Descriptive characterisation of the variational Henstock-Kurzweil-Stieltjes integral and applications

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Diese Dissertation widme ich in liebevoller und dankbarer Erinnerung

meinem Großvater Egon Volz (1930-2012)

und meinem Großonkel Theo Schmitt (1928-2013).

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Contents

1 Introduction

1.1 Overview of this work

Without doubt, the fundamental theorem of calculus is one of the most important cornerstones in the history of modern analysis. Actually, the desire to reconstruct a function only by the information of some kind of derivative and a given initial value was an important impetus to the architects of modern integration theory - with H. Lebesgue leading the way as the title of his monography “Leçons sur l’intégration et la recherche des fonctions primitives” distinctly testifies. However, within Lebesgue’s integration theory one cannot recover every differentiable function from its derivative. So the question arises in which sense this is possible.
The first complete solution to this problem (without any integrability conditions imposed on the derivative) was given by A. Denjoy using his so-called totalization process. This approach led to the so-called Denjoy integral in the restricted sense. Loosely speaking, Denjoy constructed a transfinite sequence $(I_\xi)_{\xi\leq\Omega}$ of more and more general integrals, where I_0 is the Lebesgue integral and Ω denotes the first uncountable ordinal. Then I_Ω is the Denjoy integral in the restricted sense (for a fairly detailed account to Denjoy’s approach we refer to Chapter XVI of [Nat60]). Apparently, this approach is not that easily accessible and moreover, and due to work of R. Dougherty and A. S. Kechris (see [DK91]), we know today that the constructive flavour of Denjoy’s approach can only be purchased at the cost of this complexity, or to put it another way, this complexity is an inherent characteristic of Denjoy’s constructive solution.
Nevertheless, Lebesgue was so amazed at Denjoy’s solution of his own “problème des fonctions primitives” that Lebesgue incorporated Denjoy’s solution into the second edition of his *Leçons*.
Despite the complexity of Denjoy’s solution it is quite surprising that soon after Denjoy’s solution, N. Lusin found a strikingly simple characterisation of Denjoy integrable functions, which has by now become the usual access to this integral within modern treatises (see, e.g., [Gor94]) and which is very close to the fundamental theorem of calculus for the Lebesgue integral. It reads as follows:

A function $f : [0,1] \to \mathbb{R}$ *is Denjoy-integrable in the restricted sense if and only if there is a continuous function* $F : [0,1] \to \mathbb{R}$ *of generalized absolute continuity in the restricted sense such that* $F' = f$ *Lebesgue-a.e. on* $[0,1]$.

The notion of functions of generalized absolute continuity in the restricted sense will be explained later on in Chapter 2 and at this point we content ourselves with the hint that this notion is a reasonable generalisation of the classical notion of absolute continuity.

Another solution to the problem of recovering a function from its derivative was given by O. Perron, who used a notion of integral which is in its flavour close to Darboux’s approach to the Riemann integral (see, e.g., [Gor94] or [Sak41] for details).

It took some time, but finally around 1925 it was recognized that the Denjoy and Perron integral are equivalent. This is the so-called Hake-Alexandroff-Looman Theorem (see, e.g., Chapter VIII in [Sak41]). However, it took further more than thirty years till R. Henstock and J. Kurzweil came up independently of each other with a new notion of Integral which is based on an alteration of the classical Riemann integral as simple as ingenious: Instead of considering in the definition tagged partitions of uniform mesh, the mesh is localised and depends on the tags of the partition. The result is an integral using a formalism almost as simple as the formalism of the Riemann integral, but, as it finally turned out, with the same strength as the Denjoy integral: The Henstock-Kurzweil integral and the Denjoy integral in the restricted sense are equivalent. In particular, there is a Riemann-type integration process that integrates all derivatives and recovers the corresponding function up to an additive constant. Moreover, Lusin's theorem also characterizes the Henstock-Kurzweil integral.

Now a natural question arises: Are there similar results concerning Stieltjes-type integrals? Are there in particular results comparable to Lusin's theorem? These questions are not only interesting from a purely theoretical point of view, but also with respect to potential applications in complex analysis (we will come back to this issue later on). Explorations of this kind date back to the early days of modern integration theory where initially functions of bounded variation serving as integrators were mainly in the focus of research (see, e.g., [Dan18, Rid36, You17]; see also [Gar92] for a more recent treatise of these questions). Even Lebesgue himself seized this question and devoted a great part of the eleventh chapter of the second edition of his *Leçons* to this topic based on a generalisation of Denjoy's totalization method.
Advanced considerations taking as their starting point either Denjoy's totalization method (see, e.g., G. Choquet's paper [Cho47]) or Perron's approach to the restricted Denjoy integral (see, e.g., J. Ridder's work [Rid38, Rid39] and A. J. Ward's paper [War36]) succeeded in overcoming the premise of bounded variation; in particular, in [Rid38, Rid39, War36] theorems are proved giving Lusin-type descriptive characterisations of the respective Perron-Stieltjes integrals defined there (see, e.g., [Rid38, Satz 20], [Rid39, Satz 6] and [War36, Theorem 7, 12, 13]).
Now it stands to reason to reflect about these questions in the Henstock-Kurzweil setting and, further proceeding, to take into consideration vector-valued functions. Indeed, there is a huge amount of works devoted to these question, see, e.g., [BP92, BGP92, BP91, Pfe83, Pfe95, Nar04, Fed99, Fau97, Sch96, SY05] and the references therein to name but only a very few. However, although the problem of giving a so-called descriptive characterisation of Lusin-type for a given kind of integral has become a recurrent object of mathematical research and a lively topic up to date in particular in connection with the Henstock-Kurzweil integral, see, e.g., [BPP95, BPT96, Pia01, Fau95, Lee05, Lee03, LL99, Ye06] to name but a few, there is surprisingly only a small number of results of this kind concerning the Henstock-Kurzweil-Stieltjes integral, see, e.g., Theorem 3.4 in [BP92], Theorem 3.2 in [BP91], Theorem 1 and 2 in [Fed99] and [Fau97]. However, apart from [Fau97] all the aforementioned works are afflicted with the same flaw: Due to the absence of an appropriate notion of differentiation with respect to a given function (one might speak of an differentiator in the style of the terminology in the context of Stieltjes integration), they invoke differentiability (in the ordinary sense) of the integrator. In particular, all these works consider integrators that satisfy one kind or another of generalized absolute continuity. In contrast to that, Faure could treat in [Fau97] continuous integrators of so-called generalized bounded variation (in the restricted sense) and moreover, in [War36] Ward was even able, using the Perron approach, to handle the discontinuous case and in [Cho47] Choquet could treat, based on the Denjoy approach, continuous functions without any condition imposed on them concerning their variational behaviour. So there seems to be some hope that one can strengthen the results for the Henstock-Kurzweil-Stieltjes integral and perhaps extend them to the vector-valued case. However, the things are more delicate. If we want to explore some kind of vector-valued bilinear integral as, e.g., proposed by Š. Schwabik in [Sch96], where both the integrand and the integrator take their values in (possibly different) Banach spaces and to derive a descriptive characterisation of this integral, we have to face the problem that none of the approaches in the existing literature seems to be well-suited in order to achieve this aim:
As we said, the works [BP92] and [BP91] use differentiability properties of functions that are in some generalized sense of absolute continuity. But this causes heavy problems if we want to proceed to the vector-valued case since even a Lipschitz function with values in an infinite-dimensional space which does not possess the so-called Radon-Nikodým property may fail to have at least a single point of differentiablility: The function

$$f : [0,1] \to L_1([0,1]);\ t \mapsto \mathbb{1}_{[0,t]}$$

is a classical text book example. For this reason in [Fed99] Federson cannot avoid to impose differentiablility assumptions on the integrator.

Ridder's and Ward's approach based on a Perron-type integration process uses minorant and majorant functions, for which there is no natural replacement if the considered functions do not take their values, e.g., in a lattice. And since their proofs heavily rely on these concepts, there is no hope that one can extend offhand these proofs to our situation. Moreover, Ward uses at several decisive points the outer Lebesgue measure of the range of a given function, which cannot be done just like that for functions with values in an infinite-dimensional space as we do not have there a canonical analogon to the outer Lebesgue measure.
A similar remark applies to Choquet's approach; indeed, in section 5.1 we will see that an assertion as general as Choquet's Théorème 10 in [Cho47] cannot hold true even if complex-valued functions are considered and that it might dramatically fail. In fact, the reason for this is that at its heart Choquet's approach is a real-valued approach as it is essentially based on the idea to consider the points in the plane with the one component arising from the integrand and the other arising from the integrator, or to have Choquet's own say: "*Le méthode que nous utiliserons repose essentiellement sur la considération de l'ensemble plan paramétré:* $x = \alpha(m)$; $y = F(m)$." (see [Cho47], page 146).
So the most promising ansatz seems at the first glance to be Faure's approach where a notion of differentiability with respect to a function is used and which makes it obsolete to consider ordinary derivatives. Unfortunately, Faure's methods are essentially *real* methods: For example, some of Faure's results essentially rest on his Proposition 3.10 in [Fau97] which also cannot have an extension to the general vector-valued situation where the considered Banach spaces are not supposed to possess the Radon-Nikodým property. Apart from this, Faure considers, as we already said, continuous integrators and uses the continuity at some decisive points (Lemma 4.2, which is used to prove the crucial Lemma 4.3), while we want to capture discontinuous integrators, too.

However, the good news is that a combination of Ward's concept of φ-continuity and Faure's ideas concerning absolute continuity finally leads to the desired Lusin-type characterisation. The main goal of the present work is to establish such a characterisation.

We now describe the content of this work in greater detail.

As in [War36] we want so-called BVG_*-functions to serve as integrators. Therefore the first section of Chapter 2 is devoted to the study of this class of functions in the vector-valued situation and we collect some basic facts, which are at least partly known in the scalar case. But nevertheless there is also a novel aspect: Our Lemma 2.4 in its present form seems to have not yet appeared explicitely in the existing literature. Indeed, an invalid version of it appears as Lemma 6.15 in [Lee89], but we do not know of any other reference. Moreover, we use Lemma 2.4 together with Lemma 2.8, an extension of a result originally due to Ward in the scalar case, as a starting point in order to systematically work out some of the most important properties of BVG_*-functions (we will come back to this issue later on).
The next section explores BVG_*-functions in connection with the so-called ACG_*-functions and examines in particular the basic properties of the latter ones. Once again some of the results are more or less known in the scalar or even in the vector-valued case, but despite this fact we provide detailed proofs of these results for several reasons, which we would like to shortly quote. First, in consideration of the sheer number of results it seems to be, at least in our opinion, all too audacious to claim that all proofs go through in the vector-valued case; in particular, in view of the fact that this is indeed not true! As an example, the proof of Lemma 2.16 as given, e.g., in [Sak41] (Lemma 8.1) or [Gor94] (Lemma 6.3) essentially rests on the assumption that the functions considered are real-valued. The same remark applies to Theorem 5.12 (cf. Theorem 6 in [Gor89]). Moreover, existing proofs as, e.g., in [Sak41] are at least in parts rather concisely written and so providing fairly detailed proofs removes doubts whether in the omitted details of the existing proof the real-valuedness is unnoticeably used. For this reason it is also no real alternative to cite [SY05], although they treat the vector-valued case, as they often just refer to the proofs of the scalar-valued case leaving it to the reader to check whether the proofs of the real-valued case are still valid in the vector-valued situation. This is finally aggravated by the fact that several proofs or statements in the existing literature are erroneous. Above we have already mentioned Lemma 6.15 in [Lee89]. In [Gor94] it is frequently

used that AC_*-functions are BV_*-functions, but this is (given the definitions in [Gor94]) not true! As a result, a whole series of statements in [Gor94] in its present form is false. This seems to have been noticed in [SY05] where the statements are more carefully formulated. However, in comparison to the results we shall present the formulations are in fact too careful in the sense that they partly use stronger assumptions than really needed. As a consequence, our results are distinctly preciser.
The last section of the second chapter is devoted to the proof of the Banach-Zarecki theorem for vector-valued functions of generalized absolute continuity. It is based on the quite recently established main result in [Dc05] and to our best knowledge it seems to be completely new. Apart from the fact that this result is of some interest in its own, we will use it later on to establish an extension of Theorem 6 in [Gor89].

Chapter 3 introduces and explores the so-called Henstock-Kurzweil variational measures. These measures are well-suited to describe the variational behaviour of a function and are deeply related to BVG_*-functions. In fact, after an excursion in the second section of Chapter 3 on the question when these measures are σ-finite (where we prove distinct extensions of existing results), we will explore this close connection in the third section and prove Thomson's characterisation of σ-finite variational measures (see, e.g., Theorem (40.1) in [Tho85], Theorem 1 in [Tho81]). Since false versions of this theorem have appeared, unfortunately, several times in the literature, it seems to be worth to provide an ab ovo proof of this result. In addition, our proof is completely elementary (in particular, we avoid the notion of local systems used in [Tho85] or the language of derivation basis employed in [Tho81]) and presents a completely novel approach to this characterisation because we apply, as already mentioned above, systematically Lemma 2.4 and Lemma 2.8, whereas in the existing literataure these results seem to have carved out so far a rather stepmotherly existence. In addition, this systematisation comes along with rather transparent and simple proofs.
The last section additionally introduces the so-called fine measures. They are, as it turns out, well-suited to describe differential properties of functions. The close relation between fine measures and Henstock-Thomson variational measure is in the centre of attention in this section. Later on, it will be revealed that this relation is in fact of extremely high importance for the differential properties of the variational Henstock-Kurzweil-Stieltjes integrals under consideration.

Chapter 4 finally introduces the variational Henstock-Kurzweil-Stieltjes integral, which we are interested in, as well as various notions of differentiability with respect to a function. The main results then concern differentiability properties of the Henstock-Kurzweil-Stieltjes integral and the integrability properties of the derivatives introduced before. Combining these two kind of results, we finally arrive in the last section at the desired Lusin-type characterisation of the variational Henstock-Kurzweil-Stieltjes integral with BVG_*-integrators thus giving improvements and far-reaching extensions of some of the results in [War36] and [Fau97].

The last chapter is devoted to various applications of the results obtained so far. In the first section we come back to the problem of recovering a function from a relative derivative and we completely solve this problem for bounded differentiators of generalized bounded variation (in the restricted sense).
Afterwards, we revisit ACG_*-functions and derive another characterisation for them, which is in the real-valued case originally due to Gordon (see [Gor89]). Using this result we shall reprove the classical descriptive characterisation of the Henstock-Kurzweil integral due to Lusin.
As we will explain later on, this last mentioned result motivates to study Henstock-Kurzweil integrals of functions with values in a space having the Radon-Nikodým property. This will be done in the third section where we prove a far reaching extension of a result originally due to Bongiorno, Di Piazza and Musiał (see Theorem 3.6 in [BPM09b]) and we will also fill a gap in their proof.
The fourth section demonstrates how one can obtain "integration by parts"-results for variational Henstock-Kurzweil-Stieltjes integral using the characterisations proved in Chapter 4.
In the last section we apply, as we indicated above, our results to complex analysis as we utilize our results in order to study certain normed algebras of differentiable functions on compact plane sets.

1.2 Notation

Throughout the entire work we fix real numbers $a < b$. And we denote by $\mathfrak{I}$ the set of all non-degenerated (i.e., with nonvoid interior) closed subintervals of $[a,b]$. For $t,s \in \mathbb{R}$ we put $\langle t,s\rangle := [\min\{t,s\},\max\{t,s\}]$. By $\mathbb{N}$ we denote the set of natural numbers without 0. If $A \subseteq \mathbb{R}$, then $\lambda(A)$ or occasionally $\lambda^*(A)$ denotes the outer Lebesgue measure of A.

We further consider Banach spaces $(V,\|\cdot\|_V)$, $(X,\|\cdot\|_X)$, $(Y,\|\cdot\|_Y)$ and $(Z,\|\cdot\|_Z)$ over the common field $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$, where we usually supress the index in $\|\cdot\|_{\mathcal{X}}$ and just write $\|\cdot\|$ for $\mathcal{X} \in \{V,X,Y,Z\}$ if no confusion is to be expected. We further assume that there is a bilinear mapping $B : X \times Y \to Z$ such that $\|B(x,y)\|_Z \leq \|x\|_X \cdot \|y\|_Y$ holds for all $x \in X$ and $y \in Y$. We put $x \cdot y := B(x,y)$ for $(x,y) \in X \times Y$. This notation also contains the convention that $\cdot$ bounds more strongly than other algebraic operations. If we have, e.g., $Z = X$ and $x,z \in X$ and $y \in Y$, then $z - x \cdot y$ is to be read as $z - (x \cdot y) = z - B(x,y)$. The topological dual space of $\mathcal{X}$ is denoted by $\mathcal{X}^\star$. If $x^\star \in \mathcal{X}^\star$ and $x \in \mathcal{X}$, then we usually write $\langle x, x^\star\rangle$ for $x^\star(x)$. (Note that this notation will not conflict with $\langle t,s\rangle$ introduced above for $t,s \in \mathbb{R}$ as the respective meaning will be always completely clear by the context.) Moreover, throughout the entire work, the symbols f, F and φ are exclusively reserved for functions $f : [a,b] \to X$, $\varphi : [a,b] \to Y$ and $F : [a,b] \to Z$.

Let $(\mathfrak{X},d)$ denote a metric space. For a non-empty subset $E \subseteq \mathfrak{X}$ we denote by d_E the restriction $d|_{E\times E}$ of d to E. By $\mathrm{Bor}(\mathfrak{X})$ we denote the Borel σ-algebra associated with $(\mathfrak{X},d)$ and by $\mathcal{T}(\mathfrak{X})$ the set of all open subsets of $(\mathfrak{X},d)$. For $x \in \mathfrak{X}$ and $\varepsilon > 0$ we set

$$U_\varepsilon(x) := \{y \in \mathfrak{X} :\ d(x,y) < \varepsilon\} \qquad \text{and} \qquad \Delta(x,\varepsilon) := \{y \in \mathfrak{X} :\ d(x,y) \leq \varepsilon\}.$$

Moreover, we set $\dot{U}_\varepsilon(x) := U_\varepsilon(x) \setminus \{x\}$. For any nonvoid subset $E \subseteq \mathfrak{X}$ we denote by $\mathrm{diam}(E)$, ∂E, $\mathrm{int}(E) = E^\circ$ and $\overline{E}$ the diameter, the boundary, the interior and the closure of E in the space $(\mathfrak{X},d)$, respectively. Moreover, we put $\mathrm{dist}(x,E) := \inf\{d(x,e) :\ e \in E\}$ for $x \in \mathfrak{X}$. If $\mathfrak{X} = \mathbb{K}$, then we always assume that d is the usual Euclidean metric. If $(\mathfrak{Y},\rho)$ is another metric space, we denote by $\mathrm{C}(\mathfrak{X},\mathfrak{Y})$ the set of all continuous functions from $\mathfrak{X}$ to $\mathfrak{Y}$, where we supress the mention of $\mathfrak{Y}$ provided that $\mathfrak{Y} = \mathbb{K}$. By $\mathcal{H}^1_{\mathfrak{X}}$ we denote the one-dimensional outer Hausdorff measure on $\mathfrak{X}$ (once again, we drop the index $\mathfrak{X}$ if the meaning of $\mathcal{H}^1$ is clear by the context). For $\varepsilon > 0$ and $A \subseteq \mathfrak{X}$ we set

$$\mathcal{H}^1_\varepsilon(A) := \inf\left\{\sum_{B\in\mathcal{D}} \mathrm{diam}(B) :\ \mathcal{D} \subseteq \mathcal{A}_\varepsilon \text{ countable with } A \subseteq \bigcup \mathcal{D}\right\},$$

where $\mathcal{A}_\varepsilon := \{B \subseteq \mathfrak{X} :\ \mathrm{diam}(B) < \varepsilon\}$. Then $\mathcal{H}^1(A) = \lim_{\varepsilon\to 0^+} \mathcal{H}^1_\varepsilon(A) = \sup_{\varepsilon>0} \mathcal{H}^1_\varepsilon(A)$.

If A is a set we denote by $\sharp A$ or $|A|$ the cardinality of A and by $\mathfrak{P}(A)$ its power set. Moreover, we let $B(A,V)$ denote the subspace of all bounded elements in V^A and denote by $\|\cdot\|_{A,\infty}$ or (if no confusion is to be expected) simply by $\|\cdot\|_\infty$ the uniform norm on $B(A,V)$ (arising from $\|\cdot\|_V$). The indicator function of a set A is denoted by $\mathbb{1}_A$ and the identity function on A is denoted by id_A or simply by id if no confusion is to be expected.

2 Functions of generalized bounded variation and absolute continuity

In this chapter we introduce the important class of functions of generalized bounded variation (in the restricted and wide sense) and the class of functions of generalized absolute continuity (in the restricted and wide sense) and explore their basic and deeper properties. Later on we will benefit from these studies for the examination of Henstock-Kurzweil-Stieltjes integrals with integrators that are of generalized bounded variation in the restricted sense.

First we must fix some notation.
For $\emptyset \neq E \subseteq [a,b]$ we set

$$\mathcal{A}(E) := \left\{ \{[a_j, b_j]\}_{j=1}^r : \begin{array}{c} r \in \mathbb{N},\ a \leq a_1 < b_1 \leq a_2 < b_2 \leq \ldots \leq a_r < b_r \leq b \\ \text{with } a_j, b_j \in E \text{ for all } j \in \{1, \ldots, r\} \end{array} \right\}.$$

Notice that the notation $\{[a_j, b_j]\}_{j=1}^r$ is always meant to indicate that the respective intervals are listed in "increasing order" as in the defintion of $\mathcal{A}(E)$. In addition, note that in what follows a generic element of $\mathcal{A}(E)$ will always be written in the form $\{[a_j, b_j]\}_{j=1}^r$ and we speak of *a partition* (*on* E).

For a function $g : [a,b] \to V$ and a set $\emptyset \neq E \subseteq [a,b]$ we set

$$\omega(g, E) := \operatorname{diam}(g(E)) = \sup\{\|g(t) - g(s)\| :\ t, s \in E\} \in [0, \infty],$$

and for $S = \{[a_j, b_j]\}_{j=1}^r \in \mathcal{A}([a,b])$ we put

$$V_g^*(S) := \sum_{j=1}^r \omega(g, [a_j, b_j]) \qquad \text{and} \qquad W_g(S) := \sum_{j=1}^r \|g(b_j) - g(a_j)\|.$$

In particular, $W_{\mathrm{id}}(S) = \sum_{j=1}^r (b_j - a_j)$. We further define

$$V_*(g, E) := \begin{cases} 0, & \text{if } E \text{ is a singleton,} \\ \sup\{V_g^*(S) :\ S \in \mathcal{A}(E)\}, & \text{elsewise,} \end{cases}$$

and

$$V(g, E) := \begin{cases} 0, & \text{if } E \text{ is a singleton,} \\ \sup\{W_g(S) :\ S \in \mathcal{A}(E)\}, & \text{elsewise.} \end{cases}$$

Now we can define functions of generalized bounded variation and of generalized absolute continuity.

2.1 Definition. *Let $\emptyset \neq E \subseteq [a,b]$ and $g \in V^{[a,b]}$. We call g a function of*

(a) bounded weak variation on E *or of* bounded variation in the wide sense on E *or a* BV-function on E, *if $V(g, E) < \infty$, and we denote by* $\mathrm{BV}(E, V)$ *the set of all those functions;*

(b) generalized bounded weak variation on E *or of* generalized bounded variation in the wide sense on E *or a* BVG-function on E, *if there exists a sequence* $(E_n)_{n=1}^{\infty}$ *of subsets of* E *with* $\bigcup_{n\in\mathbb{N}} E_n = E$ *such that* $V(g, E_n) < \infty$ *for all* n*, and we denote by* $\mathrm{BVG}(E, V)$ *the set of all those functions;*

(c) bounded strong variation on E *or of* bounded variation in the restricted sense on E *or a* BV$*$-function on E, *if* $V_*(g, E) < \infty$*, and we denote by* $\mathrm{BV}*(E, V)$ *the set of all those functions;*

(d) generalized bounded strong variation on E *or of* generalized bounded variation in the restricted sense on E *or a* BVG$*$-function on E, *if there exists a sequence* $(E_n)_{n=1}^{\infty}$ *of subsets of* E *with* $\bigcup_{n\in\mathbb{N}} E_n = E$ *such that* $V_*(g, E_n) < \infty$ *for all* n*, and we denote by* $\mathrm{BVG}(E, V)$ *the set of all those functions;*

(e) absolute continuity on E in the wide sense *or an* AC-function on E, *if* $|E| = 1$ *or for each* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that* $W_g(S) < \varepsilon$ *for each* $S \in \mathcal{A}(E)$ *with* $W_{\mathrm{id}}(S) < \delta$*, and we denote by* $\mathrm{AC}(E, V)$ *the set of all those functions;*

(f) generalized absolute continuity on E in the wide sense *or an* ACG-function on E, *if there exists a sequence* $(E_n)_{n=1}^{\infty}$ *of subsets of* E *with* $\bigcup_{n\in\mathbb{N}} E_n = E$ *such that* $g \in \mathrm{AC}(E_n, V)$ *for all* n*, and we denote by* $\mathrm{ACG}(E, V)$ *the set of all those functions;*

(g) absolute continuity on E in the restricted sense *or an* AC$*$-function on E, *if* $|E| = 1$ *or for each* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that* $V_g^*(S) < \varepsilon$ *for each* $S \in \mathcal{A}(E)$ *with* $W_{\mathrm{id}}(S) < \delta$*, and we denote by* $\mathrm{AC}*(E, V)$ *the set of all those functions;*

(h) generalized absolute continuity on E in the restricted sense *or an* ACG$*$-function on E, *if there exists a sequence* $(E_n)_{n=1}^{\infty}$ *of subsets of* E *with* $\bigcup_{n\in\mathbb{N}} E_n = E$ *such that* $g \in \mathrm{AC}*(E_n, V)$ *for all* n*, and we denote by* $\mathrm{ACG}*(E, V)$ *the set of all those functions.*

2.2 Remark Let $\emptyset \neq E \subseteq [a, b]$.

(a) One immediately verifies that for $E = [a, b]$ the above notions of bounded variation and absolute continuity coincide with the classical ones and for $g \in \mathrm{BV}([a, b]) = \mathrm{BV}*([a, b])$ the values $V(g, [a, b])$ and $V_*(g, [a, b])$ both equal the classical variation.

(b) It is easy to show that the sets $\mathrm{BV}(E, V)$, $\mathrm{BVG}(E, V)$, $\mathrm{BV}*(E, V)$, $\mathrm{BVG}*(E, V)$, $\mathrm{AC}(E, V)$, $\mathrm{ACG}(E, V)$, $\mathrm{AC}*(E, V)$ and $\mathrm{ACG}*(E, V)$ endowed with the usual algebraic operations form vector spaces over $\mathbb{K}$.

(c) We always have $\mathrm{BV}*(E, V) \subseteq \mathrm{BV}(E, V) \subseteq \mathrm{BVG}(E, V)$ and thus $\mathrm{BVG}*(E, V) \subseteq \mathrm{BVG}(E, V)$.

(d) We always have $\mathrm{AC}*(E, V) \subseteq \mathrm{AC}(E, V) \subseteq \mathrm{ACG}(E, V)$ and hence $\mathrm{ACG}*(E, V) \subseteq \mathrm{ACG}(E, V)$.

(e) Some authors (as, e.g., Saks, see p. 223 and 231 in [Sak41]) require additional continuity or boundedness conditions in the definition of generalized absolute continuity (in the wide or restricted sense). The reason for this will be explained later on in Remark 2.13 and Remark 2.15. As we refrain from doing so, the subsequent results are preciser than those given in [Sak41] in so far as they work out in more detail, which boundedness and continuity conditions are actually needed in order to obtain the fundamental properties of functions of generalized bounded variation and generalized absolute continuity.

(f) One has $\mathrm{BV}(E, V) \subseteq B(E, V)$. In fact, let $g \in \mathrm{BV}(E, V)$ and fix $t \in E$. Then for each $s \in E$ we estimate $\|g(s)\| \leq \|g(s) - g(t)\| + \|g(t)\| \leq V(g, E) + \|g(t)\|$ and obtain $\|g\|_E \leq V(g, E) + \|g(t)\| < \infty$.

(g) If $g \in \mathrm{AC}(E, V)$, then clearly $g|_E$ is uniformly continuous and so it is bounded on E and possesses a unique continuous extension to $\overline{E}$. However note that this extension does not need to coincide with $g|_{\overline{E}}$ and that g may not be bounded on $\overline{E}$.

After these preliminary definitions we start the exploration of functions of generalized bounded variation and of generalized absolute continuity in the next section.

2.1 Basic properties of functions of generalized bounded variation

In this section we collect some of the basic properties of functions of generalized bounded variation.

2.3 Lemma. *Let $\emptyset \neq E \subseteq [a,b]$ with $c := \inf E$ and $d := \sup E$ and let $\varphi \in \mathrm{BV}_*(E,Y)$. Then φ is bounded on $[c,d]$.*

Proof. For $E = \{c,d\}$ there is nothing to be shown. So assume $c < d$ as well as $(c,d) \cap E \neq \emptyset$ and fix $t_0 \in E \cap (c,d)$. Consider an arbitrary $t \in (c,t_0]$ and choose $x \in E$ with $x \leq t$. Then we obtain

$$\begin{aligned} \|\varphi(t)\| &\leq \|\varphi(t) - \varphi(t_0)\| + \|\varphi(t_0)\| \leq \sup_{s,\sigma \in [x,t_0]} \|\varphi(s) - \varphi(\sigma)\| + \|\varphi(t_0)\| \\ &= \omega(\varphi, [x,t_0]) + \|\varphi(t_0)\| \leq V_*(\varphi, E) + \|\varphi(t_0)\|. \end{aligned}$$

Similarly, we get

$$\|\varphi(t)\| \leq V_*(\varphi, E) + \|\varphi(t_0)\|$$

for $t \in [t_0, d)$. Hence, we arrive at

$$\sup_{c \leq t \leq d} \|\varphi(t)\| \leq \max\{\|\varphi(c)\|, \|\varphi(d)\|, V_*(\varphi, E) + \|\varphi(t_0)\|\} < \infty.$$

So φ is bounded on $[c,d]$. □

2.4 Lemma. *Let $\emptyset \neq E \subseteq [a,b]$ be not a singleton and put $c := \inf E$ and $d := \sup E$. The following are equivalent.*

(a) $\varphi \in \mathrm{BV}_*(E,Y)$.

(b) There is a strictly increasing function $\chi : [a,b] \to \mathbb{R}$ such that $\|\varphi(x) - \varphi(y)\| \leq |\chi(x) - \chi(y)|$ for all $x \in E$ and all $y \in [c,d]$.

(c) There exists an $M > 0$ such that $V_\varphi^*(S) \leq M$ for all $S = \{[a_j, b_j]\}_{j=1}^r \in \mathcal{A}([c,d])$ with $\{a_j, b_j\} \cap E \neq \emptyset$ for each $j \in \{1, \ldots, r\}$.

(d) There exists an $M > 0$ such that $V_\varphi^*(S) \leq M$ for all $S = \{[a_j, b_j]\}_{j=1}^r \in \mathcal{A}([c,d])$ with $[a_j, b_j] \cap E \neq \emptyset$ for every $j \in \{1, \ldots, r\}$.

Proof. (a) $\Longrightarrow$ (b): (cf. Lemma 2 in [War36]) For $x \in [c,d]$ we set

$$E_x^- := (E \cap [a,x]) \cup \{x\}, \qquad E_x^+ := (E \cap [x,b]) \cup \{x\} \qquad \text{and} \qquad \chi(x) := x + V_*(\varphi, E_x^-) - V_*(\varphi, E_x^+).$$

We first show that χ is in fact real-valued, i.e., we never have $V_*(\varphi, E_x^-) = \infty$ or $V_*(\varphi, E_x^+) = \infty$. Indeed, let $x \in [c,d]$, assume that E_x^- is not a singleton and let $S = \{[a_j, b_j]\}_{j=1}^r \in \mathcal{A}(E_x^-)$. If $b_r < x$, then we even have $S \in \mathcal{A}(E)$ and we derive

$$V_\varphi^*(S) = \sum_{j=1}^r \omega(\varphi, [a_j, b_j]) \leq V_*(\varphi, E).$$

If $b_r = x$, then $\{[a_j, b_j]\}_{j=1}^{r-1}$ belongs to $\mathcal{A}(E)$ provided that $r > 1$. We can thus compute

$$V_\varphi^*(S) = \sum_{j=1}^r \omega(\varphi, [a_j, b_j]) = \sum_{j=1}^{r-1} \omega(\varphi, [a_j, b_j]) + \omega(\varphi, [a_r, x]) \leq V_*(\varphi, E) + 2 \sup_{c \leq t \leq d} \|\varphi(t)\|;$$

note that the quantity $\sup_{c\le t\le d}\|\varphi(t)\|$ is finite by Lemma 2.3.

In any case, we infer

$$V_*(\varphi, E_x^-) = \sup_{S\in\mathcal{A}(E_x^-)} V_\varphi^*(S) \le V_*(\varphi, E) + 2\sup_{c\le t\le d}\|\varphi(t)\| < \infty.$$

Analogously, one verifies $V_*(\varphi, E_x^+) < \infty$.

We next show that χ strictly increases on $[c,d]$. For this purpose, let $x, y \in [c,d]$ with $x < y$. Obviously, $V_*(\varphi, E_y^-) \ge V_*(\varphi, E_x^-)$ if E_x^- is a singleton. So let us suppose that E_x^- is not a singleton. Then E_y^- is not a singleton either. Take a partition $S = \{[a_j, b_j]\}_{j=1}^r$ in $\mathcal{A}(E_x^-)$ and put

$$S' := \{[a_1, b_1], \dots, [a_{r-1}, b_{r-1}], [a_r, y]\} \in \mathcal{A}(E_y^-),$$

where this is to be read as $S' := \{[a_1, y]\}$ for $r = 1$. We obtain

$$V_*(\varphi, E_y^-) - V_\varphi^*(S) \ge V_\varphi^*(S') - V_\varphi^*(S) = \omega(\varphi, [a_r, y]) - \omega(\varphi, [a_r, b_r]) \ge 0$$

and hence

$$V_*(\varphi, E_y^-) \ge \sup_{S\in\mathcal{A}(E_x^-)} V_\varphi^*(S) = V_*(\varphi, E_x^-).$$

A similar reasoning shows $V_*(\varphi, E_y^+) \le V_*(\varphi, E_x^+)$, and we thus conclude $\chi(y) > \chi(x)$.

Let $x \in E$ and $y \in [c,d]$. For $x = y$ we trivially have $\|\varphi(x) - \varphi(y)\| \le |\chi(x) - \chi(y)|$. So let $x \ne y$ and assume $x < y$ as well as that E_x^- is not a singleton. Pick $S = \{[a_j, b_j]\}_{j=1}^r \in \mathcal{A}(E_x^-)$ and set, using $x \in E$,

$$\widetilde{S} := \{[a_1, b_1], \dots, [a_r, b_r], [x, y]\} \in \mathcal{A}(E_y^-).$$

We then estimate

$$V_*(\varphi, E_y^-) - V_\varphi^*(S) \ge V_\varphi^*(\widetilde{S}) - V_\varphi^*(S) = \omega(\varphi, [x, y]) \ge \|\varphi(x) - \varphi(y)\|$$

and arrive at

$$V_*(\varphi, E_y^-) - V_*(\varphi, E_x^-) \ge \|\varphi(x) - \varphi(y)\|.$$

This last inequality is certainly also true if E_x^- is a singleton (because $\{[x,y]\} \in \mathcal{A}(E_y^-)$). These relations lead to

$$|\chi(x) - \chi(y)| = y - x + V_*(\varphi, E_y^-) - V_*(\varphi, E_x^-) + V_*(\varphi, E_x^+) - V_*(\varphi, E_y^+) \ge \|\varphi(x) - \varphi(y)\|.$$

The case $y < x$ can be treated similarly.

Finally we extend χ to a strictly increasing function on $[a,b]$.

(b) $\Longrightarrow$ (c): Let $\{[a_j, b_j]\}_{j=1}^r \in \mathcal{A}([c,d])$ with $\{a_j, b_j\} \cap E \ne \emptyset$ for all j and pick $t_j \in \{a_j, b_j\} \cap E$. For $s, \widetilde{s} \in [a_j, b_j]$ we then obtain

$$\begin{aligned}\|\varphi(s) - \varphi(\widetilde{s})\| &\le \|\varphi(s) - \varphi(t_j)\| + \|\varphi(t_j) - \varphi(\widetilde{s})\| \\ &\le |\chi(s) - \chi(t_j)| + |\chi(t_j) - \chi(\widetilde{s})| \le 2(\chi(b_j) - \chi(a_j))\end{aligned}$$

and, hence, $\omega(\varphi, [a_j, b_j]) \le 2(\chi(b_j) - \chi(a_j))$. This yields

$$\sum_{j=1}^r \omega(\varphi, [a_j, b_j]) \le 2\sum_{j=1}^r (\chi(b_j) - \chi(a_j)) = 2(\chi(b) - \chi(a)).$$

(c) $\Longrightarrow$ (d): Take $\{[a_j,b_j]\}_{j=1}^r \in \mathcal{A}([c,d])$ with $[a_j,b_j]\cap E \neq \emptyset$ for all j and fix $t_j \in [a_j,b_j]\cap E$. By hypothesis,

$$\sum_{j=1}^{r}(\omega(\varphi,[a_j,t_j])+\omega(\varphi,[t_j,b_j])) \leq M.$$

For $s,\widetilde{s}\in[a_j,b_j]$ we have

$$\|\varphi(s)-\varphi(\widetilde{s})\| \leq \|\varphi(s)-\varphi(t_j)\|+\|\varphi(t_j)-\varphi(\widetilde{s})\| \leq 2(\omega(\varphi,[a_j,t_j])+\omega(\varphi,[t_j,b_j]))$$

and, consequently, $\sum_{j=1}^r \omega(\varphi,[a_j,b_j]) \leq 2M$.

(d) $\Longrightarrow$ (a) is obvious. ☐

We want to not let go unmentioned that it is also possible to prove the implication (a) $\Longrightarrow$ (c) in Lemma 2.4 more directly.

2nd proof of (a) $\Longrightarrow$ (c) in Lemma 2.4. Assume that (a) holds and consider $S=\{[a_j,b_j]\}_{j=1}^r \in \mathcal{A}([c,d])$ with $\{a_j,b_j\}\cap E\neq\emptyset$ for all $j\in\{1,\ldots,r\}$. We put $L:=\sup_{c\leq t\leq d}\|\varphi(t)\|$ (recall that L is indeed finite due to Lemma 2.3).
Let $x_1<\ldots<x_s$ be the points of the set $E\cap\bigcup_{j=1}^r\{a_j,b_j\}$. If $s=1$, then we have $S=\{[a_1,b_1]\}$ or $S=\{[a_1,b_1],[a_2,b_2]\}$ with $b_1=a_2\in E$. In both cases we obtain $V_\varphi^*(S)\leq 4L$.
Now suppose $s\geq 2$. We then put $I_\nu := [x_\nu,x_{\nu+1}]$ for $\nu\in\{1,\ldots,s-1\}$. If $x_1=b_1$, then define $I_0:=[c,b_1]$. If $x_s=a_r$, then put $I_{s+1}:=[a_r,d]$. It is easy to see that every interval $[a_j,b_j]$ is contained in some I_ν and that conversely each I_ν contains at most two of the intervals $[a_j,b_j]$. Therefore, we obtain

$$V_\varphi^*(S)\leq 4L+\sum_{\nu=1}^{s}2\omega(\varphi,I_\nu)\leq 4L+2V_*(\varphi,E)$$

since $\omega(\varphi,I_\nu)\leq 2L$ for $\nu\in\{0,s+1\}$. The result follows with $M:=4L+2V_*(\varphi,E)$. ☐

As a simple corollary we mark down the following result.

2.5 Lemma. *For $\varphi:[a,b]\to Y$ and $\emptyset\neq E\subseteq[a,b]$ not a singleton the following statements are equivalent.*

- *(a) We have $\varphi\in \mathrm{BV}_*(E,Y)$ and φ is bounded on $[a,b]$.*
- *(b) There exists an $M>0$ such that $V_\varphi^*(S)\leq M$ for each $S=\{[a_j,b_j]\}_{j=1}^r\in\mathcal{A}([a,b])$ with $\{a_j,b_j\}\cap E\neq\emptyset$ for all $j\in\{1,\ldots,r\}$.*

Proof. First suppose that condition (a) is satisfied. Lemma 2.4 implies that there is an $L>0$ with $V_\varphi^*(S)\leq L$ for each $S=\{[a_j,b_j]\}_{j=1}^r\in\mathcal{A}([c,d])$ with $\{a_j,b_j\}\cap E\neq\emptyset$ for all $\in\{1,\ldots,r\}$, where $c:=\inf E$ and $d:=\sup E$. Take $S=\{[a_j,b_j]\}_{j=1}^r\in\mathcal{A}([a,b])$ with $\{a_j,b_j\}\cap E\neq\emptyset$ for all $\in\{1,\ldots,r\}$. Put $M:=L+4\sup_{a\leq x\leq b}\|\varphi(x)\|$. As the intervals $[a_j,b_j]$ are non-overlapping and have at least one endpoint in E, we conclude $a_j,b_j\in[c,d]$ at least for $j\in\{1,\ldots,r\}\setminus\{1,r\}$. In addition, we have $\omega(\varphi,[a_j,b_j])\leq 2\sup_{a\leq x\leq b}\|\varphi(x)\|$ for each j. Therefore, we immediately obtain $V_\varphi^*(S)\leq M$.

Conversely assume that (b) holds. Clearly, (b) implies $\varphi\in\mathrm{BV}_*(E,Y)$. Fix $x\in E$ and take $t\in[a,b]$. Assertion (b) then yields

$$\|\varphi(t)\|\leq\|\varphi(t)-\varphi(x)\|+\|\varphi(x)\|\leq\omega(\varphi,\langle t,x\rangle)+\|\varphi(x)\|\leq M+\|\varphi(x)\|.$$

Hence, φ is bounded on $[a,b]$. ☐

2.6 Remark The equivalence (a) $\iff$ (c) in Lemma 2.4 and Corollary 2.5 both give correct versions of the false Lemma 6.15 in [Lee89].

2.7 Lemma. *Let $\emptyset \neq E \subseteq [a,b]$ and let $\varphi \in \mathrm{BV}_*(E,Y)$. Then we also have $\varphi \in \mathrm{BV}_*(\overline{E},Y)$.*

Proof. This can be shown essentially as Theorem 6.2 (c) in [Gor94]. But here is an alternative proof, which is a natural outflow of Lemma 2.4.
We put

$$M := \sup\{V_\varphi^*(S) : \; S = \{[a_j,b_j]\}_{j=1}^r \in \mathcal{A}([c,d]) \text{ with } [a_j,b_j] \cap E \neq \emptyset \text{ for every } j \in \{1,\dots,r\}\},$$

which is finite due to Lemma 2.4. Let $S = (I_j)_{j=1}^r \in \mathcal{A}(\overline{E})$ and set $A := \{j \in \{1,\dots,r\} : \; I_j \cap E = \emptyset\}$. Then $V_\varphi^*(\{I_j\}_{j\notin A}) \leq M$. The intervals in the set $\{I_j : \; j \in A\}$ are pairwise disjoint. In fact, let $i,j \in A$ distinct and suppose to the contrary that $I_i \cap I_j$ is nonvoid. As the intervals of S are non-overlapping, I_i and I_j intersect at a boundary point, say t, which belongs to $\overline{E}$ by the choice of S. Due to $t \in (I_i \cup I_j)^\circ \cap \overline{E}$, we infer $(I_i \cup I_j)^\circ \cap E \neq \emptyset$, which implies $I_i \cap E \neq \emptyset$ or $I_j \cap E \neq \emptyset$ contradicting the choice of i and j. Due to the pairwise disjointness of the closed intervals in the set $\{I_j : \; j \in A\}$, whose endpoints lie in $\overline{E}$, we can find closed, pairwise disjoint intervals $\{\widetilde{I}_j\}_{j\in A}$ such that $I_j \subseteq \widetilde{I}_j \subseteq [c,d]$ and $\widetilde{I}_j \cap E \neq \emptyset$ for each $j \in A$. This yields $V_\varphi^*(\{I_j\}_{j\in A}) \leq V_\varphi^*(\{\widetilde{I}_j\}_{j\in A}) \leq M$. Altogether, we obtain $V_\varphi^*(S) = V_\varphi^*(\{I_j\}_{j\in A}) + V_\varphi^*(\{I_j\}_{j\notin A}) \leq 2M$. □

The next lemma plays together with Lemma 2.4 a crucial and vital part in the proofs of some of our main results. It gives a complete characterisation of BVG_*-functions on a set E and extends Lemma 6 in [War36].

2.8 Lemma. *Let $\emptyset \neq E \subseteq [a,b]$.The following statements are equivalent for a function $\varphi : [a,b] \to Y$.*

(a) The function φ belongs to $\mathrm{BVG}_*(E,Y)$.

(b) There is a strictly increasing function $\chi : [a,b] \to \mathbb{R}$ and a countable set $A \subseteq [a,b]$ such that

$$\overline{\lim_{y\to x}} \frac{\|\varphi(x) - \varphi(y)\|}{|\chi(x) - \chi(y)|} < \infty$$

for all $x \in E \setminus A$.

Proof. We closely follow the proof of Lemma 6 in [War36]. We first show that (a) implies (b). We choose sets $\emptyset \neq E_n \subseteq [a,b]$ for $n \in \mathbb{N}$ with $\bigcup_{n\in\mathbb{N}} E_n = E$ such that $\varphi \in \mathrm{BV}_*(E_n,Y)$ is satisfied for all $n \in \mathbb{N}$. For each E_n we take a function χ_n as in Lemma 2.4 and we put

$$\chi(x) := \sum_{n=1}^\infty 2^{-n} \cdot \frac{\chi_n(x) - \chi_n(a)}{\chi_n(b) - \chi_n(a)}$$

for $x \in [a,b]$. Clearly, χ is well-defined and strictly increasing.
For $m \in \mathbb{N}$ we set $c_m := \inf E_m$ and $d_m := \sup E_m$. Take $m \in \mathbb{N}$ with $E_m \cap (c_m,d_m) \neq \emptyset$ and $x \in E_m \cap (c_m,d_m)$. The choice of χ_m yields

$$\begin{aligned}|\chi(x) - \chi(y)| = \sum_{n=1}^\infty \frac{|\chi_n(x) - \chi_n(y)|}{2^n(\chi_n(b) - \chi_n(a))} &\geq \frac{|\chi_m(x) - \chi_m(y)|}{2^m(\chi_m(b) - \chi_m(a))} \\ &\geq \frac{\|\varphi(x) - \varphi(y)\|}{2^m(\chi_m(b) - \chi_m(a))}\end{aligned}$$

and thus

$$\frac{\|\varphi(x) - \varphi(y)\|}{|\chi(x) - \chi(y)|} \leq 2^m(\chi_m(b) - \chi_m(a))$$

for all $y \in (c_m, d_m) \setminus \{x\}$. (For the first equation above note that we either have $\chi_n(x) > \chi_n(y)$ for all $n \in \mathbb{N}$ or $\chi_n(x) < \chi_n(y)$ for all $n \in \mathbb{N}$.) Statement (b) now follows with $A := \bigcup_{m\in\mathbb{N}}\{c_m, d_m\}$.

Conversely, assume that (b) holds. For $n \in \mathbb{N}$ and $k \in \mathbb{Z}$ we define E_n as the set of all $x \in E$ with the property

$$\forall\, y \in [a,b]:\ |\chi(x) - \chi(y)| \leq \frac{1}{n} \implies \|\varphi(x) - \varphi(y)\| \leq n|\chi(x) - \chi(y)|,$$

and we put

$$E_{n,k} := \left\{ x \in E_n :\ \frac{k}{n} \leq \chi(x) < \frac{k+1}{n} \right\}.$$

Since the inverse of a strictly increasing function defined on an interval is everywhere continuous (see Proposition D.2), we easily infer

$$\bigcup_{n\in\mathbb{N}} \bigcup_{k\in\mathbb{Z}} E_{n,k} = \bigcup_{n\in\mathbb{N}} E_n \supseteq E \setminus A.$$

Since A is countable and since every function is of class BV_* on a singleton, it suffices to verify that φ belongs to $\mathrm{BV}_*(E_{n,k}, Y)$ for all $n \in \mathbb{N}$ and $k \in \mathbb{Z}$.
So fix $n \in \mathbb{N}$ and $k \in \mathbb{Z}$, assume without loss of generality that $E_{n,k}$ is not a singleton and consider a partition $S = \{[a_j, b_j]\}_{j=1}^r$ in $\mathcal{A}(E_{n,k})$. For each $j \in \{1, \dots, r\}$ we then have

$$\|\varphi(z) - \varphi(b_j)\| \leq n|\chi(z) - \chi(b_j)| \tag{2.1}$$

for every $z \in [a_j, b_j]$ because $a_j, b_j \in E_{n,k} \subseteq E_n$ and

$$0 \leq \chi(b_j) - \chi(z) \leq \chi(b_j) - \chi(a_j) < \frac{k+1}{n} - \frac{k}{n} = \frac{1}{n}.$$

Inequality (2.1) implies

$$\begin{aligned}
\omega(\varphi, [a_j, b_j]) &= \sup_{a_j \leq x \leq y \leq b_j} \|\varphi(x) - \varphi(y)\| \\
&\leq \sup_{a_j \leq x \leq y \leq b_j} (\|\varphi(y) - \varphi(b_j)\| + \|\varphi(b_j) - \varphi(x)\|) \\
&\leq \sup_{a_j \leq x \leq y \leq b_j} (n|\chi(b_j) - \chi(y)| + n|\chi(b_j) - \chi(x)|) \\
&\leq 2n(\chi(b_j) - \chi(a_j)).
\end{aligned}$$

As a result,

$$V_\varphi^*(S) = \sum_{j=1}^r \omega(\varphi, [a_j, b_j]) \leq 2n \sum_{j=1}^r (\chi(b_j) - \chi(a_j)) \leq 2n(\chi(b) - \chi(a)),$$

from which we conclude

$$V_*(\varphi, E_{n,k}) = \sup_{S \in \mathcal{A}(E_{n,k})} V_\varphi^*(S) \leq 2n(\chi(b) - \chi(a)) < \infty.$$

Thus (a) holds. □

2.9 Remark As a references for a proof of Lemma 2.8 and for the implication (a) $\Longrightarrow$ (b) in Lemma 2.4 in the case of real-valued functions we only know [War36] and [Sak41, p. 236] (but see also Lemma 3.6 in [Fau97] for a proof for bounded functions). For the sake of completeness and since the existing proofs are a little bit sketchy (especially Ward's original one), we provided complete proofs.

Since any monotone function has at most countably many discontinuities, we immediately infer the following corollary from Lemma 2.8.

2.10 Corollary. *Every function* $\varphi \in \mathrm{BVG}_*([a,b],Y)$ *has at most countably many discontinuities.*

At the close of this section we state a result analogous to Lemma 2.7 for BV-functions.

2.11 Lemma. *Let* $\emptyset \neq E \subseteq [a,b]$*. Then* $\mathrm{C}(\overline{E},Y) \cap \mathrm{BV}(E,Y) \subseteq \mathrm{BV}(\overline{E},Y)$*; more precisely: if* $\varphi|_{\overline{E}} \in \mathrm{C}(\overline{E},Y)$ *and if* $\varphi \in \mathrm{BV}(E,Y)$*, then* $\varphi \in \mathrm{BV}(\overline{E},Y)$*.*

Proof. At this point we refrain from a detailed proof, but refer to the proof of Lemma 2.14 below, which is quite similar. □

2.2 Basic properties of functions of generalized absolute continuity

In this section we collect basic properties of functions of generalized absolute continuity. With these preliminaries at our disposal we shall prove the Banach-Zarecki theorem for ACG_*-functions in the next section and a generalisation of Theorem 6 of [Gor89] in chapter 5 below. A vital point in the now upcoming examinations consists of the interplay between generalized bounded variation and generalized absolute continuity.

Throughout the entire section let $\emptyset \neq E \subseteq [a,b]$, $c := \inf E$ and $d := \sup E$.

2.12 Lemma. *(a)* $\mathrm{AC}(E,Y) \subseteq \mathrm{BV}(E,Y)$ *and* $\mathrm{ACG}(E,Y) \subseteq \mathrm{BVG}(E,Y)$*.*

(b) $\mathrm{AC}_*(E,Y) \cap B([c,d],Y) \subseteq \mathrm{BV}_*(E,Y)$ *and* $\mathrm{ACG}_*(E,Y) \cap B([c,d],Y) \subseteq \mathrm{BVG}_*(E,Y)$*.*

Proof. In (a) and (b) the corresponding second assertion is an immediate consequence of the corresponding first one. We now only prove the first assertion of part (b) as the first assertion of part (a) may be proved similarly using that every function in $\mathrm{AC}(E,Y)$ is necessarily bounded on E. Our proof is a detailed exposition of the proof on page 231 in [Sak41].
Clearly, the assertion is true if E is a singleton. For this reason we may and will assume w.l.o.g. that E is not a singleton.
Let $\varphi \in \mathrm{AC}_*(E,Y) \cap B([c,d],Y)$. We put $M := \|\varphi\|_{[c,d],\infty}$ and we choose $\eta > 0$ such that $V_\varphi^*(S) < 1$ for all $S \in \mathcal{A}(E)$ with $W_{\mathrm{id}}(S) < \eta$. Furthermore, choose $n \in \mathbb{N}$ so large that $n \geq 2$ and $\frac{d-c}{n} < \eta$. Let $S = \{[a_j,b_j]\}_{j=1}^r \in \mathcal{A}(E)$. We now define

$$A_k := \left\{ j \in \{1,\dots,r\} :\ c + \frac{k-1}{n}(d-c) \leq a_j < b_j \leq c + \frac{k}{n}(d-c) \right\}$$

for $k \in \{1,\dots,n\}$. We have $j \notin \bigcup_{k=1}^n A_k$ if and only if (a_j,b_j) contains at least one of the points of the set $\{c + \frac{k}{n}(d-c) :\ k \in \{1,\dots,n-1\}\}$. Since the sets $\{(a_j,b_j)\}_{j=1}^r$ are pairwise disjoint, we therefore derive

$$\sharp\left(\{1,\dots,r\} \setminus \bigcup_{k=1}^n A_k\right) \leq \sum_{k=1}^{n-1} \sharp\left\{ j \in \{1,\dots,r\} : c + \frac{k}{n}(d-c) \in (a_j,b_j) \right\} \leq n-1.$$

As a consequence, we get

$$V_\varphi^*(S) = \sum_{k=1}^n \sum_{j\in A_k} \omega(\varphi,[a_j,b_j]) + \sum_{j \notin \bigcup_{k=1}^n A_k} \omega(\varphi,[a_j,b_j]) \leq n + 2M(n-1),$$

where we used that $\sum_{j\in A_k} \omega(\varphi,[a_j,b_j]) < 1$ because of $\sum_{j\in A_k}(b_j - a_j) < \eta$ for each k. □

2.13 Remark The boundedness condition in part (b) of Lemma 2.12 is actually needed. If, e.g., E consists only of two points, then it is trivial that every function $\varphi \in Y^{[a,b]}$ belongs to $\mathrm{AC}_*(E,Y)$ as in this case $\mathcal{A}(E)$ contains no intervals with length smaller than the distance between these two points so that the defining condition in the definition of AC_*-functions becomes empty. But clearly not every function $\varphi \in Y^{[a,b]}$ belongs to $\mathrm{BV}_*(E,Y)$ (take any function unbounded on the convex hull of these two points). It seems that for this reason Saks has incorporated a corresponding boundedness condition in his definition of ACG_*-functions. Unfortunately, such a condition is missing in [Gor94] which leads to the unpleasant situation that several results as stated in [Gor94] are not correct (the root of all evil is part (b) of Theorem 6.2 in [Gor94], which is stated without proof).

2.14 Lemma. *We have* $\mathrm{C}(\overline{E},Y) \cap \mathrm{AC}(E,X) \subseteq \mathrm{AC}(\overline{E},Y)$.

Proof. (cf. the proof of Theorem 6.2 in [Gor94]) Let $\varepsilon > 0$ and choose $\delta > 0$ such that $W_\varphi(S) < \frac{\varepsilon}{2}$ for all $S \in \mathcal{A}(E)$ with $W_{\mathrm{id}}(S) < \delta$. Let $\{[u_k, v_k]\}_{k=1}^r \in \mathcal{A}(\overline{E})$ with $\sum_{k=1}^r (v_k - u_k) < \frac{\delta}{2}$. We now define $\{[a_j, b_j]\}_{j=1}^r$ iteratively as follows: Choose $a_1 \in [a, v_1) \cap E$ (this set is nonvoid as it contains $u_1 \in \overline{E}$) with $|a_1 - u_1| < \frac{\delta}{4r}$ and $\|\varphi(a_1) - \varphi(u_1)\| < \frac{\varepsilon}{4r}$. If $u_2 = v_1$, choose $b_1 \in (a_1, v_2) \cap E$ with

$$|b_1 - v_1| < \frac{\delta}{4r} \qquad \text{and} \qquad \|\varphi(b_1) - \varphi(v_1)\| < \frac{\varepsilon}{4r}. \tag{2.2}$$

If $v_1 < u_2$ choose $b_1 \in (a_1, u_2) \cap E$ with (2.2). Next, if $u_2 = v_1$ set $a_2 := b_1 \in (a_1, v_2) \cap E$ and observe that in this case (2.2) becomes

$$|a_2 - u_2| < \frac{\delta}{4r} \qquad \text{and} \qquad \|\varphi(a_2) - \varphi(u_2)\| < \frac{\varepsilon}{4r}.$$

If, however, $v_1 < u_2$ pick $a_2 \in (b_1, v_2) \cap E$ with $|a_2 - u_2| < \frac{\delta}{4r}$ and $\|\varphi(a_2) - \varphi(u_2)\| < \frac{\varepsilon}{4r}$. If $u_3 = v_2$, choose $b_2 \in (a_2, v_3) \cap E$ with

$$|b_2 - v_2| < \frac{\delta}{4r} \qquad \text{and} \qquad \|\varphi(b_2) - \varphi(v_2)\| < \frac{\varepsilon}{4r}. \tag{2.3}$$

If $v_2 < u_3$ choose $b_2 \in (a_2, u_3) \cap E$ with (2.3). Next, if $u_3 = v_2$ set $a_3 := b_2 \in (a_2, v_3) \cap E$ and observe that in this case (2.3) becomes

$$|a_3 - u_3| < \frac{\delta}{4r} \qquad \text{and} \qquad \|\varphi(a_3) - \varphi(u_3)\| < \frac{\varepsilon}{4r}.$$

If, however, $v_2 < u_3$ pick $a_3 \in (b_2, v_3) \cap E$ with $|a_3 - u_3| < \frac{\delta}{4r}$ and $\|\varphi(a_3) - \varphi(u_3)\| < \frac{\varepsilon}{4r}$ and so on and so forth (where we finally choose $b_r \in (a_r, b] \cap E$). In this way we obtain a finite sequence $\{[a_j, b_j]\}_{j=1}^r \in \mathcal{A}(E)$ with $|u_j - a_j| < \frac{\delta}{4r}$, $|v_j - b_j| < \frac{\delta}{4r}$, $\|\varphi(a_j) - \varphi(u_j)\| < \frac{\varepsilon}{4r}$ and $\|\varphi(b_j) - \varphi(v_j)\| < \frac{\varepsilon}{4r}$ for $j \in \{1, \ldots, r\}$. Thus, we get

$$\sum_{j=1}^r (b_j - a_j) \le \sum_{j=1}^r |b_j - v_j| + \sum_{j=1}^r |v_j - u_j| + \sum_{j=1}^r |u_j - a_j| < \frac{\delta}{4} + \frac{\delta}{2} + \frac{\delta}{4} = \delta,$$

which implies $\sum_{j=1}^r \|\varphi(b_j) - \varphi(a_j)| < \frac{\varepsilon}{2}$ by the choice of δ. In addition, we have

$$\sum_{j=1}^r \|\varphi(b_j) - \varphi(v_j)\| < \frac{\varepsilon}{4} \qquad \text{and} \qquad \sum_{j=1}^r \|\varphi(a_j) - \varphi(u_j)\| < \frac{\varepsilon}{2}.$$

Altogether this yields

$$\begin{aligned} \sum_{j=1}^r \|\varphi(u_j) - \varphi(v_j)\| &\le \sum_{j=1}^r \|\varphi(u_j) - \varphi(a_j)\| + \sum_{j=1}^r \|\varphi(a_j) - \varphi(b_j)\| + \sum_{j=1}^r \|\varphi(b_j) - \varphi(v_j)\| \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon, \end{aligned}$$

which finishes the proof. □

2.15 Remark The continuity condition in Lemma 2.14 is actually needed in so far as $\varphi \in \mathrm{AC}(E,Y)$ only implies that $\varphi|_E$ is continuous on E (this is obvious), while φ can be discontinuous at points of $\overline{E}$. For example, consider the function

$$\varphi : [0,1] \to \mathbb{R};\ t \mapsto \begin{cases} 1, & \text{if } t \neq 0, \\ 0, & \text{elsewise.} \end{cases}$$

and the set $E = (0,1]$. Then $\varphi \in \mathrm{AC}(E,\mathbb{R})$, but φ is not continuous at 0.

We next establish a technical auxiliary result which is an example par excellence for a statement whose existing proofs for real-valued functions (see, e.g., Lemma 8.1 in [Sak41] or Lemma 6.3 in [Gor94]) cannot be transferred to the vector-valued case, which necessitates a finer analysis. Indeed, this finer analysis even leads to a slight improvement in comparison to Lemma 8.1 in [Sak41] resp. Lemma 6.3 in [Gor94].

2.16 Lemma. *Assume that E is closed and let $(I_k)_k$ be the connected components of $[c,d] \setminus E$. Then*

$$\omega(\varphi,[c,d]) \leq V(\varphi,E) + \sum_k \omega(\varphi,\overline{I_k}).$$

Proof. The assertion is clear if $V(\varphi,E) = \infty$ or $\sum_k \omega(\varphi,\overline{I_k}) = \infty$. For this reason we assume that $V(\varphi,E) < \infty$ and $\sum_k \omega(\varphi,\overline{I_k}) < \infty$. Let $t,s \in [c,d]$ with $t \neq s$. We distinguish between several cases.
1st case: $t \in I_k$ and $s \in I_l$ with $k \neq l$. Pick $r \in \partial I_k \subseteq E$ and $\rho \in \partial I_l \subseteq E$. We then estimate

$$\begin{aligned} \|\varphi(t) - \varphi(s)\| &\leq \|\varphi(t) - \varphi(r)\| + \|\varphi(r) - \varphi(\rho)\| + \|\varphi(\rho) - \varphi(s)\| \\ &\leq \omega(\varphi,\overline{I_k}) + V(\varphi,E) + \omega(\varphi,\overline{I_l}) \leq V(\varphi,E) + \sum_j \omega(\varphi,\overline{I_j}). \end{aligned}$$

2nd case: $t,s \in I_k$. In this case we have

$$\|\varphi(t) - \varphi(s)\| \leq \omega(\varphi,\overline{I_k}) \leq V(\varphi,E) + \sum_j \omega(\varphi,\overline{I_j}).$$

3rd case: $t,s \in E$. Then one gets

$$\|\varphi(t) - \varphi(s)\| \leq V(\varphi,E) \leq V(\varphi,E) + \sum_j \omega(\varphi,\overline{I_j}).$$

4th case: $t \in E$ and $s \in I_k$. In this situation we take $r \in \partial I_k$ and we estimate

$$\|\varphi(t) - \varphi(s)\| \leq \|\varphi(t) - \varphi(r)\| + \|\varphi(r) - \varphi(s)\| \leq V(\varphi,E) + \omega(\varphi,\overline{I_k}) \leq V(\varphi,E) + \sum_j \omega(\varphi,\overline{I_j}).$$

Altogether these inequalities lead to $\sup_{t,s\in[c,d]} \|\varphi(t) - \varphi(s)\| = \omega(\varphi,[c,d]) \leq V(\varphi,E) + \sum_k \omega(\varphi,\overline{I_k})$ as claimed. □

By means of the preceding result, we next show the following lemma, which contains a useful sufficient condition for the membership of a function to the space $\mathrm{BV}_*(E,Y)$ resp. $\mathrm{AC}_*(E,Y)$, which is well-known in the scalar-valued case (see, e.g., Theorem 8.5 in [Sak41]). Our proof follows the one given in [Gor94].

2.17 Lemma. *Assume that E is closed and let $(I_k)_k$ be the connected components of $[c,d] \setminus E$. We consider the subsequent statements.*

(a) $\varphi \in \mathrm{BV}_*(E,Y)$ *(*$\varphi \in \mathrm{AC}_*(E,Y)$*).*

(b) $\varphi \in \mathrm{BV}(E,Y)$ *(*$\varphi \in \mathrm{AC}(E,Y)$*) and* $\sum_k \omega(\varphi, \overline{I_k}) < \infty$.

We have (b) $\Longrightarrow$ *(a) and in the* BV_**-case the converse is always valid, while it is also true in the* AC_**-case provided that* φ *is bounded on* $[c,d]$.

Proof. We first show the assertions concerning the implication (a) $\Longrightarrow$ (b). We have $\mathrm{BV}_*(E,Y) \subseteq \mathrm{BV}(E,Y)$ and $\mathrm{AC}_*(E,Y) \subseteq \mathrm{AC}(E,Y)$ and, provided that φ is bounded on $[c,d]$, also $\mathrm{AC}_*(E,Y) \subseteq \mathrm{BV}_*(E,Y)$ by Lemma 2.12. Therefore, it suffices to verify $\sum_k \omega(\varphi, \overline{I_k}) < \infty$ for $\varphi \in \mathrm{BV}_*(E,Y)$. But this is clear as the closed intervals $\overline{I_k}$ (if there are any at all) are mutually non-overlapping with endpoints in E.

We now turn to the proof of (b) $\Longrightarrow$ (a) starting with the case $\varphi \in \mathrm{BV}(E,Y)$. Let $S = \{K_j\}_{j=1}^p \in \mathcal{A}(E)$. Since $c,d \in E$ and since we wish to find an upper bound on $V_\varphi^*(S)$ independent of S, we can assume w.l.o.g. $\bigcup_{j=1} K_j = [c,d]$. Thanks to Lemma 2.16, we estimate

$$\omega(\varphi, K_j) \le V(\varphi, E \cap K_j) + \sum_{\substack{k \\ I_k \subseteq K_j}} \omega(\varphi, \overline{I_k})$$

for every j. Now observe that one has $\bigcup_{j=1}^p S_j \in \mathcal{A}(E)$ if $S_j \in \mathcal{A}(E \cap K_j)$ for each j because the intervals $K_1, \ldots, K_p$ are pairwise non-overlapping. Hence, $\sum_{j=1}^p V(\varphi, E \cap K_j) \le V(\varphi, E)$. As a result, we obtain

$$\sum_{j=1}^p \omega(\varphi, K_j) \le \sum_{j=1}^p V(\varphi, E \cap K_j) + \sum_{j=1}^p \sum_{\substack{k \\ I_k \subseteq K_j}} \omega(\varphi, \overline{I_k}) \le V(\varphi, E) + \sum_k \omega(\varphi, \overline{I_k}) < \infty$$

and $\varphi \in \mathrm{BV}_*(E,Y)$.

Now we assume that $\varphi \in \mathrm{AC}(E,Y)$. Let $\varepsilon > 0$ and choose $\delta_0 > 0$ such that we have $W_\varphi(S) < \frac{\varepsilon}{2}$ for all $S \in \mathcal{A}(E)$ with $W_{\mathrm{id}}(S) < \delta_0$. Choose $N \in \mathbb{N}$ sucht that $\sum_{k>N} \omega(\varphi, \overline{I_k}) < \frac{\varepsilon}{2}$ and put $\delta := \min\{\delta_0, \lambda(I_1), \ldots, \lambda(I_N)\}$ provided, of course, that the set $\{I_k\}_k$ is non-empty, otherwise just set $\delta := \delta_0$. Furthermore, let $(K_j)_{j=1}^p \in \mathcal{A}(E)$ with $\sum_{j=1}^p \lambda(K_j) = \lambda(\bigcup_{j=1}^p K_j) < \delta$. Using that we have $I_k \not\subseteq K_j$ due to $\lambda(K_j) < \delta \le \lambda(I_k)$ for each $j \in \{1, \ldots, p\}$ and each $k \in \{1, \ldots, N\}$ we estimate as above

$$\begin{aligned}\sum_{j=1}^p \omega(\varphi, K_j) &\le \sum_{j=1}^p V(\varphi, E \cap K_j) + \sum_{j=1}^p \sum_{\substack{k \\ I_k \subseteq K_j}} \omega(\varphi, \overline{I_k}) = \sum_{j=1}^p V(\varphi, E \cap K_j) + \sum_{j=1}^p \sum_{\substack{k>N \\ I_k \subseteq K_j}} \omega(\varphi, \overline{I_k}) \\ &\le \sum_{j=1}^p V(\varphi, E \cap K_j) + \sum_{k>N} \omega(\varphi, \overline{I_k}) \le \sum_{j=1}^p V(\varphi, E \cap K_j) + \frac{\varepsilon}{2}.\end{aligned}$$

Let $S_j \in \mathcal{A}(E \cap K_j)$. Then $\bigcup_{j=1}^p S_j \in \mathcal{A}(E)$ with

$$W_{\mathrm{id}}\left(\bigcup_{j=1}^p S_j\right) = \sum_{j=1}^p W_{\mathrm{id}}(S_j) \le \sum_{j=1}^p \lambda(K_j) < \delta.$$

Consequently, $W_\varphi\left(\bigcup_{j=1}^p S_j\right) = \sum_{j=1}^p W_\varphi(S_j) < \frac{\varepsilon}{2}$. Taking the supremum with respect to $S_j \in \mathcal{A}(E \cap K_j)$ then leads to the inequality $\sum_{j=1}^p V(\varphi, E \cap K_j) \le \frac{\varepsilon}{2}$. This finally implies

$$\sum_{j=1}^p \omega(\varphi, K_j) \le \sum_{j=1}^p V(\varphi, E \cap K_j) + \frac{\varepsilon}{2} \le \varepsilon$$

and we deduce $\varphi \in \mathrm{AC}_*(E,Y)$. □

2.18 Corollary. *Assume that E is closed. Then* $\mathrm{AC}_*(E,Y) \cap B([c,d],Y) = \mathrm{BV}_*(E,Y) \cap \mathrm{AC}(E,Y)$.

Proof. The inclusion $\subseteq$ is clear by Lemma 2.12. Conversely, let $\varphi \in \mathrm{BV}_*(E,Y) \cap \mathrm{AC}(E,Y)$ and denote by $(I_k)_k$ the connected components of $[c,d] \setminus E$. By Lemma 2.3, we have $\varphi \in B([c,d],Y)$. Moreover, due to $\varphi \in \mathrm{BV}_*(E,Y)$ we also have $\sum_k \omega(\varphi, \overline{I_k}) < \infty$. Hence, Lemma 2.17 gives the assertion. □

2.19 Corollary. *Let $\varphi \in \mathrm{AC}_*(E,Y)$ be bounded on $[c,d]$ and assume that $\varphi|_{\overline{E}} \in \mathrm{C}(\overline{E},Y)$. Then $\varphi \in \mathrm{AC}_*(\overline{E},Y)$.*

Proof. By Lemma 2.12 (b) and Lemma 2.7, we have $\varphi \in \mathrm{BV}_*(\overline{E},Y)$. Furthermore, Lemma 2.14 yields $\varphi \in \mathrm{AC}(\overline{E},Y)$. As a consequence, the assertion results from Corollary 2.18. □

2.20 Corollary. *Assume that E is closed and that $\varphi|_E \in \mathrm{C}(E,Y)$. Then* $\mathrm{ACG}_*(E,Y) \cap B([c,d],Y) \subseteq \mathrm{BVG}_*(E,Y) \cap \mathrm{ACG}(E,Y) \subseteq \mathrm{ACG}_*(E,Y)$.

Proof. First, let $\varphi \in \mathrm{ACG}_*(E,Y) \cap B([c,d],Y)$ and choose a sequence $(E_n)_n$ such that $\bigcup_{n\in\mathbb{N}} E_n = E$ and $\varphi \in \mathrm{AC}_*(E_n,Y)$ for all n. Thanks to Corollary 2.19 and 2.18 (note that $\overline{E_n} \subseteq E$, hence, φ is bounded on $[\inf E_n, \sup E_n] \subseteq [c,d]$), we infer $\varphi \in \mathrm{BV}_*(\overline{E_n},Y) \cap \mathrm{AC}(\overline{E_n},Y)$ for all n. Hence, $\varphi \in \mathrm{BVG}_*(E,Y) \cap \mathrm{ACG}(E,Y)$.

Now, let $\varphi \in \mathrm{BVG}_*(E,Y) \cap \mathrm{ACG}(E,Y)$. Write $E = \bigcup_n E_n = \bigcup_m \widetilde{E}_m$ with $\varphi \in \mathrm{BV}_*(E_n,Y)$ for all n and $\varphi \in \mathrm{AC}(\widetilde{E}_m,Y)$ for all m. Applying Lemma 2.7 and Lemma 2.14, we derive $\varphi \in \mathrm{BV}_*(\overline{E_n},Y)$ for all n and $\varphi \in \mathrm{AC}(\overline{\widetilde{E}_m},Y)$ for all m. Let $\{F_k\}_k$ be an enumeration of the countable set $\{\overline{E_n} \cap \overline{\widetilde{E}_m} : n,m\}$. Each F_k is closed with $\varphi \in \mathrm{BV}_*(F_k,Y) \cap \mathrm{AC}(F_k,Y)$ and $\varphi|_{F_k} \in \mathrm{C}(F_k,Y)$. Corollary 2.18 implies $\varphi \in \mathrm{AC}_*(F_k,Y)$ for all k and because of $\bigcup_k F_k = E$ we finally conlude $\varphi \in \mathrm{ACG}_*(E,Y)$. □

Let $\emptyset \neq E \subseteq [a,b]$ be closed. By *a linear extension of $\varphi|_E$* to the whole of $[a,b]$ we understand any function $G : [a,b] \to Y$ that extends $\varphi|_E$ and is linear on the subintervals of $[a,b]$ contiguous to E (note that such a G is only uniquely determined on $[c,d]$).

2.21 Lemma. *Assume that E is closed and let G be a linear extension of $\varphi|_E$ to $[a,b]$.*

(a) The function G is continuous on $[a,b] \setminus E$ and each continuity point of $\varphi|_E$ is also a continuity point of G.

(b) If $\varphi \in \mathrm{BV}(E,Y)$, then $G \in \mathrm{BV}([a,b],Y)$.

Proof. We start with the proof of (a). Clearly, G is continuous on $[a,b] \setminus E$. Now assume that there is a $t \in E$ where $\varphi|_E$ is continuous, whereas G is not. Then $t \notin [a,b] \setminus E$ and, obviously $t \notin E^\circ$, so $t \in \partial E$. Furthermore, there is an $\varepsilon > 0$ and a sequence $(t_n)_n$ in $[a,b]$ converging to t with $\|G(t) - G(t_n)\| \geq \varepsilon$ for all $n \in \mathbb{N}$. Due to the continuity of $\varphi|_E$, at most finitely many t_n can belong to E and we may therefore assume w.l.o.g. that $t_n \notin E$ for all n. By defintion G is continuous on $[a, \min E]$ and on $[\max E, b]$. Thus we may further assume that $t_n \in [\min E, \max E] = [c,d]$. Then for each $n \in \mathbb{N}$ there are $c_n, d_n \in E$ with $t_n \in (c_n, d_n)$ and (c_n, d_n) is a connected component of $[c,d] \setminus E$. We now distinguish between two cases.

1st case: There are $C, D \in E$ with $(c_n, d_n) = (C, D)$ for infinitely many $n \in \mathbb{N}$. Then we can extract a subsequence $(t_{n_k})_k$ with $t_{n_k} \in (C,D)$ for all $k \in \mathbb{N}$ and we derive $t \in \{C, D\}$. One then gets

$$\varepsilon \leq \|G(t) - G(t_{n_k})\| = \left\| \varphi(t) - \left(\frac{D - t_{n_k}}{D - C} \cdot \varphi(C) + \frac{t_{n_k} - C}{D - C} \cdot \varphi(D) \right) \right\| \xrightarrow[k\to\infty]{} 0,$$

which is absurd.

2nd case: For each $m \in \mathbb{N}$ the set $\{n \in \mathbb{N} : (c_n, d_n) = (c_m, d_m)\}$ is finite. So we can extract a subsequence $(t_{n_k})_k$ with $(c_{n_k}, d_{n_k}) \neq (c_{n_l}, d_{n_l})$ for $k \neq l$. By considering another subsequence, we may further assume that either $t_{n_k} < t$ for all k or $t > t_{n_k}$ for all k. We only treat the first case (the second one is analogous). By taking once again a subsequence, we can further assume w.l.o.g. that $(t_{n_k})_k$ is strictly increasing. Thus we obtain $c_{n_k} < t_{n_k} < d_{n_k} \leq c_{n_{k+1}} < t_{n_{k+1}} < d_{n_{k+1}} \leq t$ and taking the limit $k \to \infty$ yields $\lim_{k\to\infty} c_{n_k} = \lim_{k\to\infty} d_{n_k} = t$. We thus obtain an index $k_0 \in \mathbb{N}$ with $\|\varphi(c_{n_k}) - \varphi(t)\| < \varepsilon$ and $\|\varphi(d_{n_k}) - \varphi(t)\| < \varepsilon$ for all $k \geq k_0$. For $k \geq k_0$ we now infer

$$
\begin{aligned}
\varepsilon \leq \|G(t) - G(t_{n_k})\| &= \left\| \varphi(t) - \frac{d_{n_k} - t_{n_k}}{d_{n_k} - c_{n_k}} \cdot \varphi(c_{n_k}) - \frac{t_{n_k} - c_{n_k}}{d_{n_k} - c_{n_k}} \cdot \varphi(d_{n_k}) \right\| \\
&= \left\| \frac{d_{n_k} - t_{n_k}}{d_{n_k} - c_{n_k}} \cdot (\varphi(t) - \varphi(c_{n_k})) + \frac{t_{n_k} - c_{n_k}}{d_{n_k} - c_{n_k}} \cdot (\varphi(t) - \varphi(d_{n_k})) \right\| \\
&\leq \frac{d_{n_k} - t_{n_k}}{d_{n_k} - c_{n_k}} \cdot \|\varphi(t) - \varphi(c_{n_k})\| + \frac{t_{n_k} - c_{n_k}}{d_{n_k} - c_{n_k}} \cdot \|\varphi(t) - \varphi(d_{n_k})\| < \varepsilon,
\end{aligned}
$$

which is impossible.

As both cases have led to a contradiction, we conclude that our assumption must be wrong, i.e., part (a) is true.

Part (b) can be essentially shown as in the solution to Exercise 6.2 in [Gor94]. □

2.3 The Banach-Zarecki theorem for ACG$_*$-functions

The main objective of this section is the proof of the Banach-Zarecki theorem for ACG$_*$- and ACG-functions. In order to formulate this theorem we need the following definition.

As before let $\emptyset \neq E \subseteq [a, b]$, $c := \inf E$ and $d := \sup E$ throughout the entire section.

2.22 Definition. *We say that $\varphi : [a, b] \to Y$ satisfies on E Lusin's condition*

- *(a) $(N)_E$ (we then write $\varphi \in (N)_E$) if we have $\mathcal{H}^1_Y(f(A)) = 0$ for each set $A \subseteq E$ with $\lambda^*(A) = 0$;*
- *(b) $[N]_E$ (we then write $\varphi \in [N]_E$) if we have $\mathcal{H}^1_Y(f(A)) = 0$ for each closed set $A \subseteq E$ with $\lambda^*(A) = 0$.*

The following lemma shows that there are plenty examples of functions satisfying Lusin's condition $(N)_E$.

2.23 Lemma. *If $\varphi \in \mathrm{ACG}(E, Y)$, then $\varphi \in (N)_E$.*

Proof. Let $\eta > 0$ and let us first assume that $\varphi \in \mathrm{AC}(E, Y)$. Furhtermore, let $A \subseteq E$ with $\lambda(A) = 0$ and let $\varepsilon > 0$ be arbitrary. Then there is a $\delta > 0$ such that $W_\varphi(S) < \min\{\varepsilon, \frac{\eta}{2}\}$ for all $S \in \mathcal{A}(E)$ with $W_{\mathrm{id}}(S) < \delta$. Choose an open set $G \subseteq \mathbb{R}$ with $A \subseteq G$ and $\lambda(G) < \delta$ and denote by $(I_k)_k$ the countable family of those connected components of G that have non-empty intersection with A. Then $\{\varphi(A \cap I_k)\}_k$ is a countable cover of $\varphi(A)$ and for each k and all $s, t \in I_k \cap A$ we have $|s - t| \leq \lambda(I_k) \leq \lambda(G) < \delta$. Because of $\langle s, t\rangle \in \mathcal{A}(A) \subseteq \mathcal{A}(E)$, this implies $\|\varphi(s) - \varphi(t)\| < \frac{\eta}{2}$. As a result, we infer

$$
\mathrm{diam}(\varphi(I_k \cap A)) \leq \frac{\eta}{2} < \eta. \tag{2.4}
$$

Now pick $s_k, t_k \in I_k \cap A$ for each k. For each finite family $S = \{\langle s_{k_1}, t_{k_1}\rangle, \ldots, \langle s_{k_\nu}, t_{k_\nu}\rangle\}$ ($k_j \neq k_i$ for $i \neq j$) we have $S \in \mathcal{A}(A) \subseteq \mathcal{A}(E)$ (note that the intervals in S do not overlap as they are subintervals of the intervals I_k, which are even pairwise disjoint) and $\sum_{j=1}^{\nu} |s_{k_j} - t_{k_j}| \leq \sum_{j=1}^{\nu} \lambda(I_{k_j}) \leq \lambda(G) < \delta$

and thus $\sum_{j=1}^{\nu} \|\varphi(s_{k_j})-\varphi(t_{k_j})\| < \varepsilon$. This estimate yields $\sum_{j=1}^{\nu} \omega(\varphi, I_{k_j} \cap A) \leq \varepsilon$. As a consequence, we derive

$$\sum_k \operatorname{diam}(\varphi(I_k \cap A)) = \sum_k \omega(\varphi, I_k \cap A) \leq \varepsilon. \tag{2.5}$$

The both inequalities (2.4) and (2.5) together lead to the estimate $\mathcal{H}^1_\eta(\varphi(A)) \leq \varepsilon$. Letting first $\eta \to 0^+$ and afterwards $\varepsilon \to 0^+$ then yields $\mathcal{H}^1_Y(\varphi(A)) = 0$ and we finally derive $\varphi \in (N)_E$.
Now we consider the general case $\varphi \in \mathrm{ACG}(E, Y)$ and we choose a sequence $(E_n)_n$ with $\bigcup_n E_n = E$ and $\varphi \in \mathrm{AC}(E_n, Y)$ for all n. Let $A \subseteq E$ with $\lambda(A) = 0$. Then $\lambda(A_n) = 0$ for all n, where $A_n := A \cap E_n$, and hence $\mathcal{H}^1_Y(\varphi(A_n)) = 0$ by what we have shown so far. Since $\mathcal{H}^1_Y$ is an outer measure, we deduce $\mathcal{H}^1_Y(\varphi(A)) = 0$ and obtain $\varphi \in (N)_E$ as claimed. □

Results of the Banach-Zarecki-type are engaged with partial converses to Lemma 2.23. One important (and surprisingly rather recent result) is the following Theorem due to J. Duda and L. Zajíček that extends the classical Banach-Zarecki theorem for real-valued functions to the vector-valued case.

2.24 Theorem. *The following statements are equivalent.*

(a) $\varphi \in \mathrm{AC}([a, b], Y)$.

(b) $\varphi \in \mathrm{C}([a, b], Y) \cap \mathrm{BV}([a, b], Y)$ *and* $\varphi \in (N)_{[a,b]}$.

(c) $\varphi \in \mathrm{C}([a, b], Y) \cap \mathrm{BV}([a, b], Y)$ *and* $\varphi \in [N]_{[a,b]}$.

Proof. (a) $\Longrightarrow$ (b) follows from Lemma 2.12 (a) and Lemma 2.23. The implication (b) $\Longrightarrow$ (c) is trivial and the remaining implication (c) $\Longrightarrow$ (a) emerges from the proof of the main result in [Dc05]. □

2.25 Corollary. *Assume that E is closed and let G be a linear extension of $\varphi|_E$ to $[a, b]$. If $\varphi \in \mathrm{AC}(E, Y)$, then $G \in \mathrm{AC}([a, b], Y)$.*

Proof. Lemma 2.21 yields $G \in \mathrm{C}([a, b], Y) \cap \mathrm{BV}([a, b], Y)$ and by Lemma 2.23 we have $\varphi \in (N)_E$. Clearly, G is Lipschitz continuous, hence, an AC-function on the closure of each interval contiguous to E in $[a, b]$. Consequently, G fulfills Lusin's condition $(N)_I$ on each such interval I thanks to Lemma 2.23. Since $\mathcal{H}^1_Y$ is an outer measure, the countable union of $\mathcal{H}^1_Y$-zero sets is again a $\mathcal{H}^1_Y$-null set. It is now easy to conclude that $G \in (N)_{[a,b]}$. As a result, we infer that G belongs to $\mathrm{AC}([a, b], Y)$ by employing Theorem 2.24. □

We now come to the announced Banach-Zarecki theorem for functions of generalized absolute continuity.

2.26 Theorem. *Let E be a nonvoid, closed subset of $[a, b]$ and assume $\varphi|_E \in \mathrm{C}(E, Y)$. Then we have:*

(a) $\varphi \in \mathrm{AC}(E, Y) \iff \varphi \in \mathrm{BV}(E, Y)$ *and* $\varphi \in (N)_E \iff \varphi \in \mathrm{BV}(E, Y)$ *and* $\varphi \in [N]_E$,

(b) $\varphi \in \mathrm{ACG}(E, Y) \iff \varphi \in \mathrm{BVG}(E, Y)$ *and* $\varphi \in (N)_E \iff \varphi \in \mathrm{BVG}(E, Y)$ *and* $\varphi \in [N]_E$,

(c) $\varphi \in \mathrm{AC}_*(E, Y) \Longleftarrow \varphi \in \mathrm{BV}_*(E, Y)$ *and* $\varphi \in (N)_E \iff \varphi \in \mathrm{BV}_*(E, Y)$ *and* $\varphi \in [N]_E$,

(d) $\varphi \in \mathrm{ACG}_*(E, Y) \Longleftarrow \varphi \in \mathrm{BVG}_*(E, Y)$ *and* $\varphi \in (N)_E \iff \varphi \in \mathrm{BVG}_*(E, Y)$ *and* $\varphi \in [N]_E$.

In addition, in (c) and (d) the implication $\Longrightarrow$ holds true provided that φ is bounded on the interval $[c, d] = [\min E, \max E]$.

Proof. W.l.o.g. we assume that $\sharp E \geq 2$.

In all cases the second implication $\Longrightarrow$ follows by definition. The addendum and the first implication $\Longrightarrow$ in (a) and (b) result from from Lemma 2.12 and Lemma 2.23.

We now first prove the remaining implications in part (a). Let G be the linear extension of $\varphi|_E$ to $[c,d]$. Lemma 2.21 and the proof of Corollary 2.25 yield $G \in \mathrm{C}([c,d],Y) \cap \mathrm{BV}([c,d],Y)$ and $G \in (N)_{[c,d]}$ if $\varphi \in (N)_E$. We further show that $G \in [N]_{[c,d]}$ if $\varphi \in [N]_E$. For this purpose let $\{I_j\}_{j\in J}$ be the countable family of all connected components of $[c,d] \setminus E$ and assume $\varphi \in [N]_E$. Let $N = \overline{N} \subseteq [c,d]$ with $\lambda(N) = 0$. Then the sets $N \cap E$ and $N \cap \overline{I_j}$ are also closed Lebesgue-null sets and we obtain $\mathcal{H}^1(G(N \cap E)) = \mathcal{H}^1(\varphi(N \cap E)) = 0$ as well as $\mathcal{H}^1(G(N \cap \overline{I_j})) = 0$, and consequently $G \in [N]_{[c,d]}$. Thanks to Theorem 2.24, we derive $G \in \mathrm{AC}([c,d],Y)$ in both cases $\varphi \in (N)_E$ or $\varphi \in [N]_E$. Because of $\varphi|_E = G|_E$, we conclude $\varphi \in \mathrm{AC}(E,Y)$.

We now turn to the proof of part (c). Assume that $\varphi \in \mathrm{BV}_*(E,Y)$ and $\varphi \in [N]_E$. Then we have in particular $\varphi \in \mathrm{BV}(E,Y)$ and $\varphi \in [N]_E$. Hence $\varphi \in \mathrm{AC}(E,Y)$ by part (a) and thus $\varphi \in (N)_E$ due to Lemma 2.23. So we are now in the case $\varphi \in \mathrm{BV}_*(E,Y)$ and $\varphi \in (N)_E$. In particular $\varphi \in \mathrm{BV}(E,Y)$ and $\varphi \in (N)_E$. Consequently, part (a) gives us $\varphi \in \mathrm{AC}(E,Y) \cap \mathrm{BV}_*(E,Y)$, which yields $\varphi \in \mathrm{AC}_*(E,Y)$ by Corollary 2.18.

Finally, the implications $\Longleftarrow$ in part (b) resp. (d) follow from part (a) resp. (c) using Lemma 2.11 resp. Lemma 2.7. □

2.27 Corollary. *Let E be a nonvoid, F_σ-set contained in $[a,b]$ and assume $\varphi|_E \in \mathrm{C}(E,Y)$. Then we have:*

- *(a)* $\varphi \in \mathrm{BV}(E,Y)$ *and* $\varphi \in (N)_E \iff \varphi \in \mathrm{BV}(E,Y)$ *and* $\varphi \in [N]_E$,
- *(b)* $\varphi \in \mathrm{ACG}(E,Y) \iff \varphi \in \mathrm{BVG}(E,Y)$ *and* $\varphi \in (N)_E \iff \varphi \in \mathrm{BVG}(E,Y)$ *and* $\varphi \in [N]_E$,
- *(c)* $\varphi \in \mathrm{BV}_*(E,Y)$ *and* $\varphi \in (N)_E \iff \varphi \in \mathrm{BV}_*(E,Y)$ *and* $\varphi \in [N]_E$,
- *(d)* $\varphi \in \mathrm{ACG}_*(E,Y) \Longleftarrow \varphi \in \mathrm{BVG}_*(E,Y)$ *and* $\varphi \in (N)_E \iff \varphi \in \mathrm{BVG}_*(E,Y)$ *and* $\varphi \in [N]_E$.

In addition, in (d) the implication $\Longrightarrow$ holds true provided that φ is bounded on the interval $[c,d] = [\inf E, \sup E]$.

Proof. The addendum and the implications $\Longrightarrow$ are clear by Lemma 2.12 and the definition of Lusin's conditions. In what follows let $(E_n)_n$ be sequence of closed sets such that $E = \bigcup_{n\in\mathbb{N}} E_n$.

We first complete the proof of part (a). For this purpose, suppose that $\varphi \in \mathrm{BV}(E,Y)$ and $\varphi \in [N]_E$. Then $\varphi \in \mathrm{BV}(E_n,Y) \cap \mathrm{C}(E_n,Y)$ and $\varphi \in [N]_{E_n}$ for each $n \in \mathbb{N}$ and consequently $\varphi \in \mathrm{BV}(E_n,Y)$ and $\varphi \in (N)_{E_n}$ for each $n \in \mathbb{N}$ by part (a) of Theorem 2.26. Since the countable union of $\mathcal{H}^1_Y$-null sets is itself a $\mathcal{H}^1_Y$-null set, we derive $\varphi \in (N)_E$.
A similar argument establishes part (c).
We now turn to the proof of part (b) and we assume that $\varphi \in \mathrm{BVG}(E,Y)$ and $\varphi \in [N]_E$. Let $(F_m)_m$ be a sequence of sets such that $E = \bigcup_m F_m$ and $\varphi \in \mathrm{BV}(F_m,Y)$ for all m. We put $E_{n,m} := \overline{E_n \cap F_m} \subseteq E_n \subseteq E$ for all n and m. Then $\varphi|_{E_{n,m}} \in \mathrm{C}(E_{n,m},Y)$ and hence $\varphi \in \mathrm{BV}(E_{n,m},Y)$ due to Lemma 2.11 and $\varphi \in \mathrm{BV}(E_n \cap F_m,Y)$. Moreover, $\varphi \in [N]_{E_{n,m}}$, which now yields $\varphi \in \mathrm{AC}(E_{n,m},Y)$ for all n and m thanks to part (a) of Theorem 2.26. We therefore conclude $\varphi \in \mathrm{ACG}(E,Y)$.
An analogous argument employing Lemma 2.7 establishes part (d). □

2.28 Remark To our best knowledge Theorem 2.26 and Corollary 2.27, both well-known in the real-valued case, are completely new in the vector-valued case. Nevertheless, our line of argument follows the general strategy in Theorem 6.16 of [Gor94].

2.29 Remark Notice that in general one has neither "$\varphi \in \mathrm{BV}(E,Y)$ and $\varphi \in (N)_E \implies \varphi \in \mathrm{AC}(E,Y)$" nor "$\varphi \in \mathrm{BV}_*(E,Y)$ and $\varphi \in (N)_E \implies \varphi \in \mathrm{AC}_*(E,Y)$". In fact, assume that $Y \neq \{0\}$, pick $y \in Y$ with $\|y\| = 1$, choose an increasing function $\chi : [a,b] \to \mathbb{R}$ which is not absolutely continuous (e.g., the Cantor function) and put $\varphi(t) := \chi(t)y$ for $t \in [a,b]$. Since χ is not absolutely continuous, there exists an $\varepsilon > 0$ such that we can find for each $n \in \mathbb{N}$ a partition $S_n = \{[a_{j,n}, b_{j,n}]\}_{j=1}^{r_n} \in \mathcal{A}([a,b])$ with $W_{\mathrm{id}}(S_n) < \frac{1}{n}$ and $V_\chi^*(S_n) \geq \varepsilon$. The set $E := \bigcup_{n\in\mathbb{N}} \bigcup_{j=1}^{r_n} \{a_{j,n}, b_{j,n}\}$ is countable (hence an F_σ-set) and φ belongs to $\mathrm{BV}_*(E,Y)$ and satisfies Lusin's condition $(N)_E$, but φ is not even an element of $\mathrm{AC}(E,Y)$.

3 Variational measures

In this chapter we introduce the so-called Henstock-Thomson variational measures and study some of their most important properties.

3.1 Definition of variational measures

Let $(\mathfrak{X}, d)$ be a metric space and $\mu : \mathrm{Bor}(\mathfrak{X}) \to [0, \infty]$ a fixed measure.

A non-empty subset $\mathcal{B} \subseteq (\mathrm{Bor}(\mathfrak{X}) \setminus \{\emptyset\}) \times \mathfrak{X}$ is called a *differentiation basis* or just *basis* on $(\mathfrak{X}, d)$ if we have $x \in U$ for all $(U, x) \in \mathcal{B}$ (note that some authors call this a Perron basis, see, e.g., [Pia01]). We call $U \in \mathrm{Bor}(\mathfrak{X})$ a $\mathcal{B}$-set if there exists some $x \in \mathfrak{X}$ such that $(U, x) \in \mathcal{B}$; by $\mathcal{B}(\mathfrak{X})$ we denote the set of all $\mathcal{B}$-sets.

For a non-empty subset E of $\mathfrak{X}$ and a so-called *gauge* $\delta \in (0, \infty)^{\mathfrak{X}}$ on $\mathfrak{X}$, we define

$$\mathcal{B}[E] := \{(U, x) \in \mathcal{B} : \ x \in E\} \qquad \text{and} \qquad \mathcal{B}_\delta[E] := \{(U, x) \in \mathcal{B}[E] : \ U \subseteq U_{\delta(x)}(x)\}.$$

Note that we will sometimes define gauges only on certain subsets of $\mathfrak{X}$; in this case one may set by default the gauge, e.g., equal to 1 on the set where it is not explicitely specified.

Let δ be a gauge on $\mathfrak{X}$. A finite sequence $(U_1, x_1), \ldots, (U_r, x_r)$, $r \in \mathbb{N}$, in $\mathcal{B}_\delta[E]$ such that $U_i^\circ \cap U_j^\circ = \emptyset$, whenever $i, j \in \{1, \ldots, r\}$ are distinct, is called a *δ-fine partition on E* and a *δ-fine partition of E* if even $\bigcup_{j=1}^r U_j = E$ holds. We denote by $\mathcal{P}_\delta(E)$ the set of all δ-fine partitions on E. The points $x_1, \ldots, x_r$ are called the *tags of the partition* $\{(U_j, x_j)\}_{j=1}^r$. In what follows a generic element of $\mathcal{P}_\delta(E)$ will always be written in the form $\{(U_j, x_j)\}_{j=1}^r$. We further put

$$\mathcal{S}(E, \delta) := \left\{P = \{(U_j, x_j)\}_{j=1}^r \in \mathcal{P}_\delta(E) : \ \mu(\overline{U}_i \cap \overline{U}_j) = 0 \text{ for all distinct } i, j \in \{1, \ldots, r\}\right\}.$$

We call $\mathcal{B}$

- an *open basis* if U is open for all $(U, x) \in \mathcal{B}$,
- a *semi-open basis* if every $\mathcal{B}$-set has non-empty interior,
- a *Vitali-basis* if we have $\mathcal{B}_\delta[\{x\}] \neq \emptyset$ for each $x \in \mathfrak{X}$ and $\delta \in (0, \infty)^{\mathfrak{X}}$,
- a *Busemann-Feller-basis* or simply *BF-basis* if we have $(U, x) \in \mathcal{B}$ whenever U is a $\mathcal{B}$-set and $x \in U$.

In what follows, a function $\tau : \mathcal{B}(\mathfrak{X}) \to [0, \infty]$ is called a *$\mathcal{B}$-set function*. For $\emptyset \neq E \subseteq \mathfrak{X}$, $\delta \in (0, \infty)^{\mathfrak{X}}$, $P = \{(U_j, x_j)\}_{j=1}^r \in \mathcal{P}_\delta(E)$ and a $\mathcal{B}$-set function τ we define

$$\tau(P) := \sum_{j=1}^r \tau(U_j),$$

$$W_\delta(\tau, E) := \begin{cases} 0, & \text{if } \mathcal{P}_\delta(E) = \emptyset, \\ \sup\{\tau(P) : \ P \in \mathcal{S}(E, \delta)\}, & \text{elsewise,} \end{cases}$$

and
$$\mu_\tau^*(E) := \inf_{\delta\in(0,\infty)^{\mathfrak{X}}} W_\delta(\tau, E) \qquad \text{and} \qquad \mu_\tau^*(\emptyset) := 0.$$
If we consider different bases simultaneously, we write $\mathcal{P}_\delta(E;\mathcal{B})$, $W_\delta(\tau, E;\mathcal{B})$, $\mu_{\tau,\mathcal{B}}^*$ and so on instead of simply $\mathcal{P}_\delta(E)$, $W_\delta(\tau, E)$, μ_τ^*.
Following well-known patterns like, e.g., in [Tho76] or in [Fau95] one can show the subsequent result.

3.1 Lemma. *The set function μ_τ^* is a metric outer measure on $(\mathfrak{X}, d)$. In particular, the restriction $\mu_\tau := \mu_\tau^*|_{\mathrm{Bor}(\mathfrak{X})}$ is a measure in the usual sense. For this reason, one calls μ_τ^** the $(\mu,\mathcal{B})$-Henstock-Thomson variational measure of/induced by/associated with τ *or simply* the Henstock-Thomson variational measure of/induced by/associated with τ.

Proof. We clearly have $\mu_\tau^*(E) \geq 0$ for all $E \subseteq \mathfrak{X}$ and by definition $\mu_\tau^*(\emptyset) = 0$.

For $E \subseteq F \subseteq \mathfrak{X}$ and any $\delta \in (0,\infty)^{\mathfrak{X}}$ we obviously have $\mathcal{P}_\delta(E) \subseteq \mathcal{P}_\delta(F)$ and thus $W_\delta(\tau, E) \leq W_\delta(\tau, F)$ as well as $\mu_\tau^*(E) \leq \mu_\tau^*(F)$.

Let $(A_n)_n$ be any sequence of subsets of $\mathfrak{X}$ and set $A := \bigcup_{n=1}^\infty A_n$. To see that μ_τ is an outer measure, we have to show $\mu_\tau^*(A) \leq \sum_{n=1}^\infty \mu_\tau^*(A_n)$.

In order to do this we first observe that we have $\mu_\tau^*(E) = 0$ whenever there is some $\delta_0 \in (0,\infty)^{\mathfrak{X}}$ such that $\mathcal{P}_{\delta_0}(E) = \emptyset$; this is clear by definition. In this case we obtain $\mathcal{P}_\delta(F) = \mathcal{P}_\delta(F \setminus E)$ for every $F \supseteq E$ and each $\delta \in (0,\infty)^{\mathfrak{X}}$ satisfying $\delta|_E \leq \delta_0|_E$.

Let us additionally assume that the sets A_n are pairwise disjoint and let $\varepsilon > 0$ be arbitrary. We put $I := \{n \in \mathbb{N} : \exists\, \delta \in (0,\infty)^{\mathfrak{X}} \text{ with } \mathcal{P}_\delta(A_n) = \emptyset\}$ and $J := \mathbb{N}\setminus I$.

For each $n \in I$ we choose a gauge δ_n with $\mathcal{P}_{\delta_n}(A_n) = \emptyset$ and for each $n \in J$ we choose a gauge δ_n with $W_{\delta_n}(\tau, A_n) \leq \mu_\tau^*(A_n) + \frac{\varepsilon}{2^n}$. Next we define $\delta : \mathfrak{X} \to [0,\infty)$ by putting $\delta(x) := 1$ for $x \in \mathfrak{X} \setminus A$ and by setting $\delta(x) := \delta_n(x)$ if $x \in A_n$. (As the sets A_n are pairwise disjoint, δ is well-defined.) If $\mathcal{P}_\delta(A) = \emptyset$, then we clearly have $\mu_\tau^*(A) = 0 \leq \sum_{n=1}^\infty \mu_\tau^*(A_n)$. We thus assume that $\mathcal{P}_\delta(A) \neq \emptyset$ holds. For an arbitrary partition $P = \{(U_j, x_j)\}_{j=1}^r \in \mathcal{P}_\delta(A) = \mathcal{P}_\delta(A \setminus \bigcup_{n\in I} A_n)$ with $\mu(\overline{U_i} \cap \overline{U_j}) = 0$, we then estimate
$$\begin{aligned}\tau(P) &= \sum_{n=1}^\infty \sum_{\substack{j=1\\ x_j\in A_n}}^r \tau(U_j) = \sum_{n\in J} \sum_{\substack{j=1\\ x_j\in A_n}}^r \tau(U_j) \leq \sum_{n\in J} W_{\delta_n}(\tau, A_n)\\ &\leq \sum_{n\in J}\Big(\mu_\tau^*(A_n) + \frac{\varepsilon}{2^n}\Big) \leq \varepsilon + \sum_{n=1}^\infty \mu_\tau^*(A_n).\end{aligned}$$
Taking the supremum over such P, we infer
$$W_\delta(\tau, A) \leq \varepsilon + \sum_{n=1}^\infty \mu_\tau^*(A_n)$$
and thus
$$\mu_\tau^*(A) \leq \varepsilon + \sum_{n=1}^\infty \mu_\tau^*(A_n).$$
Letting $\varepsilon \to 0^+$, we arrive at $\mu_\tau^*(A) \leq \sum_{n=1}^\infty \mu_\tau^*(A_n)$.

If the sets A_n are not pairwise disjoint, we put $B_1 := A_1$ and $B_n := A_n \setminus \bigcup_{j=1}^{n-1} A_j$ for $n > 1$. We then obtain
$$\mu_\tau^*(A) = \mu_\tau^*\Big(\bigcup_{n=1}^\infty B_n\Big) \leq \sum_{n=1}^\infty \mu_\tau^*(B_n) \leq \sum_{n=1}^\infty \mu_\tau^*(A_n),$$

i.e., μ_τ^* is an outer measure.

In order to complete the proof, we now show that we have $\mu_\tau^*(A_1 \cup A_2) = \mu_\tau^*(A_1) + \mu_\tau^*(A_2)$ for $A_1, A_2 \subseteq \mathfrak{X}$ provided that there are disjoint open sets $V_1, V_2 \in \mathcal{T}(\mathfrak{X})$ with $A_j \subseteq V_j$. Since we already know $\mu_\tau^*(A_1 \cup A_2) \leq \mu_\tau^*(A_1) + \mu_\tau^*(A_2)$, it only remains to establish $\mu_\tau^*(A_1) + \mu_\tau^*(A_2) \leq \mu_\tau^*(A_1 \cup A_2)$. This is clear for $\mu_\tau^*(A_1 \cup A_2) = \infty$. Therefore we may assume that $\mu_\tau^*(A_1 \cup A_2)$ is finite, which also yields $\mu_\tau^*(A_j) < \infty$ for $j \in \{1, 2\}$.

Moreover, the assertion is also true if we have $\mu_\tau^*(A_j) = 0$ for some $j \in \{1, 2\}$. Hence, we may assume that $\mu_\tau^*(A_j) > 0$ for $j \in \{1, 2\}$, which yields in particular $\mathcal{P}_\delta(A_j) \neq \emptyset$ for $j \in \{1, 2\}$ and thus $\mathcal{P}_\delta(A_1 \cup A_2) \neq \emptyset$ for each $\delta \in (0, \infty)^{\mathfrak{X}}$.

Now let $\varepsilon > 0$ be arbitrary and take a gauge $\delta_0 \in (0, \infty)^{\mathfrak{X}}$ with $W_{\delta_0}(\tau, A_1 \cup A_2) \leq \mu_\tau^*(A_1 \cup A_2) + \varepsilon$. For $x \in V_j$ and $j \in \{1, 2\}$ we choose a positive number $\rho(x) > 0$ with $\Delta(x, \rho(x)) \subseteq V_j$. We put $\delta(x) := \delta_0(x)$ for $x \in \mathfrak{X} \setminus (V_1 \cup V_2)$ and $\delta(x) := \min\{\delta_0(x), \rho(x)\}$ for $x \in V_1 \cup V_2$. For $j \in \{1, 2\}$ we take a sequence $P_j = \{(U_{i,j}, x_{i,j})\}_{i=1}^{r_j}$ in $\mathcal{P}_\delta(A_j)$ with $\mu(\overline{U_{i,j}} \cap \overline{U_{k,j}}) = 0$ for distinct $i, k \in \{1, \ldots, r_j\}$ such that $\tau(P_j) \geq W_\delta(\tau, A_j) - \varepsilon$. (Note that $W_\delta(\tau, A_j)$ is finite because $W_{\delta_0}(\tau, A_1 \cup A_2)$ is finite and because $\delta \leq \delta_0$.) Thanks to $U_{i,j} \subseteq U_{\delta(x_{i,j})}(x_{i,j}) \subseteq \overline{U_{\delta(x_{i,j})}(x_{i,j})} \subseteq \Delta(x_{i,j}, \delta(x_{i,j})) \subseteq V_j$, we have $U_{i,j}^\circ \subseteq \overline{U_{i,j}} \subseteq V_j$ and we see that $P := \{(U_{i,j}, x_{i,j})\}_{\substack{i=1,\ldots,r_j \\ j \in \{1,2\}}}$ is a δ-fine partition on $A_1 \cup A_2$ with $\mu(\overline{U_{i,j}} \cap \overline{U_{k,l}}) = 0$ for distinct pairs $(i, j), (k, l) \in (\{1, \ldots, r_1\} \times \{1\}) \cup (\{1, \ldots, r_2\} \times \{2\})$. Therefore we can now estimate

$$\begin{aligned}\mu_\tau^*(A_1) + \mu_\tau^*(A_2) &\leq W_\delta(\tau, A_1) + W_\delta(\tau, A_2) \leq \tau(P_1) + \tau(P_2) + 2\varepsilon = \tau(P) + 2\varepsilon \\ &\leq W_{\delta_0}(\tau, A_1 \cup A_2) + 2\varepsilon \leq \mu_\tau^*(A_1 \cup A_2) + 3\varepsilon,\end{aligned}$$

and deduce $\mu_\tau^*(A_1) + \mu_\tau^*(A_2) \leq \mu_\tau^*(A_1 \cup A_2)$ by letting $\varepsilon \to 0$. □

3.2 Remark If no confusion is to be expected and if the difference does not matter, we frequently write just μ_τ instead of μ_τ^* and we use the notation μ_τ^* especially in cases where we want to emphasize the character of an outer measure (as, e.g., in Corollary 3.14 and 3.16) to illustrate how certain measure theoretical properties of the outer measure μ_τ^* are related to corresponding properties of μ_τ in order to feature special particularities inherent in the nature of variational measures.

3.2 σ-finite variational measures

In 1997 Z. Buczolich and W. F. Pfeffer asked the question (see Question 5.4 in [BP97]) whether or not certain variational measures on $\mathbb{R}^d$ are automatically σ-finite provided that they are absolutely continuous with respect to d-dimensional Lebesgue measure. Since then there has been a lot of research on this topic starting from the affirmative answer to this question in the one-dimensional case already given in [BPS96, Theorem 1] and then producing strengthened results (see, e.g., [Tho98, Zhe07a]), extensions to higher dimensions (see, e.g., [BP98, Pia01, SZ04]) as well as extensions to the case of more general differentiation bases (see, e.g., [BPS00, BPS02, BPS06, Zhe07b]), to name but a few. Note that this affirmative answer is a very characteristic feature of variational measures. For example, consider the (outer) measure that vanishes on all Lebesgue-null sets and assigns the value ∞ to all other sets. This measure is clearly absolutely continuous with respect to Lebesgue measure, but it is not even semifinite. Moreover, see Theorem 1.1 in [EK06] for a much more elaborate example of this kind.
However, to our best knowledge the analogous question where absolute continuity with respect to the Lebesgue measure is replaced by absolute continuity with respect to some more general measure has only been treated in [Ene00] and [Cap03], although such studies are well motivated, e.g., by the intention to obtain full descriptive characterisations of Henstock-Kurzweil-Stieltjes integrals, see [Ene00, Theorem 5.1] combined with [Fau97, Theorem 4.7]) and see also Lemma 5.19 and Lemma

5.20 below.
In the papers [Ene00] and [Cap03], the authors use an approach essentially based on properties of the functions of class BVG_* or ACG_* so that it is not "clear that this is a feature of the method used to construct the measures and not a property merely of functions", where we borrowed these words from Thomson (see [Tho98]).
In the spirit of these words, we want to make a contribution to this question. In the next section we first prove an extension of Thomson's theorem [Tho98, Theorem 1], where the real line is replaced by a metric space that is complete or locally compact and the differentiation basis of open intervals is replaced by an open differentiation basis.
Afterwards, we focus our attention on the corresponding situation for the real line where we consider Busemann-Feller interval bases. We shall derive an extension of [BPS00, Theorem 3.1 (α)] due to Bongiorno, Di Piazza and Skvortsov, which entails (as we will see in the next section) as special cases extensions of the results obtained by Ene ([Ene00, Theorem 3.2]) and Caponetti ([Cap03, Theorem 3.4])
The scheme of proof has been appeared several times in the literature (cf., e.g., [BPS96, BPS00, BPS02, Cap03, Pia01, SZ04, Tho98, Zhe07b, Zhe07a]) and is well-known to experts. Therefore our contribution is not based on inventing a new approach, but on the try to exhaust this well-known method of proof as effectively as possible. Indeed, there is one new small, but decisive feature consisting of the definition of the variational measure in the preceding section: It is now related to the given reference measure μ. It is this slight modification that allows us to decouple measure and distance within the framework of the afore-mentioned method of proof and to derive the mentioned generalisations.

3.2.1 An extension of Thomson's theorem

The aim of this section is the proof of the following result, which generalises Thomson's theorem [Tho98, Theorem 1]. In particular, this extension is applicable to situations in $\mathbb{R}^d$, although for $d > 1$ usually quite different arguments are needed (see, e.g., [BP98, BP97]).

3.3 Theorem. *Assume that $(\mathfrak{X}, d)$ is complete or locally compact and let $\emptyset \neq E \subseteq \mathfrak{X}$ be a subset such that (E, d_E) is separable. Let $\mathcal{B}$ be an open Vitali-BF-basis and τ a $\mathcal{B}$-set function. Moreover, assume that μ vanishes on all singletons contained in E and that μ is* semi-moderate *on E, i.e., there is a sequence $(E_n)_n$ of closed sets with $\bigcup_{n\in\mathbb{N}} E_n = E$ such that $\mu(E_n) < \infty$ for all $n \in \mathbb{N}$ (in particular, E is an F_σ-set and so μ is indeed defined on the Borel set E). Suppose that μ_τ is σ-finite on singletons (i.e., finite on singletons) and on each perfect compact μ-nullset contained in E. Then μ_τ is σ-finite on E.*

Proof. We divide the proof into several steps.
step 1. Suppose that the assertion is already established in the special case that E is closed and μ is finite on E. Then we may deduce the general assertion as follows: For every $n \in \mathbb{N}$ the measure μ_τ is clearly σ-finite on each perfect compact μ-nullset contained in E_n and μ is finite on E_n; in addition, each space (E_n, d_{E_n}) ($n \in \mathbb{N}$) is separable. Therefore the special case yields that μ_τ is σ-finite on E_n for each $n \in \mathbb{N}$ and thus σ-finite on E. Consequently, we may and will assume in what follows that E is closed and μ is finite on E.
step 2. We define

$$\mathfrak{U} := \{U \in \mathcal{T}(\mathfrak{X}) : \ \mu_\tau \text{ is } \sigma\text{-finite on } E \cap U\}$$

and $\mathcal{O} := \bigcup \mathfrak{U}$, which is clearly open. We note that the measure μ_τ is σ-finite on $E \cap \mathcal{O}$. Indeed, for each $x \in E \cap \mathcal{O}$ we can find a set $U(x)$ in $\mathfrak{U}$ with $x \in U(x)$. Then $\{U(x) \cap E\}_{x \in E\cap\mathcal{O}} = \{U(x) \cap E \cap \mathcal{O}\}_{x\in E\cap\mathcal{O}}$ is an open (relative to $(E \cap \mathcal{O}, d_{E\cap\mathcal{O}})$) cover of the separable metric space $(E \cap \mathcal{O}, d_{E\cap\mathcal{O}})$. Since every separable metric space is a Lindelöf space, there is a sequence $(x_n)_n$ in $E \cap \mathcal{O}$ with $E \cap \mathcal{O} = \bigcup_{n\in\mathbb{N}} (U(x_n) \cap E)$. Since μ_τ is σ-finite on each of the sets $U(x_n) \cap E$, it is σ-finite on $E \cap \mathcal{O}$ as asserted.

Now we put $P := E \setminus \mathcal{O}$. Obviously, P is closed. Furthermore, P has no isolated points: If $x_0 \in P$ were an isolated point of P, then there would be a number $r > 0$ with $U_r(x_0) \cap P = \{x_0\}$. The set $U := U_r(x_0) \setminus \{x_0\} \subseteq \mathfrak{X} \setminus P = (\mathfrak{X} \setminus E) \cup \mathcal{O}$ would satisfy $U \cap E \subseteq E \cap \mathcal{O}$, and hence μ_τ would be σ-finite on $U \cap E$. As a result, μ_τ would be σ-finite on $(U \cup \{x_0\}) \cap E = U_r(x_0) \cap E$. But this would imply $U_r(x_0) \in \mathfrak{U}$ and consequently $x_0 \in \mathcal{O}$ contradicting $x_0 \in P \subseteq \mathfrak{X} \setminus \mathcal{O}$.
We next observe that μ_τ is not σ-finite on $P \cap U$ for any $U \in \mathcal{T}(\mathfrak{X})$ with $P \cap U \neq \emptyset$. Indeed, if there were such a U, then μ_τ would be σ-finite on $E \cap U$ thanks to the decomposition $E \cap U = (U \cap P) \dot\cup (U \cap E \cap \mathcal{O})$ and as a consequence we could derive $U \in \mathfrak{U}$, which would yield the contradiction $\emptyset = P \cap \mathcal{O} \supseteq P \cap U \neq \emptyset$.
step 3 a). We claim that P is empty. As soon as we will have shown this, we obtain $E \subseteq \mathcal{O}$ and we see that μ_τ is σ-finite on $E = E \cap \mathcal{O}$.
In order to verify this claim we suppose to the contrary that P is nonvoid. Since (P, d_P) is a perfect separable metric space which is complete or locally compact (as P is a non-empty closed subset of the closed set E), P is infinite (even uncountable, see, e.g., [Kec95, Theorem (5.3) and Theorem (6.2)]) and we may choose two distinct elements x_1, x_2 of P. Let $j \in \{1, 2\}$. Because of $\Delta(x_j, \frac{1}{n}) \cap E \downarrow \{x_j\}$ and because of the finiteness of μ on E, we obtain

$$0 = \mu(\{x_j\}) = \mu\left(\bigcap_{n \in \mathbb{N}} \left(\Delta(x_j, 1/n) \cap E\right)\right) = \lim_{n \to \infty} \mu(\Delta(x_j, 1/n) \cap E).$$

There thus exists an index $n_j \in \mathbb{N}$ with $\mu(\Delta(x_j, \frac{1}{n}) \cap E) \leq \frac{1}{4}$ for all $n \geq n_j$. We may additionally assume that each n_j is so large that $\Delta(x_1, \frac{1}{n_1}) \cap \Delta(x_2, \frac{1}{n_2}) = \emptyset$ and that both $\Delta(x_1, \frac{1}{n_1})$ and $\Delta(x_2, \frac{1}{n_2})$ are compact in the case that $(\mathfrak{X}, d)$ is locally compact. For $x \in U_{\frac{1}{n_j}}(x_j)$ with $j \in \{1, 2\}$ we define the gauge

$$\delta(x) := \begin{cases} \frac{1}{2} \min\{\operatorname{dist}(x, \mathfrak{X} \setminus U_{\frac{1}{n_j}}(x_j)), d(x, x_j), \frac{1}{2}\}, & \text{if } x \neq x_j, \\ \frac{1}{2} \min\{\operatorname{dist}(x, \mathfrak{X} \setminus U_{\frac{1}{n_j}}(x_j)), \frac{1}{2}\}, & \text{if } x = x_j. \end{cases}$$

Due to $\mu_\tau(U_{\frac{1}{n_j}}(x_j) \cap P) = \infty$, we have $W_{\delta'}(\tau, U_{\frac{1}{n_j}}(x_j) \cap P) = \infty$ for each gauge δ'. Therefore we can find a a partition $\{(U_{i,j}, \xi_{i,j})\}_{i=1}^{r_j} \in \mathcal{P}_\delta(U_{\frac{1}{n_j}}(x_j) \cap P)$ with $r_j \in \mathbb{N}$ and $\mu(\overline{U}_{i,j} \cap \overline{U}_{k,j}) = 0$ for all distinct $i, k \in \{1, \ldots, r_j\}$ such that $\sum_{i=1}^{r_j} \tau(U_{i,j}) > 2$. We observe that $U_{i,j} \subseteq U_{\delta(\xi_{i,j})}(\xi_{i,j}) \subseteq U_{\frac{1}{n_j}}(x_j)$ and thus $\operatorname{diam}(U_{i,j}) \leq 2\delta(\xi_{i,j}) \leq \frac{1}{2}$. If $x_j \neq \xi_{i,j}$ for some $i \in \{1, \ldots, r_j\}$, then x_j does not even belong to $\Delta(\xi_{i,j}, \delta(\xi_{i,j}))$ because otherwise $0 < d(x_j, \xi_{i,j}) \leq \delta(\xi_{i,j}) \leq \frac{1}{2} d(x_j, \xi_{i,j})$ would follow. Assume for a moment that $x_j \neq \xi_{i,j}$ holds indeed for every $i \in \{1, \ldots, r_j\}$. According to what we have just said, we may then choose a radius $\varepsilon \in (0, \min\{\frac{1}{4}, \frac{1}{n_j}, \delta(x_j)\})$ so small that $\Delta(x_j, \varepsilon)$ and $\Delta(\xi_{i,j}, \delta(\xi_{i,j}))$ are disjoint for all $i \in \{1, \ldots, r_j\}$ and $\Delta(x_j, \varepsilon)$ is compact provided that $(\mathfrak{X}, d)$ is locally compact. Thanks to the premise that $\mathcal{B}$ is a Vitali-basis, there exists a set V in $\mathcal{B}(\mathfrak{X})$ with $(V, x_j) \in \mathcal{B}_\varepsilon[\{x_j\}]$; hence, $V \subseteq U_\varepsilon(x_j)$. We clearly have $\overline{V} \cap \overline{U_{i,j}} = \emptyset$ for each $i \in \{1, \ldots, r_j\}$. As a consequence, we may add (V, x_j) to the sequence $\{(U_{i,j}, \xi_{i,j})\}_{i=1}^{r_j}$ without violating the subsequent conditions:

- $\xi_{i,j} \in U_{i,j} \subseteq U_{\frac{1}{n_j}}(x_j)$ and $\xi_{i,j} \in P$ for all $i \in \{1, \ldots, r_j\}$,
- $U_{i,j}$ is open for all $i \in \{1, \ldots, r_j\}$,
- $\operatorname{diam}(U_{i,j}) \leq \frac{1}{2}$ for all $i \in \{1, \ldots, r_j\}$,
- $\sum_{i=1}^{r_j} \tau(U_{i,j}) > 2$,
- $U_{i,j} \cap U_{k,j} = \emptyset$ for all $i, k \in \{1, \ldots, r_j\}$ with $i \neq k$,
- $\mu(\overline{U_{i,j}} \cap \overline{U_{k,j}}) = 0$ for all $i, k \in \{1, \ldots, r_j\}$ with $i \neq k$,
- $\overline{U_{i,j}}$ is compact for all $i \in \{1, \ldots, r_j\}$ if $(\mathfrak{X}, d)$ is locally compact.

Therefore we assume without loss of generality that x_j is an element of $\{\xi_{i,j} : i \in \{1, \dots, r_j\}\}$.

Using the inclusion-exclusion principle for finite measures, we arrive at

$$\sum_{i=1}^{r_j} \mu(\overline{U_{i,j}} \cap E) = \mu\left(\bigcup_{i=1}^{r_j} \overline{U_{i,j}} \cap E\right) \leq \mu(\Delta(x_j, 1/n_j) \cap E) \leq \frac{1}{4}.$$

We next relabel the pairs $\{(U_{i,j}, \xi_{i,j})\}_{\substack{i=1,\dots,r_j \\ j\in\{1,2\}}}$ by

$$(V_{1,1}, x_{1,1}),\ (V_{1,2}, x_{1,2}),\ (V_{1,3}, x_{1,3}), \dots$$

and observe that

$$\sum_k \mu(\overline{V_{1,k}} \cap E) \leq \sum_{j=1}^{2} \mu(\Delta(x_j, 1/n_j) \cap E) \leq \frac{2}{4} = \frac{1}{2}.$$

We further obtain $V_{1,p} \cap V_{1,q} = \emptyset$ as well as $\mu(\overline{V_{1,p}} \cap \overline{V_{1,q}}) = 0$ for $p \neq q$ due to the construction and due to $\Delta(x_1, \frac{1}{n_1}) \cap \Delta(x_2, \frac{1}{n_2}) = \emptyset$.

We repeat the preceding procedure as follows. Take any of the indices k. Because $x_{1,k} \in V_{1,k} \cap P$, $V_{1,k}$ is open and P is perfect, we can find a point $\zeta_{2,k}$ in $V_{1,k} \cap P$ such that $\zeta_{1,k} := x_{1,k}$ and $\zeta_{2,k}$ are distinct. As above we can find integers $n_{j,k} \in \mathbb{N}$ for $j \in \{1,2\}$ such that $\Delta(\zeta_{j,k}, \frac{1}{n}) \subseteq V_{1,k}$ and $\mu(\Delta(\zeta_{j,k}, \frac{1}{n}) \cap E) \leq \frac{1}{4}\mu(\overline{V_{1,k}} \cap E)$ for all $n \geq n_{j,k}$, $\Delta(\zeta_{1,k}, \frac{1}{n_{1,k}})$ and $\Delta(\zeta_{2,k}, \frac{1}{n_{2,k}})$ are disjoint and such that both $\Delta(\zeta_{1,k}, \frac{1}{n_{1,k}})$ and $\Delta(\zeta_{2,k}, \frac{1}{n_{2,k}})$ are compact in the case that $(\mathfrak{X}, d)$ is locally compact.
For $x \in U_{\frac{1}{n_{j,k}}}(\zeta_{j,k})$ and $j \in \{1,2\}$ we define the gauge

$$\delta(x) := \begin{cases} \frac{1}{2}\min\{\operatorname{dist}(x, \mathfrak{X} \setminus U_{\frac{1}{n_{j,k}}}(\zeta_{j,k})), d(x, \zeta_{j,k}), \frac{1}{4}\}, & \text{if } x \neq \zeta_{j,k} \\ \frac{1}{2}\min\{\operatorname{dist}(x, \mathfrak{X} \setminus U_{\frac{1}{n_{j,k}}}(\zeta_{j,k})), \frac{1}{4}\}, & \text{if } x = \zeta_{j,k}. \end{cases}$$

Again there exists a partition $\{(U_{i,j,k}, \xi_{i,j,k})\}_{i=1}^{r(j,k)}$ in $\mathcal{B}_\delta[U_{\frac{1}{n_{j,k}}}(\zeta_{j,k}) \cap P]$ satisfying

- $\xi_{i,j,k} \in U_{i,j,k} \subseteq U_{\delta(\xi_{i,j,k})}(\xi_{i,j,k}) \subseteq U_{\frac{1}{n_{j,k}}}(\zeta_{j,k}) \subseteq V_{1,k}$ and $\xi_{i,j,k} \in P$ for all $i \in \{1, \dots, r(j,k)\}$,
- $U_{i,j,k}$ is open for all $i \in \{1, \dots, r(j,k)\}$,
- $\operatorname{diam}(U_{i,j,k}) \leq \frac{1}{4}$ for all $i \in \{1, \dots, r(j,k)\}$,
- $\sum_{i=1}^{r(j,k)} \tau(U_{i,j,k}) > 4$,
- $U_{i,j,k} \cap U_{\iota,j,k} = \emptyset$ for all $i, \iota \in \{1, \dots, r(j,k)\}$ with $i \neq \iota$,
- $\mu(\overline{U_{i,j,k}} \cap \overline{U_{\iota,j,k}}) = 0$ for all $i, \iota \in \{1, \dots, r(j,k)\}$ with $i \neq \iota$
- $\zeta_{j,k} \in \{\xi_{i,j,k} : i \in \{1, \dots, r(j,k)\}\}$,
- $\overline{U_{i,j,k}}$ is compact for all $i \in \{1, \dots, r(j,k)\}$ if $(\mathfrak{X}, d)$ is locally compact.

It follows

$$\begin{aligned}\sum_{i=1}^{r(j,k)} \mu(\overline{U_{i,j,k}} \cap E) &= \mu\left(\bigcup_{i=1}^{r(j,k)} \overline{U_{i,j,k}} \cap E\right) \\ &\leq \mu(\Delta(\zeta_{j,k}, 1/n_{j,k}) \cap E) \leq \frac{1}{4}\mu(\overline{V_{1,k}} \cap E)\end{aligned}$$

as well as

$$\sum_{j=1}^{2} \sum_{i=1}^{r(j,k)} \mu(\overline{U_{i,j,k}} \cap E) \leq \frac{1}{2}\mu(\overline{V_{1,k}} \cap E).$$

We thus infer

$$\sum_k \sum_{j=1}^{2} \sum_{i=1}^{r(j,k)} \mu(\overline{U_{i,j,k}} \cap E) \leq \frac{1}{2} \sum_k \mu(\overline{V_{1,k}} \cap E) \leq \frac{1}{4}.$$

We next relabel the pairs $\{(U_{i,j,k}, \xi_{i,j,k})\}_{\substack{i=1,\dots,r(j,k)\\ j\in\{1,2\},\ k}}$ by

$$(V_{2,1}, x_{2,1}),\ (V_{2,2}, x_{2,2}), \dots .$$

Observe that for every k we have $x_{1,k} = x_{2,l}$ for some l.
So, proceeding by induction we construct for each $i \in \mathbb{N}$ a finite sequence

$$(V_{i,1}, x_{i,1}),\ (V_{i,2}, x_{i,2}), \dots$$

at least of length 2 with $\mathcal{B}$-sets (in particular open sets) $V_{i,k}$ fulfilling the subsequent requirements:

(a) $x_{i,k} \in P \cap V_{i,k}$,

(b) for each $j > i$ there is some l such that $x_{i,k} = x_{j,l}$,

(c) each set $V_{i,k}$ is contained in some set $V_{i-1,l}$ if $i > 1$),

(d) each set $V_{i-1,l}$ contains at least two of the sets $V_{i,k}$ if $i > 1$,

(e) $\sum_{\substack{k\\ V_{i,k}\subseteq V_{i-1,l}}} \tau(V_{i,k}) > 2^i$ for each l and $i > 1$ and $\sum_k \tau(V_{1,k}) > 2$,

(f) $\operatorname{diam}(V_{i,k}) \leq \frac{1}{2^i}$,

(g) $\sum_k \mu(\overline{V_{i,k}} \cap E) \leq \frac{1}{2^i}$,

(h) $\mu(\overline{V_{i,k}} \cap \overline{V_{i,l}}) = 0$ for distinct k, l,

(i) $V_{i,k} \cap V_{i,l} = \emptyset$ for distinct k, l,

(j) $\overline{V_{i,k}}$ is compact for all i and k provided that $(\mathfrak{X}, d)$ is locally compact.

step 3 b). We now define $N := \bigcap_{i\in\mathbb{N}} \bigcup_k \left(\overline{V_{i,k}} \cap E\right) \subseteq E$. Clearly, N is closed. Thanks to condition (c) above we have

$$\bigcup_k \left(\overline{V_{i,k}} \cap E\right) \subseteq \bigcup_l \left(\overline{V_{i-1,l}} \cap E\right)$$

and hence $\bigcup_k \left(\overline{V_{i,k}} \cap E\right) \downarrow N$. Together with conditions (a) and (b) this fact yields $x_{i,k} \in N$ for every i and k. In particular, N is not empty and each set $V_{i,k}$ contains a point of N.
If $(\mathfrak{X}, d)$ is locally compact, then N is compact since all sets $\overline{V_{i,k}} \cap E$ are compact by construction. If $(\mathfrak{X}, d)$ is complete, then (N, d_N) is complete, too, and N is covered by the finitely many sets $\{\overline{V_{i,k}} \cap E\}_k$ each of which has diameter of at most $\frac{1}{2^i}$ according to (f). Using a diagonalisation argument we infer that each sequence in N possesses a subsequence which is a Cauchy sequence. Hence, it converges by the completeness of (N, d_N) and N is compact also in this case.
Now we show that N is perfect. Take $x \in N$ and $r > 0$ and choose an integer $i \in \mathbb{N}$ with $\frac{1}{2^i} < r$. Because of $x \in N$ there exists an index k such that $x \in \overline{V_{i,k}}$. Using condition (f) we deduce $\overline{V_{i,k}} \subseteq U_r(x)$. Thanks to conditions (d) and (i) there are distinct $x_{i+1,l}$ and $x_{i+1,l'}$ contained in $\overline{V_{i,k}} \subseteq U_r(x)$. But as we already mentioned above these points both belong to N and clearly at least one of them is distinct from x. Therefore we conclude that $(U_r(x) \cap N) \setminus \{x\}$ is nonvoid.
Conditions (g) and (h) further imply

$$\mu(N) = \lim_{i\to\infty} \mu\left(\bigcup_k \left(\overline{V_{i,k}} \cap E\right)\right) = \lim_{i\to\infty} \sum_k \mu\left(\overline{V_{i,k}} \cap E\right) \leq \lim_{i\to\infty} 2^{-i} = 0.$$

Consequently, μ_τ is σ-finite on N by assumption and we can choose a finite or infinite sequence $(N_p)_p$ of pairwise disjoint non-empty Borel sets with $\bigcup_p N_p = N$ and $\mu_\tau(N_p) < \infty$ for each p. Due

to $\mu_\tau(N_p) < \infty$ and the pairwise disjointness of the sets N_p, we can thus find a gauge δ on $\mathfrak{X}$ with $W_\delta(\tau, N_p) < \infty$ for each p. We next set $E_m := \{x \in N : \delta(x) > \frac{1}{m}\}$ for each $m \in \mathbb{N}$. Clearly $N = \bigcup_{m,p}(N_p \cap E_m)$ and the closedness of N yields $N = \bigcup_{m,p} \overline{N_p \cap E_m}$. Since the space (N, d_N) it compact, Baire's theorem applies and gives indices p and m such that $\overline{N_p \cap E_m}$ has non-empty interior in the metric space (N, d_N). There thus exists an open set U in $\mathcal{T}(\mathfrak{X})$ with $\emptyset \neq N \cap U \subseteq \overline{N_p \cap E_m}$, where we may assume without loss of generality $\operatorname{diam}(U) < \frac{1}{m}$. (Just take some $x \in N \cap U$ and choose $\varepsilon \in (0, \frac{1}{2m})$ such that $U_\varepsilon(x) \subseteq U$ holds and replace U by $U_\varepsilon(x)$.) One easily verifies that $N_p \cap E_m \cap U$ is dense in $N \cap U$.
We finally claim that for all sufficiently large indices i there is an index k_i with $\overline{V_{i-1,k_i}} \subseteq U$. Indeed, fix $x \in N \cap U$, choose $r > 0$ with $\Delta(x, r) \subseteq U$ and take $i \in \mathbb{N}$ with $i > 1$ and $\frac{1}{2^{i-1}} < r$. Since $x \in N$, for each such i there is an index k_i with $x \in \overline{V_{i-1,k_i}}$. Due to $\operatorname{diam}(\overline{V_{i-1,k_i}}) = \operatorname{diam}(V_{i-1,k_i}) \leq \frac{1}{2^{i-1}} < r$, we thus infer $\overline{V_{i-1,k_i}} \subseteq \Delta(x, r) \subseteq U$.
As we mentioned above, each set $V_{i,k}$ contains a point of N. Therefore for all sufficiently large i, each set $V_{i,l}$ contained in $V_{i-1,k_i} \subseteq U$ possesses an element of $N \cap U$. Since $N_p \cap E_m \cap U$ is dense in $N \cap U$ and $V_{i,l} \cap N \cap U$ is open in $N \cap U$, we can find a point $y_{i,l}$ belonging to $N_p \cap E_m \cap V_{i,l}$. The collection $\{(V_{i,l}, y_{i,l})\}_{l, V_{i,l} \subseteq V_{i-1,k_i}}$ belongs to $\mathcal{B}$ because $\mathcal{B}$ is a BF-basis. Moreover, we have $\delta(y_{i,l}) \geq \frac{1}{m}$ and $\operatorname{diam}(V_{i,l}) < \frac{1}{m}$ thanks to $y_{i,l} \in E_m$, and $V_{i,l} \subseteq U$ respectively. As a result, $V_{i,l} \subseteq U_{\delta(y_{i,l})}(y_{i,l})$ and $\{(V_{i,l}, y_{i,l})\}_{l, V_{i,l} \subseteq V_{i-1,k_i}}$ is a δ-fine partition on N_p with $\mu(\overline{V_{i,l}} \cap \overline{V_{i,k}}) = 0$ for distinct l and k (recall conditions (h) and (i) from above). Consequently, we obtain (by condition (e) from above)

$$\infty > W_\delta(\tau, N_p) \geq \sum_{\substack{l \\ V_{i,l} \subseteq V_{i-1,k_i}}} \tau(V_{i,l}) > 2^i$$

for all sufficiently large i, which is absurd.
As a result, we infer that our assumption "$P \neq \emptyset$" must be wrong and the proof is completed. □

Note that instead of verifying that N is perfect we could have equally applied the Cantor-Bendixson Theorem (see, e.g., [Kec95, Theorem (6.4)]) to N by observing that (N, d_N) is polish since N is a closed subset of the closed set E and (E, d_E) is a separable metric space which is complete or locally compact, hence, polish (see, e.g., [Kec95, Theorem (5.3)]). Then we could have also concluded that μ_τ is σ-finite on N.

3.2.2 Variational measures on the real line

Now we turn our attention to variational measures on the real line. In this section we let $\mathfrak{X} \subseteq \mathbb{R}$ be closed and endowed with the usual Euclidean metric. As before $\mu : \operatorname{Bor}(\mathfrak{X}) \to [0, \infty]$ is a fixed measure. We further recall at this point that we form the interior, the boundary and so on with respect to $\mathfrak{X}$.
Usually the bases considered in this context are not open. In order to capture this case, the main aim of this section is to establish the following theorem, which generalises [BPS00, Theorem 3.1 (α)].

3.4 Theorem. *Let $\emptyset \neq E \subseteq \mathfrak{X} \subseteq \mathbb{R}$ and $\mathcal{B}$ a semi-open BF-basis on $\mathfrak{X}$ satisfying*

- *$\overline{U} = \overline{U^\circ}$ for each $U \in \mathcal{B}(\mathfrak{X})$;*
- *if $U \in \mathcal{B}(\mathfrak{X})$, then each boundary point of U° (equivalently, boundary point of U) is a boundary point of some connected component (with respect ot $\mathfrak{X}$) of U°.*

Moreover, let τ be a $\mathcal{B}$-set function and assume that μ is semi-moderate on E (in particular E is a F_σ-set in $\mathfrak{X}$, hence, in $\mathbb{R}$) and that we have $\mu(\{x\}) = 0$ for all $x \in E$. If μ_τ is σ-finite on each singleton and on each perfect compact μ-nullset contained in E, then μ_τ is σ-finite on E.

Note that these two additional properties imposed on the basis $\mathcal{B}$ are, e.g., satisfied if each $\mathcal{B}$-set is the finite union of intervals of positive length (where we do not assume that each single interval is closed or open).

As one may expect, the proof of this result follows very closely the lines of the proof to Theorem 3.3. However, there are some crucial differences.
In the proof of Theorem 3.3 we constructed finite sequences

$$(V_{i,1}, x_{i,1}),\ (V_{i,2}, x_{i,2}), \dots$$

inter alia with the two properties

- $x_{i,k} \in P \cap V_{i,k}$,
- for each $j > i$ there is some l such that $x_{i,k} = x_{j,l}$.

We used these properties, e.g., to ensure that N was nonvoid and that each set $V_{i,k}$ contains some element of N. But to fulfill these two requirements, it was essential that we considered an *open* basis. Moreover, the first condition was most decisive for the inductive contruction of those finite sequences above since it guaranteed - thanks to the *openness* of $V_{i,k}$ (and the perfectness of P) - that $V_{i,k}$ contained enough points belonging to P so that we could construct at least two sets $V_{i+1,l}$ contained in $V_{i,k}$.
In Theorem 3.4 however, we do not consider an open basis. Thus it may happen that some of those tags $x_{i,k}$ from above are boundary points of the respective set $V_{i,k}$; and then the construction in the proof of Theorem 3.3 does not work anymore this way! As a consequence, we have to modify the construction in order to make sure that even in that case $V_{i,k}^\circ$ contains sufficiently many points of P.
In the papers [BPS00, BPS02, SZ04, Zhe07b, Zhe07a] the Lebesgue measure was considered, which allowed to solve this problem by means of Lebesgue density points. This approach was extended in [Cap03] in order to include measures defined by monotone functions. But, we cannot apply such arguments in the general situation of Theorem 3.4.
The following simple, but useful observation is the key to modify the above inductive construction in the situation of Theorem 3.4. (A similar reasoning is used in the proof of [BPS02, Theorem 4.3].)
If $P \subseteq \mathbb{R}$ is a closed set, then the set of all points that are boundary points of a connected component of $\mathbb{R} \setminus P$ is countable because $\mathbb{R} \setminus P$ has only countably many connected components and each such connected component is an interval with at most two boundary points.
We now give a proof for Theorem 3.4, where we omit those details that are completely analogous to the corresponding arguments used in the proof of Theorem 3.3.

Proof of Theorem 3.4. As in the proof of Theorem 3.3, we may assume w.l.o.g. that E is closed and that μ is finite on E.
Let $\mathfrak{U}$, $\mathcal{O}$ and P be defined as before. Then μ_τ is σ-finite on $E \cap \mathcal{O}$, P is closed (in $\mathfrak{X}$ and, hence, in $\mathbb{R}$ as $\mathfrak{X}$ is closed in $\mathbb{R}$) without isolated points and μ_τ is not σ-finite on $P \cap U$ for any $U \in \mathcal{T}(\mathfrak{X})$ with $P \cap U \neq \emptyset$. Once again we claim that P is void and assume to the contrary $P \neq \emptyset$.
We define

$$\widehat{P} := \{x \in P : \exists\, V \in \mathcal{B}(\mathfrak{X}) \text{ with } x \in V \text{ and } V^\circ \cap P = \emptyset\}.$$

Let $x \in \widehat{P}$ and $V \in \mathcal{B}(\mathfrak{X})$ be as in the definition of $\widehat{P}$. The point x then belongs to $V \setminus V^\circ \subseteq \overline{V} \setminus V^\circ = \overline{V^\circ} \setminus V^\circ = \partial V^\circ$. Let G be a connected component of V° with $x \in \partial G$. Since $G \subseteq \mathbb{R} \setminus P$ is connected, it is contained in a connected component H of $\mathbb{R} \setminus P$. We conclude x is contained in ∂H since $x \in P$ and in each neighbourhood of x there is point of $G \subseteq H$. However, $\mathbb{R} \setminus P$ has only countably many connected components (since $\mathbb{R} \setminus P$ is open in $\mathbb{R}$) and each such connected component is an interval with at most two boundary points. As a result, $\widehat{P}$ is countable, and hence an F_σ-μ-nullset. The assumption thus yields that μ_τ is σ-finite on $\widehat{P}$.
Consider any $U \in \mathcal{T}(\mathfrak{X})$ with $P \cap U \neq \emptyset$. From the composition

$$U \cap P = \Big(U \cap (P \setminus \widehat{P})\Big)\, \dot{\cup} (U \cap \widehat{P})$$

and the fact that μ_τ is not σ-finite on $P \cap U$, we infer that μ_τ is not σ-finite on $U \cap (P \setminus \widehat{P})$ either. Take $V \in \mathcal{B}(\mathfrak{X})$ with $V \cap (P \setminus \widehat{P}) \neq \emptyset$. We claim that $V^\circ \cap P$ is nonvoid. Indeed, pick $x \in V \cap (P \setminus \widehat{P})$. If $V^\circ \cap P$ were empty, then x would belong to $\widehat{P}$ by definition. But this contradicts the choice of x. We can now proceed similarly as in the proof of Theorem 3.3. We first choose two distinct elements x_1, x_2 of P. Let $j \in \{1,2\}$. We find $n_j \in \mathbb{N}$ with $\mu(\Delta(x_j, \frac{1}{n}) \cap E) \leq \frac{1}{4}$ for all $n \geq n_j$ and with $\Delta(x_1, \frac{1}{n_1}) \cap \Delta(x_2, \frac{1}{n_2}) = \emptyset$. For $x \in U_{\frac{1}{n_j}}(x_j)$ $(j \in \{1,2\})$ we define the gauge

$$\delta(x) := \frac{1}{2} \min \left\{ \operatorname{dist}(x, \mathfrak{X} \setminus U_{\frac{1}{n_j}}(x_j)), \frac{1}{2} \right\}.$$

Since μ_τ ist not σ-finite on $U_{\frac{1}{n_j}}(x_j) \cap P$, it is not σ-finite on $U_{\frac{1}{n_j}}(x_j) \cap (P \setminus \widehat{P})$ as observed above; in particular $\mu_\tau(U_{\frac{1}{n_j}}(x_j) \cap (P \setminus \widehat{P})) = \infty$. It follows that $W_{\delta'}(\tau, U_{\frac{1}{n_j}}(x_j) \cap (P \setminus \widehat{P})) = \infty$ for each gauge δ'. We can thus find a partition $\{(U_{i,j}, \xi_{i,j})\}_{i=1}^{r_j}$ in $\mathcal{B}_\delta[U_{\frac{1}{n_j}}(x_j) \cap (P \setminus \widehat{P})]$ with $r_j \in \mathbb{N}$ such that $U_{i,j}^\circ \cap U_{k,j}^\circ = \emptyset$, $\mu(\overline{U_{i,j}} \cap \overline{U_{k,j}}) = 0$ for distinct $i, k \in \{1, \ldots, r_j\}$ and $\sum_{i=1}^{r_j} \tau(U_{i,j}) > 2$. We observe that $U_{i,j} \subseteq U_{\delta(\xi_{i,j})}(\xi_{i,j}) \subseteq U_{1/n_j}(x_j)$ and hence $\operatorname{diam}(U_{i,j}) \leq 2\delta(\xi_{i,j}) \leq \frac{1}{2}$. As before we obtain that

$$\sum_{i=1}^{r_j} \mu(\overline{U_{i,j}} \cap E) \leq \mu(\Delta(x_j, 1/n_j) \cap E) \leq \frac{1}{4}.$$

as well as

$$\sum_{j=1}^{2} \sum_{i=1}^{r_j} \mu(\overline{U_{i,j}} \cap E) \leq \frac{1}{2}.$$

Clearly, all sets $\{U_{i,j}^\circ\}_{i,j}$ are pairwise disjoint with $\mu(\overline{U_{i,j}} \cap \overline{U_{k,l}}) = 0$ for distinct pairs (i,j) and (k,l). We finally relabel the pairs $\{(U_{i,j}, \xi_{i,j})\}_{\substack{i=1,\ldots,r_j \\ j \in \{1,2\}}}$ as

$$(V_{1,1}, x_{1,1}),\ (V_{1,2}, x_{1,2}), \ldots.$$

Because of $x_{1,k} \in V_{1,k} \cap (P \setminus \widehat{P})$, the intersection $V_{1,k}^\circ \cap P$ is non-empty. Since P has no isolated points, we are able to choose in each set $V_{k,1}^\circ$ two distinct points $\zeta_{1,k}$ and $\zeta_{2,k}$ of P and to repeat the above procedure in a similar manner as in the proof of Theorem 3.3. As in the induction step in the proof of Theorem 3.3 we can construct (using the notation of the proof of Theorem 3.3) points $\xi_{i,j,k}$, sets $U_{i,j,k}$ and indices $n_{j,k} \in \mathbb{N}$ such that

$$\xi_{i,j,k} \in U_{i,j,k} \subseteq U_{\frac{1}{n_{j,k}}}(\zeta_{j,k}) \subseteq \Delta(\zeta_{j,k}, 1/n_{j,k}) \subseteq V_{1,k}^\circ \subseteq V_{1,k}.$$

Hence $\overline{U_{i,j,k}} \subseteq \Delta(\zeta_{j,k}, 1/n_{j,k}) \subseteq V_{1,k}^\circ$. Arguing as above, we obtain for each $i \in \mathbb{N}$ a finite sequence

$$(V_{i,1}, x_{i,1}),\ (V_{i,2}, x_{i,2}), \ldots$$

at least of length 2 with $\mathcal{B}$-sets $V_{i,k}$ fulfilling the requirements:

(a) $x_{i,k} \in V_{i,k} \cap (P \setminus \widehat{P})$,

(b) each set $\overline{V_{i,k}}$ is contained in some set $V_{i-1,l}^\circ$ if $i > 1$,

(c) each set $V_{i-1,l}^\circ$ contains at least two of the sets $\overline{V_{i,k}}$ if $i > 1$,

(d) $\sum_{\substack{k \\ V_{i,k} \subseteq V_{i-1,l}}} \tau(V_{i,k}) > 2^i$ for all l and $i > 1$ and $\sum_k \tau(V_{1,k}) > 2$,

(e) $\operatorname{diam}(V_{i,k}) \leq \frac{1}{2^i}$,

(f) $\sum_k \mu(\overline{V_{i,k}} \cap E) \leq \frac{1}{2^i}$,

(g) $\mu(\overline{V_{i,k}} \cap \overline{V_{i,l}}) = 0$ for distinct k, l,

(h) $V_{i,k}^\circ \cap V_{i,l}^\circ = \emptyset$ for distinct k, l.

We again define $N := \bigcap_{i\in\mathbb{N}} \bigcup_k \left(\overline{V_{i,k}} \cap E\right)$. Then $\mu(N) = 0$, and the set N is compact since the sets $\bigcup_k \left(\overline{V_{i,k}} \cap E\right)$ are bounded and closed subsets of the closed set $\mathfrak{X}$. Moreover, the set N is non-empty by the finite intersection property of compact sets because we have

$$\emptyset \neq \bigcup_k \left(\overline{V_{i,k}} \cap E\right) \subseteq \bigcup_l \left(\overline{V_{i-1,l}} \cap E\right).$$

We claim that each set $V_{i,k}^\circ$ contains a point of N. In order to see this, we first choose a sequence of indices $(\kappa_j)_{j=0}^\infty$ with $\kappa_0 := k$ and with $\overline{V_{i+n,\kappa_n}} \subseteq V_{i+n-1,\kappa_{n-1}}^\circ$ for $n \in \mathbb{N}$, which is possible due to condition (c). Thanks to the compactness of $\overline{V_{i+1,\kappa_1}}$, the sequence $(x_{i+n,\kappa_n})_{n=1}^\infty$ possesses a limit point. One easily verifies that each such limit point belongs to $N \cap \overline{V_{i+1,\kappa_1}} \subseteq N \cap V_{i,k}^\circ$.
As in the proof of Theorem 3.3 one can show that N is perfect and, hence, that μ_τ is σ-finite on N (alternative: use the Cantor-Bendixson Theorem). Therefore we can choose $\{N_p\}_p$, δ and $\{E_m\}_m$ as in the proof of Theorem 3.3 and we find indices p and $m \in \mathbb{N}$ and an open set $U \in \mathcal{T}(\mathfrak{X})$ with $\mathrm{diam}(U) < \frac{1}{m}$ such that $N_p \cap E_m \cap U$ is dense in the non-empty set $N \cap U$. Moreover, $N \cap U \subseteq \overline{N_p \cap E_m}$ and for all sufficiently large indices i there is an index k_i with $\overline{V_{i-1,k_i}} \subseteq U$.
Because every set $V_{i,k}^\circ$ contains some point of N, for all sufficiently large i, we have $V_{i,l}^\circ \cap N \cap U \neq \emptyset$ for each set $V_{i,l}$ contained in $V_{i-1,k_i} \subseteq U$. Since $N_p \cap E_m \cap U$ is dense in $N \cap U$ and $V_{i,l}^\circ \cap N \cap U$ is non-empty and open in $N \cap U$, there is an element $y_{i,l}$ of $N_p \cap E_m \cap V_{i,l}^\circ$. The collection $\{(V_{i,l}, y_{i,l})\}_{l, V_{i,l} \subseteq V_{i-1,k_i}}$ belongs to the BF-basis $\mathcal{B}$. Furthermore, we have $\delta(y_{i,l}) \geq \frac{1}{m}$ and $\mathrm{diam}(V_{i,l}) < \frac{1}{m}$ thanks to $y_{i,l} \in E_m$ and $V_{i,l} \subseteq U$, respectively. As a result, $V_{i,l} \subseteq U_{\delta(y_{i,l})}(y_{i,l})$ and $\{(V_{i,l}, y_{i,l})\}_{l, V_{i,l} \subseteq V_{i-1,k_i}}$ is a δ-fine partition on N_p with $\mu(\overline{V_{i,l}} \cap \overline{V_{i,k}}) = 0$ for distinct l and k (recall conditions (g) and (h)). Condition (d) thus implies

$$\infty > W_\delta(\tau, N_p) \geq \sum_{\substack{l \\ V_{i,l} \subseteq V_{i-1,k_i}}} \tau(V_{i,l}) > 2^i$$

for all sufficiently large i, which is absurd. □

We notice that in contrast to [BPS00, Theorem 3.1 (α)] we do not need to assume in Theorem 3.4 that $\mathcal{B}$ is a Vitali-basis; this is essentially due to our definition of μ_τ, which allows the situation $\mathcal{P}_\delta(E) = \emptyset$.

Amongst other things, we shall apply in the next section our Theorem 3.4 to measures μ_τ derived from functions.

3.3 BVG$_*$-functions and Henstock-Thomson variational measures

In this section we explore the intimate relation between BVG$_*$-functions and the so-called Henstock-Thomson variational measures associated with them. In particular, we give a proof of Thomson's characterisation of σ-finite variational measures induced by functions (see [Tho85, Theorem (40.1)]). Since false versions of this theorem have appeared, unfortunately, several times in the literature, it seems to be worth to provide an ab ovo and elementary proof of this result merely based on the most elementary properties of BVG$_*$-functions and avoiding the notion and machinery of local systems as used by Thomson. En passant, we reprove results essentially due to Yu. A. Zhereb'ev (cf.

[Zhe07b, Theorem 2] and [Zhe07a, Corollary 1]) and we shall use these results to give a complete characterisation of BVG_*-functions solely in terms of their associated variational measure. In addition, we will derive, as announced at the beginning of this chapter, extensions of the results obtained by Ene ([Ene00, Theorem 3.2]) and Caponetti ([Cap03, Theorem 3.4]).

We start with the definition of Henstock-Thomson variational measures. In fact, the definition just instantiates a special case of the general framework considered in the preceding section. To see how this special case is embedded in the general framework, let $\mathfrak{X} = [a, b]$ (endowed as before with the usual Euclidean metric). We denote by $\mathfrak{S}$ the set of all non-degenerate (not necessarily closed) subintervals of $[a, b]$ and, as before, by $\mathfrak{I}$ the set of all non-degenerate, closed subintervals of $[a, b]$. For a function $f : [a, b] \to X$ (recall that $(X, \|\cdot\|)$ is a Banach space) and for a basis $\mathcal{B} \subseteq \mathfrak{S} \times [a, b]$ on $\mathfrak{X} = [a, b]$ (a so-called interval basis) we consider the $\mathcal{B}$-set function

$$\tau_{f,\mathcal{B}} : \mathcal{B} \to [0, \infty);\ (I, t) \mapsto \|f(\sup I) - f(\inf I)\|.$$

We put $\mu^*_{f,\mathcal{B}} := \mu^*_{\tau_{f,\mathcal{B}}}$ and $\mu_{f,\mathcal{B}} := \mu_{\tau_{f,\mathcal{B}}}$. We write $\mathcal{B}_\mathfrak{I}$ for the unique BF-basis $\mathcal{B} \subseteq \mathfrak{S} \times [a, b]$ on $\mathfrak{X}$ with $\mathcal{B}(\mathfrak{X}) = \mathfrak{I}$ (the so-called full interval basis). We further set $\tau_f := \tau_{f,\mathcal{B}_\mathfrak{I}}$ and $W_\delta(f, \cdot) := W_\delta(\tau_{f,\mathcal{B}_\mathfrak{I}}, \cdot)$ for every gauge δ on $[a, b]$ as well as $\mu^*_f := \mu^*_{\tau_{f,\mathcal{B}_\mathfrak{I}}}$ and $\mu_f := \mu_{\tau_{f,\mathcal{B}_\mathfrak{I}}}$, where we frequently also just write μ_f instead of μ^*_f if no confusion is to be expected.
Moreover, we assume for the remainder of this section that our measure $\mu : \mathrm{Bor}(\mathfrak{X}) \to [0, \infty]$ is the one-dimensional Lebesgue measure λ. We then write m_f instead of $\mu_f = \lambda_f$. The set function m_f is known as the *Henstock-Thomson variational measure induced by/ associated with f*. At this point, we note that we could also treat measures μ that vanish on all singletons. Here one obtains precisely the same results since such μ yields the same measures $\mu_{f,\mathcal{B}}$ as the Lebesgue measure because $\mathcal{B}$ is an interval basis and thus two $\mathcal{B}$-sets whose interiors have empty intersection intersect in at most one point. This is a simple, but important observation, which we shall use later on.
As a consequence, for $\emptyset \neq E \subseteq [a, b]$ and for $\delta \in (0, \infty)^E$ we have

$$\mathcal{S}(E, \delta; \mathcal{B}_\mathfrak{I}) = \left\{ \{([a_j, b_j], x_j)\}_{j=1}^r : \begin{array}{c} r \in \mathbb{N},\ a \le a_1 < b_1 \le a_2 < b_2 \le \ldots \le a_r < b_r \le b, \\ x_j \in [a_j, b_j] \subseteq U_{\delta(x_j)}(x_j),\ x_j \in E \text{ for all } j \in \{1, \ldots, r\} \end{array} \right\}.$$

We usually write a generic element of $\mathcal{S}(E, \delta)$ in the form $\{([a_j, b_j], t_j)\}_{j=1}^r$ or $\{([a_j, b_j], x_j)\}_{j=1}^r$, where we simply write $\mathcal{S}(E, \delta)$ in lieu of $\mathcal{S}(E, \delta; \mathcal{B}_\mathfrak{I})$. Notice that the notation $\{([a_j, b_j], t_j)\}_{j=1}^r$ resp.$\{([a_j, b_j], x_j)\}_{j=1}^r$ always means that the respective intervals are listed in "increasing order". For $\delta \in (0, \infty)^E$ and $S = \{([a_j, b_j], x_j)\}_{j=1}^r \in \mathcal{S}(E, \delta)$ we write

$$W_g(S) = \sum_{j=1}^r \|g(b_j) - g(a_j)\| \qquad \text{and} \qquad W_\delta(g, E) = \sup\{W_g(S) :\ S \in \mathcal{S}(E, \delta)\}.$$

Note that the notation of this chapter is consistent with notation of Chapter 2. For $a \le c < d \le b$ and $\delta \in (0, \infty)^{[c,d]}$, we call, as above, $S = \{([a_j, b_j], x_j)\}_{j=1}^r \in \mathcal{S}([c, d], \delta)$ a δ-fine partition on $[c, d]$ and we call it a δ-fine partition of $[c, d]$ if $\bigcup_{j=1}^r [a_j, b_j] = [c, d]$ additionally holds. If A is an arbitrary set, then a δ-fine partition of A does not need to exist. However, for intervals the situation is different. This is the content of the following result, known as Cousin's lemma (see, e.g., Lemma 9.2 in [Gor94] for a proof).

3.5 Lemma (Cousin's lemma). *If $\delta : [a, b] \to (0, \infty)$ is a gauge on $[a, b]$ and $[c, d]$ an element of $\mathfrak{I}$, then there exists a δ-fine partition of $[c, d]$, i.e., an interval-point sequence $\{([a_j, b_j], x_j)\}_{j=1}^r$ belonging $\mathcal{S}([c, d], \delta; \mathcal{B}_\mathfrak{I})$ with $\bigcup_{j=1}^r [a_j, b_j] = [c, d]$.*

The next lemma shows that the Henstock-Thomson variational measure of a function is assigned to an exposed position among variational measures (which is one aspect that justifies to give a specific name to it).

3.6 Lemma. *Let $f : [a, b] \to X$ be a function. For each basis $\mathcal{B} \subseteq \mathfrak{S} \times [a, b]$ on $\mathfrak{X}$ we have $\mu^*_{f,\mathcal{B}} \le m^*_f$. In particular, we have $\mu^*_{f,\mathcal{B}} = m^*_f$ whenever $\mathcal{B}$ is a BF-basis with $\mathcal{B}(\mathfrak{X}) \supseteq \mathfrak{I}$.*

Proof. Let $\emptyset \neq E \subseteq X$ and consider any gauge δ on X. If there is no δ-fine partition on E with respect to the basis $\mathcal{B}$, then we clearly have $\mu^*_{f,\mathcal{B}}(E) = 0 \le m^*_f(E)$. Therefore we may assume without loss of generality that a δ-fine partition on E with respect to the basis $\mathcal{B}$ exists. Let $\{(I_j, x_j)\}_{j=1}^r$ be such a δ-fine partition on E with respect to $\mathcal{B}$. Then $\{(\overline{I}_j, x_j)\}_{j=1}^r$ is a 2δ-fine partition on E with respect to the basis $\mathcal{B}_{\mathfrak{I}}$. Due to $\sum_{j=1}^r \tau_{f,\mathcal{B}}(I_j) = \sum_{j=1}^r \tau_f(\overline{I}_j)$, this yields $W_\delta(\tau_{f,\mathcal{B}}, E; \mathcal{B}) \le W_{2\delta}(f, E)$ and consequently $\mu^*_{f,\mathcal{B}}(E) \le m^*_f(E)$ as asserted. □

We now start the exploration of the relation between BVG∗-functions and the Henstock-Thomson variational measures associated with them. The first results of this kind also distinctly explain the adjective "variational", as they demonstrate how closely the Henstock-Thomson variational measure is related to the classical notion of variation.

3.7 Lemma (see also Lemma 39.1 in [Tho85]). *Let $J \subseteq [a, b]$ be a non-empty interval which is open relative to $[a, b]$. Then $m_\varphi(J) = V(\varphi, J)$. In particular, $\varphi \in \mathrm{BV}([a, b], Y)$ if and only if $m_\varphi([a, b]) < \infty$, and φ is constant on J if and only if $m_\varphi(J) = 0$.*

Proof. Let $\delta \in (0, \infty)^J$ with $\overline{U_{\delta(t)}(t)} \cap [a, b] \subseteq J$ for all $t \in J$. Furthermore, let $\{([a_j, b_j], t_j)\}_{j=1}^r \in \mathcal{S}(J, \delta)$. We then have $[a_j, b_j] \subseteq \overline{U_{\delta(t_j)}(t_j)} \cap [a, b] \subseteq J$; in particular $\{[a_j, b_j]\}_{j=1}^r \in \mathcal{A}(J)$, which yields $\sum_{j=1}^r \|\varphi(b_j) - \varphi(a_j)\| \le V(\varphi, J)$ and thus $m_\varphi(J) \le \sup_{S \in \mathcal{S}(J,\delta)} W_\varphi(S) \le V(\varphi, J)$.
Conversely, take $\{[a_j, b_j]\}_{j=1}^r \in \mathcal{A}(J)$ and let $\varepsilon > 0$. Pick $\delta \in (0, \infty)^J$ such that $W_\delta(\varphi, J) \le m_\varphi(J) + \varepsilon$. Thanks to Cousin's lemma, we may choose for each $j \in \{1, \ldots, r\}$ a δ-fine partition $\{([a_{\nu j}, b_{\nu j}], t_{\nu j})\}_{\nu=1}^{r_j}$ of $[a_j, b_j]$. As J is an interval, we obtain $\bigcup_{j=1}^r \{([a_{\nu j}, b_{\nu j}], t_{\nu j})\}_{\nu=1}^{r_j} \in \mathcal{S}(J, \delta)$. This implies

$$\sum_{j=1}^{r} \|\varphi(b_j) - \varphi(a_j)\| \le \sum_{j=1}^{r} \sum_{\nu=1}^{r_j} \|\varphi(b_{\nu j}) - \varphi(a_{\nu j})\| \le W_\delta(\varphi, J) \le m_\varphi(J) + \varepsilon,$$

hence $V(\varphi, J) \le m_\varphi(J) + \varepsilon$. Letting $\varepsilon \to 0^+$ yields the assertion.
The addendum is clear. □

Furthermore, it is also possible to determine the value of the Henstock Thomson variational measure on singletones as the next result shows.

3.8 Lemma (see [Tho85, Example 37.5]). *We have*

- $m_\varphi(\{x\}) = \overline{\lim}_{h\to 0^+} \|\varphi(x) - \varphi(x+h)\| + \overline{\lim}_{h\to 0^+} \|\varphi(x) - \varphi(x-h)\|$ *for all $x \in (a, b)$ as well as*
- $m_\varphi(\{a\}) = \overline{\lim}_{h\to 0^+} \|\varphi(a) - \varphi(a+h)\|$ *and*
- $m_\varphi(\{b\}) = \overline{\lim}_{h\to 0^+} \|\varphi(b) - \varphi(b-h)\|$.

Moreover, we have

$$\omega_\varphi(x) \le 2m_\varphi(\{x\}) \le 4\omega_\varphi(x). \tag{3.1}$$

for any $x \in [a, b]$, where $\omega_\varphi(x) := \lim_{\varepsilon\to 0^+} \omega(\varphi, U_\varepsilon(x) \cap [a, b]) = \inf_{\epsilon>0} \omega(\varphi, U_\epsilon(x) \cap [a, b]) \in [0, \infty]$. In particular, φ is continuous at $x \in [a, b]$ if and only if $m_\varphi(\{x\}) = 0$.

Proof. Let $x \in (a,b)$ be arbitrary and put $\rho := \min\{x-a, b-x\}$. (The cases $x = a$ and $x = b$ are a little bit easier and may be similarly treated.) We then compute

$$\begin{aligned}
&m_\varphi(\{x\}) \\
&= \inf_{\delta \in (0,\infty)^{[a,b]}} W_\delta(\varphi, \{x\}) \\
&= \inf_{\delta \in (0,\infty)^{[a,b]}} \sup \left\{ \sum_{j=1}^{r} \|\varphi(b_j) - \varphi(a_j)\| : \begin{array}{c} r \in \mathbb{N},\ a_j \le x \le b_j \le a_{j+1}, \\ a \le a_j < b_j \le b,\ [a_j, b_j] \subseteq U_{\delta(x)}(x) \end{array} \right\} \\
&= \inf_{\delta > 0} \sup \left\{ \sum_{j=1}^{r} \|\varphi(b_j) - \varphi(a_j)\| : \begin{array}{c} r \in \mathbb{N},\ a_j \le x \le b_j \le a_{j+1}, \\ a \le a_j < b_j \le b,\ [a_j, b_j] \subseteq U_\delta(x) \end{array} \right\} \\
&= \inf_{0<\delta\le\rho} \max \left\{ \begin{array}{c} \sup \left\{ \|\varphi(t) - \varphi(s)\| : \begin{smallmatrix} x-\delta<s\le x\le t<x+\delta, \\ a\le s<t\le b \end{smallmatrix} \right\}, \\ \sup \left\{ \|\varphi(x) - \varphi(s)\| + \|\varphi(t) - \varphi(x)\| : \begin{smallmatrix} x-\delta<s<x<t<x+\delta, \\ a\le s<t\le b \end{smallmatrix} \right\} \end{array} \right\} \\
&= \inf_{0<\delta\le\rho} \sup \left\{ \|\varphi(x) - \varphi(s)\| + \|\varphi(t) - \varphi(x)\| : x-\delta < s < x < t < x+\delta \right\} \\
&= \inf_{0<\delta\le\rho} \left(\sup_{0<h<\delta} \|\varphi(x) - \varphi(x+h)\| + \sup_{0<h<\delta} \|\varphi(x) - \varphi(x-h)\| \right). \qquad (3.2)
\end{aligned}$$

This last line immediately gives $m_\varphi(\{x\}) \le 2\omega_\varphi(x)$.
Conversely, for each $\delta \in (0, \rho]$ and all $s, t \in U_\delta(x)$, we clearly have

$$\begin{aligned}
\|\varphi(t) - \varphi(s)\| &\le \|\varphi(t) - \varphi(x)\| + \|\varphi(x) - \varphi(s)\| \\
&\le 2 \left(\sup_{0<h<\delta} \|\varphi(x) - \varphi(x+h)\| + \sup_{0<h<\delta} \|\varphi(x) - \varphi(x-h)\| \right),
\end{aligned}$$

and hence

$$\omega(\varphi, U_\delta(x)) \le 2 \left(\sup_{0<h<\delta} \|\varphi(x) - \varphi(x+h)\| + \sup_{0<h<\delta} \|\varphi(x) - \varphi(x-h)\| \right).$$

This means that $\omega_\varphi(x) \le 2m_\varphi(\{x\})$ and thus assertion (3.1) holds.
Starting from (3.2), we derive the lower estimate

$$\begin{aligned}
&m_\varphi(\{x\}) \\
&= \inf_{0<\delta\le\rho} \left(\sup_{0<h<\delta} \|\varphi(x) - \varphi(x+h)\| + \sup_{0<h<\delta} \|\varphi(x) - \varphi(x-h)\| \right) \\
&\ge \inf_{0<\delta\le\rho} \sup_{0<h<\delta} \|\varphi(x) - \varphi(x+h)\| + \inf_{0<\delta\le\rho} \sup_{0<h<\delta} \|\varphi(x) - \varphi(x-h)\| \\
&= \overline{\lim_{h\to 0^+}} \|\varphi(x) - \varphi(x+h)\| + \overline{\lim_{h\to 0^+}} \|\varphi(x) - \varphi(x-h)\|.
\end{aligned}$$

Take $\varepsilon > 0$ and choose $\delta_0 \in (0, \rho)$ such that

$$\sup_{0<h<\delta} \|\varphi(x) - \varphi(x+h)\| \le \overline{\lim_{h\to 0^+}} \|\varphi(x) - \varphi(x+h)\| + \frac{\varepsilon}{2}$$

and

$$\sup_{0<h<\delta} \|\varphi(x) - \varphi(x-h)\| \le \overline{\lim_{h\to 0^+}} \|\varphi(x) - \varphi(x-h)\| + \frac{\varepsilon}{2}$$

both hold for all $\delta \in (0, \delta_0)$. These inequalities yield the upper estimate

$$\begin{aligned}
&m_\varphi(\{x\}) \\
&= \inf_{0<\delta\le\rho} \left(\sup_{0<h<\delta} \|\varphi(x) - \varphi(x+h)\| + \sup_{0<h<\delta} \|\varphi(x) - \varphi(x-h)\| \right) \\
&= \inf_{0<\delta<\delta_0} \left(\sup_{0<h<\delta} \|\varphi(x) - \varphi(x+h)\| + \sup_{0<h<\delta} \|\varphi(x) - \varphi(x-h)\| \right) \\
&\le \inf_{0<\delta<\delta_0} \left(\overline{\lim_{h\to 0^+}} \|\varphi(x) - \varphi(x+h)\| + \overline{\lim_{h\to 0^+}} \|\varphi(x) - \varphi(x-h)\| + \varepsilon \right) \\
&= \overline{\lim_{h\to 0^+}} \|\varphi(x) - \varphi(x+h)\| + \overline{\lim_{h\to 0^+}} \|\varphi(x) - \varphi(x-h)\| + \varepsilon.
\end{aligned}$$

The lemma follows letting $\varepsilon \to 0^+$. □

A natural question arising at this point is whether it possible to transcend Lemma 3.7 and 3.8 in order to identify m_φ in its entirety. Indeed it is often stated (without proof) that for a real-valued, continuous function φ of bounded variation on $[a, b]$, the set function m_φ coincides with the Lebesgue-Stieltjes measure associated with the total variation. We refer to Appendix A for a proof of this claim in the case of a vector-valued function of bounded variation that is continuous from the right.

Now we prove the announced results linking BVG_*-functions with their variational measures. In the existing literature there are different approaches to these results (mostly for real-valued functions). Here we choose a completely different and new approach that systematically employs Lemma 2.4 and 2.8. This approach has a methodological merit as it reveals the subsequent results as a completely natural outflow of the characterisations established in Lemma 2.4 and 2.8 and provides a unified framework to the following results. As a consequence, the proofs are, in comparison, e.g., to [Gor94], simplified.

3.9 Lemma (cf. [Tho81]). *Let $\emptyset \neq E \subseteq [a, b]$, $c := \inf E$, $d := \sup E$ and $\varphi \in \mathrm{BV}_*(E, Y)$.*

- *(a) We have $m_\varphi(E \cap (c, d)) < \infty$.*
- *(b) If for each $t \in \{c, d\}$ there is a $\rho(t) > 0$ such that φ is bounded on $U_{\rho(t)}(t) \cap [a, b]$, then $m_\varphi(E) < \infty$.*

Proof. We first prove (a). If $(c, d) \cap E = \emptyset$, then $m_\varphi((c, d) \cap E) < \infty$ is clear. Therefore we may assume without loss of generality $(c, d) \cap E \neq \emptyset$. For $t \in (c, d) \cap E$ we set $\delta(t) := \min\{d - t, t - c\}$ and we choose a strictly increasing function $\chi : [c, d] \to \mathbb{R}$ according to Lemma 2.4. For each $S = \{([a_j, b_j], x_j)\}_{j=1}^r \in \mathcal{S}((c, d) \cap E, \delta)$ we obtain

$$\begin{aligned}
W_\varphi(S) &\le \sum_{j=1}^r \|\varphi(b_j) - \varphi(x_j)\| + \sum_{j=1}^r \|\varphi(x_j) - \varphi(a_j)\| \\
&\le \sum_{j=1}^r |\chi(b_j) - \chi(x_j)| + \sum_{j=1}^r |\chi(x_j) - \chi(a_j)| = \sum_{j=1}^r (\chi(b_j) - \chi(a_j)) \le \chi(d) - \chi(c).
\end{aligned}$$

This implies $W_\delta(\varphi, (c, d) \cap E) \le \chi(d) - \chi(c) < \infty$ and hence $m_\varphi((c, d) \cap E) < \infty$ as asserted.

In order to prove part (b) it suffices to observe that the hypothesis yields $m_\varphi(\{c, d\}) < \infty$ thanks to Lemma 3.8. □

3.10 Corollary. *Let $\emptyset \neq E \subseteq [a, b]$ and $\varphi \in \mathrm{BVG}_*(E, Y)$.*

- *(a) There is a sequence $(E_n)_{n=0}^\infty$ with $\bigcup_{n\in\mathbb{N}_0} E_n = E$ such that E_0 is countable and $m_\varphi(E_n) < \infty$ for all $n \in \mathbb{N}$.*

(b) *If φ is locally bounded at all points of E, i.e., for each $t \in E$ there is a $\rho(t) > 0$ such that φ is bounded on $U_{\rho(t)}(t) \cap [a,b]$, then we may choose $E_0 = \emptyset$ in part (a).*

(c) *If E is an F_σ-set, then we can additionally achieve that the sets E_n in part (a) and (b) are closed for every $n \in \mathbb{N}$.*

Proof. Due to $\varphi \in \mathrm{BVG}_*(E,Y)$, there exists a sequence of sets $(F_n)_{n=1}^\infty$ with $\bigcup_{n=1}^\infty F_n = E$ such that $\varphi \in \mathrm{BV}_*(F_n,Y)$ for all $n \in \mathbb{N}$. Put $c_n := \inf F_n$ and $d_n := \sup F_n$ for $n \in \mathbb{N}$. If we set $E_0 := \bigcup_{n\in\mathbb{N}}\{c_n,d_n\}$ and $E_n := F_n \setminus \{c_n,d_n\}$ for $n \in \mathbb{N}$, part (a) follows from part (a) of Lemma 3.9.

Moreover, if φ is locally bounded at all points of E, then part (b) of Lemma 3.9 permits that we may take $E_0 = \emptyset$ and $E_n = F_n$ for $n \in \mathbb{N}$.

Finally, suppose that E is an F_σ-set and write $E = \bigcup_{k=1}^\infty A_k$ with closed sets A_k. Applying Lemma 2.7, we derive $\varphi \in \mathrm{BV}_*(\overline{(F_n \cap A_k)},Y)$ for all k and n. For this reason we may assume w.l.o.g. that the sets F_n themselves are closed. For those $n \in \mathbb{N}$ such that c_n and d_n are distinct, we choose sequences $(c_{\nu,n})_\nu$ and $(d_{\nu,n})_\nu$ in (c_n,d_n) converging to c_n and d_n, respectively, with $c_{\nu,n} < d_{\nu,n}$. Then $m_\varphi(F_n \cap [c_{\nu,n},d_{\nu,n}]) \leq m_\varphi(F_n \cap (c_n,d_n)) < \infty$ by part (a) of Lemma 3.9. Hence, in the situation of part (a) we may take as before $E_0 := \bigcup_{n\in\mathbb{N}}\{c_n,d_n\}$ and we may choose $(E_n)_{n=1}^\infty$ as an enumeration of the set $\{F_n \cap [c_{\nu,n},d_{\nu,n}] :\ n,\nu \in \mathbb{N} \text{ with } c_n \neq d_n\}$. Finally, in the situation of part (b) we can put $E_0 := \emptyset$ and $E_n := F_n$ for $n \in \mathbb{N}$ once again because of part (b) of Lemma 3.9. ☐

Now we prove a partial converse to part (b) of Lemma 3.9.

3.11 Lemma. *Let $\emptyset \neq E \subseteq [a,b]$ and assume that $m_\varphi(E) < \infty$. Then φ belongs to* $\mathrm{BVG}_*(E,Y)$.

Proof. We adapt the proof of Lemma 3.5 in [Fau97] and provide necessary details omitted there.

We choose $\delta \in (0,\infty)^E$ such that $W_\delta(\varphi,E) \leq m_\varphi(E)+1$ and for $t \in [a,b]$ we put

$$\mathcal{S}_t := \left\{ S = \{([a_j,b_j],t_j)\}_{j=1}^r \in \mathcal{S}(E,\delta) :\ \bigcup_{j=1}^r [a_j,b_j] \subseteq [a,t] \right\}$$

and

$$\psi(t) := \begin{cases} \sup\{W_\varphi(S) :\ S \in \mathcal{S}_t\}, & \text{if } \mathcal{S}_t \neq \emptyset, \\ 0, & \text{if } \mathcal{S}_t = \emptyset. \end{cases}$$

We now consider

$$\chi : [a,b] \to \mathbb{R};\ t \mapsto t + \psi(t).$$

For $t,s \in [a,b]$ with $s < t$ we have $\mathcal{S}_s \subseteq \mathcal{S}_t$ and hence $\psi(s) \leq \psi(t)$ and we conclude that χ is strictly increasing. Now let $t \in E$ and $s \in \dot{U}_{\delta(t)}(t)$. We shall show that $\|\varphi(s)-\varphi(t)\| \leq |\chi(t)-\chi(s)|$. Then we obtain $\overline{\lim}_{s\to t} \frac{\|\varphi(s)-\varphi(t)\|}{|\chi(t)-\chi(s)|} < \infty$ for all $t \in E$ and Lemma 2.8 yields the assertion.
We first consider the case where $s < t$ and $\mathcal{S}_s = \emptyset$. Because of $([s,t],t) \in \mathcal{S}_t$ and $\psi(s) = 0$, we derive

$$\|\varphi(t)-\varphi(s)\| \leq \psi(t) \leq t - s + \psi(t) - \psi(s) = \chi(t) - \chi(s) = |\chi(t)-\chi(s)|.$$

Next we consider the case where $s < t$ and $\mathcal{S}_s \neq \emptyset$. Let $S = \{([a_j,b_j],t_j)\}_{j=1}^r \in \mathcal{S}_s$. Then $S' := \{([a_j,b_j],t_j)\}_{j=1}^r \cup \{([s,t],t)\} \in \mathcal{S}_t$ and we estimate

$$W_\varphi(S) + \|\varphi(s)-\varphi(t)\| = W_\varphi(S') \leq \psi(t),$$

$$\psi(s) + \|\varphi(s)-\varphi(t)\| \leq \psi(t),$$

which yields

$$\|\varphi(s) - \varphi(t)\| \leq \psi(t) - \psi(s) \leq t - s + \psi(t) - \psi(s) = |\chi(t) - \chi(s)|.$$

Finally, let $t < s$. If $t = a$, then $\mathcal{S}_t = \emptyset$ and $\psi(t) = 0$. Due to $([t,s], t) \in \mathcal{S}_s$ we then obtain

$$\|\varphi(t) - \varphi(s)\| \leq \psi(s) \leq s - t + \psi(s) - \psi(t) = \chi(s) - \chi(t) = |\chi(t) - \chi(s)|.$$

Now assume that $t \neq a$. Then $\mathcal{S}_t \neq \emptyset$. Let $S = \{([a_j, b_j], t_j)\}_{j=1}^r \in \mathcal{S}_t$. Then $S' := \{([a_j, b_j], t_j)\}_{j=1}^r \cup \{([t,s], t)\} \in \mathcal{S}_s$ and we estimate

$$W_\varphi(S) + \|\varphi(s) - \varphi(t)\| = W_\varphi(S') \leq \psi(s),$$

$$\psi(t) + \|\varphi(s) - \varphi(t)\| \leq \psi(s),$$

which yields

$$\|\varphi(s) - \varphi(t)\| \leq \psi(s) - \psi(t) \leq s - t + \psi(s) - \psi(t) = |\chi(t) - \chi(s)|.$$

This finishes the proof. □

Combining Corollary 3.10 and Lemma 3.11 we arrive at the subsequent result that completely characterises the membership of φ to the class $\mathrm{BVG}_*(E, Y)$ in terms of the Henstock-Thomson variational measure associated with φ.

3.12 Theorem (cf. Theorem 1 in [Tho81] and Theorem 40.1 in [Tho85]). *Let $\emptyset \neq E \subseteq [a,b]$.*

(a) The following statements are equivalent

- *(i) $\varphi \in \mathrm{BVG}_*(E, Y)$.*
- *(ii) There is a sequence $(E_n)_{n=0}^\infty$ with $\bigcup_{n \in \mathbb{N}_0} E_n = E$ such that E_0 is countable and $m_\varphi(E_n) < \infty$ for all $n \in \mathbb{N}$.*

(b) If E is an F_σ-set, the following statements are equivalent.

- *(i) $\varphi \in \mathrm{BVG}_*(E, Y)$.*
- *(ii) There is a sequence $(E_n)_{n=0}^\infty$ with $\bigcup_{n \in \mathbb{N}_0} E_n = E$ such that E_0 is countable, each set E_n is closed and $m_\varphi(E_n) < \infty$ for all $n \in \mathbb{N}$.*

(c) If φ is locally bounded at each point of E, then we may choose $E_0 = \emptyset$ in part (a) and part (b).

3.13 Definition. *Let $\emptyset \neq E \subseteq [a,b]$ and $\varphi \in \mathrm{BVG}_*(E, Y)$. Then we call each sequence $(E_n)_{n=0}^\infty$ with $\bigcup_{n \in \mathbb{N}_0} E_n = E$ such that E_0 is countable and $m_\varphi(E_n) < \infty$ for all $n \in \mathbb{N}$ a* decomposition admissible for (φ, E) *or just a* decomposition admissible for φ *if $E = [a,b]$. We call a decomposition admissible for (φ, E)* closed *respectively* Borel *respectively* measurable *provided that for each $n \in \mathbb{N}$ the set E_n is closed, Borel or m_φ-measurable (i.e., measurable in the sense of Carathéodory), respectively. Clearly, each closed admissible decomposition is also a Borel admissible decomposition and every Borel admissible decomposition is a measurable admissible decomposition (as m_φ is a metric outer measure). Due to Theorem 3.12, there always exists a closed admissible decomposition for (φ, E) if $\varphi \in \mathrm{BVG}_*(E, Y)$ and E is an F_σ-set.*

We now derive two corollaries to Theorem 3.12.

3.14 Corollary. *Let $\emptyset \neq E \subseteq [a,b]$ be closed. Then the following assertions are equivalent.*

(a) We have $\varphi \in \mathrm{BVG}_*(E, Y)$ and $\omega_\varphi(t) < \infty$ for all $t \in E$.

(b) The measure m_φ is semi-moderate on E.

(c) The measure m_φ is σ-finite on E.

(d) The outer measure m_φ^* is σ-finite on E.

(e) *We have $\varphi \in \mathrm{BVG}_*(E,Y)$ and φ is bounded on E.*

Proof. Theorem 3.12 gives the equivalence of (a), (b), (c) and (d). Finally, it is clear that (e) implies (a), while the converse statement is easily deduced using a compactness argument. □

3.15 Remark Propostion 3.12 respectively Corollary 3.14 is the correct version of [BPM09a, Proposition 3.4] and [BPM09b, Theorem 2.5]. In the final analysis, the proof given there suffers (as Gordon's solution to his Exercise 11.3 in [Gor94]) from the fact that it is taken for granted that the implication "(a) $\Longrightarrow$ (c)" in Corollary 3.14 always holds even without the boundedness assumption provided that E is a singleton (or countable); in some sense this error corresponds to the difference between the oscillation of a function on a singleton and the oscillation of this function at the respective point. But Lemma 3.8 shows that this is not correct. Let us give a very simple example. We take $[a,b]=[0,1]$ and we put $\varphi(x):=\frac{1}{x}$ for $x\in(0,1]$ and $\varphi(0):=0$. Then we have $\varphi\in\mathrm{BV}_*(E_n,\mathbb{R})$ for all $n\in\mathbb{N}_0$, where $E_n:=[\frac{1}{n+1},1]$ for $n\in\mathbb{N}$ and $E_0:=\{0\}$. However, m_φ is not σ-finite because of $m_\varphi(\{0\})=\infty$.

As a further corollary to Theorem 3.12 and to Theorem 3.4 we obtain the following result essentially due to Yu. A. Zhereb'ev (cf. [Zhe07b, Theorem 2] and [Zhe07a, Corollary 1]).

3.16 Corollary. *Let $\emptyset\neq E\subseteq[a,b]$ be an F_σ-set. Then the following assertions are equivalent.*

(a) *We have $\varphi\in\mathrm{BVG}_*(E,Y)$ and $\omega_\varphi(t)<\infty$ for all $t\in E$.*

(b) *The measure m_φ is semi-moderate on E.*

(c) *The measure m_φ is σ-finite on E.*

(d) *The measure m_φ is σ-finite on each compact Lebesgue-null set contained in E.*

(e) *The outer measure m_φ^* is σ-finite on E.*

Proof. Assertions (a), (b), (c) and (e) are equivalent thanks to Theorem 3.12 and obviously (c) implies (d). Finally, assertion (d) yields (c) thanks to Theorem 3.4. □

3.17 Remark Corollary 3.16 is an instance where it is very worth to distinguish between m_φ and m_φ^* as there exist an example of a σ-finite measure defined on a σ-algebra on $\mathbb{R}$ containing the Borel sets such that its restriction to the Borel σ-algbra is not σ-finite (see Theorem 1.1 in [EK06]). Therefore, it is a priori not clear whether the σ-finiteness of m_φ^* yields that m_φ itself is σ-finite.

As a special case of Theorem 3.4 we obtain the following result generalising [Cap03, Theorem 3.4] and [Ene00, Theorem 3.2].

3.18 Corollary. *Let $f:[a,b]\to X$ and $\varphi:[a,b]\to Y$ be two functions and $\emptyset\neq E\subseteq[a,b]$ be an F_σ-set. Assume furthermore that $\varphi\in\mathrm{BVG}_*(E,Y)$ with (countable) discontinuity set D. Finally, suppose that $\mathcal{B}\subseteq\mathfrak{S}\times[a,b]$ is a BF-basis and let μ be the Lebesgue measure (recall the introductory part at the beginning of this section). Then the following assertions hold.*

(a) *Assume that $D=\emptyset$. If $\mu_{f,\mathcal{B}}$ is σ-finite on each closed $\mu_{\varphi,\mathcal{B}}$-nullset contained in E, then $\mu_{f,\mathcal{B}}$ is σ-finite on E. (cf. [Cap03, Theorem 3.4])*

(b) *Assume that D is closed. If m_f is absolutely continuous on the set $E\setminus D$ with respect to m_φ, then f is a BVG_*-function on E being continuous at each point of $E\setminus D$ and m_f is semi-moderate on $E\setminus D$. (cf. [Ene00, Theorem 3.2])*

Proof. We start with the proof of part (a). By Corollary 3.16, the measure m_φ is semi-moderate on E. Thanks to Lemma 3.6, the measure $\mu_{\varphi,\mathcal{B}}$ is semi-moderate on E, too. The crucial point is now the observation that we just have $\left(\mu_{\varphi,\mathcal{B}}\right)_{\tau_{f,\mathcal{B}}} = \mu_{f,\mathcal{B}}$. This is due to the nature of the basis $\mathcal{B}$ being an interval basis and the fact that $\mu_{\varphi,\mathcal{B}}$ vanishes on singletons thanks to Lemma 3.6 and the continuity of φ (use Lemma 3.8). As a result, Theorem 3.4 implies that $\mu_{f,\mathcal{B}}$ is σ-finite on E.

Now we turn to the proof of part (b). First of all we note that $m_\varphi(\{t\}) = 0$ for $t \in [a,b] \setminus D$ by Lemma 3.8 and consequently $m_f(\{t\}) = 0$ for all $t \in E \setminus D$, which implies (once again by Lemma 3.8) that f is continuous at each point of $E \setminus D$.
By hypothesis, D is closed. Therefore we can find a sequence $(I_n)_n$ of closed intervals such that $\bigcup_n I_n = [a,b] \setminus D$. Then each of the sets $E \cap I_n$ is also an F_σ-set. Now we consider the variational Henstock-Thomson measures $m_{f|_{I_n}}$ and $m_{\varphi|_{I_n}}$ formed with respect to I_n (and the full interval basis on I_n). It is easy to see that $m_{f|_{I_n}}$ and m_f resp $m_{\varphi|_{I_n}}$ and m_φ coincide on the interior of I_n (relative to $[a,b]$). Since f resp. φ is continuous at each point of I_n, we further have $m_f(\{t\}) = 0 = m_{f|_{I_n}}(\{t\})$, resp. $m_\varphi(\{t\}) = 0 = m_{\varphi|_{I_n}}(\{t\})$ for $t \in \{\min I_n, \max I_n\}$ and we conclude that $m_{f|_{I_n}}$ and m_f resp $m_{\varphi|_{I_n}}$ and m_φ coincide on I_n. Applying part (a) with $[a,b]$ replaced by I_n and E replaced by $E \cap I_n$, we infer that $m_{f|_{I_n}} = m_f$ is σ-finite on $E \cap I_n$, which yields that m_f is semi-moderate on $E \cap I_n$ by Corollary 3.16 and that f is a BVG_*-function on $E \cap I_n$ thanks to Theorem 3.12. As a result, μ_f is semi-moderate on $E \setminus D$ and f is a BVG_*-function on $E \setminus D$. Since D is countable, f is even a BVG_*-function on E. $\square$

3.4 Full and fine variational measures

In this section we introduce the notion of full and fine variational measures. The notion of fine variational measure plays a crucial part in the differentiation theory of Henstock-Kurzweil integrals as we shall see in the next chapter. The exposition closely follows the arguments in chapter 4 of [Tho85] and the proof of Theorem 6.29 in [Tho13]; see the remark at the end of this section for a more detailed comparison.

We start with some definitions and notations.

3.19 Definition. *Let $\emptyset \neq E \subseteq [a,b]$.*

(a) A nonvoid subset $\beta \subseteq \mathfrak{I} \times [a,b]$ is called a covering relation *if $t \in I$ for all $(I,t) \in \beta$.*

(b) A covering relation β is called

- *a* full cover of E *if for each $t \in E$ there exists a $\delta(t) > 0$ such that $(I,t) \in \beta$ for all $I \in \mathfrak{I}$ with $\mathrm{diam}(I) < \delta(t)$ and with $t \in I$;*
- *a* restricted full cover of E *if for each $t \in E$ there exists a $\delta(t) > 0$ such that $(I,t) \in \beta$ for all $I \in \mathfrak{I}$ with $\mathrm{diam}(I) < \delta(t)$ and with $t \in \partial I$;*
- *a* fine cover *or* Vitali cover of E *if for each $\varepsilon > 0$ and each $t \in E$ there exists an element $(I,t) \in \beta$ with $\mathrm{diam}(I) < \varepsilon$ and $t \in \partial I$.*

We denote by $\mathcal{C}(E)$, $\widetilde{\mathcal{C}}(E)$ resp. $\mathcal{C}^\star(E)$ the set of all full covers, restricted full covers resp. fine covers of E.

For a covering relation β and a non-empty set $E \subseteq [a,b]$ we introduce

$$\mathcal{P}(\beta) := \{\{(I_j,t_j)\}_{j=1}^r :\ r \in \mathbb{N},\ (I_1,t_1)\ldots,(I_r,t_r) \in \beta,\ I_1,\ldots,I_r \text{ are non-overlapping}\}$$

and

$$\mathcal{P}(\beta, E) := \{\{(I_j,t_j)\}_{j=1}^r \in \mathcal{P}(\beta) :\ t_1,\ldots,t_r \in E\}.$$

For $\tau : \mathfrak{I} \to X$, $\varphi \in Y^{[a,b]}$ and $\pi = \{(I_j, t_j)\}_{j=1}^r \in \mathcal{P}(\beta)$ we put $W_\tau(\pi) := \sum_{j=1}^r \|\tau(I_j)\|$ and we set $W_\varphi(\pi) := W_{\tau_\varphi}(\pi)$, where

$$\tau_\varphi : \mathfrak{I} \to Y;\ I \mapsto \varphi(\max I) - \varphi(\min I).$$

Notice that we always have $W_\tau(\pi) = W_{\|\tau\|}(\pi)$ and that the notation used here is consistent with the notation of the preceding sections. We further define set functions by

$$\lambda_\tau(E) := \inf_{\beta \in \mathcal{C}(E)} \sup_{\pi \in \mathcal{P}(\beta)} W_\tau(\pi),$$

$$\widetilde{\lambda}_\tau(E) := \inf_{\beta \in \widetilde{\mathcal{C}}(E)} \sup_{\pi \in \mathcal{P}(\beta)} W_\tau(\pi),$$

$$m^\tau(E) := \inf_{\beta \in \mathcal{C}^\star(E)} \sup_{\pi \in \mathcal{P}(\beta)} W_\tau(\pi),$$

and $\lambda_\tau(\emptyset) := \widetilde{\lambda}_\tau(\emptyset) := m^\tau(\emptyset) := 0$. Note that $\lambda_\tau = \lambda_{\|\tau\|}$, $\widetilde{\lambda}_\tau = \widetilde{\lambda}_{\|\tau\|}$ and $m^\tau = m^{\|\tau\|}$.

The following lemma collects some of the basic properties of the set functions just defined and relates them to the variational measures considered in the previous sections.

3.20 Lemma. *(a) We have $m^\tau \le \widetilde{\lambda}_\tau \le \lambda_\tau$ and the second inequality is even an equality provided that*

$$\|\tau([c,d])\| \le \|\tau([c,e])\| + \|\tau([e,d])\| \tag{3.3}$$

for all $a \le c < e < d \le b$. Note that for $\varphi \in Y^{[a,b]}$ the set function τ_φ satisfies (3.3).

(b) We have

$$m^\tau(E) = \inf_{\beta \in \mathcal{C}^\star(E)} \sup_{\pi \in \mathcal{P}(\beta, E)} W_\tau(\pi)$$

and

$$\widetilde{\lambda}_\tau(E) = \inf_{\beta \in \widetilde{\mathcal{C}}(E)} \sup_{\pi \in \mathcal{P}(\beta, E)} W_\tau(\pi)$$

for all $\emptyset \ne E \subseteq [a,b]$.

(c) We have $\lambda_\tau = m_\tau := \mu_{\tau, \mathcal{B}_\mathfrak{I}}$ (with μ the Lebesgue measure, recall the notation introduced at the respective beginning of sections 3.1 and 3.3).

(d) The set functions $\widetilde{\lambda}_\tau, m^\tau : \mathfrak{P}([a,b]) \to [0,\infty]$ are metric outer measures. One calls $\widetilde{\lambda}_\tau$ the full variational measure of τ *and m^τ* the fine variational measure of τ

Proof. on (a): Let $\emptyset \ne E \subseteq [a,b]$. Obviously, we have $\mathcal{C}(E) \subseteq \widetilde{\mathcal{C}}(E) \subseteq \mathcal{C}^\star(E)$, which yields the claimed chain of inequalities.
Now assume that τ satisfies (3.3). For each $\beta \in \mathcal{C}(E)$ we put

$$\widetilde{\beta} := \{([\min(I), t], t) :\ (I,t) \in \beta \text{ with } \min(I) < t\} \cup \{([t, \max(I)], t) :\ (I,t) \in \beta \text{ with } \max(I) > t\}.$$

Clearly, $\widetilde{\beta} \in \widetilde{\mathcal{C}}(E)$. Pick $\pi = \{(I_j, t_j)\}_{j=1}^r \in \mathcal{P}(\beta)$. Using (3.3) we obtain

$$\begin{aligned} W_\tau(\pi) = \sum_{j=1}^r \|\tau(I)\| &= \sum_{\substack{j=1 \\ t_j \in I_j^\circ}}^r \|\tau(I_j)\| + \sum_{\substack{j=1 \\ t_j = \max(I_j)}}^r \|\tau(I_j)\| + \sum_{\substack{j=1 \\ t_j = \min(I_j)}}^r \|\tau(I_j)\| \\ &\le \sum_{\substack{j=1 \\ t_j \in I_j^\circ}}^r \|\tau([\min(t_j), t_j])\| + \sum_{\substack{j=1 \\ t_j \in I_j^\circ}}^r \|\tau([t_j, \max(t_j)])\| + s_1 + s_2 = W_\tau(\widetilde{\pi}), \end{aligned}$$

where we define

$$s_1 := \sum_{\substack{j=1 \\ t_j=\max(I_j)}}^{r} \|\tau(I_j)\|,$$

$$s_2 := \sum_{\substack{j=1 \\ t_j=\min(I_j)}}^{r} \|\tau(I_j)\|$$

and

$$\widetilde{\pi} := \{([\min(I_j), t_j], t_j)\}_{\substack{j\in\{1,\ldots,r\} \\ t_j\in I_j^\circ}} \cup \{([t_j, \max(I_j)], t_j)\}_{\substack{j\in\{1,\ldots,r\} \\ t_j\in I_j^\circ}} \cup \{(I_j, t_j)\}_{\substack{j\in\{1,\ldots,r\} \\ t_j\in\partial I_j}}.$$

Clearly, $\widetilde{\pi} \in \mathcal{P}(\widetilde{\beta})$, and hence $W_\tau(\widetilde{\pi}) \leq \sup_{\widehat{\pi}\in\mathcal{P}(\widetilde{\beta})} W_\tau(\widehat{\pi})$. This implies

$$\lambda_\tau(E) \leq \sup_{\pi\in\mathcal{P}(\beta)} W_\tau(\pi) \leq \sup_{\widetilde{\pi}\in\mathcal{P}(\widetilde{\beta})} W_\tau(\widetilde{\pi})$$

and thus

$$\lambda_\tau(E) \leq \inf_{\beta\in\mathcal{C}(E)} \sup_{\widetilde{\pi}\in\mathcal{P}(\widetilde{\beta})} W_\tau(\widetilde{\pi}).$$

For $\gamma \in \widetilde{\mathcal{C}}(E)$ we define

$$\Gamma := \{(I \cup J, t) :\ (I,t), (J,t) \in \gamma \text{ with } t \in \partial I \cap \partial J\}.$$

One easily verifies that Γ belongs to $\mathcal{C}(E)$ and that $\widetilde{\Gamma}$ is contained in γ. These facts imply

$$\lambda_\tau(E) \leq \inf_{\beta\in\mathcal{C}(E)} \sup_{\widetilde{\pi}\in\mathcal{P}(\widetilde{\beta})} W_\tau(\widetilde{\pi}) \leq \inf_{\gamma\in\widetilde{\mathcal{C}}(E)} \sup_{\widetilde{\pi}\in\mathcal{P}(\widetilde{\Gamma})} W_\tau(\widetilde{\pi}) \leq \inf_{\gamma\in\widetilde{\mathcal{C}}(E)} \sup_{\pi\in\mathcal{P}(\gamma)} W_\tau(\pi) = \widetilde{\lambda}_\tau(E).$$

The addendum is clear.

on (b): We only show the first equation as the second one can be derived analogously. Let $\beta \in \mathcal{C}^\star(E)$ and define $\beta_E := \{([u,v], t) \in \beta :\ t \in E\}$. We have $\beta_E \in \mathrm{C}^\star(E)$ and $\mathcal{P}(\beta_E) = \mathcal{P}(\beta, E)$. Since $\beta_E \subseteq \beta$, we obtain

$$m^\tau(E) \leq \sup_{\pi\in\beta_E} W_\tau(\pi) \leq \sup_{\pi\in\beta} W_\tau(\pi),$$

$$m^\tau(E) \leq \inf_{\beta\in\mathcal{C}^\star(E)} \sup_{\pi\in\beta_E} W_\tau(\pi) \leq \inf_{\beta\in\mathcal{C}^\star(E)} \sup_{\pi\in\beta} W_\tau(\pi) = m^\tau(E)$$

and consequently $m^\tau(E) = \inf_{\beta\in\mathcal{C}^\star(E)} \sup_{\pi\in\mathcal{P}(\beta,E)} W_\tau(\pi)$ using the definitions.

on (c): Let $\emptyset \neq E \subseteq [a,b]$ and $\delta \in (0,\infty)^E$ and put

$$\beta_\delta := \{([u,v], t) \in \mathfrak{I} \times E :\ ([u,v], t) \in \mathcal{S}(E,\delta)\} \in \mathcal{C}(E)$$

and

$$\mathcal{F}(E) := \{\beta_\delta :\ \delta \in (0,\infty)^E\} \subseteq \mathcal{C}(E).$$

We then have $\mathcal{S}(E,\delta) = \mathcal{P}(\beta_\delta)$ and consequently

$$\begin{aligned} m_\tau(E) &= \inf_{\delta\in(0,\infty)^E} \sup_{S\in\mathcal{S}(E,\delta)} W_\tau(S) = \inf_{\delta\in(0,\infty)^E} \sup_{\pi\in\mathcal{P}(\beta_\delta)} W_\tau(S) \\ &= \inf_{\beta\in\mathcal{F}(E)} \sup_{\pi\in\mathcal{P}(\beta_\delta)} W_\tau(S) \geq \inf_{\beta\in\mathcal{C}(E)} \sup_{\pi\in\mathcal{P}(\beta_\delta)} W_\tau(S) = \lambda_\tau(E). \end{aligned}$$

Now let $\beta \in \mathcal{C}(E)$ and fix for each $t \in E$ a number $\delta(t) > 0$ such that $([u,v],t) \in \beta$ for every $[u,v]$ with $t \in [u,v]$ and $v - u < \delta(t)$. It follows $\mathcal{S}(E, \frac{\delta}{2}) \subseteq \mathcal{P}(\beta)$ and therefore

$$m_\tau(E) \leq \sup_{S \in \mathcal{S}(E, \frac{\delta}{2})} W_\tau(S) \leq \sup_{\pi \in \mathcal{P}(\beta)} W_\tau(S),$$

which implies

$$m_\tau(E) \leq \inf_{\beta \in \mathcal{C}(E)} \sup_{\pi \in \mathcal{P}(\beta)} W_\tau(S) = \lambda_\tau(E).$$

on (d): We only show that m^τ is a metric outer measure; the proof for $\widetilde{\lambda}_\tau$ is similar. The definition yields $m^\tau(E) \geq 0$ for all $E \subseteq [a,b]$ and $m^\tau(\emptyset) = 0$. For $\emptyset \neq E \subseteq F \subseteq [a,b]$ we obtain $\mathcal{C}^\star(F) \subseteq \mathcal{C}^\star(E)$ and thus

$$m^\tau(E) = \inf_{\beta \in \mathcal{C}^\star(E)} \sup_{\pi \in \mathcal{P}(\beta)} W_\tau(\pi) \leq \inf_{\beta \in \mathcal{C}^\star(F)} \sup_{\pi \in \mathcal{P}(\beta)} W_\tau(\pi) = m^\tau(F).$$

Let $(A_n)_n$ be sequence of subsets of $[a,b]$ and put $A := \bigcup_{n \in \mathbb{N}} A_n$. At first, we additinally assume that the sets are pairwise disjoint and we put $I := \{n \in \mathbb{N} : A_n \neq \emptyset\}$. If $I = \emptyset$, we trivially have $m^\tau(A) \leq \sum_{n=1}^\infty m^\tau(A_n)$. So let I be nonvoid. Let $\varepsilon > 0$ and choose for each $n \in I$ a cover relation $\beta_n \in \mathcal{C}^\star(A_n)$ with $\sup_{\pi \in \mathcal{P}(\beta_n)} W_\tau(\pi) \leq m^\tau(A_n) + \frac{\varepsilon}{2^n}$. One easily verifies that $\beta := \dot{\bigcup}_{n \in I} \widehat{\beta}_n$ belongs to $\mathcal{C}^\star(A)$, where $\widehat{\beta}_n := \{(I,t) \in \beta_n : t \in A_n\}$. Consider $\pi = \{(I_j, t_j)\}_{j=1}^r \in \mathcal{P}(\beta, A)$. Observe that due to the pairwise disjointness of the sets $\{A_j\}_{j \in I}$ we have automatically $(I,t) \in \widehat{\beta}_n$ provided that $(I,t) \in \beta$ and $t \in A_n$. For this reason $\{(I_j, t_j) : j \in \{1, \ldots, r\} \text{ with } t_j \in A_n\} \in \mathcal{P}(\widehat{\beta}_n)$. Further note that we have $\mathcal{P}(\widehat{\beta}_n) \subseteq \mathcal{P}(\beta_n)$ for all $n \in I$, which gives us

$$\sup_{\pi \in \mathcal{P}(\widehat{\beta}_n)} W_\tau(\pi) \leq \sup_{\pi \in \mathcal{P}(\beta_n)} W_\tau(\pi) \leq m^\tau(A_n) + \frac{\varepsilon}{2^n}.$$

Combining these obserbvations we deduce

$$W_\tau(\pi) = \sum_{n \in I} \sum_{\substack{j=1 \\ t_j \in A_n}}^r \|\tau(I_j)\| \leq \sum_{n \in I} \sup_{\pi \in \mathcal{P}(\widehat{\beta}_n)} W_\tau(\pi) \leq \sum_{n \in I} \left(m^\tau(A_n) + \frac{\varepsilon}{2^n} \right) \leq \sum_{n \in I} m^\tau(A_n) + \varepsilon$$

and

$$\sup_{\pi \in \mathcal{P}(\beta, A)} W_\tau(\pi) \leq \sum_{n \in I} m^\tau(A_n) + \varepsilon.$$

Thanks to part (b) we obtain

$$m^\tau(A) = \inf_{\beta \in \mathcal{C}^\star(A)} \sup_{\pi \in \mathcal{P}(\beta, A)} W_\tau(\pi) \leq \sum_{n \in I} m^\tau(A_n) + \varepsilon$$

and taking the limit $\varepsilon \to 0$ we arrive at $m^\tau(A) \leq \sum_{n \in I} m^\tau(A_n) = \sum_{n=1}^\infty m^\tau(A_n)$. If the sets $(A_n)_n$ are not pairwise disjoint, then we put $B_1 := A_1$ and $B_n := A_n \setminus \bigcup_{j=1}^{n-1} A_j$ for $n > 1$. We then have (note that $m^\tau(B_n) \leq m^\tau(A_n)$ because of $B_n \subseteq A_n$)

$$m^\tau(A) = m^\tau \left(\dot{\bigcup}_{n \in \mathbb{N}} B_n \right) \leq \sum_{n=1}^\infty m^\tau(B_n) \leq \sum_{n=1}^\infty m^\tau(A_n)$$

and we are done.

Now let $A_1, A_2 \subseteq [a,b]$ and let $V_1, V_1 \subseteq [a,b]$ be open (relative to $[a,b]$) with $A_j \subseteq V_j$ and $V_1 \cap V_2 = \emptyset$. We want to show that $m^\tau(A_1 \cup A_2) = m^\tau(A_1) + m^\tau(A_2)$. Since we have already realized that m^τ is an outer measure, the inequality $\leq$ is clear by now. In particular, we have equality if $m^\tau(A_1 \cup A_2) = \infty$.

Moreover, equality surely holds if $A_1 = \emptyset$ or $A_2 = \emptyset$. So let us assume that the sets A_1 and A_2 are non-empty with $m^\tau(A_1 \cup A_2) < \infty$, which further implies $m^\tau(A_j) < \infty$ because m^τ is isotone as an outer measure. Let $\varepsilon > 0$ and choose $\beta \in \mathcal{C}^\star(A_1 \cup A_2)$ with $\sup_{\pi \in \mathcal{P}(\beta)} W_\tau(\pi) \le m^\tau(A_1 \cup A_2) + \varepsilon$. We put

$$\beta_j := \{(I,t) \in \beta : \ t \in A_j,\, I \subseteq V_j\} \in \mathcal{C}^\star(A_j)$$

and $\beta' := \beta_1 \dot\cup \beta_2$. Note that $\beta \supseteq \beta' \in \mathcal{C}^\star(A_1 \cup A_2)$. Let $\pi_j \in \mathcal{P}(\beta_j)$. Then $\pi_1 \dot\cup \pi_2 \in \mathcal{P}(\beta')$ as each interval of π_1 and each interval of π_2 are disjoint due to $V_1 \cap V_2 = \emptyset$. We estimate

$$W_\tau(\pi_1) + W_\tau(\pi_2) = W_\tau(\pi_1 \cup \pi_2) \le \sup_{\pi \in \mathcal{P}(\beta')} W_\tau(\pi),$$

and hence

$$m^\tau(A_1) + m^\tau(A_2) \le \sup_{\pi_1 \in \mathcal{P}(\beta_1)} W_\tau(\pi_1) + \sup_{\pi_2 \in \mathcal{P}(\beta_2)} W_\tau(\pi_2) \le \sup_{\pi \in \mathcal{P}(\beta')} W_\tau(\pi) \le m^\tau(A_1 \cup A_2) + \varepsilon.$$

Letting $\varepsilon \to 0^+$ we arrive at the conclusion $m^\tau(A_1) + m^\tau(A_2) \le m^\tau(A_1 \cup A_2)$. □

The remaining part of this section is now devoted to the aim to work out the intimate relation between m_φ and m^φ for BVG$_*$-functions φ. As a first step, the next lemma gives us an important and very useful sufficient condition for two interval functions $\tau, \sigma \in Y^{\mathfrak{I}}$ to produce the same set functions.

3.21 Lemma. *Let $\tau, \sigma \in Y^{\mathfrak{I}}$ and $\emptyset \ne E \subseteq [a,b]$ with $\widetilde{\lambda}_{\tau-\sigma}(E) = 0$. Then we have $\widetilde{\lambda}_\tau(F) = \widetilde{\lambda}_\sigma(F)$ and $m^\tau(F) = m^\sigma(F)$ for all $F \subseteq E$.*

Proof. For $F = \emptyset$ there is nothing to be shown and for $\emptyset \ne F \subseteq E$ we have $0 \le \widetilde{\lambda}_{\tau-\sigma}(F) \le \widetilde{\lambda}_{\tau-\sigma}(E) = 0$ and so the assumptions of this lemma are also satisfied for F in lieu of E. Therefore it suffices to establish the result for $F = E$.
Let $\varepsilon > 0$, choose a covering relation $\beta_1 \in \widetilde{\mathcal{C}}(E)$ with $\sup_{\pi \in \mathcal{P}(\beta_1)} W_{\tau-\sigma}(\pi) < \varepsilon$ and pick $\beta_2 \in \widetilde{\mathcal{C}}(E)$ such that $\sup_{\pi \in \mathcal{P}(\beta_2)} W_\tau(\pi) \le \widetilde{\lambda}_\tau(E) + \varepsilon$. We put $\beta := \beta_1 \cap \beta_2 \in \widetilde{\mathcal{C}}(E)$. Notice that for $\pi \in \mathcal{P}(\beta)$ we have $W_\sigma(\pi) \le W_{\tau-\sigma}(\pi) + W_\tau(\pi)$ and $W_{\tau-\sigma}(\pi) \le \varepsilon$ due to $\beta \subseteq \beta_1$. Moreover, because of $\beta \subseteq \beta_2$ we have $\sup_{\pi \in \mathcal{P}(\beta)} W_\tau(\pi) \le \sup_{\pi \in \mathcal{P}(\beta_2)} W_\tau(\pi) \le \widetilde{\lambda}_\tau(E) + \varepsilon$. These observations together yield

$$\widetilde{\lambda}_\sigma(E) \le \sup_{\pi \in \mathcal{P}(\beta)} W_\sigma(\pi) \le \sup_{\pi \in \mathcal{P}(\beta)} \left(W_{\tau-\sigma}(\pi) + W_\tau(\pi)\right) \le \varepsilon + \sup_{\pi \in \mathcal{P}(\beta)} W_\tau(\pi) \le \widetilde{\lambda}_\tau(E) + 2\varepsilon.$$

Letting $\varepsilon \to 0^+$ we obtain $\widetilde{\lambda}_\sigma(E) \le \widetilde{\lambda}_\tau(E)$. By symmetry, $\widetilde{\lambda}_\tau(E) \le \widetilde{\lambda}_\sigma(E)$, which proves the first assertion.

Next pick $\gamma \in \mathcal{C}^\star(E)$ with $\sup_{\pi \in \mathcal{P}(\gamma)} W_\tau(\pi) \le m^\tau(E) + \varepsilon$ and consider $\gamma \cap \beta_1 \in \mathcal{C}^\star(E)$. Similarly as above, we estimate $W_\sigma(\pi) \le W_{\tau-\sigma}(\pi) + W_\tau(\pi)$ and $W_{\tau-\sigma}(\pi) \le \varepsilon$ due to $\gamma \cap \beta_1 \subseteq \beta_1$ and, using $\gamma \cap \beta_1 \subseteq \gamma$, also $\sup_{\pi \in \mathcal{P}(\gamma \cap \beta_1)} W_\tau(\pi) \le \sup_{\pi \in \mathcal{P}(\gamma)} W_\tau(\pi) \le m^\tau(E) + \varepsilon$. Consequently, we derive as above $m^\sigma(E) \le m^\tau(E) + 2\varepsilon$ and may complete the proof as before. □

We now need some additional notation. For $\varphi \in Y^{[a,b]}$ we consider the function

$$V_\varphi : [a,b] \to [0,\infty];\ t \mapsto \begin{cases} 0, & \text{if } t = a, \\ V(\varphi, [a,t]), & \text{if } t > a. \end{cases}$$

We need some elementary properties of V_φ, which are well-known for scalar-valued functions. For the sake of completeness we provide a proof for these properties. As an auxiliary result we need the subsequent lemma.

3.22 Lemma. *Let $\varphi \in \mathrm{BV}([a,b],Y)$ and $c \in (a,b)$. Then $\varphi \in \mathrm{BV}([a,c],Y)$ and $\varphi \in \mathrm{BV}([c,b],Y)$ and $V(\varphi,[a,c]) + V(\varphi,[c,b]) = V(\varphi,[a,b])$.*

Proof. As the proof of Theorem 6.11 in [Apo74]. □

3.23 Lemma. *Let $\varphi \in \mathrm{BV}([a,b],Y)$. The function V_φ is an increasing, real-valued function, whose continuity points coincide with the continuity points of φ, and we have $\|\varphi(t)-\varphi(s)\| \le V(\varphi,[t,s]) = V_\varphi(s)-V_\varphi(t)$ for all $a \le t \le s \le b$.*

Proof. Lemma 3.22 yields that V_φ is real-valued and the inequality $\|\varphi(t) - \varphi(s)\| \le V(\varphi,[t,s])$ is obvious. For $t = s$ or $t = a$ we clearly have $V(\varphi,[t,s]) = V_\varphi(s) - V_\varphi(t)$. Let $a < t < s$. Lemma 3.22 implies

$$V_\varphi(s) = V(\varphi,[a,s]) = V(\varphi,[a,t]) + V(\varphi,[t,s]) = V_\varphi(t) + V(\varphi,[t,s]),$$

$$0 \le V(\varphi,[t,s]) = V_\varphi(s) - V_\varphi(t).$$

This shows the asserted relations as well as the increasing monotonicity of V_φ. In addition, this also yields that each continuity point of V_φ is also a continuity point of φ. Next let $t \in (a,b)$ be a continuity point of φ. Let $\varepsilon > 0$ and pick $\delta > 0$ with $\|\varphi(s) - \varphi(s)\| < \frac{\varepsilon}{2}$ for all $s \in U_\delta(t) \cap [a,b]$. Take a partition $S = \{[a_j,b_j]\}_{j=1}^r \in \mathcal{A}([t,b])$ of $[t,b]$ such that $V(\varphi,[t,b]) - \sum_{j=1}^r \|\varphi(b_j) - \varphi(a_j)\| < \frac{\varepsilon}{2}$ and w.l.o.g. $b_1 - t < \delta$. For $s \in (t,b_1)$ we then obtain applying once again Lemma 3.22

$$\begin{aligned}
0 \le V_\varphi(s) - V_\varphi(t) &= V(\varphi,[t,s]) = V(\varphi,[t,b]) - V(\varphi,[s,b]) \\
&= V(\varphi,[t,b]) - \sum_{j=1}^r \|\varphi(b_j) - \varphi(a_j)\| + \sum_{j=1}^r \|\varphi(b_j) - \varphi(a_j)\| - V(\varphi,[s,b]) \\
&\le \frac{\varepsilon}{2} + \|\varphi(s) - \varphi(t)\| + \|\varphi(b_1) - \varphi(s)\| + \sum_{j=2}^r \|\varphi(b_j) - \varphi(a_j)\| - V(\varphi,[s,b]) \\
&\le \varepsilon + V(\varphi,[s,b]) - V(\varphi,[s,b]) = \varepsilon.
\end{aligned}$$

This shows $\lim_{s\to t^+} V_\varphi(s) = V_\varphi(t)$. Analogously, we deduce $\lim_{s\to t^-} V_\varphi(s) = V_\varphi(t)$, and the case $t \in \{a,b\}$ is treated similarly. □

Now we come to a very crucial lemma, which constitutes an important ingredient in order to relate m_φ and m^φ.

3.24 Lemma. *Let $\varphi \in \mathrm{BV}([a,b],Y)$. Then we have $\widetilde{\lambda}_{\tau_{V_\varphi} - \|\tau_\varphi\|}([a,b]) = 0$.*

Proof. Fix $\varepsilon > 0$ and choose a partition $\{[a_j,b_j]\}_{j=1}^r$ of $[a,b]$ with $V(\varphi,[a,b]) - \sum_{j=1}^r \|\varphi(b_j) - \varphi(a_j)\| < \varepsilon$. We not put

$$\beta := \{(I,t) \in \mathfrak{I} \times [a,b] :\ t \in \partial I,\ \exists\, j \in \{1,\dots,r\} :\ I \subseteq [a_j,b_j]\}.$$

It is easy to confirm that $\beta \in \widetilde{\mathcal{C}}([a,b])$. Consider an arbitrary element $\pi = \{([c_k,d_k],t_k)\}_{k=1}^\nu$ of $\mathcal{P}(\beta)$ with the additional property that $\bigcup_{k=1}^\nu [c_k,d_k] = [a,b]$ (where as usual the intervals $[c_k,d_k]$ are in increasing order). Using Lemma 3.23 and observing that

$$\bigcup\{[c_k,d_k] :\ k \in \{1,\dots,\nu\} \text{ with } [c_k,d_k] \subseteq [a_j,b_j]\} = [a_j,b_j] \tag{3.4}$$

for each $j \in \{1, \dots, r\}$, we then compute

$$\begin{aligned}\sum_{k=1}^{\nu} \left|(\tau_{V_\varphi} - \|\tau_\varphi\|)([c_k, d_k])\right| &= \sum_{k=1}^{\nu} |V_\varphi(d_k) - V_\varphi(c_k) - \|\varphi(d_k) - \varphi(c_k)\|| \\ &= \sum_{k=1}^{\nu} (V(\varphi, [c_k, d_k]) - \|\varphi(d_k) - \varphi(c_k)\|) \\ &= \sum_{j=1}^{r} \sum_{\substack{k=1 \\ [c_k,d_k]\subseteq[a_j,b_j]}}^{\nu} (V(\varphi, [c_k, d_k]) - \|\varphi(d_k) - \varphi(c_k)\|) \\ &= \sum_{j=1}^{r} \left(V(\varphi, [a_j, b_j]) - \sum_{\substack{k=1 \\ [c_k,d_k]\subseteq[a_j,b_j]}}^{\nu} \|\varphi(d_k) - \varphi(c_k)\| \right) \\ &\le \sum_{j=1}^{r} V(\varphi, [a_j, b_j]) - \sum_{j=1}^{r} \|\varphi(b_j) - \varphi(a_j)\| \\ &= V(\varphi, [a, b]) - \sum_{j=1}^{r} \|\varphi(b_j) - \varphi(a_j)\| < \varepsilon,\end{aligned}$$

where we used

$$\|\varphi(b_j) - \varphi(a_j)\| \le \sum_{\substack{k=1 \\ [c_k,d_k]\subseteq[a_j,b_j]}}^{\nu} \|\varphi(d_k) - \varphi(c_k)\|,$$

which results from (3.4). Now let $\pi \in \mathcal{P}(\beta)$ be arbitrary. Using the definition of β we easily find a partition $\widetilde{\pi} \in \mathcal{P}(\beta)$ of $[a, b]$ (i.e., $\bigcup_{(I,t)\in\widetilde{\pi}} I = [a, b]$) with $\pi \subseteq \widetilde{\pi}$. So the above estimate yields

$$\sum_{(I,t)\in\pi} \left|(\tau_{V_\varphi} - \|\tau_\varphi\|)(I)\right| \le \sum_{(I,t)\in\widetilde{\pi}} \left|(\tau_{V_\varphi} - \|\tau_\varphi\|)(I)\right| < \varepsilon.$$

We conclude

$$\widetilde{\lambda}_{\tau_{V_\varphi} - \|\tau_\varphi\|}([a, b]) \le \sup_{\pi\in\mathcal{P}(\beta)} W_{\tau_{V_\varphi} - \|\tau_\varphi\|}(\pi) \le \varepsilon$$

and hence $\widetilde{\lambda}_{\tau_{V_\varphi} - \|\tau_\varphi\|}([a, b]) = 0$. □

Now we use the preceding lemma to make a further important step towards our aim to relate m_φ and m^φ.

3.25 Lemma. *Let $\varphi \in \mathrm{BV}([a, b], Y)$ and denote by D the countable set of all discontinuities of φ. Then we have $m_\varphi(E) = \widetilde{\lambda}_\varphi(E) = m^\varphi(E)$ for all $E \subseteq [a, b] \setminus D$.*

Proof. Let $E \subseteq [a, b] \setminus D$. Combining Lemma 3.21, Lemma 3.24 and Lemma 3.20 we derive

$$m^\varphi(E) = m^{\|\tau_\varphi\|}(E) = m^{V_\varphi}(E)$$

and

$$\widetilde{\lambda}_\varphi(E) = \widetilde{\lambda}_{\|\tau_\varphi\|}(E) = \widetilde{\lambda}_{V_\varphi}(E) = \lambda_{V_\varphi}(E) = m_{V_\varphi}(E).$$

Furthermore, Lemma 3.20 gives us

$$m^\varphi(E) \le \widetilde{\lambda}_\varphi(E) = \lambda_\varphi(E) = m_\varphi(E).$$

Putting these equations together we obtain

$$m^{V_\varphi}(E) = m^\varphi(E) \le m_\varphi(E) = m_{V_\varphi}(E).$$

So it remains to establish $m_{V_\varphi}(E) \le m^{V_\varphi}(E)$. For this purpose pick $\beta \in \mathcal{C}^\star(E)$ and put $\Lambda := \{I \in \mathfrak{I} : \exists\, t \in [a,b] : (I,t) \in \beta\}$. Observe that Λ forms a Vitali cover of E in the classical sense, i.e., for each $t \in E$ and each $\varepsilon > 0$ there exists an interval $I \in \Lambda$ with $t \in I$ and $\operatorname{diam}(I) < \varepsilon$. Thanks to Proposition B.1 we can extract from Λ a countable family $(I_n)_n$ of pairwise disjoint intervals such tha $m_{V_\varphi}(E \setminus \bigcup_n I_n) = 0$ (notice that m_{V_φ} is finite due to Lemma 3.7 as φ belongs to $\mathrm{BV}([a,b],Y)$). As as result,

$$m_{V_\varphi}(E) \le m_{V_\varphi}\left(E \cap \bigcup_n I_n\right) + m_{V_\varphi}\left(E \setminus \bigcup_n I_n\right) = m_{V_\varphi}\left(\bigcup_n (E \cap I_n)\right) \le \sum_n m_{V_\varphi}(E \cap I_n).$$

Consider $t \in \partial I_n$. If $t \in D$, then $t \notin E \cap I_n$. If, however, $t \in [a,b] \setminus D$, then V_φ is continuous at t by Lemma 3.23 and thus $m_{V_\varphi}(\{t\}) = 0$ due to Lemma 3.8. This observation leads to $m_{V_\varphi}(E \cap I_n) = m_{V_\varphi}(E \cap I_n^\circ)$, where we form I_n° in $\mathbb{R}$. Applying once again Lemma 3.7 and Lemma 3.23, we estimate

$$\begin{aligned} m_{V_\varphi}(E) &\le \sum_n m_{V_\varphi}(E \cap I_n) = \sum_n m_{V_\varphi}(E \cap I_n^\circ) \\ &\le \sum_n m_{V_\varphi}(I_n^\circ) = \sum_n V(V_\varphi, I_n^\circ) = \sum_n \left(\lim_{t \to \max(I_n)^-} V_\varphi(t) - \lim_{t \to \min(I_n)^+} V_\varphi(t) \right) \\ &\le \sum_n \left(V_\varphi(\max(I_n)) - V_\varphi(\min(I_n))\right) = \sum_n \tau_{V_\varphi}(I_n). \end{aligned}$$

Fix $t_n \in [a,b]$ with $(I_n, t_n) \in \beta$ for each n. As the intervals in the countable family $\{I_n\}_n$ are pairwise disjoint, every finite subset of $\{(I_n,t_n)\}_n$ is an element of $\mathcal{P}(\beta)$. This yields $\sum_n \tau_{V_\varphi}(I_n) \le \sup_{\pi \in \mathcal{P}(\beta)} W_{\tau_{V_\varphi}}(\pi)$. As a consequence, we deduce

$$m_{V_\varphi}(E) \le \inf_{\beta \in \mathcal{C}^\star(E)} \sup_{\pi \in \mathcal{P}(\beta)} W_{\tau_{V_\varphi}}(\pi) = m^{V_\varphi}(E)$$

and this finishes the proof. □

Taking the preceding lemma as a starting point, our next objective to extend exactly this lemma to BVG$_*$-functions. In order to achieve this aim we shall need the next three technial lemmata.

3.26 Lemma. *Let $\emptyset \ne E \subseteq [a,b]$ be closed, $\varphi \in Y^{[a,b]}$ and let $(I_n)_n$ denote the finite or infinite sequence of the connected components of $[c,d] \setminus E$, where $c =: \min E$ and $d := \max E$. Assume that $\sum_n \omega(\varphi, \overline{I}_n) < \infty$ and $\varphi|_E = 0$. Denote by D the set of discontinuities of φ. Then $m_\varphi(E \setminus D) = 0$.*

Proof. W.l.o.g. we may assume that $E \ne [c,d]$ (hence the family $(I_n)_n$ is nonvoid). Let $\varepsilon > 0$ and choose k such that $\sum_{n>k} \omega(\varphi, \overline{I}_n) < \varepsilon$. Write $\overline{I}_n = [c_n, d_n]$. If $n \le k$ and $c_n \notin D$ resp. $d_n \notin D$, take $\delta(c_n) > 0$ resp. $\delta(d_n) > 0$ with $\|\varphi(t) - \varphi(c_n)\| < \frac{\varepsilon}{k}$ for all $t \in U_{\delta(c_n)}(c_n) \cap [a,b]$ resp. with $\|\varphi(t) - \varphi(d_n)\| < \frac{\varepsilon}{k}$ for all $t \in U_{\delta(d_n)}(d_n) \cap [a,b]$. For $t \in E \setminus (D \cup \bigcup_{n \le k}\{c_n, d_n\})$ choose $\delta(t) > 0$ with $U_{\delta(t)}(t) \subseteq (c,d) \setminus \bigcup_{n\le k}\{c_n,d_n\}$ if $t \in (c,d)$ resp. with $U_{\delta(t)}(t) \subseteq \mathbb{R} \setminus \bigcup_{n \le k}\{c_n, d_n\}$ if $t \in \{c,d\}$ and in both of these two cases with $\|\varphi(t) - \varphi(s)\| < \varepsilon$ for all $s \in U_{\delta(t)}(t) \cap [a,b]$. We now consider an arbitrary element $\{([a_j,b_j],t_j)\}_{j=1}^r$ of $\mathcal{S}(E \setminus D, \delta)$. Let $1 \le j < \nu \le r$ with $a_j, a_\nu \in [c,d] \setminus E$. Then a_j and a_ν belong to different components of $[c,d] \setminus E$, for otherwise we would obtain $t_j \in [a_j,b_j] \subseteq [a_j, a_\nu] \subseteq I_n \subseteq \mathbb{R} \setminus \{t_j\}$ for an n, but this is not possible. For $a_j \in [c,d] \setminus E$ let $I_{n(j)}$ denote that component of $[c,d] \setminus E$ that contains a_j. Analogously, one sees that for $1 \le j < \nu \le r$ with $b_j, b_\nu \in [c,d] \setminus E$ the points b_j and b_ν also belong to different components of $[c,d] \setminus E$ and we let $J_{m(j)}$ denote the corresponding component of $[c,d] \setminus E$ containing b_j. If $a_j \in [c,d] \setminus E$ and $t_j \notin \bigcup_{n \le k}\{c_n, d_n\}$, then $a_j \le d_{n(j)} \le t_j$ and hence $d_{n(j)} \in U_{\delta(t_j)}(t_j)$, so that by the definition of δ

we have $d_{n(j)} \notin \bigcup_{n\le k}\{c_n,d_n\}$, i.e., $n(j) > k$. Analogously, one can derive $m(j) > k$ provided that $b_j \in [c,d]\setminus E$ and $t_j \notin \bigcup_{n\le k}\{c_n,d_n\}$. We now estimate

$$\begin{aligned}\sum_{j=1}^{r} \|\varphi(b_j)-\varphi(a_j)\| \le & \|\varphi(b_1)-\varphi(t_1)\| + \|\varphi(t_1)-\varphi(a_1)\| + \|\varphi(t_r)-\varphi(a_r)\| + \|\varphi(b_r)-\varphi(t_r)\| \\ & + \sum_{\substack{1<j<r\\ b_j\in E}} \|\varphi(b_j)-\varphi(t_j)\| + \sum_{\substack{1<j<r\\ b_j\notin E}} \|\varphi(b_j)-\varphi(t_j)\| \\ & + \sum_{\substack{1<j<r\\ a_j\in E}} \|\varphi(t_j)-\varphi(a_j)\| + \sum_{\substack{1<j<r\\ a_j\notin E}} \|\varphi(t_j)-\varphi(a_j)\|.\end{aligned}$$

By the definition of δ and the hypothesis $\varphi|_E = 0$, we have

$$\|\varphi(b_1)-\varphi(t_1)\| + \|\varphi(t_1)-\varphi(a_1)\| + \|\varphi(t_r)-\varphi(a_r)\| + \|\varphi(b_r)-\varphi(t_r)\| < 4\varepsilon,$$

$$\sum_{\substack{1<j<r\\ b_j\in E}} \|\varphi(b_j)-\varphi(t_j)\| = 0$$

and

$$\sum_{\substack{1<j<r\\ a_j\in E}} \|\varphi(t_j)-\varphi(a_j)\| = 0.$$

Let $1<j<r$, $a_j \notin E$ and $t_j \notin \bigcup_{n\le k}\{c_n,d_n\}$. Due to $a_j \in U_{\delta(t_j)}(t_j)$ and $t_j \in (c,d)$ (as $1<j<r$), we conclude $a_j \in [c,d]\setminus E$. Therefore we obtain

$$\begin{aligned}\sum_{\substack{1<j<r\\ a_j\notin E,\, t_j\notin\bigcup_{n\le k}\{c_n,d_n\}}} \|\varphi(t_j)-\varphi(a_j)\| &= \sum_{\substack{1<j<r\\ a_j\notin E,\, t_j\notin\bigcup_{n\le k}\{c_n,d_n\}}} \|0-\varphi(a_j)\| \\ &= \sum_{\substack{1<j<r\\ a_j\notin E,\, t_j\notin\bigcup_{n\le k}\{c_n,d_n\}}} \|\varphi(d_{n(j)})-\varphi(a_j)\| \\ &\le \sum_{\substack{1<j<r\\ a_j\notin E,\, t_j\notin\bigcup_{n\le k}\{c_n,d_n\}}} \omega(\varphi,\overline{I}_{n(j)}) \le \sum_{n>k}\omega(\varphi,\overline{I}_n) < \varepsilon,\end{aligned}$$

where the inequality

$$\sum_{\substack{1<j<r\\ a_j\notin E,\, t_j\notin\bigcup_{n\le k}\{c_n,d_n\}}} \omega(\varphi,\overline{I}_{n(j)}) \le \sum_{n>k}\omega(\varphi,\overline{I}_n)$$

follows from $n(j) > k$ (see above) and $I_{n(j)} \ne I_{n(\nu)}$ for $j \ne \nu$ (see also above). We further estimate

$$\sum_{\substack{1<j<r\\ a_j\in E,\, t_j\in\bigcup_{n\le k}\{c_n,d_n\}}} \|\varphi(t_j)-\varphi(a_j)\| \le \sum_{\substack{1<j<r\\ a_j\in E,\, t_j\in\bigcup_{n\le k}\{c_n,d_n\}}} \frac{\varepsilon}{k} \le 4\varepsilon,$$

where we used

$$\sharp\left\{j\in\{1,\dots,r\} :\ a_j\in E,\ t_j\in\bigcup_{n\le k}\{c_n,d_n\}\right\} \le \sharp\left\{j\in\{1,\dots,r\} :\ t_j\in\bigcup_{n\le k}\{c_n,d_n\}\right\} \le 4k,$$

which results from the observation that $\sharp\{j\in\{1,\dots,r\} :\ t_j = s\} \le 2$ for all $s \in \bigcup_{n\le k}\{c_n,d_n\}$. Summarizing, we arrive at

$$\sum_{\substack{1<j<r\\ a_j\in E}} \|\varphi(t_j)-\varphi(a_j)\| + \sum_{\substack{1<j<r\\ a_j\notin E}} \|\varphi(t_j)-\varphi(a_j)\| < 5\varepsilon.$$

Analogously, one can deduce

$$\sum_{\substack{1<j<r \\ b_j\in E}} \|\varphi(b_j)-\varphi(t_j)\| + \sum_{\substack{1<j<r \\ b_j\notin E}} \|\varphi(b_j)-\varphi(t_j)\| < 5\varepsilon.$$

Altogether we derive

$$\sum_{j=1}^{r} \|\varphi(b_j)-\varphi(a_j)\| < 14\varepsilon,$$

which completes the proof. □

3.27 Lemma. *Let $\varphi \in \mathrm{BVG}_*([a,b],Y)$ and denote by D the countable set of all discontinuities of φ. Then there are sequences $(E_n)_n$ and $(\varphi_n)_n$ possessing the following properties:*

(a) *E_n is closed for all $n \in \mathbb{N}$;*

(b) *$\bigcup_n E_n = [a,b]$ and $\varphi \in \mathrm{BV}_*(E_n, Y)$ for all $n \in \mathbb{N}$;*

(c) *$\varphi_n \in \mathrm{BV}([a,b],Y)$ and the point set D_n of discontinuities of φ_n is contained in D for all $n \in \mathbb{N}$;*

(d) *$m_{\varphi-\varphi_n}(E_n \setminus D) = 0$ for all $n \in \mathbb{N}$.*

Proof. We choose nonvoid closed sets $(E_n)_n$ such that $\bigcup_n E_n = [a,b]$ and $\varphi \in \mathrm{BV}_*(E_n,Y)$ for all $n \in \mathbb{N}$. Then $\sum_k \omega(\varphi, \overline{I_{k,n}}) < \infty$ for all n, where $(I_{k,n})_k$ is the (finite or infinite) sequence of the connected components of $[\min E_n, \max E_n] \setminus E_n$ (see Lemma 2.17). Furthermore, let φ_n be a linear extension of $\varphi|_{E_n}$ to the whole of $[a,b]$. Then $\varphi_n \in \mathrm{BV}([a,b],Y)$ and $D_n \subseteq D$ thanks to Lemma 2.21, where D_n is the point set of discontinuities of φ_n. Moreover, we have

$$\sum_k \omega(\varphi-\varphi_n, \overline{I_{k,n}}) \le \sum_k \omega(\varphi, \overline{I_{k,n}}) + \sum_k \omega(\varphi_n, \overline{I_{k,n}}) < \infty$$

and $(\varphi-\varphi_n)|_{E_n} = 0$ and so Lemma 3.26 implies $m_{\varphi-\varphi_n}(E_n \setminus D) = 0$. □

3.28 Lemma. *Let $(\mathfrak{X}, d)$ be a metric space and let $\mu : \mathfrak{P}(\mathfrak{X}) \to [0,\infty]$ be a metric outer measure. Let $(A_n)_n$ be pairwise disjoint Borel sets with $\mu(A_n) < \infty$ for all $n \in \mathbb{N}$ and let $E \subseteq \bigcup_n A_n$. Then $\mu(E) = \sum_n \mu(E \cap A_n)$.*

Proof. First, let $A, B \in \mathrm{Bor}(\mathfrak{X})$ with $A \cap B = \emptyset$, $\mu(A) < \infty$ and $\mu(B) < \infty$. Since A, B and $A \cup B \in \mathrm{Bor}(\mathfrak{X})$ are μ-measurable (in the sense of Carathéodory), we obtain

$$\begin{aligned}\mu(A\setminus E)+\mu(A\cap E)+\mu(B\setminus E)+\mu(B\cap E) &= \mu(A)+\mu(B) = \mu(A\cup B)\\ &= \mu((A\cup B)\setminus E)+\mu((A\cup B)\cap E)\\ &\le \mu(A\setminus E)+\mu(B\setminus E)+\mu((A\cup B)\cap E),\end{aligned}$$

hence

$$\mu(A\cap E)+\mu(B\cap E) \le \mu((A\cap E)\cup(B\cap E)) \le \mu(A\cap E)+\mu(B\cap E),$$

i.e., $\mu(A\cap E)+\mu(B\cap E) = \mu((A\cup B)\cap E)$. Using this equality, we derive

$$\begin{aligned}\mu(E) = \mu\left(\bigcup_n (A_n\cap E)\right) &\le \sum_n \mu(A_n\cap E) = \sup_k \sum_{n\le k} \mu(A_n\cap E)\\ &= \sup_k \mu\left(E \cap \bigcup_{n\le k} A_n\right) \le \sup_k \mu(E) = \mu(E)\end{aligned}$$

as claimed. □

We finally arrive at the announced main result of this section that relates m_φ and m^φ for BVG_*-functions φ extending Lemma 3.25.

3.29 Theorem. *Let $\varphi \in \mathrm{BVG}_*([a,b],Y)$ and let D denote the set of discontinuities of φ. Then we have*

$$m_\varphi(E) = m^\varphi(E)$$

for all $E \subseteq [a,b] \setminus D$.

Proof. Choose sequences $(E_n)_n$ and $(\varphi_n)_n$ according to Lemma 3.27. By Lemma 3.25, we have $m_{\varphi_n}(E) = m^{\varphi_n}(E)$ for all $E \subseteq [a,b]\setminus D \subseteq [a,b]\setminus D_n$. Lemma 3.20 (a) and (c) and Lemma 3.21 together imply $m_\varphi(F) = m_{\varphi_n}(F)$ and $m^\varphi(F) = m^{\varphi_n}(F)$ for all $F \subseteq E_n \setminus D$. Let $A_1 := E_1 \in \mathrm{Bor}([a,b])$, $A_n := E_n \setminus \bigcup_{k<n} E_k \in \mathrm{Bor}([a,b])$ for $n > 1$ and let $E \subseteq [a,b] \setminus D$. Using that $E \cap A_n \subseteq E_n \setminus D$, we infer $m_\varphi(E \cap A_n) = m_{\varphi_n}(E \cap A_n) \le m_{\varphi_n}([a,b]) < \infty$ (recall Lemma 3.7) and $m^\varphi(E \cap A_n) = m^{\varphi_n}(E \cap A_n) = m_{\varphi_n}(E \cap A_n) < \infty$ for all n. Applying Lemma 3.28, we therefore deduce

$$m_\varphi(E) = \sum_n m_\varphi(E\cap A_n) = \sum_n m_{\varphi_n}(E\cap A_n) = \sum_n m^{\varphi_n}(E\cap A_n) = \sum_n m^\varphi(E\cap A_n) = m^\varphi(E)$$

as asserted. □

3.30 Remark For real-valued functions with an additional continuity condition results comparable to Theorem 3.29 appear as Theorem 4 in [Hen79] and as Theorem 15.10 in [Hen88]. However, for real-valued functions Theorem 3.29 seems to appear only in [Tho85] as Theorem 41.4. Apart from the fact that the proof given there is rather concisely written (several details are not carried out), it has a considerable gap (for the subsequent considerations cf. the proof of Theorem 41.4 in [Tho85]): At a decisive point of his proof, Thomson refers to his previous Corollary 41.2, but this result as well as the preceding results from which 41.2 is deduced consider continuous functions on $\mathbb{R}$, whereas the function g under consideration in the proof of Theorem 41.4 is in general not continuous. Admittedly, Thomson wants to apply his Corollray 41.2 on a set, namely C_f, where g is continuous, but it can't be helped that the assumptions of Corollary 41.2 are not satisfied, as this set C_f does not fit into the framework of this result. This is due to the fact that C_f is the continuity set of a BVG_*-function and as every countable set set can appear as the discontinuity set of a function of bounded variation, the set set C_f may have such an unpleasant structure (e.g., it might be totally disconnected and neither open nor closed) that it is at least very questionable whether it is possible to make Corollary 41.2 applicable without any great additional efforts. Nevertheless, the question arises if continuity is really that essential in Thomson's proof. In fact, it is more or less: In order (in our notation) to move from m_φ to m^φ and vice versa the Lebesgue-Stieltjes measure ν_{V_φ} (see Lemma A.1) associated with V_φ plays a central role. Hence without assuming that φ resp. V_φ (φ of bounded variation in the classical sense) is at least everywhere continuous from the right or from the left, one has to face the problem to give a reasonable meaning to ν_{V_φ}. Therefore we were compelled to deeply enter the proofs of Thomson and to furnish refinements of them from the very beginning on in order to obtain a full and complete proof of our Theorem 3.29 resp. Thomson's Theorem 41.4 in [Tho85]. In [Ene98] V. Ene made similar efforts in the real-valued case and Theorem 6.1 in [Ene98] is closest to our Theorem 3.29, but Ene does not proceed to the full version of Theorem 3.29 and poises at a preliminary version of it (he considers only BV_*-functions). Moreover, there are two methodological differences between Ene's approach and ours. First, Ene's arguments rely on the conceptual framework and the machinery of the so-called local systems introduced by Thomson, while we avoid them, and it should not go unmentioned that this is not at the cost of a loss of generality. Indeed, using Corollary 37.2 in [Tho85], it is easy to obtain Ene's seemingly more general results from our approach. Second, Ene handles the afore-mentioned complication of giving a meaning to ν_{V_φ} by considering $\lambda(V_\varphi(\cdot))$ as a substitute (recall that λ is the one-dimensional outer Lebesgue-measure), whereas in contrast to that we do not need such a substitute and hence we give a directer deduction for Theorem 3.29. Consequently, Ene's and our approach (although both following the route of Thomson's line of argument) differ in several details distinctly.

4 The variational Henstock-Kurzweil-Stieltjes integral

4.1 Definition and basic properties

In this section we finally introduce the (variational) Henstock-Kurzweil-Stieltjes integral and collect some of its basic properties.

4.1 Definition. *Let $\varphi \in Y^{[a,b]}$. A function $f : [a,b] \to X$ is called*

(a) Riemann-Stieltjes integrable with respect to $\varphi : [a,b] \to Y$ *if there is a point $z \in Z$ such that for each $\varepsilon > 0$ there exists a constant $\delta > 0$ such that*

$$\left\| \sum_{j=1}^{r} f(x_j) \cdot [\varphi(b_j) - \varphi(a_j)] - z \right\|_Z < \varepsilon$$

for all $\{([a_j,b_j],x_j)\}_{j=1}^r \in \mathcal{S}([a,b],\delta)$ with $\bigcup_{j=1}^r [a_j,b_j] = [a,b]$. In this case z is unique and we write $z = (\mathcal{R}) \int_a^b f(s) \cdot \mathrm{d}\varphi(s)$.

(b) Henstock-Kurzweil-Stieltjes integrable with respect to $\varphi : [a,b] \to Y$ *if there is a point $z \in Z$ such that for each $\varepsilon > 0$ there exists a gauge $\delta : [a,b] \to (0,\infty)$ such that*

$$\left\| \sum_{j=1}^{r} f(x_j) \cdot [\varphi(b_j) - \varphi(a_j)] - z \right\|_Z < \varepsilon$$

for all $\{([a_j,b_j],x_j)\}_{j=1}^r \in \mathcal{S}([a,b],\delta)$ with $\bigcup_{j=1}^r [a_j,b_j] = [a,b]$. In this case z is unique and we write $z = (\mathcal{HK}) \int_a^b f(s) \cdot \mathrm{d}\varphi(s)$.

(c) variationally/strongly Henstock-Kurzweil-Stieltjes integrable with respect to $\varphi : [a,b] \to Y$ *if there is a function $F : [a,b] \to Z$ such that for each $\varepsilon > 0$ there exists a gauge $\delta : [a,b] \to (0,\infty)$ such that the inequality*

$$\sum_{j=1}^{r} \|(F(b_j) - F(a_j)) - f(x_j) \cdot [\varphi(b_j) - \varphi(a_j)]\|_Z < \varepsilon$$

is fulfilled whenever $\{([a_j,b_j],x_j)\}_{j=1}^r \in \mathcal{S}([a,b],\delta)$. In this case we say that F is an indefinite variational Henstock-Kurzweil-Stieltjes integral of f with respect to φ.

(d) variationally/strongly McShane-Stieltjes integrable with respect to $\varphi : [a,b] \to Y$ *if there is a function $F : [a,b] \to Z$ such that for each $\varepsilon > 0$ there exists a gauge $\delta : [a,b] \to (0,\infty)$ such that*

$$\sum_{j=1}^{r} \|(F(b_j) - F(a_j)) - f(x_j) \cdot [\varphi(b_j) - \varphi(a_j)]\|_Z < \varepsilon$$

is satisfied for all finite sequences $\{([a_j,b_j],x_j)\}_{j=1}^r$, $r \in \mathbb{N}$, with $x_1,\dots,x_r,a_1,\dots,a_r,b_1,\dots,b_r \in [a,b]$, $a_j < b_j$ and $[a_j,b_j] \subseteq U_{\delta(x_j)}(x_j)$ for all $j \in \{1,\dots,r\}$.

4.2 Remark (a) For basic properties of the Riemann-Stieltjes and the Henstock-Kurzweil-Stieltjes integral we refer to Chapter 2 of [DN11].

(b) Note that the given definition for the variational Henstock-Kurzweil-Stieltjes integral is equivalent to the definition where only sequences $\{([a_j,b_j],x_j)\}_{j=1}^r \in \mathcal{S}([a,b],\delta)$ with $\bigcup_{j=1}^r [a_j,b_j] = [a,b]$ are taken into consideration; indeed, one implication is trivial and the converse follows by means of Cousin's lemma.

(c) Observe that the crucial point in the definition of the variational McShane-integral is the fact that is not assumed that $x_j \in [a_j,b_j]$.

(d) Every function that is Riemann-Stieltjes integrable with respect to φ is clearly also Henstock-Kurzweil-Stieltjes integrable with respect to φ with the same integral. In general the converse is false, as the classical Henstock-Kurzweil integral (i.e., $X = Y = Z = \mathbb{R}$ and $\varphi = \mathrm{id}_{[a,b]}$) integrates every ordinary derivative (see, e.g., Theorem 4.24 below combined with Lemma 4.3), whereas the classical Riemann integral does not.

(e) Every function that is variationally McShane-Stieltjes integrable with respect to φ is obviously also variationally Henstock-Kurzweil-Stieltjes integrable with respect to φ. The converse statement, however, is not valid, since, e.g., the classical variational McShane integral (i.e. $X = Y = Z = \mathbb{R}$ and $\varphi = \mathrm{id}_{[a,b]}$) is equivalent to the Lebesgue-integral, which does not integrate all ordinary derivatives, while the classical Henstock-Kurzweil integral does.

(f) Every function that is variationally Henstock-Kurzweil-Stieltjes integrable with respect to φ is clearly also Henstock-Kurzweil-Stieltjes integrable with respect to φ and we then have $F(b) - F(a) = (\mathcal{HK}) \int_a^b f(t) \cdot \mathrm{d}\varphi(t)$. In general the converse fails (we will expose this in a moment).

Let f be variationally Henstock-Kurzweil-Stieltjes integrable with respect to φ and let F be an indefinite variational Henstock-Kurzweil-Stieltjes integral of f with respect to φ. Clearly, F is not unique as $F + z$ does the same for any $z \in Z$. However, F is indeed uniquely determined if we additionally demand $F(a) = 0$. In particular, two indefinite variational Henstock-Kurzweil-Stieltjes integrals of f with respect to φ differ from each other only by an additive constant. In fact, consider F and $\widetilde{F}$ as above with $F(a) = 0 = \widetilde{F}(a)$. Fix $t \in [a,b]$ and take $\varepsilon > 0$. There is a gauge $\delta_G : [a,b] \to (0,\infty)$ with

$$\sum_{j=1}^r \|f(x_j) \cdot [\varphi(b_j) - \varphi(a_j)] - (G(b_j) - G(a_j))\|_Z < \frac{\varepsilon}{2}$$

for all $\{([a_j,b_j],x_j)\}_{j=1}^r$ in $\mathcal{S}([a,b],\delta_G)$ where $G \in \{F, \widetilde{F}\}$. We then put $\delta := \min\{\delta_F, \delta_{\widetilde{F}}\}$ and choose a $\delta|_{[a,t]}$-fine partition $\{([a_j,b_j],x_j)\}_{j=1}^r \in \mathcal{S}([a,t],\delta|_{[a,t]})$ of $[a,t]$ by means of Cousin's lemma. In particular, $a_j = b_{j-1}$ for $j \in \{1,\ldots,r\} \setminus \{1\}$, $b_r = t$ and $a_1 = a$. Due to $F(a) = 0 = \widetilde{F}(a)$, we thus obtain

$$\begin{aligned}
\|F(t) - \widetilde{F}(t)\|_Z &= \left\| \sum_{j=1}^r (F(b_j) - F(a_j)) - \sum_{j=1}^r (\widetilde{F}(b_j) - \widetilde{F}(a_j)) \right\|_Z \\
&\leq \sum_{j=1}^r \|f(x_j) \cdot [\varphi(b_j) - \varphi(a_j)] - (F(b_j) - F(a_j))\|_Z \\
&\quad + \sum_{j=1}^r \|f(x_j) \cdot [\varphi(b_j) - \varphi(a_j)] - (\widetilde{F}(b_j) - \widetilde{F}(a_j))\|_Z \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Hence, $\|F(t) - \widetilde{F}(t)\|_Z < \varepsilon$ for every $\varepsilon > 0$, which yields $F(t) = \widetilde{F}(t)$. We call this uniquely determined function F with $F(a) = 0$ the *primitive (function) of f (with respect to φ)* and we write

$$F(t) = \int_a^t f(\tau) \cdot \mathrm{d}\varphi(\tau)$$

for $t \in [a, b]$. We denote by $\mathcal{HK}([a, b], \varphi, X)$ the set of all X-valued functions that are variationally Henstock-Kurzweil-Stieltjes integrable with respect to φ. In the special case $X = Z$, $Y = \mathbb{K}$, where φ is the identity on $[a, b]$ and the bilinear mapping B is the usual multiplication with scalars, we simply write $\mathcal{HK}([a, b], X)$ instead of $\mathcal{HK}([a, b], \varphi, X)$.

One easily verifies that $\mathcal{HK}([a, b], \varphi, X)$ is a $\mathbb{K}$-vector space and that the mapping

$$\int_a^t \cdot \, \mathrm{d}\varphi : \mathcal{HK}([a, b], \varphi, X) \to Z;\ f \mapsto \int_a^t f(\tau) \cdot \mathrm{d}\varphi(\tau)$$

is linear for each $t \in [a, b]$.

If $f \in \mathcal{HK}([a, b], \varphi, X)$ and $a < c < b$, then the restriction $f|_{[a,c]}$ belongs to $\mathcal{HK}([a, c], \varphi|_{[a,c]})$ and $f|_{[c,b]}$ to $\mathcal{HK}([c, b], \varphi|_{[c,b]})$ with

$$\int_a^c f|_{[a,c]}(\tau) \cdot \mathrm{d}\varphi|_{[a,c]}(\tau) + \int_c^b f|_{[c,b]}(\tau) \cdot \mathrm{d}\varphi|_{[c,b]}(\tau) = \int_a^b f(\tau) \cdot \mathrm{d}\varphi(\tau).$$

These facts easily follow from the definition. Therefore we simply write, e.g., $\int_a^c f(\tau) \cdot \mathrm{d}\varphi(\tau)$ instead of $\int_a^c f|_{[a,c]}(\tau) \cdot \mathrm{d}\varphi|_{[a,c]}(\tau)$. Furthermore, f is contained in $\mathcal{HK}([c, d], \varphi, X)$ for $a \le c < d \le b$ and we finally put $\int_c^c f(t) \cdot \mathrm{d}\varphi(t) := 0$ for every $c \in [a, b]$ and $f \in \mathcal{HK}([a, b], \varphi, X)$.

We said above that in general the Henstock-Kurzweil-Stieltjes integral strictly contains the variational Henstock-Kurzweil-Stieltjes integral. Indeed, A. P. Solodov (see [Sol99]) was the first to show that in the case where $X = Z$, $Y = \mathbb{R}$, φ is the identity map on $[a, b]$ and B is the usual multiplication with scalars these both notions of integral coincide if and only if X is a finite dimensional Banach space. The if-part of this statement is a consequence of the well-known Henstock-Saks lemma (see, e.g., Lemma 2.59 in [DN11]). The following lemma gives us a version of this Henstock-Saks lemma in our framework.

4.3 Lemma. *Assume that Z is finite-dimensional. Then the following assertions are equivalent.*

(a) The function f is Henstock-Kurzweil-Stieltjes integrable with respect to φ.

(b) We have $f \in \mathcal{HK}([a, b], \varphi, X)$.

If (a) or (b) is satisfied, we have $\int_a^t f(s) \cdot \mathrm{d}\varphi(s) = (\mathcal{HK}) \int_a^t f(s) \cdot \mathrm{d}\varphi(s)$ for all $t \in [a, b]$.

Proof. We have already noticed that (b) implies (a) and that in this case the addendum holds. We now conversely assume that (a) is satisfied. Considering Z as a Banach space over $\mathbb{R}$, one easily verifies that it suffices to consider the case $Z = \mathbb{R}^d$ (endowed with the usual Euclidean norm) for a $d \in \mathbb{N}$ and then one reduces equally easily the assertion to the case $d = 1$.
By Theorem 2.73 in [DN11], f is Henstock-Kurzweil-Stieltjes integrable with respect to φ on each closed subinterval of $[a, b]$ and the Henstock-Kurzweil-Stieltjes integral is additive with respect to the integration domain. We can therefore set $F(t) := (\mathcal{HK}) \int_a^t f(\tau) \cdot \mathrm{d}\varphi(\tau)$ for $t \in [a, b]$. Thanks to Lemma 2.59 in [DN11], we infer that for each $\varepsilon > 0$ there exists a gauge $\delta : [a, b] \to (0, \infty)$ such that

$$\left| \sum_{j=1}^r (F(b_j) - F(a_j) - f(x_j) \cdot [\varphi(b_j) - \varphi(a_j)]) \right| < \varepsilon,$$

whenever $\{([a_j,b_j],x_j)\}_{j=1}^r \in \mathcal{S}([a,b],\delta)$. To finish the proof, one may thus proceed as in the proof of the classical Henstock-Saks lemma (cf., e.g., the proof of Lemma 2.3.1 in [Pfe93]). Fix $\varepsilon > 0$ and choose a gauge δ with

$$\left|\sum_{j=1}^r (F(b_j) - F(a_j) - f(x_j) \cdot [\varphi(b_j) - \varphi(a_j)])\right| < \frac{\varepsilon}{2}$$

for each $\{([a_j,b_j],x_j)\}_{j=1}^r \in \mathcal{S}([a,b],\delta)$. Consider an arbitrary δ-fine partition $\{([a_j,b_j],x_j)\}_{j=1}^r \in \mathcal{S}([a,b],\delta)$ on $[a,b]$ and put $I_+ := \{j \in \{1,\ldots,r\} : F(b_j) - F(a_j) - f(x_j) \cdot [\varphi(b_j) - \varphi(a_j)] \geq 0\}$. We then compute

$$\begin{aligned}
&\sum_{j=1}^r |F(b_j) - F(a_j) - f(x_j) \cdot [\varphi(b_j) - \varphi(a_j)]| \\
&= \sum_{\substack{j=1,\ldots,r \\ j \in I_+}} (F(b_j) - F(a_j) - f(x_j) \cdot [\varphi(b_j) - \varphi(a_j)]) \\
&\quad - \sum_{\substack{j=1,\ldots,r \\ j \notin I_+}} (F(b_j) - F(a_j) - f(x_j) \cdot [\varphi(b_j) - \varphi(a_j)]) \\
&= \left| \sum_{\substack{j=1,\ldots,r \\ j \in I_+}} (F(b_j) - F(a_j) - f(x_j) \cdot [\varphi(b_j) - \varphi(a_j)]) \right| \\
&\quad + \left| \sum_{\substack{j=1,\ldots,r \\ j \notin I_+}} (F(b_j) - F(a_j) - f(x_j) \cdot [\varphi(b_j) - \varphi(a_j)]) \right| < 2 \cdot \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Consequently, $f \in \mathcal{HK}([a,b],\varphi,X)$ with $\int_a^t f(s) \cdot \mathrm{d}\varphi(s) = F(t)$ for all $t \in [a,b]$. □

We now start to collect the deeper properties of the variational Henstock-Kurzweil-Stieltjes integral. In doing so it turns out that the following two notions take up dominant roles; they are inspired by concept formations due to Ward in [War36] and Faure in [Fau97].

4.4 Definition. *(a) A set $A \subseteq [a,b]$ is called a φ-*null set *if there is an m_φ-null set $N \subseteq [a,b]$ (i.e., $m_\varphi(N) = 0$) and a countable set $D \subseteq [a,b]$ with $A = N \cup D$.*

(b) We call F absolutely continuous with respect to φ *if each m_φ-null set is also an m_F-null set. In this case we write $m_F \ll m_\varphi$.*

4.5 Remark We obviously have $m_F \ll m_\varphi$ if and only if every φ-null set is also a F-null set.

4.6 Definition. *Let $F : [a,b] \to Z$ and $\varphi : [a,b] \to Y$ be functions and $t \in [a,b]$. We say that F is* φ-continuous at t *if there is a $x \in X$ such that*

$$\lim_{\substack{h \to 0 \\ t+h \in [a,b]}} \big(F(t+h) - F(t) - x \cdot (\varphi(t+h) - \varphi(t))\big) = 0.$$

Such an $x \in X$ is called a φ-continuity value of g at the point t.

Observe that the point x in the definition of φ-continuity is not unique in general: if, e.g., F and φ are both continuous at t, then each $x \in X$ is a φ-continuity value of F at t. So we denote by $\mathcal{C}(F,\varphi,t)$ the set of all φ-continuity values of F at t (in particular, $\mathcal{C}(F,\varphi,t) = \emptyset$ precisely means that F is not φ-continuous at t); notice that $\mathcal{C}(F,\varphi,t)$ is always convex.

4.7 Lemma (cf. [Fau97, Proposition 4.4]). *Let $f \in \mathcal{HK}([a,b],\varphi,X)$ with primitive F. Then $m_F \ll m_\varphi$.*

Proof. Let $\emptyset \neq E \subseteq [a,b]$ with $m_\varphi(E) = 0$ and set $E_n := \{x \in E : \|f(x)\| \leq n\}$. It suffices to verify $m_F(E_n) = 0$ for each $n \in \mathbb{N}$ since m_F is an outer measure on $[a,b]$. For this purpose, fix $n \in \mathbb{N}$, assume $E_n \neq \emptyset$ and let $\varepsilon > 0$ be arbitrary. Then there is a gauge $\delta_1 : [a,b] \to (0,\infty)$ such that

$$\sum_{j=1}^{r} \|f(x_j) \cdot [\varphi(b_j) - \varphi(a_j)] - (F(b_j) - F(a_j))\| < \frac{\varepsilon}{2}$$

for every $\{([a_j,b_j],x_j)\}_{j=1}^r \in \mathcal{S}([a,b],\delta_1)$ and a gauge $\delta_2 : E_n \to (0,\infty)$ with

$$\sum_{j=1}^{r'} \|\varphi(b'_j) - \varphi(a'_j)\| < \frac{\varepsilon}{2n}$$

for all $\{([a'_j,b'_j],x'_j)\}_{j=1}^{r'} \in \mathcal{S}(E_n,\delta_2)$. Take a collection $\{([a_j,b_j],x_j)\}_{j=1}^r$ in $\mathcal{S}(E_n, \min\{\delta_1|_{E_n},\delta_2\})$. We can estimate

$$\begin{aligned}\sum_{j=1}^{r} \|F(b_j) - F(a_j)\| \leq& \sum_{j=1}^{r} \|f(x_j) \cdot [\varphi(b_j) - \varphi(a_j)] - (F(b_j) - F(a_j))\| \\ &+ \sum_{j=1}^{r} \|f(x_j)\| \cdot \|\varphi(b_j) - \varphi(a_j)\| \\ <& \frac{\varepsilon}{2} + n \sum_{j=1}^{r} \|\varphi(b_j) - \varphi(a_j)\| < \varepsilon\end{aligned}$$

and we conclude $m_F(E_n) = 0$. □

4.8 Corollary. *Let $f \in \mathcal{HK}([a,b],\varphi,X)$ with primitive F and let $E \subseteq [a,b]$ be an F_σ-set such that m_φ is σ-finite on E. If φ is continuous, then m_F is also σ-finite on E and we have $F \in \mathrm{BVG}_*(E,Z)$.*

Proof. This is a direct consequence of Lemma 4.7 and Corollary 3.18. □

4.9 Lemma. *Let $f \in \mathcal{HK}([a,b],\varphi,X)$ with primitive F. Then $f(x) \in \mathcal{C}(F,\varphi,x)$ for all $x \in [a,b]$.*

Proof. Fix $x \in [a,b]$ and let $\varepsilon > 0$. Choose a gauge $\delta \in (0,\infty)^{[a,b]}$ such that

$$\sum_{j=1}^{r} \|f(x_j) \cdot [\varphi(b_j) - \varphi(a_j)] - (F(b_j) - F(a_j))\| < \varepsilon$$

for every $\{([a_j,b_j],x_j)\}_{j=1}^r \in \mathcal{S}([a,b],\delta)$. For each $y \in [a,b] \cap U_{\delta(x)}(x)$ we have $(\langle y,x\rangle, x) \in \mathcal{S}([a,b],\delta)$ and consequently

$$\|F(y) - F(x) - f(x) \cdot [\varphi(y) - \varphi(x)]\| < \varepsilon.$$

As a result, $f(x)$ is contained in $\mathcal{C}(F,\varphi,x)$. □

4.10 Corollary. *Let f belong to $\mathcal{HK}([a,b],\varphi,X)$ with primitive F and assume that φ is continuous at some point $x_0 \in [a,b]$. Then F is also continuous at x_0.*

Proof. Lemma 4.9 and the continuity of φ yield

$$\begin{aligned}&\|F(x) - F(x_0)\| \\ \leq&\|F(x) - F(x_0) - f(x_0) \cdot [\varphi(x) - \varphi(x_0)]\| + \|f(x_0) \cdot [\varphi(x) - \varphi(x_0)]\| \\ \leq&\|F(x) - F(x_0) - f(x_0) \cdot [\varphi(x) - \varphi(x_0)]\| + \|f(x_0)\| \cdot \|\varphi(x) - \varphi(x_0)\| \longrightarrow 0\end{aligned}$$

as $x \to x_0$. □

4.11 Lemma. *Let $N := \{x \in [a,b] :\ f(x) \neq 0\}$ and assume that $m_\varphi(N) = 0$. Then $f \in \mathcal{HK}([a,b],\varphi,X)$ with $\int_a^x f(t)\cdot \mathrm{d}\varphi(t) = 0$ for all $x \in [a,b]$.*

Proof. We set $N_j := \{x \in [a,b] : j-1 < \|f(x)\| \leq j\}$ for $j \in \mathbb{N}$. Because of $N_j \subseteq N$ we have $m_\varphi(N_j) = 0$ for all $j \in \mathbb{N}$. Let $\varepsilon > 0$. Put $A := \{j \in \mathbb{N} :\ N_j \neq \emptyset\}$. For each $j \in A$ there exists a gauge $\delta_j \in (0,\infty)^{N_j}$ with

$$\sum_{k=1}^{r} \|\varphi(b_k) - \varphi(a_k)\| < \frac{\varepsilon}{j2^j}$$

for all $\{([a_k,b_k],x_k)\}_{k=1}^r \in \mathcal{S}(N_j,\delta_j)$. We now define

$$\delta(x) := \begin{cases} 1, & \text{if } f(x) = 0,\\ \delta_j(x), & \text{if } x \in N_j \text{ for a (unique) } j \in A. \end{cases}$$

For every $\{([a_k,b_k],x_k)\}_{k=1}^r$ in $\mathcal{S}([a,b],\delta)$ we estimate

$$\begin{aligned}\sum_{k=1}^{r} \|f(x_k)\cdot[\varphi(b_k) - \varphi(a_k)]\| &\leq \sum_{j=1}^{\infty} \sum_{\substack{k=1\\ x_k \in N_j}}^{r} \|f(x_k)\| \cdot \|[\varphi(b_k) - \varphi(a_k)]\|\\ &\leq \sum_{j=1}^{\infty} j \sum_{\substack{k=1\\ x_k \in N_j}}^{r} \|[\varphi(b_k) - \varphi(a_k)]\| < \sum_{j=1}^{\infty} j \cdot \frac{\varepsilon}{j2^j} = \varepsilon,\end{aligned}$$

using in the last (strict) inequality that $\{([a_k,b_k],x_k)\}_{\substack{k=1,\dots,r\\ x_k \in N_j}}$ belongs to $\mathcal{S}(N_j,\delta_j)$ for $j \in A$. As a consequence, we get f is an element of $\mathcal{HK}([a,b],\varphi,X)$ with $\int_a^x f(t)\cdot\mathrm{d}\varphi(t) = 0$ for all $x \in [a,b]$. □

4.2 Notions of differentiation

The next two sections contain our main results concerning the variational Henstock-Kurzweil-Stieltjes integral. These results deal on the one side with the question in which sense the variational Henstock-Kurzweil-Stieltjes integral is differentiable and whether certain derivatives are variationally Henstock-Kurzweil-Stieltjes integrable. In this short section we want to specify these "certain derivatives".

The following definition is inspired by [War36].

4.12 Definition. *Let $t \in [a,b]$. We say that F is* φ-Roussel-differentiable *at t provided the following two conditions are satisfied*

- *$\mathcal{C}(F,\varphi,t) \neq \emptyset$ and*
- *$\exists\, x \in \mathcal{C}(F,\varphi,t)\ \forall\, \varepsilon > 0\ \exists\, \delta > 0\ \forall\, 0 < |h| < \delta$ with $t+h \in [a,b]$:*

$$\|F(t+h) - F(t) - x\cdot(\varphi(t+h) - \varphi(t))\|_Z \leq \varepsilon\,\omega(\varphi;t,h),$$

where

$$\omega(\varphi;t,h) := \begin{cases} \omega(\varphi,[t,t+h]), & \text{if } h > 0,\\ \omega(\varphi,[t+h,t]), & \text{if } h < 0. \end{cases}$$

If $x \in \mathcal{C}(F,\varphi,t)$ satisfies the second condition, we say that x is a φ-Roussel-differentiability value *of F at t. The set of all φ-Roussel-differentiability values of F at t is denoted by $\mathcal{D}(F,\varphi,t)$; using that $\mathcal{C}(F,\varphi,t)$ is* *convex, one easily deduces that $\mathcal{D}(F,\varphi,t)$ is convex, too. Note that the second condition is trivially fulfilled if $\omega(\varphi;t,h)=\infty$ holds. Any point $x\in X$ (not necessarily belonging to $\mathcal{C}(F,\varphi,t)$) satisfying the second condition is called a* pseudo-φ-Roussel-differentiability value of F at t*; if such an x exists, we then say that F is* pseudo-φ-Roussel-differentiable *at t.*

4.13 Remark (a) Observe that if x is a pseudo-φ-Roussel-differentiability value of F at t and if F is bounded in a neighbourhood of t, then x is even a φ-Roussel-differentiability value of F at t since in this case one has $x\in\mathcal{C}(F,\varphi,t)$ automatically.

(b) Furthermore, notice that in the special case $X=Z$, $Y=\mathbb{K}$, where φ is the identity on $[a,b]$ and the bilinear mapping B is the usual multiplication with scalars, the notion of φ-Roussel-differentiability coincides with usual notion of differentiability.

If there is an m_φ-null set $N\subseteq[a,b]$ such that F is pseudo-φ-Roussel-differentiable on $[a,b]\setminus N$, then we say that F is m_φ-almost everywhere (m_φ-a.e.) pseudo-φ-Roussel-differentiable and any function $f:[a,b]\to X$ with the property that $f(t)$ is a pseudo-φ-Roussel-differentiability value of F at t for all $t\in[a,b]\setminus N$ is called an *m_φ-Roussel derivative of F*.

Let $F:[a,b]\to Z$ be a φ-continuous function and $A\subseteq[a,b]$ a φ-null set such that F is φ-Roussel-differentiable on $[a,b]\setminus A$. Then any function $f:[a,b]\to X$ satisfying

- $\forall\, t\in[a,b]:\ f(t)\in\mathcal{C}(F,\varphi,t)$ and
- $\forall\, t\in[a,b]\setminus A:\ f(t)\in\mathcal{D}(F,\varphi,t)$

is called *a φ-Roussel derivative of F*. Note that, if we say that F possesses a φ-Roussel derivative, then this means in particular that F is assumed to be φ-continuous.

The following two examples show that in general none of the notions "φ-Roussel derivative" and "m_φ-Roussel derivative" includes the respective other one.

4.14 Example We consider

$$f:[0,1]\to\mathbb{R};\ t\mapsto \mathbb{1}_{[0,1)}(t),$$

$$F:[0,1]\to\mathbb{R};\ t\mapsto \mathbb{1}_{[0,1)}(t)\cdot t$$

and

$$\varphi:[0,1]\to\mathbb{R};\ t\mapsto t.$$

Clearly, f is an m_φ-Roussel derivative of F, but it is not a φ-Roussel derivative because $f(1)\notin\mathcal{C}(F,\varphi,1)$. In fact, $F(1)-F(s)-f(1)(\varphi(1)-\varphi(s))=-s\to-1$ as $s\to 1^+$.

4.15 Example We consider

$$f:[0,1]\to\mathbb{R};\ t\mapsto \mathbb{1}_{[-1,1]\setminus\{0\}}(t),$$

$$F:[0,1]\to\mathbb{R};\ t\mapsto t$$

and

$$\varphi:[0,1]\to\mathbb{R};\ t\mapsto\begin{cases}t, & \text{if } -1\le t\le 0,\\ t+1, & \text{if } 0<t\le 1.\end{cases}$$

We first verify that $f(t)\in\mathcal{D}(F,\varphi,t)$ for each $t\in[-1,1]\setminus\{0\}$. Let $t\in[-1,0)$ and $h\in\mathbb{R}\setminus\{0\}$ with $t+h\in[-1,0)$. Then we have $|F(t)-F(t+h)-f(t)(\varphi(t)-\varphi(t+h))|=|t-(t+h)-(t-(t+h))|=0$. For $t\in(0,1]$ and $h\in\mathbb{R}\setminus\{0\}$ with $t+h\in(0,1]$ one obtains $|F(t)-F(t+h)-f(t)(\varphi(t)-\varphi(t+h))|=|t-(t+h)-(t+1-(t+h+1))|=0$. Moreover, we calculate

$$|F(0)-F(\pm h)-f(0)(\varphi(0)-\varphi(\pm h))|=|0-(\pm h)-0|=h \tag{4.1}$$

for $h \in (0,1)$ and we conclude $f(t) \in \mathcal{C}(F,\varphi,t)$ for all $t \in [-1,1]$. So f is a φ-Roussel derivative of F. Furthermore, for $h \in (0,1)$ we have $\omega(\varphi,[-h,0]) = h$. Consequently, if we have

$$|F(0) - F(-h) - f(0)(\varphi(0) - \varphi(-h))| \leq \varepsilon\omega(\varphi; 0, -h)$$

for an $\varepsilon > 0$, then equation (4.1) yields $\varepsilon \geq 1$ and therefore $f(0) \notin \mathcal{D}(F,\varphi,0)$. As a consequence, f is not an m_φ-Roussel derivative as for the exceptional set $\{0\}$ we have $m_\varphi(\{0\}) > 0$ since φ is discontinuous at 0.

In contrast to the above two examples there are important cases where the notions "φ-Roussel derivative" and "m_φ-Roussel derivative" coincide as the subsequent result shows.

4.16 Lemma. *Let $\varphi : [a,b] \to Y$ be continuous with $m_F \ll m_\varphi$. Then the following assertions are equivalent.*

(a) The function f is a φ-Roussel derivative of F.

(b) The function f is an m_φ-Roussel derivative of F.

Proof. First suppose that (a) is satisfied. We have $m_\varphi(\{x\}) = 0$ for all $x \in [a,b]$ because φ is continuous and we therefore conclude that f is an m_φ-Roussel derivative of F. Now we conversely assume that (b) holds. Since φ is continuous and $m_F \ll m_\varphi$, we have $m_F(\{x\}) = 0$ for all $x \in [a,b]$, which implies that F is continuous. Thus F and φ are both continuous, which yields $\mathcal{C}(F,\varphi,t) = X$ for all $t \in [a,b]$. As a result, we infer that f is a φ-Roussel derivative of F. □

We shall need two further notions of differentiation.

4.17 Definition. *We say that F is φ-*Fréchet differentiable at *t if there exists an $x \in X$ such that for all $\varepsilon > 0$ there exists a $\rho > 0$ such that we have*

$$\|F(s) - F(t) - x \cdot [\varphi(s) - \varphi(t)]\| \leq \varepsilon\|\varphi(s) - \varphi(t)\|$$

*for all $s \in [a,b]$ with $|s-t| < \rho$. We then call x a φ-*Fréchet-differentiability value of *F* at *t. The set of all φ-Fréchet-differentiability values of F at t is denoted by $\mathcal{FD}(F,\varphi,t)$.*

Let $F : [a,b] \to Z$ be a φ-continuous function and $A \subseteq [a,b]$ a φ-null set such that F is φ-Fréchet differentiable off A. A function $f : [a,b] \to X$ satisfying

- $\forall\, t \in [a,b] : \; f(t) \in \mathcal{C}(F,\varphi,t)$ and
- $\forall\, t \in [a,b] \setminus A : \; f(t) \in \mathcal{FD}(F,\varphi,t)$

is then called *a φ-Fréchet derivative of F*.

4.18 Remark It is obvious that each φ-Fréchet-differentiability value is also a pseudo-φ-Roussel-differentiability value and each φ-Fréchet derivative is also a φ-Roussel derivative.

The last notion of differentiation is adapted to the situation where BVG_*-functions serve as "differentiators".

4.19 Definition. *Let $\varphi \in \mathrm{BVG}_*([a,b],Y)$ with admissible decomposition $(E_n)_{n=0}^\infty$. We say that F is* $(\varphi,(E_n)_n)$-differentiable at *t if there exists an $x \in X$ such that for all $\varepsilon > 0$ and all $n \in \{m \in \mathbb{N} : \; t \in E_m\}$ there exists a $\rho > 0$ such that we have*

$$\|F(s) - F(t) - x \cdot [\varphi(s) - \varphi(t)]\| \leq \varepsilon\, m_\varphi(\langle s,t\rangle \cap E_n)$$

for all $s \in [a,b]$ with $|s-t| < \rho$. The set of all those x is denoted by $\mathcal{D}(F,\varphi,t,(E_n)_n)$.

It seems to be quite a delicate matter to fathom how this last notion of differentiability depends on the choosen admissible decomposition $(E_n)_{n=0}^{\infty}$ and how it is related to the derivatives introduced above. Using the results from the next two sections we will obtain a clearer picture concerning these questions. Nevertheless we can give already at this point a partial result.

4.20 Lemma. *Let $\varphi \in \mathrm{BV}([a,b],Y)$ be continuous from the right and choose the admissible decomposition $(E_n)_{n=0}^{\infty}$ given by $E_0 = \emptyset$ and $E_n = [a,b]$ for $n \in \mathbb{N}$. If F is φ-Roussel-differentiable at $t \in [a,b]$, then F is also $(\varphi,(E_n)_n)$-differentiable at t.*

Proof. Let $x \in \mathcal{D}(F,\varphi,t)$, $\varepsilon > 0$ and pick $\rho > 0$ with

$$\|F(t+h) - F(t) - x \cdot (\varphi(t+h) - \varphi(t))\|_Z \le \varepsilon\, \omega(\varphi; t, h)$$

for all $0 < |h| < \rho$ with $t + h \in [a,b]$. Using Corollary A.4 we obtain

$$\|\varphi(u) - \varphi(v)\| \le m_\varphi([u,v]) \le m_\varphi(\langle t, t+h\rangle)$$

for all $u, v \in \langle t, t+h\rangle$, hence $\omega(\varphi; t, h) \le m_\varphi(\langle t, t+h\rangle \cap E_n)$, $n \in \mathbb{N}$. □

4.3 Differentiation properties

We now come to our main results concerning the variational Henstock-Kurzweil-Stieltjes integral. In this section we establish two theorems concerning its differentiability properties.
The first result concerns differentiation in the sense of Definition 4.19.

4.21 Theorem. *Let $\varphi \in \mathrm{BVG}_*([a,b],Y)$ with an admissible decomposition $(E_n)_{n=0}^{\infty}$. Furthermore, let $f \in \mathcal{HK}([a,b],\varphi,X)$ with primitive F. Then there exists an m_φ-null set $N \subseteq [a,b]$ such that $f(t) \in \mathcal{D}(F,t,\varphi,(E_n)_n)$ for all $t \in [a,b] \setminus N$.*

Proof. For $k, \nu \in \mathbb{N}$ we set

$$N_{k,\nu} := \left\{ t \in E_\nu : \begin{array}{c} \forall\, \rho > 0\ \exists\, s_\rho \in U_\rho(t) \cap [a,b] : \\ \|F(s_\rho) - F(t) - f(t) \cdot [\varphi(s_\rho) - \varphi(t)]\| > \frac{1}{k} \cdot m_\varphi(\langle s_\rho, t\rangle \cap E_\nu) \end{array} \right\}.$$

Let $t \in [a,b] \setminus \bigcup_{k,\nu \in \mathbb{N}} N_{k,\nu}$, $\varepsilon > 0$ and $m \in \mathbb{N}$ with $t \in E_m$. Take $l \in \mathbb{N}$ with $\frac{1}{l} \le \varepsilon$. Because of $t \notin N_{l,m}$, there exists a $\rho > 0$ such that

$$\|F(s) - F(t) - f(t) \cdot [\varphi(s) - \varphi(t)]\| \le \frac{1}{l} \cdot m_\varphi(\langle s_\rho, t\rangle \cap E_\nu) \le \varepsilon\, m_\varphi(\langle s_\rho, t\rangle \cap E_\nu)$$

for all $s \in U_\rho(t) \cap [a,b]$ and, as a consequence, $f(t) \in \mathcal{D}(F,t,\varphi,(E_n)_n)$. Therefore it suffices to show $m_\varphi(N_{k,\nu}) = 0$ for all $k, \nu \in \mathbb{N}$. For this purpose fix $k, \nu \in \mathbb{N}$ and choose $\delta_1 \in (0,\infty)^{[a,b]}$ with

$$\sum_{j=1}^{r} \|F(b_j) - F(a_j) - f(t_j) \cdot [\varphi(b_j) - \varphi(a_j)]\| < \frac{\varepsilon}{2k} \tag{4.2}$$

for all $\{([a_j,b_j],t_j)\}_{j=1}^r \in \mathcal{S}([a,b],\delta_1)$, which is possible by hypothesis. For every $t \in N_{k,\nu}$ we pick a sequence $(s_{n,t})_n$ in $[a,b]$ with $|t - s_{n,t}| < \min\{1/n, \delta_1(t)\}$ and with

$$\|F(s_{n,t}) - F(t) - f(t) \cdot [\varphi(s_{n,t}) - \varphi(t)]\| > \frac{1}{k} \cdot m_\varphi(\langle s_{n,t}, t\rangle \cap E_\nu) \tag{4.3}$$

for all $n \in \mathbb{N}$, which is possible by the definition of $N_{k,\nu}$. Then $\mathcal{J} := \{\langle s_{n,t}, t\rangle : \ t \in N_{k,\nu},\ n \in \mathbb{N}\}$ covers $N_{k,\nu}$ in the (classical) Vitali-sense and the elements of $\mathcal{J}$ are closed intervals having non-trivial interior as the inequality in (4.3) is strict. We now consider the finite set function

$$\mu : \mathfrak{P}(\mathbb{R}) \to [0,\infty);\ A \mapsto m_\varphi(B \cap E_\nu).$$

One easily verifies that μ is a metric outer measure on $\mathbb{R}$ using that m_φ is a metric outer measure on $[a,b]$. Thanks to Proposition B.1, we can extract pairwise disjoint intervals $\{\langle s_{n_j}, t_j\rangle\}_{j=1}^r$, where $s_{n_j} := s_{n_j,t_j}$ and $n_j \in \mathbb{N}$ such that $\mu(N_{k,\nu} \setminus \bigcup_{j=1}^r \langle s_{n_j}, t_j\rangle) < \frac{\varepsilon}{2}$. Employing (4.3) we now estimate

$$\begin{aligned}
m_\varphi(N_{k,\nu}) = \mu(N_{k,\nu}) &\le \mu\Big(N_{k,\nu} \setminus \bigcup_{j=1}^r \langle s_{n_j}, t_j\rangle\Big) + \mu\Big(N_{k,\nu} \cap \bigcup_{j=1}^r \langle s_{n_j}, t_j\rangle\Big) \\
&< \frac{\varepsilon}{2} + \mu\Big(E_\nu \cap \bigcup_{j=1}^r \langle s_{n_j}, t_j\rangle\Big) = \frac{\varepsilon}{2} + m_\varphi\Big(\bigcup_{j=1}^r (E_\nu \cap \langle s_{n_j}, t_j\rangle)\Big) \\
&\le \frac{\varepsilon}{2} + \sum_{j=1}^r m_\varphi(E_\nu \cap \langle s_{n_j}, t_j\rangle) \\
&\le \frac{\varepsilon}{2} + \sum_{j=1}^r k\|F(s_{n_j}) - F(t_j) - f(t_j)\cdot[\varphi(s_{n_j}) - \varphi(t_j)]\| < \frac{\varepsilon}{2} + k \cdot \frac{\varepsilon}{2k} = \varepsilon,
\end{aligned}$$

where the last inequality results from applying (4.2), which is allowed because $\{(\langle s_{n_j}, t_j\rangle, t_j)\}_{j=1}^r \in \mathcal{S}([a,b],\delta_1)$ due to the pairwise disjointness of the intervals $\{\langle s_{n_j}, t_j\rangle\}_{j=1}^r$. We thus arrive at the conclusion $m_\varphi(N_{k,\nu}) = 0$. □

The next theorem treats the Fréchet-differentiability of the variational Henstock-Kurzweil-Stieltjes integral and gives an improvement and indeed far-reaching extension of Theorem 7 in [War36] and of the implication 1) $\Longrightarrow$ 2) of Faure's Theorem 4.7 in [Fau97] (note that Faure's theorem is based on his Proposition 3.10 which cannot have an vector-valued extension beyond the scope of spaces with the Radon-Nikodým property). At this point our exploration on the relation between fine and full variational measure becomes fruitful.

4.22 Theorem. *Let $\varphi \in \mathrm{BVG}_*([a,b],Y)$ and $f \in \mathcal{HK}([a,b],\varphi,X)$ with primitive F. Then there exists an m_φ-null set $N \subseteq [a,b]$ and a countable set D such that $f(t) \in \mathcal{FD}(F,t,\varphi)$ for all $t \in [a,b] \setminus (N \cup D)$.*

Proof. For $k \in \mathbb{N}$ we put

$$N_k := \left\{ t \in [a,b] : \begin{array}{c} \forall\, \rho > 0\ \exists\, s_\rho \in U_\rho(t) \cap [a,b] : \\ \|F(s_\rho) - F(t) - f(t)\cdot[\varphi(s_\rho) - \varphi(t)]\| > \frac{1}{k}\cdot\|\varphi(s_\rho) - \varphi(t)\| \end{array} \right\}.$$

Let D denote the countable point set of all discontinuities of φ. It suffices to verify $m_\varphi(N_k \setminus D) = 0$ for all $k \in \mathbb{N}$. For this purpose fix $k \in \mathbb{N}$ and choose $\delta \in (0,\infty)^{[a,b]}$ with

$$\sum_{j=1}^r \|F(b_j) - F(a_j) - f(t_j)\cdot[\varphi(b_j) - \varphi(a_j)]\| < \frac{\varepsilon}{k} \tag{4.4}$$

for all $\{([a_j,b_j],t_j)\}_{j=1}^r \in \mathcal{S}([a,b],\delta)$, which is possible by assumption. For each $t \in N_k$ we take a sequence $(s_{n,t})_n$ in $[a,b]$ with $|t - s_{n,t}| < \min\{1/n, \delta(t)\}$ and with

$$\|F(s_{n,t}) - F(t) - f(t)\cdot[\varphi(s_{n,t}) - \varphi(t)]\| > \frac{1}{k}\cdot\|\varphi(s_\rho) - \varphi(t)]\|. \tag{4.5}$$

We set $\beta := \{(\langle s_{n,t}, t\rangle, t) : t \in N_k,\ n \in \mathbb{N}\}$. As the inequality (4.4) is strict, we always have $s_{n,t} \neq t$ and we conclude that $\beta \in \mathcal{C}^\star(N_k)$. Now let $\pi = \{(\langle s_{n_j}, t_j\rangle, t_j)\}_{j=1}^r \in \mathcal{P}(\beta)$, where $s_{n_j} := s_{n_j,t_j}$ and $n_j \in \mathbb{N}$. From (4.4) and(4.5) we deduce

$$W_\varphi(\pi) = \sum_{j=1}^r \|\varphi(s_{n_j}) - \varphi(t_j)\| \le k \sum_{j=1}^r \|F(s_{n_j}) - F(t) - f(t)\cdot[\varphi(s_{n_j}) - \varphi(t)]\| < k \cdot \frac{\varepsilon}{k} = \varepsilon,$$

using $\{(\langle s_{n_j}, t_j\rangle, t_j)\}_{j=1}^r \in \mathcal{S}(N_k, \delta) \subseteq \mathcal{S}([a,b], \delta)$. Hence, $m^\varphi(N_k) \le \sup_{\pi\in\mathcal{P}(\beta)} W_\varphi(\pi) \le \varepsilon$ and thus $m^\varphi(N_k) = 0$. Now Theorem 3.29 yields $m_\varphi(N_k \setminus D) = m^\varphi(N_k \setminus D) = 0$. □

4.4 Integration of derivatives

In this section we explore the integrability properties of some of the derivatives introduced above.

4.23 Theorem. *Let $\varphi \in \mathrm{BVG}_*([a,b], Y)$ with measurable admissible decomposition $(E_n)_{n=0}^\infty$ and let $F : [a,b] \to Z$ be a φ-continuous function such that there exists a set $N \subseteq [a,b]$ with $m_F(N) = 0$ and a countable set $A \subseteq [a,b]$ such that F is $(\varphi, (E_n)_n)$-differentiable on $[a,b] \setminus (N \cup A)$. Furthermore let $f : [a,b] \to X$ be any function with the following properties:*

- *$f(t) \in \mathcal{C}(F, \varphi, t)$ for all $t \in A \setminus N$;*
- *$f(t) \in \mathcal{D}(F, \varphi, t, (E_n)_n)$ for all $t \in [a,b] \setminus (N \cup A)$;*
- *$f(t) = 0$ for all $t \in N$.*

Then we have $f \in \mathcal{HK}([a,b], \varphi)$ with $\int_a^t f(s)\cdot \mathrm{d}\varphi(s) = F(t) - F(a)$ for all $t \in [a,b]$.

Proof. Let D denote the countable set of all discontinuities of φ and let $\varepsilon > 0$. If $B := (A \cup D) \setminus N$ is nonvoid, let $(s_n)_{n\in I}$ be an enumeration of B, where $I = \{1, \dots, \sharp B\}$, if B is finite, and $I = \mathbb{N}$, if B is infinite. We further set $\widetilde{E}_1 := E_1 \setminus (B \cup N)$ and $\widetilde{E}_n := E_n \setminus \left(\bigcup_{k<n} E_k \cup B \cup N\right)$ for $n > 1$. For each $n \in I$ take $\delta_n > 0$ with

$$\|F(s) - F(s_n) - f(s_n)\cdot[\varphi(s) - \varphi(s_n)]\| < \frac{\varepsilon}{2^{n+4}}$$

for all $s \in [a,b] \cap U_{\delta_n}(s_n)$, using that $f(s_n) \in \mathcal{C}(F, \varphi, s_n)$. Because of $m_F(N) = 0$, there is a gauge $\delta_0 \in (0,\infty)^N$ with $W_F(S) < \frac{\varepsilon}{4}$ for all $S \in \mathcal{S}(N, \delta_0)$. Finally, for $t \in \widetilde{E}_n$ we choose $\delta_{0,n}(t) > 0$ with

$$\|F(s) - F(t) - f(t)\cdot[\varphi(s) - \varphi(t)]\| \le \frac{\varepsilon}{2^{n+2}(m_\varphi(E_n) + 1)} \cdot m_\varphi(\langle s, t\rangle \cap E_n)$$

for all $s \in [a,b] \cap U_{\delta_{0,n}(t)}(t)$. We now define

$$\delta : [a,b] \to (0,\infty);\ t \mapsto \begin{cases} \delta_n, & \text{if } t = s_n \text{ for a (unique) } n \in I, \\ \delta_0(t), & \text{if } t \in N, \\ \delta_{0,n}(t), & \text{if } t \in \widetilde{E}_n \text{ for a (unique) } n \in \mathbb{N}\,. \end{cases}$$

Let $\{([a_j, b_j], t_j)\}_{j=1}^r$ be an arbitrary element of $\mathcal{S}([a,b], \delta)$. We have

$$\sum_{j=1}^r \|F(b_j) - F(a_j) - f(t_j)\cdot[\varphi(b_j) - \varphi(a_j)]\| \le \sum_{\substack{j=1 \\ t_j\in N}}^r \|F(b_j) - F(a_j)\| + \Sigma_1 + \Sigma_2 + \Sigma_3,$$

where

$$\Sigma_1 := \sum_{\substack{j=1 \\ t_j \in B}}^{r} \|F(b_j) - F(a_j) - f(t_j) \cdot [\varphi(b_j) - \varphi(a_j)]\|,$$

$$\Sigma_2 := \sum_{\substack{j=1 \\ t_j \notin B \cup N}}^{r} \|F(b_j) - F(t_j) - f(t_j) \cdot [\varphi(b_j) - \varphi(t_j)]\|,$$

and

$$\Sigma_3 := \sum_{\substack{j=1 \\ t_j \notin B \cup N}}^{r} \|F(t_j) - F(a_j) - f(t_j) \cdot [\varphi(t_j) - \varphi(a_j)]\|.$$

By the definition of δ we obtain

$$\sum_{\substack{j=1 \\ t_j \in N}}^{r} \|F(b_j) - F(a_j)\| < \frac{\varepsilon}{4}.$$

For $t_j \in B$ let $k_j \in I$ with $s_{k_j} = t_j$. Using that $\sharp\{j \in \{1, \ldots, r\} : t_j = s_k\} \leq 2$ for each $k \in I$ we calculate

$$\begin{aligned}
&\Sigma_1 \\
&\leq \sum_{\substack{j=1 \\ t_j \in B}}^{r} \|F(b_j) - F(s_{k_j}) - f(t_j) \cdot [\varphi(b_j) - \varphi(s_{k_j})]\| + \sum_{\substack{j=1 \\ t_j \in B}}^{r} \|F(s_{k_j}) - F(a_j) - f(t_j) \cdot [\varphi(s_{k_j}) - \varphi(a_j)]\| \\
&\leq 2 \sum_{\substack{j=1 \\ t_j \in B}}^{r} \frac{\varepsilon}{2^{k_j+4}} \leq \frac{4\varepsilon}{2^4} \sum_{k \in I} \frac{1}{2^k} \leq \frac{\varepsilon}{4} \sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{\varepsilon}{4}.
\end{aligned}$$

Next observe that for $t_j, t_k \in \widetilde{E}_n$ with $j \neq k$ we have $[t_j, b_j] \cap [t_k, b_k] \subseteq \{t_j, t_k\}$ and $m_\varphi(\{t_k\}) = 0$ due to $\widetilde{E}_n \subseteq [a, b] \setminus D$. This yields (recall that $(E_n)_{n=0}^{\infty}$ is a measurable admissible decomposition)

$$\sum_{\substack{j=1 \\ t_j \in \widetilde{E}_n}}^{r} m_\varphi([t_j, b_j] \cap E_n) = m_\varphi\Bigg(\bigcup_{\substack{j=1 \\ t_j \in \widetilde{E}_n}}^{r} [t_j, b_j] \cap E_n \Bigg) \leq m_\varphi(E_n).$$

Utilising this inequality we further estimate

$$\begin{aligned}
\Sigma_2 &= \sum_{n=1}^{\infty} \sum_{\substack{j=1 \\ t_j \in \widetilde{E}_n}}^{r} \|F(b_j) - F(t_j) - f(t_j) \cdot [\varphi(b_j) - \varphi(t_j)]\| \\
&\leq \sum_{n=1}^{\infty} \sum_{\substack{j=1 \\ t_j \in \widetilde{E}_n}}^{r} \frac{\varepsilon}{2^{n+2}(m_\varphi(E_n) + 1)} \cdot m_\varphi([t_j, b_j] \cap E_n) \\
&\leq \sum_{n=1}^{\infty} \frac{\varepsilon m_\varphi(E_n)}{2^{n+2}(m_\varphi(E_n) + 1)} < \frac{\varepsilon}{4}.
\end{aligned}$$

Analogously, we see $\Sigma_3 < \frac{\varepsilon}{4}$. Summarizing, we arrive at

$$\sum_{j=1}^{r} \|F(b_j) - F(a_j) - f(t_j) \cdot [\varphi(b_j) - \varphi(a_j)]\| < 4 \cdot \frac{\varepsilon}{4} = \varepsilon$$

for each $\{([a_j, b_j], t_j)\}_{j=1}^{r} \in \mathcal{S}([a, b], \delta)$. This completes the proof. □

Analogous to Theorem 4.23, the next result states a corresponding statement essentially concerning the integrability properties of the Roussel-derivative.

4.24 Theorem. *Let $\varphi \in \mathrm{BVG}_*([a,b],Y)$ and let $F : [a,b] \to Z$ be a φ-continuous function such that there exists a set $N \subseteq [a,b]$ with $m_F(N) = 0$ and a countable set $A \subseteq [a,b]$ such that F is φ-Roussel differentiable on $[a,b] \setminus (N \cup A)$. Furthermore let $f : [a,b] \to X$ be any function with the following properties:*

- *$f(x) \in \mathcal{C}(F,\varphi,x)$ for all $x \in A \setminus N$;*
- *$f(x) \in \mathcal{D}(F,\varphi,x)$ for all $x \in [a,b] \setminus (N \cup A)$;*
- *$f(x) = 0$ for all $x \in N$.*

Then we have $f \in \mathcal{HK}([a,b],\varphi)$ with $\int_a^x f(t) \cdot \mathrm{d}\varphi(t) = F(x) - F(a)$ for all $x \in [a,b]$.

We shall give two different proofs for this important result. The first one rests on an immediate application of Lemma 2.8, while the second proof is based on Lemma 2.4

1st proof of Theorem 4.24. Lemma 2.8 gives a strictly increasing function $\chi : [a,b] \to \mathbb{R}$ and a countable set $M \subseteq [a,b]$ such that

$$\overline{\lim_{y \to x}} \frac{\|\varphi(x) - \varphi(y)\|}{|\chi(x) - \chi(y)|} < \infty$$

for all $x \in [a,b] \setminus M$. In view of this fact, for each $n \in \mathbb{N}$ we define $\widetilde{E}_n$ as the set of all $x \in [a,b]$ with the property

$$\forall\, y \in [a,b] : \ |\chi(x) - \chi(y)| < \frac{1}{n} \implies \|\varphi(x) - \varphi(y)\| \le n|\chi(x) - \chi(y)|.$$

Let $\varepsilon > 0$. We set

$$\varepsilon_n := \frac{\varepsilon}{n 2^{n+3}(\chi(b) - \chi(a))}$$

for $n \in \mathbb{N}$. Let D denote the countable set of all discontinuities of χ and put $B := N \cup A \cup M \cup D \cup \{a,b\}$. For each $x \in \widetilde{E}_n \setminus B$ there exists a number $\delta_n(x) > 0$ such that the two implications

$$|x - y| < \delta_n(x) \implies |\chi(x) - \chi(y)| < \frac{1}{n}$$

and

$$\begin{aligned} & x - \delta_n(x) < y \le x \le z < x + \delta_n(x) \\ \implies & \|F(z) - F(y) - f(x) \cdot [\varphi(z) - \varphi(y)]\| \le \varepsilon_n \omega(\varphi, [y,z]) \end{aligned} \tag{4.6}$$

are satisfied for all $y, z \in [a,b]$. Indeed, the first condition can be fulfilled because of $x \notin D$. Due to $x \notin N \cup A$, F is φ-Roussel-differentiable at x with $f(x) \in \mathcal{D}(F,\varphi,x)$. There thus exists a radius $r > 0$ such that $x + h \in [a,b]$ and

$$\|F(x+h) - F(x) - f(x) \cdot [\varphi(x+h) - \varphi(x)]\| \le \frac{\varepsilon_n}{2} \omega(\varphi; x, h)$$

for all $h \in (-r, r)$. For $y, z \in (x - r, x + r)$ with $y \le x \le z$, we deduce

$$\begin{aligned} & \|F(z) - F(y) - f(x) \cdot [\varphi(z) - \varphi(y)]\| \\ \le & \|F(z) - F(x) - f(x) \cdot [\varphi(z) - \varphi(x)]\| + \|F(x) - F(y) - f(x) \cdot [\varphi(x) - \varphi(y)]\| \\ \le & \frac{\varepsilon_n}{2} \left(\omega(\varphi, [x,z]) + \omega(\varphi, [y,x])\right) \le \varepsilon_n \omega(\varphi, [y,z]). \end{aligned}$$

Hence, we can choose $\delta_n(x) > 0$ as claimed. Observe that if $x \in \widetilde{E}_n \setminus B$, $y, z \in [a, b]$ and $x - \delta_n(x) < y \leq x \leq z < x + \delta_n(x)$, then we obtain $|\xi - x| < \delta_n(x)$ for each $\xi \in [y, z]$. Hence, $|\chi(\xi) - \chi(x)| < \frac{1}{n}$, which in turn yields $\|\varphi(x) - \varphi(\xi)\| \leq n|\chi(x) - \chi(\xi)|$ because $x \in \widetilde{E}_n$. We thus derive

$$\begin{aligned}\|\varphi(\xi) - \varphi(\zeta)\| &\leq \|\varphi(\xi) - \varphi(x)\| + \|\varphi(x) - \varphi(\zeta)\| \\ &\leq n(|\chi(\xi) - \chi(x)| + |\chi(x) - \chi(\zeta)|) \\ &\leq 2n\,\omega(\chi, [y, z]) = 2n(\chi(z) - \chi(y))\end{aligned}$$

for all $\xi, \zeta \in [y, z]$, i.e.,

$$\omega(\varphi, [y, z]) \leq 2n(\chi(z) - \chi(y)). \tag{4.7}$$

We next put $C := B \setminus N = (A \cup M \cup D \cup \{a, b\}) \setminus N$. If $C \neq \emptyset$, let $(y_n)_{n \in I}$ be a (bijective) enumeration of C, where $I := \{1, \ldots, \sharp C\}$ for finite C and $I := \mathbb{N}$ for infinite C. (Note that C is countable.) Since F is φ-continuous with $f(x) \in \mathcal{C}(F, \varphi, x)$ for all $x \in [a, b] \setminus N$, we can choose a number $\delta_n > 0$ for each $n \in I$ such that

$$\|F(y) - F(y_n) - f(y_n) \cdot [\varphi(y) - \varphi(y_n)]\| < \frac{\varepsilon}{2^{n+4}}$$

for all $y \in [a, b]$ with $|y - y_n| < \delta_n$.

Thanks to $m_F(N) = 0$, we finally find a gauge $\widetilde{\delta} : N \to (0, \infty)$ with

$$\sum_{j=1}^{r} \|F(b_j) - F(a_j)\| < \frac{\varepsilon}{4}$$

for each $\{([a_j, b_j], x_j)\}_{j=1}^r \in \mathcal{S}(N, \widetilde{\delta})$.

Now, we define $E_1 := \widetilde{E}_1 \setminus B$, $E_{n+1} := \widetilde{E}_{n+1} \setminus \left(B \cup \bigcup_{j=1}^n \widetilde{E}_j\right)$ for $n \in \mathbb{N}$ and

$$\delta(x) := \begin{cases} \delta_n(x), & \text{if } x \in E_n \text{ for some (unique) } n \in \mathbb{N}, \\ \delta_n, & \text{if } x = y_n \text{ for some (unique) } n \in I, \\ \widetilde{\delta}(x), & \text{if } x \in N, \end{cases}$$

for $x \in [a, b]$. Note that $\dot{\bigcup}_{n=1}^\infty E_n \dot{\cup} N \dot{\cup} C = \left(\bigcup_{n=1}^\infty \widetilde{E}_n \setminus B\right) \cup B = \bigcup_{n=1}^\infty \widetilde{E}_n \cup B = [a, b]$ since $[a, b] \setminus M \subseteq \bigcup_{n=1}^\infty \widetilde{E}_n$ and $M \subseteq B$.

Now let $\{([a_j, b_j], x_j)\}_{j=1}^r \in \mathcal{S}([a, b], \delta)$ be arbitrary. We have

$$\begin{aligned}&\sum_{j=1}^{r} \|F(b_j) - F(a_j) - f(x_j) \cdot [\varphi(b_j) - \varphi(a_j)]\| \\ &= \sum_{\substack{j=1 \\ x_j \in N}}^{r} \|F(b_j) - F(a_j)\| + \sum_{\substack{j=1 \\ x_j \in C}}^{r} \|F(b_j) - F(a_j) - f(x_j) \cdot [\varphi(b_j) - \varphi(a_j)]\| \\ &\quad + \sum_{\substack{j=1 \\ x_j \notin B}}^{r} \|F(b_j) - F(a_j) - f(x_j) \cdot [\varphi(b_j) - \varphi(a_j)]\| \\ &=: \sum_{\substack{j=1 \\ x_j \in N}}^{r} \|F(b_j) - F(a_j)\| + \Sigma_1 + \Sigma_2 < \frac{\varepsilon}{4} + \Sigma_1 + \Sigma_2\end{aligned}$$

For each $x_j \in C$ let $n_j \in I$ be the uniquely determined number with $y_{n_j} = x_j$. Then we estimate

$$\Sigma_1 = \sum_{\substack{j=1 \\ x_j \in C}}^{r} \|F(b_j) - F(a_j) - f(y_{n_j}) \cdot [\varphi(b_j) - \varphi(a_j)]\|$$

$$\leq \sum_{\substack{j=1\\x_j\in C}}^{r} \|F(b_j)-F(y_{n_j})-f(y_{n_j})\cdot[\varphi(b_j)-\varphi(y_{n_j})]\|$$
$$+\sum_{\substack{j=1\\x_j\in C}}^{r} \|F(y_{n_j})-F(a_j)-f(y_{n_j})\cdot[\varphi(y_{n_j})-\varphi(a_j)]\|$$
$$<2\sum_{\substack{j=1\\x_j\in C}}^{r} \frac{\varepsilon}{2^{n_j+4}} \leq \frac{\varepsilon}{4}\sum_{n=1}^{\infty}\frac{1}{2^n}=\frac{\varepsilon}{4},$$

where we used that for each $k\in I$ the set $\{j\in\{1,\dots,r\}:\ x_j=y_k\}$ has at most two elements. We next estimate

$$\Sigma_2=\sum_{n=1}^{\infty}\sum_{\substack{j=1\\x_j\in E_n}}^{r} \|F(b_j)-F(a_j)-f(x_j)\cdot[\varphi(b_j)-\varphi(a_j)]\|$$
$$\leq \sum_{n=1}^{\infty}\sum_{\substack{j=1\\x_j\in E_n}}^{r} \varepsilon_n\omega(\varphi,[a_j,b_j])$$
$$\leq \sum_{n=1}^{\infty}\sum_{\substack{j=1\\x_j\in E_n}}^{r} 2n\varepsilon_n(\chi(b_j)-\chi(a_j)),$$

where we applied the inequalities (4.6) and (4.7). The monotonicity of χ and the definition of ε_n then yield

$$\Sigma_2\leq\sum_{n=1}^{\infty}\left(2n\varepsilon_n\sum_{\substack{j=1\\x_j\in E_n}}^{r}(\chi(b_j)-\chi(a_j))\right)$$
$$\leq\sum_{n=1}^{\infty}\left(2n\cdot\frac{\varepsilon}{n2^{n+3}(\chi(b)-\chi(a))}\cdot(\chi(b)-\chi(a))\right)$$
$$=\frac{\varepsilon}{4}\sum_{n=1}^{\infty}\frac{1}{2^n}=\frac{\varepsilon}{4}.$$

Summarizing, we arrive at

$$\sum_{j=1}^{r}\|F(b_j)-F(a_j)-f(x_j)\cdot[\varphi(b_j)-\varphi(a_j)]\|\leq\frac{\varepsilon}{4}+\Sigma_1+\Sigma_2\leq\frac{3\varepsilon}{4}<\varepsilon$$

and conclude $f\in\mathcal{HK}([a,b],\varphi)$ with $\int_a^x f(t)\cdot\mathrm{d}\varphi(t)=F(x)-F(a)$ for all $x\in[a,b]$. □

2nd proof of Theorem 4.24. We choose a sequence $(E_n)_{n=1}^{\infty}$ of closed sets with $\bigcup_{n=1}^{\infty}E_n=[a,b]$ and $\varphi\in\mathrm{BV}_*(E_n,Y)$ for all $n\in\mathbb{N}$. We set $c_n:=\min E_n$ and $d_n:=\max E_n$. By Lemma 2.4 the quantity

$$M_n:=\sup\{V_\varphi^*(S):\ S=\{([a_j,b_j],t_j)\}_{j=1}^{r}\in\mathcal{A}([c_n,d_n])\text{ with }\{a_j,b_j\}\cap E_n\neq\emptyset\text{ for }j\in\{1,\dots,r\}\}$$

is finite for each $n\in\mathbb{N}$. Let $(s_k)_{k\in I}$ be an enumeration of $B:=(\bigcup_{n=1}^{\infty}\{c_n,d_n\}\cup A)\setminus N$, where $I=\{1,\dots,\sharp B\}$, if B is finite, and $I=\mathbb{N}$, if B is infinite (provided that B is non-empty at all). We further set $\widetilde{E}_1:=E_1\setminus(B\cup N)$ and $\widetilde{E}_n:=E_n\setminus\left(\bigcup_{k<n}E_k\cup B\cup N\right)$ for $n>1$. For each $k\in I$ take $\delta_k>0$ with

$$\|F(s)-F(s_n)-f(s_n)\cdot[\varphi(s)-\varphi(s_n)]\|<\frac{\varepsilon}{2^{n+4}}$$

for all $s \in [a,b] \cap U_{\delta_k}(s_k)$, using that $f(s_k) \in \mathcal{C}(F,\varphi,s_k)$. Because of $m_F(N) = 0$, there is a gauge $\delta_0 \in (0,\infty)^N$ with $W_F(S) < \frac{\varepsilon}{4}$ for all $S \in \mathcal{S}(N,\delta_0)$. Finally, for $t \in \widetilde{E}_n \subseteq E_n \setminus \{c_n, d_n\}$ we choose $\delta_{0,n}(t) > 0$ with $U_{\delta_{0,n}(t)}(t) \subseteq (c_n, d_n)$

$$\|F(s) - F(t) - f(t) \cdot [\varphi(s) - \varphi(t)]\| \leq \frac{\varepsilon}{2^{n+2}(M_n+1)} \cdot \omega(\varphi, \langle s,t \rangle)$$

for all $s \in [a,b] \cap U_{\delta_{0,n}(t)}(t)$. We now define

$$\delta : [a,b] \to (0,\infty);\ t \mapsto \begin{cases} \delta_n, & \text{if } t = s_n \text{ for a (unique) } n \in I, \\ \delta_0(t), & \text{if } t \in N, \\ \delta_{0,n}(t), & \text{if } t \in \widetilde{E}_n \text{ for a (unique) } n \in \mathbb{N}\,. \end{cases}$$

Let $\{([a_j,b_j],t_j)\}_{j=1}^r$ be an arbitrary element of $\mathcal{S}([a,b],\delta)$. We have

$$\sum_{j=1}^{r} \|F(b_j) - F(a_j) - f(t_j) \cdot [\varphi(b_j) - \varphi(a_j)]\| \leq \sum_{\substack{j=1 \\ t_j \in N}}^{r} \|F(b_j) - F(a_j)\| + \Sigma_1 + \Sigma_2 + \Sigma_3,$$

where

$$\Sigma_1 := \sum_{\substack{j=1 \\ t_j \in B}}^{r} \|F(b_j) - F(a_j) - f(t_j) \cdot [\varphi(b_j) - \varphi(a_j)]\|,$$

$$\Sigma_2 := \sum_{\substack{j=1 \\ t_j \notin B \cup N}}^{r} \|F(b_j) - F(t_j) - f(t_j) \cdot [\varphi(b_j) - \varphi(t_j)]\|,$$

and

$$\Sigma_3 := \sum_{\substack{j=1 \\ t_j \notin B \cup N}}^{r} \|F(t_j) - F(a_j) - f(t_j) \cdot [\varphi(t_j) - \varphi(a_j)]\|.$$

By the definition of δ we obtain

$$\sum_{\substack{j=1 \\ t_j \in N}}^{r} \|F(b_j) - F(a_j)\| < \frac{\varepsilon}{4}.$$

As in the proof of Theorem 4.23 one shows $\Sigma_1 < \frac{\varepsilon}{4}$. Next notice that the system $\{[t_j,b_j] : j \in \{1,\ldots,r\}$ with $t_j \in \widetilde{E}_n\}$ belongs to $\mathcal{A}([c_n,d_n])$ and satisfies $\{t_j,b_j\} \cap E_n \supseteq \{t_j\} \neq \emptyset$ for each $j \in \{1,\ldots,r\}$ with $t_j \in \widetilde{E}_n$. Hence,

$$\sum_{\substack{j=1 \\ t_j \in \widetilde{E}_n}}^{r} \omega(\varphi,[t_j,b_j]) \leq M_n.$$

Using this observation we derive

$$\begin{aligned} \Sigma_2 &= \sum_{n=1}^{\infty} \sum_{\substack{j=1 \\ t_j \in \widetilde{E}_n}}^{r} \|F(b_j) - F(t_j) - f(t_j) \cdot [\varphi(b_j) - \varphi(t_j)]\| \\ &\leq \sum_{n=1}^{\infty} \sum_{\substack{j=1 \\ t_j \in \widetilde{E}_n}}^{r} \frac{\varepsilon}{2^{n+2}(M_n+1)} \cdot \omega(\varphi,[t_j,b_j]) \\ &\leq \sum_{n=1}^{\infty} \frac{\varepsilon M_n}{2^{n+2}(M_n+1)} < \frac{\varepsilon}{4}. \end{aligned}$$

Analogously, we see $\Sigma_3 < \frac{\varepsilon}{4}$. Summarizing, we arrive at

$$\sum_{j=1}^{r} \|F(b_j) - F(a_j) - f(t_j) \cdot [\varphi(b_j) - \varphi(a_j)]\| < 4 \cdot \frac{\varepsilon}{4} = \varepsilon$$

for each $\{([a_j, b_j], t_j)\}_{j=1}^{r} \in \mathcal{S}([a,b], \delta)$. This completes the proof. □

4.5 Descriptive characterisation of the variational Henstock-Kurzweil-Stieltjes integral

In the final section of this chapter we bring together the results from the previous two sections in order to obtain descriptive characterisations of functions Stieltjes-integrable in the variational sense of Henstock-Kurzweil resp. of functions being the indefinite integral of such functions. As a consequence of these characterisations, we shall obtain, as announced before, a clearer picture of the relations between the various notions of derivatives introduced above.

4.25 Theorem. *Let $\varphi \in \mathrm{BVG}_*([a,b], Y)$ with measurable admissible decompostion $(E_n)_{n=0}^{\infty}$. For $f \in X^{[a,b]}$ the following statements are equivalent.*

(a) We have $f \in \mathcal{HK}([a,b], \varphi)$.

(b) There exists a function $F : [a,b] \to Z$ with $m_F \ll m_\varphi$ such that f is a φ-Fréchet derivative of F.

(c) There exists a function $F : [a,b] \to Z$ with $m_F \ll m_\varphi$ such that f is a φ-Roussel derivative of F.

(d) There exists a φ-null set A and a φ-continuous function $F : [a,b] \to Z$ with $m_F \ll m_\varphi$ such that $f(t) \in \mathcal{C}(F, t, \varphi)$ for all $t \in [a,b]$ and $f(t) \in \mathcal{D}(F, \varphi, t, (E_n)_n)$ for all $t \in [a,b] \setminus (N \cup A)$;

If one (and thus all) of these conditions is satisfied, we have $\int_a^t f(s) \cdot \mathrm{d}\varphi(s) = F(t) - F(a)$ for all $t \in [a,b]$ and any function F as in condition (b)-(d).

Proof. The implication (a) $\Longrightarrow$ (b) follows from Theorem 4.22, Lemma 4.7 and Lemma 4.9.
The implication (a) $\Longrightarrow$ (d) follows from Theorem 4.21, Lemma 4.7 and Lemma 4.9.
The implication (b) $\Longrightarrow$ (c) is trivial.
Now assume that (c) holds. Then $f(t)$ is contained in $\mathcal{C}(F, \varphi, t)$ for all $t \in [a,b]$ and there exists a countable set $A \subseteq [a,b]$ and a set $N \subseteq [a,b]$ with $m_\varphi(N) = 0$ such that $f(t)$ belongs to $\mathcal{D}(F, \varphi, t)$ for all $t \in [a,b] \setminus (N \cup A)$. We now define

$$\widetilde{f}(t) := \begin{cases} 0, & \text{if } t \in N, \\ f(t), & \text{elsewise,} \end{cases}$$

for $t \in [a,b]$. Since $m_\varphi(N) = 0$ and $m_F \ll m_\varphi$, Theorem 4.24 yields $\widetilde{f} \in \mathcal{HK}([a,b], \varphi)$ with $\int_a^t \widetilde{f}(s) \cdot \mathrm{d}\varphi(s) = F(t) - F(a)$ for all $t \in [a,b]$. The set $M := \{x \in [a,b] :\ f(t) \neq \widetilde{f}(t)\}$ is a subset of N. Hence, $m_\varphi(M) = 0$ and Lemma 4.11 implies $f - \widetilde{f} \in \mathcal{HK}([a,b], \varphi)$ with $\int_a^t (f(s) - \widetilde{f}(s)) \cdot \mathrm{d}\varphi(s) = 0$ for all $t \in [a,b]$. These facts then imply $f = \widetilde{f} + (f - \widetilde{f}) \in \mathcal{HK}([a,b], \varphi)$ with

$$\int_a^t f(s) \cdot \mathrm{d}\varphi(s) = \int_a^t \widetilde{f}(s) \cdot \mathrm{d}\varphi(s) + \int_a^t (f(s) - \widetilde{f}(s)) \cdot \mathrm{d}\varphi(s) = F(t) - F(a)$$

for all $t \in [a,b]$. This establishes (a) and the addendum in the cases (b) and (c).
Finally, the remaining implication (d) $\Longrightarrow$ (a) and the addendum in the case (d) can be proved as the implication (c) $\Longrightarrow$ (a) before replacing the use of Theorem 4.24 by an application of Theorem 4.23. □

Now we can derive several corollaries from our main results above.

The first two show how the characterisation in Theorem 4.25 simplifies under additional assumptions imposed on φ.

4.26 Corollary. *Let $\varphi \in \mathrm{BVG}_*([a,b],Y)$ be bounded and $f \in X^{[a,b]}$. The following assertions are equivalent.*

(a) We have $f \in \mathcal{HK}([a,b],\varphi)$.

(b) There exists a function $F : [a,b] \to Z$ with $m_F \ll m_\varphi$ and a φ-null set N such that $f(x)$ is a pseudo-φ-Roussel-differentiability value of F at x for all $x \in [a,b] \setminus N$ and $f(x)$ is a φ-continuity value of F at x for all $x \in N$.

(c) There exists a function $F : [a,b] \to Z$ with $m_F \ll m_\varphi$ and a φ-null set N such that $f(t) \in \mathcal{FD}(F,\varphi,t)$ for all $x \in [a,b] \setminus N$ and $f(t) \in \mathcal{C}(F,\varphi,t)$ for all $t \in N$.

Proof. By Theorem 4.25 it is clear that (a) implies (c) and obviously (c) implies (b).
Now assume that (b) holds. Since φ is a bounded function, any pseudo-φ-Roussel-differentiability value is also a φ-continuity value and thus even a φ-Roussel-differentiability value. Consequently, F is a φ-continuous function with $m_F \ll m_\varphi$ and f is a φ-Roussel-derivative of F. Theorem 4.25 now yields (a). □

4.27 Corollary. *Let $\varphi \in \mathrm{BVG}_*([a,b],Y)$ be continuous and $f \in X^{[a,b]}$. The following assertions are equivalent.*

(a) We have $f \in \mathcal{HK}([a,b],\varphi)$.

(b) There exists a function $F : [a,b] \to Z$ with $m_F \ll m_\varphi$ such that f is an m_φ-Roussel-derivative of F.

(c) There exists a function $F : [a,b] \to Z$ with $m_F \ll m_\varphi$ and a m_φ-null set N such that $f(t) \in \mathcal{FD}(F,\varphi,t)$ for all $t \in [a,b] \setminus N$.

Proof. Statement (a) and (b) are equivalent thanks to Theorem 4.25 and Lemma 4.16. Clearly, (c) yields (b) and (c) itself follows from (a) once again by means of Theorem 4.25. □

Of course, we also obtain a characterisation of indefinite variational Henstock-Kurzweil-Stieltjes integrals with respect to φ.

4.28 Corollary. *Let $\varphi \in \mathrm{BVG}_*([a,b],Y)$ and $F \in Z^{[a,b]}$. The following assertions are equivalent.*

(a) The function F is an indefinite variational Henstock-Kurzweil-Stieltjes integral with respect to φ.

(b) The function F satisfies $m_F \ll m_\varphi$ and possesses a φ-Roussel derivative.

(c) The function F satisfies $m_F \ll m_\varphi$ and possesses a φ-Fréchet derivative.

In fact, if (b) resp. (c) holds, then F is an indefinite variational Henstock-Kurzweil-Stieltjes integral of each of its φ-Roussel derivatives resp. φ-Fréchet derivatives with respect to φ.

Proof. Since two indefinite variational Henstock-Kurzweil-Stieltjes integrals of the same function with respect to φ differ from each other only by an additive constant, the implication "(a) $\Longrightarrow$ (c)" follows from Lemma 4.7, Lemma 4.9, Theorem 4.22 and the simple observations that $\mathcal{C}(F,\varphi,t) = \mathcal{C}(F+z,\varphi,t)$ and $\mathcal{FD}(F,\varphi,t) = \mathcal{FD}(F+z,\varphi,t)$ and $m_{F+z} = m_F$ for all $t \in [a,b]$ and all $z \in Z$.
Obviously, (c) implies (b).
Assume finally that (b) holds and let $f : [a,b] \to X$ be a φ-Roussel derivative of F. Theorem 4.25 implies $f \in \mathcal{HK}([a,b],\varphi)$ with $\int_a^x f(t) \cdot \mathrm{d}\varphi(t) = F(x) - F(a)$ for all $x \in [a,b]$ and we conclude that F is an indefinite variational Henstock-Kurzweil-Stieltjes integral with respect to φ. □

As an immediate corollary to the preceding result we gain a new insight into the relation between the various notions of derivatives defined above.

4.29 Corollary. *Let $\varphi \in \mathrm{BVG}_*([a,b],Y)$ and $F \in Z^{[a,b]}$ with $m_F \ll m_\varphi$. Then F possesses a φ-Roussel derivative if and only if F possesses a φ-Fréchet derivative.*

4.30 Corollary. *Let $\varphi \in \mathrm{BVG}_*([a,b],Y)$ and let $F, \widetilde{F} : [a,b] \to Z$ be two functions with $m_F \ll m_\varphi$ resp. $m_{\widetilde{F}} \ll m_\varphi$ possessing a common φ-Roussel derivative or a common φ-Fréchet derivative. Then F and $\widetilde{F}$ only differ by an additive constant.*

In the special case $X = Z, Y = \mathbb{K}$, where φ is the identity on $[a,b]$ and the bilinear mapping B is the usual multiplication with scalars, we obtain the following version of Lusin's classical characterisation of the Henstock-Kurzweil integral.

4.31 Corollary. *Let $X = Z$, $Y = \mathbb{K}$, let φ be the identity on $[a,b]$ and let the bilinear mapping B be the usual multiplication with scalars and let $f \in X^{[a,b]}$. The following assertions are equivalent.*

(a) We have $f \in \mathcal{HK}([a,b],\varphi)$.

(b) There exists a function $F : [a,b] \to X$ with $m_F \ll \lambda$ that is Lebesgue-almost everywhere differentiable with $F'(x) = f(x)$ for Lebesgue-almost all $x \in [a,b]$.

Proof. This follows from Corollary 4.27 and the fact that $m_\varphi = m_{\mathrm{id}}$ coincides in this situation with the usual outer Lebesgue measure by Proposition A.3. □

4.32 Remark The preceding result does not completely coincide with Lusin's classical theorem in view of two aspects. First, Corollary 4.31 uses functions $F : [a,b] \to X$ with $m_F \ll \lambda$ to characterize Henstock-Kurzweil integrable function, while Lusin's result uses (continuous) ACG_*-functions. As an application of our main results of this chapter, we shall show in the second section of the next chapter that we may indeed utilise (continuous) ACG_*-functions in order to get the same characterisation. Second, Corollary 4.31 incorporates differentiability properties of the indefinite integral F. In contrast to that, the classical Lusin theorem can refrain from that because real-valued ACG_*-functions are automatically Lebesgue-a.e. differentiable. Therefore, Corollary 4.31 is a so-called partial descriptive characterisation, whereas a so-called full descriptive characterisation would only invoke the condition $m_F \ll m_\varphi$. As a consequence, the question arises whether or not it is indispensable to incorporate this differentiability condition. This was the main motivation for Bongiorno, Di Piazza, Musiał for their paper [BPM09a], where they examine this question and its relation to the Radon-Nikodým property. We shall continue their explorations in the third section of the next chapter and significantly extend their results.

The last result of this section concerns the question to which extent the derivative introduced in Definition 4.19 depends on the chosen admissible decomposition. It turns out that under appropriate conditions the existence of this derivative is in a global sense independent of the chosen admissible decomposition.

4.33 Corollary. *Let φ be an element of $\mathrm{BVG}_*([a,b],Y)$ with measurable admissible decompositions $(E_n)_{n=0}^\infty$ and $(\widetilde{E}_n)_{n=0}^\infty$. Let $F \in Z^{[a,b]}$ be φ-continuous with $m_F \ll m_\varphi$. Then the following statements are equivalent.*

(a) There is a countable set A and an m_φ-null set N with $\mathcal{D}(F,t,\varphi,(E_n)_{n=0}^\infty) \neq \emptyset$ for all $t \in [a,b] \setminus (A \cup N)$.

(b) There is a countable set $\widetilde{A}$ and an m_φ-null set $\widetilde{N}$ with $\mathcal{D}(F,t,\varphi,(\widetilde{E}_n)_{n=0}^\infty) \neq \emptyset$ for all $t \in [a,b] \setminus (\widetilde{A} \cup \widetilde{N})$.

Proof. By symmetry it suffices to establish that (a) yields (b). So assume (a), set $B := A \cup N \cup E_0$, take a choice function $f_1 : [a,b] \setminus B \to X$ with $f_1(t) \in \mathcal{D}(F,t,\varphi,(E_n)_{n=0}^{\infty})$ for all $t \in [a,b] \setminus (A \cup N)$, a choice function $f_2 : B \to X$ with $f(t) \in \mathcal{C}(F,\varphi,t)$ for all $t \in B$ and for $t \in [a,b]$ define $f(t) := f_1(t)$ for $t \in [a,b] \setminus B$ and $f(t) := f_2(t)$ for $t \in B$. Note that $\mathcal{D}(F,t,\varphi,(E_n)_{n=0}^{\infty}) \subseteq \mathcal{C}(F,\varphi,t)$ for each $t \in [a,b] \setminus E_0$. Indeed, pick $n \in \mathbb{N}$ such that $t \in E_n$ and let $\varepsilon > 0$. Then there exists by definition a $\rho > 0$ such that

$$\|F(s) - F(t) - x \cdot [\varphi(s) - \varphi(t)]\| \leq \frac{\varepsilon}{m_\varphi(E_n) + 1} \cdot m_\varphi(\langle s,t\rangle \cap E_n) < \varepsilon$$

for all $s \in [a,b]$ with $|s-t| < \rho$. Theorem 4.25 now yields $f \in \mathcal{HK}([a,b],\varphi,X)$ with $\int_a^t f(s) \cdot \mathrm{d}\varphi(s) = F(t) - F(a)$ for all $t \in [a,b]$. With $(\widetilde{E}_n)_{n=0}^{\infty}$ in lieu of $(E_n)_{n=0}^{\infty}$ Theorem 4.25 gives us now a φ-continuous function $\widetilde{F} : [a,b] \to Z$ with $m_{\widetilde{F}} \ll m_\varphi$, a countable set $\widetilde{A}$ and an m_φ-null set $\widetilde{N}$ such that $f(t) \in \mathcal{C}(\widetilde{F},t,\varphi)$ for all $t \in [a,b]$ and $f(t) \in \mathcal{D}(\widetilde{F},\varphi,t,(\widetilde{E}_n)_{n=0}^{\infty})$ for all $t \in [a,b] \setminus (N \cup A)$ and $\int_a^t f(s) \cdot \mathrm{d}\varphi(s) = \widetilde{F}(t) - \widetilde{F}(a)$ for all $t \in [a,b]$. In particular, F and $\widetilde{F}$ differ from each other only by a constant. This yields part (b). $\square$

5 Applications

This chapter is devoted to several applications of the results of the previous chapters. In the first section we revisit one of our starting points of our explorations, namely the question of recovering a function from a given relative derivative. As one can expect at this point in the text the most satisfactory results can be achieved using the Henstock-Kurzweil integral. Nevertheless the question arises whether Riemann- or Lebesgue-integration suffice if we additionally impose a corresponding integrability condition on the given relative derivative. It will turn out that this is indeed the case.

Afterwards, we revisit ACG_*-functions and derive another characterisation for them, which is in the real-valued case originally due to Gordon (see [Gor89]). Using this result we shall reprove the classical descriptive characterisation of the Henstock-Kurzweil integral due to Lusin.

Remark 4.32 motivates to study Henstock-Kurzweil integrals of functions with values in a space having the Radon-Nikodým property. This will be done in the third section where we prove a far reaching extension of a result due to Bongiorno, Di Piazza and Musiał (see Theorem 3.6 in [BPM09b]) and also fill a gap in their proof.

The fourth section demonstrates how one can obtain "integration by parts"-results for variational Henstock-Kurzweil-Stieltjes integrals using the characterisations proved in the previous chapter.

In the last section we apply our results to the study of certain normed algebras of differentiable functions on compact plane sets.

5.1 Recovering a function from relative derivatives

We now return to the question that motivated, as we indicated in the introduction, Lebesgue's development of his integration theory: How can one recover a function from its derivative? But instead of the ordinary derivative we want to consider a relativized version of it with respect to another function. However, before doing this we have to explain what we mean by a relativized derivative - in particular in view of the fact that there is no established definition within the literature. We refer to, e.g., [AP99, CC08, Dan18, Gar92, Gra90, Jef32, Leb50, Rid36, Rid38, Rid39, Sak41, War36, You17] for similar notions of differentiability of a function relative to another function, as well as for results related to ours.

Let $f : [a, b] \to X$ and $\varphi : [a, b] \to Y = \mathbb{K}$ be two functions and $t_0 \in [a, b]$. One natural candidate for the definition of the derivative of f at t_0 with respect to φ is

$$\lim_{t \to t_0} \frac{f(t) - f(t_0)}{\varphi(t) - \varphi(t_0)}.$$

At this point, two problems rise. First, one has either to explain how the above expression is to be understood if the denominator equals 0 for $t \neq t_0$ or one should replace the limit $\lim_{t \to t_0}$ by taking the limit $t \to t_0$ with respect to those t that fulfill $\varphi(t) \neq \varphi(t_0)$; this, however, requires to demand the existence of a non-degenerate interval $I \subseteq [a, b]$ containing t_0 such that φ is not constant on any non-degenerate subinterval of I containing t_0.

Second, one might argue that the above working definition is not a good one if φ has a discontinuity at t_0. Therefore, many authors have treated the discontinuities of φ separately or they have altered

the above definition by incorporating left-sided and right-sided limits of φ; but in the last case one has to require that φ is at least a so-called regular function.
In order to avoid all these questions, we have decided to follow [AP99] and to use our Definition 4.17.

5.1 Definition. *We say that f is* differentiable at t_0 with respect to φ *if there is an $x \in X$ such that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$\|f(t) - f(t_0) - x(\varphi(t) - \varphi(t_0))\| \leq \varepsilon|\varphi(t) - \varphi(t_0)|$$

for all $t \in [a, b]$ with $|t - t_0| < \delta$.

5.2 Remark (a) Definition 5.1 precisely coincides with the definition of our notion of Fréchet-differentiability at the point t_0, where $X = Z$, $Y = \mathbb{C}$ and the bilinear mapping B is the multiplication by scalars.

(b) If f is differentiable at t_0 with respect to φ and if φ is constant on some neighbourhood (relative to $[a, b]$) of t_0, then so is f on this neighbourhood and we can take any $x \in X$ in Definition 5.1.

(c) If f is differentiable at t_0 with respect to φ and if there exists a sequence $(t_n)_n$ in $[a, b]$ converging to t_0 with $\varphi(t_n) \neq \varphi(t_0)$ for all $n \in \mathbb{N}$, then the x in Definition 5.1 is unique.

Proof for part (c). Let $\widetilde{x} \in X$ also fulfill the above condition in the definition, $\varepsilon > 0$ and $\delta > 0$ be as above in the definition. We then obtain $|t_n - t_0| < \delta$ for all sufficiently large $n \in \mathbb{N}$ and thus

$$\begin{aligned}
&\|x - \widetilde{x}\| \\
=&\frac{\|x(\varphi(t_n) - \varphi(t_0)) - \widetilde{x}(\varphi(t_n) - \varphi(t_0))\|}{|\varphi(t_n) - \varphi(t_0)|} \\
=&\frac{\|f(t_n) - f(t_0) - \widetilde{x}(\varphi(t_n) - \varphi(t_0)) - (f(t_n) - f(t_0) - x(\varphi(t_n) - \varphi(t_0)))\|}{|\varphi(t_n) - \varphi(t_0)|} \\
\leq&\frac{\|f(t_n) - f(t_0) - \widetilde{x}(\varphi(t_n) - \varphi(t_0))\|}{|\varphi(t_n) - \varphi(t_0)|} + \frac{\|f(t_n) - f(t_0) - x(\varphi(t_n) - \varphi(t_0))\|}{|\varphi(t_n) - \varphi(t_0)|} \\
\leq&\frac{\varepsilon|\varphi(t) - \varphi(t_0)|}{|\varphi(t) - \varphi(t_0)|} + \frac{\varepsilon|\varphi(t) - \varphi(t_0)|}{|\varphi(t) - \varphi(t_0)|} = 2\varepsilon
\end{aligned}$$

for all these sufficiently large $n \in \mathbb{N}$. Hence, $x = \widetilde{x}$. □

Due to the observations in Remark 5.2 we make the following convention: if f is differentiable at t_0 with respect to φ, then $f'_\varphi(t_0)$ denotes any x as in Definition 5.1.

Now we want to explore the following question. Assume that f is differentiable at each point $t \in [a, b]$ and that we know f'_φ on $[a, b]$ as well as $f(t_0)$ for some $t_0 \in [a, b]$. Can we reconstruct the function f from these data? In contrast to far reaching positive results (see, e.g., Théorème 10 in [Cho47] and page 307 in [Leb50]) in the case of real-valued φ, the answer is in general "no", even under quite "good" circumstances and even for complex-valued φ and very tame f. To see this, let $\theta \in (0, \frac{\pi}{4})$ and consider Koch curves Γ_θ constructed in the usual iterative manner starting with the isosceles triangle having base angle θ with vertices 0, 1 and $\frac{1}{2}(1 + i\tan(\theta))$ (see [Pon07, §1] for this construction). Then there is a homeomorphism $\varphi : [0, 1] \to \Gamma_\theta$ satisfying

$$\frac{\sin 3\theta}{8\cos^3\theta}|t - s|^{\log_2(2\cos\theta)} \leq |\varphi(t) - \varphi(s)| \leq 4|t - s|^{\log_2(2\cos\theta)}$$

for all $s, t \in [0, 1]$ (see, e.g., [Pon07, Theorem 1]). We now set $f(t) := t$ for $t \in [0, 1]$. For distinct $t_0, t \in [0, 1]$, we then obtain by means of the above inequality for φ

$$\frac{|t - t_0|}{|\varphi(t) - \varphi(t_0)|} \leq \frac{8\cos^3\theta}{\sin 3\theta} \cdot |t - t_0|^{1 - \log_2(2\cos\theta)} \xrightarrow[t \to t_0]{} 0$$

since $1 - \log_2(2\cos\theta) > 0$. Therefore f is differentiable at each point of $[0,1]$ with respect ot φ and with $f'_\varphi(t) = 0$, but f is not constant. In particular, Choquet's Théorème 10 in [Cho47] fails dramatically if we pass from the real-valued to the complex-valued situation, i.e., Choquet's result is of genuinely real nature and does not allow an extension to Banach-space valued functions φ. Furthermore, the above example prompts to impose additional conditions on φ in order to obtain positive results.

Using Theorem 4.24 it is now very easy to derive the following result, which solves Lebesgue's "problème des fonctions primitives" in a much more general framework (cf. [Leb50, p. 307]).

5.3 Theorem. *Let $\varphi \in \mathrm{BVG}_*([a,b],\mathbb{K})$ be bounded.*

(a) If $f : [a,b] \to X$ is differentiable at each point $t \in [a,b]$ with respect to φ, then we have $f'_\varphi \in \mathcal{HK}([a,b],\varphi,X)$ with

$$\int_a^t f'_\varphi(s) \cdot \mathrm{d}\varphi(s) = f(t) - f(a)$$

for all $t \in [a,b]$.

(b) Assume that $f : [a,b] \to X$ is φ-continuous and differentiable at each point $t \in [a,b] \setminus N$ with respect to φ and put $f'_\varphi(t) := 0$ for all $t \in N$, where $N \subseteq [a,b]$ with $m_f(N) = 0$. Then we have $f'_\varphi \in \mathcal{HK}([a,b],\varphi,X)$ with

$$\int_a^t f'_\varphi(s) \cdot \mathrm{d}\varphi(s) = f(t) - f(a)$$

for all $t \in [a,b]$.

The remaining part of this section is devoted to the question to which extent it is possible to recover a function from relative derivatives using Riemann-Stieltjes and Lebesgue-Stieltjes integration (for the latter notion see Definition C.1). Clearly, one has at least to require additionally that the relative derivative is integrable in the Riemann-Stieltjes respectively Lebesgue-Stieltjes sense. It turns out that under mild conditions these minimum demands are sufficient.

5.4 Theorem. *Let $\varphi \in \mathrm{BVG}_*([a,b],\mathbb{K})$ be bounded and assume that $f : [a,b] \to X$ is differentiable at each point $t \in [a,b]$ with respect to φ.*

(a) If f'_φ is Riemann-Stieltjes integrable with respect to φ, then

$$(\mathcal{R})\int_a^t f'_\varphi(s) \cdot \mathrm{d}\varphi(s) = f(t) - f(a)$$

for all $t \in [a,b]$.

(b) Assume additionally that φ is even of bounded variation (i.e., $\varphi \in \mathrm{BV}([a,b],\mathbb{C})$) and continuous from the right and that f'_φ is Lebesgue-Stieltjes integrable with respect to φ. Then

$$(\mathcal{L})\int_a^t f'_\varphi(s) \cdot \mathrm{d}\varphi(s) = f(t) - f(a)$$

for all $t \in [a,b]$.

Proof. Part (b) follows immediately from Theorem 5.3 and Proposition C.2 and part (a) is a very special instance of the subsequent Theorem 5.5. □

5.5 Theorem. *Let $\varphi \in \mathrm{BVG}_*([a,b],Y)$ and assume that $F : [a,b] \to Z$ possesses a φ-Roussel derivative, say f, and that $m_F \ll m_\varphi$ or that $f(t) \in \mathcal{D}(F,\varphi,t)$ for all $t \in [a,b]$. If f is Riemann-Stieltjes integrable with respect to φ, then*

$$(\mathcal{R})\int_a^t f \cdot \mathrm{d}\varphi(s) = F(t) - F(a)$$

for all $t \in [a,b]$.

Proof. We consider X, Y and Z as real Banach spaces. Let $z^\star$ be an $\mathbb{R}$-linear functional on Z with $\|z^\star\| \le 1$. We consider the bilinear mapping

$$B_{z^\star} : X \times Y \to \mathbb{R};\ (x, y) \mapsto \langle B(x, y), z^\star\rangle$$

and we observe that

$$|B_{z^\star}(x, y)| \le \|B(x, y)\| \cdot \|z^\star\| \le \|x\| \cdot \|y\|$$

for all $x \in X$ and $y \in Y$. We put $z := (\mathcal{R}) \int_a^b f(s) \cdot \mathrm{d}\varphi(s)$. Let $\varepsilon > 0$ and choose a constant $\delta > 0$ such that

$$\left\| z - \sum_{j=1}^{r} f(t_j) \cdot [\varphi(b_j) - \varphi(a_j)] \right\| < \varepsilon$$

for all $\{([a_j, b_j], t_j)\}_{j=1}^r \in \mathcal{S}([a, b], \delta)$ with $\bigcup_{j=1}^r [a_j, b_j] = [a, b]$. For such an element $\{([a_j, b_j], t_j)\}_{j=1}^r$ of $\mathcal{S}([a, b], \delta)$ we then estimate

$$\begin{aligned} \left| \langle z, z^\star\rangle - \sum_{j=1}^{r} B_{z^\star}(f(t_j), \varphi(b_j) - \varphi(a_j)) \right| &= \left| \left\langle z - \sum_{j=1}^{r} f(t_j) \cdot [\varphi(b_j) - \varphi(a_j)], z^\star \right\rangle \right| \\ &\le \left\| z - \sum_{j=1}^{r} f(t_j) \cdot [\varphi(b_j) - \varphi(a_j)] \right\| < \varepsilon. \end{aligned}$$

As a consequence, we infer that f is Riemann-Stieltjes integrable with respect to φ and $B_{z^\star}$ and we have $(\mathcal{R}) \int_a^b B_{z^\star}(f(s), \mathrm{d}\varphi(s)) = \langle z, z^\star\rangle$. Then f is also Henstock-Kurzweil integrable with respect to φ and $B_{z^\star}$. Lemma 4.3 now implies that f is also variationally Henstock-Kurzweil integrable with respect to φ and $B_{z^\star}$ with

$$\begin{aligned} \int_a^t B_{z^\star}(f(s), \mathrm{d}\varphi(s)) = (\mathcal{HK}) \int_a^t B_{z^\star}(f(s), \mathrm{d}\varphi(s)) &= (\mathcal{R}) \int_a^t B_{z^\star}(f(s), \mathrm{d}\varphi(s)) \\ &= \left\langle (\mathcal{R}) \int_a^t f(s) \cdot \mathrm{d}\varphi(s), z^\star \right\rangle \end{aligned} \tag{5.1}$$

for all $t \in [a, b]$. On the other side, we know $f \in \mathcal{HK}([a, b], \varphi, X)$ with $\int_a^t f \cdot \mathrm{d}\varphi(s) = F(t) - F(a)$ for all $t \in [a, b]$ by Theorem 4.25 or Theorem 4.24, respectively. Moreover, observe that for $u, v, w \in [a, b]$ we have

$$\begin{aligned} \|\langle F(u), z^\star\rangle - \langle F(v), z^\star\rangle - B_{z^\star}(f(w), \varphi(u) - \varphi(v))\| &= \|\langle F(u) - F(v) - f(w) \cdot [\varphi(u) - \varphi(v)], z^\star\rangle\| \\ &\le \|F(u) - F(v) - f(w) \cdot [\varphi(u) - \varphi(v)]\|. \end{aligned}$$

Using this estimate one easily verfies

$$\int_a^t B_{z^\star}(f(s), \mathrm{d}\varphi(s)) = \langle F(t) - F(a), z^\star\rangle \tag{5.2}$$

for all $t \in [a, b]$. Combining the two equations (5.1) and (5.2) and applying the Hahn-Banach theorem we deduce $(\mathcal{R}) \int_a^t f \cdot \mathrm{d}\varphi(s) = F(t) - F(a)$ for all $t \in [a, b]$ as claimed. □

5.2 ACG$_*$- and ACG$_\delta$-functions

In this section we turn our attention once again to ACG$_*$-functions and extend from the real-valued case to the vector-valued case a result originally due to Gordon providing another characterisation of ACG$_*$-function.

We start with a definition due to Gordon (see [Gor89])

5.6 Definition. *Let $\emptyset \neq E \subseteq [a,b]$. We say that a function $f : [a,b] \to X$ is of* variational absolute continuity on E *if for each $\varepsilon > 0$ there exists an $\eta > 0$ and a gauge $\delta \in (0,\infty)^E$ on E such that $W_f(S) < \varepsilon$ for all $S = \{([a_j,b_j],x_j)\}_{j=1}^r \in \mathcal{S}(E,\delta)$ with $W_{\mathrm{id}}(S) = \sum_{j=1}^r (b_j - a_j) < \eta$. We denote by $\mathrm{AC}_\delta(E,X)$ the set of all X-valued functions of variational absolute continuity on E. Moreover, we say that a function $f : [a,b] \to X$ is of* generalized variational absolute continuity on E *if E is the countable union of sets on each of which f is of variational absolute continuity. We denote by $\mathrm{ACG}_\delta(E,X)$ the set of all X-valued functions of generalized variational absolute continuity on E.*

5.7 Remark Note that in [Gor89] Gordon defines the notion of variational absolute continuity using the condition $\left\|\sum_{j=1}^r F(b_j) - F(a_j)\right\| < \varepsilon$ instead of our condition $W_f(S) < \varepsilon$. But using the same reasoning as in the proof Lemma 4.3, it is easy to see that these two concepts coincide in the case $X = \mathbb{R}$, which is considered by Gordon.

Now the question rises how the notions of generalized absolute continuity in the restricted sense and of generalized variational absolute continuity are related to each other. As a first step we have the following result.

5.8 Lemma. *Let $\emptyset \neq E \subseteq [a,b]$ be closed and $f \in \mathrm{AC}_*(E,X)$. If f is continuous at each point of E and bounded on $[\min(E),\max(E)]$, then $f \in \mathrm{AC}_\delta(E,X)$ holds.*

The preceding result is origionally due to R. A. Gordon (cf. the solution to Exercise 11.7 in [Gor94] or Theorem 5 of Gordon's original paper [Gor89]). Unfortunately, his proof is not quite correct (for the details we now refer to see the solution to Exercise 11.7 in [Gor94]): When considering $\mathcal{P}_1$, Gordon claims that for any i there exists a unique index $k_i \geq K$ such that $c_{k_i} < u_i < d_{k_i}$. But this requires $u_i \in [c,d]$, which may fail since we cannot guarantee that $v_i - \delta(v_i) \geq c$ always holds and we only know $v_i - \delta(v_i) < u_i < v_i$ (the same mistake appears in the solution to Exercise 11.3 in [Gor94]). Nevertheless, it is not hard to repair Gordon's proof, but for the sake of completeness we have attached a correct version.

Proof of Lemma 5.8. Let $\varepsilon > 0$. By assumption we can find a number $\eta > 0$ such that $V_f^*(S) = \sum_{j=1}^r \omega(f,[a_j,b_j]) < \varepsilon$ for all $S = \{[a_j,b_j]\}_{j=1}^r \in \mathcal{A}(E)$ with $\sum_{j=1}^r |b_j - a_j| < \eta$.

We put $c := \inf E \in E$ and $d := \sup E \in E$. Since f is continuous at each point of E, there is a $\rho > 0$ such that $\|f(x) - f(c)\| < \varepsilon$ for all $x \in [a,b]$ with $|x - c| < \rho$ and such that $\|f(x) - f(d)\| < \varepsilon$ for all $x \in [a,b]$ with $|x - d| < \rho$.
If $c = d$, then we obviously have $W_f(S) < 2\varepsilon$ for every $S \in \mathcal{S}(E,\delta)$ with $\delta \equiv \rho$, as asserted. So let $c < d$.
Since E is closed, $[c,d] \setminus E$ is an open set. We dinstinguish between two cases.

1st case: $E = [c,d]$. For $x \in E \setminus \{c,d\}$ we put $\delta(x) := \min\{x - c, d - x\}$ and for $x \in \{c,d\}$ we set $\delta(x) := \min\{\rho, c - d\}$. Let $S = \{([a_j,b_j],x_j)\}_{j=1}^r$ be a δ-fine partition on E with $\sum_{j=1}^r |b_j - a_j| < \eta$. As $[a_j,b_j] \subseteq U_{\delta(x_j)}(x_j)$, we have $a_j, b_j \in E = [c,d]$ at least for $j \in \{1,\ldots,r\} \setminus \{1,r\}$. If $a_1 \notin E = [c,d]$, then $x_1 = c$ and hence $|a_1 - c| < \rho$ as well as $|c - b_1| < \rho$ and thus $\|f(b_1) - f(a_1)\| < 2\varepsilon$. Similarly, one sees $\|f(b_r) - f(a_r)\| < 2\varepsilon$ for $b_r \notin E$. The choice of η now implies

$$W_f(S) = \sum_{j=1}^r \|f(b_j) - f(a_j)\| \leq 5\varepsilon.$$

2nd case: $E \neq [c,d]$. In this case $[c,d] \setminus E$ is a non-empty open subset of (c,d). Therefore we may write $[c,d] \setminus E = \bigcup_{k \in I}(c_k,d_k)$ with pairwise disjoint intervals (c_k,d_k), where $c_k, d_k \in E$ and I is either $\mathbb{N}$ or $\{1,\ldots,\sharp I\}$ if I is finite (and nonvoid). If I is finite, we put $A := \bigcup_{k \in I}\{c_k,d_k\}$ and $J := I$. Otherwise we choose $K \in \mathbb{N}$ with $K \geq 2$ such that $\sum_{k=K}^\infty \omega(f,[c_k,d_k]) < \varepsilon$. This is indeed possible since the series $\sum_{k=1}^\infty \omega(f,[c_k,d_k])$ converges because $f \in \mathrm{AC}_*(E,X)$ is bounded on E (due to the compactness of E and the continuity of f on E) and thus belongs to $\mathrm{BV}_*(E, X)$ (see Lemma 2.12). Then we put $A = \bigcup_{k=1}^{K-1} \{c_k, d_k\}$ and $J := \{1, \ldots, K-1\}$. In each of these two cases we set $J' := \{1, \ldots, 2 \cdot (\sharp J)\}$.

Since f is continuous at each point of E, we can find a function $\delta_1 : A \to (0, \infty)$ with $W_f(S) < \varepsilon$ for each $S \in \mathcal{S}(A, \delta_1)$ (cf. the solution to Exercise 9.9 in [Gor94]): Write $A = \{y_n : n \in J'\}$. For each $n \in J'$ there exists a number $\delta_1(y_n) > 0$ such that $\|f(y_n) - f(x)\| < \frac{\varepsilon}{2^{n+2}}$ for all $x \in [a, b]$ with $|y_n - x| < \delta_1(y_n)$. For each $S = \{([a_j, b_j], x_j)\}_{j=1}^r \in \mathcal{S}(A, \delta_1)$, we then obtain

$$\begin{aligned} W_f(S) &= \sum_{j=1}^r \|f(b_j) - f(a_j)\| \le \sum_{j=1}^r \|f(b_j) - f(x_j)\| + \sum_{j=1}^r \|f(x_j) - f(a_j)\| \\ &= \sum_{n \in J} \sum_{\substack{j=1 \\ x_j = y_n}}^r \|f(b_j) - f(x_j)\| + \sum_{n \in J} \sum_{\substack{j=1 \\ x_j = y_n}}^r \|f(x_j) - f(a_j)\| \\ &< 2 \sum_{n=1}^\infty \frac{\varepsilon}{2^{n+2}} + 2 \sum_{n=1}^\infty \frac{\varepsilon}{2^{n+2}} = \varepsilon, \end{aligned}$$

where we used that every set $\{j \in \{1, \ldots, r\} : x_j = y_n\}$ has at most two elements.

We now put $\delta(x) := \min\{x - c, d - x, \min_{y \in A} |x - y|\}$ for $x \in E \setminus (A \cup \{c, d\})$, $\delta(x) := \delta_1(x)$ for $x \in A$ and $\delta(x) := \min\{\rho, c - d, \min_{y \in A} |x - y|\}$ for $x \in \{c, d\} \setminus A$.

Take a partition $S = \{([a_j, b_j], x_j)\}_{j=1}^r$ in $\mathcal{S}(E, \delta)$ with $\sum_{j=1}^r |b_j - a_j| < \eta$. Because of

$$\sum_{j=1}^r \|f(b_j) - f(a_j)\| \le \sum_{j=1}^r \|f(x_j) - f(a_j)\| + \sum_{j=1}^r \|f(b_j) - f(x_j)\|,$$

$\{([a_j, x_j], x_j)\}_{j=1}^r \cup \{([x_j, b_j], x_j)\}_{j=1}^r \in \mathcal{S}(E, \delta)$ and $\sum_{j=1}^r (|b_j - x_j| + |x_j - a_j|) = \sum_{j=1}^r |b_j - a_j| < \eta$, we may assume without loss of generality that each tag x_j is an endpoint of its respective interval $[a_j, b_j]$.

Let S_A be that subset of S for which every tag x_j belongs to A. We denote by S_0 that subset of $S \setminus S_A$ for which both endpoints belong to E, by S_1 that subset of $S \setminus S_A$ for which the left endpoint does not belong to E and by S_2 that subset of $S \setminus S_A$ for which the right endpoint does not belong to E. The choice of δ (respectively δ_1) yields $W_f(S_A) < \varepsilon$ if $S_A \neq \emptyset$ (see above).

If $\emptyset \neq S_0 = \{([\alpha_i, \beta_i], \xi_i)\}_{i=1}^q$, we deduce

$$W_f(S_0) = \sum_{i=1}^q \|f(\beta_i) - f(\alpha_i)\| \le \sum_{i=1}^q \omega(f, [\alpha_i, \beta_i]) = V_f^*(S_0) < \varepsilon.$$

If I is finite, then a_j and b_j belong to E at least for each $([a_j, b_j], x_j) \in S \setminus S_A$ with $j \in \{1, \ldots, r\} \setminus \{1, r\}$ because of $U_{\delta(x_j)}(x_j) \cap A = \emptyset$. One can now proceed as in the 1st case above in order to obtain $W_f(S \setminus S_A) < 3\varepsilon$ (recall that each tag is a endpoint of its respective interval), provided that $S \setminus S_A \neq \emptyset$. For this reason we henceforth consider the case of an infinite I.

Let $\emptyset \neq S_1 = \{([u_i, v_i], v_i)\}_{i=1}^s$. The point u_i then belongs to $[c, d]$ at least for all $i \in \{1, \ldots, s\} \setminus \{1\}$. We distinguish between two cases.

Case 2.1: $u_i \in [c, d]$ for all $i \in \{1, \ldots, s\}$. In this case there exists for each i a unique index $k_i \in I \setminus J$ such that $c_{k_i} < u_i < d_{k_i}$ because $U_{\delta(x_j)}(x_j) \cap A = \emptyset$ and $u_i \notin E$. For $i, j \in \{1, \ldots, s\}$ with $i \neq j$ we have $k_i \neq k_j$ since the intervals $\{[u_i, v_i]\}_{i=1}^s$ do not overlap and since $d_{k_i} \le v_i$ and $d_{k_j} \le v_j$ due to $v_i, v_j \in E$. These facts further imply $d_{k_i} \le v_i \le c_{k_{i+1}}$ as well as

$$c_{k_i} < u_i < d_{k_i} \le v_i \le c_{k_{i+1}} < u_{i+1} < d_{k_{i+1}} \le v_{i+1}$$

for $i \in \{1, \ldots, s-1\}$ due to $v_i < u_{i+1}$, provided that $s > 1$. Hence, the intervals $\{[d_{k_i}, v_i]\}_{i=1}^s$ do not overlap, have endpoints in E and satisfy $\sum_{i=1}^{s}|v_i - d_{k_i}| \leq \sum_{i=1}^{s}|v_i - u_i| < \eta$. We derive

$$\begin{aligned}\sum_{i=1}^{s}\|f(v_i)-f(u_i)\| &\leq \sum_{i=1}^{s}\left(\|f(v_i)-f(d_{k_i})\| + \|f(d_{k_i})-f(u_i)\|\right)\\ &\leq \sum_{i=1}^{s}\left(\omega(f,[d_{k_i},v_i]) + \omega(f,[c_{k_i},d_{k_i}])\right)\\ &\leq \sum_{i=1}^{s}\omega(f,[d_{k_i},v_i]) + \sum_{k\in I\setminus J}\omega(f,[c_k,d_k]) < 2\varepsilon.\end{aligned}$$

Case 2.2: $u_i \in [c,d]$ for all $i \in \{1,\dots,s\}\setminus\{1\}$ and $u_1 \notin [c,d]$. Note that then $v_1 = c$. Applying case 2.1 to $\{([u_i,v_i],v_i)\}_{i=2}^{s}$, we estimate

$$\sum_{i=1}^{s}\|f(v_i)-f(u_i)\| = \|f(v_1)-f(u_1)\| + \sum_{i=2}^{s}\|f(v_i)-f(u_i)\| < 3\varepsilon.$$

Anyway, we obtain

$$\sum_{i=1}^{s}\|f(v_i)-f(u_i)\| < 3\varepsilon.$$

in each of the above cases.

The set S_2 can be treated analogously.

Summarizing, we arrive at

$$\sum_{j=1}^{r}\|f(b_j)-f(a_j)\| < \varepsilon + \varepsilon + 3\varepsilon + 3\varepsilon = 8\varepsilon.$$

This completes the proof. □

As an immediate corollary to the preceding lemma we get the subsequent result (for $E = [a,b]$ this result is due to Gordon, see, e.g., [Gor89, Theorem 5]).

5.9 Corollary. *Let $\emptyset \neq E \subseteq [a,b]$ be an F_σ-set and $f \in \mathrm{ACG}_*(E,X)$. If f is continuous at each point of E and bounded on the interval $[\inf E, \sup E]$, then $f \in \mathrm{ACG}_\delta(E,X)$.*

Proof. We choose a sequence $(E_n)_{n=1}^{\infty}$ of closed sets with $E = \bigcup_{n=1}^{\infty} E_n$ and a sequence $(A_n)_{n=1}^{\infty}$ of subsets of E with $\bigcup_{n=1}^{\infty} A_n = E$ such that $f \in \mathrm{AC}_*(A_n,X)$ for every $n \in \mathbb{N}$. Then also $f \in \mathrm{AC}_*(E_n \cap A_m, X)$ for all $n,m \in \mathbb{N}$. Thanks to Corollary 2.19, we also have $f \in \mathrm{AC}_*(B_{n,m},X)$ for all $n,m \in \mathbb{N}$, where $B_{n,m} \subseteq E_n$ is the closure of $E_n \cap A_m$. Lemma 5.8 now yields $f \in \mathrm{AC}_\delta(B_{n,m},X)$, which implies $f \in \mathrm{ACG}_\delta(E,X)$ because of $\bigcup_{n,m\in\mathbb{N}} B_{n,m} = E$. □

As a consequence of the preceding corollary we infer

$$\mathrm{ACG}_*([a,b],X) \cap \mathrm{C}([a,b],X) \subseteq \mathrm{ACG}_\delta([a,b],X)$$

But what about the converse inclusion? In [Gor89] Gordon proves that also $\mathrm{ACG}_\delta([a,b],\mathbb{R}) \subseteq \mathrm{ACG}_*([a,b],\mathbb{R})$ holds (see Theorem 6 in [Gor89]); however, his proof essentially relies on the fact that the functions considered there are real-valued. Therefore we have to invest some extra work in order to obtain results concerning this question. For this purpose we need some properties of ACG_δ-functions. The first result links the generalized variational absolute continuity to the (measure theoretical) absolute continuity with respect to Lebesgue measure.

5.10 Lemma. *Let $f \in X^{[a,b]}$ and $\emptyset \neq E \subseteq [a,b]$. Recall that λ denotes the one-dimensional (outer) Lebesgue measure.*

(a) If $f \in \mathrm{AC}_\delta(E,X)$, then $m_f \ll \lambda$ on E. The converse is also true, provided that $m_f(E) < \infty$ holds.

(b) If $f \in \mathrm{ACG}_\delta(E,X)$, then $m_f \ll \lambda$ on E. The converse is also true provided that m_f is σ-finite on E.

(c) Assume that E is an F_σ-set. We then have $f \in \mathrm{ACG}_\delta(E,X)$ if and only if $m_f \ll \lambda$ holds on E. In particular, f is continuous at each point of E in this case.

Proof. In order to verify part (a) one can use mutatis mutandis Skvortsov's and Zherebyov's proof of Lemma 1 in [SZ04] combined with Remark 1 there or one can directly apply the results just cited to the interval function τ_f given by $\tau_f(I) := \|f(\sup I) - f(\inf I)\|$. Part (b) easily follows from part (a) (see the proof of [SZ04, Theorem 1]; see also the proof of [Gor89, Lemma 2] for the first assertion in part (b)). Finally, assertion (c) is a direct consequence of part (b) combined with Corollary 3.16 and of Lemma 3.8. □

The next result is of interest in its own as it exposes a connection between variational Henstock-Thomson measures of continuous functions and the one-dimensional outer Hausdorff measure (cf. Theorem 43.1 in [Tho85] for the easier scalar-valued case).

5.11 Proposition. *Let $f \in \mathrm{C}([a,b],X)$. Then we have $\mathcal{H}^1_X(f(E)) \le 2m_f(E)$ for each $E \subseteq [a,b]$.*

Proof. The assertion is trivial if $E = \emptyset$ or $m_f(E) = \infty$. For this reason we may assume that E is non-empty and that $m_f(E) < \infty$. Let $\varepsilon > 0$ and $\eta > 0$. Then there exists a constant $\delta_1 > 0$ such that $\|f(t) - f(s)\| < \frac{\varepsilon}{2}$ for all $s,t \in [a,b]$ with $|s-t| < 2\delta_1$ and there exists a gauge $\delta_2 \in (0,\infty)^{[a,b]}$ such that $W_f(S) \le m_f(E) + \eta$ for all $S \in \mathcal{S}(E,\delta_2)$. We now put $\delta := \min\{\delta_1,\delta_2\}$. Applying the so-called covering lemma in [McL80] (see page 143), we obtain a (finite or infinite) sequence $\{([a_j,b_j],\xi_j)\}_{j\in J}$ (where $J = \{1,\dots,\sharp J\}$, if J is finite, and $J = \mathbb{N}$, if J is infinite) with

- $\xi_j \in [a_j,b_j] \cap E$ for all $j \in J$,
- $[a_j,b_j] \subseteq U_{\delta(\xi_j)}(\xi_j) \cap [a,b]$ for all $j \in J$,
- $(a_j,b_j) \cap (a_i,b_i) = \emptyset$ for for all $i,j \in J$ with $i \neq j$ and
- $E \subseteq \bigcup_j [a_j,b_j]$.

Clearly, $f(E) \subseteq \bigcup_j f([a_j,b_j])$ and for each $j \in J$ and all $s,t \in [a_j,b_j]$ we have $\|f(s) - f(t)\| < \frac{\varepsilon}{2}$ as $|s-t| < 2\delta(\xi_j) \le 2\delta_1$. Hence, $\operatorname{diam} f([a_j,b_j]) \le \frac{\varepsilon}{2} < \varepsilon$ for all $j \in J$. Let $k \in J$ and fix $s_j,t_j \in [a_j,b_j]$ for $j \in \{1,\dots,k\}$. We then estimate

$$\sum_{j=1}^k \|f(s_j) - f(t_j)\| \le \sum_{j=1}^k \|f(s_j) - f(\xi_j)\| + \sum_{j=1}^k \|f(\xi_j) - f(t_j)\| \le 2m_f(E) + 2\eta; \tag{5.3}$$

to see the last inequality notice that $\langle s_j,\xi_j\rangle \subseteq [a_j,b_j]$ and the intervals $[a_j,b_j]$ are non-overlapping. Therefore $\{(\langle s_j,\xi_j\rangle,\xi_j)\}_{j=1}^k \in \mathcal{S}(E,\delta) \subseteq \mathcal{S}(E,\delta_2)$. The same reasoning also shows that the system $\{(\langle t_j,\xi_j\rangle,\xi_j)\}_{j=1}^k$ is an element of $\mathcal{S}(E,\delta) \subseteq \mathcal{S}(E,\delta_2)$. Taking in (5.3) the supremum with respect to $s_j,t_j \in [a_j,b_j]$, we arrive at

$$\sum_{j=1}^k \omega(f,[a_j,b_j]) \le 2m_f(E) + 2\eta,$$

which yields $\sum_{j\in J} \omega(f,[a_j,b_j]) \le 2m_f(E) + 2\eta$. Therefore we have shown now

$$\mathcal{H}^1_\varepsilon(f(E)) \le \sum_{j\in J} \omega(f,[a_j,b_j]) \le 2m_f(E) + 2\eta.$$

Taking first the limit $\eta \to 0^+$ and afterwards $\varepsilon \to 0^+$, the assertion follows. □

Now we come to the announced result connecting ACG∗- and ACG$_\delta$-functions.

5.12 Theorem. *Let $E \subseteq [a,b]$ be a non-empty F_σ-set and assume that $f : [a,b] \to X$ is continuous at each point of E and bounde on the interval $[\inf(E), \sup(E)]$. Then we have $f \in \mathrm{ACG}*(E,X)$ if and only if $f \in \mathrm{ACG}_\delta(E,X)$. In particular, we obtain $\mathrm{ACG}*([a,b],X) \cap \mathrm{C}([a,b],X) = \mathrm{ACG}_\delta([a,b],X)$.*

Proof. The only-if part follows from Corollary 5.9. Let conversely $f \in \mathrm{ACG}_\delta(E,X)$. Lemma 5.10 implies $m_f \ll \lambda$ on E. This in turn yields $f \in (N)_E$ thanks to Proposition 5.11 as well as $f \in \mathrm{BVG}*(E,X)$ by Corollary 3.16. Now Corollary 2.27 implies $f \in \mathrm{ACG}*(E,X)$ as claimed.
The addendum is clear. □

We next consider a generalisation of ACG$_\delta$-functions.

5.13 Definition. *Let $F \in Z^{[a,b]}$ and $\varphi \in Y^{[a,b]}$. We say that a function F is of* variational absolute continuity on E with respect to φ *and we write $F \in \mathrm{AC}_{\delta,\varphi}(E,Z)$ if if for each $\varepsilon > 0$ there exists an $\eta > 0$ and a gauge $\delta \in (0,\infty)^E$ on E such that $W_F(S) < \varepsilon$ for all $S = \{([a_j,b_j],x_j)\}_{j=1}^r \in \mathcal{S}(E,\delta)$ with $W_\varphi(S) < \eta$. Moreover, we say that a function $F : [a,b] \to Z$ is of* generalized variational absolute continuity on E with respect to φ *and we write $F \in \mathrm{ACG}_{\delta,\varphi}(E,Z)$ if E is the countable union of sets on each of which f is of variational absolute continuity with respect to φ.*

We first note the following lemma, which gives an analogon to the first assertions of part a) and b) in Lemma 5.10 above.

5.14 Lemma. *Let $\emptyset \neq E \subseteq [a,b]$. Then the following statements hold.*

- *(a) If $F \in \mathrm{AC}_{\delta,\varphi}(E,Z)$, then $m_F \ll m_\varphi$ on E.*
- *(b) If $F \in \mathrm{ACG}_{\delta,\varphi}(E,Z)$, then $m_F \ll m_\varphi$ on E.*

Proof. Part b) follows easily from part a), so we turn to the proof of part a). Consider $\emptyset \neq A \subseteq E$ with $m_\varphi(A) = 0$ and let $\varepsilon > 0$. Because of $F \in \mathrm{AC}_{\delta,\varphi}(E,Z)$ we may find an $\eta > 0$ and a gauge $\delta_1 \in (0,\infty)^E$ with $W_F(S) < \varepsilon$ for all $S = \{([a_j,b_j],x_j)\}_{j=1}^r \in \mathcal{S}(E,\delta_1)$ with $W_\varphi(S) < \eta$. Due to $m_\varphi(A) = 0$, there exists $\delta_2 \in (0,\infty)^A$ such that $W_\varphi(S) < \eta$ for all $S = \{([a_j,b_j],x_j)\}_{j=1}^r \in \mathcal{S}(A,\delta_2)$. As a consequence, we deduce $W_F(S) < \varepsilon$ for each $S = \{([a_j,b_j],x_j)\}_{j=1}^r \in \mathcal{S}(A,\min\{\delta_1,\delta_2\})$, which shows $m_F(A) = 0$. □

With the preceding lemma at disposal we arrive at the following descriptive characterisation of the variational Henstock-Kurzweil-Stieltjes integral.

5.15 Theorem. *Let $\varphi \in \mathrm{BVG}*([a,b],Y)$. For $f \in X^{[a,b]}$ the following statements are equivalent.*

- *(a) The function f belongs to $\mathcal{HK}([a,b],\varphi,X)$.*
- *(b) There exists a function $F \in \mathrm{ACG}_{\delta,\varphi}([a,b],Z)$ such that f is a φ-Fréchet-derivative of F.*

Proof. Part (b) implies (a) by Lemma 5.14 and, e.g., Theorem 4.25. Conversely assume that (a) holds and let F be the primitive of f. We set $E_n := \{t \in [a,b] : \ n-1 \le \|f(t)\| < n\}$ for $n \in \mathbb{N}$. It suffices to verify $F \in \mathrm{AC}_{\delta,\varphi}(E_n,Z)$ for all $n \in \mathbb{N}$ in order to finish the proof. For this purpose fix $n \in \mathbb{N}$, let $\varepsilon > 0$ and pick $\delta \in (0,\infty)^{[a,b]}$ with $\sum_{j=1}^r \|F(b_j) - F(a_j) - f(t_j)\cdot[\varphi(b_j) - \varphi(a_j)]\| < \frac{\varepsilon}{2}$ for each $S = \{([a_j,b_j],t_j)\}_{j=1}^r \in \mathcal{S}([a,b],\delta)$. For every $S = \{([a_j,b_j],t_j)\}_{j=1}^r \in \mathcal{S}(E_n,\delta)$ with $W_\varphi(S) < \eta := \frac{\varepsilon}{2n}$ we now obtain

$$\begin{aligned} W_F(S) &\le \sum_{j=1}^r \|F(b_j) - F(a_j) - f(t_j)\cdot[\varphi(b_j) - \varphi(a_j)]\| + \sum_{j=1}^r \|f(t_j)\cdot[\varphi(b_j) - \varphi(a_j)]\| \\ &< \frac{\varepsilon}{2} + nW_\varphi(S) < \varepsilon, \end{aligned}$$

using that $t_j \in E_n$. □

Combining the preceding theorem with Theorem 5.12 we obtain the classical Lusin-type characterisation of the Henstock-Kurzweil integral in the vector-valued case.

5.16 Theorem. *For $f \in X^{[a,b]}$ the following statements are equivalent.*

(a) The function f belongs to $\mathcal{HK}([a,b],\varphi,X)$.

(b) There is a continuous function $F \in \mathrm{ACG}_*([a,b],Z)$ differentiable (in the ordinary sense) Lebesgue-a.e. such that $F' = f$ Lebesgue-a.e. on $[a,b]$.

5.17 Remark Theorem 5.16 is also stated in [SY05] as Theorem 7.4.5, but the proof given there has a gap: It rests on Theorem 7.4.3 whose proof is invalid (without going to far into details we content ourselves with the hint that the sequence $\{\overline{E}_{n,i}\}_{n,i}$ constructed in the proof of this Theorem 7.4.3 depends on a gauge δ, which itself depends on a fixed $\varepsilon > 0$, but the sequence must not depend on $\varepsilon > 0$!). The authors of [SY05] claim that the result is known and cite Proposition 4 in [PM02], however the notion of an ACG_*-function in [PM02] is indeed our notion of an ACG_δ-function and our notion of an ACG_*-function coincides with the corresponding notion in [SY05]. The classical proof for Theorem 5.16 in the real-valued function depends on the equivalence between the Henstock-Kurzweil integral and the Perron integral. But the latter one makes in general no sense for vector-valued function. Consequently, this classical proof cannot be transposed offhand to the vector-valued case. In [Gor89] Gordon has developed a proof strategy avoiding the Perron integral; this is indeed the source for the introduction of ACG_δ-functions. Although several proofs of [Gor89] use that the considered functions are real-valued (e.g., Gordon uses Corrollary 2.27 in the real-valued case, which seems to be completely new in the vector-valued case, and properties of the outer Lebesgue-measure that seem to have no analogon on the level of one-dimensional outer Hausdorff measures), his principal strategy served as a blue print for our approach. Moreover, there seems to be only one other reference where Theorem 5.16 is proved, namely [Sol01] (Theorem 2.2). Note however that the approach in [Sol01] is different from ours.

5.3 Banach spaces with the Radon-Nikodým property

Recall that a Banach space X has the so-called Radon-Nikodým property if and only if every Lipschitz continuous function defined on $[0,1]$ with values in X is Lebesgue-a.e. differentiable (in the ordinary sense) on $[0,1]$; consult [DU77] for further information. In this section we are concerned with a characterisation of Banach spaces possessing the Radon-Nikodým property, which is originally due to B. Bongiorno, L. Di Piazza and K. Musiał (see [BPM09b, Theorem 3.6]). It reads as follows.

5.18 Theorem. *Let λ be the one-dimensional (outer) Lebesgue measure on $[0,1]$ (which coincides with m_φ for φ the identity on $[0,1]$). For a Banach space $(X,\|\cdot\|)$ the following statements are equivalent.*

(a) The space $(X,\|\cdot\|)$ has the Radon-Nikodým property.

(b) Every function in $\mathrm{BVG}_*([0,1],X)$ is Lebesgue-a.e. differentiable.

(c) If $h:[0,1]\to X$ is a function such that m_h is σ-finite, then h is Lebesgue-a.e. differentiable.

(d) If $h:[0,1]\to X$ is a function with $m_h \ll \lambda$, then h is Lebesgue-a.e. differentiable.

(e) If $h:[0,1]\to X$ is a function with $m_h \ll \lambda$, then h is Lebesgue-a.e. differentiable with derivative $h' \in \mathcal{HK}([a,b],X)$ and with $\int_0^x h'(t)\,\mathrm{d}t = h(x)-h(0)$ for all $x\in[0,1]$ (where we put $h'(x)=0$ at non-differentiability points).

(f) If $h:[0,1]\to X$ is a function with $m_h \ll \lambda$, then h is an indefinite variational Henstock-Kurzweil integral.

Before proving this theorem, we want to explain why the proof of B. Bongiorno, L. Di Piazza and K. Musiał in [BPM09b] has a gap. This gap concerns the implication "(a) $\Longrightarrow$ (b)" above. For the following details consult the proof of the implication "(i) $\Longrightarrow$ (ii)" of Theorem 3.6 in [BPM09b].

In [BPM09b] the authors arrive at the inequality

$$\sum_i \|F[t_i, u_i]\| = \sum_i \|f(u_i) - f(t_i)\| \geq n \sum_i |u_i - t_i| > n\frac{|E_k \cap G|_e}{2} > M \quad \text{[sic!]}$$

(note that on the left side, there is a misprint: read Φ in lieu of F; moreover the intervals $[t_i, u_i]$ appearing at this point in the proof should be replaced by $[\min\{t_i, u_i\}, \max\{t_i, u_i\}]$; the same remark applies to the definition of the set $\mathcal{F}$ appearing there some lines before). Then the authors conclude that Φ is not of bounded variation in the restricted sense on E_k, which gives the desired contradiction completing their proof. However, we know that t_i belongs to E_k, but we only know that u_i is an element of $[0, 1]$ with $|u_i - t_i| < \frac{1}{n}$ and $\|f(t_i) - f(u_i)\| \geq n|u_i - t_i|$, where n is a certain fixed natural number. Thus one needs a criterion that guarantees that if Φ belongs to $\mathrm{BV}_*(E_k, X)$, then there exists some constant $M > 0$ such that $W_\Phi(S) \leq M$ holds for each $S = \{[a_j, b_j]\}_{j=1}^r \in \mathcal{A}([a, b])$ with $\{a_j, b_j\} \cap E_k \neq \emptyset$ for all $j \in \{1, \ldots, r\}$. One might think that this could be achieved by constructing intervals with both endpoints in E_k starting from such an S. Indeed, Lemma 6.15 in [Lee89] claims that this is a successful approach, but this lemma is false; see Lemma 2.4 and Corollary 2.5 above for correct versions. In fact, the lemmata just mentioned provide such criterions. However, Lemma 2.4 requires $u_i \in [\inf E_k, \sup E_k]$ and Lemma 2.4 requires Φ to be bounded and neither of these two conditions can be assumed to be satisfied in this general situation. Moreover, if we really want $W_\Phi(S) \leq M$ to hold for each $S = \{[a_j, b_j]\}_{j=1}^r \in \mathcal{A}([a, b])$ with $\{a_j, b_j\} \cap E_k \neq \emptyset$ for all $j \in \{1, \ldots, r\}$, then Φ is indeed necessarily bounded (cf. the proof of Corollary 2.5).
Therefore, we can rescue the current proof if we additionally assume Φ to be bounded. Then we can complete the proof of the implication "(i) $\Longrightarrow$ (ii)" using some extra reasoning; see the proof of the implication "(b) $\Longrightarrow$ (a)" of Lemma 5.19 below.
Another way to rescue the proof works as follows (here we use the notation of [BPM09b]): Set $c_k := \inf E_k$ and $d_k := \sup E_k$. If $|E_k \cap G|_e > 0$, then $|E_k \cap G \cap (c_k, d_k)|_e > 0$. Now work with $E_k \cap G \cap (c_k, d_k)$ instead of $E_k \cap G$ and choose u in the proof of Theorem 3.6 in [BPM09b] with the additional property $u \in (c_k, d_k)$, which is indeed possible. Then the proof goes through using our Lemma 2.4.

Now we come to our proof of Theorem 5.18. In comparison with the proof due to B. Bongiorno, L. Di Piazza and K. Musiał we make two observations, which are of some interest. First, our proof of the implication "(a) $\Longrightarrow$ (b)" is more elementary than the proof of the corresponding implication "(i) $\Longrightarrow$ (ii)" of Theorem 3.6 in [BPM09b] in so far as we can avoid the Vitali covering argument applied there by using one time more Lemma 2.8 (but we also use [Bon98, Proposition 1]).
Second, the next lemma shows that the above assertions (d), (e) and (f) are equivalent in a more general context; indeed, this is an almost immediate and natural outflow of our results obtained so far.

5.19 Lemma. *Let $\varphi \in \mathrm{BVG}_*([0, 1], Y)$ be continuous. Consider the following assertions.*

(a) Every function in $\mathrm{BVG}_*([0, 1], Z)$ is pseudo-φ-Roussel-differentiable m_φ-almost everywhere.

(b) Every bounded function in $\mathrm{BVG}_*([0, 1], Z)$ is m_φ-a.e. pseudo-φ-Roussel-differentiable.

(c) If $F : [0, 1] \to Z$ is a function such that m_F is σ-finite, then F is m_φ-a.e. pseudo-φ-Roussel-differentiable.

(d) If $F : [0, 1] \to Z$ is a function with $m_F \ll m_\varphi$, then F is m_φ-a.e. pseudo-φ-Roussel-differentiable.

(e) If $F : [0, 1] \to Z$ is a function with $m_F \ll m_\varphi$, then F possesses a m_φ-Roussel derivative $f \in \mathcal{HK}([a, b], \varphi, X)$ with $\int_0^x f(t) \cdot \mathrm{d}\varphi(t) = F(x) - F(0)$ for all $x \in [0, 1]$.

(f) If $F : [0, 1] \to Z$ is a function with $m_F \ll m_\varphi$, then F is an indefinite variational Henstock-Kurzweil-Stieltjes integral with respect to φ.

Then we have (a) $\iff$ (b) $\iff$ (c) $\implies$ (d) $\iff$ (e) $\iff$ (f).

Proof. (a) $\implies$ (b) and (e) $\implies$ (f) are clear.

(b) $\iff$ (c) is a direct consequence of Corollary 3.14.

Assume (b), let $F \in \mathrm{BVG}_*([0,1], Z)$ and choose a sequence $\{E_n\}_{n=1}^{\infty}$ of sets with $\bigcup_{n=1}^{\infty} E_n = [0,1]$ such that $F \in \mathrm{BV}_*(E_n, Z)$ for all $n \in \mathbb{N}$. Then F is bounded on each interval $[c_n, d_n] := [\inf E_n, \sup E_n]$ by Lemma 2.3. Extend $F_n := F|_{[c_n,d_n]}$ to $[0,1]$ by setting $F_n(t) := 0$ for $t \in [0,1] \setminus [c_n, d_n]$. Because of $F \in \mathrm{BVG}_*([c_n, d_n], Z)$ for each $n \in \mathbb{N}$ (notice that $\bigcup_{m\in\mathbb{N}}(E_m \cap [c_n, d_n]) = [c_n, d_n]$), we derive $F_n \in \mathrm{BVG}_*([0,1], Z)$. Since F_n is bounded, F_n is m_φ-a.e. pseudo-φ-Roussel-differentiable on $[0,1]$ due to (b) and thus a fortiori on each interval (c_n, d_n). As a result, F is m_φ-a.e. pseudo-φ-Roussel-differentiable on (c_n, d_n) and consequently on $[0,1]$ because by Lemma 3.8 the countable set $\bigcup_{n\in\mathbb{N}}\{c_n, d_n\}$ is an m_φ-null set due to the continuity of φ. So we have shown (a).

(c) $\implies$ (d) follows from Corollary 3.18.

Now assume that (d) holds and let $F : [0,1] \to Z$ be any function with $m_F \ll m_\varphi$. Due to (d) F has an m_φ-Roussel derivative f. Lemma 4.16 and Corollary 4.27 show that f is a φ-Roussel derivative of F with $f \in \mathcal{HK}([a,b], \varphi, X)$ and $\int_0^x f(t) \cdot \mathrm{d}\varphi(t) = F(x) - F(0)$ for all $x \in [0,1]$ (because of Theorem 4.25). Hence, (e) holds.

Finally, Corollary 4.28 and Lemma 4.16 yield (f) $\implies$ (d). □

Essentially the same arguments as in the preceding proof also establish the following result.

5.20 Lemma. *Let $\varphi \in \mathrm{BVG}_*([0,1], Y)$ be continuous. Consider the following assertions.*

(a) We have $m_\varphi(\{t \in [a,b] : \mathcal{FD}(F, \varphi, t) = \emptyset\}) = 0$ for every function $F \in \mathrm{BVG}_*([0,1], Z)$.

(b) We have $m_\varphi(\{t \in [a,b] : \mathcal{FD}(F, \varphi, t) = \emptyset\}) = 0$ for every bounded function $F \in \mathrm{BVG}_*([0,1], Z)$.

(c) If $F : [0,1] \to Z$ is any function such that m_F is σ-finite, then $m_\varphi(\{t \in [a,b] : \mathcal{FD}(F, \varphi, t) = \emptyset\}) = 0$.

(d) If $F : [0,1] \to Z$ is any function with $m_F \ll m_\varphi$, then $m_\varphi(\{t \in [a,b] : \mathcal{FD}(F, \varphi, t) = \emptyset\}) = 0$.

(e) If $F : [0,1] \to Z$ is any function with $m_F \ll m_\varphi$, then F possesses a φ-Fréchet derivative $f \in \mathcal{HK}([a,b], \varphi, X)$ with $\int_0^x f(t) \cdot \mathrm{d}\varphi(t) = F(x) - F(0)$ for all $x \in [0,1]$.

(f) If $F : [0,1] \to Z$ is any function with $m_F \ll m_\varphi$, then F is an indefinite variational Henstock-Kurzweil-Stieltjes integral with respect to φ.

Then we have (a) $\iff$ (b) $\iff$ (c) $\implies$ (d) $\iff$ (e) $\iff$ (f).

The next lemma gives a proof for the implication (a) $\implies$ (b) in Theorem 5.18.

5.21 Lemma. *If the Banach space $(X, \|\cdot\|)$ has the Radon-Nikodým property, then every function in $\mathrm{BVG}_*([0,1], X)$ is Lebesgue-a.e. differentiable.*

Proof. Let $f \in \mathrm{BVG}_*([0,1], X)$ be arbitrary and choose a strictly increasing function $\chi : [0,1] \to \mathbb{R}$ according to Lemma 2.8 and let A be a countable set with $\overline{\lim}_{y\to x} \frac{\|f(x)-f(y)\|}{|\chi(x)-\chi(y)|} < \infty$ for all $x \in [0,1] \setminus A$. By Lebesgue's differentiation theorem there exists a Lebesgue-null set N such that χ is differentiable on $[0,1] \setminus N$. As a consequence, we conclude that for every $x \in [0,1] \setminus (N \cup A)$ there exists a $\delta > 0$ and an $L > 0$ such that

$$\|f(x) - f(y)\| = \frac{\|f(x) - f(y)\|}{|\chi(x) - \chi(y)|} \cdot \frac{|\chi(x) - \chi(y)|}{|x - y|} \cdot |x - y| \le L\,|x - y|$$

for all $y \in [0,1]$ with $0 < |x - y| < \delta$. Thus the Stepanoff-type result Proposition 1 in [Bon98] implies that f is Lebesgue-a.e. differentiable on $[0,1] \setminus (N \cup A)$, hence on $[0,1]$. □

Now we have all ingredients to prove Theorem 5.18.

Proof of Theorem 5.18. Thanks to Lemma 5.19 and 5.21, it suffices to prove that (d) implies (a). One easily verifies $m_h \ll \lambda$ whenever h is a Lipschitz function. So (d) yields that every Lipschitz function $h : [0,1] \to X$ is Lebesgue-a.e. differentiable, which is equivalent to the Radon-Nikodým property. □

As an immediate corollary to Theorem 5.18 we obtain the following result.

5.22 Corollary. *Let $X = Z$, $Y = \mathbb{K}$, $F \in X^{[a,b]}$, let φ be the identity on $[a,b]$, let the bilinear mapping B be the usual multiplication with scalars and finally assume that X has the Radon-Nikodým property. The following assertions are equivalent.*

- *(a) The function F is an indefinite variational Henstock-Kurzweil integral.*
- *(b) We have $m_F \ll m_\varphi$.*

The remaining part of this section is now devoted to a substantial improvement of Lemma 5.21 giving a vector-valued extension of Proposition 3.10 in [Fau97].

5.23 Theorem. *Let $X = Z$, $Y = \mathbb{K}$, $\varphi \in \mathrm{BVG}_*([0,1], Y) \cap \mathrm{C}([0,1], Y)$, let the bilinear mapping B be the usual multiplication with scalars and finally assume that X has the Radon-Nikodým property. Then we have $m_\varphi(\{t \in [a,b] : \mathcal{FD}(F,\varphi,t) = \emptyset\}) = 0$ for all $F \in \mathrm{BVG}_*([0,1], X)$. In particular, as φ is bounded, F is m_φ-a.e. φ-Roussel differentiable.*

Proof. We divide the proof into several steps.

step 1: There exists a strictly increasing, continuous function $\chi : [0,1] \to \mathbb{R}$ and a countable subset $A \subseteq [0,1]$ such that

$$\overline{\lim_{s\to t}} \left| \frac{\varphi(t) - \varphi(s)}{\chi(t) - \chi(s)} \right| < \infty$$

for all $t \in [a,b] \setminus A$.
If φ is real-valued, the result may be found on page 237 of [Sak41]. According to this result choose χ_1 and A_1 resp. χ_2 and A_2 for $\mathrm{Re}\,\varphi$ resp. $\mathrm{Im}\,\varphi$; note that $\mathrm{Re}\,\varphi$, $\mathrm{Im}\,\varphi \in \mathrm{BVG}_*([0,1], Y) \cap \mathrm{C}([0,1], Y)$. Now put $\chi := \chi_1 + \chi_2$ and $A := A_1 \cup A_2$.

step 2: For χ as in step 1 we have $m_\varphi \ll m_\chi$.
Let $N \subseteq [0,1]$ with $m_\chi(N) = 0$ and let A be as in step 1. For $k \in \mathbb{N}$ we set

$$N_k := \left\{ t \in N \setminus A : \overline{\lim_{s\to t}} \left| \frac{\varphi(t) - \varphi(s)}{\chi(t) - \chi(s)} \right| < k \right\}.$$

Since A is countable and φ is continuous, we have $m_\varphi(A) = 0$. Due to $N = A \cup \bigcup_{k\in\mathbb{N}} N_k$, it suffices to establish $m_\varphi(N_k) = 0$ for each $k \in \mathbb{N}$. For this purpose fix $k \in \mathbb{N}$, let $\varepsilon > 0$ and choose a gauge $\widetilde{\delta}$ on N with $\sum_{j=1}^r |\chi(b_j) - \chi(a_j)| < \frac{\varepsilon}{k}$ for each $\{([a_j,b_j],t_j)\}_{j=1}^r \in \mathcal{S}(N,\widetilde{\delta})$. For every $t \in N_k$ pick $\delta(t) \in (0, \widetilde{\delta}(t))$ with $|\varphi(s) - \varphi(t)| < k|\chi(s) - \chi(t)|$ for all $s \in [0,1] \cap U_{\delta(t)}(t)$. For every system $\{([a_j,b_j],t_j)\}_{j=1}^r \in \mathcal{S}(N_k,\delta)$ we then have

$$\begin{aligned}
\sum_{j=1}^r |\varphi(b_j) - \varphi(a_j)| &\le \sum_{j=1}^r |\varphi(b_j) - \varphi(t_j)| + \sum_{j=1}^r |\varphi(t_j) - \varphi(a_j)| \\
&\le k \left(\sum_{j=1}^r |\chi(b_j) - \chi(t_j)| + \sum_{j=1}^r |\chi(t_j) - \chi(a_j)| \right) \\
&= k \sum_{j=1}^r |\chi(b_j) - \chi(a_j)| < \varepsilon,
\end{aligned}$$

which yields $m_\varphi(N_k) = 0$.

step 3: Let $G : [0,1] \to X$ be a function and let $\Psi : [0,1] \to \mathbb{R}$ be continuous and strictly increasing. Set

$$L := \{t \in [0,1] :\ \exists\, h > 0, C > 0\ \forall\, s \in [0,1] \cap U_h(t) :\ \|G(t) - G(s)\| \le C|\Psi(t) - \Psi(s)|\}.$$

There exists a set $N \subseteq L$ with $m_\Psi(N) = 0$ such that the limit

$$D_\Psi G(t) := \lim_{s\to\infty} \frac{G(t) - G(s)}{\Psi(t) - \Psi(s)}$$

exists in X for all $t \in L \setminus N$.
We introduce the function

$$H : [\Psi(0), \Psi(1)] \to X;\ \tau \mapsto G(\Psi^{-1}(\tau)).$$

Let $t \in L$ with the corresponding $h, C > 0$. Then there exists $\widetilde{h} \in L$ such that $|\Psi^{-1}(\sigma) - t| = |\Psi^{-1}(\sigma) - \Psi^{-1}(\Psi(t))| < h$ for all $\sigma \in [\Psi(0), \Psi(1)] \cap U_{\widetilde{h}}(\Psi(t))$. For each $\sigma \in [\Psi(0), \Psi(1)] \cap U_{\widetilde{h}}(\Psi(t))$ we thus obtain

$$\|H(\sigma) - H(\Psi(t))\| = \|G(\Psi^{-1}(\sigma)) - G(t)\| \le C|\Psi(\Psi^{-1}(\sigma)) - \Psi(t)| = C|\sigma - \Psi(t)|. \tag{5.4}$$

We put

$$\widetilde{L} := \{\tau \in [\Psi(0), \Psi(1)] :\ \exists\, \widetilde{h} > 0, \widetilde{C} > 0\ \forall\, \sigma \in [\Psi(0), \Psi(1)] \cap U_{\widetilde{h}}(\tau) :\ \|H(\tau) - H(\sigma)\| \le \widetilde{C}|\tau - \sigma|\}.$$

Thanks to the Stepanoff-type result Proposition 1 in [Bon98], there is a set $\widetilde{N} \subseteq [\Psi(0), \Psi(1)]$ with $\lambda(\widetilde{N}) = 0$ such that H is differentiable (in the ordinary sense) at each point of $\widetilde{L} \setminus \widetilde{N}$. Inequality (5.4) implies $\Psi(L) \subseteq \widetilde{L}$. We set $N := \Psi^{-1}(\widetilde{N})$. We then have $\lambda(\Psi(N)) = \lambda(\widetilde{N}) = 0$ and

$$\frac{G(t) - G(s)}{\Psi(t) - \Psi(s)} = \frac{H(\Psi(t)) - H(\Psi(s))}{\Psi(t) - \Psi(s)} \xrightarrow[s\to t]{} H'(\Psi(t))$$

for all $t \in L \setminus N$. Since Ψ is strictly increasing and continuous, one can easily verify that the set function $\lambda(\Psi(\cdot))$ coincides with the outer Lebesgue-Stieltjes measure ν_Ψ associated with Ψ. Applying Proposition A.3 therefore yields $m_\Psi(N) = 0$. More precisely, for $\varepsilon > 0$ choose intervals $I_1, \ldots, I_n$ covering N such that $\sum_{j=1}^n \nu_\Psi(I_j) < \varepsilon$. Using that m_Ψ is an outer measure, Proposition A.3 gives us $m_\Psi(N) \le \sum_{j=1}^n m_\Psi(I_j) = \sum_{j=1}^n \nu_\Psi(I_j) < \varepsilon$ and we obtain $m_\Psi(N) = 0$ as claimed.

step 4: Finishing the proof
Let χ be as in step 1. Furthermore, let $F \in \mathrm{BVG}_*([0,1], X)$ and choose for F a strictly increasing function χ_F and a countable set A_F according to Lemma 2.8. By Proposition 3.10 in [Fau97] there is a set $N_1 \subseteq [0,1]$ with $m_\chi(N_1) = 0$ such that the limit

$$D_\chi \chi_F(t) := \lim_{s\to t} \frac{\chi_F(s) - \chi_F(t)}{\chi(s) - \chi(t)}$$

exists for every $t \in [0,1] \setminus N_1$. As a consequence, for $t \in [0,1] \setminus (N_1 \cup A_F)$ there are $h > 0$ and $C > 0$ with

$$\|F(s) - F(t)\| = \frac{\|F(s) - F(t)\|}{|\chi_F(s) - \chi_F(t)|} \cdot \frac{|\chi_F(s) - \chi_F(t)|}{|\chi(s) - \chi(t)|} \cdot |\chi(s) - \chi(t)| \le C|\chi(s) - \chi(t)|$$

for all $s \in [0,1] \cap \dot{U}_h(t)$. By step 3 there exists $N_2 \subseteq [0,1]$ with $m_\chi(N_2) = 0$ such that the limit

$$D_\chi F(t) := \lim_{s\to t} \frac{F(s) - F(t)}{\chi(s) - \chi(t)}$$

exists in X for all $t \in [0,1] \setminus (N_1 \cup N_2 \cup A_F)$. We set $N_3 := N_1 \cup N_2 \cup A_F$ and note that $m_\chi(N_3) = 0$. Moreover, combining Lemma 3.9 and Proposition 3.10 both in [Fau97] with step 2 and the simple observation that $m_\varphi \leq m_{\mathrm{Re}\,\varphi} + m_{\mathrm{Im}\,\varphi}$, one can find an m_φ-null set $N_4 \subseteq [0,1]$ such that

$$D_\chi \varphi(t) := \lim_{s \to t} \frac{\varphi(s) - \varphi(t)}{\chi(s) - \chi(t)}$$

exists in $\mathbb{C} \setminus \{0\}$ for every $t \in [0,1] \setminus N_4$. In particular, for each $t \in [0,1] \setminus N_4$ we can find a $\delta(t) > 0$ such that $\varphi(s) \neq \varphi(t)$ for all $s \in \dot{U}_{\delta(t)}(t) \cap [0,1]$. We finally put $N := N_3 \cup N_4$ and observe that $m_\varphi(N) = 0$ by step 2. For $t \in [0,1] \setminus N$ and $s \in \dot{U}_{\delta(t)}(t) \cap [0,1]$ we now obtain

$$\frac{F(s) - F(t)}{\varphi(s) - \varphi(t)} = \frac{\chi(s) - \chi(t)}{\varphi(s) - \varphi(t)} \cdot \frac{F(s) - F(t)}{\chi(s) - \chi(t)} \xrightarrow[s \to t]{} (D_\chi \varphi(t))^{-1} \cdot D_\chi F(t).$$

We thus conclude $(D_\chi \varphi(t))^{-1} \cdot D_\chi F(t) \in \mathcal{FD}(F, \varphi, t)$ for all $t \in [0,1] \setminus N$. □

As corollary we now obtain a partial generalisation of Lemma 5.10.

5.24 Corollary. *Suppose that we are in the situation of Theorem 5.23. Then we have $F \in \mathrm{ACG}_{\delta,\varphi}([a,b], X)$ if and only of $m_F \ll m_\varphi$.*

Proof. The only-if part holds always true by Lemma 5.14. So assume conversely that $m_F \ll m_\varphi$. Since φ is continuous, this implies that F is continuous, too, as $m_F(\{t\}) = 0$ for all $t \in [a,b]$ (see Lemma 3.8); in particular $\mathcal{C}(F, \varphi, t) = X$ for all $t \in \mathbb{R}$. Thanks to Corollary 3.18, the hypothesis $m_F \ll m_\varphi$ further yields $F \in \mathrm{BVG}_*([a,b], X)$. Because of Theorem 5.23 we now conclude that F possesses a φ-Fréchet derivative. Therefore F is an indefinite Henstock-Kurzweil-Stieltjes integral with respect to φ due to Theorem 4.28. The proof of Theorem 5.15 now shows $F \in \mathrm{ACG}_{\delta,\varphi}([a,b], X)$ as claimed. □

5.4 Integration by parts

In this section we use Theorem 4.25 to obtain an integration by parts formula. For this we consider besides the bilinear mapping B two further bilinear mappings

$$B_1 : X \times Z \to X \qquad \text{and} \qquad B_2 : Z \times Z \to Z.$$

We write $x \cdot_1 z := B_1(x,z)$ and $z \cdot_2 \widetilde{z} := B_2(z, \widetilde{z})$ for $x \in X$ and $z, \widetilde{z} \in Z$. We assume that

- $\|x \cdot_1 z\| \leq \|x\| \cdot \|z\|$ for all $x \in X$ and $z \in Z$,
- $\|z \cdot_2 \widetilde{z}\| \leq \|z\| \cdot \|\widetilde{z}\|$ for all $z, \widetilde{z} \in Z$,
- $(x \cdot_1 z) \cdot y = (x \cdot y) \cdot_2 z$ for all $x \in X$, $y \in Y$ and $z \in Z$ and finally
- $z \cdot_2 \widetilde{z} = \widetilde{z} \cdot_2 z$ for all $z, \widetilde{z} \in Z$.

5.25 Example In the following cases the above conditions are satsified.

(a) $X = Y = Z$ is a commutative Banach algebra and $B = B_1 = B_2$ is the multiplication on X.

(b) $Z = \mathbb{K}$, B_1 is the multiplication by scalars and B_2 is the multiplication on $\mathbb{K}$.

(c) $X = Z$, $Y = \mathbb{K}$, B is the multiplication by scalars and $B_1 = B_2$ is a symmetric bilinear mapping with $\|B_1(x, \widetilde{x})\| \leq \|x\| \cdot \|\widetilde{x}\|$ for all $x, \widetilde{x} \in X$.

Here comes the announced result concerning integration by parts.

5.26 Theorem. *Let* $\varphi \in \mathrm{BVG}_*([a,b],Y)$*,* $f \in \mathcal{HK}([a,b],\varphi,X)$ *with primitive* F*, let* $g \in \mathcal{HK}([a,b],\varphi,X)$ *with primitive* G *and let* $h \in Z^{[a,b]}$*.*

(a) We have $h \in \mathcal{HK}([a,b],F,Z,B_2)$ *(this notation indicates that integration with respect to the bilinear mapping* B_2 *is used) if and only if* $f \cdot_1 h \in \mathcal{HK}([a,b],\varphi,X)$*. In this case we have*

$$\int_a^t h(\tau) \cdot_2 \mathrm{d}F(\tau) = \int_a^t (f \cdot_1 h)(\tau) \cdot \mathrm{d}\varphi(\tau)$$

for all $t \in [a,b]$*.*

(b) Assume that φ *is bounded and denote by* D *the countable point set of all discontinuities of* φ*. Furthermore, suppose that*

(i) for all $t \in D \setminus \{b\}$ *we have* $\lim_{\tau\to t^+} F(\tau) = F(t)$ *or* $\lim_{\tau\to t^+} G(\tau) = G(t)$*, and*

(ii) for all $t \in D \setminus \{a\}$ *we have* $\lim_{\tau\to t^-} F(\tau) = F(t)$ *or* $\lim_{\tau\to t^-} G(\tau) = G(t)$*.*

Then $(f \cdot_1 G) + (g \cdot_1 F) \in \mathcal{HK}([a,b],\varphi,X)$ *with*

$$\int_a^t [(f(\tau) \cdot_1 G(\tau)) + (g(\tau) \cdot_1 F(\tau))] \cdot \mathrm{d}\varphi(\tau) = F(t) \cdot_2 G(t)$$

for all $t \in [a,b]$*.*

(c) Under the hypotheses of part (b), $F \in \mathcal{HK}([a,b],G,Z,B_2)$ *if and only if* $G \in \mathcal{HK}([a,b],F,Z,B_2)$ *and if either of these two cases occur (and thus both of them), then*

$$\int_a^t F(\tau) \cdot_2 \mathrm{d}G(\tau) + \int_a^t G(\tau) \cdot_2 \mathrm{d}F(\tau) = F(t) \cdot_2 G(t)$$

for all $t \in [a,b]$*.*

Proof. on (a): Using the above assumed relations between B, B_1 and B_2, it easy to adjust the proof on page 264 of [McL80] for the scalar-valued case to our framework. We therefore omit the proof.

on b): We consider the two functions

$$H : [a,b] \to Z;\ t \mapsto F(t) \cdot_2 G(t)$$

and

$$\eta : [a,b] \to X;\ t \mapsto (f(t) \cdot_1 G(t)) + (g(t) \cdot_1 F(t)).$$

We shall show now that $m_H \ll m_\varphi$ and that η is a φ-Fréchet-derivative of H. Then Theorem 4.25 yields $\eta \in \mathcal{HK}([a,b],\varphi,X)$ with $\int_a^t \eta(s) \cdot \mathrm{d}\varphi(s) = H(t)$ for all $t \in [a,b]$, which is precisely the claim.

step 1: We have $\eta(t) \in \mathcal{C}(H,\varphi,t)$ *for all* $t \in [a,b]$*.* Fix $t \in [a,b]$ and pick $h \in \mathbb{R}\setminus\{0\}$ with $t+h \in [a,b]$. Employing the above relations between B, B_1 and B_2 we estimate

$$\begin{aligned}
&\|H(t+h) - H(t) - \eta(t) \cdot [\varphi(t+h) - \varphi(t)]\| \\
\le &\|F(t+h) \cdot_2 G(t+h) - F(t) \cdot_2 G(t+h) - [f(t) \cdot_1 G(t+h)] \cdot [\varphi(t+h) - \varphi(t)]\| \\
&+ \|[f(t) \cdot_1 G(t+h) - f(t) \cdot_1 G(t)] \cdot [\varphi(t+h) - \varphi(t)]\| \\
&+ \|F(t) \cdot_2 G(t+h) - F(t) \cdot_2 G(t) - [g(t) \cdot_1 F(t)] \cdot [\varphi(t+h) - \varphi(t)]\| \\
= &\|F(t+h) \cdot_2 G(t+h) - F(t) \cdot_2 G(t+h) - (f(t) \cdot [\varphi(t+h) - \varphi(t)]) \cdot_2 G(t+h)\| \\
&+ \|[f(t) \cdot_1 (G(t+h) - G(t))] \cdot [\varphi(t+h) - \varphi(t)]\| \\
&+ \|G(t+h) \cdot_2 F(t) - G(t) \cdot_2 F(t) - (g(t) \cdot [\varphi(t+h) - \varphi(t)]) \cdot_2 F(t)\|.
\end{aligned}$$

We further estimate

$$\begin{aligned} &\|H(t+h)-H(t)-\eta(t)\cdot[\varphi(t+h)-\varphi(t)]\| \\ \le&\|F(t+h)-F(t)-f(t)\cdot[\varphi(t+h)-\varphi(t)]\|\cdot\|G(t+h)\| \\ &+\|f(t)\|\cdot\|G(t+h)-G(t)\|\cdot\|\varphi(t+h)-\varphi(t)\| \\ &+\|G(t+h)-G(t)-g(t)\cdot[\varphi(t+h)-\varphi(t)]\|\cdot\|F(t)\|. \end{aligned} \tag{5.5}$$

By symmetry, we also obtain

$$\begin{aligned} &\|H(t+h)-H(t)-\eta(t)\cdot[\varphi(t+h)-\varphi(t)]\| \\ \le&\|G(t+h)-G(t)-g(t)\cdot[\varphi(t+h)-\varphi(t)]\|\cdot\|F(t+h)\| \\ &+\|g(t)\|\cdot\|F(t+h)-F(t)\|\cdot\|\varphi(t+h)-\varphi(t)\| \\ &+\|F(t+h)-F(t)-f(t)\cdot[\varphi(t+h)-\varphi(t)]\|\cdot\|G(t)\|. \end{aligned} \tag{5.6}$$

Recall from Corollary 4.10 that continuity points of φ are also continuity points of F and G and also recall that both F and G are φ-continuous (see Lemma 4.9). Hence, using condition (b) (i) or (b) (ii) and the hypothesis that φ is bounded, we conclude from the inequalities (5.5) and (5.6) $\eta(t)\in\mathcal{C}(H,\varphi,t)$ for all $t\in[a,b]$.

step 2: η is a Fréchet-derivative of H. Using, e.g., Theorem 4.22 we may choose an m_φ-null set N and a countable set D with $f(t)\in\mathcal{FD}(F,\varphi,t)$ and $g(t)\in\mathcal{FD}(G,\varphi,t)$ for all $t\in[a,b]\setminus(N\cup D)$. It suffices to verify $\eta(t)\in\mathcal{FD}(H,\varphi,t)$ for all $t\in[a,b]\setminus(N\cup D)$. Let $\varepsilon>0$, consider $t\in[a,b)\setminus(N\cup D)$ and assume that $\lim_{\tau\to t^+}F(\tau)=F(t)$. Then there is a $\rho>0$ such that

$$\|F(t+h)-F(t)-f(t)\cdot[\varphi(t+h)-\varphi(t)]\|\le\frac{\varepsilon}{3(\|G(t)\|+1)}\cdot\|\varphi(t+h)-\varphi(t)\|,$$

$$\|F(t+h)-F(t)\|<\min\left\{1,\frac{\varepsilon}{3(\|g(t)\|+1)}\right\}$$

and

$$\|G(t+h)-G(t)-g(t)\cdot[\varphi(t+h)-\varphi(t)]\|\le\frac{\varepsilon}{6(\|F(t)\|+1)}\cdot\|\varphi(t+h)-\varphi(t)\|$$

for all $h\in(0,\rho)$ with $t+h\in[a,b]$. For those h inequality (5.6) yields

$$\begin{aligned} &\|H(t+h)-H(t)-\eta(t)\cdot[\varphi(t+h)-\varphi(t)]\| \\ \le&\|G(t+h)-G(t)-g(t)\cdot[\varphi(t+h)-\varphi(t)]\|\cdot\|F(t+h)-F(t)\| \\ &+\|G(t+h)-G(t)-g(t)\cdot[\varphi(t+h)-\varphi(t)]\|\cdot\|F(t)\| \\ &+\|g(t)\|\cdot\|F(t+h)-F(t)\|\cdot\|\varphi(t+h)-\varphi(t)\| \\ &+\|F(t+h)-F(t)-f(t)\cdot[\varphi(t+h)-\varphi(t)]\|\cdot\|G(t)\| \\ \le&\frac{\varepsilon}{6(\|F(t)\|+1)}\cdot\|\varphi(t+h)-\varphi(t)\|+\frac{\varepsilon}{6(\|F(t)\|+1)}\cdot\|\varphi(t+h)-\varphi(t)\|\cdot\|F(t)\| \\ &+\|g(t)\|\cdot\frac{\varepsilon}{3(\|g(t)\|+1)}\cdot\|\varphi(t+h)-\varphi(t)\|+\frac{\varepsilon}{3(\|G(t)\|+1)}\cdot\|\varphi(t+h)-\varphi(t)\|\cdot\|G(t)\| \\ \le&\varepsilon\|\varphi(t+h)-\varphi(t)\|. \end{aligned}$$

Similarly, one treats the remaining cases $\lim_{\tau\to t^-}F(\tau)=F(t)$ and $\lim_{\tau\to t^\pm}G(\tau)=G(t)$.

step 3: $m_H\ll m_\varphi$. Let $\emptyset\neq M\subseteq[a,b]$ with $m_\varphi(M)=0$. In particular, $m_\varphi(\{t\})=0$ for all $t\in M$ and φ is therefore continuous at each point of M. Hence, F and G are also continuous at every point of M. In particular, for all $t\in M$ there exists $\delta_0(t)>0$ such that F and G are both bounded on

$U_{\delta_0(t)}(t) \cap [a,b]$. The family $\{U_{\delta_0(t)}(t)\}_{t\in M}$ forms an open cover of M. Since M is a Lindelöf space (as a separable metric space), there is a sequence $(\tau_n)_n$ in M such that $M \subseteq \bigcup_n U_{\delta_0(\tau_n)}(\tau_n)$. We put $U_1 := U_{\delta_0(\tau_1)}(\tau_1)$, $U_n := U_{\delta_0(\tau_n)}(\tau_n) \setminus \bigcup_{k=1}^{n-1} U_{\delta_0(\tau_k)}(\tau_k)$ for $n > 1$ and $M_n := M \cap U_n$. Furthermore, we set $C_n := \sup\{\max\{\|F(s)\|, \|G(s)\|\} : \ s \in U_{\delta_0(\tau_n)}(\tau_n) \cap [a,b]\}$ for $n \in \mathbb{N}$. Notice that $m_\varphi(M_n) = 0$ for all n because $M_n \subseteq M$. Hence, $m_F(M_n) = 0$ and $m_G(M_n) = 0$ for all $n \in \mathbb{N}$ since $m_F \ll m_\varphi$ and $m_G \ll m_\varphi$. Now let $\varepsilon > 0$ be arbitrary. For each $n \in \mathbb{N}$ and $\Phi \in \{F, G\}$ choose a gauge $\delta_{n,\Phi} \in (0,\infty)^{M_n}$ such that

$$\sum_{j=1}^{r} \|\Phi(b_j) - \Phi(a_j)\| < \frac{\varepsilon}{2^{n+1}(C_n + 1)}$$

for all $\{([a_j, b_j], t_j)\}_{j=1}^r \in \mathcal{S}(M_n, \delta_{n,\Phi})$. Moreover, for $n \in \mathbb{N}$ take $\delta_n \in (0,\infty)^{M_n}$ with $U_{\delta_n(t)}(t) \subseteq U_{\delta_0(\tau_n)}(\tau_n)$ for all $t \in M_n$. Finally, we define for $t \in M$ the gauge $\delta(t) := \min\{\delta_{n,F}(t), \delta_{n,G}(t), \delta_n(t)\}$ if $t \in M_n$ for a (unique) $n \in \mathbb{N}$. For each $\{([a_j, b_j], t_j)\}_{j=1}^r \in \mathcal{S}(M, \delta)$ we now derive

$$\begin{aligned}
&\sum_{j=1}^{r} \|H(b_j) - H(a_j)\| \\
&= \sum_{j=1}^{r} \|F(b_j) \cdot_2 G(b_j) - F(a_j) \cdot_2 G(a_j)\| \\
&\leq \sum_{j=1}^{r} \|F(b_j) - F(a_j)\| \cdot \|G(b_j)\| + \sum_{j=1}^{r} \|G(b_j) - G(a_j)\| \cdot \|F(a_j)\| \\
&= \sum_{n=1}^{\infty} \sum_{\substack{j=1 \\ t_j \in M_n}}^{r} \|F(b_j) - F(a_j)\| \cdot \|G(b_j)\| + \sum_{n=1}^{\infty} \sum_{\substack{j=1 \\ t_j \in M_n}}^{r} \|G(b_j) - G(a_j)\| \cdot \|F(a_j)\| \\
&\leq \sum_{n=1}^{\infty} C_n \sum_{\substack{j=1 \\ t_j \in M_n}}^{r} \|F(b_j) - F(a_j)\| + \sum_{n=1}^{\infty} C_n \sum_{\substack{j=1 \\ t_j \in M_n}}^{r} \|G(b_j) - G(a_j)\| \\
&< \sum_{n=1}^{\infty} C_n \cdot \frac{\varepsilon}{2^{n+1}(C_n + 1)} + \sum_{n=1}^{\infty} C_n \cdot \frac{\varepsilon}{2^{n+1}(C_n + 1)} < \varepsilon,
\end{aligned}$$

where we use that the systems $\{([a_j, b_j], t_j) : \ j \in \{1, \ldots, r\} \text{ with } t_j \in M_n\}$ belong to $\mathcal{S}(M_n, \delta_F)$ and $\mathcal{S}(M_n, \delta_G)$. As a result, we deduce $m_H(M) = 0$ as claimed.

on (c): Let $F \in \mathcal{HK}([a,b], G, Z, B_2)$. Then part (a) implies $g \cdot_1 F \in \mathcal{HK}([a,b], \varphi, X)$ with

$$\int_a^t F(s) \cdot_2 \mathrm{d}G(s) = \int_a^t (g \cdot_1 F)(s) \cdot \mathrm{d}\varphi(s)$$

for all $t \in [a,b]$. From part (b) we now deduce $f \cdot_1 G = ((f \cdot_1 G) + (g \cdot_1 F)) - (g \cdot_1 F) \in \mathcal{HK}([a,b], \varphi, X)$ and

$$\begin{aligned}
\int_a^t (f(s) \cdot_1 G(s)) \cdot \mathrm{d}\varphi(s) &= \int_a^t ((f(s) \cdot_1 G(s)) + (g(s) \cdot_1 F(s))) \cdot \mathrm{d}\varphi(s) - \int_a^t (g(s) \cdot_1 F(s)) \cdot \mathrm{d}\varphi(s) \\
&= F(t) \cdot_2 G(t) - \int_a^t F(s) \cdot_2 \mathrm{d}G(s)
\end{aligned}$$

for all $t \in [a,b]$. Due to part (a), the function G thus belongs to $\mathcal{HK}([a,b], F, Z, B_2)$ and

$$\int_a^t G(s) \cdot_2 \mathrm{d}F(s) = \int_a^t (f(s) \cdot_1 G(s)) \cdot \mathrm{d}\varphi(s) = F(t) \cdot_2 G(t) - \int_a^t F(s) \cdot_2 \mathrm{d}G(s)$$

for all $t \in [a,b]$. This establishes the only-if part and the addendum. Interchanging of the roles of F and G gives the converse implication. □

5.27 Remark Theorem 5.26 is a far reaching extension of the main theorem in [Pfe83], which needs much stronger assumptions (e.g., it is assumed that $\varphi \in \mathrm{BV}([a,b],\mathbb{R})$) actually needed in the proof given there. Moreover, Theorem 5.26 also weakens the hypotheses of Corollary 1 in [Nar04]. Furthermore, note that the proof of Corollary 1 in [Nar04] cannot be transposed offhand to the vector-valued case as it relies on Ward's Stieltjes-version of the Perron integral.

5.5 Normed algebras of differentiable functions on complact plane sets

In this last subsection we apply the results obtained so far to the study of certain normed algebras of differentiable functions. Throughout this section assume that $X = Y = Z = \mathbb{C}$ and B is the usual multiplication with scalars.

Let K be a non-empty, compact, perfect (i.e., K possesses no isolated points) subset of $\mathbb{C}$. A function $h : K \to X$ is called differentiable at $z_0 \in K$ if the limit

$$h'(z_0) := \lim_{K\setminus\{z_0\}\ni z\to z_0} \frac{h(z)-h(z_0)}{z-z_0}$$

exists in $\mathbb{C}$. We denote by $\mathcal{D}^1(K)$ the set of all functions $h : K \to \mathbb{C}$ that are everywhere differentiable on K such that h' is continuous. Then $\|h\|_{\mathcal{D}^1} := \|h\|_{\infty,K} + \|h'\|_{\infty,K}$ defines a norm on $\mathcal{D}^1(K)$ such that $(\mathcal{D}^1(K), \|\cdot\|_{\mathcal{D}^1})$ is a normed commutative algebra, which is in general not complete (see, e.g., [BF05] and [DF10]). These normed algebras and related ones have been extensively examined, see, e.g.,[BF05], [DF10], [Hof11] and the references therein. In the cited works a version of the fundamental theorem of calculus on rectifiable paths has proved very advantageous. More precisely, let $\gamma : [0,1] \to \mathbb{C}$ be a continuous path with $\gamma \in \mathrm{BVG}_*([0,1],\mathbb{C})$ such that $K := \gamma([0,1])$ is not a singleton (and then K is a perfect set) and let $h : K \to \mathbb{C}$ be a function differentiable everywhere on K. Now we want to integrate h' along the path γ; we can do this by defining

$$\int_\gamma h'(z)\,\mathrm{d}z := \int_0^1 h'(\gamma(t))\,\mathrm{d}\gamma(t),$$

$$(\mathcal{R})\int_\gamma h'(z)\,\mathrm{d}z := (\mathcal{R})\int_0^1 h'(\gamma(t))\,\mathrm{d}\gamma(t)$$

or

$$(\mathcal{L})\int_\gamma h'(z)\,\mathrm{d}z := (\mathcal{L})\int_0^1 h'(\gamma(t))\,\mathrm{d}\gamma(t),$$

respectively, provided the corresponding integral on the right-hand side exists.
Let $t_0 \in [0,1]$ and set $z_0 := \gamma(t_0)$. Then there is a $\delta' > 0$ such that $\|h(z)-h(z_0)-h'(z_0)(z-z_0)\| \leq \varepsilon|z-z_0|$ for all $z \in K$ with $|z-z_0| < \delta'$. Since γ is continuous, there exists a $\delta > 0$ such that $|\gamma(t)-\gamma(t_0)| < \delta'$ for all $t \in [0,1]$ with $|t-t_0| < \delta$. Hence,

$$\|h(\gamma(t)) - h(\gamma(t_0)) - h'((\gamma(t_0)))(\gamma(t)-\gamma(t_0))\| \leq \varepsilon|\gamma(t)-\gamma(t_0)|$$

for all $t \in [0,1]$ with $|t-t_0| < \delta$. As a consequence, $h\circ\gamma$ is differentiable at t_0 with respect to γ (see Definition 5.1) and we have $(h\circ\gamma)'_\gamma(t_0) = h'(\gamma(t_0))$.
Theorem 5.3 and Theorem 5.4 thus directly imply the following result.

5.28 Theorem. *Let $\gamma \in \mathrm{BVG}_*([0,1],\mathbb{K})$ be continuous such that $K := \gamma([0,1])$ is not a singleton and let $h : K \to \mathbb{C}$ be a function differentiable everywhere on K. Then the following assertions hold.*

(a) The integral $\int_\gamma h'(z)\,\mathrm{d}z$ exists with $\int_\gamma h'(z)\,\mathrm{d}z = h(\gamma(1)) - h(\gamma(0))$.

(b) *Assume that γ is rectifiable and that $(\mathcal{L})\int_0^1 h'(\gamma(t))\,\mathrm{d}\gamma(t)$ exists. Then*

$$(\mathcal{L})\int_\gamma h'(z)\,\mathrm{d}z = h(\gamma(1)) - h(\gamma(0))$$

holds.

(c) *Assume that $(\mathcal{R})\int_0^1 h'(\gamma(t))\,\mathrm{d}\gamma(t)$ exists. Then*

$$(\mathcal{R})\int_\gamma h'(z)\,\mathrm{d}z = h(\gamma(1)) - h(\gamma(0))$$

holds.

We add an immediate corollary to part (a) of Theorem 5.28.

5.29 Corollary. *Let $\emptyset \neq K \subseteq \mathbb{C}$ be a compact and perfect set such that for any two points $z, w \in K$ there exists a continuous path $\gamma \in \mathrm{BVG}_*([0,1],\mathbb{C})$ with $\gamma([0,1]) \subseteq K$ and with $\gamma(0) = z$ and $\gamma(1) = w$. Furthermore, let $h : K \to \mathbb{C}$ be differentiable everywhere on K with $h' = 0$. Then h is constant.*

5.30 Remark (a) Note that the conclusion of Corollary 5.29 was known so far only in the situation of Lemma 9.2 in [DF10] and for so-called rectifiably connected (see, e.g., [DF10, Definition 2.5]) compacta. But using Theorem 10.5 in [DF10] it is easy to find compact sets that are not captured by these two situations, to which, however, Corollary 5.29 may be applied.

(b) Part (c) of Theorem 5.28 for rectifiable γ and continuously differentiable h is stated in [BF05, Theorem 3.3] and in [Hof11, Theorem 2.6], however, apart from quite general hints without a proof. In [BF05], one can read that "Elegant proofs of this general result using the method of repeated bisection" (see [BF05, p. 95]) were communicated to W. J. Bland and J. F. Feinstein by G. R. Allan, T. W. Körner and W. K. Hayman, and that the proof of Allan can be found in the PhD thesis of Bland. Moreover, a weaker – although the proof also works in this situation – version of Theorem 5.28 (c) for rectifiable γ and continuously differentiable h is proved by D. Gaier in [Gai98] (Theorem 4).

(c) For rectifiable γ there is another elementary proof of part (c) of Theorem 5.28 hidden in Theorem 2.1 of [FR79].

(d) The prementioned proofs of Allan, Gaier and Fixman and Rubel are more elementary than ours because they do not invoke Henstock-Kurzweil integration theory, but their proofs are limited to rectifiable paths. In addition, the statements in 5.28 are optimal with regard to integrability conditions. Moreover, the value of our proof is also constituted by the demonstration that Theorem 5.28 is a natural and direct outflow of our results.

Although we needed, in contrast to the afore-mentioned simpler proofs, a bigger machinery to arrive at part (c) of Theorem 5.28, our results may nevertheless even lead to proofs of known results, which are from some point of view more natural than the existing proofs as our proofs feature the flavour of the ideas of elementary calculus in a greater measure. We want to illustrate this by considering the example of the so-called $\mathcal{F}$-derivative introduced in [BF05]. Indeed, compared to our proofs of [BF05, Theorem 4.9] and [BF05, Theorem 4.12] below, the proofs given in [BF05] seem to be a little bit ad-hoc, while our proofs are of more conceptual nature and well integrated in the abstract framework of our results.

For this purpose we fix a compact set $\emptyset \neq K \subseteq \mathbb{C}$ supporting a nonvoid system $\mathcal{F}$ of rectifiable paths in K such that for any path $\gamma \in \mathcal{F}$ defined on some Interval I the mapping $\gamma|_J$ is non-constant and we have $\gamma|_J \in \mathcal{F}$ for any non-degenerate subinterval J of I.
Let $H, h : K \to \mathbb{C}$ be two continuous functions. Then h is called an $\mathcal{F}$-derivative of H if

$$(\mathcal{R})\int_\gamma h(z)\,\mathrm{d}z = H(\gamma^+) - H(\gamma^-)$$

holds for each $\gamma \in \mathcal{F}$, where γ^- denotes the starting point and γ^+ denotes the endpoint of γ. We denote by $\mathcal{D}^1_{\mathcal{F}}(K)$ the set of all continuous functions $H : K \to \mathbb{C}$ having an $\mathcal{F}$-derivative. We then have the following result.

5.31 Proposition. *Let $H_1, H_2 \in \mathcal{D}^1_{\mathcal{F}}(K)$ with $\mathcal{F}$-derivatives h_1 and h_2, respectively. Then $h_1 H_2 + H_1 h_2$ is an $\mathcal{F}$-derivative of $H_1 H_2$.*

Proof. Consider $\gamma : [a, b] \to K$ in $\mathcal{F}$. By Lemma 4.3, Remark 4.2 (d) (or Theorem 5.3 and Theorem 5.4) and by assumption we have $h_j \circ \gamma \in \mathcal{HK}([a, b], \gamma, \mathbb{C})$ with

$$\int_a^t h_j(\gamma(s))\,\mathrm{d}\gamma(s) = (\mathcal{R})\int_a^t h_j(\gamma(s))\,\mathrm{d}\gamma(s) = H_j(\gamma(t)) - H_j(\gamma(a))$$

for all $t \in [a, b]$. Therefore part (b) of Theorem 5.26 combined with Lemma 4.3 and Remark 4.2 (d) (or combined with Theorem 5.3 and Theorem 5.4) yields

$$\begin{aligned}(\mathcal{R})\int_\gamma (h_1 H_2 + H_1 h_2)(z)\,\mathrm{d}z =&(\mathcal{R})\int_\gamma (h_1(H_2 - H_2(\gamma(a)) + (H_1 - H_1(\gamma(a))h_2)(z)\,\mathrm{d}z\\ &+ (\mathcal{R})\int_\gamma h_1(z)H_2(\gamma(a)) + H_1(\gamma(a))h_2(z)\,\mathrm{d}z\\ =&(H_1(\gamma(b)) - H_1(\gamma(a))(H_2(\gamma(b)) - H_2(\gamma(a))\\ &+ H_2(\gamma(a))(H_1(\gamma(b)) - H_1(\gamma(a))\\ &+ H_1(\gamma(a))(H_2(\gamma(b)) - H_2(\gamma(a))\\ =&H_1(\gamma(b))H_2(\gamma(b)) - H_1(\gamma(a))H_2(\gamma(a)).\end{aligned}$$

This finishes the proof. □

5.32 Remark Proposition 5.31 was proved in [BF05, Theorem 4.9] using the non-trivial Mergelyan's resp. Lavrentiev's approximation theorem in order to reduce the assertion to the special case of polynomials. In contrast to this approach our proof (taking into consideration the proof of Theorem 5.26) is in the spirit of the classical (and easy proof) for the product rule of differentiation.

The next result was first proved in [BF05, Theorem 4.12] using a bisection argument based on geometric properties of rectifiable paths.

5.33 Proposition. *We set*

$$\mathcal{F}(K) := \bigcup\{\gamma([c, d]) : \ \gamma : [c, d] \to K \text{ in } \mathcal{F}\}$$

and we assume that $\mathcal{F}(K)$ is dense in K. Let $h : K \to \mathbb{C}$ be a continuous function with $(\mathcal{R})\int_\gamma h(z)\,\mathrm{d}z = 0$ for all $\gamma \in \mathcal{F}$, then $h = 0$ holds.

Proof. Since $\mathcal{F}(K)$ is dense in K and since h is continuous, it suffices to show that $h|_{\gamma([a,b])} = 0$ is fulfilled for each $\gamma : [a, b] \to K$ in $\mathcal{F}$. Using Lemma 4.3 and Remark 4.2 (d) (or Theorem 5.3 and Theorem 5.4), we infer $h \circ \gamma \in \mathcal{HK}([a, b], \gamma, \mathbb{C})$ with

$$\int_a^t (h \circ \gamma)(s)\,\mathrm{d}\gamma(s) = (\mathcal{R})\int_a^t (h \circ \gamma)(s)\,\mathrm{d}\gamma(s) = 0$$

for all $t \in [a, b]$. We now consider the function

$$H_\gamma : [a, b] \to \mathbb{C};\ t \mapsto \int_a^t (h \circ \gamma)(s)\,\mathrm{d}\gamma(s).$$

We have $H_\gamma = 0$ and hence $0 \in \mathcal{FD}(H_\gamma, \gamma, t)$ and thus $\mathcal{FD}(H_\gamma, \gamma, t) = \{0\}$ for all $t \in [a, b]$ by part (c) of Remark 5.2, where part (c) of Remark 5.2 is applicable since γ does not vanish on any non-degenerate subinterval. On the other side, Theorem 4.22 gives us (taking into consideration that γ is continuous) a set $N \subseteq [a, b]$ with $m_\gamma(N) = 0$ such that $h(\gamma(t)) \in \mathcal{FD}(H_\gamma, \gamma, t) = \{0\}$ for all $t \in [a, b] \setminus N$. Thanks to Lemma 3.7 and the assumption that γ is not constant on any non-degenerate subinterval, we infer that N must have empty interior; in particular, $D \setminus N$ is dense in $[a, b]$. Hence, the continuity of $h \circ \gamma$ implies $h \circ \gamma = 0$. □

Appendix

A Henstock-Thomson variational measures and Lebesgue-Stieltjes measures

In this section our aim is to prove that for a function $\varphi \in \mathrm{BV}([a,b],Y)$ which is continuous from the right, the Henstock-Thomson variational measure coincides with the Lebesgue-Stieltjes measure associated with the total variation of φ.

We start with a result belonging to mathematical folklore, for which we could not find an appropriate reference. For this reason we provide a fairly complete proof. But before doing so we must fix some notation: Let Ω be a nonvoid set, $\mathcal{H}$ a semi-ring on it and $\nu : \mathcal{H} \to Y$ a set function on it. Then let $|\nu|$ denote the total variation of ν.

A.1 Lemma. *We consider $\Omega := (a,b]$ and the semi-ring $\mathcal{H} := \{(c,d] : a \le c \le d \le b\}$ on Ω. Let $\varphi \in \mathrm{BV}([a,b],Y)$ be continuous from the right. Then the vector-valued set function*

$$\tau_\varphi : \mathcal{H} \to Y;\ (c,d] \mapsto \varphi(d) - \varphi(c)$$

can be uniquely extended to a σ-additive measure

$$\nu_\varphi : \mathrm{Bor}(\Omega) \to Y,$$

which is of bounded variation. Moreover, an analogous statement is true for the total variation $|\tau_\varphi|$ of τ_φ and we have $\overline{|\tau_\varphi|} = |\nu_\varphi|$, where $\overline{|\tau_\varphi|}$ denotes the unique extension of $|\tau_\varphi|$ to $\mathrm{Bor}(\Omega)$

Proof. Clearly, τ_φ is finitely additive on $\mathcal{H}$. Let $\mathcal{R}$ be the ring generated by $\mathcal{H}$. It is known that

$$\mathcal{R} = \left\{ \bigcup_{j=1}^{n} A_j :\ n \in \mathbb{N},\ A_1,\ldots,A_n \in \mathcal{H} \text{ pairwise disjoint} \right\}. \tag{A.1}$$

Thus τ_φ can be extended to $\mathcal{R}$ by setting

$$\widetilde{\tau}_\varphi\left(\bigcup_{j=1}^{n} A_j\right) := \sum_{j=1}^{n} \tau_\varphi(A_j);$$

indeed, this gives a well-defined, finitely additive vector measure $\widetilde{\tau}_\varphi : \mathcal{R} \to Y$ (see, e.g., the proof of Proposition 1 in §5 of Chapter I in [Din67]) and, obviously, this extension is unique. Note that $\mathcal{R}$ is even an algebra because of $\Omega \in \mathcal{H} \subseteq \mathcal{R}$. The vector measure $\widetilde{\tau}_\varphi$ is of bounded variation (on $\mathcal{R}$): In fact, let $A \in \mathcal{R}$. Then

$$\begin{aligned} |\widetilde{\tau}_\varphi|(A) &= \sup\left\{ \sum_{j=1}^{n} \|\widetilde{\tau}_\varphi(A_j)\| :\ n \in \mathbb{N},\ A_1,\ldots,A_n \in \mathcal{R} \cap \mathfrak{P}(A) \text{ pairwise disjoint} \right\} \quad (\mathrm{A.2}) \\ &= \sup\left\{ \sum_{j=1}^{n} \|\tau_\varphi(I_j)\| :\ n \in \mathbb{N},\ I_1,\ldots,I_n \in \mathcal{H} \cap \mathfrak{P}(A) \text{ pairwise disjoint} \right\} \\ &\le \sup\left\{ \sum_{j=1}^{n} \|\varphi(t_j) - \varphi(t_{j-1})\| :\ n \in \mathbb{N},\ a = t_0 < \ldots < t_n = b \right\} = V(\varphi,[a,b]) < \infty, \end{aligned}$$

where we used (A.1) in the second equality. In particular, we further see that $|\widetilde{\tau}_\varphi|(A) = |\tau_\varphi|(A)$ for each $A \in \mathcal{H}$. Now let $y^\star \in Y^\star$. Then $\langle \varphi(\cdot), y^\star\rangle \in \mathrm{BV}([a,b],\mathbb{C})$ is right continuous. Hence, there are non-decreasing, right continuous functions $\varphi_1, \dots, \varphi_4 \in \mathrm{BV}([a,b],\mathbb{R})$ with $\langle \varphi(\cdot), y^\star\rangle = \varphi_1 - \varphi_2 + i(\varphi_3 - \varphi_4)$. Each of the Stieltjes contents

$$\tau_{\varphi_j} : \mathcal{H} \to [0,\infty);\ (c,d] \mapsto \varphi_j(d) - \varphi_j(c)$$

is σ-additive, thus a premeasure (cf., e.g., II.2.2 in [Els07]) and we have

$$\langle \tau_\varphi, y^\star\rangle = \tau_{\langle \varphi, y^\star\rangle} = \tau_{\varphi_1} - \tau_{\varphi_2} + i(\tau_{\varphi_3} - \tau_{\varphi_4})$$

on $\mathcal{H}$. Consequently, $\langle \tau_\varphi, y^\star\rangle$ is σ-additive on $\mathcal{H}$ with $\widetilde{\langle \tau_\varphi, y^\star\rangle} = \langle \widetilde{\tau}_\varphi, y^\star\rangle$, where $\widetilde{\langle \tau_\varphi, y^\star\rangle}$ denotes the unique finitely additive extension of $\langle \tau_\varphi, y^\star\rangle$ to $\mathcal{R}$. Thanks to Proposition 1 in §5 of Chapter I in [Din67], the set function $\widetilde{\langle \tau_\varphi, y^\star\rangle}$ is σ-additive and so is $\langle \widetilde{\tau}_\varphi, y^\star\rangle$, too. We conclude that $\widetilde{\tau}_\varphi$ is a weakly σ-additive vector measure. As $\widetilde{\tau}_\varphi$ is of bounded variation, it is even strongly additive due to Proposition I.1.15 in [DU77]. Now the Carathéodory-Hahn-Kluvanek extension theorem (see, e.g., Theorem I.5.2 in [DU77]) implies that $\widetilde{\tau}_\varphi$ possesses a unique σ-additive extension $\nu_\varphi : \sigma(\mathcal{R}) \to Y$, where $\sigma(\mathcal{R})$ is the σ-algebra generated by $\mathcal{R}$; notice that $\sigma(\mathcal{R}) = \mathrm{Bor}(\Omega)$. Because of $\nu_\varphi|_{\mathcal{R}} = \widetilde{\tau}_\varphi$, the set function $\widetilde{\tau}_\varphi$ is σ-additive on $\mathcal{R}$. Therefore $|\widetilde{\tau}_\varphi|$ is also σ-additive on $\mathcal{R}$ (see, e.g., §3 of Chapter I in [Din67], property 9) and it is of bounded variation, since it is even a finite content on $\mathcal{R}$ due to (A.2). As a consequence, $|\widetilde{\tau}_\varphi|$ is a finite premeasure on $\mathcal{R}$ and thus has a unique, σ-additive extension $\overline{|\widetilde{\tau}_\varphi|} : \mathrm{Bor}(\Omega) \to [0,\infty)$ by the usual extension procedure. Furthermore, we obtain $\|\widetilde{\tau}_\varphi(A)\| \le |\widetilde{\tau}_\varphi|(A) = \overline{|\widetilde{\tau}_\varphi|}(A)$ for all $A \in \mathcal{R}$. Employing Satz 2.14 in [Dec06] and Theorem 1 in §5 of [Din67] and using the fact that ν_φ is the unique σ-additive extension of $\widetilde{\tau}_\varphi$ to $\mathrm{Bor}(\Omega)$, we derive that $|\nu_\varphi|$ is of bounded variation (on $\mathrm{Bor}(\Omega)$) and extends $|\widetilde{\tau}_\varphi|$. Moreover, $|\nu_\varphi|$ is σ-additive on $\mathrm{Bor}(\Omega)$ by property 9 listed in §3 of [Din67]. The uniqueness of $\overline{|\widetilde{\tau}_\varphi|}$ now yields $|\nu_\varphi|(A) = \overline{|\widetilde{\tau}_\varphi|}(A)$ for all $A \in \mathrm{Bor}(\Omega)$. As τ_φ is additive on $\mathcal{H}$, so is $|\tau_\varphi|$ and $|\tau_\varphi|$ is finite. By the usual extension procedure for finite non-negative finitely additive set functions on semi-rings, $|\tau_\varphi|$ possesses a unique finite σ-additive extension $\overline{|\tau_\varphi|} : \mathrm{Bor}(\Omega) \to [0,\infty)$ (more precisely, we should write $\overline{\widetilde{|\tau_\varphi|}}$ instead of $\overline{|\tau_\varphi|}$). We next observe that

$$\overline{|\tau_\varphi|}(A) = |\tau_\varphi|(A) = |\widetilde{\tau}_\varphi|(A) = \overline{|\widetilde{\tau}_\varphi|}(A)$$

for all $A \in \mathcal{H}$, which yields $\overline{|\tau_\varphi|}(A) = \overline{|\widetilde{\tau}_\varphi|}(A)$ for all $A \in \mathcal{R}$. By once again utilising uniqueness, we conclude $\overline{|\tau_\varphi|} = \overline{|\widetilde{\tau}_\varphi|} = |\nu_\varphi|$ on $\mathrm{Bor}(\Omega)$. □

A.2 Corollary. *Under the hypotheses of Lemma A.1 and refering to the notation used there we obtain*

$$|\nu_\varphi|((c,d]) = \sup\left\{ \sum_{j=1}^n \|\varphi(d_j) - \varphi(c_j)\| : \begin{array}{c} n \in \mathbb{N},\ (c_1,d_1], \dots, (c_n,d_n] \in \mathcal{H} \text{ pairwise disjoint with} \\ \bigcup_{j=1}^n (c_j,d_j] = (c,d] \end{array} \right\}$$

for all $a \le c \le d \le b$.

Proof. The assertion follows from

$$|\nu_\varphi|((c,d]) = \overline{|\tau_\varphi|}((c,d]) = |\tau_\varphi|((c,d]).$$

□

A.3 Proposition. *Let* $\varphi \in \mathrm{BV}([a,b],Y)$ *be continuous from the right. Then* $|\nu_\varphi| = m_\varphi$ *on* $\mathrm{Bor}((a,b])$.

Proof. Let $a \le c < d \le b$. For $\varepsilon > 0$ we put

$$\mathcal{G}_\varepsilon := \{\delta \in (0,\infty)^{(c,d]} : \forall\, t \in (c,d) : U_{\delta(t)}(t) \subseteq (c,d) \text{ and } \forall\, s \in (d, d+\delta(d)) : \|\varphi(d) - \varphi(s)\| < \varepsilon\}.$$

Due to the right continuity of φ we infer $\mathcal{G}_\varepsilon \neq \emptyset$ for each $\varepsilon > 0$. Furthermore, one easily verifies

$$m_\varphi((c,d]) = \inf_{\delta \in (0,\infty)^{(c,d]}} \sup_{S \in \mathcal{S}((c,d],\delta)} W_\varphi(S) = \inf_{\varepsilon>0} \inf_{\delta \in \mathcal{G}_\varepsilon} \sup_{S \in \mathcal{S}((c,d],\delta)} W_\varphi(S).$$

Let $\varepsilon > 0$, $\delta \in \mathcal{G}_\varepsilon$ and $S = \{([a_j,b_j],t_j)\}_{j=1}^r \in \mathcal{S}((c,d],\delta)$. Then $d \in \bigcup_{j=1}^r [a_j,b_j]$ if and only if $t_r = d$. Hence, $\bigcup_{j=1}^r [a_j,b_j] \subseteq (c,d]$, if $t_r < d$, and $[a_r,d] \cup \bigcup_{j=1}^{r-1} [a_j,b_j] \subseteq (c,d]$, if $t_r = d$. In the first case, we have $S' := \{([a_j,b_j],t_j)\}_{j=1}^r \cup \{([d,d+\frac{\delta(d)}{2}],d)\} \in \mathcal{S}((c,d],\delta)$ with $W_\varphi(S) \leq W_\varphi(S')$. Setting $\mathcal{S}'((c,d],\delta) := \{S \in \mathcal{S}((c,d],\delta) : t_r = d\}$, we therefore conclude $\sup_{S\in\mathcal{S}((c,d],\delta)} W_\varphi(S) = \sup_{S\in\mathcal{S}'((c,d],\delta)} W_\varphi(S)$. As a consequence, $m_\varphi((c,d]) = \inf_{\varepsilon>0} \inf_{\delta\in\mathcal{G}_\varepsilon} \sup_{S\in\mathcal{S}'((c,d],\delta)} W_\varphi(S)$. Now fix $\varepsilon > 0$, let $\delta \in \mathcal{G}_\varepsilon$ and pick $S = \{([a_j,b_j],t_j)\}_{j=1}^r \in \mathcal{S}'((c,d],\delta)$. We choose closed intervals $[c_1,d_1],\dots,[c_m,d_m]$ such that $[a_r,d] \cup \bigcup_{k=1}^m [c_k,d_k] \cup \bigcup_{j=1}^{r-1} [a_j,b_j] = [c,d]$ and such that the intervals of the system $\{[a_j,b_j]\}_{j=1}^r \cup \{[c_k,d_k]\}_{k=1}^m$ are mutually non-overlapping. We estimate

$$\begin{aligned}
W_\varphi(S) &= \sum_{j=1}^r \|\varphi(b_j) - \varphi(a_j)\| \\
&\leq \sum_{j=1}^{r-1} \|\varphi(b_j) - \varphi(a_j)\| + \sum_{k=1}^m \|\varphi(d_k) - \varphi(c_k)\| + \|\varphi(d) - \varphi(a_r)\| + \|\varphi(d) - \varphi(b_r)\| \\
&\leq |\tau_\varphi|((c,d]) + \|\varphi(d) - \varphi(b_r)\| \leq |\tau_\varphi|((c,d]) + \varepsilon,
\end{aligned}$$

$$m_\varphi((c,d]) \leq \sup_{S\in\mathcal{S}'((c,d],\delta)} W_\varphi(S) \leq |\tau_\varphi|((c,d]) + \varepsilon,$$

which leads to $m_\varphi((c,d]) \leq \nu_\varphi|((c,d])$ by letting ε tend to 0.

Let now $\{(c_k,d_k]\}_{k=1}^m$ be pairwise disjoint elements (listed in increasing order) with $\bigcup_{k=1}^m (c_k,d_k] = (c,d]$; in particular, $c_1 = c$ and $d_m = d$. Fix $\varepsilon > 0$ and choose $\delta \in (0,\infty)^{(c,d]}$ with $W_\delta(\varphi,(c,d]) \leq m_\varphi((c,d]) + \varepsilon$. Since φ is continuous from the right, we can pick $\widetilde{c} \in (c,d_1)$ such that $\|\varphi(c) - \varphi(\widetilde{c})\| < \varepsilon$. Thanks to Cousin's lemma we can find $\{([a_{j,k},b_{j,k}],t_{j,k})\}_{j=1}^{r_k} \in \mathcal{S}(E_k,\delta|_{E_k})$ with $\bigcup_{j=1}^{r_k} [a_{j,k},b_{j,k}] = E_k$, where $E_1 := [\widetilde{c},d_1]$ and $E_k := [c_k,d_k]$ for $k \in \{2,\dots,m\}$. We then have

$$\{([a_{j,k},b_{j,k}],t_{j,k}) : \ k \in \{1,\dots,m\},\ j \in \{1,\dots,r_k\}\} \in \mathcal{S}\left(\bigcup_{k=1}^m E_k, \delta\right) = \mathcal{S}([\widetilde{c},d],\delta) \subseteq \mathcal{S}((c,d],\delta).$$

For this reason we obtain

$$\begin{aligned}
\sum_{j=1}^m \|\varphi(d_k) - \varphi(c_k)\| &\leq \|\varphi(\widetilde{c}) - \varphi(c)\| + \sum_{k=1}^m \sum_{j=1}^{r_k} \|\varphi(b_{j,k}) - \varphi(a_{j,k})\| \\
&\leq \varepsilon + W_\delta(\varphi,(c,d]) \leq m_\varphi((c,d]) + 2\varepsilon,
\end{aligned}$$

$$|\tau_\varphi|((c,d]) \leq m_\varphi((c,d]) + 2\varepsilon,$$

which implies $|\tau_\varphi|((c,d]) \leq m_\varphi((c,d])$ by letting $\varepsilon \to 0^+$. Consequently, $m_\varphi : \mathrm{Bor}((a,b]) \to [0,\infty)$ is a finite (due to Lemma 3.7), σ-additive measure that extends $|\tau_\varphi|$. By uniqueness we conclude $m_\varphi(A) = |\nu_\varphi|(A)$ for all $A \in \mathrm{Bor}((a,b])$. □

A.4 Corollary. *Let $\varphi \in \mathrm{BV}([a,b],Y)$ be continuous from the right. Then we have $\|\varphi(d) - \varphi(c)\| \leq m_\varphi((c,d])$ for all $a \leq c \leq d \leq b$.*

Proof. Using Proposition A.3 we obtain

$$\|\varphi(d) - \varphi(c)\| = \|\tau_\varphi((c,d])\| = \|\nu_\varphi((c,d])\| \leq |\nu_\varphi|((c,d]) = m_\varphi((c,d])$$

for all $a \leq c \leq d \leq b$. □

A.5 Remark. Analogous versions of the above results (with analogous proofs) hold true if $\varphi \in \mathrm{BV}([a,b],Y)$ is continuous from the left.

B The Vitali covering theorem for finite metric outer measures on the real line

The aim of this section is to establish the following very useful and general Vitali-type covering theorem.

B.1 Proposition. *Let $\mathcal{J}$ be a collection of closed intervals with non-empty interior that cover a nonvoid subset A of $\mathbb{R}$ in the (classical) Vitali sense, i.e., for each $t \in A$ and each $\varepsilon > 0$ there exists an interval $I \in \mathcal{J}$ with $t \in I$ and $\operatorname{diam}(I) < \varepsilon$. Moreover, let $\mu : \mathfrak{P}(\mathbb{R}) \to [0, \infty)$ be a finite metric outer measure, where $\mathfrak{P}(\mathbb{R})$ denotes the power set of $\mathbb{R}$. Then there exists a finite or infinite sequence $(I_n)_n$ of pairwise disjoint intervals belonging to $\mathcal{J}$ such that*

$$\lim_{N\to\infty} \mu\left(A \setminus \bigcup_{n=1}^{N} I_n\right) = 0,$$

if $(I_n)_n$ is infinite, and

$$\mu\left(A \setminus \bigcup_{n} I_n\right) = 0,$$

if $(I_n)_n$ is finite. In addition, one has $\mu\left(A \setminus \bigcup_n I_n\right) = 0$ in any case.

B.2 Remark. The addendum of the preceding proposition (resp. a generalisation of it) appears, e.g., in [dG75] or [Ise60]. But note that our assertion for infinite $(I_n)_n$ is a little bit preciser, in so far as one cannot conclude from the validity of the equation $\mu\left(A \setminus \bigcup_n I_n\right) = 0$ the validity of the relation $\lim_{N\to\infty} \mu\left(A \setminus \bigcup_{n=1}^{N} I_n\right) = 0$ if μ is not continuous from above. This preciser variant also appears implicitely in [dG75], but we cannot cite this result because de Guzman (in contrast to Iseki in [Ise60]) considers in [dG75] outer measures associated to usual measures. Therefore we shall give a detailed proof of the above proposition, where we closely follow the lines of the proof of Theorem 3 in Chapter X of [Doo94].

For the proof of Proposition B.1 we need the following auxiliary result due to Aldaz (for a proof consult, e.g., [Ald91]).

B.3 Lemma. *Let $\Sigma \subseteq \mathfrak{P}(\mathbb{R})$ be σ-algebra containing $\operatorname{Bor}(\mathbb{R})$ and let $\nu : \Sigma \to [0, \infty]$ be a measure. Furthermore, let Λ be a collection of intervals (not necessarily closed) with non-empty interior. Then for each $c \in (0, \frac{1}{2})$ there exists a finite subcollection $\Gamma \subseteq \Lambda$ of pairwise disjoint intervals such that $\nu(\bigcup \Gamma) \geq c \cdot \nu(\bigcup \Lambda)$. (Notice that $\bigcup \Lambda \in \operatorname{Bor}(\mathbb{R})$; see, e.g., Section 2 of Chapter X in [Doo94].)*

Proof of Proposition B.1. Denote by Σ_μ the σ-algebra of all μ-measurable (i.e., measurable in the sense of Carathéodory) subsets of $\mathbb{R}$. We have $\operatorname{Bor}(\mathbb{R}) \subseteq \Sigma_\mu$ as μ is a metric outer measure and we put $\nu := \mu|_{\operatorname{Bor}(\mathbb{R})}$. The set function $\nu : \operatorname{Bor}(\mathbb{R}) \to [0, \infty)$ is a finite measure and for $B \subseteq \mathbb{R}$ we put

$$\nu^*(B) := \inf\{\nu(C) :\ B \subseteq C \in \operatorname{Bor}(\mathbb{R})\} \leq \nu(\mathbb{R}) < \infty.$$

It is known that ν^* is a metric outer measure with $\nu^*|_{\operatorname{Bor}(\mathbb{R})} = \nu$. For $B \subseteq \mathbb{R}$ and $C \in \operatorname{Bor}(\mathbb{R})$ with $B \subseteq C$ we have $\mu(B) \leq \mu(C) = \nu(C)$, which implies $\mu(B) \leq \nu^*(B)$. Hence, it sufficies to establish the assertion for ν^* in lieu of μ.

If $\nu^*(A) = 0$, we have $\nu^*(A \setminus I) = 0$ for each $I \in \mathcal{J}$ and we are done in this case. So assume that $\nu^*(A) > 0$ and choose $\widetilde{A} \in \mathrm{Bor}(\mathbb{R})$ with $A \subseteq \widetilde{A}$ and $\nu(\widetilde{A}) < \frac{10}{9}\nu^*(A)$. Thanks to Ulam's theorem, each (locally finite) Borel measure is regular; in particular, there exists an open set G_1 satisfying $A \subseteq \widetilde{A} \subseteq G_1$ and $\nu(G_1) < \frac{10}{9} \cdot \nu^*(A)$. Let $\mathcal{J}_1 := \{I \in \mathcal{J} : \ I \subseteq G_1\}$ and observe that $\mathcal{J}_1$ still covers A in the Vitali sense. Applying Aldaz's lemma, we obtain a finite subcollection $\mathcal{F}_1 \subseteq \mathcal{J}_1$ of pairwise disjoint intervals such that $\nu(\bigcup \mathcal{F}_1) \geq \frac{1}{3}\nu(\bigcup \mathcal{J}_1) \geq \frac{1}{3}\nu^*(A)$. This leads to

$$\begin{aligned}\frac{1}{3}\nu^*(A) + \nu^*(A \setminus \bigcup \mathcal{F}_1) &\leq \nu(\bigcup \mathcal{F}_1) + \nu^*(A \setminus \bigcup \mathcal{F}_1) \leq \nu(\bigcup \mathcal{F}_1) + \nu^*(G_1 \setminus \bigcup \mathcal{F}_1)\\ &= \nu(G_1 \cap \bigcup \mathcal{F}_1) + \nu(G_1 \setminus \bigcup \mathcal{F}_1) = \nu(G_1) < \frac{10}{9} \cdot \nu^*(A)\end{aligned}$$

and consequently to $\nu^*(A \setminus \bigcup \mathcal{F}_1) \leq \frac{7}{9} \cdot \nu^*(A)$. Since $\bigcup \mathcal{F}_1$ is a finite union of closed sets, it is itself closed and therefore $\mathcal{K}_1 := \{I \in \mathcal{J}_1 : \ I \cap \bigcup \mathcal{F}_1 = \emptyset\}$ is a Vitali cover of the set $A \setminus \bigcup \mathcal{F}_1$, provided that $A \setminus \bigcup \mathcal{F}_1$ is non-empty. If $\nu^*(A \setminus \bigcup \mathcal{F}_1) = 0$, we are done. Otherwide we can repeat the above procedure: Choose an open set $G_2 \supseteq A \setminus \bigcup \mathcal{F}_1$ with $\nu(G_2) < \frac{10}{9}\nu^*(A \setminus \bigcup \mathcal{F}_1)$, put $\mathcal{J}_2 := \{I \in \mathcal{K}_1 : I \subseteq G_2\}$, apply Aldaz's lemma in order to obtain a finite, pairwise disjoint subcollection $\mathcal{F}_2 \subseteq \mathcal{J}_2$ with $\nu(\bigcup \mathcal{F}_2) \geq \frac{1}{3}\nu(\bigcup \mathcal{J}_2) \geq \frac{1}{3}\nu^*(A \setminus \bigcup \mathcal{F}_1)$. We then arrive at

$$\frac{1}{3}\nu^*(A \setminus \bigcup \mathcal{F}_1) + \nu^*((A \setminus \bigcup \mathcal{F}_1) \setminus \bigcup \mathcal{F}_2) \leq \nu(\bigcup \mathcal{F}_2) + \nu(G_2 \setminus \bigcup \mathcal{F}_2) = \nu(G_2) < \frac{10}{9}\nu^*(A \setminus \bigcup \mathcal{F}_1),$$

which implies

$$\nu^*(A \setminus \bigcup(\mathcal{F}_1 \cup \mathcal{F}_2)) \leq \frac{7}{9}\nu^*(A \setminus \bigcup \mathcal{F}_1) \leq \left(\frac{7}{9}\right)^2 \nu^*(A).$$

Notice that by construction all intervals in the finite collection $\mathcal{F}_1 \cup \mathcal{F}_2$ are mutually disjoint. It is now clear how itering this procedure finally produces the sequence $(I_n)_n$ in Proposition B.1.

The addendum is trivial if $(I_n)_n$ is finite and for infinite $(I_n)_n$ it follows from $\mu(A \setminus \bigcup_{n \in \mathbb{N}} I_n) \leq \mu(A \setminus \bigcup_{n=1}^{N} I_n)$, $N \in \mathbb{N}$. □

C The Bochner and the Henstock-Kurzweil-Stieltjes integral

The aim of this section is to prove that the variational Henstock-Kurzweil-Stieltjes integral contains the Lebesgue-Bochner integral. More precisely, we want to prove Proposition C.2 below. In order to formulate it, we need some notation.

C.1 Definition. *Let $\varphi \in \mathrm{BV}([a,b],\mathbb{C})$ be of bounded variation and continuous from the right. We say that $f : [a,b] \to X$ is* Lebesgue-Stieltjes integrable with respect to φ, *if f is Bochner-integrable with respect to the complex measure ν_φ associated with φ, i.e., f is Bochner-integrable with respect to the (positive) Lebesgue-Stieltjes measures $\nu_{\varphi_1}, \ldots, \nu_{\varphi_4}$, where $\varphi = \varphi_1 - \varphi_2 + i(\varphi_3 - \varphi_4)$ is the Jordan decomposition of φ (i.e., $\varphi_1, \ldots, \varphi_4$ are monotonically increasing and continuous from the right), see also Lemma A.1. We write $(\mathcal{L}) \int_a^b f(s)\,\mathrm{d}\varphi(s)$ for the Lebesgue-Stieltjes integral of f (over $[a,b]$) with respect to φ.*

The result we want to establish now reads as follows.

C.2 Proposition. *Assume that $\varphi \in \mathrm{BV}([a,b],\mathbb{C})$ is of bounded variation and continuous from the right and let $f : [a,b] \to X$ be Lebesgue-Stieltjes integrable with respect to φ. Then we have $f \in \mathcal{HK}([a,b],\varphi,X)$ with*

$$(\mathcal{L}) \int_a^t f(s)\,\mathrm{d}\varphi(s) = \int_a^t f(s)\,\mathrm{d}\varphi(s)$$

for all $t \in [a,b]$.

The crucial ingredient for the proof of Proposition C.2 is the following result.

C.3 Proposition. *Let Σ be a σ-algebra on $[a,b]$ which contains all Borel sets and let $\mu : \Sigma \to [0,\infty)$ be a (finite) measure which is outer and inner regular. Let $f : [a,b] \to X$ be Bochner-integrable with respect to μ. Then for every $\varepsilon > 0$ there exists a gauge $\delta : [a,b] \to (0,\infty)$ such that*

$$\sum_{j=1}^{r} \left\| \mu((a_j,b_j]) f(t_j) - \int_{(a_j,b_j]} f(t)\,\mathrm{d}\mu(t) \right\| < \varepsilon$$

for each $\{([a_j,b_j],t_j)\}_{j=1}^r \in \mathcal{S}([a,b],\delta)$.

It is tempting to simply cite [PM01, Lemma 1] in order to justify Proposition C.3, but this would be inopportune because the work [PM01] lacks a proof that the variational McShane integral considered in [PM01] reduces to the ordinary variational McShane integral if the measure space is just $[a,b]$ with an appropriate regular quasi-Radon measure. Indeed, for this assertion one has to verify that under these special circumstances one can replace the infinite sequence $(E_n)_n$ appearing in [PM01, Definition 1] by a finite sequence of intervals. At least for a generalized McShane integral results of this kind exist, see Proposition 1E and 1F in D. H. Fremlin's paper [Fre95]. However, at this point we refrain from establishing such general results (as well as from discussing far-reaching generalisations of Proposition C.3) and confine ourselves to a direct proof of Proposition C.3. We closely follow the lines of the proof of [Fre95, Theorem 1K].

Proof of Proposition C.3. We proceed in several steps.

step 1: The assertion is valid for integrable simple functions.
We first consider the case $f = \mathbb{1}_E x$, where $E \in \Sigma$ and $x \in X$. Let $\varepsilon > 0$ and choose an open set G and a closed set F in $[a,b]$ with $F \subseteq E \subseteq G$ such that $\mu(G \setminus F) < \frac{\varepsilon}{2(\|x\|+1)}$. For $t \in F$ we choose $\delta(t) > 0$ with $U_{\delta(t)}(t) \subseteq G$, for $t \in G \setminus F$ we take $\delta(t) > 0$ such that $U_{\delta(t)}(t) \subseteq G \setminus F$ and for $t \in [a,b] \setminus G$ we choose $\delta(t) > 0$ such that $U_{\delta(t)}(t) \subseteq [a,b] \setminus F$. For any $\{([a_j,b_j],t_j)\}_{j=1}^r \in \mathcal{S}([a,b],\delta)$ we then obtain

$$\begin{aligned}&\sum_{\substack{j=1\\t_j\notin G}}^r \left\|\mu((a_j,b_j])f(t_j) - \int_{(a_j,b_j]} f(t)\,\mathrm{d}\mu(t)\right\|\\ &= \sum_{\substack{j=1\\t_j\notin G}}^r \mu((a_j,b_j]\cap E)\|x\| = \mu\Bigg(\bigcup_{\substack{j=1\\t_j\notin G}}^r (a_j,b_j]\cap E\Bigg)\|x\| \le \mu(E\setminus F)\|x\|,\end{aligned}$$

due to $(a_j,b_j] \subseteq U_{\delta(t_j)}(t_j) \subseteq [a,b]\setminus F$ for $t_j \notin G$,

$$\begin{aligned}&\sum_{\substack{j=1\\t_j\in G\setminus E}}^r \left\|\mu((a_j,b_j])f(t_j) - \int_{(a_j,b_j]} f(t)\,\mathrm{d}\mu(t)\right\|\\ &= \sum_{\substack{j=1\\t_j\in G\setminus E}}^r \mu((a_j,b_j]\cap E)\|x\| = \mu\Bigg(\bigcup_{\substack{j=1\\t_j\in G\setminus E}}^r (a_j,b_j]\cap E\Bigg)\|x\|\\ &\le \mu((G\setminus F)\cap E)\|x\| = \mu(E\setminus F)\|x\|,\end{aligned}$$

because of $(a_j,b_j] \subseteq U_{\delta(t_j)}(t_j) \subseteq G\setminus F$ for $t_j \in G\setminus E \subseteq G\setminus F$,

$$\begin{aligned}&\sum_{\substack{j=1\\t_j\in E\setminus F}}^r \left\|\mu((a_j,b_j])f(t_j) - \int_{(a_j,b_j]} f(t)\,\mathrm{d}\mu(t)\right\|\\ &= \sum_{\substack{j=1\\t_j\in E\setminus F}}^r \mu((a_j,b_j]\setminus((a_j,b_j]\cap E))\|x\| = \mu\Bigg(\bigcup_{\substack{j=1\\t_j\in E\setminus F}}^r (a_j,b_j]\setminus E\Bigg)\|x\|\\ &\le \mu((G\setminus F)\setminus E)\|x\| = \mu(G\setminus E)\|x\|,\end{aligned}$$

since $(a_j,b_j] \subseteq U_{\delta(t_j)}(t_j) \subseteq G\setminus F$ for $t_j \in E\setminus F \subseteq G\setminus F$, and

$$\begin{aligned}&\sum_{\substack{j=1\\t_j\in F}}^r \left\|\mu((a_j,b_j])f(t_j) - \int_{(a_j,b_j]} f(t)\,\mathrm{d}\mu(t)\right\|\\ &= \sum_{\substack{j=1\\t_j\in F}}^r \mu((a_j,b_j]\setminus((a_j,b_j]\cap E))\|x\| = \mu\Bigg(\bigcup_{\substack{j=1\\t_j\in F}}^r (a_j,b_j]\setminus E\Bigg)\|x\| \le \mu(G\setminus E)\|x\|,\end{aligned}$$

because $(a_j,b_j] \subseteq U_{\delta(t_j)}(t_j) \subseteq G$ for $t_j \in F$. Summarizing, we arrive at

$$\sum_{j=1}^r \left\|\mu((a_j,b_j])f(t_j) - \int_{(a_j,b_j]} f(t)\,\mathrm{d}\mu(t)\right\| \le 2\|x\|(\mu(G\setminus E)+\mu(E\setminus F)) = 2\|x\|\mu(G\setminus F) < \varepsilon.$$

It is now easy to verify that the assertion of Proposition C.3 holds for every integrable simple function, i.e., for any linear combination of functions just considered.

step 2: Let $h : [a,b] \to [0,\infty)$ be μ-integrable and $\varepsilon > 0$ be arbitrary. Then there exists a gauge $\delta : [a,b] \to (0,\infty)$ with

$$\sum_{j=1}^{r} \mu((a_j,b_j])h(t_j) \leq \int_{[a,b]} h(t)\,\mathrm{d}\mu(t) + \varepsilon$$

for every $\{([a_j,b_j],t_j)\}_{j=1}^r \in \mathcal{S}([a,b],\delta)$.
The Vitali-Carathéodory Theorem (see, e.g., [Rud87, 2.24]) yields a lower semicontinuous function $v : [a,b] \to [0,\infty)$ with $h < h + \frac{\varepsilon}{2(\mu([a,b])+1)} \leq v$ and $\int_{[a,b]}(v - h - \frac{\varepsilon}{2(\mu([a,b])+1)})\,\mathrm{d}\mu < \frac{\varepsilon}{2}$. Hence $h < v$ everywhere and $\int_{[a,b]}(v-h)\,\mathrm{d}\mu < \varepsilon$. For $t \in [a,b]$ we take $\delta(t) > 0$ such that $U_{\delta(t)}(t) \subseteq \{s \in [a,b] : v(s) > h(t)\} \neq \emptyset$, which is possible because v is lower semicontinuous. Take $\{([a_j,b_j],t_j)\}_{j=1}^r \in \mathcal{S}([a,b],\delta)$. We then obtain

$$\begin{aligned}\sum_{j=1}^{r} \mu((a_j,b_j])h(t_j) &= \sum_{j=1}^{r}\int_{(a_j,b_j]} h(t_j)\,\mathrm{d}\mu(t) \leq \sum_{j=1}^{r}\int_{(a_j,b_j]} v(t)\,\mathrm{d}\mu(t)\\ &\leq \int_{[a,b]} v(t)\,\mathrm{d}\mu(t) \leq \int_{[a,b]} h(t)\,\mathrm{d}\mu(t) + \varepsilon\end{aligned}$$

as claimed.

step 3: Finishing the proof
Let f be a Bochner-integrable function and $\varepsilon > 0$. Choose an integrable simple function $g : [a,b] \to X$ such that $\int_{[a,b]} \|f-g\|\,\mathrm{d}\mu < \frac{\varepsilon}{4}$. Thanks to the step 1 and step 2, there is a gauge $\delta : [a,b] \to (0,\infty)$ such that

$$\sum_{j=1}^{r} \left\| \mu((a_j,b_j])g(t_j) - \int_{(a_j,b_j]} g(t)\,\mathrm{d}\mu(t) \right\| < \frac{\varepsilon}{4}$$

and

$$\sum_{j=1}^{r} \mu((a_j,b_j])h(t_j) \leq \int_{[a,b]} h(t)\,\mathrm{d}\mu(t) + \frac{\varepsilon}{4}$$

for every $\{([a_j,b_j],t_j)\}_{j=1}^r \in \mathcal{S}([a,b],\delta)$, where $h(t) := \|f(t)-g(t)\|$ for $t \in [a,b]$. For each sequence $\{([a_j,b_j],t_j)\}_{j=1}^r \in \mathcal{S}([a,b],\delta)$ we can now estimate

$$\begin{aligned}&\sum_{j=1}^{r} \left\| \mu((a_j,b_j])f(t_j) - \int_{(a_j,b_j]} f(t)\,\mathrm{d}\mu(t) \right\|\\ &\leq \sum_{j=1}^{r} \|\mu((a_j,b_j])f(t_j) - \mu((a_j,b_j])g(t_j)\| + \sum_{j=1}^{r} \left\| \mu((a_j,b_j])g(t_j) - \int_{(a_j,b_j]} g(t)\,\mathrm{d}\mu(t) \right\|\\ &\quad + \sum_{j=1}^{r} \left\| \int_{(a_j,b_j]} g(t)\,\mathrm{d}\mu(t) - \int_{(a_j,b_j]} f(t)\,\mathrm{d}\mu(t) \right\|\\ &< \int_{[a,b]} \|f(t)-g(t)\|\,\mathrm{d}\mu(t) + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \sum_{j=1}^{r}\int_{(a_j,b_j]} \|f(t)-g(t)\|\,\mathrm{d}\mu(t)\\ &< \frac{3\varepsilon}{4} + \int_{[a,b]} \|f(t)-g(t)\|\,\mathrm{d}\mu(t) < \varepsilon.\end{aligned}$$

This completes the proof. □

Proof of Proposition C.2. We write $\varphi = \varphi_1 - \varphi_2 + i(\varphi_3 - \varphi_4)$ with monotonically increasing functions $\varphi_1, \ldots, \varphi_4$ continuous from the right. Since each Lebesgue-Stieltjes measure ν_{φ_j} is a regular Radon measure with $\nu_{\varphi_j}((c,d]) = \varphi_j(d) - \varphi_j(c)$ for every $[c,d] \subseteq [a,b]$, Proposition C.3 therefore shows that

$$F_j : [a,b] \to X;\ t \mapsto (\mathcal{L}) \int_{(a,t]} f(s)\,\mathrm{d}\nu_{\varphi_j}$$

is a Henstock-Kurzweil primitive for f with respect to φ_j. Hence, $f \in \mathcal{HK}([a,b], \varphi_j)$ with

$$(\mathcal{L}) \int_a^t f(s)\,\mathrm{d}\varphi_j(s) = \int_a^t f(s)\,\mathrm{d}\varphi_j(s)$$

for all $t \in [a,b]$ and $j \in \{1, \dots, 4\}$. It is now easy to show that $f \in \mathcal{HK}([a,b], \varphi, X)$ with

$$\int_a^t f(s)\,\mathrm{d}\varphi(s) = (\mathcal{L}) \int_a^t f(s)\,\mathrm{d}\varphi(s)$$

for all $t \in [a,b]$. □

C.4 Remark.

(a) For the proof of Proposition C.3 it is irrelevant that the tag t_j belongs to the respective interval $[a_j, b_j]$; therefore Fremlin's proof shows in fact (as indicated by Di Piazza and Musiał in [PM01]) that Bochner-integrable functions are (under approriate circumstances) variationally McShane integrable (in a generalized sense).

(b) If X is finite-dimensional, then one easily reduces the assertion of Proposition C.3 to the case $X = \mathbb{R}$, which can be treated by an obvious modification of the proof in [DS70].

(c) Instead of interval-point sequences $\{([a_j, b_j], t_j)\}_{j=1}^r$ one might take any finite set-point sequence $\{(E_j, t_j)\}_{j=1}^r$ such that $E_j \in \Sigma$, $E_j \subseteq U_{\delta(t_j)}(t_j)$ (or $E_j \subseteq G(t_j)$, where $t_j \in G(t_j)$ is open) and $\mu(E_j \cap E_i) = 0$ for $i \neq j$ (cf. [PM01, Fre95, DS70]); then one can state and prove a result analogous to Proposition C.3.

D The continuity of the inverses of strictly monotonic functions

It is a fundamental question in analysis under which conditions the inverse of a continuous bijection, say between two topological spaces, is itself continuous. There are well-known results like the invariance of domain theorem or the classical (and easy to prove) result that the inverse of a continuous bijection from a compact space onto a Hausdorff space is also continuous.

It seems that results like the ones just mentioned have influenced the presentation of similar results on the level of undergraduate courses. So it seems that the following statement is most wide-spread in such courses.

If $\emptyset \neq I \subseteq \mathbb{R}$ is an interval and if $f : I \to \mathbb{R}$ is continuous and injective, then $f^{-1} : f(I) \to \mathbb{R}$ is continuous, too.

Usually, the proofs given for this result make use of the continuity of f in such a way that the continuity assumption appears to be indispensable at a first cursory glance. However, there is a more general result (see, e.g., [Heu00, 37.1]), which, unfortunately, seems to be seldom taught in undergraduate courses.

If $\emptyset \neq I \subseteq \mathbb{R}$ is an interval and if $f : I \to \mathbb{R}$ is strictly monotonic, then $f^{-1} : f(I) \to \mathbb{R}$ is continuous, too.

This statement demonstrates that the premise of the continuity of f is entirely superfluous and proofs based on this premise might disguise the deeper reason for this phenomenon. Indeed, from the point of view of topology, the true reason lies in the observation that a strictly monotonic function $f : I \to f(I)$ is a homeomorphism if I and $f(I)$ both carry the order topology induced by the order inherited from $\mathbb{R}$ instead of the usual subspace topology (see below). Since the subspace topology is finer than the order topology the mapping $f^{-1} : f(I) \to I$ is continuous if $f(I)$ is endowed with the subspace topology and I carries the order topology. But since for intervals the order and subspace topology coincide (see Lemma D.1 below), we conclude that $f^{-1} : f(I) \to I$ is continous where I and $f(I)$ now both carry the usual subspace topology.

Clearly, the same argument works for every strictly monotonic function $f : A \to \mathbb{R}$ ($\emptyset \neq A \subseteq \mathbb{R}$) whenever the order and subspace topology of A coincide. In this section we state a known characterisation (but for the sake of completeness and because we failed to find a good reference for it, we also provide a proof for it) of those non-empty subsets A of $\mathbb{R}$ for which the order and subspace topology coincide in order to complete our picture and to relate Proposition D.2 and Proposition D.5 below to the topological point of view just described. Moreover, we present an elementary proof for the assertion that $f^{-1} : f(A) \to A$ is always continuous provided that $f : A \to \mathbb{R}$ is strictly monotonic and the order and subspace topology of A coincide. Finally, we show that this result is optimal in the sense that on each non-empty subset A of $\mathbb{R}$ for which order and subspace topology differ there exists a strictly monotonic function f whose inverse is not continuous.

First of all, recall that the order topology on A is by definition the topology generated by all sets of the form $(-\infty, a) \cap A$ or $(a, \infty) \cap A$ where $a \in A$, while the subspace topology is, as one easily verifies, generated by all sets of the form $(-\infty, x) \cap A$ or $(x, \infty) \cap A$ where $x \in \mathbb{R}$.

D.1 Lemma. *Let $\emptyset \neq A \subseteq \mathbb{R}$. Then the order and subspace topology of A coincide if and only if every bounded component of $\mathbb{R} \setminus A$ is either closed or open.*

Proof. Let us suppose that $\mathbb{R} \setminus A$ possesses a bounded component which is neither closed nor open, i.e., there are $a, b \in \mathbb{R}$ with $a < b$ such that either $[a, b)$ or $(a, b]$ is a component of $\mathbb{R} \setminus A$. We only treat the first case since the second one can be handled analogously. In this case we have $a \notin A$, $b \in A$ and there exists a sequence $(x_n)_n$ in A that converges to a in $\mathbb{R}$. In particular, for every $x \in A$ with $x < b$ (which implies $x < a$), resp. for each $x' \in A$ with $x' > b$, there is an $n_0 \in \mathbb{N}$ with $x_n \in (x, \infty) \cap A$, resp. with $x_n \in (-\infty, x') \cap A$, for all $n \geq n_0$. Therefore $(x_n)_n$ converges to b with respect to the order topology on A.

If the order topology and the subspace topology of A coincided, then we could infer that $(x_n)_n$ converges to b in $\mathbb{R}$, which would yield $a = b$ in contrast to $a < b$. As a result, we deduce that the subspace topology of A is strictly finer than the order topology of A.

Now we conversely assume that each bounded component of $\mathbb{R} \setminus A$ is either closed or open. In order to show that in this case the order and subspace topology of A coincide, it sufficies to verify that each set of the form $(-\infty, \xi) \cap A$ or $(\xi, \infty) \cap A$, where $\xi \in \mathbb{R}$, is open with respect to the order topology on A. We show this only for $(-\infty, \xi) \cap A$ because the remaining case can be treated similarly.

In the cases $\xi \in A$, $(-\infty, \xi) \cap A = \emptyset$ or $(-\infty, \xi) \cap A = A$ the assertion is clear. Therefore we may assume that $\xi \notin A$ and $(-\infty, \xi) \cap A \neq \emptyset$ and $(-\infty, \xi) \cap A \neq A$ or equivalently that $\xi \notin A$ and $(-\infty, \xi) \cap A \neq \emptyset$ and $(\xi, \infty) \cap A \neq \emptyset$ hold. We denote by I that component of $\mathbb{R} \setminus A$ that contains ξ. Due to $(-\infty, \xi) \cap A \neq \emptyset$ and $(\xi, \infty) \cap A \neq \emptyset$, the set I is bounded. By assumption we therefore either have $I = [a, b]$ with $a \leq \xi \leq b$ and $a, b \in \mathbb{R} \setminus A$ or $I = (a, b)$ with $a < \xi < b$ and $a, b \in A$.

In the first case we can choose a strictly increasing sequence $(x_n)_n$ in A converging to a in $\mathbb{R}$. We then obtain

$$(-\infty, \xi) \cap A = (-\infty, a) \cap A = \bigcup_{n=1}^{\infty} (-\infty, x_n) \cap A,$$

so that $(-\infty, \xi) \cap A$ is a union of sets open with respect to the order topology on A and consequently itself open with respect to the order topology on A.

In the second case we observe that $[b, \infty) \cap A$ is (because of $b \in A$) closed with respect to the order topology on A. Therefore

$$(-\infty, \xi) \cap A = A \setminus ([b, \infty) \cap A)$$

is open with respect to the order topology on A. □

Now we come to the main result of this section.

D.2 Proposition. *Let $\emptyset \neq A \subseteq \mathbb{R}$ such that every bounded component of $\mathbb{R} \setminus A$ is either closed or open. Furthermore, let $f : A \to \mathbb{R}$ be a strictly monotonic function on A. Then the function $f^{-1} : f(A) \to \mathbb{R}$ is continuous.*

Proof. We suppose that f is strictly increasing (the case that f is strictly decreasing can be treated in a similar way).

Let $y_0 \in f(A)$ be arbitrary. We want to show that $f^{-1} : f(A) \to \mathbb{R}$ is continuous at y_0. For this purpose, let $(y_n)_n$ be an arbitrary convergent sequence in $f(A)$ with limit y_0. We then have to show that $(x_n)_n := (f^{-1}(y_n))_n \in A^{\mathbb{N}}$ converges to $x_0 := f^{-1}(y_0) \in A$.

It is easy to verify that there are $u, v \in f(A)$ with $u \leq v$ such that $y_n \in [u, v]$ for all $n \in \mathbb{N}_0$. We put $a := f^{-1}(u)$ and $b := f^{-1}(v)$. Then we have $x_n \in [a, b] \cap A$ for all $n \in \mathbb{N}_0$. In particular, the sequence $(x_n)_n$ is bounded and it thus suffices to verify that x_0 is its only possible limit point in order to conclude that $(x_n)_n$ converges to x_0, which completes the proof.

Suppose now that $(x_n)_n$ possesses a limit point ξ different from x_0 and let $(x_{n_k})_k$ be a subsequence converging to ξ. We then either have $\xi > x_0$ or $\xi < x_0$. We only treat the first case (the second one is analogous) and we shall show that we obtain a contradiction.

First, assume additionally that ξ does not belong to A and denote by I that component of $\mathbb{R} \setminus A$ that contains ξ. Observe that we have $\xi \in \partial I$ because of $\xi \in \partial A$, where ∂M denotes the boundary of a set $M \subseteq \mathbb{R}$.

If ξ is the left endpoint of I and if I is not a singleton, then there exists a $k_0 \in \mathbb{N}$ with $x_{n_{k_0}} \in (x_0, \xi)$ and an index $k_1 \in \mathbb{N}$ with $x_{n_k} \in (x_{n_{k_0}}, \xi)$ for all $k \geq k_1$. This yields

$$y_{n_k} = f(x_{n_k}) \geq f(x_{n_{k_0}}) > f(x_0) = y_0$$

for all $k \geq k_1$. As $k \to \infty$ we obtain the contradiction $y_0 \geq f(x_{n_{k_0}}) > y_0$.

If ξ is the right endpoint of I (which includes the case that I is a singleton), then I must be bounded due to $x_0 < \xi$. By assumption I is either closed or open, but due to $\xi \in \partial I \cap (\mathbb{R} \setminus A)$, the set I must be closed. Therefore we then have $I = [\alpha, \xi]$ with an $\alpha \leq \xi$ such that $\alpha \notin A$.

We may now choose an element $z \in (x_0, \alpha) \cap A$. (Note that this is indeed possible: If $\alpha < \xi$, this follows from $\alpha \in \partial A$ and $x_0 < \xi$, which yields $x_0 < \alpha$. If however $\alpha = \xi$, then $(x_0, \xi) \cap A$ is nonvoid since otherwise we would obtain $(x_0, \xi] \subseteq I = \{\xi\}$, which is impossible.) There exists a $k_0 \in \mathbb{N}$ such that $x_{n_k} > z$ for all $k \geq k_0$. This implies

$$y_{n_k} = f(x_{n_k}) \geq f(z) > f(x_0) = y_0$$

for all $k \geq k_0$ and we arrive at the contradition $y_0 \geq f(z) > y_0$.

Summarizing, we infer that ξ must be an element of A. Here we distinguish between two cases: $(x_0, \xi) \cap A \neq \emptyset$ or $(x_0, \xi) \cap A = \emptyset$. In the first case we choose $z \in (x_0, \xi) \cap A$ and proceed as in the above case where ξ was a right endpoint of the above I to arrive at a contradiction.

So let us assume that $(x_0, \xi) \cap A = \emptyset$. Then there exists a $k_0 \in \mathbb{N}$ such that $x_{n_k} \geq \xi$ for every $k \geq k_0$. This yields $y_{n_k} \in [f(\xi), \infty)$ for each $k \geq k_0$, which leads to the contradiction $y_0 \geq f(\xi) > f(x_0) = y_0$.

Altogether we arrive at the conclusion that $\xi > x_0$ ist not possible. □

Proposition D.2 gives rise to the following characterisation of the continuity of a strictly monotonic function.

D.3 Corollary. *Let $\emptyset \neq A \subseteq \mathbb{R}$ be such that every bounded component of $\mathbb{R} \setminus A$ is either closed or open. Then for a strictly monotonic function $f : A \to \mathbb{R}$ the following assertions are equivalent.*

- *(a) The function $f : A \to \mathbb{R}$ is continuous (for the subspace topology).*
- *(b) Each bounded component of $\mathbb{R} \setminus f(A)$ is either closed or open.*

In the case of equivalence the sets A and $f(A)$ are homeomorphic. Moreover, the implication "(b) $\Longrightarrow$ (a)" is still true if we drop the assumption imposed on A.

Proof. Applying Proposition D.2 to the function $f^{-1} : f(A) \to \mathbb{R}$ gives us the implication "(b) $\Longrightarrow$ (a)"; even without the assumption imposed on A.

Now assume that f is continuous as well as, without loss of generality, that f strictly increases. Furthermore, suppose to the contrary that $\mathbb{R} \setminus f(A)$ possesses a bounded component that is neither closed nor open, thus having the form $(u, v]$ or $[u, v)$. We only treat the first case.

Then $u \in f(A)$, $v \notin f(A)$ and there is a strictly decreasing sequence $(y_n)_n$ in $f(A)$ with limit v. We set $x_n := f^{-1}(y_n)$ for $n \in \mathbb{N}$ and $x := f^{-1}(u)$. The sequence $(x_n)_n$ is strictly decreasing and bounded from below by x, thus it converges to $\xi := \inf_{n \in \mathbb{N}} x_n$ in $\mathbb{R}$. The number ξ does not belong to A since otherwise the continuity of f would imply

$$v = \lim_{n \to \infty} y_n = \lim_{n \to \infty} f(x_n) = f(\xi) \in f(A),$$

which is impossible because of $v \notin f(A)$. Now consider an arbitrary $z \in A$ with $z > x$. We then have $f(A) \ni f(z) > f(x) = u$ and thus $f(z) > v$. Consequently, there exists an index $n \in \mathbb{N}$ with $v < y_n = f(x_n) < f(z)$, which implies $\xi < x_n < z$. We conclude that $(x, \xi]$ is a component of $\mathbb{R} \setminus A$ (because $x \in A$ and $A \ni x_n \to \xi \notin A$ as $n \to \infty$), which contradicts the assumption on A.

The first part of addendum is clear by Proposition D.2. □

D.4 Remark The characterisation of the continuity of strictly monotonic functions obtained in the preceding corollary fails if the adverb "strictly" is dropped. Indeed, just consider the function $f : \{\frac{1}{n};\ n \in \mathbb{N}\} \cup \{0\} \to \mathbb{R}$ given by $f(0) := 0$ and $f(\frac{1}{n}) = 1$ $(n \in \mathbb{N})$.

As announced we now demonstrate that Proposition D.2 is in some sense optimal.

D.5 Proposition. *Let $\emptyset \neq A \subseteq \mathbb{R}$ be a set such that $\mathbb{R} \setminus A$ possesses a bounded component that is neither closed nor open. Then there exists a strictly monotonic, continuous function $f : A \to \mathbb{R}$ such that the function $f^{-1} : f(A) \to \mathbb{R}$ is discontinuous.*

Proof. By assumption $\mathbb{R} \setminus A$ possesses a bounded component having the form $(a, b]$ or $[a, b)$ (with $a < b$). We only consider the first case since the second case is analogous.

Clearly, b is a cluster point of $(b, \infty) \cap A$. Therefore we can choose a strictly decreasing sequence $(x_n)_n$ in A converging to b. Moreover, we choose a strictly decreasing sequence $(y_n)_n$ in $\mathbb{R}$ with limit a. Now we put $g(x_n) := y_n$ for $n \in \mathbb{N}$ and $g(a) := a$ and we extend g on (x_{n+1}, x_n) linearly. This gives us a strictly increasing, continuous function $g : \{a\} \cup (b, x_1] \to \mathbb{R}$, which we extend to a strictly increasing, continuous function $g : (-\infty, a] \cup (b, \infty) \to \mathbb{R}$ in any way. Then the function $f := g|_A$ (note that $A \subseteq (-\infty, a] \cup (b, \infty)$) is strictly increasing and continuous, but its inverse is discontinuous at a. Indeed, we calculate $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} y_n = a = f(a)$, while $\lim_{n\to\infty} f^{-1}(y_n) = \lim_{n\to\infty} x_n = b \neq a = f^{-1}(a)$. □

D.6 Remark (a) By Proposition D.2, the function $g|_{(-\infty,a)\cup(b,\infty)}$ (where g is as in the proof of Proposition D.5) has a continuous inverse. Therefore the point a is the only discontinuity of the above f^{-1}.

(b) Combined with the order topological considerations in the introduction, Proposition D.5 furnishes a slightly different proof that the order and subspace topology of A do not coincide whenever $\mathbb{R} \setminus A$ possesses a bounded component that is neither closed nor open.

(c) If we combine Proposition D.2, D.5 and Lemma D.1, we arrive at the following result:
Let $\emptyset \neq A \subseteq \mathbb{R}$. Then the order and subspace topology of A coincide if and only if every (continuous) strictly monotonic function $f : A \to \mathbb{R}$ possesses a continuous inverse $f^{-1} : f(A) \to A$, where A and $f(A)$ are endowed with their respective subspace topologies.

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