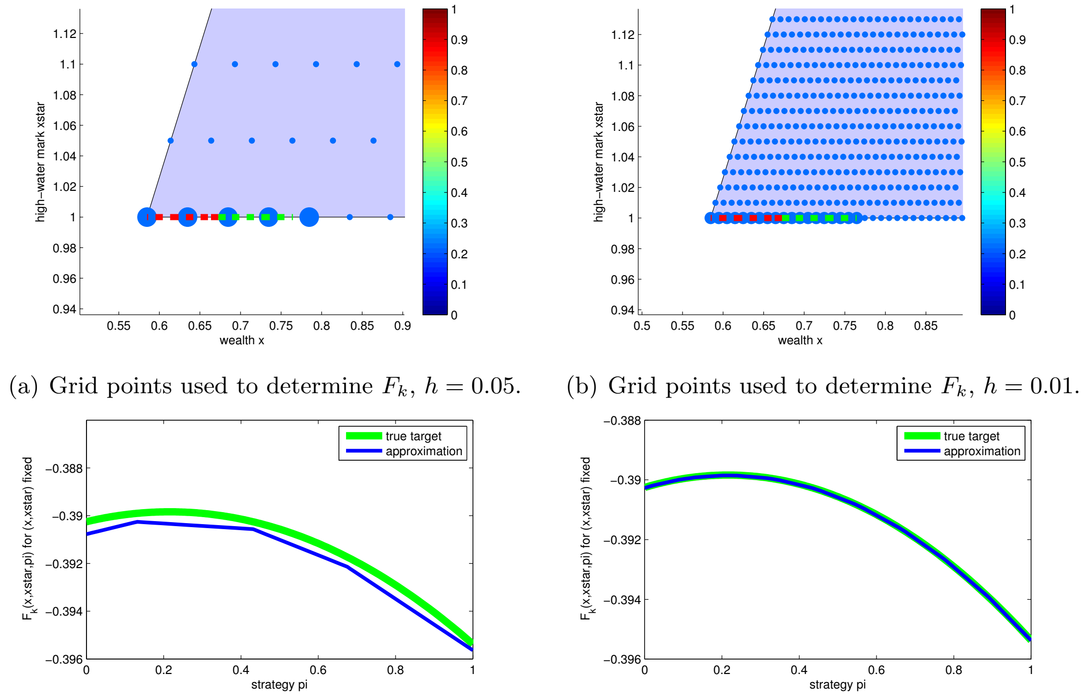
$$\begin{aligned} r &\approx 0.0059, \\ d &\approx 0.8744, \\ p &\approx 0.5029, \quad \text{and} \\ \pi^*_{\mathrm{Merton}} &\approx 0.2155 \in [0,1]. \end{aligned} \tag{5.30}$$

The investor is equipped with initial endowment $x_0 = 1$.

Numerical Issues

By this the admissible states $\mathcal{X}_k$ and their enlargements $\tilde{\mathcal{X}}_k$, $k = 0, \ldots, n$, are completely determined. The question arises how dense the grids $\Delta\mathcal{X}_k$, $k = 0, \ldots, n-1$, need to be. The choice of the grid width h is crucial:

At step $k = n-1$ the analytically known utility function is employed to compute the optimal strategy for each $(x, x^*) \in \Delta\mathcal{X}_{n-1}$. Hence, the value function J_{n-1} is accurate over the grid. Going back another step we need to determine the value function at arbitrary points $(x, x^*) \in \mathcal{X}_{n-1}$ in order to derive an (approximately) optimal strategy over the set $\Delta\mathcal{X}_{n-2}$. At each point $(x, x^*) \in \mathcal{X}_{n-1} \setminus \Delta\mathcal{X}_{n-1}$ the value function is interpolated linearly. It is clear that the interpolated values are the better the denser the grid is. Figure 5.11 illustrates how this directly affects the target functions at step $k-2$:



(a) Grid points used to determine F_k, $h = 0.05$.

(b) Grid points used to determine F_k, $h = 0.01$.

(c) Approximation F_k and true target, $h = 0.05$.

(d) Approximation F_k and true target, $h = 0.01$.

Figure 5.11: Approximation (blue) of the target functions $F_k(x, x^*, \pi)$ (green).

Denser grids yield a better approximation of the target functions $\pi \mapsto F_k(x, x^*, \pi)$ and by that a better approximation of the optimal strategies $\pi_k^*(x, x^*)$.

When deciding about a sensible value for h one has to keep in mind another effect: The one-dimensional subset of $\mathcal{X}_{k+1}$ looked up when determining an optimizer $\pi_{t_k}^*(x, x^*)$ at some point $(x, x^*) \in \mathcal{X}_k$ is

$$\begin{aligned} & \hat{X}(\{(x, x^*)\}) \cup \mathring{X}(\{(x, x^*)\}) \\ = \; & \{(\hat{x}(x, \pi), \max\{\hat{x}(x, \pi), x^*\}) : \pi \in [0, 1]\} \cup \{(\mathring{x}(x, \pi), \max\{\mathring{x}(x, \pi), x^*\}) : \pi \in [0, 1]\} \,. \end{aligned}$$

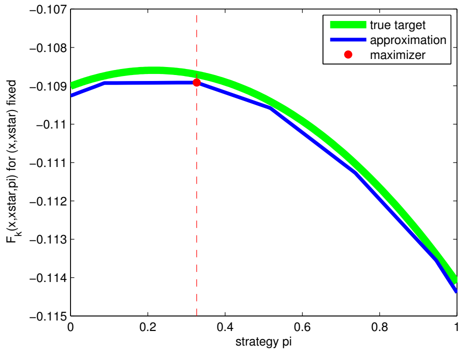
Both $\hat{x}$ and $\mathring{x}$ are linear in x, thus the above set is larger when x is larger. As larger lookup sets mean more grid points are used, the approximated target function $\pi \mapsto F_k(x, x^*, \pi)$ and the optimizer found are more precise on the right edge of $\tilde{\mathcal{X}}_k$ than on the left edge. So to ensure a specific quality of the approximation one has to adjust the grid width using the first point in each $\Delta_k(x^*)$, $x^* \in \Delta_k^*$, $k = 1, \ldots, n-1$.

As we can see in figure 5.11, for all steps from $n-2$ down to 0 the target function is glued together from linear pieces and each two neighboring segments meet in a corner. When now the interpolation point (x, x^*) is shifted along the x-axis, the relative position of the line segments in the target function shifts along (though, by the definition of the interpolation in (5.23) and (5.24), in the opposite direction). By the linearity of the approximated target function there is always a corner with maximal value and as the corners shift along with the edges, the position of the maximum erroneously drafts by the true value. When x has shifted far enough, the next corner will adopt the maximum property and again draft by the true value. Figure 5.12 gives an example.

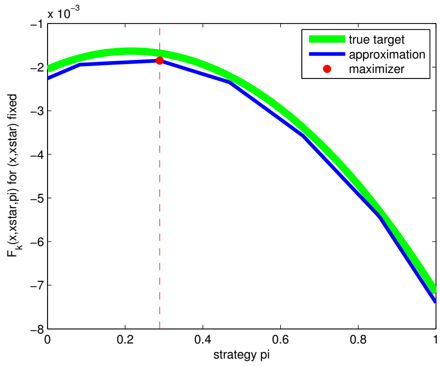
When examining the optimal strategy values for each grid point the same effect is very well recognizable as waves in the color pattern, as figure 5.13 illustrates:

The waves reflect the triangulation used to interpolate values of $J_k(x, x^*)$ at non-grid $(x, x^*) \in \tilde{\mathcal{X}}_k \setminus \Delta\mathcal{X}_k$. Especially in part b of figure 5.13 the triangles can be easily recognized. With a sharper eye one also finds the irregularities in the triangulation when approaching the right edge of $\tilde{\mathcal{X}}_k$. As soon as the x value is large enough such that $\hat{x}(x, \pi)$ enters the irregularly triangulated region on the right edge for $\pi = 1$, the waves pattern attributes irregularities, too.

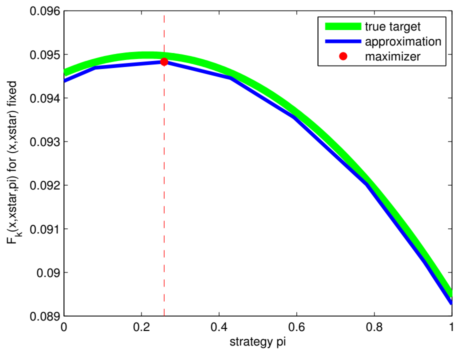
All effects vanish when the grid gets denser.



(a) $x = 0.891 + 0h$.

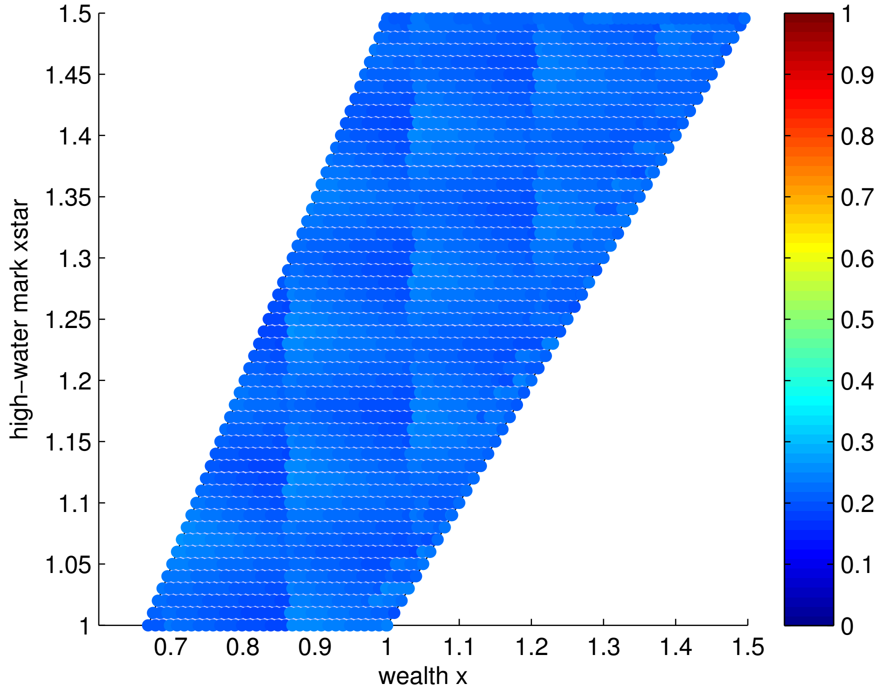


(b) $x = 0.891 + 2h$.

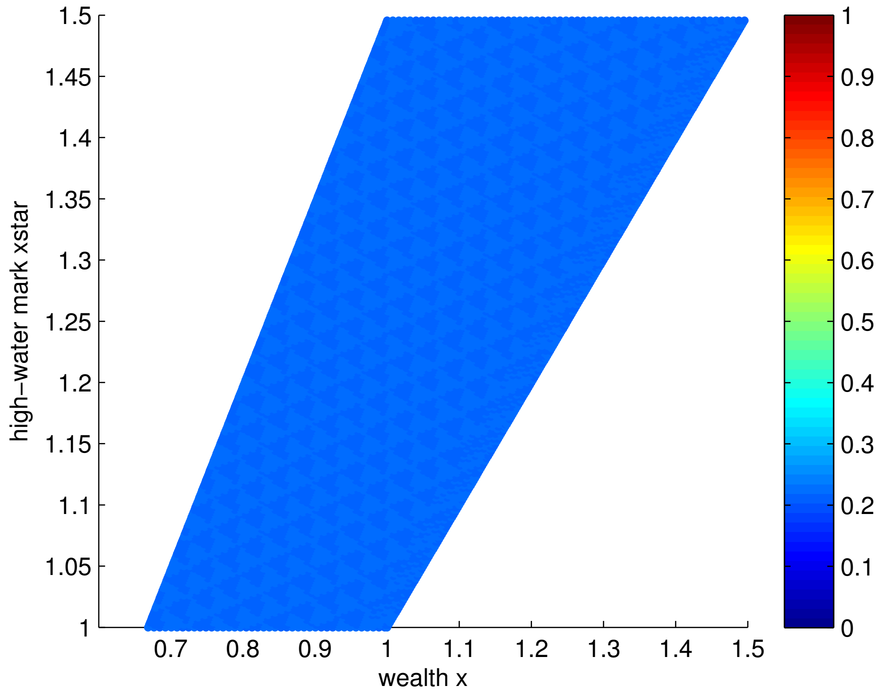


(c) $x = 0.891 + 4h$.

Figure 5.12: The approximated target function $\pi \mapsto F_k(x, x^*, \pi)$ (blue) and the true target $\pi \mapsto J_k(x, x^*, \pi)$ (green) at step $k = 3$ of $n = 5$. The high-water mark is fixed at $x^* = 1.25$ while x increases. Highlighted in red is the optimal strategy as found by optimizing F_k. The grid width is $h = 0.05$.



(a) Grid width $h = 0.01$.



(b) Grid width $h = 0.002$.

Figure 5.13: The waves effect in the optimizers at step $k = 3$ of $n = 5$ for different grid widths h.

5.4.2 An Investor with Shifted Log Preference

The log portfolio problem discussed in the previous section is well-defined only if the wage or wages plugged into the log are positive. One way to ensure this property is to examine the wage schedules and pose sufficient conditions on all parameters involved there. For more elaborate schedules that may be inconvenient or yield technical constraints that would not naturally arise from the problem structure.

As second benchmark let us therefore consider a slight modification of the utility function that ensures well-definedness in another way: For $C > 0$ let $U : (-C, \infty)$, $y \mapsto U(y) := \log(C + y)$ denote the shifted logarithm. The shift enlarges the pre-image such that zero or even negative amounts can be evaluated.

The associated portfolio problem of maximizing the expected utility of terminal wealth is

$$\left(P^{[0,1],n}_{1,0,\log(C+\cdot)}\right) \quad \begin{cases} \sup_\pi \left\{\mathbb{E}\left[\log\left(C + X_{t_n}\right)\right]\right\} \\ \pi = (\pi_{t_0}, \ldots, \pi_{t_{n-1}}), \\ \pi_{t_k} = \pi_{t_k}(x) \in [0,1], \; k = 0, \ldots, n-1 \end{cases} . \tag{5.31}$$

We will later encounter problems that are equivalent or closely related to the last step of $\left(P^{[0,1],n}_{1,0,\log(C+\cdot)}\right)$. Proposition 5.5 provides the solution for last step of the shifted log problem and the affine variation of the latter:

Proposition 5.5 Let $C > 0$.

a) Then the optimal strategy $\pi^* = (\pi^*_0, \ldots, \pi^*_{n-1})$ of $\left(P^{[0,1],n}_{1,0,\log(C+\cdot)}\right)$ satisfies

$$\begin{aligned} \pi^*_{n-1} = \pi^*_{n-1}(x) &= \max\{0, \min\{1, \pi^{C,*}_{\text{Merton}}(x)\}\}, \; x > 0, \text{ where} \\ \pi^{C,*}_{\text{Merton}}(x) &:= \frac{(C + (1+r)x)\,(p(u-1-r) + (1-p)(d-1-r))}{(u-1-r)(1+r-d)x}, \; x > 0. \end{aligned}$$

The shifted discretized Merton ratio can be represented as

$$\pi^{C,*}_{\text{Merton}} = \pi^*_{\text{Merton}} + \frac{C}{x} \cdot \frac{p(u-1-r) + (1-p)(d-1-r)}{(u-1-r)(1+r-d)}, \; x > 0. \tag{5.32}$$

For $x \to 0$ the optimal strategies can be continued to $\pi^*_k(0) := 1$, $k = 0, \ldots, n-1$.

b) Let $\lambda > 0$ and consider $U : (-\lambda^{-1}C, \infty) \to \mathbb{R}$, $y \mapsto \log(C + \lambda y)$. Then the optimal strategy for the portfolio problem $\left(P^{[0,1],n}_{1,0,\log(C+\lambda(\cdot))}\right)$ is $\pi^* = (\pi^*_0, \ldots, \pi^*_{n-1})$ with

$$\begin{aligned} \pi^*_{n-1} = \pi^*_{n-1}(x) &= \max\{0, \min\{1, \pi^{C,\lambda,*}_{\text{Merton}}(x)\}\}, \; x > 0, \text{ where} \\ \pi^{C,\lambda,*}_{\text{Merton}}(x) &:= \frac{(C + (1+r)\lambda x)\,(p(u-1-r) + (1-p)(d-1-r))}{(u-1-r)(1+r-d)\lambda x}, \; x > 0, \end{aligned}$$

and it holds

$$\begin{aligned} \pi^{C,\lambda,*}_{\text{Merton}}(x) &= \pi^*_{\text{Merton}} + \frac{C}{\lambda x} \cdot \frac{p(u-1-r) + (1-p)(d-1-r)}{(u-1-r)(1+r-d)} \\ &= \frac{1}{\lambda} \pi^{C,*}_{\text{Merton}}(x) + \left(1 - \frac{1}{\lambda}\right) \pi^*_{\text{Merton}}, \; x > 0. \end{aligned} \tag{5.33}$$

For $x \to 0$ the optimal strategy can be continued to $\pi^*_{n-1}(0) := 1$. □

Proof. Part a) is a direct consequence of b) for $\lambda = 1$, so let $C > 0$, $\lambda > 0$ and consider $\left(P^{[0,1],n}_{1,0,\log(C+\lambda(\cdot))}\right)$ with $U : (-\lambda^{-1}C, \infty) \to \mathbb{R}$, $y \mapsto \log(C + \lambda y)$ for $x > 0$. An application of (5.29) yields

$$\begin{aligned}
J_{n-1}(x) &= \sup_{\pi_{n-1}} \left\{\mathbb{E}^{n-1,x}\left[\log(C + \lambda X_{t_n-})\right]\right\} \\
&= \sup_{\pi_{n-1}} \left\{p\log(C + \lambda\hat{x}(x,\pi)) + (1-p)\log(C + \lambda\mathring{x}(x,\pi))\right\}, \ x > 0; \\
\frac{\partial}{\partial\pi} J_{n-1}(x;\pi) &= p\frac{\lambda x(u-1-r)}{C + \lambda\hat{x}(x,\pi)} + (1-p)\frac{\lambda x(d-1-r)}{C + \lambda\mathring{x}(x,\pi)} \\
&= 0 \quad \Longleftrightarrow \\
\pi &= \frac{(C + (1+r)\lambda x)\left(p(u-1-r) + (1-p)(d-1-r)\right)}{(u-1-r)(1+r-d)\lambda x} \\
&= \pi^{C,\lambda,*}_{\text{Merton}}.
\end{aligned}$$

As $y \mapsto U(C + \lambda y)$ still is a strictly concave and strictly increasing utility function, the value $J_{n-1}(x;\pi)$ given wealth x at step $k = n-1$ is strictly concave in π and thus the maximum is global over $\mathbb{R}$. As $J_{n-1}(x;0) < J_{n-1}(x;1)$ when $\pi^{C,\lambda,*}_{\text{Merton}}(x) > 1$ and vice versa $J_{n-1}(x;0) > J_{n-1}(x;1)$ when $\pi^{C,\lambda,*}_{\text{Merton}}(x) < 0$ the nearest boundary maximizes in these cases.

When $x \to 0$, $\pi^{C,\lambda,*}_{\text{Merton}}(x) \to +\inf$. As $x \mapsto \pi^{C,\lambda,*}_{\text{Merton}}(x)$ is continuous, $\pi^*_{n-1}(x)$ is capped in a (positive real) neighborhood of $x = 0$ and $\pi^{C,\lambda,*}_{\text{Merton}}(0) = 1$ is the unique continuous choice.

The representation holds as

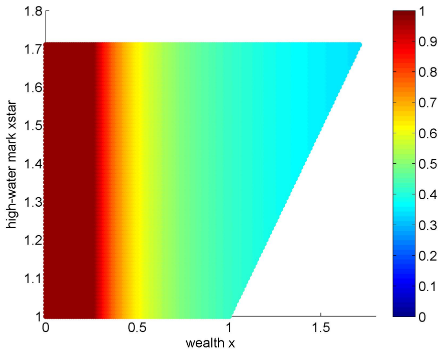
$$\begin{aligned}
\pi^{C,\lambda,*}_{\text{Merton}}(x) &= \frac{(C + (1+r)\lambda x)\left(p(u-1-r) + (1-p)(d-1-r)\right)}{(u-1-r)(1+r-d)\lambda x} \\
&= \frac{(1+r)\left(p(u-1-r) + (1-p)(d-1-r)\right)}{(u-1-r)(1+r-d)} + \\
&\quad \frac{C\left(p(u-1-r) + (1-p)(d-1-r)\right)}{(u-1-r)(1+r-d)\lambda x} \\
&= \pi^*_{\text{Merton}} + \frac{C}{\lambda x} \cdot \frac{p(u-1-r) + (1-p)(d-1-r)}{(u-1-r)(1+r-d)} \\
&= \frac{1}{\lambda}\pi^{C,*}_{\text{Merton}} + \left(1 - \frac{1}{\lambda}\right)\pi^*_{\text{Merton}}
\end{aligned}$$

for $x > 0$. ∎

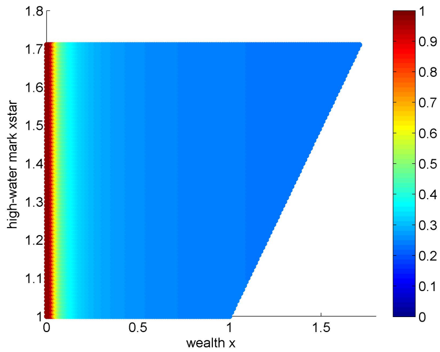
Note that due to the wealth being shifted along the real axis, the optimal strategy is no longer constant in the wealth. In (5.32) the discretized Merton ratio is positively translated if and only if $p > q$ and negatively if and only if $p < q$. The translation is linear in the off-set C and hyperbolic in the wealth x, particularly π^* converges to π_{Merton} when x increases.

When $\lambda < 1$ the translation in (5.33) is larger and implies a slower convergence to the discretized Merton ratio when $x \to \infty$. When $\lambda > 1$ the convergence is accelerated.

Economically, shifting the wealth makes less difference in the agent's utility when the wealth is larger and more severely affects the agent when lower wealths are to be traded. Figure 5.14 illustrates the effect of the shift on the optimal strategy for $\lambda = 1$:



(a) $C = 1$.



(b) $C = 0.1$.

Figure 5.14: The effect of the shift C in the last step of $\left(P^{[0,1],n}_{1,0,\log(C+\cdot)}\right)$. Plotted are the optimal strategies at step $k = 4$ of $n = 5$ for the same market as described in (5.30).

As (5.32) and (5.33) provide, the optimal strategy $\pi^*_{n-1} = \pi^*_{n-1}(x)$ converges to the discretized Merton ratio at each $x > 0$ when $C \to 0$. In part a) of figure 5.14 the shift $C = 1$ is comparatively large and accordingly π^* is clearly larger (note $p > q$ in this numerical example) than π^*_{Merton}. Close to the left boundary, i.e. for small x, π^* is larger than the upper no-shortselling bound and therefore cut at 1. For x increasing π^* enters the no-shortselling domain and slowly approaches π^*_{Merton}. Part b) illustrates the effect for a smaller shift $C = \frac{1}{10}$. Here π^*_{n-1} is already nearly constant for larger wealths x and the discretized Merton ratio's shade of blue (c.f. the colorbar at approximately 0.22) is closely approximated. Nevertheless $\pi^*_{n-1}(0) = 1$ and, regardless of C, there always exists a (positive) neighborhood of $x = 0$ with the same property and $\pi^*_{n-1}(x)$ converges to 1 when x approaches that neighborhood.

As the plain log investor, an agent with affine shifted log utility $U(y) = U(C + \lambda y)$ attributes hyperbolic absolute risk aversion (HARA):

$$\mathrm{ARA}_U(y) = -\frac{U''(y)}{U'(y)} = \frac{\lambda}{C + \lambda y}.$$

But she is no longer of constant relative risk aversion type as

$$\mathrm{RRA}_U(y) = -\frac{yU''(y)}{U'(y)} = \frac{\lambda y}{C + \lambda y}.$$

For increasing welfare y or increasing scale λ or decreasing shift C the constant relative risk aversion of the log investor is approached:

$$\lim_{y\to\infty} \mathrm{RRA}_U(y) = \lim_{\lambda\to\infty} \mathrm{RRA}_U(y) = \lim_{C\to 0+} \mathrm{RRA}_U(y) = 1 = \mathrm{RRA}_{\log}(y).$$

Despite of all mathematical similarity to the plain logarithm, economists would consider the affine logarithmic utility a fundamentally different type of preference.

5.5 Incentives of High-Water Mark Wages in One Period

Let us first consider the special case of pure high-water mark schedules, i.e. terminal wages of the form $\beta X^*_{t_n}$. This section discusses the agent's and the principal's problem and their solution in $n = 1$ period.

5.5.1 The Agent's Problem

In these setups the first step is the only one we need to consider. To make things more interesting, let us consider an agent with general utility U. The portfolio problem then reads as

$$\left(P^{[0,1],1}_{0,\beta,U}\right) \quad \begin{cases} \sup_{\pi_{t_0}} \left\{\mathbb{E}\left[U\left(\beta X^*_{t_1}\right)\right]\right\} \\ \pi_{t_0} \in [0,1] \end{cases} .$$

Although with the general utility U this problem covers a large class of setups, we can solve it explicitly:

Theorem 5.6 Let $U : (0,\infty) \to \mathbb{R}$ a utility function, $\beta \in (0,1)$, and $p \geq q$ or, equivalently, $\mu > r_{\text{BS}}$ in the approximated Black-Scholes market.

a) Then $\pi_{t_0} \equiv 1$ is an optimal strategy for $\left(P^{[0,1],1}_{0,\beta,U}\right)$ and its value is

$$J_0(x_0, x_0) = pU(\beta u x_0) + (1-p)U(\beta x_0)$$

b) Furthermore, if the admissibility constraints are relaxed to $\pi_{t_0} \in (-\infty, a]$ for some $a > 0$, then $\pi_{t_0} \equiv a$ is an optimal strategy and $\left(P^{(-\infty,a],1}_{0,\beta,U}\right)$ has the value

$$\begin{cases} pU(\beta x_0[au + (1-a)(1+r)]) + (1-p)U(\beta x_0[ad + (1-a)(1+r)]), & a \leq \mathring{\pi} \\ pU(\beta x_0[au + (1-a)(1+r)]) + (1-p)U(\beta x_0), & a \geq \mathring{\pi} \end{cases} .$$

◇

Proof. As part a) follows from b) with $a := 1$, consider the extended problem

$$\left(P^{(-\infty,a],1}_{0,\beta,U}\right) \quad \begin{cases} \sup_{\pi_{t_0}} \left\{\mathbb{E}\left[U\left(\beta X^*_{t_1}\right)\right]\right\} \\ \pi_{t_0} \in (-\infty, a] \end{cases} .$$

Here it is $J_1(x, x^*) = U(\beta x^*)$ and with Bellman

$$J_0(x_0, x_0) = \sup_{\pi_{t_0} \in (-\infty,a]} \left\{ pF_1(\pi_{t_0}) + (1-p)F_2(\pi_{t_0}) \right\}$$

holds, where now

$$\begin{aligned} F_1(\pi) &:= J_1(\hat{x}(x_0,\pi), \max\{\hat{x}(x_0,\pi), x_0\}) = U(\beta \max\{\hat{x}(x_0,\pi), x_0\}), \\ F_2(\pi) &:= J_1(\mathring{x}(x_0,\pi), \max\{\mathring{x}(x_0,\pi), x_0\} = U(\beta \max\{\mathring{x}(x_0,\pi), x_0\}), \\ F(\pi) &:= pF_1(\pi) + (1-p)F_2(\pi). \end{aligned}$$

For brevity drop the arguments in $\hat{x} := \hat{x}(x_0,\pi)$ and $\mathring{x} := \mathring{x}(x_0,\pi)$. As at time t_0 by definition $x = x_0$ and $x^* = x_0$, the performance ratio $z = 1$ is the only case we need to consider. It follows $\hat{\pi} = \frac{r}{1+r-u} < 0$ and $\mathring{\pi} = \frac{r}{1+r-d} > 0$.

Let initially $a > \mathring{\pi}$. In this situation the current high-water mark will be beaten if prices go up and it still can be beaten if prices go down and $\pi \le \mathring{\pi}$, yielding

$$\begin{aligned} F_1(\pi) &= J_1(\hat{x}(x_0,\pi),\hat{x}(x_0,\pi)) = U(\beta\hat{x}(x_0,\pi)),\ \pi \in (-\infty,a], \\ F_2(\pi) &= \begin{cases} J_1(\mathring{x}(x_0,\pi),\mathring{x}(x_0,\pi)) = U(\beta\mathring{x}(x_0,\pi)), & -\infty < \pi \le \mathring{\pi} \\ J_1(\mathring{x}(x_0,\pi),x_0) = U(\beta x_0), & \mathring{\pi} \le \pi \le a \end{cases}, \\ F(\pi) &= \begin{cases} pU(\beta\hat{x}(x_0,\pi)) + (1-p)U(\beta\mathring{x}(x_0,\pi)), & 0 \le \pi \le \mathring{\pi} \\ pU(\beta\hat{x}(x_0,\pi)) + (1-p)U(\beta x_0), & \mathring{\pi} \le \pi \le a \end{cases}. \end{aligned}$$

The second branch of F increases with π as

$$\frac{\partial}{\partial\pi}F(\pi)|_{\mathring{\pi}\le\pi\le a} = pU'(\beta\hat{x}(x_0,\pi))(u-1-r)x_0 > 0$$

due to U being a strictly increasing (utility) function.

For the first branch it holds

$$\frac{\partial}{\partial\pi}F(\pi)|_{-\infty<\pi\le\mathring{\pi}} = pU'(\beta\hat{x}(x_0,\pi))(u-1-r)x_0 + (1-p)U'(\beta\mathring{x}(x_0,\pi))(d-1-r)x_0.$$

It holds $\hat{x}(x_0,\pi) \ge \mathring{x}(x_0,\pi) \iff \pi \ge 0$, so utilizing U being a strictly concave (utility) function, for $\pi \ge 0$ one has $U'(\beta\hat{x}(x_0,\pi)) \le U'(\beta\mathring{x}(x_0,\pi))$. By $p \ge q$ it additionally holds $p(u-1-r)+(1-p)(d-1-r) \ge 0$ and thus

$$\frac{\partial}{\partial\pi}F(\pi)|_{0\le\pi\le\mathring{\pi}} \ge x_0U'(\beta\hat{x}(x_0,\pi))(p(u-1-r)+(1-p)(d-1-r)) \ge 0.$$

If on the other hand $\pi \le 0$, $U'(\beta\hat{x}(x_0,\pi)) \ge U'(\beta\mathring{x}(x_0,\pi))$ and

$$\frac{\partial}{\partial\pi}F(\pi)|_{-\infty<\pi\le 0} \ge x_0U'(\beta\mathring{x}(x_0,\pi))(p(u-1-r)+(1-p)(d-1-r)) \ge 0.$$

So the first branch of F is non-decreasing in π over the complete admissible set $(-\infty,a]$.

Together with the second branch of F increasing, the optimal strategy is uniquely determined by $\pi^*_{t_0} \equiv a$.

It remains to show that the same strategy is optimal for $0 < a \le \mathring{\pi}$. This is quickly established as in this case

$$\begin{aligned} F_1(\pi) &= J_1(\hat{x}(x_0,\pi),\hat{x}(x_0,\pi)) = U(\beta\hat{x}(x_0,\pi)),\ \pi \in (-\infty,a], \\ F_2(\pi) &= J_1(\mathring{x}(x_0,\pi),\mathring{x}(x_0,\pi)) = U(\beta\mathring{x}(x_0,\pi)),\ \pi \in (-\infty,a], \\ F(\pi) &= pU(\beta\hat{x}(x_0,\pi)) + (1-p)U(\beta\mathring{x}(x_0,\pi)),\ \pi \in (-\infty,a]. \end{aligned}$$

This F coincides with the second branch of the F above, which is non-decreasing. So $\pi^*_{t_0} \equiv a$ is one (of, if $p = q$, potentially many) optimal strategies.

The value induced is

$$\begin{aligned} & J_0(x_0,x_0) \\ =\ & F(a) \\ =\ & \begin{cases} pU(\beta x_0[au+(1-a)(1+r)]) + (1-p)U(\beta x_0[ad+(1-a)(1+r)]), & 0 < a \le \mathring{\pi} \\ pU(\beta x_0[au+(1-a)(1+r)]) + (1-p)U(\beta x_0), & \mathring{\pi} \le a \end{cases}. \end{aligned}$$

For $a = 1$ the value simplifies to (note $\mathring{\pi} \le 1 = a$ in this case)

$$J_0(x_0,x_0) = F(1) = pU(\beta u x_0) + (1-p)U(\beta x_0)$$

as asserted. ∎

The condition $p \geq q$ employed in theorem 5.6 can be translated to the approximated Black-Scholes market:

$$p \geq q = \frac{1+r-d}{u-d} \iff \mu \geq r_{\mathrm{BS}}. \tag{5.34}$$

Economically, $p \geq q$ means that the expected return of the stock is as least as large as the riskless rate. In such markets it is favorable to go long in the stock (assuming one is willing to take risk at all).

We can reformulate theorem 5.6 in terms of risk-taking:

Conclusion 5.7 In discrete approximations of a Black-Scholes market with a favorable stock it holds: Rewarding the agent with a share of the terminal high-water mark incites her to take deliberately high risks, regardless of her own utility. □

Note that allowing short positions and borrowing from the bond as in the extended problem $\left(P^{(-\infty,a],1}_{0,\beta,U}\right)$ will potentially yield a negative terminal wealth. As theorem 5.6 tells us, a negative terminal wealth will occur if and only if with $a > \mathring{\pi} > 1$ borrowing more than the riskless rate can accommodate is admitted and stock prices go down. From the agent's perspective this is not a problem as her wage never undercuts the fixed positive level βx_0. The principal's viewpoint is of course different as in the unfortunate case he is not only broke but left indebted to the bank.

5.5.2 The Principal's Problem

The principal's means of influencing his expected wealth remaining after wages are to define the rules – in this case by setting the level a and declaring no larger fraction than a may be invested in the stock. Let us assume the principal brings along his own preference, expressed via the strictly increasing and strictly concave (utility) function V. His optimization problem then reads as

$$(P^*) \qquad \begin{cases} \sup_a \left\{\mathbb{E}\left[V(X_{t_1} - \beta X^*_{t_1})\right]\right\} \\ a \in \mathbb{R} \end{cases}. \tag{5.35}$$

Note that $X_{t_1-} = X_{t_1}$ as no intermediate wages are paid. The principal's target is two-fold, depending on whether the enforced upper bound a is less or larger than the critical strategy value $\mathring{\pi}$ below which an improved high-water mark is achievable even when stock prices go down:

$$\begin{aligned} &\mathbb{E}[V(X_{t_1} - \beta X^*_{t_1})] \\ = &\begin{cases} pV(\hat{x}(x_0,a) - \beta\hat{x}(x_0,a)) + (1-p)V(\mathring{x}(x_0,a) - \beta\mathring{x}(x_0,a)), & 0 < a \leq \mathring{\pi} \\ pV(\hat{x}(x_0,a) - \beta\hat{x}(x_0,a)) + (1-p)V(\mathring{x}(x_0,a) - \beta x_0), & \mathring{\pi} < a \end{cases}. \end{aligned}$$

As $\mathring{x}(x_0, \mathring{\pi}) = x_0$ by definition of the update margins, the target is continuous in a. Both branches are continuously differentiable because $\hat{x}(x_0,\cdot)$ and $\mathring{x}(x_0,\cdot)$ have that property as well as V, being a utility function. Gluing the branches together, however, yields a singular point of non-differentiability in $a = \mathring{\pi}$.

We can solve (P^*) explicitly:

Theorem 5.8 Let $V : \mathbb{R} \to \mathbb{R}$ a utility function, $\beta \in (0,1)$, and $p \geq q$ or, equivalently, $\mu \geq r_{\mathrm{BS}}$ in the approximated Black-Scholes market.

a) Then (P^*) is finite and admits a unique solution a^* if $\beta \in (\beta^*, 1)$ or $\beta = \beta^*$ and additionally $p > q$, where

$$\beta^* := 1 - \frac{1-p}{p} \cdot \frac{1+r-d}{u-1-r}.$$

b) The optimal strategy in the finite case is $a^* = \mathring{\pi}$ and (P^*) has the value

$$pV((1-\beta)\hat{x}(x_0, \mathring{\pi})) + (1-p)V((1-\beta)x_0).$$

c) If $p > q$ and $\beta \in (0, \beta^*)$, the value of (P^*) strictly increases with $a \in \mathbb{R}$. ◇

Proof. As we know from theorem 5.6, the agent's wage depends on a being less or larger than $\mathring{\pi}$. Thus we need to discuss both cases also for the principal's wealth after wages:

- For $0 < a \leq \mathring{\pi}$ it holds

$$\begin{aligned} \mathbb{E}[V(X_{t_1} - \beta X^*_{t_1})] &= pV(\hat{x}(x_0,a) - \beta\hat{x}(x_0,a)) + (1-p)V(\mathring{x}(x_0,a) - \beta\mathring{x}(x_0,a)) \\ &= pV((1-\beta)\hat{x}(x_0,a)) + (1-p)V((1-\beta)\mathring{x}(x_0,a)). \end{aligned}$$

As $\hat{x}$ and $\mathring{x}$ are differentiable in their second argument, it holds

$$\begin{aligned} \frac{\partial}{\partial a}\mathbb{E}[V(X_{t_1} - \beta X^*_{t_1})] &= pV'((1-\beta)\hat{x}(x_0,a))(1-\beta)(u-1-r)x_0 \\ &+ (1-p)V'((1-\beta)\mathring{x}(x_0,a))(1-\beta)(d-1-r)x_0 \\ &= (1-\beta)[V'((1-\beta)\hat{x}(x_0,a))p(u-1-r) \\ &+ V'((1-\beta)\mathring{x}(x_0,a))(1-p)(d-1-r)]x_0. \end{aligned}$$

As $a > 0$, $(1-\beta)\hat{x}(x_0,a) > (1-\beta)\mathring{x}(x_0,a)$ and due to V strictly concave

$$V'((1-\beta)\hat{x}(x_0,a)) < V'((1-\beta)\mathring{x}(x_0,a)).$$

Hence,

$$\frac{\partial}{\partial a}\mathbb{E}[V(X_{t_1} - \beta X^*_{t_1})] \geq (1-\beta)V'((1-\beta)\hat{x}(x_0,a))[p(u-1-r) + (1-p)(d-1-r)]x_0 \geq 0,$$

as V is strictly increasing and $p \geq q$ implies $[p(u-1-r) + (1-p)(d-1-r)] \geq 0$. For $p > q$, the latter is strictly positive and thus $\frac{\partial}{\partial a}\mathbb{E}[V(X_{t_1} - \beta X^*_{t_1})] > 0$.

- For $\mathring{\pi} < a$ it holds

$$\begin{aligned} \mathbb{E}[V(X_{t_1} - \beta X^*_{t_1})] &= pV(\hat{x}(x_0,a) - \beta\hat{x}(x_0,a)) + (1-p)V(\mathring{x}(x_0,a) - \beta x_0) \\ &= pV((1-\beta)\hat{x}(x_0,a)) + (1-p)V(\mathring{x}(x_0,a) - \beta x_0) \end{aligned}$$

and

$$\begin{aligned} \frac{\partial}{\partial a}\mathbb{E}[V(X_{t_1} - \beta X^*_{t_1})] &= pV'((1-\beta)\hat{x}(x_0,a))(1-\beta)(u-1-r)x_0 \\ &+ (1-p)V'(\mathring{x}(x_0,a) - \beta x_0)(d-1-r)x_0. \end{aligned} \tag{5.36}$$

Due to $a > \mathring{\pi}$ it holds $\mathring{x}(x_0, a) < x_0$ and thus

$$(1-\beta)\hat{x}(x_0, a) > \mathring{x}(x_0, a) - \beta x_0.$$

Use this and the strict concavity of V to conclude

$$V'((1-\beta)\hat{x}(x_0, a)) < V'(\mathring{x}(x_0, a) - \beta x_0).$$

In the next step we will assess the sign of (5.36) by replacing $V'((1-\beta)\hat{x}(x_0, a))$ with $V'(\mathring{x}(x_0, a) - \beta x_0)$ and vice versa. In each case the sign of the derivative (5.36) is determined by a common factor whose sign we asses first:

$$\begin{aligned} & [p(1-\beta)(u-1-r) + (1-p)(d-1-r)]x_0 > 0 \\ \iff \quad & \beta < \frac{p(u-1-r) + (1-p)(d-1-r)}{p(u-1-r)} = 1 - \frac{1-p}{p} \cdot \frac{1+r-d}{u-1-r} =: \beta^* < 1. \end{aligned}$$

Due to $p > q$ we can deduce

$$p(u-1-r) + (1-p)(d-1-r) > 0 \Rightarrow \beta^* = \frac{p(u-1-r) + (1-p)(d-1-r)}{p(u-1-r)} > 0.$$

- For $\beta \in (0, \beta^*)$

$$\begin{aligned} & \frac{\partial}{\partial a}\mathbb{E}[V(X_{t_1} - \beta X^*_{t_1})] \\ > \quad & V'((1-\beta)\hat{x}(x_0, a))[p(1-\beta)(u-1-r) + (1-p)(d-1-r)]x_0 > 0. \end{aligned}$$

- For $\beta \in (\beta^*, 1)$

$$\begin{aligned} & \frac{\partial}{\partial a}\mathbb{E}[V(X_{t_1} - \beta X^*_{t_1})] \\ < \quad & V'(\mathring{x}(x_0, a) - \beta x_0)[p(1-\beta)(u-1-r) + (1-p)(d-1-r)]x_0 < 0. \end{aligned}$$

- For $\beta = \beta^*$ we have $\frac{\partial}{\partial a}\mathbb{E}[V(X_{t_1} - \beta X^*_{t_1})] = 0$ by the sandwich criterion (replace the sharp inequalities with unsharp inequalities in the cases above).

Concluding we have found $\mathbb{E}[V(X_{t_1} - \beta X^*_{t_1})]$ is non-decreasing for $a \in (0, \mathring{\pi}]$ and, depending on β, can increase, decrease, or be constant for $a \in (\mathring{\pi}, \infty)$. For $\beta \in (0, \beta^*)$ the principal's value is first non-decreasing and then increases strictly, so no finite solution a^* exists and the value of (P^*) increases with a beyond all bounds. This is part c). For $\beta \in (\beta^*, 1)$ the value is non-decreasing up to $\mathring{\pi}$ and then decreases, so $a^* = \mathring{\pi}$ is an optimal strategy. The same conclusion holds for $\beta = \beta^*$ if $p > q$ as then the value strictly increases up to $a = \mathring{\pi}$ and is further constant. That establishes a). In the finite case plugging in the optimal strategy yields

$$\mathbb{E}[V(X_{t_1} - \beta X^*_{t_1})]_{a=\mathring{\pi}} = pV((1-\beta)\hat{x}(x_0, \mathring{\pi})) + (1-p)V((1-\beta)x_0),$$

which proves b). ∎

The value of β^* is public knowledge to both parties as it can be determined from the market parameters alone.

It increases in p and also gains when either

- the relative return $(u - (1+r))$ of an upwards stock price movement improves or
- when the relative loss $(1 + r - d)$ of a downwards price change is diminished.

Hence, β^* is coupled with the expected relative return of the stock over the bond: Whenever a change of market parameters improves the expected return of a long position in the stock (with capital loaned from the bond), β^* increases, and vice versa.

Theorem 5.8 shows that the choice of β determines what value should be chosen for a. If the principal is free in picking both parameters by his wish, he will of course choose β as low as possible and a as large as possible. This means the agent would have to work for nothing and accordingly no sensible solution for (P^*) arises.

The situation changes if the agents brings along a participations constraint, i.e. a condition of the form

$$\sup_{\pi_{t_0}} \left\{ \mathbb{E}[U(\beta X_{t_1}^*)] \right\} \geq \underline{\mathrm{u}} \tag{5.37}$$

for some $\underline{\mathrm{u}} > 0$. The constraint means that the agent demands a minimal expected utility of wage. Now the agent's best expected utility of wage depends on the set of strategies she is admitted to choose from, i.e. it is in general a function in a. As it is in the principal's discretion to determine a, an equilibrium problem arises.

Let us assume $p \geq q$ and examine the situation in more detail:

Theorem 5.6 provides that the agent's best strategy is $\pi_{t_0} = a$. Her expected utility of wage increases in a as well as in β. Hence for any fixed a there is a minimal $\beta_0 = \beta_0(a)$ such that the participation constraint (5.37) holds.

Note that β_0 also depends on the value of $\underline{u}$ where larger participation constraints imply larger values of $\beta_0 = \beta_0(a, \underline{u})$.

There are three cases:

- If $\beta_0 \geq 1$, the principal will not be willing to offer a contract.
- If $\beta_0 < \beta^*$, employing the agent is – from the principal's point of view – that inexpensive that his expected utility can be arbitrarily enlarged by picking larger and larger constraints a.
- If $\beta^* < \beta_0 < 1$ an economical balance is achieved by setting $a = a^*$ and thereby producing finite and positive expected values for both parties.

Now $\beta_0(a)$ decreases for increasing a, so without further restrictions the principal can always find a sufficiently large a such that the agent ends up with $\beta_0 < \beta^*$. In economical terms: The principal can loosen the restrictions until the agent agrees to work for virtually nothing.

But a larger a implies a riskier trading strategy and that yields a riskier position for the principal as well. The principal will not be willing to accept arbitrarily high risks, so let us express his risk constraint as

$$a \leq \overline{a} \tag{5.38}$$

for some $\overline{a} > 0$.

Then if the agent's minimal $\underline{\mathrm{u}}$ and the principal's maximal $\overline{a}$ provide $\beta^* < \beta_0(\overline{a}, \underline{u}) < 1$ both parties can achieve the balance.

Conclusion 5.9 Given the constraints (5.37) and (5.38) of both parties yield $\beta^* < \beta_0(\overline{a}, \underline{u}) < 1$, the maximum fraction a^* of wealth the principal allows his agent to put on the stake (and by that the strategy utilized by the agent) is uniquely defined. In particular, via $\beta_0 = \beta_0(a, \underline{u})$ the optimal choice a^* indirectly depends on both the principal's and the agent's preference. □

5.6 High-Water Mark Wages in Multiple Periods

Let us now turn to the multi-period generalization of pure high-water mark wage schedules. To keep the analysis simple a risk-neutral agent is assumed. This section discusses the analytical approach to the problem and utilizes the computational toolkit of section 5.3 to present an example.

5.6.1 The Analytical Approach

Let us now switch the focus to high-water marks and consider the terminal utility problem

$$\left(P_{0,\beta,\mathrm{id}}^{[0,1],n}\right) \quad \begin{cases} \sup_\pi \left\{\mathbb{E}\left[\beta X_{t_n}^*\right]\right\} \\ \pi = (\pi_{t_0}, \dots, \pi_{t_{n-1}}),\ \pi_{t_k} = \pi_{t_k}(x, x^*) \in [0,1],\ k = 0, \dots, n-1 \end{cases},$$

which is the multi-period special case of (5.7) with $\psi \equiv 0$ and $W(x, x^*) := \beta x^*$ for some $\beta \in (0,1)$. Here, the agent is rewarded with a share of the terminal high-water mark (and no share of the actual wealth produced).

The Last Step

We have $J_n(x,x^*) = \beta x^*$ and

$$\begin{aligned}
&J_{n-1}(x,x^*)\\
= &\sup_{\pi_{n-1}:\mathcal{X}_{n-1}\to[0,1]} \left\{\mathbb{E}^{t,x,x^*}\left[J_n\left(g_{n-1}(x,x^*;\pi_{n-1}), \max\{g_{n-1}(x,x^*;\pi_{n-1}),x^*\}\right)\right]\right\}\\
= &\sup_{\pi_{n-1}:\mathcal{X}_{n-1}\to[0,1]} \Big\{\mathbb{E}^{t,x,x^*}\big[J_n\big([1+r+\pi_{n-1}(x,x^*)(Y_n-(1+r))]x,\\
&\qquad\qquad \max\{[1+r+\pi_{n-1}(x,x^*)(Y_n-(1+r))]x,x^*\}\big)\big]\Big\},
\end{aligned}$$

where $Y_n := Y_{t_n}$.

Recalling $\hat{x}(x,\pi)$ and $\mathring{x}(x,\pi)$ from definition 5.1 let $F(\pi) := pF_1(\pi) + (1-p)F_2(\pi)$ with $F_1(\pi) := \max\{\hat{x}(x,\pi),x^*\}$ and $F_2(\pi) := \max\{\mathring{x}(x,\pi),x^*\}$, i.e.

$$J_{n-1}(x,x^*) = k \sup_{\pi:\mathcal{X}_{n-1}\to[0,1]} \left\{F(\pi(x,x^*))\right\}.$$

Both F_1 and F_2 are linear in their argument and again

$$\begin{aligned}
\hat{x}(x,\pi) \le x^* &\iff \pi \le \left(\frac{x^*}{x}-(1+r)\right)(u-(1+r))^{-1} = \hat{\pi}(z),\\
\mathring{x}(x,\pi) \le x^* &\iff \pi \ge \left(\frac{x^*}{x}-(1+r)\right)(d-(1+r))^{-1} = \mathring{\pi}(z),
\end{aligned}$$

where at step $k=n-1$ the performance ratio z is located in $\left[1,u^{n-1}\right]$. The next question is, how $\hat{\pi}(z)$ and $\mathring{\pi}(z)$ are aligned within the unit interval from where admissible strategy values must be drawn. It is easy to see that $\mathring{\pi}(z) \le 0 \le \hat{\pi}(z)$ if and only if $z \ge 1+r$ and $\hat{\pi}(z) \le 0 \le \mathring{\pi}(z)$ if and only if $z \le 1+r$. Further is $\hat{\pi}(z) > 1$ if and only if $z > u$, i.e. if and only if the current high-water mark x^* is unbeatable in one step. On the other hand $\mathring{\pi}(z) > 1$ if and only if $z < d$ and that can never happen as $x^* \ge x$ and $d = \frac{1}{u} < \frac{1}{1+r} < 1$.

To determine the optimal strategy we have to distinguish three cases of performance ratios:

- $u \le z \le u^{n-1}$. In this situation the current high-water mark is unbeatable and we have $\mathring{x}(x,\pi) \le \hat{x}(x,\pi) \le x^*$ for all π. It follows for all $\pi \in [0,1]$

$$\begin{aligned}
F_1(\pi) &= J_n(\hat{x}(x,\pi),x^*) = \beta x^*,\\
F_2(\pi) &= J_n(\mathring{x}(x,\pi),x^*) = \beta x^*,\\
F(\pi) &= pF_1(\pi) + (1-p)F_2(\pi) = \beta x^*.
\end{aligned}$$

  Obviously and in accordance with what is expected in this situation, the strategy has no influence on the value achieved and thus any $\pi^*_{t_{n-1}} \in [0,1]$ is optimal. The value generated is

$$J_{n-1}(x,x^*) = F(\pi^*_{t_{n-1}}) = \beta x^*.$$

- $1+r \le z < u$. In this situation we may beat the current high-water mark if we invest $\pi \ge \hat{\pi}(z)$ and stock prices go up, but we cannot reach x^* if stock prices go down. Note that as remarked above $\hat{\pi}(z) < 1$ in this case. We now have

$$\begin{aligned}
F_1(\pi) &= \begin{cases} J_n(\hat{x}(x,\pi), x^*) = \beta x^*, & 0 \le \pi \le \hat{\pi}(z) \\ J_n(\hat{x}(x,\pi), \hat{x}(x,\pi)) = \beta \hat{x}, & \hat{\pi}(z) \le \pi \le 1 \end{cases}, \\
F_2(\pi) &= J_n(\mathring{x}(x,\pi), x^*) = \beta x^*, \; \pi \in [0,1], \\
F(\pi) &= \begin{cases} \beta x^*, & 0 \le \pi \le \hat{\pi}(z) \\ p\beta \hat{x}(x,\pi) + (1-p)\beta x^*, & \hat{\pi}(z) \le \pi \le 1 \end{cases}.
\end{aligned}$$

F is flat on the first branch but as $\frac{\partial}{\partial \pi}\hat{x} = (u-1-r)x > 0$ it strictly increases on the second branch. The optimal strategy in this case is uniquely determined by $\pi^*_{t_{n-1}} \equiv 1$ and the value generated is

$$J_{n-1}(x, x^*) = F(\pi^*_{t_{n-1}}) = \beta \left(pux + (1-p)x^*\right).$$

- $1 \le z < 1+r$. In this situation we will definitely beat the current high-water mark if prices go up and still can beat it if prices go down and we are cautious enough to put $\pi \le \mathring{\pi}(z)$ on the stake. This yields

$$\begin{aligned}
F_1(\pi) &= J_n(\hat{x}(x,\pi), \hat{x}(x,\pi)) = \beta \hat{x}(x,\pi), \; \pi \in [0,1], \\
F_2(\pi) &= \begin{cases} J_n(\mathring{x}(x,\pi), \mathring{x}(x,\pi)) = \beta \mathring{x}(x,\pi), & 0 \le \pi \le \mathring{\pi}(z) \\ J_n(\mathring{x}(x,\pi), x^*) = \beta x^*, & \mathring{\pi}(z) \le \pi \le 1 \end{cases}, \\
F(\pi) &= \begin{cases} p\beta \hat{x}(x,\pi) + (1-p)\beta \mathring{x}(x,\pi), & 0 \le \pi \le \mathring{\pi}(z) \\ p\beta \hat{x}(x,\pi) + (1-p)\beta x^*, & \mathring{\pi}(z) \le \pi \le 1 \end{cases}.
\end{aligned}$$

We already know that the second branch increases in π. The first branch is linear in π and its slope is determined by

$$\frac{\partial}{\partial \pi}\beta \left(p\hat{x}(x,\pi) + (1-p)\mathring{x}(x,\pi)\right) = \beta x \left(p(u-1-r) + (1-p)(d-1-r)\right).$$

As β and x are positive, the sign of the third factor tips the balance. It holds

$$p(u-1-r) + (1-p)(d-1-r) \ge 0 \qquad \Longleftrightarrow \qquad p \ge \frac{1+r-d}{u-d} = q$$

where q is the well-known upwards probability of the Cox-Ross-Rubinstein model.

If indeed $p \ge q$, F is non-decreasing on the first branch and increases on the second branch, so $\pi_{t_{n-1}} \equiv 1$ is again optimal and the value function

$$J_{n-1}(x, x^*) = F(\pi^*_{t_{n-1}}) = \beta \left(pux + (1-p)x^*\right)$$

coincides with what we found in the previous case.

If otherwise $p < q$, the maximum is located on one of the edges of $[0,1]$. It holds

$$\begin{aligned}
& F(0) > F(1) \\
\Longleftrightarrow \quad & \beta(1+r)x > \beta\left(pux + (1-p)x^*\right) \\
\Longleftrightarrow \quad & (1+r)x - x^* > p\left(ux - x^*\right) \\
\Longleftrightarrow \quad & p < \frac{(1+r)x - x^*}{ux - x^*}
\end{aligned}$$

as $\beta > 0$ and, in the case we are examining right now, $ux > x^*$, too.

Before concluding the optimal strategy or the value function let us verify that $F(0) > F(1)$ may actually occur given all further constraints on the market parameters. Rearranging slightly different from above one may equivalently claim

$$\begin{aligned} & F(0) > F(1) \\ \iff \quad & (1+r)x - x^* > p\,(ux - x^*) \\ \iff \quad & \frac{x^*}{x} < \frac{1+r-pu}{1-p} =: f(p). \end{aligned}$$

As $p \in (0,1)$ consider $\lim_{p\to 0+} f(p) = 1+r$ and note

$$f'(p) = \frac{1+r-pu-(1-p)u}{(1-p)^2} < 0 \qquad \iff \qquad 1+r < u,$$

which is true to rule out arbitrage. Thus $f(p)$ takes values in $(-\infty, 1+r)$ over $(0,1)$. By definition $x^* \geq x$ and that requires $f(p) \geq 1$ which is true for $p \leq \frac{r}{u-1}$. Gathering all above properties we find that any $p \in (0,1) \cap (0, r(u-1)^{-1})$ will indeed yield $F(0) > F(1)$.

For $p < \min\left\{q, \frac{(1+r)x-x^*}{ux-x^*}\right\}$ we may now safely conclude the optimal strategy $\pi^*_{t_{n-1}} \equiv 0$ and the value function

$$J_{n-1}(x, x^*) = F(\pi^*_{t_{n-1}}) = \beta(1+r)x.$$

In the last case remaining it is $p < q$ but $p \geq \frac{(1+r)x-x^*}{ux-x^*}$. We then have $\pi_{t_{n-1}} \equiv 1$ and

$$J_{n-1}(x, x^*) = F(\pi^*_{t_{n-1}}) = \beta\,(pux + (1-p)x^*)$$

just as discussed for $p \geq q$.

Concluding we have the solution

$$\begin{aligned} \pi^*_{t_{n-1}} \quad &\equiv \quad 1 - \mathbb{1}_{\left[p<\min\left\{q, \frac{(1+r)x-x^*}{ux-x^*}\right\}\right]}, \\ J_{n-1}(x, x^*) \quad &= \quad \begin{cases} \beta x^*, & u \leq z \leq u^{n-1} \\ \beta\,(pux + (1-p)x^*), & (1+r) \leq z \leq u \\ \beta\,(pux + (1-p)x^*), & 1 \leq z \leq (1+r) \text{ and } p \geq \min\left\{q, \frac{(1+r)x-x^*}{ux-x^*}\right\} \\ \beta(1+r)x, & 1 \leq z \leq (1+r) \text{ and } p < \min\left\{q, \frac{(1+r)x-x^*}{ux-x^*}\right\} \end{cases}. \end{aligned}$$

The interpretation is as follows: If the running high-water mark exceeds ux, it is unreachable in one time-step, even by putting all capital in the stock. Which strategy we choose does not matter and the value produced is βx^* just as before. If the running high-water mark can certainly be exceeded by putting all capital in the bond, this action is undertaken and the value is β times the interest-charged capital $(1+r)x$. In all other cases the bold strategy is favored and the value is nothing else than the expected outcome of this action.

As last remark note that J_{n-1} is continuous in x and x^*, so the branch conditions may also be read as strict inequalities on one side. One may count 4 branches but may also consider J_{n-1} split in only 3 branches as the values of the second and third branch coincide. Anyway, for $p \geq q$, $\pi^* \equiv 1$ is a globally optimal strategy and J_{n-1} drastically simplifies to

$$J_{n-1}(x, x^*) \quad = \quad \begin{cases} \beta x^*, & u \leq z \leq u^{n-1} \\ \beta \left(pux + (1-p)x^* \right), & 1 \leq z \leq u \end{cases} .$$

Another Step Back in Time

Let us now focus on the case $p \geq q$ and go back in time another step. With similar notation as introduced above write

$$J_{n-2}(x, x^*) = \sup_{\pi : \mathcal{X}_{n-2} \to [0,1]} \left\{ F(\pi(x, x^*)) \right\},$$

where again $F(\pi) := pF_1(\pi) + (1-p)F_2(\pi)$, but of course now

$$\begin{aligned} F_1(\pi) &:= J_{n-1}\left(\hat{x}(x,\pi), \max\{\hat{x}(x,\pi), x^*\}\right) \quad \text{and} \\ F_2(\pi) &:= J_{n-1}\left(\mathring{x}(x,\pi), \max\{\mathring{x}(x,\pi), x^*\}\right) \end{aligned}$$

refer to J_{n-1}. In order to establish solutions $\pi^*(x, x^*)$ we again have to assess the maxima of $\{\hat{x}(x,\pi), x^*\}$ and $\{\mathring{x}(x,\pi), x^*\}$ first. As before it holds

$$\begin{aligned} \hat{x}(x,\pi) \geq x^* \quad &\Longleftrightarrow \quad \pi \geq \hat{\pi}(z) \quad \text{and} \\ \mathring{x}(x,\pi) \geq x^* \quad &\Longleftrightarrow \quad \pi \leq \mathring{\pi}(z), \end{aligned}$$

but the next distinction, arising when plugging this into J_{n-1}, differs from what we have already seen:

$$\begin{aligned} u\hat{x}(x,\pi) \geq x^* \quad &\Longleftrightarrow \quad \pi \geq \big(zu^{-1} - (1+r)\big)\left(u - (1+r)\right)^{-1} =: \hat{\pi}_u(z), \\ u\mathring{x}(x,\pi) \geq x^* \quad &\Longleftrightarrow \quad \pi \leq \big(zu^{-1} - (1+r)\big)\left(d - (1+r)\right)^{-1} =: \mathring{\pi}_u(z). \end{aligned}$$

It follows

$$F_1(\pi) = \begin{cases} x^*, & 0 \leq \pi < \hat{\pi}_u(z) \\ pu\hat{x}(x,\pi) + (1-p)x^*, & \hat{\pi}_u(z) \leq \pi \leq 1 \end{cases}$$

and

$$F_2(\pi) = \begin{cases} pu\mathring{x}(x,\pi) + (1-p)\mathring{x}(x,\pi), & 0 \leq \pi \leq \mathring{\pi}_u(z) \\ x^*, & \mathring{\pi}_u(z) < \pi \leq 1 \end{cases} .$$

Next, we have to distinguish six cases for the performance ratio z to determine the proper order of $\hat{\pi}(z)$, $\hat{\pi}_u(z)$, $\mathring{\pi}(z)$ and $\mathring{\pi}_u(z)$ and their alignment within the unit interval.

These are:

- $1 \leq z < (1+r)d$.
  Here $\hat{\pi}_u(z) < \hat{\pi}(z) < 0 < \mathring{\pi}(z) < \mathring{\pi}_u(z)$ and $\mathring{\pi}_u(z) > 1$. This implies
  $$F(\pi) = p(pu\hat{x}(x,\pi) + (1-p)x^*) + (1-p)(pu\mathring{x}(x,\pi) + (1-p)\mathring{x}(x,\pi)).$$
  Although this function is linear in π, the sign of its derivative is not easily established as it depends on all 3 market parameters. In general, the derivative may be non-negative (and thus implying $\pi^* \equiv 1$) as well as negative (yielding $\pi^* \equiv 0$).

- $(1+r)d \leq z < (1+r)$.
  In this case we find $\hat{\pi}_u(z) < \hat{\pi}(z) < 0 < \mathring{\pi}(z) < \mathring{\pi}_u(z)$ with $\mathring{\pi}_u(z) \leq 1$ and $F(\pi)$ is
  $$\begin{cases} p(pu\hat{x}(x,\pi) + (1-p)x^*) + (1-p)(pu\mathring{x}(x,\pi) + (1-p)\mathring{x}(x,\pi)), & 0 \leq \pi \leq \mathring{\pi}_u(z) \\ p^2u\hat{x}(x,\pi) + (1+p)(1-p)x^*, & \mathring{\pi}_u(z) < \pi \leq 1 \end{cases}.$$
  The second branch increases in π, while the first may decrease or increase in π, c.f. the discussion of case 1. The maximum is located on one of the edges (in any case – which one depends on the market parameters) or generally on the right edge (if the first branch is non-decreasing).

- $(1+r) \leq z < (1+r)\tilde{u}$, where $\tilde{u} := \frac{(1+r)(u-d)u}{u-(1+r)-u(d-(1+r))} \in (1,u)$.
  Now $\hat{\pi}_u(z) < \mathring{\pi}(z) < 0 < \hat{\pi}(z) < \mathring{\pi}_u(z)$ and $F(\pi)$ coincides with the second case.

- $(1+r)\tilde{u} \leq z < (1+r)u$.
  In this situation $\hat{\pi}_u(z) < \mathring{\pi}(z) < 0 < \mathring{\pi}_u(z) < \hat{\pi}(z)$ and $\mathring{\pi}_u(z) < 1$. Again $F(\pi)$ coincides with the second case.

- $(1+r)u \leq z < u^2$.
  For this one $\mathring{\pi}(z) < \mathring{\pi}_u(z) < 0 < \hat{\pi}_u(z) < \hat{\pi}(z)$ and $\hat{\pi}_u(z) < 1$. We have
  $$F(\pi) = \begin{cases} x^*, & 0 \leq \pi \leq \hat{\pi}_u(z) \\ p^2u\hat{x}(x,\pi) + (1+p)(1-p)x^*, & \hat{\pi}_u(z) < \pi \leq 1 \end{cases}$$
  and F increases in π. Thus, $\pi^* \equiv 1$ is optimal.

- $u^2 \leq z$.
  As in case 5, $\mathring{\pi}(z) < \mathring{\pi}_u(z) < 0 < \hat{\pi}_u(z) < \hat{\pi}(z)$, but $\hat{\pi}_u(z) \geq 1$. This directly yields $F(\pi) = x^*$ and $\pi^* \equiv c$ is optimal for any $c \in [0,1]$.

It is difficult to state sufficient and necessary conditions incorporating all three market parameters p, r, and d for the sign of the derivative in the first case (and accordingly also in cases 2, 3, 4). Without that the optimizer cannot be stated in all generality and only special cases can be considered.

Even when assuming such conditions were available the problem gets more difficult with each further step back in time: At step $n-3$ we need to perform the well-known distinctions of $\hat{x}(x,\pi)$ being less or larger than x^* and $\mathring{x}(x,\pi)$ being less or larger than x^*, both in dependence on x and π. Plugging this into J_{n-2}, we already have to distinguish between $u^2\hat{x}(x,\pi)$, $ud\hat{x}(x,\pi)$, $d^2\hat{x}(x,\pi)$, $u^2\mathring{x}(x,\pi)$, $ud\mathring{x}(x,\pi)$, and $d^2\mathring{x}(x,\pi)$ each being less or larger than x^*. Although the middle case $ud\hat{x}(x,\pi) = \hat{x}(x,\pi)$ is nothing new, at each further time step the chain of stepwise factors grows by one as does the number of branching points. The number of possible alignments in the unit interval grows even faster and each case potentially brings up a new condition on the market parameters which has not yet been taken care of by previous assumptions.

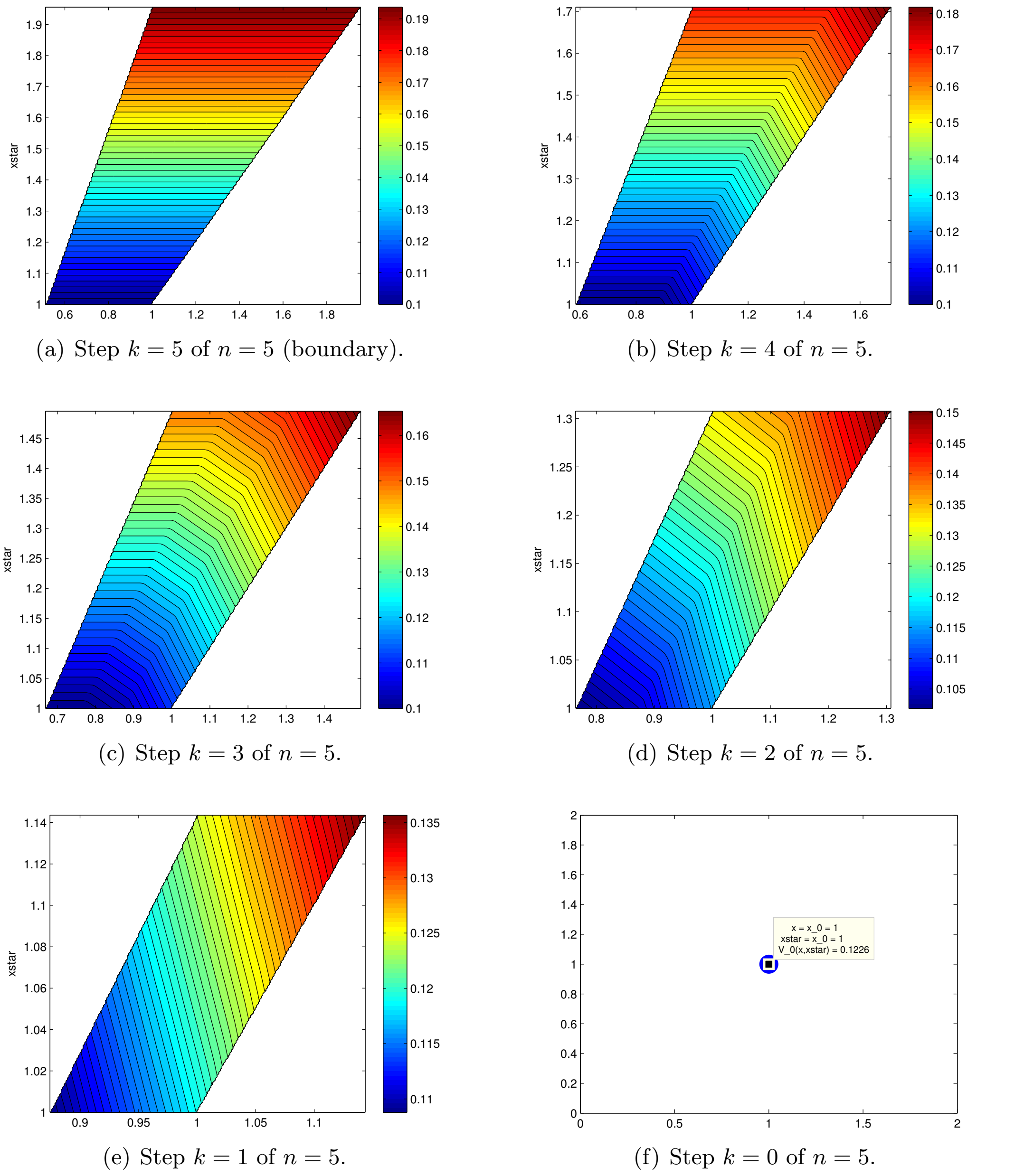
5.6.2 An Example

So let us examine this terminal high-water mark problem numerically in a specific market instead. For any market parameters we first have to assess if $p \geq q$, in which case $\pi^*_{t_{n-1}}(x, x^*) \equiv 1$ would be optimal and the third (or fourth) branch of J_{n-1} would vanish. Recalling from (5.34) that $p \geq q$ if and only if $\mu \geq r_{\mathrm{BS}}$ we can approve that for the same market parameters as in (5.30): $\mu = \log(1.05) > \log(1.03) = r_{\mathrm{BS}}$.

For $n = 5$ steps we indeed find an optimal $\pi_{4/5} \equiv 1$ along with the value function

$$J_4(x, x^*) = \frac{1}{10} \begin{cases} x^*, & x^* > 1.1436x \\ 0.5752x + 0.4971x^*, & x^* \leq 1.1436x \end{cases},$$

where we chose $\beta = 10\%$ and all real constants were rounded to 4 digits. The remaining steps are easily computed with the approach discussed in section 5.3. It turns out that also $\pi^*_{k/5}(x, x^*) \equiv 1$ for all remaining $k = 0, \ldots, 3$ and $(x, x^*) \in \mathcal{X}_k$. This is remarkable and indeed the constant choice of $\pi \equiv 1$ for all steps and all situations seems to be optimal for any market approximating a Black-Scholes model with $\mu \geq r_{\mathrm{BS}}$. We will shortly examine this hypothesis closer, but let us have a look on the value functions of this very problem first. Figure 5.15 presents the value functions over the matching (enlarged) admissible state spaces and uncovers some interesting facts:



(a) Step $k = 5$ of $n = 5$ (boundary).

(b) Step $k = 4$ of $n = 5$.

(c) Step $k = 3$ of $n = 5$.

(d) Step $k = 2$ of $n = 5$.

(e) Step $k = 1$ of $n = 5$.

(f) Step $k = 0$ of $n = 5$.

Figure 5.15: Value functions for $\Psi(X^{t_n-}, X^{t_n,*}) = \frac{1}{10}X^*_{t_n}$ as computed in an $n = 5$ step approximation over grids of width $h = 0.01$.

Each subplot reports the results of one specific step k, starting out with $k = n = 5$ in part a) and moving backwards in time to $k = 0$ in part f). In each plot the value function is laid out over the appropriate set $\tilde{\mathcal{X}}_k$ and the values are characterized by contour lines and colors, where blue symbolizes the lowest and red denotes the highest values.

$\tilde{\mathcal{X}}_0$ is just the point (x_0, x_0) representing the investor's initial endowment, so the value $J_0(x_0, x_0)$ is just reported at the proper location.

Part a) simply reports the known boundary condition $V_n(x, x^*) = \beta x^*$. As x has no influence on the value, the contour lines run horizontally.

In part b) the contour lines run horizontally from the left edge of $\tilde{\mathcal{X}}_4$ until they sharply bend somewhat downwards. The location of the bends form a straight line with slope $z = u$. Any point left of the straight satisfies $x^* > xu$, so the current high-water mark can by no admissible strategy be beaten in the last remaining step till termination and the expected value after step n is constant along the horizontal line of all points with identical x^* coordinate. For points right of the straight it holds $x^* < xu$ and x^* may be beaten by bold enough strategies when stock prices go up. The fraction of wealth the agent is required to bet on the stock tends to zero when approaching the right edge of $\tilde{\mathcal{X}}_4$. Nevertheless we know that the optimal strategy at this step is constantly 1 and thus the contour lines exhibit a constant negative slope.

Note that now the wealth x matters in that subset of $\tilde{\mathcal{X}}_k$ where the high-water mark may be beaten. Another step back to part c) grants another bend at $z = u^2$ with the analogue interpretation that now only points left of that bend can never improve x^* while points between this and the next bend may improve x^* given 2 times luck and points right of the second bend can improve x^* even in the next of two remaining steps.

The effect continuously manifests in new bends at $z = u^3, u^4, \ldots$ where in step $k = n-j$ we expect j bends. An anomaly occurs in step $k = 2$ in part d) of the figure, where only 1 instead of 4 bends appears. This is due to the fact that now all admissible states potentially increase the high-water mark. As $\tilde{\mathcal{X}}_k$ shrinks with k decreasing, some bends are located outside of it and the contour lines already start with non-zero slope at the left edge. Note that the left edge of $\tilde{\mathcal{X}}_k$ has slope u^k (for $k \geq 2$), so from $\lfloor \frac{n}{2} \rfloor$ on downwards the left edge actually is a bend, too.

In general for decreasing k the current wealth and its potential to produce a new record in the remaining steps becomes more important for the value at (x, x^*).

5.7 Incentives of Combined Rewards in One Period

We have seen that rewarding the agent only with a share of the high-water mark does not incite her to avoid risks as desired by the principal. The intended attitude towards risks can very well be conveyed to the agent when additionally contracting to a share of the terminal wealth. This section considers the agent's problem for general utility in one period and provides the boundedness of the optimal strategies. Then an agent with log utility is discussed in details and the convergence of strategy and expected terminal utility to the respective solutions in the benchmark case is lined out.

5.7.1 The Agent's Problem

For a more detailed analysis let us consider the one-period problem

$$\left(P^{\mathbb{R},1}_{\alpha,\beta,U}\right) \quad \begin{cases} \sup_\pi \left\{ \mathbb{E}\left[U\left(\alpha X_{t_1} + \beta X^*_{t_1} \right) \right] \right\} \\ \pi \in \mathbb{R} \end{cases}$$

in which we generalize the target by admitting the agent her own utility and allowing the principal to adjust the contract by weighting the wealth and high-water mark components with factors $\alpha, \beta > 0$ of his choice. Without further restricting the weights, the combined reward from wealth and high-water mark may be non-positive if the market goes down. One way to deal with that situation is to restrict the choice of α and β to ensure positivity of the reward. We will come back to that later and assume for the moment that U is defined on the complete real domain. We then can already predicate the risk associated with the agent's strategy choice:

Theorem 5.10 Let $U : \mathbb{R} \to \mathbb{R}$ an INADA utility function and $\alpha, \beta > 0$.

a) Then an optimal strategy π^* for $\left(P^{\mathbb{R},1}_{\alpha,\beta,U}\right)$ exists,

b) π^* is finite, and

c) the value of $\left(P^{\mathbb{R},1}_{\alpha,\beta,U}\right)$ is finite as well. $\diamond$

Proof. In the one-step problem $\left(P^{\mathbb{R},1}_{\alpha,\beta,U}\right)$ it holds

$$\begin{aligned} \hat{x}(x_0,\pi) \geq x^* = x_0 \quad &\Longleftrightarrow \quad \pi \geq \hat{\pi}(x_0,x_0) = \hat{\pi} \quad \text{as well as} \\ \mathring{x}(x_0,\pi) \geq x^* = x_0 \quad &\Longleftrightarrow \quad \pi \leq \mathring{\pi}(x_0,x_0) = \mathring{\pi} \end{aligned}$$

and $-\infty < \hat{\pi} < 0 < \mathring{\pi} < \infty$. As now $\pi \in \mathbb{R}$, the agent's expected utility of wage splits up in three branches:

$$\begin{aligned} &\mathbb{E}[U(\alpha X_{t_1} + \beta X^*_{t_1})] \\ =\; &\begin{cases} pU(\alpha\hat{x}(x_0,\pi) + \beta x_0) + (1-p)U(\alpha\mathring{x}(x_0,\pi) + \beta\mathring{x}(x_0,\pi)), & -\infty < \pi \leq \hat{\pi} \\ pU(\alpha\hat{x}(x_0,\pi) + \beta\hat{x}(x_0,\pi)) + (1-p)U(\alpha\mathring{x}(x_0,\pi) + \beta\mathring{x}(x_0,\pi)), & \hat{\pi} \leq \pi \leq \mathring{\pi} \\ pU(\alpha\hat{x}(x_0,\pi) + \beta\hat{x}(x_0,\pi)) + (1-p)U(\alpha\mathring{x}(x_0,\pi) + \beta x_0), & \mathring{\pi} \leq \pi < \infty \end{cases} \\ =:\; &\begin{cases} pU(f_1(\pi)) + (1-p)U(f_2(\pi)), & -\infty < \pi \leq \hat{\pi} \\ pU(g_1(\pi)) + (1-p)U(g_2(\pi)), & \hat{\pi} \leq \pi \leq \mathring{\pi} \\ pU(h_1(\pi)) + (1-p)U(h_2(\pi)), & \mathring{\pi} \leq \pi < \infty \end{cases} \\ =:\; &\begin{cases} F(\pi), & -\infty < \pi \leq \hat{\pi} \\ G(\pi), & \hat{\pi} \leq \pi \leq \mathring{\pi} \\ H(\pi), & \mathring{\pi} \leq \pi < \infty \end{cases} . \end{aligned}$$

Each target F, G, and H is a linear combination of the strictly increasing and strictly concave utility U, evaluated at two different monetary positions. Note that all three are continuously differentiable.

F and G can be continuously glued together in $\hat{\pi}$ as by definition $\hat{x}(x_0,\hat{\pi}) = x_0$ and thus $F(\hat{\pi}) = G(\hat{\pi})$. Analogously $\mathring{x}(x_0,\mathring{\pi}) = x_0$ provides $G(\mathring{\pi}) = H(\mathring{\pi})$ and we can continuously glue G and H at $\pi = \mathring{\pi}$. Hence the composed target $\mathbb{E}[U(\alpha X_{t_1} + \beta X^*_{t_1})]$, considered as a function of π, is continuous on whole real domain and continuously differentiable on $\mathbb{R} \setminus \{\hat{\pi}, \mathring{\pi}\}$.

On all branches the arguments f_i, g_i, or h_i, $i = 1, 2$, supplied to U are linear in π, where f_1, g_1, and h_1 increase in π and f_2, g_2, and h_2 decrease in π.

Let us form the derivatives of all branches:

- $F'(\pi) = p\lambda_1 U'(f_1(\pi)) + (1-p)\lambda_2 U'(f_2(\pi))$ where
$$\begin{aligned}\lambda_1 &= \alpha(u-1-r)x_0 > 0,\\ \lambda_2 &= (\alpha+\beta)(d-1-r)x_0 < 0.\end{aligned}$$
- $G'(\pi) = p\mu_1 U'(g_1(\pi)) + (1-p)\mu_2 U'(g_2(\pi))$ where
$$\begin{aligned}\mu_1 &= (\alpha+\beta)(u-1-r)x_0 > 0,\\ \mu_2 &= (\alpha+\beta)(d-1-r)x_0 < 0.\end{aligned}$$
- $H'(\pi) = p\nu_1 U'(h_1(\pi)) + (1-p)\nu_2 U'(h_2(\pi))$ where
$$\begin{aligned}\nu_1 &= (\alpha+\beta)(u-1-r)x_0 > 0,\\ \nu_2 &= \alpha(d-1-r)x_0 < 0.\end{aligned}$$

What follows now is exemplary discussed for F but analogously works for G and H:
It holds $F''(\pi) = p\lambda_1^2 U''(f_1(\pi)) + (1-p)\lambda_2^2 U''(f_2(\pi)) < 0$ as U is strictly concave, so F is strictly concave, too. We have f_1 increasing and f_2 decreasing in π. As U strictly increases, U' is strictly positive, and by the INADA properties $U'(y) \to 0+$ for $y \to \infty$ and $U'(y) \to \infty$, $y \to -\infty$. It follows

$$F'(\pi) = \underbrace{p\lambda_1}_{>0} U'(f_1(\pi)) + \underbrace{(1-p)\lambda_2}_{<0} U'(f_2(\pi)) \to \begin{cases} -\infty, & \pi \to \infty \\ \infty, & \pi \to -\infty \end{cases}.$$

As F is continuous, there exists a unique and finite $\pi_F \in \mathbb{R}$ such that $F'(\pi_F) = 0$. As F is strictly concave, π_F is the global maximizer of F.

Analogously we find unique global maximizers π_G and π_H of G and H. Without further assumptions it is not clear if π_F, π_G, and π_H are included in the restricted domain of their respective branch. But due to the strict concavity of all branches, the maximum is attained at one (or more) of the points $\{\pi_F, \pi_G, \pi_H, \hat{\pi}, \mathring{\pi}\}$ which proves a). As all candidates are finite, b) follows immediately and directly yields c) as a trivial consequence. ■

For the portfolio problem $\left(P^{\mathbb{R},1}_{\alpha,\beta,U}\right)$ with general utility a whole bunch of cases how the branches F, G, and H can be aligned and glued together arises. It is difficult to give a complete yet clearly and well arranged graphical overview. But we will encounter similar branched targets again when examining the log and the multi-period variations of latter. In the special case $U = \log$ more can be asserted and the possible relations between F, G, and H narrow down. Following the proof of theorem 5.12, which states the main findings in that situation, figure 5.16 illustrates how the branches work together.

Let us summarize the findings so far:

Conclusion 5.11 When combining wealth and high-water mark in the agent's wage, her optimal strategy is always bounded, regardless of her own preference. □

The assertion of theorem 5.10 and conclusion 5.11 for combined wage schedules stand in sharp contrast to the results of theorem 5.6 and conclusion 5.7 for pure high-water mark wage schedules. As we know now rewarding the agent only with a share of the high-water mark incites her to take deliberately high risks as long as the principal does not restrict her actions by enforcing an upper bound on the fraction of wealth to be invested in the stock. If on the other hand an arbitrarily small share of the terminal wealth is added to the wage schedule, the agent immediately and without enforcement picks a bounded optimal strategy. All results hold regardless of the agent's utility, so it is solely the involvement of the terminal wealth that tips the balance of risk.

5.7.2 The Log Case

We know about the optimal strategy being bounded in the combined case, but the question remains how generous these bounds are. To keep the risk below some maximal level, e.g. due to regulatory limitations, is the principal's key concern. The maximizers which induce the bounds depend on the weights α and β chosen by the principal, so the principal can exercise control over agent. But as the maximizers also depend on the agent's utility U, that control is limited by the principal's knowledge about his employees attitude towards risk.

Let us examine the problem $\left(P^{\mathbb{R},1}_{\alpha,\beta,U}\right)$ for an agent with log preference. To ensure the utility is well-defined we have to make sure that the reward is strictly positive for every stock price change. If stock prices go down, we have to ensure that the strategy is bounded from above:

$$\left[\alpha X_{t_1} + \beta X^*_{t_1}\right]_{Y_{t_1}=d} = \alpha\mathring{x}(x_0,\pi) + \beta x_0 > 0 \iff \pi < \frac{(1+r)\alpha+\beta}{1+r-d} > 0. \tag{5.39}$$

If otherwise stock prices go up, we have to restrict short-selling:

$$\left[\alpha X_{t_1} + \beta X^*_{t_1}\right]_{Y_{t_1}=u} = \alpha\hat{x}(x_0,\pi) + \beta x_0 > 0 \iff \pi > -\frac{(1+r)\alpha+\beta}{u-(1+r)} < 0. \tag{5.40}$$

Note that due to (NA) conditions and $\alpha, \beta \geq 0$ it holds

$$-\frac{(1+r)\alpha+\beta}{u-(1+r)} < 0 < \frac{(1+r)\alpha+\beta}{1+r-d}.$$

As discussed in section 5.4 we cannot allow π to be chosen from an open interval but have to pose some closed interval instead. To that end pick some $\varepsilon > 0$ which is small enough to guarantee

$$\mathring{\pi}_{\alpha,\beta} := \frac{(1+r)\alpha+\beta}{1+r-d} - \varepsilon > 0 \quad \text{and} \quad \hat{\pi}_{\alpha,\beta} := -\frac{(1+r)\alpha+\beta}{u-(1+r)} + \varepsilon < 0.$$

For any such ε we can safely consider the log problem

$$\left(P^{[\hat{\pi}_{\alpha,\beta},\mathring{\pi}_{\alpha,\beta}],1}_{\alpha,\beta,\log}\right) \qquad \begin{cases} \sup_\pi \left\{\mathbb{E}\left[\log\left(\alpha X_{t_1} + \beta X^*_{t_1}\right)\right]\right\} \\ \hat{\pi}_{\alpha,\beta} \leq \pi \leq \mathring{\pi}_{\alpha,\beta} \end{cases}.$$

The assignment of ε is to be small enough while positive to ensure the problem is well-posed. Apart from that it does not play a notable role in the further analysis and accordingly is not explicitly quoted in the variables and designators involved.

$\left(P^{[\hat{\pi}_{\alpha,\beta},\mathring{\pi}_{\alpha,\beta}],1}_{\alpha,\beta,\log}\right)$ is a mixture of the benchmark log problem $\left(P^{[0,1],n}_{1,0,\log}\right)$ of section 5.4 in one period and the pure high-water mark problem $\left(P^{[0,1],1}_{0,\beta,U}\right)$ from section 5.5 with log utility. We can assert some facts about its solution:

Theorem 5.12 Let $\alpha, \beta > 0$ and consider $\left(P_{\alpha,\beta,\log}^{[\hat{\pi}_{\alpha,\beta},\mathring{\pi}_{\alpha,\beta}],1}\right)$.

a) With the notation of theorem 5.10 it holds

$$\begin{aligned}
\pi_F &= \frac{\alpha(1+r)(p(u-1-r)+(1-p)(d-1-r))+\beta(1-p)(d-1-r)}{\alpha(u-1-r)(1+r-d)},\\
\pi_G &= \frac{(1+r)(p(u-1-r)+(1-p)(d-1-r))}{(u-1-r)(1+r-d)},\\
\pi_H &= \frac{\alpha(1+r)(p(u-1-r)+(1-p)(d-1-r))+\beta p(u-1-r)}{\alpha(u-1-r)(1+r-d)},
\end{aligned}$$

and $\pi_F < \pi_G < \pi_H$.

b) $\pi^*_{t_0} \in \{\pi_F, \pi_G, \pi_H\} \cup \{\hat{\pi}_{\alpha,\beta}, \mathring{\pi}_{\alpha,\beta}\}$, particularly $\hat{\pi}$ and $\mathring{\pi}$ are never optimal.

c) If α or β is large enough, the optimal strategy $\pi^*_{t_0}$ is among $\{\pi_F, \pi_G, \pi_H\}$.

d) For $p = q$ and

$$\frac{\beta}{\alpha} \le \min\left\{\frac{r(u-d)}{u-1-r}, \frac{r(u-d)}{1+r-d}\right\}$$

the strategy $\pi^*_{t_0} = \pi_G = 0$ is the unique solution and $(P_{\log})$ has the value

$$\log(x_0) + \log(\alpha+\beta) + \log(1+r).$$

Proof. a) Evaluate the derivatives F', G', and H' of all branches from theorem 5.10 for $U = \log$:

$$\begin{aligned}
F'(\pi) &= p\frac{\alpha(u-1-r)x_0}{\alpha\hat{x}(x_0,\pi)+\beta x_0} + (1-p)\frac{(\alpha+\beta)(d-1-r)x_0}{(\alpha+\beta)\mathring{x}(x_0,\pi)} = 0 \iff\\
\pi &= \frac{\alpha(1+r)(p(u-1-r)+(1-p)(d-1-r))+\beta(1-p)(d-1-r)}{\alpha(u-1-r)(1+r-d)} = \pi_F,\\
G'(\pi) &= p\frac{(\alpha+\beta)(u-1-r)x_0}{(\alpha+\beta)\hat{x}(x_0,\pi)} + (1-p)\frac{(\alpha+\beta)(d-1-r)x_0}{(\alpha+\beta)\mathring{x}(x_0,\pi)} = 0 \iff\\
\pi &= \frac{(1+r)(p(u-1-r)+(1-p)(d-1-r))}{(u-1-r)(1+r-d)} = \pi_G,\\
H'(\pi) &= p\frac{(\alpha+\beta)(u-1-r)x_0}{(\alpha+\beta)\hat{x}(x_0,\pi)} + (1-p)\frac{\alpha(d-1-r)x_0}{\alpha\mathring{x}(x_0,\pi)+\beta x_0} = 0 \iff\\
\pi &= \frac{\alpha(1+r)(p(u-1-r)+(1-p)(d-1-r))+\beta p(u-1-r)}{\alpha(u-1-r)(1+r-d)} = \pi_H.
\end{aligned}$$

Rewrite

$$\begin{aligned}
\pi_F &= \pi_G - \frac{\beta}{\alpha}(1-p)(u-1-r)^{-1} < \pi_G \quad \text{and}\\
\pi_H &= \pi_G + \frac{\beta}{\alpha}p(1+r-d)^{-1} > \pi_G
\end{aligned}$$

to derive the claimed alignment.

b) From theorem 5.10 it is already clear that in the unbounded case the optimizer is among $\{\pi_F, \pi_G, \pi_H\} \cup \{\hat{\pi}, \mathring{\pi}\}$. The boundaries $\hat{\pi}_{\alpha,\beta}$ and $\mathring{\pi}_{\alpha,\beta}$ for π which were introduced by the choice of $U = \log$ may be optimizers, too:

As F increases on $(-\infty, \pi_F)$ and H decreases on (π_H, ∞), the optimal strategy must be located on one of the edges $\hat{\pi}_{\alpha,\beta}$ or $\mathring{\pi}_{\alpha,\beta}$ if the choice of α and β yields $\{\pi_F, \pi_G, \pi_H\} \cap [\hat{\pi}_{\alpha,\beta}, \mathring{\pi}_{\alpha,\beta}] = \emptyset$ (which may happen for small α).

It remains to show that $\hat{\pi}$ and $\mathring{\pi}$ are never optimal in the log case.

Consider the alignment of $\hat{\pi}$ and π_F: Nothing is to show for $\pi_F = \hat{\pi}$. If $\pi_F < \hat{\pi}$, F decreases on $(\pi_F, \hat{\pi})$, so $F(\pi_F) > F(\hat{\pi})$ and $\hat{\pi}$ is no optimizer. If otherwise $\hat{\pi} < \pi_F$, F increases on $(-\infty, \hat{\pi})$ and as $\pi_F < \pi_G$, F also increases on $(\hat{\pi}, \pi_G)$. The continuity of the target yields $F(\hat{\pi}) = G(\hat{\pi}) < G(\pi_G)$ and thus again $\hat{\pi}$ is no optimizer.

Analogue arguments work for $\mathring{\pi}$ and π_H exploiting $\pi_G < \pi_H$.

c) With (5.39) and (5.40)

$$\begin{aligned}
\hat{\pi}_{\alpha,\beta} &= \frac{(1+r)\alpha + \beta}{1+r-u} + \varepsilon = \frac{1+r}{1+r-u}\alpha + \frac{1}{1+r-u}\beta + \varepsilon, \\
\mathring{\pi}_{\alpha,\beta} &= \frac{(1+r)\alpha + \beta}{1+r-d} - \varepsilon = \frac{1+r}{1+r-d}\alpha + \frac{1}{1+r-d}\beta - \varepsilon.
\end{aligned}$$

As $d < 1+r < u$, $\hat{\pi}_{\alpha,\beta}$ linearly decreases in α and β and $\mathring{\pi}_{\alpha,\beta}$ linearly increases in α and β. π_G is constant in terms of α and β, so there exist α_0 and β_0 such that $\hat{\pi}_{\alpha,\beta} < \pi_G < \mathring{\pi}_{\alpha,\beta}$ for all $\alpha \geq \alpha_0$ (with β fixed) as well as for all $\beta \geq \beta_0$ (with α fixed). π_F or π_H may also lie within the same bounds for some of these α and β, so the maximizer is among $\{\pi_F, \pi_G, \pi_H\}$.

d) $p = q$ yields $p(u-1-r) + (1-p)(d-1-r) = 0$ and thus $\pi_G = 0$. It holds

$$\begin{aligned}
\hat{\pi} \leq \pi_F \quad &\Longleftrightarrow \quad \frac{\beta}{\alpha} \leq \frac{r(u-d)}{u-1-r} \quad \text{and} \\
\mathring{\pi} \geq \pi_H \quad &\Longleftrightarrow \quad \frac{\beta}{\alpha} \leq \frac{r(u-d)}{1+r-d},
\end{aligned}$$

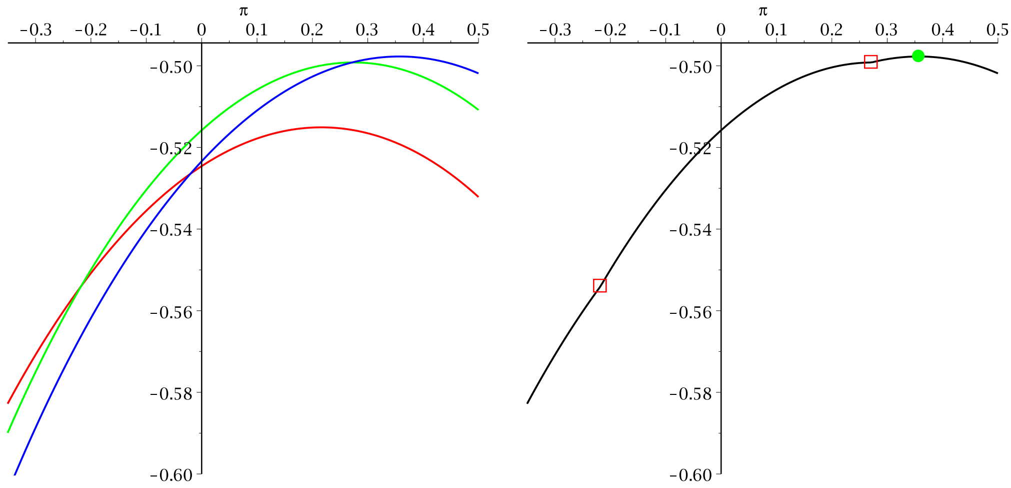
so under the assumptions of the theorem, F increases on $(-\infty, \hat{\pi})$ and H decreases on $(\mathring{\pi}, \infty)$. With $\pi_F < \hat{\pi} < \pi_G < \mathring{\pi} < \pi_H$ and G being strictly concave (c.f. proof of theorem 5.10) it follows $G(\hat{\pi}) < G(\pi_G)$ and $G(\pi_G) > G(\mathring{\pi})$, so π_G is the unique optimizer.

Evaluating G at the optimizer yields

$$G(\pi_G) = G(0) = \log(x_0) + \log(\alpha + \beta) + \log(1+r).$$

■

For a better intuition of the concave branches F, G, and H employed in the proof of theorem 5.10 and their working together let us examine a numerical example. Figure 5.16 presents all three branches as functions of π:



(a) Individual branches. (b) Branches glued together.

Figure 5.16: Numerical example for the branches F (red), G (green), and H (blue) as defined in the proof of theorem 5.10. Deviating from (5.30) the horizon is $T = 30$ while riskless rate, drift, and volatility concur; $\alpha = 45\%$ and $\beta = 5\%$. The red boxes denote the values at $\hat{\pi}$ and $\mathring{\pi}$; the green circle marks the global maximum at π_H. Gluing together F on $(-\infty, \hat{\pi}]$, G on $[\hat{\pi}, \mathring{\pi}]$, and H on $[\mathring{\pi}, \infty)$ yields the black graph.

First note that $F(\hat{\pi}) = G(\hat{\pi})$ and $G(\mathring{\pi}) = H(\mathring{\pi})$, thus the branches can be continuously glued together. Each branch is continuously differentiable on the real domain, so the same property holds for the glued branches at all π except $\hat{\pi}$ and $\mathring{\pi}$.

Following theorem 5.10 the maximizers are properly aligned to provide $\pi_F < \pi_G < \pi_H$, as indeed the vertices of F, G, and H are in subfigure a). This property ensures that both branches F and G increase in a neighborhood of $\hat{\pi}$ while G and H decrease in a neighborhood of $\mathring{\pi}$. Hence neither $\hat{\pi}$ nor $\mathring{\pi}$ can be optimizers.

The global maximizer in this case is $\pi_H \approx 0.36$. For the chosen shares and $\varepsilon := 10^{-6}$ it holds $[\hat{\pi}_{\alpha,\beta}, \mathring{\pi}_{\alpha,\beta}] \approx [-0.66, 0.82]$, so π_H is admissible.

Other market parameterizations may yield that none of the vertices π_F, π_G, and π_H is admissible. In that case one of the boundaries $\hat{\pi}_{\alpha,\beta}$ and $\mathring{\pi}_{\alpha,\beta}$ is optimal.

5.7.3 Convergence to the Benchmark

In the proof of theorem 5.12 we have seen that π_F and π_H enclose π_G and that all are closely related to each other. A second look on π_G reveals that it coincides with the (uncapped) discretized Merton ratio π^*_{Merton}. This is not surprising as the middle branch G corresponds to the situation where any stock price change will improve the high-water mark. Thus the high-water mark matches the new wealth and the target is just the log of a multiple of the wealth or, using the log-rules, an additive shift of the target of the pure log wealth problem. Hence this branch's optimal strategy necessarily is the discretized Merton ratio.

In figure 5.17 the values of π_F, π_G, and π_H together with the boundaries $\hat{\pi}_{\alpha,\beta}$ and $\mathring{\pi}_{\alpha,\beta}$ for ε very close to zero are plotted in the α-β plane. The figure illustrates part c) of theorem 5.12 and at the same time reveals another effect:

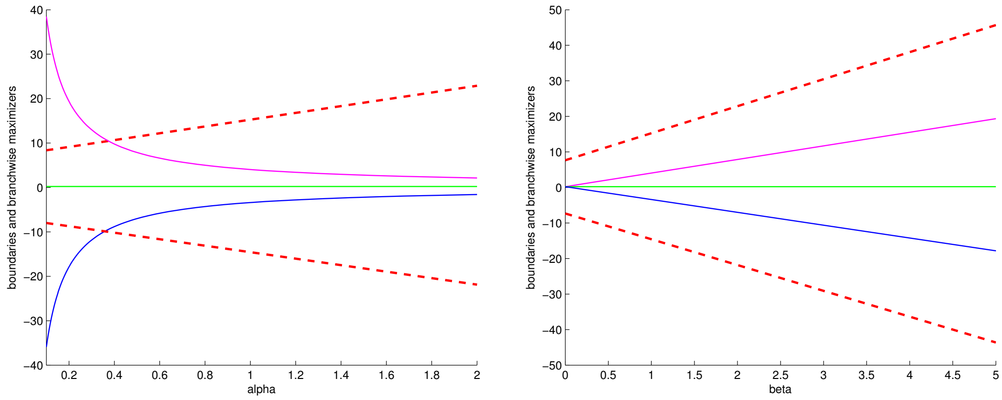
π_F and π_H approach the discretized Merton ratio for $\alpha \to \infty$ or $\beta \to 0$ as we can easily deduce from theorem 5.12:

$$\lim_{\alpha\to\infty} \pi_F = \pi_G - \lim_{\alpha\to\infty} \frac{\beta}{\alpha}(1-p)(u-1-r)^{-1} = \pi_G, \tag{5.41}$$

$$\lim_{\alpha\to\infty} \pi_H = \pi_G + \lim_{\alpha\to\infty} \frac{\beta}{\alpha}p(d-1-r)^{-1} = \pi_G, \tag{5.42}$$

$$\lim_{\beta\to 0} \pi_F = \pi_G - \lim_{\beta\to 0} \frac{\beta}{\alpha}(1-p)(u-1-r)^{-1} = \pi_G, \tag{5.43}$$

$$\lim_{\beta\to 0} \pi_H = \pi_G + \lim_{\beta\to 0} \frac{\beta}{\alpha}p(d-1-r)^{-1} = \pi_G. \tag{5.44}$$



(a) Dependence on α with $\beta = 1$.

(b) Dependence on β with $\alpha = 1$.

Figure 5.17: Shown are the boundaries $\hat{\pi}_{\alpha,\beta}$ and $\mathring{\pi}_{\alpha,\beta}$ (the lower and upper dashed red lines) and how the locations of π_F (blue), π_G (green), and π_H (violet) relate to them. In both plots the market parameters p, r, and d were chosen as in section 5.4. Part a) illustrates the dependence of all aforementioned on α while $\beta = 1$ is fixed, part b) sets $\alpha = 1$ and features the dependence on β.

Figure 5.17 suggests that all candidates for an optimal strategy boil down to the capped discretized Merton ratio when α dominates β. This and more is true:

This log problem $\left(P^{[\hat{\pi}_{\alpha,\beta},\mathring{\pi}_{\alpha,\beta}],1}_{\alpha,\beta,\log}\right)$ admits strategies from the (up to ε) maximal set that still ensures the problem is well-posed. Let us slightly diminish the choices by with $0 \le \pi_{t_0} \le \min\{1, \mathring{\pi}_{\alpha,\beta}\}$ additionally enforcing no-shortselling. The problem is

$$\left(P^{[0,\mathring{\pi}_{\alpha,\beta}\wedge 1],1}_{\alpha,\beta,\log}\right) \quad \begin{cases} \sup_\pi \left\{\mathbb{E}\left[\log\left(\alpha X_{t_1} + \beta X^*_{t_1}\right)\right]\right\} \\ 0 \le \pi \le \min\{\mathring{\pi}_{\alpha,\beta}, 1\} \end{cases} .$$

It is closely related to the one period variation

$$\left(P^{[0,1],1}_{1,0,\log}\right) \quad \begin{cases} \sup_\pi \left\{\mathbb{E}\left[\log\left(X_{t_1}\right)\right]\right\} \\ \pi_{t_0} \in [0,1] \end{cases}$$

of the benchmark problem, as theorem 5.13 asserts.

Theorem 5.13 Let $(\alpha_n)_{n\in\mathbb{N}}$ and $(\beta_n)_{n\in\mathbb{N}}$ sequences in $\mathbb{R}^+$ such that $\frac{\beta_n}{\alpha_n} \to 0$ $(n \to \infty)$.

a) Then the optimal strategy $\pi^*_{t_0}$ of $\left(P^{[0,\mathring{\pi}_{\alpha,\beta}\wedge 1],1}_{\alpha,\beta,\log}\right)$ converges to the capped discretized Merton ratio π^* which is optimal for $\left(P^{[0,1],1}_{1,0,\log}\right)$.

b) If further $\alpha \to c > 0$, the value of $\left(P^{[0,\mathring{\pi}_{\alpha,\beta}\wedge 1],1}_{\alpha,\beta,\log}\right)$ converges to the value of $\left(P^{[0,1],1}_{1,0,\log}\right)$, shifted by the logarithm of c, i.e. to

$$\log(x_0) + \log(1+r) + \mathbb{E}\left[\log\left(1 + \pi^* R_{t_1}\right)\right] + \log(c).$$

Proof. First note that with $\alpha > 0$ it holds

$$\alpha X_{t_1} + \beta X^*_{t_1} > 0 \iff X_{t_1} + \frac{\beta}{\alpha} X^*_{t_1} > 0.$$

As $X^*_{t_1} > 0$ and $X_{t_1} > 0$ if and only if $\pi \in (\hat{\pi}, \mathring{\pi})$, the boundaries $(\hat{\pi}, \mathring{\pi})$ are sharper than $(\hat{\pi}_{\alpha,\beta}, \mathring{\pi}_{\alpha,\beta})$. Further note that $\hat{\pi} < 0 < 1 < \mathring{\pi}$.

Now pick arbitrary sequences $(\alpha_n)_{n\in\mathbb{N}}$ and $(\beta_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^+$ such that $\frac{\beta_n}{\alpha_n} \to 0$ $(n \to \infty)$. As observed above there exists $n_0 \in \mathbb{N}$ such that $\hat{\pi}_{\alpha_n,\beta_n} < 0 < 1 < \mathring{\pi}_{\alpha_n,\beta_n}$ for all $n \geq n_0$.

Using part b) of the proof of theorem 5.12 it holds $\pi_F = \pi_F(\alpha,\beta) \to \pi_G$ and $\pi_H = \pi_H(\alpha,\beta) \to \pi_G$ for $\frac{\beta}{\alpha} \to 0$. Consider three cases:

- If $\pi_G > 1$ then $\pi_F(\alpha_n,\beta_n) > 1$ and $\pi_H(\alpha_n,\beta_n) > 1$ for n large enough and due to the constraints the optimal strategy is capped to $\pi^*_{t_0} = \pi^*_{t_0}(\alpha_n,\beta_n) \equiv 1$ those large enough n.
- Analogously if $\pi_G < 0$ then $\pi_{t_0}(\alpha_n,\beta_n) \equiv 0$ for n large enough.
- If finally $0 \leq \pi_G \leq 1$ then $\pi_F(\alpha_n,\beta_n) \to \pi_G$ and $\pi_H(\alpha_n,\beta_n) \to \pi_G$, so $\pi_{t_0}(\alpha_n,\beta_n) = \operatorname{argmax}\{F(\pi_F), G(\pi_G), H(\pi_H)\} \to \pi_G$.

Concluding $\pi^*_{t_0}$ converges to the capped Merton ratio π^* from (5.26). That establishes part a).

The convergence of the value function claimed in part b) immediately follows from the convergence of the optimal strategy. Observing

$$\log(\alpha X_{t_1} + \beta X^*_{t_1}) = \log(\alpha) + \log\left(X_{t_1} + \frac{\beta}{\alpha} X^*_{t_1}\right),$$

(5.26) and (5.28) provide

$$\begin{aligned} & \lim_{n\to\infty} \mathbb{E}\left[\log(\alpha_n X_{t_1} + \beta_n X^*_{t_1})\right]_{\pi=\pi^*_{t_0}(\alpha_n,\beta_n)} \\ = \; & \mathbb{E}\left[\log(X_{t_1})\right]_{\pi=\pi^*} + \lim_{n\to\infty} \log(\alpha_n) \\ = \; & \log(x_0) + \log(1+r) + \mathbb{E}\left[\log\left(1 + \pi^* R_{t_1}\right)\right] + \log(c) \end{aligned}$$

which establishes the remaining assertion of part b). ■

The next figure with designation 5.18 illustrates theorem 5.13: For various α and β close to zero the values of $\pi_F(\alpha,\beta)$, π_G, and $\pi_H(\alpha,\beta)$ have been calculated as well as the boundaries $\hat{\pi}_{\alpha,\beta}$ and $\mathring{\pi}_{\alpha,\beta}$. For all strategies $\pi_F(\alpha,\beta)$, π_G, and $\pi_H(\alpha,\beta)$ that lay within $[\hat{\pi}_{\alpha,\beta}, \mathring{\pi}_{\alpha,\beta}]$ the values $F(\pi_F(\alpha,\beta))$, $G(\pi_G)$, and $H(\pi_H(\alpha,\beta))$ were calculated and compared. That strategy with the highest value was picked and plotted at the respective location in the α-β-plane as colored dot. If none of these strategies was admissible, the larger boundary value $F(\hat{\pi}_{\alpha,\beta})$ or $H(\mathring{\pi}_{\alpha,\beta})$ was taken instead. The same colors as in figure 5.17 were used.

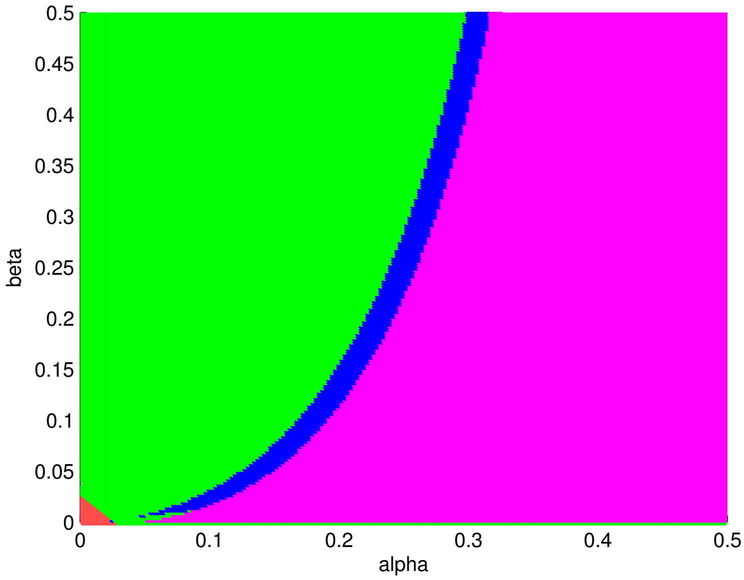


Figure 5.18: Which strategy among π_F, π_G, and π_H optimizes the value of $\left(P^{[\hat{\pi}_{\alpha,\beta},\mathring{\pi}_{\alpha,\beta}],1}_{\alpha,\beta,\log}\right)$? π_F is coded blue, π_G green, and π_H violet. If none of these is admissible, the region is colored red. p, r, and d are the usual ones from section 5.4.

Theorem 5.12 covers the general case of $\frac{\beta}{\alpha} \to 0$. This is true for α diverging quicker than β (where in this case β may be bounded as well), but of course that can never be depicted. The second case can be examined in figure 5.18: β being a zero sequence and α being bounded (or even converging).

In this situation the lower edge is approached. Note that though the path to the lower edges can lead through regions of any color, apart from the lower left corner the very edge is green, denoting $\pi_G = \pi^*_{\text{Merton}}$ is the maximizer. The sudden change of color suggests a discontinuity but, c.f. (5.41), that is not true as both π_F and π_H converge to π_G for $\beta \to 0$. On the contrary: the values associated with the maximizers are continuous in α and β.

If as in the lower left corner both α and β converge to zero, the interval for admissible strategies shrinks and finally vanishes, c.f. (5.39) and (5.40). Consequentially from some point the Merton ratio π_G is no longer admissible (note that for the market parameters used $\pi_G \neq 0$) and one of the boundaries maximizes. The effect becomes apparent in the red region in the lower left edge. The region being delimited from the adjacent green region by a straight line is due to both $\hat{\pi}_{\alpha,\beta}$ and $\mathring{\pi}_{\alpha,\beta}$ being linear in α and β.

5.8 Combined Rewards in Multiple Periods

Now let us transfer the log problem with combined rewards to multiple periods. In this section the analytical approach is lined out and the dependence of the optimal behavior on the current performance ratio is investigated. Then the solution in an example market is computed and discussed.

5.8.1 The Analytical Approach

In section 5.7 we have investigated the role of the weights α and β for wealth and high-water mark in one period. Essentially, a non-zero reward for terminal wealth ensures the boundedness of the agent's optimal strategy and the ratio of β to α largely determines it. We have also seen that the agent's utility to a great degree influences the scale of the candidate solutions.

In this multi-period setup let $\alpha = 1 = \beta$ and $U = \log$ to facilitate the analysis. Hence the problem under consideration is

$$\left(P^{[0,1],n}_{1,1,\log}\right) \quad \begin{cases} \sup_\pi \left\{\mathbb{E}\left[\log\left(X_{t_n} + X^*_{t_n}\right)\right]\right\} \\ \pi = (\pi_{t_0}, \ldots, \pi_{t_{n-1}}),\ \pi_{t_k} = \pi_{t_k}(x, x^*) \in [0,1],\ k = 0, \ldots, n-1 \end{cases}$$

Requiring the no-shortselling property ensures the problem is well-posed.

Starting with $J_n(x, x^*) = \log(x + x^*)$ we work backwards by employing (5.10). At step $k = n-1$ the question whether or not the current high-water mark x^* can be beaten in the next step grants three branches to consider:

$$\begin{aligned} L(\pi) &:= p\log\left(\hat{x}(x,\pi) + x^*\right) + (1-p)\log\left(\mathring{x}(x,\pi) + x^*\right), \\ M(\pi) &:= p\log\left(2\hat{x}(x,\pi)\right) + (1-p)\log\left(\mathring{x}(x,\pi) + x^*\right), \quad \text{and} \\ N(\pi) &:= p\log\left(2\hat{x}(x,\pi)\right) + (1-p)\log\left(2\mathring{x}(x,\pi)\right) \end{aligned}$$

where L corresponds to the situation in which the current high-water mark x^* is unreachable by any admitted strategies, M collects the cases where x^* can be beaten only if stock prices go up, and N is the case of x^* being beatable even if stock prices go down.

Note that though the resemblance to the branches F, G, and H from theorem 5.10 and the proof of theorem 5.12 for the one-period models with general or log utility the correspondence is not one-to-one: Due to the additional no-shortselling constraint which guarantees $\mathring{x}(x,\pi) \leq \hat{x}(x,\pi)$ for any admissible x and π, the branch L is introduced while F is excluded. N and G are more closely related as they both model the situation where the high-water mark is improved for any strategy choice and any stock price movement.

Each branch is strictly concave and exhibits a global maximum. The maximizing strategies are

$$\begin{aligned} \pi_L &= \frac{(x^* + (1+r)x)(p(u-1-r) + (1-p)(d-1-r))}{x(u-1-r)(1+r-d)}, \\ \pi_M &= \frac{p(u-1-r)x^* + (1+r)(p(u-1-r) + (1-p)(d-1-r))x}{x(u-1-r)(1+r-d)}, \quad \text{and} \\ \pi_N &= \frac{(1+r)(p(u-1-r) + (1-p)(d-1-r))}{(u-1-r)(1+r-d)}. \end{aligned}$$

Not unexpectedly the maximizers π_G and π_N coincide and are identical to the discretized Merton ratio π^*_{Merton}. The cause for π_N to resemble the Merton ratio is the independence of x^* in the market situation N characterizes. Accordingly the current wealth x can be taken out of both logarithms and vanishes when forming the derivative.

We can also put π_L and π_M in terms of the discretized Merton ratio:

$$\pi_L = \left(\frac{x^*}{x} - 1 - r\right)\frac{p(u-1-r) + (1-p)(d-1-r)}{(u-1-r)(1+r-d)} = (z-1-r)\pi^*_{\text{Merton}},$$

$$\pi_M = z \cdot \frac{p}{1+r-d} + \frac{(1+r)(p(u-1-r) + (1-p)(d-1-r))}{(u-1-r)(1+r-d)} = z \cdot \frac{p}{1+r-d} + \pi^*_{\text{Merton}}.$$

π_L is a scaled and translated version of the discretized Merton ratio while π_M is only (positively) translated to it. As an immediate consequence $\pi_M > \pi_N$ in any case and $\pi_L > \pi_N$ if and only if $z > 2 + r$.

Note that the update margins $\hat{\pi}$ and $\mathring{\pi}$ now depend on the performance ratio and the alignment $\hat{\pi}(z) < 0 < \mathring{\pi}(z)$ is no longer true for all z. Consequently and as demonstrated in the multi-period problem with pure high-water mark of section 5.6 we have to consider various cases of the value of the performance ratio at time step $n-1$. These are

- $u \leq z \leq u^{n-1}$ in which situation the current high-water mark is unbeatable by any admissible strategy and we have only one branch to consider. For $z > u > 1 + r$ it holds $\pi_L \geq 0$ if and only if $\pi^*_{\text{Merton}} \geq 0$ and $\pi_L = \pi_L(z) \leq 1$ if and only if $(z - 1 - r)^{-1} \geq \pi^*_{\text{Merton}}$. Consequently

$$
\begin{aligned}
&J_{n-1}(x, x^*)\\
= \;& \sup_{\pi_{t_{n-1}} \in [0,1]} \left\{ L(\pi_{t_{n-1}}) \right\}\\
= \;& \begin{cases} L(0), & \pi^*_{\text{Merton}} < 0 \\ L(\pi_L), & 0 \leq \pi^*_{\text{Merton}} \leq \frac{1}{z-1-r} \\ L(1), & 0 \leq \pi^*_{\text{Merton}} \text{ and } \frac{1}{z-1-r} < \pi^*_{\text{Merton}} \end{cases}\\
= \;& \begin{cases} \log((1+r)x + x^*), & \pi^*_{\text{Merton}} < 0 \\ p\log(\hat{x}(x, \pi_L) + x^*) + (1-p)\log(\mathring{x}(x, \pi_L) + x^*), & 0 \leq \pi^*_{\text{Merton}} \leq \frac{1}{z-1-r} \\ p\log(ux + x^*) + (1-p)\log(dx + x^*), & \begin{array}{l} 0 \leq \pi^*_{\text{Merton}} \text{ and} \\ \frac{1}{z-1-r} < \pi^*_{\text{Merton}} \end{array} \end{cases}.
\end{aligned}
$$

  Note that π_L increases with the current high-water mark and decreases in the current wealth. This produces a remarkable effect as we will discover later in the numerical example.

- $1 + r \leq z \leq u$ yielding $0 \leq \hat{\pi}(z) \leq 1$ and $\mathring{\pi}(z) < 0$. Here the current high-water mark can be improved when stock prices go up by placing a fraction larger than $\hat{\pi}(z)$ in the stock, but can never be reached when stock prices go down (without selling the stock short). Hence

$$
J_{n-1}(x, x^*) \quad = \quad \sup_{\pi_{t_{n-1}} \in [0,1]} \left\{ \begin{array}{l} L(\pi_{t_{n-1}}),\ 0 \leq \pi_{t_{n-1}} \leq \hat{\pi}(z) \\ M(\pi_{t_{n-1}}),\ \hat{\pi}(z) \leq \pi_{t_{n-1}} \leq 1 \end{array} \right\}.
$$

  Which strategy optimizes? π_L is a candidate as long as it is located left of $\hat{\pi}(z)$ which is true as long as $\pi^*_{\text{Merton}} < (u-1-r)^{-1}$. Otherwise $\hat{\pi}(z)$ becomes a nominee. The left border can be ruled out[6] as $z \geq 1 + r$ provides $\pi_L \geq 0$. The question how π_M and $\hat{\pi}(z)$ are aligned cannot be answered by such a simple linear inequality. Both cases can occur, so we have to add π_M to the list as well as the right border. Altogether

$$
\operatorname*{argsup}_{\pi_{t_{n-1}} \in [0,1]} \left\{ \begin{array}{l} L(\pi_{t_{n-1}}),\ 0 \leq \pi_{t_{n-1}} \leq \hat{\pi}(z) \\ M(\pi_{t_{n-1}}),\ \hat{\pi}(z) \leq \pi_{t_{n-1}} \leq 1 \end{array} \right\} \in \{\pi_L, \hat{\pi}, \pi_M, 1\}.
$$

[6] If $z = 1 + r$ we have $\pi_L = 0$. so the left border can be optimal, too, but it is then already covered by π_L and does not have to be enumerated individually.

- $1 \le z \le 1+r$ means the current high-water mark can be beaten even if stock prices go down if the strategy chosen invests no higher fraction of wealth than $\mathring{\pi}(z)$ in the stock. Note $\hat{\pi}(z) \le 0$ and $0 \le \mathring{\pi}(z) < 1$ for the range of performance ratios considered here. This case also resembles to the situation we have discussed in the one-step problem as there $x = x_0 = x^*$ reduced the discussion to a performance ratio of $z = 1$. It follows

$$J_{n-1}(x, x^*) \quad = \quad \sup_{\pi_{t_{n-1}} \in [0,1]} \left\{ \begin{matrix} N(\pi_{t_{n-1}}),\ 0 \le \pi_{t_{n-1}} \le \mathring{\pi}(z) \\ M(\pi_{t_{n-1}}),\ \mathring{\pi}(z) \le \pi_{t_{n-1}} \le 1 \end{matrix} \right\}.$$

  Now π_N is a candidate only if it is located left of $\mathring{\pi}$ which in turn is true if and only if $z < (1+r)(1-p)\frac{u-d}{u-1-r}$. Unfortunately the factors $(1-p) > 0$ and $\frac{u-d}{u-1-r} > 1$ can be adjusted independently of each other so there is no general answer to that question and we have to add π_N to the list. A discretized Merton ratio less than zero allows the left border to be a candidate, too.

  For the right branch we have to asses whether $\pi_M \le \mathring{\pi}(z)$ in which case neither π_M nor the right border could be optimizers. An equivalent condition is $z \le (1+r) + (d-1-r)\pi^*_{\text{Merton}} - p$ which is certainly true by the standing assumptions for small enough (and negative) discretized Merton ratios. But as we cannot assert the sign of π^*_{Merton} nothing is won and we can only claim

$$\operatorname*{argsup}_{\pi_{t_{n-1}} \in [0,1]} \left\{ \begin{matrix} N(\pi_{t_{n-1}}),\ 0 \le \pi_{t_{n-1}} \le \mathring{\pi}(z) \\ M(\pi_{t_{n-1}}),\ \mathring{\pi}(z) \le \pi_{t_{n-1}} \le 1 \end{matrix} \right\} \in \{0, \pi_N, \mathring{\pi}(z), \pi_M, 1\}.$$

The analysis of the first step proved to be difficult as the solution eminently depends on all market parameters. A possible next step is to postulate further assumptions until a unique solution arises and then to continue the analysis with the next step backwards in time.

On the other hand that would require a considerable number of necessary additional assumptions and thereby rather degrade the insight one may gain. Also experiences with the far less elaborate pure high-water mark model of section 5.6 show that the discussion of step $k = n-2$ will surely require additional assumptions in order to derive an explicit solution, as will every further step back in time.

Let us instead turn our attention to a numerical example that sheds some light on the analytical solution.

5.8.2 An Example

Let us examine the combined multi-period model in the usual example market (5.30).

Optimal Strategies

The first thing calculated are the pointwise optimal strategies over the grid $\tilde{\mathcal{X}}_4$. The branchwise maximizers employed above now read

$$\begin{aligned} \pi_L = \pi_L(z) &\approx 0.2155 + 0.2141z, \\ \pi_M = \pi_M(z) &\approx 0.2155 + 3.8252z, \quad \text{and} \\ \pi_N = \pi^*_{\text{Merton}} &\approx 0.2155. \end{aligned} \tag{5.45}$$

The values at these positions (if admissible) need to be compared with the values at the update margins

$$\hat{\pi} = \hat{\pi}(z) \approx 7.2647z - 7.3078,$$
$$\mathring{\pi} = \mathring{\pi}(z) \approx -7.6056z + 7.6507,$$

and the values at the borders 0 and 1.

Figure 5.19 presents the optimal strategies computed at step $k = n - 1 = 4$:

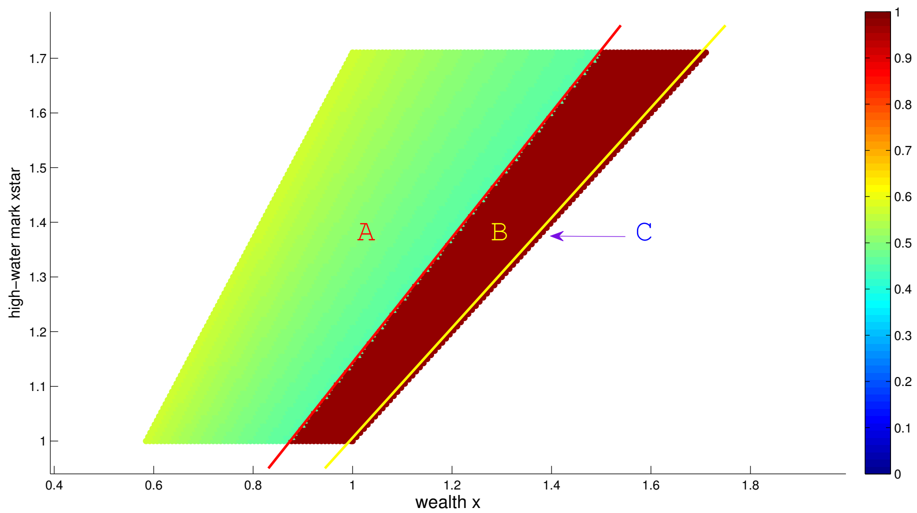


Figure 5.19: The optimal strategy at step $k = 4$ of $n = 5$ for the combined wage $\Psi(X^{t_5-}, X^{t_5,*}) = X_{t_5-} + X^*_{t_5}$ and log utility. At each $(x, x^*) \in \Delta\mathcal{X}_4$ the strategy $\pi^*_{t_4}(x, x^*)$ is plotted as point in the $x - x^*$-plane where its color denotes the value. The regions A (left of the red line), B (between the red and the yellow line), and C (right of the yellow line) correspond to the three cases of performance ratios discussed above.

For a closer look let us partition $\tilde{\mathcal{X}}_4$ in the same regions that were distinguished in the analytical discussion:

Region A corresponds to the set $\{(x, x^*) \in \tilde{\mathcal{X}}_4 \;:\; u \leq z \leq u^{n-1}\}$ of those states where the high-water mark cannot be improved in one step by any admissible strategy. With the given market parameters $\pi_L(z) \in [0, 1]$ for all $(x, x^*) \in A$, so it is the unique optimizer and we can directly observe its values. Note that for fixed high-water mark the optimal strategy decreases in the wealth. This means that, given the current high-water mark is unreachable, the agent becomes the more cautious the more wealth she has. Or, viewed the other way around, she becomes the more desperate the less wealth she commands.

That effect may be somewhat surprising at first glance, but offers a simple explanation: As the high-water mark is unreachable, we are effectively optimizing the target function

$$\mathbb{E}\left[\log\left(X_{t_5-} + x^*\right) \mid X_{t_4} = x\right] = p \log(\hat{x}(x, \pi) + x^*) + (1 - p) \log(\mathring{x}(x, \pi) + x^*).$$

For any fixed value x^* of the current high-water mark this target coincides with the second benchmark (5.31) when we set $C := x^*$. As proposition 5.5 states, the optimal strategy on that line is

$$\pi^*_{t_4}(x) = \pi^*_{\text{Merton}} + \frac{x^*}{x} \cdot \frac{p(u-1-r) + (1-p)(d-1-r)}{(u-1-r)(1+r-d)}, \ x > 0.$$

And that strategy obviously decreases in x.

Consider figure 5.20, which offers a zoom of figure 5.19. Here it is better recognizable what happens in region $B = \{(x, x^*) \in \tilde{\mathcal{X}}_4 : \ 1 + r \leq z \leq u\}$:

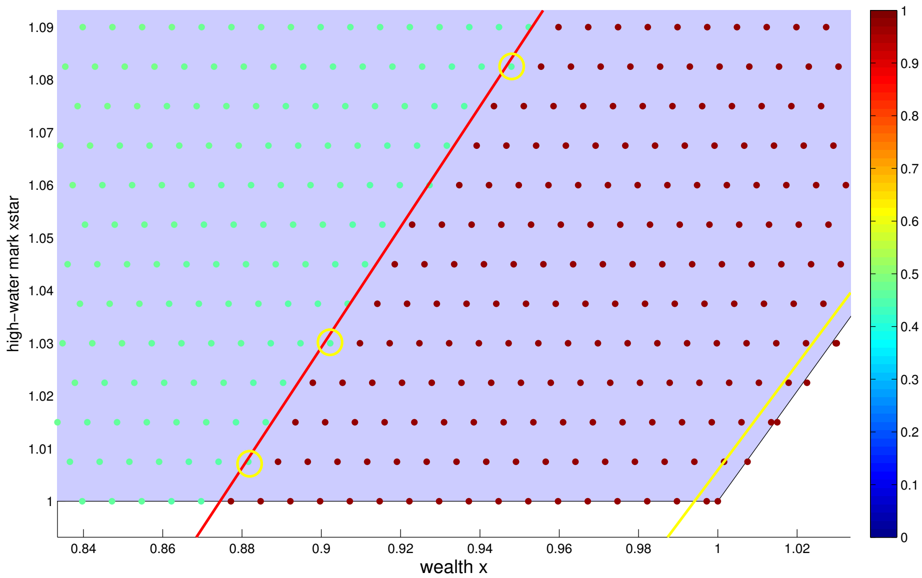


Figure 5.20: A magnified excerpt of figure 5.19. The yellow circles point out grid points $(x, x^*) \in \Delta\mathcal{X}_4 \cap B$ where the optimal value is attained at π_L.

We see that π_L is still optimal for (x, x^*) located close to the border to region A. Though these points seem to be isolated in figure 5.20 this is not true and merely due to the relatively coarse grid. In fact there is a small interval of performance ratios z within B for which π_L optimizes and that subregion would be better distinguishable for denser grids. For smaller values of z (which means farther to the right in the plot) the point $\hat{\pi}(z)$ where L and M are glued together shifts to the left. Consequently a larger part of the latter branch needs to be considered. In our example market π_M is distinctly larger than the right border, so M increases on $[\hat{\pi}(z), 1]$. As here also $M(\pi_M) > L(\pi_L)$, for each z there exists some $\pi_0(z) \in [\hat{\pi}(z), \pi_M]$ such that $M(\pi) > L(\pi_L)$ for all $\pi \geq \pi_0(z)$. This $\pi_0(z)$ now happens to be located within $[\hat{\pi}(z), 1]$ for z slightly smaller than $1 + r$ where regions A and B meet. Accordingly the maximizer jumps to the next branch and, recall $\pi_M > 1$, is capped at the right border.

Region $C = \{(x, x^*) \in \tilde{\mathcal{X}}_4 : \ 1 \leq z \leq 1 + r\}$ is the smallest. What happens here is analogue to the proceedings in B: The branches N and M are glued together at $\mathring{\pi}(z)$. A glance at (5.45) verifies that $\pi_M(z)$ is larger than 1 and significantly larger than π_N for all ratios z in C. The values $M(\pi_M(z))$ also that drastically exceed $N(\pi_N)$, that we here have $N(\pi_N) < M(1) < M(\pi_M(z))$ for all $z = \frac{x^*}{x}$ with $(x, x^*) \in C$. Thus the right border is always the best choice.

Let us examine what behavior is optimal at step $k = 3$. Figure 5.19 displays the results of the computation:

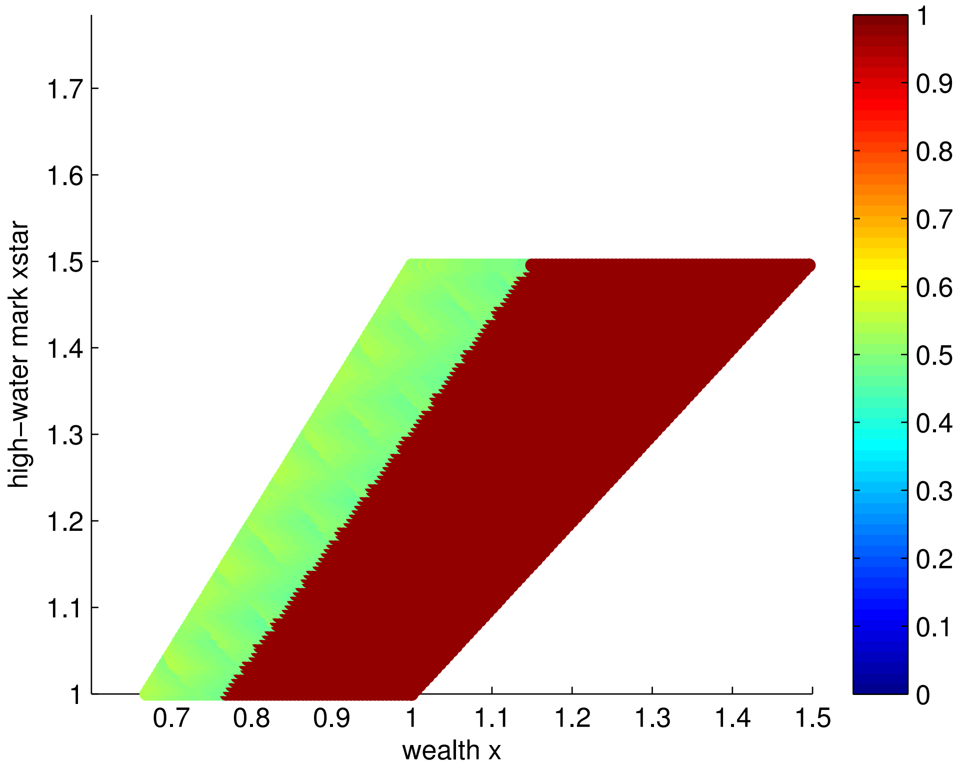


Figure 5.21: The optimal strategy for $\Psi(X^{t_5-}, X^{t_5,*}) = X_{t_5-} + X^*_{t_5}$ and $U = \log$ at step $k = 3$, plotted over $\Delta\mathcal{X}_3$ as described in figure 5.19.

What first meets the eye is the sharp change from a moderate investment (around 50%) in the stock to full risk. The jump occurs at the straight $\{z = u^2\}$ and admits a sensible explanation: The left region is exactly that subset of $\tilde{\mathcal{X}}_3$ in which the current high-water mark x^* cannot be beaten in the remaining 2 steps while the right side corresponds to those states in which improving x^* is possible before termination.

The left part renders the same situation as the region A in step $k = 4$. Using the fact that π_L from step 4 lies within the no-shortselling bounds we can find the target of this region in step 3 by collecting all cases:

$$\begin{aligned}
J_3(x, x^*, \pi) &= pJ_4(\hat{x}(x, \pi), x^*) + (1 - p)J_4(\mathring{x}(x, \pi), x^*) \\
&= p^2 \log\left(\hat{x}(\hat{x}(x, \pi), \pi_L) + x^*\right) \\
&+ p(1 - p) \log\left(\mathring{x}(\hat{x}(x, \pi), \pi_L) + x^*\right) \\
&+ (1 - p)p \log\left(\hat{x}(\mathring{x}(x, \pi), \pi_L) + x^*\right) \\
&+ (1 - p)^2 \log(\mathring{x}(\mathring{x}(x, \pi), \pi_L) + x^*).
\end{aligned}$$

This is a strictly concave function[7] in π and hence exhibits a global maximizer. The location of that maximizer does in general depend on x and x^* and – obviously – is within

[7] The strict concavity can be established analogously to the steps in the proof of theorem 5.10.

the no-shortselling bounds for the market under examination. Also note the waves effect as described in figure 5.13.

For the right part an analytical approach would at least have to consider the sets $\{u^2 \geq z \geq u(1+r)\}$, $\{u(1+r) \geq z \geq u\}$, $\{u \geq z \geq 1+r\}$, and $\{1+r \leq z \leq 1\}$ with 2 or 3 branches each (c.f. the analysis of the multi-period problem with pure high-water mark terminal wage in section 5.6). For our parameter choice it turns out that as in step 4 the branch of the left region is still optimal at the far left part of the right region. But almost immediately the right-most branch dominates the target and has an apex larger than the right border. In economical terms, this market encourages riskier strategies (almost) as soon the high-water mark can be improved in the remaining steps.

As a matter of fact the strategy $\pi_{t_k}(x, x^*) \equiv 1$ is calculated as optimal for $k = 2, 1, 0$ which goes along with all states in $\tilde{\mathcal{X}}_k$ admitting an improvement of the high-water mark for these k (c.f. the discussion of the bends in figure 5.22).

The Value Function

Talking about value functions, figure 5.22 states what is yielded here:

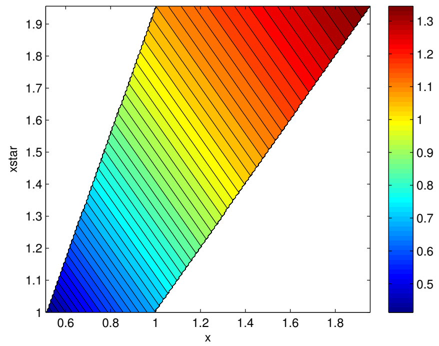
As this wage, in contrast to the pure high-water mark wage, is sensitive to changes in both axes, with $h = 0.001$ the grid is laid out much denser than in figure 5.15, where $h = 0.01$ is used. This yields approximately 99 times more grid points and the calculations are quite precise, c.f. figure 5.11 where the better approximation in parts b) and d) here is still refined by the factor 10.

$J_5(x, x^*) = \log(x + x^*)$ rewards wealth and high-water mark with the same weight, hence the contour lines $\{\log(x + x^*) = c\}$ in sub-figure a) have slope -1 instead of running horizontally.

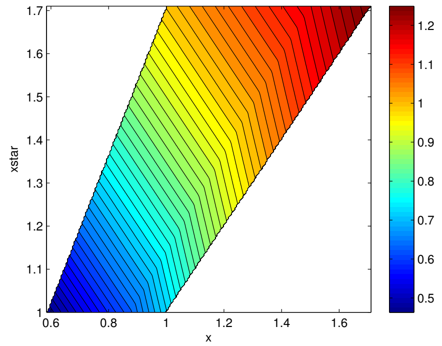
The optimal strategy at step $k = 4$ suggests an investment as bold as admitted in the regions B (mostly) and C as figure 5.19 illustrates. In this favorable market the consequence is a larger expected utility of terminal wage over those regions which manifests in sub-figure b) as bend at the (approximate) border between A and B.

The second bend visible in sub-figure c) for $k = 3$ analogously arises where the now even larger set of states with potential to improve x^* in the remaining steps meets the left part with the remaining states.

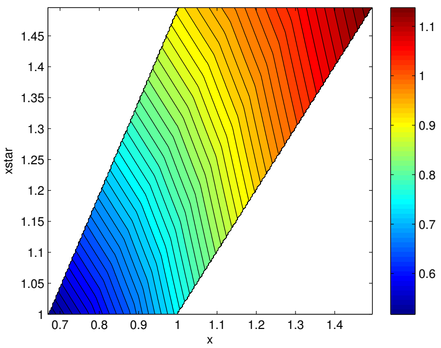
From then on the optimal strategy coincides with what was optimal in the pure high-water mark problem and thus the effects are akin with those described in parts d), e), and f) of figure 5.15.



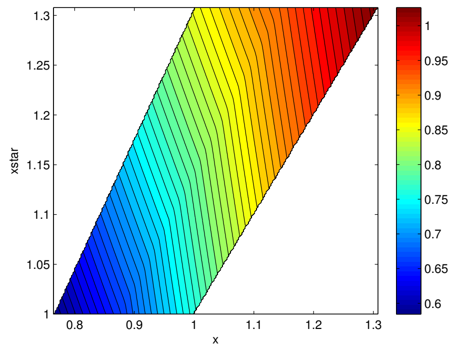
(a) Step $k = 5$ of $n = 5$ (boundary).



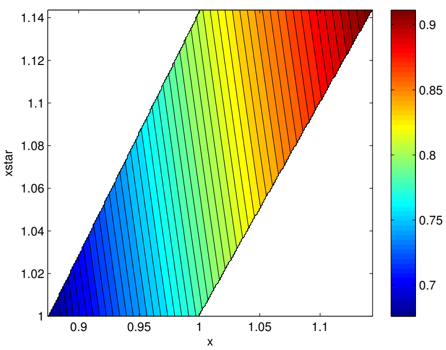
(b) Step $k = 4$ of $n = 5$.



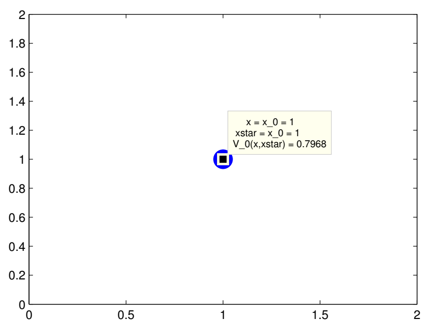
(c) Step $k = 3$ of $n = 5$.



(d) Step $k = 2$ of $n = 5$.



(e) Step $k = 1$ of $n = 5$.



(f) Step $k = 0$ of $n = 5$.

Figure 5.22: Value functions for $\Psi(X^{t_n-}, X^{t_n,*}) = \log(X_{t_n-} + X^*_{t_n})$ as computed in an $n = 5$ step approximation over grids of width $h = 0.001$.

Time-Dependence of the Optimal Strategy

The concavity of the logarithmic utility can be recognized in any of the sub-figures as the contour lines are denser for smaller $x + x^*$ in the lower left and more spread out for larger values of $x + x^*$. Note that the more steps remain the more equally-spaced the contour lines are, meaning that the expected utility of wage averages out over the remaining path to a larger value (recall the market being in favor) where the logarithm is flatter.

Among other properties we are interested in the time-dependence of the optimal strategy. As in discrete-time models the domain of admissible intermediate wealth (and high-water mark) values grows with each time-step, we have to explain what shall be meant by time-independence. It seems sensible to call the optimal strategy time-independent from step $k-1$ to step k if it coincides on the common domain, i.e. if

$$\pi^*_{t_{k-1}}(x, x^*) = \pi^*_{t_k}(x, x*) \quad \forall (x, x^*) \in \mathcal{X}_{k-1} \cap \mathcal{X}_k = \mathcal{X}_{k-1},$$

where here we have assumed the portfolio problem requires no-shortselling and no intermediate wages are deduced. More generally for $\pi \in [\pi_{\min}, \pi_{\max}]$ admissible and arbitrary intermediate wages the required property is

$$\pi^*_{t_{k-1}}(x, x^*) = \pi^*_{t_k}(x, x*) \quad \forall (x, x^*) \in \mathcal{Y}^{\pi_{\min},\pi_{\max}}_{k-1} \cap \mathcal{Y}^{\pi_{\min},\pi_{\max}}_{k} = \mathcal{Y}^{\pi_{\min},\pi_{\max}}_{k-1}.$$

As in any case the admissible states form an increasing sequence (in terms of inclusion), the only state common to all steps is (x_0, x_0). There is not much to satisfy if we understand the strategy to be globally time-independent if its values coincide on the domain common to all steps. Let us instead agree on the following:

Definition 5.6 *The strategy $(\pi_{t_k})_{k=0,\dots,n-1}$ is* ***globally time-independent*** *if and only if it is time-independent at each step, i.e. if and only if for $k = 1, \dots, n-1$*

$$\pi_{t_{k-1}}(x, x^*) = \pi_{t_k}(x, x^*) \quad \forall (x, x^*) \in \mathcal{Y}^{\pi_{\min},\pi_{\max}}_{k-1}.$$

Note that in particular the terminal log problem $\left(P^{[0,1],n}_{1,0,\log}\right)$ admits the globally time-independent solution π^* from (5.26). Also the uncapped Merton ratio π^*_{Merton} from (5.27) is a globally time-independent strategy.

Standing on safely defined grounds we can now claim: The optimal strategy for $\left(P^{[0,1],n}_{1,1,\log}\right)$ is not globally time-independent. This is easily established by comparing figures 5.19 and 5.21.

The more analytical reason is that, as long as the discretized Merton ratio is in $[0, 1]$, the solution in the left-most region of step $k = n-1$ is that ratio. In the remaining part of $\mathcal{X}_{n-1}$ that is in general not true as additional branches need to be considered. If only one of these produces a higher value for any admissible strategy, the discretized Merton ratio is discarded there.

Now in the first step in which optimal strategies need to be calculated, at $k = n-1$, the region with no hope of increasing the high-water mark is $\{u^{n-1} \geq z \geq u,\ x_0 \leq x^* \leq x_0 u^{n-1}\}$ and there the discretized Merton ratio is the unique solution. One step back in time, at $k = n-2$, there is more time to improve the current record and the Merton-only region shrinks to $\{u^{n-2} \geq z \geq u^2,\ x_0 \leq x^* \leq x_0 u^{n-2}\}$. The states

$$\{u^2 \geq z \geq u,\ x_0 \leq x^* \leq x_0 u^{n-2}\} \subset \tilde{\mathcal{X}}_{n-2}$$

remain and there the optimal strategy does, in general, not coincide.

The above arguments hold as long as $n \geq 5$ and are equally true for weights $\alpha \geq 0$ and $\beta > 0$.

5.9 Incentives of Intermediate Wages

Let us now examine principal-agent systems in which intermediate wages are paid. This broadens the class of contracts the principal can choose from and offers him more means to influence the agent's implied behavior.

When now intermediate wages are paid, the values X_{t_k-} and X_{t_k} no longer coincide. The wealth evolution scheme (5.5) explains the dynamics for that case, too. But note that the previously considered situation with terminal wages only resembles the time-continuous dynamic (5.2) while now we approach (5.3). The process X^* of recorded high-water marks adapts to whatever wealth evolution is considered and accordingly changes along with X when intermediate wages are admitted. We have to keep that in mind when comparing effects encountered by agents with intermediate wages with previously derived results.

Throughout the following assume the agent is required to observe no-shortselling restrictions.

The Agent's Linear-Linear Problem Revisited

To approximate the time-continuous wages ψ from (5.3) in discrete time put

$$\begin{aligned}\psi(X^{t_k-}, X^{t_k,*}, t_k) &= \gamma X_{t_k}(t_k - t_{k-1}) + \delta\left(X^*_{t_k} - X^*_{t_{k-1}}\right)\\ &= \gamma h X_{t_k} + \delta\left(X^*_{t_k} - X^*_{t_{k-1}}\right),\end{aligned} \tag{5.46}$$

where an equidistant time-grid with $t_k - t_{k-1} \equiv h = \frac{T}{n}$ is assumed in the last reformulation.

A risk-neutral agent accumulates the wages

$$\sum_{k=1}^{n} \psi(X^{t_k-}, X^{t_k,*}, t_k) = \gamma h \left(\sum_{k=1}^{n} X_{t_k-}\right) + \delta\left(X^*_{t_n} - x_0\right). \tag{5.47}$$

For this type of investor the intermediate utilities of shares of the high-water mark cancel out. The right-hand side of (5.47) is, up to a constant translation, the same as when paying only a terminal share of the high-water mark. Recall that the processes X and X^* in (5.47) differ from their terminal wage only counterparts. Thus the high-water mark effects may only be comparable for small shares δ and a moderate number n of time-steps.

The wealth accrued over the whole period in (5.47) however cannot be simplified and has potential to set new incentives.

We will more generally allow the agent to apply a utility U as defined in (5.6). Then neither the shares of intermediate wealth nor the shares of intermediate high-water mark cancel out and each may evolve its unique effect on the agent's risk-behavior.

Let us get back to theorem 5.2 which cites an effect discussed by Panageas & Westerfield (2009). In their setup an agent continuously collects a (discounted) share of the high-water mark. Panageas & Westerfield (2009) show that close to termination the agent's optimal strategy approaches infinity and cannot be bounded.

We want to examine the question whether this is true in a model with finite time-horizon and to that end consider the last step $k = n-1$. As now a state (x, x^*) condenses the complete path information up to t_{n-1}, the different wealth evolutions with or without intermediate wages are covered by the state. For the last step we can explicitly take into account the deduction of the last intermediate wage and reformulate the target in the last step to

$$\begin{aligned}
&\mathbb{E}^{t_{n-1},x,x^*}\left[U(\alpha X_{t_n}+\beta X^*_{t_n})+U(\gamma h X_{t_n-}+\delta(X^*_{t_n}-x^*))\right]\\
=\;&\mathbb{E}^{t_{n-1},x,x^*}\big[U(\alpha[X_{t_n-}-\psi(X^{t_n-},X^{t_n,*},t_n)]+\beta X^*_{t_n})+\\
&\qquad U(\gamma h X_{t_n-}+\delta(X^*_{t_n}-x^*))\big]\\
=\;&\mathbb{E}^{t_{n-1},x,x^*}\big[U(\alpha(1-\gamma h)X_{t_n-}+(\beta-\alpha\delta)X^*_{t_n}+\alpha\delta x^*)+\\
&\qquad U(\gamma h X_{t_n-}+\delta(X^*_{t_n}-x^*)\big].
\end{aligned}\tag{5.48}$$

For a risk-neutral investor (5.48) reads

$$\mathbb{E}^{t_{n-1},x,x^*}\left[(\alpha(1-\gamma h)+\gamma h)X_{t_n-}+(\beta+\delta(1-\alpha))X^*_{t_n}\right]+(\alpha-1)\delta x^*.$$

The structure is identical to the approach of section 5.7 where combined wages are discussed. But as theorem 5.10 assumes the presence of a strictly concave utility, we cannot directly deduce the optimal strategy.

A Log Investor's Problem Variations

It suggests itself to work in direction of theorem 5.10 by applying a utility. The logarithmic utility is very suitable as it allows the separation factors but at the same time we have to take care that all arguments passed to the logarithm are strictly positive. Let us at first pose the problems and preserve the discussion of the positivity of all arguments for later, when with the knowledge of all individual problems a thorough solution is easier formulated.

The simplest variation of (5.46) and (5.48) is offering to the agent one of the intermediate shares only (and none of the terminal shares). For a share of wealth we have the target

$$\mathbb{E}^{t_{n-1},x,x^*}\left[\log(\gamma h X_{t_n-})\right]=\log(\gamma h)+\mathbb{E}^{t_{n-1},x,x^*}\left[\log(X_{t_n-})\right]\tag{5.49}$$

which is nothing else than the (shifted) first benchmark of section 5.4. If vice versa only a share of the running high-water mark is offered, the optimization yields the target

$$\mathbb{E}^{t_{n-1},x,x^*}\left[\log(\delta(X^*_{t_n}-x^*))\right]=\log(\delta)+\mathbb{E}^{t_{n-1},x,x^*}\left[\log(X^*_{t_n}-x^*)\right].\tag{5.50}$$

For a non-null subset of Ω it holds $X^*_{t_n}=x^*$ and the target is not well-defined. Even for states (x,x^*) that can beat x^* in the remaining step there are admitted strategies in $[0,1]$ for which $\mathring{x}(x,\pi)\leq x^*$, so this issue is intrinsic to the problem. We will run into another two variations of the latter in the following. When we have them all together we will propose a fix and also discuss the solution of (5.50).

Let us now consider wage schedules that pay one share intermediately and one share terminally. There are four combinations, namely AC, AD, BC, and BD, where the letters A, B, C, and D refer to the shares α, β, γ, and δ and denote which of those are paid. We will examine the combinations one after another:

If in AC wealth and nothing else is rewarded both intermediately and terminally, (5.48) becomes

$$\begin{aligned}
&\mathbb{E}^{t_{n-1},x,x^*}\left[\log(\alpha(1-\gamma h)X_{t_n-})+\log(\gamma h X_{t_n-})\right]\\
=\;&\log(\alpha(1-\gamma h))+\log(\gamma h)+2\mathbb{E}^{t_{n-1},x,x^*}\left[\log(X_{t_n-})\right].
\end{aligned}\tag{5.51}$$

As long as $0 < \alpha$ and $0 < \gamma h < 1$, everything is well-defined and the best strategy is the (capped and discretized) Merton ratio as proposition 5.4 provides.

If on the contrary in BD only shares of both the running and the terminal high-water mark are paid, the target reads

$$\begin{aligned} & \mathbb{E}^{t_{n-1},x,x^*}\left[\log(\beta X^*_{t_n}) + \log(\delta(X^*_{t_n} - x^*))\right] \\ = \quad & \log(\beta) + \log(\delta) + \mathbb{E}^{t_{n-1},x,x^*}\left[\log(X^*_{t_n}) + \log(X^*_{t_n} - x^*)\right]. \end{aligned} \tag{5.52}$$

The structure of (5.52) is that of the pure terminal high-water mark problem from section 5.6 plus some constant and an additional term rewarding the increase of the high-water mark. For well-posedness we require $0 < \beta$ and $0 < \delta$, but as encountered in (5.50) there is no constraint to prevent $X^*_{t_n}(\omega) - x^*$ from vanishing for some $\omega \in \Omega$. We keep this in mind, too.

In BC Mixing an intermediate share of wealth with a terminal share of the high-water mark yields

$$\begin{aligned} & \mathbb{E}^{t_{n-1},x,x^*}\left[\log(\beta X^*_{t_n}) + \log(\gamma h X_{t_n-})\right] \\ = \quad & \log(\beta) + \log(\gamma h) + \mathbb{E}^{t_{n-1},x,x^*}\left[\log(X^*_{t_n}) + \log(X_{t_n-})\right]. \end{aligned} \tag{5.53}$$

Given $0 < \beta$ and as well $0 < \gamma h$ this target is similar to the combined target of section 5.8 where here the utility is individually applied to each process. We cannot yet tell which strategy is optimal.

Flipping the effects in AD we have

$$\begin{aligned} & \mathbb{E}^{t_{n-1},x,x^*}\left[\log(\alpha X_{t_n-} - \alpha\delta(X^*_{t_n} - x^*)) + \log(\delta(X^*_{t_n} - x^*))\right] \qquad (5.54) \\ = \quad & \log(\alpha) + \log(\delta) + \mathbb{E}^{t_{n-1},x,x^*}\left[\log(X_{t_n-} - \delta(X^*_{t_n} - x^*)) + \log(X^*_{t_n} - x^*)\right] \end{aligned}$$

where now $0 < \alpha$ and $0 < \delta$ is required. We already know that $X^*_{t_n} - x^*$ needs to be taken care of later. New is the term $X_{t_n-} - \delta(X^*_{t_n} - x^*)$. For $x = 0$ this is non-positive while for $x > 0$ also $X_{t_n-} > 0$ as the intermediate wage has not yet been deduced. If in the latter case x^* cannot be improved from x by any admissible strategy, nothing is subtracted from X_{t_n-} and the term is positive. There are two situations in which x^* can be beaten: If stock prices go up and $\pi > \hat{\pi}(z)$, we have

$$X_{t_n-} - \delta(X^*_{t_n} - x^*) = \hat{x}(x,\pi) - \delta(\hat{x}(x,\pi) - x^*) > (1-\delta)\hat{x}(x,\pi)$$

and it suffices to ensure $\delta < 1$ to provide the latter being positive. If stock prices go down and $\pi < \mathring{\pi}(z)$ it holds

$$X_{t_n-} - \delta(X^*_{t_n} - x^*) = \mathring{x}(x,\pi) - \delta(\mathring{x}(x,\pi) - x^*) > (1-\delta)\mathring{x}(x,\pi)$$

and again $\delta < 1$ provides positivity.

Let us collect all constraints necessary to ensure only positive arguments are passed on to the logarithm:

- $x > 0$, i.e. the previously paid intermediate wages are required to pay strictly less than the actual wealth.
- If present in the problem under consideration, each of α, β, γ, δ needs to be positive. This is a sensible property as one would expect a share that is contracted to be paid to the agent to be positive.

- $\gamma h < 1$ is a weak constraint as due to numerical reasons (c.f. discussion in the first benchmark of section 5.4) usually very small grid widths h are chosen. Economically it means that the discretized time-continuous rate of wealth paid to the agent needs to be less than 100% of the wealth.

- $\delta < 1$ binds the rate of the high-water mark share somewhere below 100%. Note that both this is and the previous condition are already posed by requiring $x > 0$.

- The last constraint remaining is $X^*_{t_n} > x^*$. The only way to guarantee this property is to restrict the agent to strategies that ensure an increasing wealth for each step from $k = 0$ to $k = n$, i.e. to $\pi \in [0, \mathring{\pi})$.

It is sensible and has economical meaning to assume $x > 0$ and $\alpha, \beta, \gamma h, \delta \in (0, 1)$. But constraining the admissible strategies as necessary for the last property is rather strict and severely narrows the class of portfolio problems under examination.

Shifting the Agent's Domain

Let us discuss another approach: Choose a constant $C > 0$ and consider the shifted logarithm $y \mapsto \log(y + C)$ as the agent's utility function. The intention behind is that a small shift C suffices to guarantee well-posedness while proposition 5.5 provides the comparability of optimal strategies for shifted log utility with those for plain log portfolio problems, at least for small enough shifts C and large enough wealths x.

Let us revisit the problems (5.49) and (5.50) again for the shifted log utility: Paying a share of intermediate wealth only, the target is

$$\mathbb{E}^{t_{n-1},x,x^*}\left[\log(C + \gamma h X_{t_n-})\right]. \tag{5.55}$$

For $\lambda := \gamma h$ this is exactly the second benchmark discussed in part b) of proposition 5.5. The incentives are, given x is not too close to zero, just about the same as when rewarding terminal wealth. In particular the optimal strategy is intrinsically bounded.

When intermediately rewarding the high-water mark only as in (5.50), the target now reads

$$\mathbb{E}^{t_{n-1},x,x^*}\left[\log(C + \delta(X^*_{t_n} - x^*))\right]. \tag{5.56}$$

If for $(x, x^*) = (X_{t_{n-1}}(\omega), X^*_{t_{n-1}}(\omega))$ the high-water mark is unreachable (i.e. if $\hat{x}(x, \pi) < x^*$ for all $\pi \in [0, 1]$) it holds $X^*_{t_n} = x^*$ and (5.56) simplifies to $\log(C)$. Any choice of $\pi \in [0, 1]$ is optimal here.

If x^* is beatable, but only when stock prices go up (i.e. for $1 + r \le z \le u$), we have

$$\mathbb{E}^{t_{n-1},x,x^*}\left[\log(C + \delta(X^*_{t_n} - x^*))\right] = p\log(C + \delta(\max\{\hat{x}(x,\pi), x^*\} - x^*)) + (1-p)\log(C).$$

For strategies $\pi < \hat{\pi}(z)$ this is just $\log(C)$ while for riskier $\pi \ge \hat{\pi}(z)$

$$p\log(C + \delta(\hat{x}(x,\pi) - x^*)) + (1-p)\log(C) > \log(C)$$

is yielded. As $\hat{x}(x, \pi)$ increases in x, the optimal solution is the upper no-shortselling boundary $\pi^* = 1$.

The last situation with an immediately obvious solution is $x = x^*$ (i.e. $z = 1$) when $p \ge q$. In that case rewrite (5.56) as

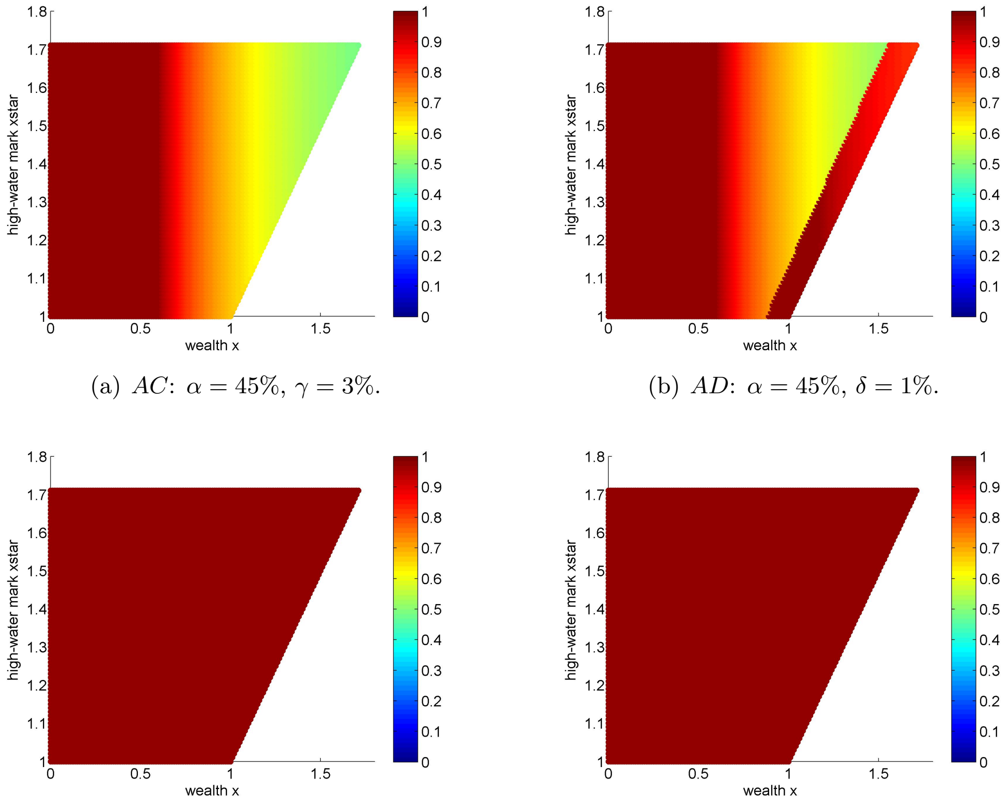
$$\mathbb{E}^{t_{n-1},x,x^*}\left[\log(C + \delta(X^*_{t_n} - x^*))\right] = \mathbb{E}^{t_{n-1},x,x^*}\left[\tilde{U}(X^*_{t_n})\right]$$

with $\tilde{U}(y) := \log((C - \delta x^*) + \delta y)$ and observe that $\tilde{U} : [x^*, \infty) \to \mathbb{R}$ is a utility function in the sense of definition 2.1. As $z = 1$ and $p \geq q$ one can work along the lines of the proof of theorem 5.6 to derive the optimal strategy $\pi^* = 1$.

These brief arguments do not cover all cases (namely $1 < z \leq 1 + r$ is missing) but we can conclude that the proposed wage schedule still incites the agent to take deliberately high risks in many[8] situations. We will therefore focus on combinations of intermediate and terminal wages.

Crossed Combinations for Agents with Shifted Logarithmic Utility

With the modification of the utility all combinations AC, AD, BC, and BD of one intermediate with one terminal effect are well-defined. Figure 5.23 presents the agent's optimal strategy in the last step for each of the above:



(a) AC: $\alpha = 45\%$, $\gamma = 3\%$.

(b) AD: $\alpha = 45\%$, $\delta = 1\%$.

(c) BC: $\beta = 5\%$, $\gamma = 3\%$.

(d) BD: $\beta = 5\%$, $\delta = 1\%$.

Figure 5.23: Optimal strategy in step $k = 4$ of $n = 5$ for $U = y \mapsto \log(1 + y)$ and all 4 combinations of one terminal share (α for the wealth or β for the high-water mark) and one intermediate share (γ for the wealth or δ for the high-water mark). The 2 shares stated in the descriptions of the individual subfigures are paid, the remaining 2 shares are omitted from the problem formulation. The market is parametrized as in (5.30).

Part a) illustrates the solution for the problem AC in which both an intermediate and

[8] More precisely: In an uncountable two-dimensional subset $A \subset \mathcal{Y}_{n-1}$ with non-vanishing area.

a terminal share of wealth are paid to the agent. The target here is

$$\mathbb{E}^{t_{n-1},x,x^*}\left[\log(C+\alpha(1-\gamma h)X_{t_n-})+\log(C+\gamma h X_{t_n-})\right]. \tag{5.57}$$

Both rewards in (5.57) apply to the same process. Hence it seems reasonable that the agent's optimal behavior apparently coincides with the solution for the second benchmark in section 5.4 as stated in proposition 5.5 where a terminal share of wealth is paid.

For the problem AD examined in part b) the target reads

$$\mathbb{E}^{t_{n-1},x,x^*}\left[\log(C+\alpha X_{t_n-}-\alpha\delta(X^*_{t_n}-x^*))+\log(C+\delta(X^*_{t_n}-x^*))\right]. \tag{5.58}$$

In states (x,x^*) with $\hat{x}(x,\pi)<x^*$ the current high-water mark is unbeatable and (5.58) simplifies to

$$\log(C)+\mathbb{E}^{t_{n-1},x,x^*}\left[\log(C+\alpha X_{t_n-})\right].$$

This target is structurally identical with (5.55) where the affine translation of wealth now reads $x\mapsto C+\alpha x$. Consequently the optimal strategy in that region (everything in the plane except for the red stripe on the right border) is $\pi^*_{n-1}(x)$ from (5.33) with $\lambda:=\alpha$.

If the current record can be beaten, the agent is willing to take a higher risk, very well recognizable in the red stripe on the red edge. The combination with a share of the terminal wealths affects the risk-taking: The higher the wealth the more the behavior is influenced by the more cautious strategy of the investor with terminal wealth only, ushering more caution. The optimal strategy is no longer unbounded as it is in the pure terminal and pure intermediate high-water mark schedules in $\left(P^{[0,1],1}_{0,\beta,U}\right)$ and (5.56).

The situation covered by part c) seems to globally incite to choose the largest admissible strategy. A small share of the running wealth does in this favorable market not prevent the agent from taking high risks to improve her terminal share in the high-water mark.

That conclusion is not intrinsic to the problem structure: By picking very large shares γ of the running wealth (for this market setting at least 2000%) one can artificially enforce a behavior similar to the results of part b). Simulations suggest that in markets with $p>q$ the risk-favoring incentives of the terminal high-water mark outweigh the moderating effects of the intermediate share of wealth for realistic choices of γ.

Combining terminal and intermediate participation in the high-water mark as in part d) amplifies the risk-taking incentives that we already know from theorem 5.6 for terminal and (5.56) for intermediate wages: If the current high-water mark is unreachable in (x,x^*), the target reads

$$\mathbb{E}^{t_{n-1},x,x^*}\left[\log(C+\beta X^*_{t_n})\right]+\log(C). \tag{5.59}$$

This corresponds to the particular situation discussed in (5.56) in which the current high-water mark was reachable (only) for rising stock prices. We have seen that the agent then will choose to invest as much as she is admitted in the stock and that is also the optimal behavior for (5.59).

If x^* is indeed beatable the target is incremented by the intermediate reward to

$$\mathbb{E}^{t_{n-1},x,x^*}\left[\log(C+\beta X^*_{t_n})+\log(C+\delta(X^*_{t_n}-x^*))\right]. \tag{5.60}$$

It seems reasonable that both effects in (5.60) further support higher risk-taking as suggested in the simulation.

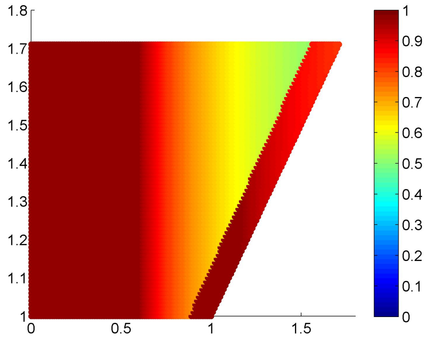
Having now had a first look into all four cross-combined wage schedules, the only one with a fresh and promising perspective is the combination AD of a running share in the high-water mark, complemented with another share of terminal wealth.

A Closer Examination of AD Type Schedules

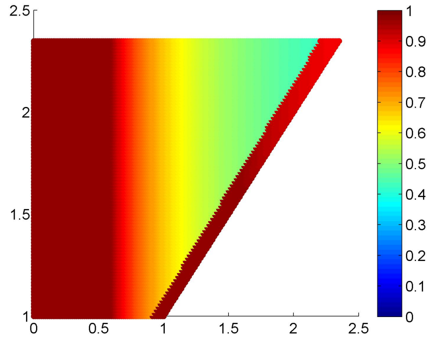
So let us focus on schedules of AD type and exploit their properties and incentives in more detail.

An obvious point of interest is how the riskier behavior on the right edge depends on the discreteness of the model – what happens when with $n \to \infty$ the approximation of the time-continuous portfolio problem with finite horizon improves?

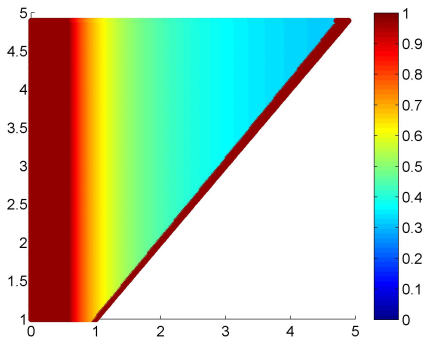
Figure 5.24 collects the optimal strategies for the last step in the scenario AD for increasing values of n:



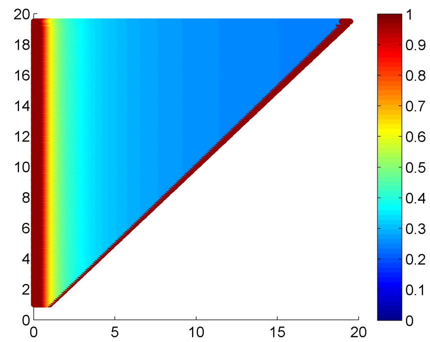
(a) $n = 5$.



(b) $n = 10$.



(c) $n = 30$.



(d) $n = 100$.

Figure 5.24: Optimal strategy in step $k = n - 1$ for $U = y \mapsto \log(1 + y)$ in scenario AD. The market is parametrized as in (5.30) and the degree n of approximation to the time-continuous model is specified in each subfigure.

Subfigure a) is identical with part b) of figure 5.23. When now with n increasing the approximation to the time-continuous model improves, one can observe various phenomena:

- With n increasing, $d = d_n$ and $u = u_n$ converge to 1 (c.f. market dynamics as specified in section 5.2). Nevertheless the upper right corner of $\mathcal{Y}_{n-1}$ is $(x_0 u^{n-1}, x_0 u^{n-1})$ and diverges to (∞, ∞) (c.f. part b) of figure 5.6 and figure 5.7), thus the depicted sets grow.
- Along with n the potential values of wealth x increase. For the region in which x^* is unbeatable the convergence of the optimal strategy π^* from (5.31) to the discretized Merton ratio (for $x \to \infty$) is clearly visible as the growing blue area.
- The latter region borders the red stripe at the right edge. The latter corresponds to the set of states (x, x^*) in which x^* can be improved in the remaining time-step. As long as $\pi^*_{\text{Merton}} < 1$ and $p > q$ the strategy jumps when crossing from one region to the other.

The red stripe gets thinner (is dilated with a scaling $\lambda \in (0, 1)$ in x-direction) when examining subplots a), b), c), and d) in sequence. But as (for obvious reasons) the scaling is different in each plot it is not immediately evident what will happen to it in the long run. Figure 5.25 discusses that effects on a common scale:

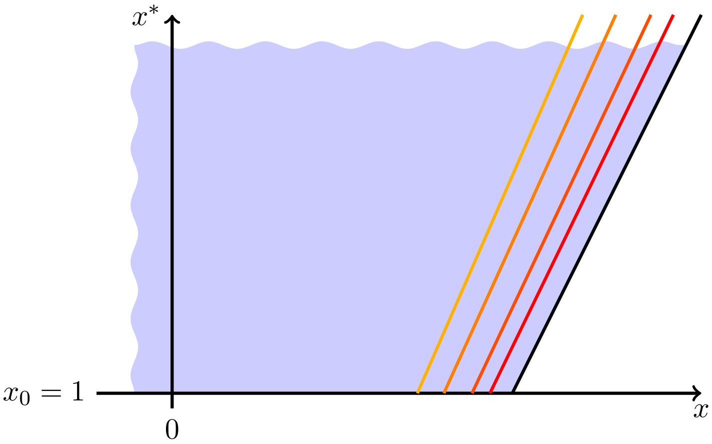


Figure 5.25: Left borders of the regions in which the current high-water mark can be improved in the remaining time-step. For $n = 5$ (yellow), $n = 10$ (lighter orange), $n = 30$ (darker orange), and $n = 100$ (red) the left border intersects $\{x^* = x_0\}$ at $(x_0 d_n, x_0)$ and has slope u_n. For n increasing the intersection point approaches the lower right corner (x_0, x_0) and the slope decreases to $\lim_{n\to\infty} u_n = 1$.

Indeed as $[x_0 d, x_0]$ vanishes with $n \to \infty$, the right region gets arbitrarily thin. But it is also stretched (is dilated with a scaling $\mu > 1$ in x^*-direction) as $[x_0, x_0 u_n^{n-1}]$ grows in $n \to \infty$. It holds that, due to the rate of stretching outweighing the rate of thinning, its area diverges. But the fraction of states belonging to that region is proportional to $(1 - d_n)$ and hence converges to zero.

Collecting all states of which we can assure the optimal strategy to be bounded in

$$\mathcal{Z}_n := \left\{ (x, x^*) \in \mathcal{Y}_n : \ \pi^{C,\alpha,*}_{\text{Merton}} < 1, \ \hat{x}\left(x, \pi^{C,\alpha,*}_{\text{Merton}}\right) < x^* \right\}$$

we can with the above considerations assert

$$\lim_{n\to\infty} \frac{\text{area}(\mathcal{Z}_n)}{\text{area}(\mathcal{Y}_n)} = 1.$$

Conclusion 5.14 For $n \to \infty$ the agent is intrinsically motivated to limit the fraction of wealth invested in the stock to the (capped) discretized Merton ratio. This holds for all states (x, x^*) that do neither belong to the (vanishing) region where the high-water mark can be beaten in the remaining step nor to the (vanishing) neighborhood of $x = 0$.$\square$

Panageas & Westerfield (2009) argue that due to the agent's rising impatience when approaching termination time (c.f. theorem 5.2) the fraction of wealth she puts in stock diverges to infinity. In their setup time is continuous, termination is random with expectation T and vanishing variance, and the risk-neutral agent is continuously rewarded with a share of the high-water mark which she discounts.

We have come to a different conclusion. Which variations between our model and theirs facilitate the difference? – With $n \to \infty$ the discrete-time model considered here approximates the proper Black-Scholes market of Panageas & Westerfield (2009). Also their random Erlang termination time τ_n converges to a Dirac delta random variable with expectation T when (their) $n \to \infty$. As our model guarantees a deterministic finite time-horizon, discounting intermediate wages is not required. One difference remaining is that our agent attributes non-constant relative risk aversion that approaches 1 for increasing wealth while Panageas & Westerfield (2009) consider a risk-neutral investor. Another, most likely crucial, difference is the additional share of terminal wealth that is offered to the agent.

The conjecture of the additional terminal share of wealth being responsible for the change in risk-behavior is further supported by the examination of the setup D in (5.56): That setup is the one resembling the approach of Panageas & Westerfield (2009) the closest. In D an agent acts optimally (at least in uncountably many situations that represent a non-zero percentage of all admitted states) when placing as much wealth in stock as admitted. And the only property discriminating between D and AD is the additional terminal share of wealth.

As we now find the last step of the portfolio problem AD with intermediate high-water mark and terminal wealth promising, let us examine the optimal strategy at the earlier steps $k = 0, \ldots, n-2$. Figure 5.26 collects the findings:

From $k = 4$ to $k = 3$ the polygon $\mathcal{Y}_k$ shrinks. Without analytical solutions at hand we can only offer conjectures about how the optimal strategy evolves over time. But up to numerical blurs the optimal strategies in the common part of the region where the current high-water mark cannot be improved in the remaining steps seems to coincide.

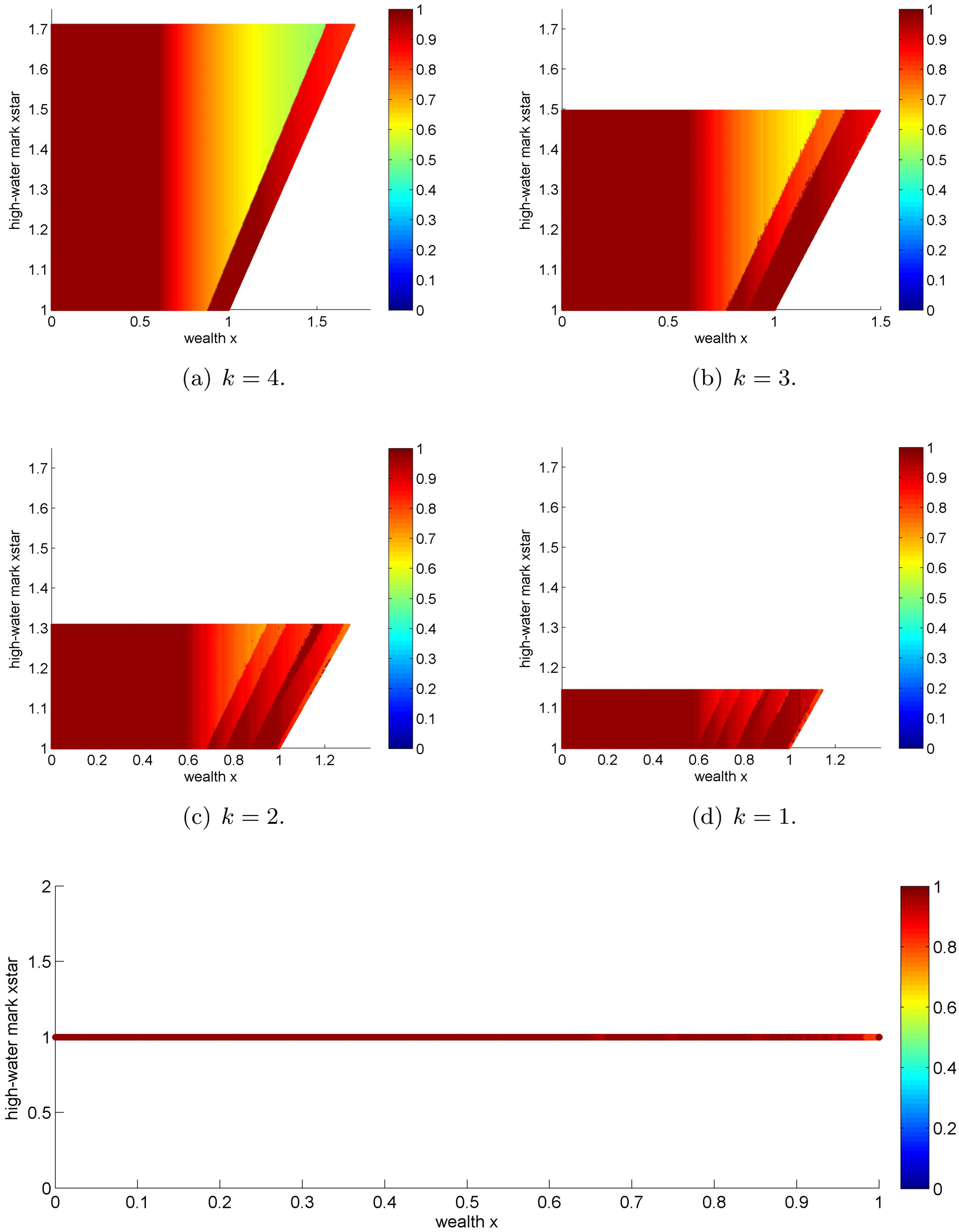
Over the set of states where x^* can be reached within another one step the strategies are very similar and as observed for $k = 4$ suggest a riskier attitude towards risk.

New at step $k = 3$ is a second stripe, located immediately left of the latter set of states, that corresponds to those tuples (x, x^*) for which x^* can be improved by two steps of trading but cannot be reached in just one.

For increasing wealth x the optimal strategy initially decreases in the region where x^* cannot be reached at all, then jumps to a higher value when entering the set where two steps of trading suffice to improve x^* and decreases again. Where x^* can be improved by just one trade π^* jumps again to a higher value and finally decreases till the right border of admissible states is reached.

When going back in time another step a third stripe, corresponding to those states that allow for an improvement of x^* in three steps, appears and all effects described above can be discovered, too.

The principle seems to continue for k further decreasing where the shrinking sets $\mathcal{Y}_k$ and self-amplifying numerical issues make them harder to verify.



(a) $k = 4$.

(b) $k = 3$.

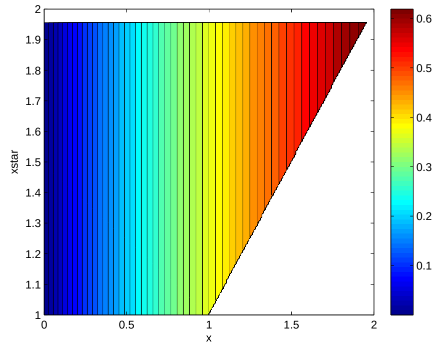
(c) $k = 2$.

(d) $k = 1$.

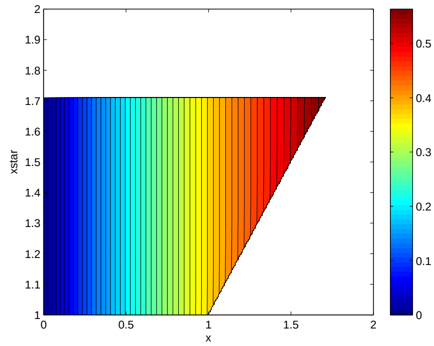
(e) $k = 0$.

Figure 5.26: Optimal strategy at steps $k = 4, 3, 2, 1$, and 0 for the problem AD. Market, approximation and wage parameters are as in part b) of figure 5.23.

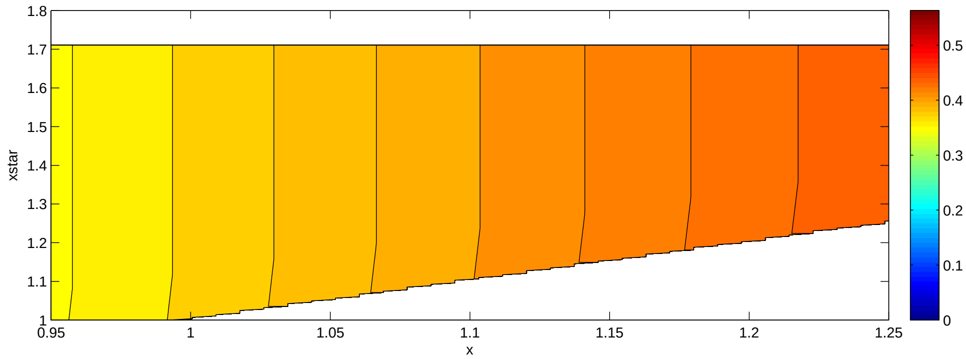
Figure 5.27 presents the corresponding value functions:



(a) $k = 5$.



(b) $k = 4$.



(c) $k = 4$, clipped and zoomed in.

Figure 5.27: Value functions for selected steps of problem AD with all parameters as in part b) of figure 5.23.

Subfigure a) states the terminal value $J_5(x, x^*) = \log(C + \alpha x)$ which is the boundary condition. Obviously, J_5 does not depend on the terminal high-water mark and all contour lines run vertically and are log-spaced.

As we know from figure 5.24 and the subsequent discussion, the optimal strategy at step $k = 4$ exhibits a jump where the regions with and without prospects of improving the current high-water mark meet. Accordingly, close to the border $\{z = u\}$ the value at step $k = 4$ (continuously but along that subspace non-differentiably) changes slope. The effect manifests in the contour lines of subfigure b) as bends to the right (when walking the lines downwards).

As the contours already run vertically that bend is hard to spot. Subfigure c) offers a zoomed view on a sector the same value function where the bends are easier to recognize.

The additional jumps in the strategies for earlier steps as discussed in figure 5.26 lead to additional bends in the value functions at those steps. As those additional jumps are of lower height and located closely together the effects are hardly perceptible to the eye and the corresponding plots have been omitted.

6 Conclusion

We have examined the implications of different compensation schemes on the agent's risk behavior.

In chapters 3 and 4 we approached the question by offering the agent various types of bonus contracts that set different rewards for different levels of outcome.

Contracts of goal-reaching type are typically discontinuous at a finite number of wealth levels. They yield discontinuous boundary conditions when approaching them with HJB. We have sketched in section 3.2 how one can verify solution candidates under these circumstances. In section 3.4 we have constructed a sequence of continuous wage functions to point-wise approximate the basic bonus contract of Browne (1999).

In section 3.5 we have replaced the discontinuous wage with a continuous contract that, interpreted as a SAHARA utility function, exhibits non-monotone absolute risk aversion. We found that a SAHARA agent will, in contrast to an agent with a simple bonus contract, always choose to put a finite amount of wealth in stock, even when termination approaches. And for increasingly large wealths the agent places the same finite fraction of wealth in the stock as an investor with power type utility.

For arbitrary contracts we have in chapter 4 employed a Markov chain approximation to solve the associated terminal portfolio problems numerically. The 2-step scheme suggested in section 4.3 provides a locally consistent set of transition probabilities that incorporate the next 2 neighbors in each direction. We have applied it to Black-Scholes markets and in addition explained how to extend that scheme to any desired neighborhood and to arbitrarily spaced grids.

In chapter 5 we have introduced the high-water mark as running maximum of portfolio wealth and considered various contracts incorporating that new variable.

To examine the agent's behavior in multi-period models we have in section 5.3 introduced a computational method for solving portfolio problems with high-water marks. In section 5.6 we employed it to conjecture the global unboundedness of a risk-neutral agent's optimal strategy in favorable markets. For combined shares of terminal wealth and high-water mark we could in section 5.8 discuss a log agent's optimal strategy and value and interpret the findings economically.

In one-period models we could establish the presence of an additional share in terminal wealth as decisive property for the agent's risk attitude: If only the terminal high-water mark is rewarded, an agent with arbitrary utility will in favorable markets always place a fraction of wealth in the stock that is as large as she is admitted to (c.f. theorem 5.6 and conclusion 5.7). If on the other hand a share of terminal wealth is added to the compensation, she is intrinsically motivated not to exceed a finite fraction of wealth (c.f. theorem 5.10 and conclusion 5.11).

When in the case of pure high-water mark wages the agent's strategy needed to be externally bounded, we discussed the principal's best choice for that boundary (c.f. theorem 5.8). Assuming a participation constraint for the agent and a risk constraint for the principal an economical balance between both parties can be achieved as long as the individual constraints do not conflict (c.f. conclusion 5.9).

For combined wages in one period we could establish the convergence of a log agent's best strategy and value to the known solution of the pure wealth benchmark if the share of wealth dominates the share of high-water mark (c.f. theorems 5.12 and 5.13).

For continuous time and a random finite time horizon, Panageas & Westerfield (2009) state the optimal strategy for an agent that continuously collects a discounted share of the high-water mark. They show (c.f. theorem 5.2) that their agent in the limit accepts arbitrarily high risks when a deterministic finite time horizon is stochastically approximated and termination approaches.

We could confirm this finding for an agent with shifted log utility who is, in discrete time, intermediately rewarded by shares of the high-water mark and faces a deterministic finite time horizon.

When replacing the share in the high-water mark with a share of running wealth, the optimal strategy is always bounded. But the high-water mark does not by itself incite unboundedness:

One can also arrive at an (in the limit) bounded strategy when adding a share of terminal wealth to the running share in the high-water mark. Table 6.1 compiles the incentives at the last step prior to termination when cross-combining one intermediately paid share with one terminal share:

Table 6.1: Boundedness of the agent's optimal strategy for all cross-combinations of one intermediate and one terminal share.

	A	B
C	globally bounded and approximately like in the log of terminal wealth case	globally unbounded for realistic market parameters
D	bounded for most states; unbounded when x^* is unreachable and x small; possibly unbounded when x^* is reachable	globally unbounded

In setup AD the agent is intrinsically motivated to invest no more than a finite fraction of her wealth in stock for states $(x, x^*) \in \mathcal{Z}_n$. $\mathcal{Z}_n$ in the limit dominates the admissible states space $\mathcal{Y}_n$ (c.f. conclusion 5.14).

Hence, whether or not the agent is incited to take arbitrarily high risks in markets with high-water marks is not only a matter of picking the right compensation scheme(s), but also depends on the time horizon of the contract.

Recapitulating all compensation schemes we have examined, we can finally give some advise on what contracts are suitable to transfer the principal's attitude towards risk to the agent:

The principal should take care to choose only such compensation schemes that ascertain the agent's strategy is bounded for all or, at least, for most states.

Simple goal-reaching contracts induce unbounded behavior when the goal cannot yet be reached and termination approaches. Thus, they do not qualify for a considerate principal. Contracting to a SAHARA wage with threshold level matching the former goal relaxes the situation. It provides an agent who will always choose to invest finite amounts of money in stock and hence is the more advisable contract.

Taking high-water marks into account, one may consider combining a terminal share of the high-water mark with another share in terminal wealth. This setup ensures bounded optimal strategies regardless of the agent's utility. When an intermediate share of the high-water mark is aspired, an overreaction when approaching the time horizon can be prevented in most situations by adding a share of terminal wealth.

SAHARA contracts imply that the amount of wealth in stock symmetrically (and approximately linearly) increases with the distance from the threshold. They are, in a sense, constructed around that target level. High-water mark contracts on the other hand do not exhibit such a prominent turning point. In any situation the incentives depend on if and how much the current high-water mark may be improved in the remaining trading steps. The closer wealth is to the current record, the more the agent is motivated to take risks.

It is now up to the principal to select that contract which better matches his own conception of risk.

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