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Geometrically Exact Theory of Contact Interactions – Further Developments and Achievements

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ABSTRACT

The focus of the current contribution is on the development of the unified geometrical formulation of contact algorithms in a covariant form for various geometrical situations of contacting bodies leading to contact pairs: surface-to-surface, line-to-surface, point-to-surface, line-to-line, point-to-line, point-to-point. The construction of the corresponding computational contact algorithms are considered in accordance with the geometry of contact bodies in a covariant form. These forms can be easily discredited within finite element methods independently of order of approximation and, therefore, the result is straightforwardly applied within iso-geometric finite element methods. This approach is recently became known as geometrically exact theory of contact interaction [10]. Application for contact between bodies with iso- and anisotropic surface, for contact between cables and curvilinear beams as well as recent development for contact between cables and bodies is straightforward. Recent developments include the improvement of the curve-to-surface (deformable) contact algorithm.

Keywords: Contact Finite Element; Surface-to-Surface Contact; Beam-to-Beam Contact; Curve-to-Surface Contact; Covariant Approach; Geometrically Exact Contact Description

1. Introduction

Computational contact mechanics has become a separated branch of computational mechanics during the last decades. A fairly large number of publications including several monographs on computational contact mechanics have been devoted to this development. Modelling of contact interactions became fairly standard in numerous finite element software packages available for engineers. Various aspects of the numerical solution such enforcement of contact conditions, possibility to apply high order and iso-geometric type of approximation has been considered. One of the important aspects, even though being obvious for everyone, – the geometrical treatment of the contact – is often remains hidden inside the computational algorithm. Contact interaction from a geometrical point of view can be seen as interaction between deformable surfaces possessing various geometrical features such as surfaces, edges and vertexes, therefore, geometrical approaches can be exploited. During the last ten years these approaches has formed a basis of the geometrically exact theory of contact interaction, recently published in monograph of Konyukhov and Schweizerhof [10] by Springer.

Current contribution is aimed on the overview of this theory with concentration on recent developments.

2. Geometrical Approaches in Computational Contact Mechanics

Only a very few publications are devoted to geometrical issues of contact interaction aiming at the final computational models. Gurtin, Wiessmueller and Larche [2] (1998) considered surface tractions on curvilinear interfaces describing them from a geometrical point of view. Jones and Papadopoulos [5] (2006) considered contact describing various mappings from the reference configuration employing the Lie derivative. Laursen and Simo [12] (1993) described some contact parameters via geometrical surface parameters.Heegaard and Curnier [3] (1996) considered geometrical properties of slip operators.

Consistent Linearization

The iterative solution of Newton type is a standard way to obtain the solution in the computational contact mechanics. However, one of the difficult parts is to obtain the full derivative of the functional which is necessary for the fast Newton solver – this procedure is known as linearization. Two approaches for linearization of the final functional representing the work of contact tractions can be distinguished in order to obtain consistent tangent matrices. *The direct approach* follows the following sequence: *functional – discretization – linearization* and *the covariant approach* follows the rule: *functional – linearization – discretization*.*The direct approach*, historically motivated by the development of the finite element method, assumes that the discretization is then involved in the process and the linearization is provided with regard to the displacement vector u and, therefore, of the discretized system. This leads to the final results containing a set of approximation matrices: for surface to surface contact it is described in Wriggers and Simo [16] (1985), Parisch and Luebbing [14](1997), Peric and Owen [15] (1992), Laursen and Simo[12] (1993), for anisotropic friction in Alart and Heege [1] (1995), for beam-to-beam type contact in Zavarise and Wriggers [17](2000), Litewka and Wriggers [13] (2002).The complexity in the derivation for curved contact interfaces led to the use ofa code containing an automatic derivation with mathematical software, see Heegeand Alart [4] (1996), Krstulovic-Opara,Wriggers and Korelc [11] (2002) and other researchers.

Open questions and drawbacks of *the direct approach* can be summarized as follows:

- A closed form for tangent matrices is available only for linear approximations of surfaces.
- The structure of the derived matrices is very complicated and often not transparent. There is no clear interpretation of each part possible.
- A specification of complex contact interface laws with properties explicitly depending on the surface geometry (e.g. **arbitrary** anisotropy) is not possible.
- A contact description of many geometrical features (curved line-to-curved line, curved line-to-surface) is almost not possible because of the necessity of convective surface coordinates.

The fully covariant approach, however, assumes only a local coordinate system associated with the deformed continuum (convective coordinates) and requires extensive application of covariant operations (derivatives etc.). This approach historically appeared with the consideration of convective variables arising from the surface approximations directly for contact traction and displacements: see Simoand Laursen and Simo [12] (1993). Two convective variables x^1, x^2 in a surface covariant basis are used as tangential measure.

This approach has many advantages:

- objectivity is straightforwardly observed because the surface coordinates are used;
- geometrical interpretation of a measure – line on a surface; geometrical interpretation of a line arized measure – relative tangent velocity of a contact point;
- the number of history variables is minimal (two for surface interaction);
- A complex constitutive law for tangent interaction can be easily formulated in a robust form for computation.
- Expressions for contact tangent matrices are by far less complex within the fully covariant approach than for direct approach.

A fully covariant approach, though, is intended for the finite element method, but does not assume approximations from the beginning and it serves to describe all necessary for solution parameters based on the geometry of the contacting bodies in the local coordinate system. The method, however, requires a lot of preliminary transformations based on differential geometry of contacting objects (surfaces or even curves) and extensive application of the tensor analysis especially for differential operation and linearization.

3. Development of the Geometrically Exact Theory

The development of the ideas started from formulation of contact algorithms in covariant form for non-frictional contact in [6], then for frictional contact in [7]. The basement of the geometrically exact theory of contact interaction with more references is summarized in monograph of Konyukhov and Schweizerhof [10]. In order to formulate goals and describe the development of geometrically exact theory we consider a model contact problem with two bodies possessing smooth surfaces as well as various geometrical features such as edges and vertexes – an example of this is a banana and a knife shown in **Figure 1**. Considering all possible geometrical situations in which knife and banana can contact each other, the following hierarchical **sequence of contact pairs** is appearing:

1. Point to point contact pair
2. Point to curve contact pair
3. Point to surface contact pair
4. Curve to curve contact pair
5. Curve to surface contact pair
6. Surface to surface contact pair

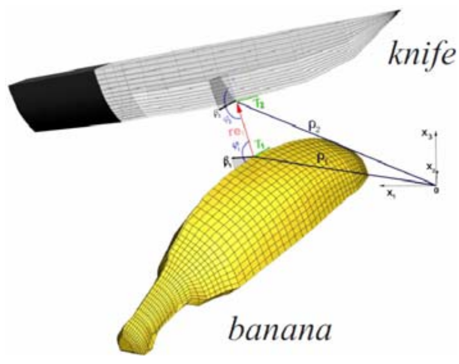

Figure 1. Contact between bodies with complex geometry. Various geometrical situations are possible: Surface-To-Surface, Curve-To-Surface, Point-To-Curve, Curve-To-Curve and Point-To-Point.

3.1. Construction of Kinematics and Numerical Algorithms for Various Contact Pairs

The following open problems are stated as goals for the geometrically exact theory

- Development of the unified geometrical formulation of contact conditions in covariant form for various geometrical situations of contacting bodies leading to contact pairs: surface-to-surface, curve-to-surface, point-to-surface, curve-to-curve, point-to-curve, point-to-point (joint). The description will be fully based on the differential geometry of specific features forming a continuum, because it is carried out in the local coordinate systems attached to this feature: this is the Gaussian surface coordinate system in the case of surface; the Serret-Frenet basis in the case of a curved line; the coordinate system standard for rigid body rotation problem (e.g. via the Euler angles) the case of a point in. **This general description is forming a geometrically exact theory for contact interaction.**
- A full set of contact pairs requires various closest point projection (CPP) procedures. Thus, fundamental problems of existence and uniqueness of closest point projection routines corresponding to the following situations are investigated: point-to-surface, point-to-line, line- to-line.
- A solution of existence and uniqueness problems of closest point projection routines leads to "projection domains" as the "maximal searching domains".
- Derivation of a unified covariant description of various applicable methods to enforce contact conditions: Lagrange multipliers methods, penalty methods, augmented Lagrange multipliers method. Consistent tangent matrices are given in closed covariant form possessing a clear geometrical structure.
- Description of all geometrical situations in a covariant form which is a-prior independent of approximations of these geometrical features leads to straightforward numerical algorithms for the implementation with any order of approximation for finite elements including iso-geometric finite elements.
- Generalization of classical Coulomb law into a complex interface laws in covariant form for arbitrary geometry of the surfaces (e.g. coupled anisotropic friction and adhesion for surfaces).
- Development of the curve-to-curve contact model allowing considering the complete set of relative motions between curves including a rotational interaction (this is a novel in the current theory and has not been possible in earlier theories).

Though, the specific points of the proposed theory are developed through many publications, they can be summarized under the unified aim, see more detail in monograph [10]. In order to construct a numerical algorithm for a certain contact pair, first of all, it is identified that the closest distance between contacting bodies is a natural measure of the contact interaction. The procedure is introduced via the closest point projection procedure (CPP), solution of which requires the differentiability of the function representing the parameterization of the surface of the contacting body. Analysis of the solvability for the CPP procedure, see more in [8], allows then to classify all types of all possible contact pairs discussed earlier. Starting with a consideration of *C2*-continuous surfaces, the concept of *the projection domain* is introduced as a domain from which any potential contact point can be uniquely projected, and therefore, the numerical contact algorithm can be further constructed. This domain can be constructed for utmost*C1*-continuous surfaces. If the surfaces contain edges and vertex then the CPP procedure should be generalized in order to include the projection onto edges and onto vertexes.

The main idea for application for the contact is then straightforward – the CPPprocedure corresponding to a certain geometrical feature gives a rise to a special, in general, curvilinear 3D coordinate system. This coordinate system is attached to a geometrical feature and its convective coordinates are directly used for further definition of the contact measures. Thus, all contact pairs listed earlier should be described in the corresponding local coordinate system. The requirement of the existence for the generalized CPP procedure leads to the transformation rule between types of contact pairs according to which the corresponding coordinate system is taken. Thus, the all contact pairs can be uniquely described in most situations.

A surface-to-surface contact pair, see **Figure 1**, is described via the well known "master-slave" contact algorithm based on the CPP procedure onto the surface. This projection allows defining a coordinate system as follows:

$$r\left(x^1, x^2\right) = \rho\left(x^1, x^2\right) + x^3 n\left(x^1, x^2\right) \qquad (1)$$

Vector $r\left(x^1, x^2\right)$ is a vector for the "slave" point, $\rho\left(x^1, x^2\right)$ is a parameterization of the "master" surface, $n\left(x^1, x^2\right)$ is a normal vector to the surface. Equation (1) describes, in fact, a coordinate transformation in which convective coordinates x^1, x^2, x^3 are used for measure of contact interaction: the first two x^1, x^2 are measures for the tangent interaction and the third coordinate x^3 is a penetration – the measure of normal on-penetrability condition. This transformation is valid applied only if the solution of the corresponding surface CPP procedure exists. Initially, the computational algorithm is constructed for non-frictional contact interaction of smooth surfaces. Here the description starts in the coordinate system given in equation (1), however, due to the small penetration it is mostly falling into the description in the Gaussian surface coordinate system arising from the surface parameterization. All contact parameters such as sliding distance and tangent forces are described then on the tangent plane. The linearization procedure is given in a form of covariant derivatives. This leads to a closed form of the tangent matrix subdivided into a main, a rotational and a curvature parts. The evolution equation for contact tangent tractions should be taken in a form of covariant derivatives in order to solve the problem with a Coulomb friction.

3.1.1. Developments within the Theory – Possibility of the Iso-geometric Modeling

Since all algorithms are formulated in covariant and, therefore, independent on coordinate system and, of course, on types of approximations, the high-order and iso-geometric formulation is just straightforwardly applicable. Both Mortar methods with penalty regularization and with Lagrange multipliers are applied. Even the anisotropic enrichment of the approximation keeping mixed linear and high order approximation in one finite element is possible. As a result a contact layer element allowing anisotropic $p-$refinement is created. A good correlation with the analytical Hertz problem is achieved even within a single contact layer element.

3.1.2. Developments within the Theory – Possibility of Anisotropic Contact Interfaces

A systematic generalization of a contact interface law from the Coulomb friction law into the anisotropic region in a covariant form including various knownvisco-elasto-plastic mechanical models is derived. Thus, *a coupled model including anisotropy for tangential adhesion and for friction* is obtained. These models formulated via the principle of maximum dissipation in a rate form. Finally, the computational model is derived via the application of the return-mapping scheme to the incremental form. As a result a frictional force is derived in a closed form including both, the adhesion and the friction tensors. The structure of structural and friction tensors are derived for various types of anisotropy: a uniform orthotropic of a plane given by the spectral decomposition, a non-uniform orthotropic of a plane inherited with the polar coordinate system and a spiral orthotropic of a cylindrical surface. The update algorithm for history variables is developed for the arbitrary coupled anisotropy. The geometrical interpretation of the return-mapping and the update algorithm is considered via the ellipse on the tangent plane onto which a contact slave point is projected in the case of elastic sticking behaviour.

3.1.3. Curve-To-Curve Contact Algorithm

If the projection onto the surface does not exist then it is necessary to consider step-by-step the existence of lower in hierarchy CPP i.e. onto the curve and then onto the point. The solution of generalized CPP including all geometrical exists in case of regular geometry. Consideration of the existence of the CPP procedure for curve allows defining then the point-to-curve contact algorithm used for the curve-to-surface contact pair in the corresponding curve Serret-Frenet coordinate system, which is constructed as follows:

$$r(s,r,\varphi) = \rho(s) + re(s,\varphi) \tag{2}$$

Here, the vector $r(s,r,\varphi)$ is describing a "slave" point from the surface, $s$is a parameterization of the "master" curve edge; a unit vector describing the shortest distance $e(s,\varphi) = \upsilon(s)\cos(\varphi) + \beta(s)\sin(\varphi)$ is written via the unit normal $\upsilon(s)$ and bi-normal $\beta(s)$ of the curve. The convective coordinates used as measures: r – for normal interaction; s – for tangential interaction; φ – for rotational interaction. The Curve-To-Curve contact pair requires the projection on both curves, therefore, there is no classical "master" and "slave" and both curves are equivalent. For the description one of two coordinate systems is taken assigned to the I-th curve:

$$\rho_2(s_1,r,\varphi_1) = \rho_1(s_1) + re_1(s_1,\varphi_1) \tag{3}$$

Here, the vector $\rho_2(s_1,r,\varphi_1)$ is a vector describing a contact point of the second curve, s_1 is a parameterization of the first curve; a unit vector describing the shortest distance $e_1(s_1,\varphi_1)$ is written again via the unit normal and bi-normal vectors of the first curve as in equation (2). Equation (3) describes the motion of the second contact point in the coordinate system attached to the first curve. Description is symmetric with respect to the choice of the curve choice 1 to 2.

The Point-To-Point contact pair is described then in a coordinate system standard for rigid body rotation problem (e.g. via the Euler angels), however in the contact situation is very seldom case, and in computations it is rather improbable unless specially treated, and therefore, because of the numerical rounding error would fall into other contact pair types.

The construction of the curve-to-curve contact pair begins consistently with the Closest Point Projection (CPP) procedure providing a shortest distance between curves as a natural measure of normal contact interaction. The CPP procedure leads to a special local coordinate system in which convective coordinates are used directly as measures of contact interaction between curves: normal, tangential and rotational. Several achievements appear to be novel for the curve-to-curve contact description:

- consideration of any relative motion separately for each curve is possible;
- Rotational interactions including corresponding rotational moments between curves can be considered consistently.

The Coulomb friction law for tangential interaction and the Teresa friction law for rotational interaction are easily considered as examples for constitutive relations between curves. All necessary linearizations for the iterative solution scheme are provided as covariant derivation in the introduced coordinate system for arbitrary large distances between curves. This leads to a closed form of tangent matrices independent of the approximation used for the finite elements. The verification of the algorithm contains the comparison between beam-to-beam and edge-to-edge finite element models as well as verification with a famous "Equilibrium of Euler elastic problem" computed via finite difference scheme see details in [9].

4. Further Development – Curve-To-Surface Contact Algorithm

The Curve-To-Surface contact pair is constructed in a dual fashion via the Surface-To-Surface contact algorithm if we consider a "slave"point on the curve and project it onto the "master" surface, see **Figure 2**. This special dual consideration of contact both in the surface coordinate system in equation (1) and in the curve coordinate system in equation (2) allows building the Curve-To-Surface contact algorithm. In this algorithm all contact parameters are defined, first, in the local surface coordinate system equation (1) attached to the surface, after fulfilling the surface CPP procedure, and then they should be projected into the curve coordinate system equation (2), see **Figure 2**. The kinematics of the Curve-To-Surface contact interaction is formulated as follows:

- A set of contact points (integration points) is set on the curve segment: all kinematical parameters are considered then in the Serret-Frenet curve coordinate system;
- The contact point (integration points) is projected onto the surface;
- At each point all kinematical parameters are considered in the surface coordinate system.

The combination of both Curve-To-Curve and Surface-To-Surface strategies leads to the Curve-To-Surface contact algorithm which is constructed as follows. The shortest distance between integration points and the surfaces

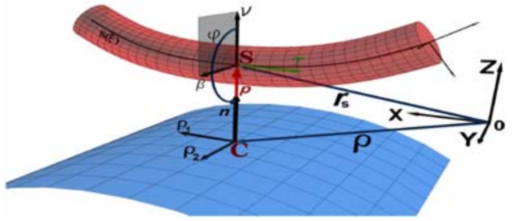

Figure 2. Both a surface coordinate system and a curve coordinate system are employed to define all characteristics of the Curve-To-Surface contact pair.

are considered as penetration. Now the Closest Point Projection (CPP) procedure as the projection onto the surface plays the main role. In general, Newton method is exploited to solve the CPP procedure defining then a point on the surface and the penetration between this surface and the selected contact (integration) point S. Kinematical relations during the contact can be obtained dually considering the relative velocity of the contact point during contact:

- normal relative velocity during contact

$$v_n = (v\_s_1 - v) \cdot n \tag{4}$$

- pulling relative velocity

$$v_p = -(\rho_i \cdot \tau) \cdot \dot{x}^i \tag{5}$$

- dragging relative velocity

$$v_d = -(\rho_i \cdot g) \cdot \dot{x}^i \tag{6}$$

The corresponding normal, pulling and dragging forces are formulated in the curve Serret-Frenet coordinate system. The result of the linearization is taken as if provided in the surface coordinate system to carry out analysis for the deformed surface parameters and as if provided in the curve Serret-Frenet coordinate system to apply for all curve parameters.

5. Conclusions

The overview of the geometrically exact theory of the contact interaction and recent development can be summarized as follows:

- Consideration of contact between bodies from geometrical point of view allows to study systematically all possible geometric contact cases: contact between surfaces, edges, beams;
- The basis of the theory is the formulation of all parameters in a local coordinate system inherited with a corresponding closest point projection (CPP) procedure;
- Surface-To-Surface contact pair is considered in the surface coordinate system of the "master" body.
- Curve-To-Curve contact pair is considered equivalently in both curveSerret-Frenet coordinate systems attached to both curves. There is no specific choice of the master and the slave in this case.
- A novel Curve-To-Surface is constructed dually in both surface and curve coordinate system. Normal, pulling and dragging velocities and corresponding forces are specified in the curve coordinate system. Linearization result should be transferred to both surface and curve coordinate system.
- All known constitutive relations (for elasticity and plasticity) can be carried into metrics giving a rise to a new contact interface laws

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