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Complex Quantum Network Manifolds in Dimension $d > 2$ are Scale-Free

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Ginestra Bianconi[1] & Christoph Rahmede[2]

In quantum gravity, several approaches have been proposed until now for the quantum description of discrete geometries. These theoretical frameworks include loop quantum gravity, causal dynamical triangulations, causal sets, quantum graphity, and energetic spin networks. Most of these approaches describe discrete spaces as homogeneous network manifolds. Here we define Complex Quantum Network Manifolds (CQNM) describing the evolution of quantum network states, and constructed from growing simplicial complexes of dimension d. We show that in $d = 2$ CQNM are homogeneous networks while for $d > 2$ they are scale-free i.e. they are characterized by large inhomogeneities of degrees like most complex networks. From the self-organized evolution of CQNM quantum statistics emerge spontaneously. Here we define the generalized degrees associated with the δ-faces of the d-dimensional CQNMs, and we show that the statistics of these generalized degrees can either follow Fermi-Dirac, Boltzmann or Bose-Einstein distributions depending on the dimension of the δ-faces.

Several theoretical approaches have been proposed in quantum gravity for the description and characterization of quantum discrete spaces including loop quantum gravity[1–3], causal dynamical triangulations[4,5], causal sets[6,7], quantum graphity[8–10], energetic spin networks[11,12], and diffusion processes on such quantum geometries[13]. In most of these approaches, the discrete spaces are network manifold with homogeneous degree distribution and do not have common features with complex networks describing complex systems such as the brain or the biological networks in the cell. Nevertheless it has been discussed[14] that a consistent theory of quantum cosmology could also be a theory of self-organization[15,16], sharing some of its dynamical properties with complex systems and biological evolution.

In the last decades, the field of network theory[17–21] has made significant advances in the understanding of the underlying network topology of complex systems as diverse as the biological networks in the cell, the brain networks, or the Internet. Therefore an increasing interest is addressed to the study of quantum gravity from the information theory and complex network perspective[22,23].

In network theory it has been found that scale-free networks[24] characterizing highly inhomogeneous network structures are ubiquitous and characterize biological, technological and social systems[17–20]. Scale-free networks have finite average degree but infinite fluctuation of the degree distribution and in these structures nodes (also called "hubs") with a number of connections much bigger than the average degree emerge. Scale-free networks are known to be robust to random perturbation and there is a significant interplay between structure and dynamics, since critical phenomena such as in the Ising model, synchronization or epidemic spreading change their phase diagram when defined on them[25,26].

Interestingly, it has been shown that such networks, when they are evolving by a dynamics inspired by biological evolution, can be described by the Bose-Einstein statistics, and they might undergo a Bose-Einstein condensation in which a node is linked to a finite fraction of all the nodes of the network[27]. Similarly evolving Cayley trees have been shown to follow a Fermi-Dirac distribution[28,29].

[1]School of Mathematical Sciences, Queen Mary University of London, London E1 4NS, United Kingdom. [2]Institute for Theoretical Physics, Karlsruhe Institute of Technology, 76128 Karlsruhe, Germany. Correspondence and requests for materials should be addressed to G.B. (email: ginestra.bianconi@gmail.com)

Recently, in the field of complex networks increasing attention is devoted to the characterization of the geometry of complex networks[30–39]. In this context, special attention has been addressed to simplicial complexes[40–44], i.e. structures formed by gluing together simplices such as triangles, tetrahedra etc.

Here we focus our attention on Complex Quantum Network Manifolds (CQNMs) of dimension d constructed by gluing together simplices of dimension d. The CQNMs grow according to a non-equilibrium dynamics determined by the energies associated to its nodes, and have an emergent geometry, i.e. the geometry of the CQNM is not imposed a priori on the network manifold, but it is determined by its stochastic dynamics. Following a similar procedure as used in several other manuscripts[8–10,41], one can show that the CQNMs characterize the time evolution of the quantum network states. In particular, each network evolution can be considered as a possible path over which the path integral characterizing the quantum network states can be calculated. Here we show that in $d=2$ CQNMs are homogeneous and have an exponential degree distribution while the CQNMs are always scale-free for $d>2$. Therefore for $d=2$ the degree distribution of the CQNM has bounded fluctuations and is homogeneous while for $d=2$ the CQNM has unbounded fluctuations in the degree distribution and its structure is dominated by hub nodes. Moreover, in CQNM quantum statistics emerges spontaneously from the network dynamics. In fact, here we define the generalized degrees of the δ-faces forming the manifold and we show that the average of the generalized degrees of the δ-faces with energy ε follows different statistics (Fermi-Dirac, Boltzmann or Bose-Einstein statistics) depending on the dimensionality δ of the faces and on the dimensionality d of the CQNM. For example in $d=2$ the average of the generalized degree of the links follows a Fermi-Dirac distribution and the average of the generalized degrees of the nodes follows a Boltzmann distribution. In $d=3$ the faces of the tetrahedra, the links and the nodes have an average of their generalized degree that follows respectively the Fermi-Dirac distribution, the Boltzmann distribution and the Bose-Einstein distribution.

Consider a d-dimensional simplicial complex formed by gluing together simplices of dimension d, i.e. a triangle for $d=2$, a tetrahedron for $d=3$ etc. A necessary requirement for obtaining a discretization of a manifold is that each simplex of dimension d can be glued to another simplex only in such a way that the $(d-1)$-faces formed by $(d-1)$-dimensional simplices (links in $d=2$, triangles in $d=3$, etc.) belong at most to two simplices of dimension d.

Here we indicate with $\mathcal{S}_{d,\delta}$ the set of all δ-faces belonging to the d-dimensional manifold with $\delta<d$. If a $(d-1)$-face α belongs to two simplices of dimension d we will say that it is "saturated" and we indicate this by an associated variable ξ_α with value $\xi_\alpha=0$; if it belongs to only one simplicial complex of dimension d we will say that it is "unsaturated" and we will indicate this by setting $\xi_\alpha=1$.

The CQNM is evolving according to a non-equilibrium dynamics described in the following.

To each node $i=1, 2\ldots, N$ an *energy* of the node ϵ_i is assigned from a distribution $g(\epsilon)$. The energy of the node is quenched and does not change during the evolution of the network. To every δ-face $\alpha\in S_{d,\delta}$ we associate an *energy* ϵ_α given by the sum of the energy of the nodes that belong to the face α,

$$\epsilon_\alpha=\sum_{i\subset\alpha}\epsilon_i. \tag{1}$$

At time $t=1$ the CQNM is formed by a single d-dimensional simplex. At each time $t>1$ we add a simplex of dimension d to an unsaturated $(d-1)$-face $\alpha\in\mathcal{S}_{d,d-1}$ of dimension $d-1$. We choose this simplex with probability $\Pi\alpha$ given by

$$\Pi_\alpha=\frac{1}{Z}e^{-\beta\epsilon_\alpha}\xi_\alpha, \tag{2}$$

where β is a parameter of the model called *inverse temperature* and Z is a normalization sum given by

$$Z=\sum_{\alpha\in\mathcal{S}_{d,d-1}}e^{-\beta\epsilon_\alpha}\xi_\alpha. \tag{3}$$

Having chosen the $(d-1)$-face α, we glue to it a new d-dimensional simplex containing all the nodes of the $(d-1)$-face α plus the new node i. It follows that the new node i is linked to each node j belonging to α.

In Fig. 1 we show few steps of the evolution of a CQNM for the case $d=2$, while in Fig. 2 we show examples of CQNM in $d=2$ and in $d=3$ for different values of β.

From the definition of the non-equilibrium dynamics described above, it is immediate to show that the network structure constructed by this non-equilibrium dynamics is connected and is a discrete manifold.

Since at time t the number of nodes of the network manifold is $N=t+d$, the evolution of the network manifold is fully determined by the sequence $\{\epsilon_i\}_{i\leq t+d}$, and the sequence $\{\alpha_{t\prime}\}_{t\prime\leq\ldots t}$, where ϵ_i, for $i\leq d+1$ indicates the energy of an initial node, while for $i=t'+d$ with $t'>1$ it indicates the energy of the node added at time t', and where $\alpha_{t\prime}$ indicates the $(d-1)$-face to which the new d-dimensional simplex is added at time $t'>1$.

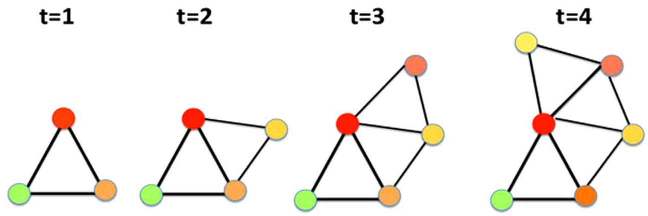

Figure 1. Evolution of the Complex Quantum Network Geometries in $d=2$. Few steps of a possible evolution of the CQNM for $d=2$. The nodes have different energies represented as different colours of the nodes. A link can be saturated (if two triangles are adjacent to it) or unsaturated (if only one triangle is incident to each). Starting from a single triangle at time $t=1$, the CQNM evolves through the addition of new triangles to unsaturated links.

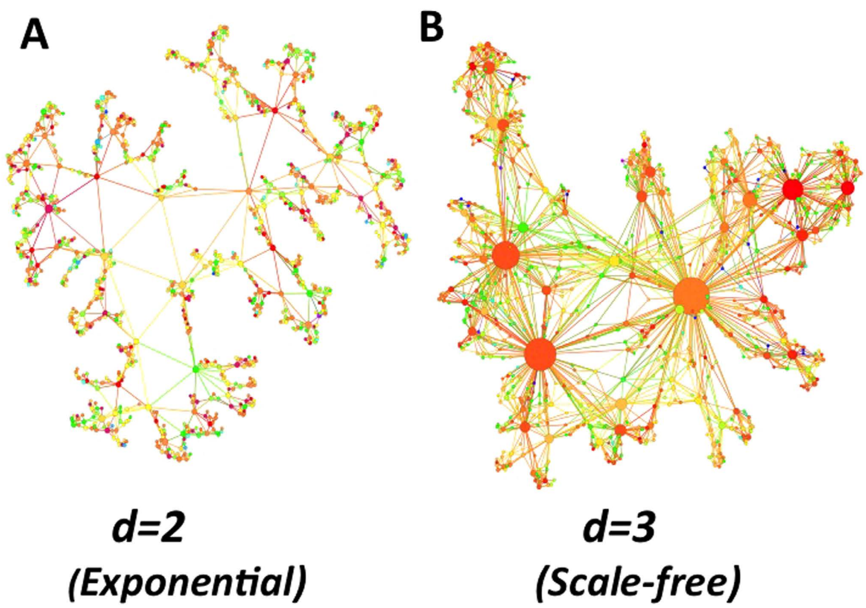

Figure 2. Visualization of Complex Quantum Network Geometries in dimensions $d=2,3$. Visualization of CQNM with $d=2$ (panel A) and $d=3$ (panel B). The colours of the nodes indicate their energy $\{\epsilon_i\}$ while their size indicates their degree $K_d(i)$. In $d=2$ the degree distribution of the CQNMs is a convolution of exponentials, in $d=3$ the CQNMs are scale-free and the presence of hubs is clearly observable from this visualization. The data shown are for CQNM with $N=10^3$ nodes, $\beta=0.2$ and Poisson distribution $g(\epsilon)$ with average $z=5$.

The dynamics described above is inspired by biological evolutionary dynamics and is related to self-organized critical models. In fact the case $\beta\to\infty$ is dictated by an extremal dynamics that can be related to invasion percolation[28,45], while the case $\beta=0$ can be identified as an Eden model[46] on a d-dimensional simplicial complex.

Here we call these network manifolds Complex Quantum Network Manifolds because using similar arguments already developed in[8–10,41] it can be shown that they describe the evolution of Quantum Network States (see Methods and Supplementary Information for details). The quantum network state is an element of an Hilbert space $\mathcal{H}_{tot}$ associated to a simplicial complex of N nodes formed by gluing d -dimensional simplices (see Methods and Supplementary Information for details). The quantum network state $|\psi(t)\rangle$ evolves through a Markovian non-equilibrium dynamics determined by the energies $\{\epsilon_i\}$ of the nodes. The quantity $\mathcal{Z}(t)$ enforcing the normalization of the quantum network state $\langle\psi(t)|\psi(t)\rangle=1$ can be interpreted as a path integral over CQNM evolutions determined by the sequences $\{\epsilon_i\}_{i\leq t+d}$, and $\{\alpha_{t'}\}_{t'\leq t}$. In fact we have

$$\mathcal{Z}(t)=\sum_{\{\epsilon_i\}_{i\leq t+d}}\sum_{\{\alpha_{t'}\}_{t'\leq t}}W(\{\epsilon_i\}_{i\leq t+d},\{\alpha_{t'}\}_{t'\leq t}),\tag{4}$$

where the explicit expression of $W(\{\epsilon_i\}_{i\leq t+d}, \{\alpha_{t'}\}_{t'\leq t})$ is given in the Supplementary Information. Moreover, $\mathcal{Z}(t)$ can be interpreted as the partition function of the statistical mechanics problem over the CQNM temporal evolutions. If we identify the sequences $\{\epsilon_i\}_{i\leq t+d}$, and $\{\alpha_{t'}\}_{t'\leq t}$, determining $\mathcal{Z}(t)$ with the sequences indicating the temporal evolution of the CQNM we have that the probability $P(\{\epsilon_i\}_{i\leq t+d}, \{\alpha_{t'}\}_{t'\leq t})$ of a given CQNM evolution is given by

$$P(\{\epsilon_i\}_{i\leq t+d}, \{\alpha_{t'}\}_{t'\leq t}) = \frac{W(\{\epsilon_i\}_{i\leq t+d}, \{\alpha_{t'}\}_{t'\leq t})}{\mathcal{Z}(t)}. \tag{5}$$

Therefore each classical evolution of the CQNM up to time t corresponds to one of the paths defining the evolution of the quantum network state up to time t.

A set of important structural properties of the CQNM are the generalized degrees $k_{d,\delta}$ of its δ-faces. Given a CQNM of dimension d, the generalized degree $k_{d,\delta}(\alpha)$ of a given δ-face α, (i.e. $\alpha \in \mathcal{S}_{d,\delta}$) is defined as the number of d-dimensional simplices incident to it. For example, in a CQNM of dimension $d=2$, the generalized degree $k_{2,1}(\alpha)$ is the number of triangles incident to a link α while the generalized degree $k_{2,0}(\alpha)$ indicates the number of triangles incident to a node α. Similarly in a CQNM of dimension $d=3$, the generalized degrees $k_{3,2,}$ $k_{3,2}$ and $k_{3,0}$ indicate the number of tetrahedra incident respectively to a triangular face, a link or a node. If from a CQNM of dimension d one extracts the underlying network, the degree $K_d(i)$ of node i is given by the generalized degree $K_{d,0}(i)$ of the same node i plus $d-1$, i.e.

$$K_d(i) = k_{d,0}(i) + d - 1. \tag{6}$$

We indicate with $P_{d,\delta}(k)$ the distribution of generalized degrees $k_{d,\delta}=k$. It follows that the degree distribution of the network $P_d(K)$ constructed from the d-dimensional CQNM is given by

$$P_d(K) = P_{d,0}(K - d + 1). \tag{7}$$

Let us consider the generalized degree distribution of CQNM in the case $\beta=0$. In this case the new d-dimensional simplex can be added with equal probability to each unsaturated $(d-1)$-face of the CQNM. Here we show that as long as the dimension d is greater than two, i.e. $d>2$, the CQNM is a scale-free network. In fact each δ-face, with $\delta < d - 1$, which has generalized degree $k_{d,\delta}(\alpha) = k$, is incident to

$$2 + (d - \delta - 2)k \tag{8}$$

unsaturated $(d-1)$-faces. Therefore the probability $\pi_{d,\delta}(\alpha)$ to attach a new d-dimensional simplex to a δ -face α with generalized degree $k_{d,\delta}(\alpha)$ and with $\delta<\mathrm{d}-1$, is given by.

$$\pi_{d,\delta}(\alpha) = \frac{2 + (d - \delta - 2)k_{d,\delta}(\alpha)}{\sum_{\alpha'\in\mathcal{S}_{d,\delta}}[2 + (d - \delta - 2)k_{d,\delta}(\alpha')]}. \tag{9}$$

Therefore, as long as $\delta<\mathrm{d}-2$, the generalized degree increases dynamically due to an effective "linear preferential attachment[24]", according to which the generalized degree of a δ-face increases at each time by one, with a probability increasing linearly with the current value of its generalized degree. Since the preferential attachment is a well-known mechanism for generating scale-free distributions, it follows, by putting $\delta=0$, that we expect that as long as $d\geq 3$ the CQNMs are scale-free. Instead, in the case $d=2$, by putting $\delta=0$ it is immediate to see that the probability $\pi_{2,0}(\alpha)$ is independent of the generalized degree $k_{2,0}(\alpha)$ of the $0-$face (node) α, and therefore there is no "effective preferential attachment". We expect therefore[17] that the CQNM in $d=2$ has an exponential degree distribution, i.e. in $d=2$ we expect to observe homogeneous CQNM with bounded fluctuations in the degree distribution. These arguments can be made rigorous by solving the master equation[19], and deriving the exact asymptotic generalized degree distributions for every $\delta<d$ (see Methods and Supplementary Information for details). For $\delta=d-1$ we find a bimodal distribution

$$P_{d,d-1}(k) = \begin{cases} \dfrac{d-1}{d} & \text{for} \quad k=1, \\ \dfrac{1}{d} & \text{for} \quad k=2. \end{cases} \tag{10}$$

For $\delta = d - 2$ instead, we find an exponential distribution, i.e.

$$P_{d,d-2}(k) = \left(\frac{2}{d+1}\right)^k \frac{d-1}{2}, \quad \text{for} \quad k \geq 1. \tag{11}$$

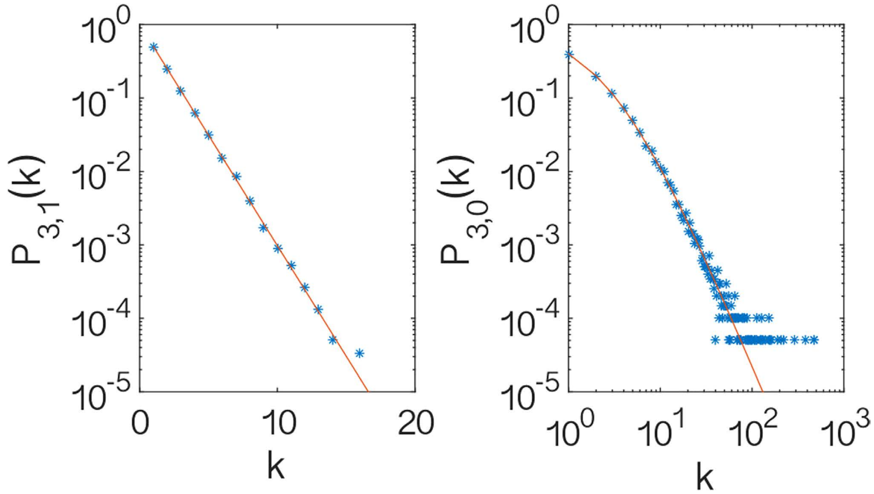

Figure 3. Distribution of the generalized degrees. The distribution of the (non-trivial) generalized degrees $P_{3,1}(k)$ and $P_{3,0}(k)$ in dimension $d=3$ are shown. The star symbols indicate the simulation results while the solid red line indicates the theoretical expectations given respectively by Eqs. (11) and (12). In particular we observe that $P_{3,1}(k)$ is exponential while $P_{3,0}(k)$ is scale-free implying that the CQNM in $d=3$ is scale-free. The simulation results are shown for a single realization of the CQNM with a total number of nodes $N=2\times 10^4$.

Therefore in $d=2$, the CQNMs have an exponential degree distribution that can be derived from Eq. (11) and Eq. (7). Finally for $0\leq\delta<d-2$ we have the distribution

$$P_{d,\delta}(k) = \frac{d-1}{d-\delta-2}\frac{\Gamma[1+(d+1)/(d-\delta-2)]}{\Gamma[1+2/(d-\delta-2)]} \times \frac{\Gamma[k+2/(d-\delta-2)]}{\Gamma[k+1+(d+1)/(d-\delta-2)]}, \quad \text{for } k\geq 1. \tag{12}$$

It follows that for $0\leq\delta<d-2$ and $k\gg 1+(d+1)/(d-\delta-2)$ the generalized degree distribution follows a power-law with exponent $\gamma_{d,\delta}$, i.e.

$$P_{d,\delta}(k) \simeq Ck^{-\gamma d,\delta} \text{ for } d-\delta>2, \tag{13}$$

and

$$\gamma_{d,\delta} = 1 + \frac{d-1}{d-\delta-2}. \tag{14}$$

The distribution $P_{d,\delta}(k)$ given by Eq. (12) is scale-free if an only if $\gamma_{d,\delta}\in(2, 3]$. Using Eq. (14) we observe that for $d\geq 3$ and $\delta=0$ we observe that the distribution of generalized degrees $P_{d,\delta}(k)$ is always scale-free. Therefore the degree distribution $P_d(K)$ given by Eq. (7), for large values of the degree K and for $d\geq 3$ is scale-free and goes like

$$P_d(K) \simeq CK^{-\gamma_d} \tag{15}$$

with

$$\gamma_d = \frac{2d-3}{d-2}. \tag{16}$$

Therefore, for $d=3$ the CQNMs have $\gamma_3=3$ and for $d\to\infty$ they have power-law exponent $\gamma_d\to 2$.

These theoretical expectations perfectly fit the simulation results of the model as can been seen in Fig. 3 where the distribution of generalized degrees $P_{3,1}(k)$ and $P_{3,0}(k)$ observed in the simulations for $\beta=0$ are compared with the theoretical expectations.

In the case $\beta>0$ the distributions of the generalized degrees depend on the density $\rho_{d,\delta}(\epsilon)$ of δ-dimensional simplices with energy ϵ in a CQNM and are parametrized by self-consistent parameters called the chemical potentials, indicated as $\mu_{d,\delta}$ and defined in the Supplementary Information.

Here we suppose that these chemical potentials $\mu_{d,\delta}$ exist and that the density $\rho_{d,\delta}(\epsilon)$ is given, and we find the self-consistent equations that they need to satisfy at the end of the derivation. Using the master equation approach[19] we obtain that for $\delta = d - 1$ the generalized degree follows the distribution

$$P_{d,d-1}(k) = \begin{cases} \rho_{d,d-1}(\epsilon)\left[1 - \frac{1}{e^{\beta(\epsilon-\mu_{d,d-1})}+1}\right], & \text{for } k=1 \\ \rho_{d,d-1}(\epsilon)\frac{1}{e^{\beta(\epsilon-\mu_{d,d-1})}+1} & \text{for } k=2, \end{cases} \tag{17}$$

while for $\delta = d-2$ it follows

$$P_{d,d-2}(k) = \sum_{\epsilon}\rho_{d,d-2}(\epsilon)\frac{e^{\beta(\epsilon-\mu_{d,d-2})}}{(e^{\beta(\epsilon-\mu_{d,d-2})}+1)^k} \quad \text{for} \quad k \geq 1. \tag{18}$$

Finally for $0 \leq \delta < d-2$ the generalized degree is given by

$$\begin{aligned} P_{d,\delta}(k) &= \sum_{\epsilon}\rho_{d,d-2}(\epsilon)e^{\beta(\epsilon-\mu_{d,\delta})} \\ &\times \frac{\Gamma\left[1+2/(d-\delta-2)+\exp\left[\beta(\epsilon-\mu_{d,\delta})\right]\right]\Gamma[k+2/(d-\delta-2)]}{\Gamma[1+2/(d-\delta-2)]\Gamma\left[k+1+2/(d-\delta-2)+\exp\left[\beta(\epsilon-\mu_{d,\delta})\right]\right]}, \\ &\text{for } k \geq 1. \end{aligned} \tag{19}$$

It follows that also for $\beta > 0$ the CQNMs in $d > 2$ are scale-free. Interestingly, we observe that the average of the generalized degrees of simplices with energy ε follows the Fermi-Dirac distribution for $\delta = d-2$, the Boltzmann distribution for $\delta = d-2$ and the Bose-Einstein distribution for $\delta = d > 2$. In fact we have,

$$\begin{aligned} \langle[k_{d,d-1}-1]|\epsilon\rangle &= n_F(\epsilon, \mu_{d,d-1}), \\ \langle[k_{d,d-2}-1]|\epsilon\rangle &= n_Z(\epsilon, \mu_{d,d-2}), \\ \langle[k_{d,\delta}-1]|\epsilon\rangle &= An_B(\epsilon, \mu_{d,\delta}), \quad \text{for } \delta < d-2. \end{aligned} \tag{20}$$

where $A = (d-\delta)/(d-\delta-2)$, $n_Z(\epsilon, \mu_{d,d-2})$ is proportional to the Boltzmann distribution and $n_F(\epsilon, \mu_{d,d-1})$, $n_B(\epsilon, \mu_{d,\delta})$ indicate respectively the Fermi-Dirac and Bose-Einstein occupation numbers[47]. In particular we have

$$\begin{aligned} n_Z(\epsilon, \mu_{d,d-2}) &= e^{-\beta(\epsilon-\mu_{d,d-2})}, \\ n_F(\epsilon, \mu_{d,d-1}) &= \frac{1}{e^{\beta(\epsilon-\mu_{d,d-1})}+1}, \\ n_B(\epsilon, \mu_{d,\delta}) &= \frac{1}{e^{\beta(\epsilon-\mu_{d,\delta})}-1}. \end{aligned} \tag{21}$$

These results suggest that the dimension $d = 3$ of CQNM is the minimal one necessary for observing at the same time scale-free CQNMs and the simultaneous emergence of the Fermi-Dirac, Boltzmann and Bose-Einstein distributions. In particular in $d = 3$ the average generalized degree of triangles of energy ϵ follows the Fermi-Dirac distribution, the average of the generalized degree of links of energy ϵ follows the Boltzmann distribution, while the generalized degree of nodes of energy ϵ follows the Bose-Einstein distribution.

Finally the chemical potentials $\mu_{d,\delta}$, if they exist, can be found self-consistently by imposing the condition.

$$\lim_{t\to\infty}\frac{\sum_{\alpha}k_{d,\delta}}{N_{d,\delta}} = \frac{d+1}{\delta+1}, \tag{22}$$

dictated by the geometry of the CQNM, which implies the following self-consistent relations for the chemical potentials $\mu_{d,\delta}$

$$\int d\epsilon\rho_{d,d-1}(\epsilon)n_F(\epsilon, \mu_{d,d-1}) = \frac{1}{d}, \tag{23}$$

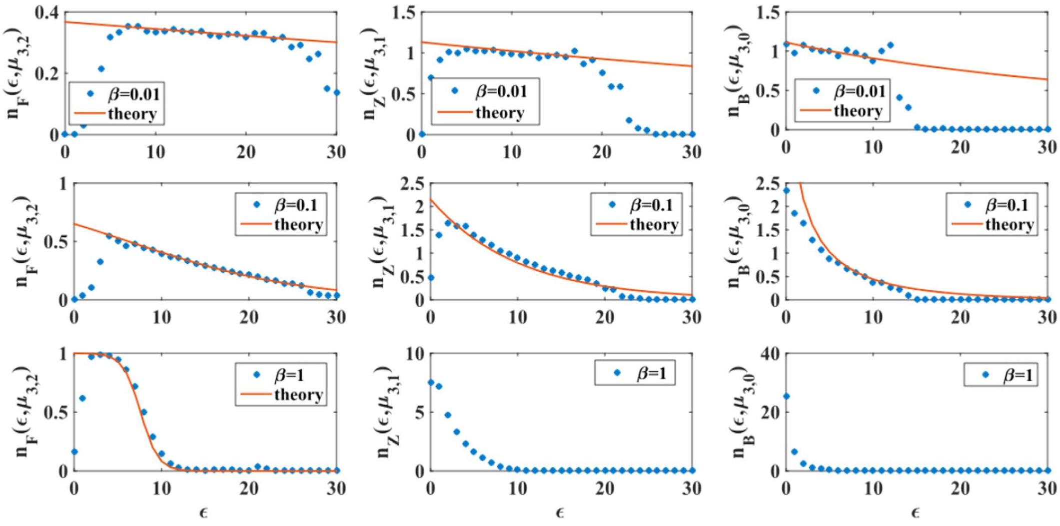

Figure 4. In $d=3$ the average of the generalized degrees of faces, links, and nodes follow respectively the Fermi-Dirac, Boltzmann and Bose-Einstein distributions. The average of the generalized degrees of δ-faces of energy ϵ, in a CQNM of dimension $d=3$, follow the Fermi-Dirac $n_F(\epsilon, \mu_{3,2})$, the Boltzmann $n_B(\epsilon, \mu_{3,1})$ or the Bose-Einstein distribution $n_B(\epsilon, \mu_{3,0})$ according to Eqs. (23)–(25) as long as the chemical potential $\mu_{d,\delta}$ is well-defined, i.e. for sufficiently low value of the inverse temperature β. Here we compare simulation results over CQNM of $N=10^3$ nodes in $d=3$ and theoretical results for $\beta=0.01$, 0.1, 1. The CQNMs in the figure have a Poisson energy distribution $g(\epsilon)$ with average $z=5$. The simulation results are averaged over 50 CQNM realizations.

$$\int d\epsilon \rho_{d,d-1}(\epsilon) n_Z(\epsilon, \mu_{d,d-2}) = \frac{2}{d-1}, \quad (24)$$

$$\int d\epsilon \rho_{d,\delta}(\epsilon) n_B(\epsilon, \mu_{d,\delta}) = \frac{d-\delta-2}{\delta+1}, \text{ for } \delta < d-2. \quad (25)$$

In Fig. 4 we compare the simulation results with the theoretical predictions given by Eqs. (20) finding very good agreement for sufficiently low values of the inverse temperature β. The disagreement occurring at large value of the inverse temperature β is due to the fact that the self-consistent Eqs. (23)–(25) do not always give a solution for the chemical potentials $\mu_{d,\delta}$. In particular the CQNM with $d \geq 3$ can undergo a Bose-Einstein condensation when Eq. (25) cannot be satisfied. When the transition occurs for the generalized degree with $\delta = 0$, the maximal degree in the network increases linearly in time similarly to the scenario described in[27].

In summary, we have shown that Complex Quantum Manifolds in dimension $d>2$ are scale-free, i. e. they are characterized by large fluctuations of the degrees of the nodes. Moreover the δ-faces with $\delta < d$ follow the Fermi-Dirac, Boltzmann or Bose-Einstein distributions depending on the dimensions d and δ. In particular for $d=3$, we find that triangular faces follow the Fermi-Dirac distribution, links follow the Boltzmann distribution and nodes follow the Bose-Einstein distribution. Interestingly, we observe that the dimension $d=3$ is not only the minimal dimension for having a scale-free CQNM, but it is also the minimal dimension for observing the simultaneous emergence of the Fermi-Dirac, Boltzmann or Bose-Einstein distributions in CQNMs.

Methods

Quantum network states. The Quantum Network State is an element of an Hilbert space $\mathcal{H}_{tot}$ associated to a simplicial complex formed by gluing d-dimensional simplices of N nodes. This Hilbert space is given by

$$\mathcal{H}_N = \overset{N}{\otimes}\mathcal{H}_{node} \overset{P}{\otimes} \mathcal{H}_{d,d-1} \overset{P}{\otimes} \widetilde{\mathcal{H}}_{d,d-1} \quad (26)$$

with $P = \binom{N}{d}$ indicating the maximum number of $(d-1)$-dimensional simplices in a network of N nodes. Here a Hilbert space $\mathcal{H}_{node}$ is associated to each possible node i of the simplicial complex, and two Hilbert spaces $\mathcal{H}_{d,d-1}$ and $\mathcal{H}_{d,d-1}$ are associated to each possible $(d-1)$-dimensional simplex of a network of N nodes. The Hilbert space $\mathcal{H}_{node}$ is the one of a fermionic oscillator of energy ϵ_i, with basis $\{|o_i, \epsilon\rangle\}$, with $o_i = 0, 1$. These states can be mapped respectively to the presence ($|o_i = 1, \epsilon\rangle$) or the absence ($|o_i = 0, \epsilon\rangle$) of a node i of energy $\epsilon_i = \epsilon$ in the simplicial complex. We indicate with $b_i^\dagger(\epsilon_i)$, $b_i(\epsilon_i)$ respectively the fermionic creation and annihilation operators acting in this space. The Hilbert space $\mathcal{H}_{d,d-1}$ associated to a $(d-1)$ simplex α is the Hilbert space of a fermionic oscillator with basis $\{|a_\alpha\rangle\}$, with $a_\alpha = 0, 1$. The quantum number $a_\alpha = 1$ is mapped to the presence of the simplex α in the network while the quantum number $a_\alpha = 0$ is mapped to the absence of such a simplex. We indicate with $c_\alpha^\dagger$, c_α respectively the fermionic creation and annihilation operators acting in this space. Finally the Hilbert space $\widetilde{\mathcal{H}}_{d,d-1}$ associated to a $(d-1)$ simplex α is the Hilbert space of a fermionic oscillator with basis $\{|n_\alpha\rangle\}$, with $n_\alpha = 0, 1$. We indicate with $h_\alpha^\dagger$, h_α respectively the fermionic creation and annihilation operators acting in this space. The quantum number $n_\alpha = 1$ is mapped to a saturated α simplex, i. e. incident to two d-dimensional simplices, while the quantum number $n_\alpha = 0$ is mapped either to an unsaturated α simplex (if also $a_\alpha = 1$) or to the absence of such a simplex (if $a_\alpha = 0$).

A quantum network state can therefore be decomposed as

$$|\psi(t)\rangle = \sum_{\{o_i,\epsilon_i,a_\alpha,n_\alpha\}} C(\{o_i, \epsilon_i, a_\alpha, n_\alpha\}) \prod_i |o_i, \epsilon_i\rangle \prod_{\alpha\in Q_{d,d-1}(N)} |a_\alpha\rangle|n_\alpha\rangle, \tag{27}$$

where with $\mathcal{Q}_{d,d-1}(N)$ we indicated all the possible $(d-1)$-faces of the CQNM of N nodes.

We assume that the quantum network state follows a Markovian evolution as it has been proposed already in the literature[8,41]. In particular we assume that at time $t=1$ the state is given by

$$|\psi(1)\rangle = \frac{1}{\sqrt{\mathcal{Z}(1)}} \sum_{\{\epsilon_i\}_{i=1,..d+1}} \prod_{i=1}^{d+1} \sqrt{g(\epsilon_i)}\, b_i^\dagger(\epsilon_i) \prod_{\alpha\in\mathcal{Q}_{d,d-1}(d+1)} c_\alpha^\dagger|0\rangle, \tag{28}$$

with $\mathcal{Z}(1)$ enforcing the normalization condition $\langle\psi(1)|\psi(1)\rangle = 1$. The quantum network state at each time $t > 1$ is updated according to the Markov chain

$$|\psi(t)\rangle = U_t|\psi(t-1)\rangle \tag{29}$$

with the unitary operator U_t given by

$$U_t = \sqrt{\frac{\mathcal{Z}(t-1)}{\mathcal{Z}(t)}} \sum_{\epsilon_{t+d}} \sqrt{g(\epsilon_{t+d})}\, b_{t+d}^\dagger(\epsilon_{t+d}) \sum_{\alpha\in\mathcal{Q}_{d,d-1}(t+d-1)} e^{-\beta\epsilon_\alpha/2} \left[\prod_{\alpha'\in\mathcal{F}(t+d,\alpha)} c_{\alpha'}^\dagger\right] h_\alpha^\dagger c_\alpha^\dagger c_\alpha$$

where $\mathcal{F}(i, \alpha)$ indicates the set of all the $(d-1)$-simplices α' formed by the node i and a subset of the nodes in $\alpha \in \mathcal{Q}_{d,d-1}(N)$. The quantity $\mathcal{Z}(t)$ present in the definition of the unitary operator U_t enforces the normalization condition $\langle\psi(t)|\psi(t)\rangle = 1$ and can be interpreted as a path integral over CQNM evolutions determined by the sequence $\{\varepsilon_i\}_{i\le t+d}$, $\{\alpha_{t'}\}_{t'\le t}$ of the energy values $\varepsilon_{t'+d}$ of the nodes added at time $t' > 1$ and the energy values $\{\varepsilon_i\}_{i=1,2,\ldots,d+1}$, together with the sequence $\alpha_{t'}$ of the $(d-1)$-faces where the new d-dimensional simplex is added at time t'. In fact we have.

$$\mathcal{Z}(t) = \sum_{\{\epsilon_i\}_{i\le t+d}} \sum_{\{\alpha_{t'}\}_{t'\le t}} W\left(\{\epsilon_i\}_{i\le t+d}, \{\alpha_{t'}\}_{t'\le t}\right) \tag{30}$$

where the probability $P(\{\epsilon_i\}_{i\le t+d}, \{\alpha_{t'}\}_{t'\le t})$ of a given CQNM evolution $\{\epsilon_i\}_{i\le t+d}$, $\{\alpha_{t'}\}_{t'\le t}$ is given by

$$P(\{\epsilon_i\}_{i\le t+d}, \{\alpha_{t'}\}_{t'\le t}) = \frac{W\left(\{\epsilon_i\}_{i\le t+d}, \{\alpha_{t'}\}_{t'\le t}\right)}{\mathcal{Z}(t)}. \tag{31}$$

For the exact expression of $W\left(\{\epsilon_i\}_{i\le t+d}, \{\alpha_{t'}\}_{t'\le t}\right)$ see the Supplementary Information.

Generalized degree distribution for $\beta=0$. The average number of δ–faces of a d-dimensional CQNM of generalized degree $k_{d,\delta} = k$ that are incident to the new d-dimensional simplex at a given time t is given, for $\delta = d-1$, by

$$n_{d,d-1}(k) = \frac{1}{(d-1)t}\delta_{k,1} \tag{32}$$

where $\delta_{x,y}$ indicates the Kronecker delta while for $\delta < d-1$ is given by,

$$n_{d,\delta}(k) = \frac{2 + (d - \delta - 2)k}{(d - 1)t}. \tag{33}$$

Using Eqs. (32), (33) and the master equation approach, it is possible to derive the exact distribution for the generalized degrees. We indicate with $N_{d,\delta}^t(k)$ the average number of δ-faces that at time t have generalized degree $k_{d,\delta} = k$. The master equation for $N_{d,\delta}^t(k)$ reads

$$N_{d,\delta}^{t+1}(k) - N_{d,\delta}^t(k) = n_{d,\delta}(k-1)N_{d,\delta}^t(k-1)(1-\delta_{k,1}) - n_{d,\delta}(k)N_{d,\delta}^t(k) + m_{d,\delta}\delta_{k,1}$$

with $k \geq 1$. The master equation can be solved by observing that for large times $t \gg 1$ we have $N_{d,\delta}^t(k) \simeq m_{d,\delta} t P_{d,\delta}(k)$ where $P_{d,\delta}(k)$ is the generalized degree distribution. In this way Eqs. (10)–(12) are obtained.

Generalized degree distribution for $\beta = 0$. For $\beta > 0$ the probability $\pi_{d,\delta}(k|\epsilon)$ that a given $\delta-$face of energy ϵ and generalized degree $k_{d,\delta} = k$ increases its generalized degree by one at time t can be expressed in terms of self-consistent parameters $\mu_{d,\delta}$ called *chemical potentials* and defined in the Supplementary Material. Using these probabilities the master equations can be written for the average number of δ-faces $N_{d,\delta}^t(k|\epsilon)$ that at time t have generalized degree $k_{d,\delta} = k$ and energy ϵ. These equations can be solved similarly to the case $\beta = 0$ obtaining for the generalized degree distributions Eqs. (17)–(19).

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G.B. and C.R. designed the research, G.B. wrote the codes, prepared figures. G.B. and C.R. wrote the main manuscript text.

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