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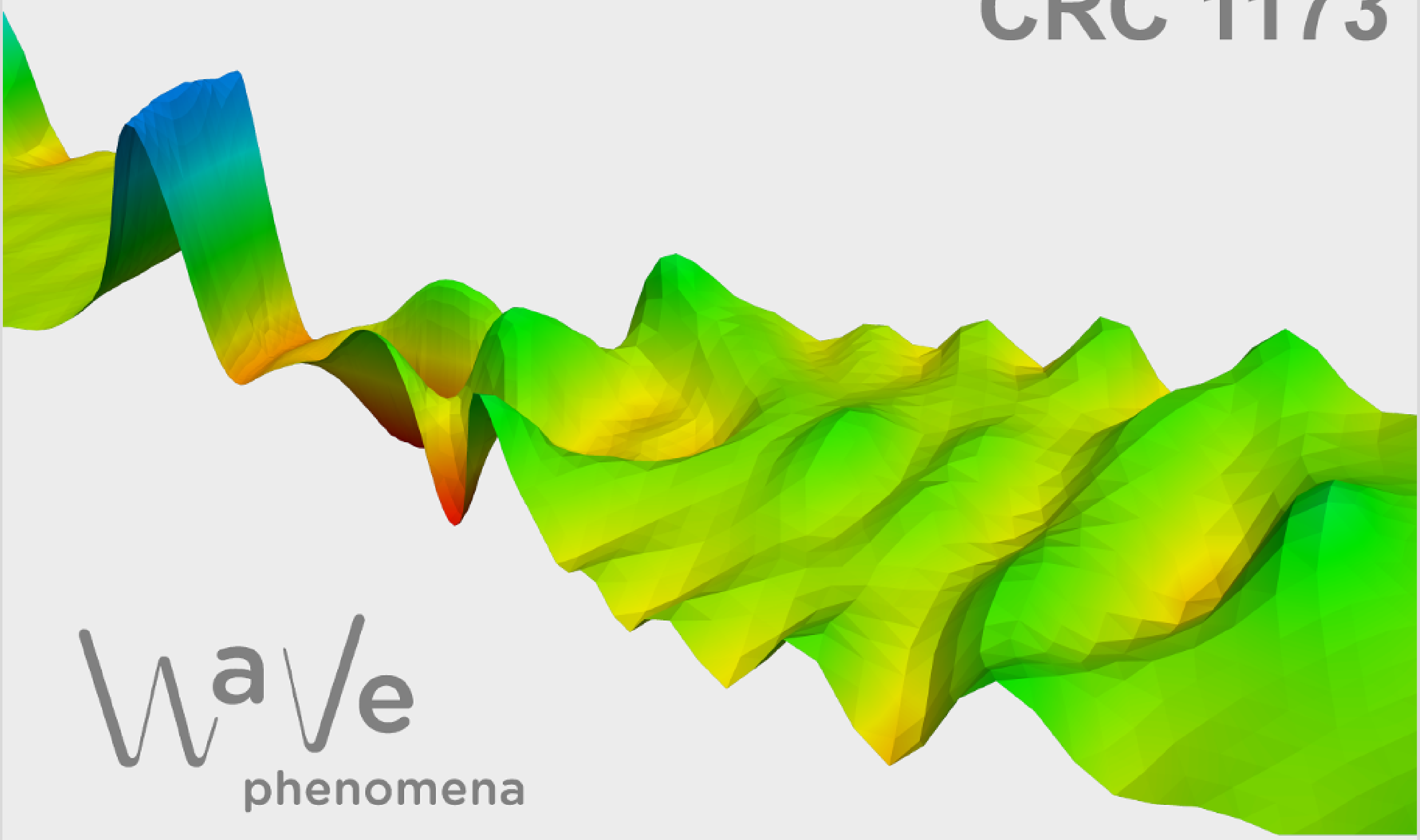

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Adiabatic Midpoint Rule for the dispersion-managed nonlinear Schrödinger Equation

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Abstract

The dispersion-managed nonlinear Schrödinger equation contains a rapidly changing discontinuous coefficient function. Approximating the solution numerically is a challenging task because typical solutions oscillate in time which imposes severe step-size restrictions for traditional methods. We present and analyze a tailor-made time integrator which attains the desired accuracy with a significantly larger step-size than traditional methods. The construction of this method is based on a favorable transformation to an equivalent problem and the explicit computation of certain integrals over highly oscillatory phases. The error analysis requires the thorough investigation of various cancellation effects which result in improved accuracy for special step-sizes.

Mathematics Subject Classification (2010): 65M12, 65M15, 65M70, 65Z05, 35B40, 35Q55

Keywords: dispersion management, nonlinear Schrödinger equation, highly oscillatory problem, discontinuous coefficients, adiabatic integrator, error bounds, limit dynamics

1 Introduction

The nonlinear Schrödinger equation (NLS) ubiquitously appears in various variants in nonlinear optics as an envelope equation which models the propagation of wave packets in optical fiber cables; [3, 32]. In this article we consider the dispersion-managed nonlinear Schrödinger equation (DMNLS)

$$\partial_t u(t,x) = \tfrac{\mathrm{i}}{\varepsilon}\gamma\left(\tfrac{t}{\varepsilon}\right)\partial_x^2 u(t,x) + \mathrm{i}\left|u(t,x)\right|^2 u(t,x)\,, \quad x\in(-\pi,\pi)\,,\ t\in[0,T]\,, \tag{1}$$

with periodic boundary conditions and a small parameter $0<\varepsilon\ll T$. The coefficient function γ is given by

$$\gamma(t)=\chi(t)+\varepsilon\alpha\,, \tag{2}$$

where

$$\chi(t)=\begin{cases} -\delta & \text{if} \quad t\in[m,m+1) \quad \text{for even} \quad m\in\mathbb{N}, \\ \delta & \text{if} \quad t\in[m,m+1) \quad \text{for odd} \quad m\in\mathbb{N} \end{cases} \tag{3}$$

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is a periodic, piecewise constant function, with constants $\alpha \geq 0$ and $\delta > 0$. We assume the relation $\delta > \varepsilon\alpha$ such that for every $t \in [0, T]$ we have $|\gamma(t)| \neq 0$. This is the appropriate setting for modeling dispersion-managed fiber cables; [3, 36].

Dispersion describes the fact that light waves of various wavelengths propagate with different speed; [3]. This effect leads typically to a broadening of wave packets over time, which is a major problem in long-haul data transmission. Especially in high speed, high volume data transfer through intercontinental fiber cables it is essential to transmit as many wave packets as possible, but at the same time interaction between different packets has to be avoided. Each pulse must be well separated from the next, thus the broadening of the wave packets due to dispersion limits the transmission rate.

The idea of dispersion management is to engineer fiber cables in alternating sections with different refractive index such that the average dispersion effect is mostly neutralized. This approach was first proposed in [27] and has developed to a very successful technique since then; cf. [2, 30, 31].

In this context, the time variable t in (1) actually corresponds to the distance along the fiber cable, whereas the space variable x represents the (retarded) time. Consequently, the coefficient function γ depending on t models the periodically changing sections and thus the varying dispersion effect along the fiber. The small parameter ε originates from the fact that cable sections with equal dispersion are small compared to the total length of the fiber, which is scaled to $O(1)$.

The parameter ε together with the function γ distinguish the DMNLS from the "classical" cubic NLS (where $\varepsilon = 1$ and $\gamma \equiv 1$) and cause considerable challenges when solutions of (1) have to be approximated by numerical methods. Typical solutions oscillate rapidly with frequency $\sim 1/\varepsilon$. As a consequence, if traditional time integration schemes are used, then a tiny step-size $\tau \ll \varepsilon$ has to be chosen to obtain an acceptable accuracy. Since γ is discontinuous, the second derivative of u with respect to time does not exist which renders higher-order Taylor expansions impossible and thus complicates the construction of numerical methods. Finally, the nonlinear term $\mathrm{i}\,|u|^2\,u$ makes implicit schemes prohibitively costly.

A popular approach to approximate solutions of the "classical" NLS are splitting methods; cf. [28, 35]. It is a well-known fact that the nonlinear part of (1) on its own can be solved exactly, see e.g. [11]. In addition, the linear part on its own can be solved exactly in Fourier space, see (6) below. Hence, we may approximate the DMNLS by a Strang splitting approach, in principle, but the rapid oscillations of the coefficient function γ impose again step-sizes $\tau \ll \varepsilon$.

These numerical and analytical difficulties have led mathematical research to consider the Gabitov-Turitsyn equation (GTE), which is obtained from the DMNLS via a transformation and averaging; cf. [14, 15]. The GTE is autonomous and does not depend on ε. It has been intensively studied in analysis [18, 23, 33, 37, 41] and by means of numerical approximations [37, 40], see also [36] for a review. Because of the averaging step in its derivation the GTE yields only an approximation to the DMNLS. The accuracy of this approximation depends on the parameter ε (cf. [33]) which is fixed in particular applications.

For this reason we avoid the averaging step and consider instead of the GTE an equivalent formulation of the DMNLS which is obtained from (1) solely by transformation. This transformed DMNLS is denoted by tDMNLS. We propose and analyze a novel numerical method for the tDMNLS which enables approximations with higher accuracy than simulating the GTE. The proposed integrator is referred to as the adiabatic midpoint rule and has the following advantageous properties:

- First-order convergence independently of ε.
- Global error of $O(\tau\varepsilon)$ for step-sizes $\tau = \varepsilon/k$ with $k \in \mathbb{N}$, with a constant independent of ε.
- Global error of $O(\tau^2)$ for step-sizes $\tau = \varepsilon \cdot k$ with $k \in \mathbb{N}$, again with a constant independent of ε.

Although the GTE is not explicitly used in the construction of our method for the tDMNLS, there is an important connection. For $\varepsilon \to 0$ the tDMNLS converges to a limit system which is equivalent to the GTE (see Section 3 for details), and we will prove the following assertions.

- In the limit $\varepsilon \to 0$ the adiabatic midpoint rule for the tDMNLS reduces to the standard explicit midpoint rule applied to the limit system; cf. section 4.
- If $\varepsilon > 0$ is fixed and we chose the step-size $\tau = \varepsilon \cdot k$ for some $k \in \mathbb{N}$, then applying the adiabatic midpoint rule to the tDMNLS yields the same approximations as applying the standard explicit midpoint rule to the limit system; cf. Theorem 4.

Hence, in order to understand the accuracy of the adiabatic midpoint rule for large step-sizes $\tau \geq \varepsilon$ we have to investigate how accurately and in which sense solutions of the limit system approximate the tDMNLS. These questions have been addressed in [14, 15, 33] for the DMNLS on $\mathbb{R}$. With other techniques we prove corresponding and slightly improved results on $\mathbb{T}$; cf. Theorem 1.

The construction and analysis of time-integrators for oscillatory problems is an active field of research, and the state of the art is documented in [9,10,20,22, 34, 39]. In the following, we mention only references where *partial* differential equations are considered. Trigonometric integrators for semilinear wave equations have been proposed and analyzed, e.g., in [16, 19, 21]. Special methods for linear Schrödinger equations in the semiclassical regime have been developed, e.g., in [5, 13] and references therein. For nonlinear Schrödinger equations, the adverse effect of oscillations caused by the semiclassical scaling on the accuracy of splitting methods has been studied, e.g., in [4]. Conversely, it was shown in [8] that for (1) with $\gamma \equiv 1$ (and optionally a more general nonlinearity) the oscillatory behavior leads to a better accuracy if the step-size is chosen in a special way. A different but related topic is the long-time conservation of geometric properties of semilinear Schrödinger equations under time-discretization with splitting methods; cf. [11, 12, 17]. The long-time behavior of a method based on stroboscopic averaging is analyzed in [7]. Last but not least, there is a rich literature on heterogeneous multiscale methods for ordinary differential equations and for partial differential equations with coefficients varying rapidy in space instead of time; see [1] for an overview.

Numerical integrators for oscillatory differential equations can only be efficient if they exploit the particular structure of the problem. For this reason, it is not surprising that most methods perform poorly when applied to a different problems class. In each of the above references the equations and assumptions differ significantly from the situation we consider here. The interplay of the unbounded differential operator, the time-dependent, discontinuous coefficient function γ, the smallness parameter ε and the nonlinearity makes the numerical integration of the DMNLS (1) a challenging problem, and the adiabatic midpoint rule proposed in this work seems to be the first problem-adapted time-integrator for the DMNLS.

The paper is organized as follows: The transformation from the DMNLS to the tDMNLS is performed in section 2, and in section 3 we derive the limit system of the tDMNLS. In Section 4 we construct the adiabatic midpoint rule by extending techniques from [25, 26]. The analytic framework provided in section 5 is adopted from [11, VII.2.]. Section 6 contains the main results regarding the accuracy of the limit system (Theorem 1) and the error analysis of the semi-discretization in time for the adiabatic midpoint rule (Theorems 2-4), whereas the proofs are postponed to section 7 and section 8. All theorems are illustrated by subsequent numerical examples.

2 The Transformation

Both the derivation of the limit system and the numerical method make use of a favorable transformation of the DMNLS (1). Our aim is to transform the DMNLS in such a way that the right-hand side of the resulting equation is bounded in the limit $\varepsilon \to 0$. We define for $m \in \mathbb{Z}$ the index set

$$I_m = \big\{(j,k,l) \in \mathbb{Z}^3 : j-k+l = m\big\} \subset \mathbb{Z}^3 \,.$$

If we represent the solution of the DMNLS by the Fourier series

$$u(t,x) = \sum_{k=-\infty}^{\infty} c_k(t) e^{\mathrm{i}kx} \,, \tag{4}$$

we obtain the infinite system of ODEs

$$c_m'(t) = -\tfrac{\mathrm{i}}{\varepsilon}\gamma\left(\tfrac{t}{\varepsilon}\right) m^2 c_m(t) + \mathrm{i}\sum_{I_m} c_j(t)\overline{c}_k(t)c_l(t) \,, \quad m = -\infty,\ldots,\infty \,. \tag{5}$$

Here and subsequently we write

$$\sum_{I_m} \quad \text{instead of} \quad \sum_{(j,k,l)\in I_m}$$

to simplify notation. Since the right-hand side of (5) is still unbounded in the limit $\varepsilon \to 0$, we consider an additional transformation. Every solution of the linear part of (5) has the form

$$c_k(t) = \exp\Big(-\mathrm{i}k^2\widehat{\phi}\left(\tfrac{t}{\varepsilon}\right)\Big)\, c_k(0) \,, \quad k \in \mathbb{Z} \,, \tag{6}$$

where the function $\widehat{\phi}$ is defined by

$$\widehat{\phi}(z) := \int_0^z \gamma(\sigma)\,\mathrm{d}\sigma = \phi(z) + \alpha\varepsilon z \quad \text{with} \quad \phi(z) = \int_0^z \chi(\sigma)\,\mathrm{d}\sigma \,. \tag{7}$$

This motivates the change of variables

$$y_k(t) := \exp\left(\mathrm{i}k^2\widehat{\phi}\left(\tfrac{t}{\varepsilon}\right)\right) c_k(t)\,, \quad k \in \mathbb{Z}\,. \tag{8}$$

Similar transformations have been used in [26] and [25] in case of oscillatory linear Schrödinger equations. Moreover, this transformation is known as the Floquet-Lyapunov transformation in physics; cf. [29, 37]. Since we have the relations

$$c_k(t) = \exp\left(-\mathrm{i}k^2\widehat{\phi}\left(\tfrac{t}{\varepsilon}\right)\right) y_k(t) \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}\widehat{\phi}\left(\tfrac{t}{\varepsilon}\right) = \tfrac{1}{\varepsilon}\gamma\left(\tfrac{t}{\varepsilon}\right)\,,$$

we obtain for $m \in \mathbb{Z}$ the equation

$$c_m'(t) = -\tfrac{\mathrm{i}}{\varepsilon}\gamma\left(\tfrac{t}{\varepsilon}\right) m^2 c_m(t) + \exp\left(\mathrm{i}m^2\widehat{\phi}\left(\tfrac{t}{\varepsilon}\right)\right) y_k'(t)\,. \tag{9}$$

Observe that the time derivative of $\widehat{\phi}$ does not exist in the classical sense. It should be understood as piece-wise derivative on the open intervals $\big(n\varepsilon, (n+1)\varepsilon\big)$ for $n \in \mathbb{N}$ with left-continuous extension. Comparing equation (9) to the corresponding equation in (5) yields the tDMNLS

$$y_m'(t) = \mathrm{i}\sum_{I_m} y_j(t)\overline{y}_k(t)y_l(t)\exp\left(-\mathrm{i}\omega_{[jklm]}\widehat{\phi}\left(\tfrac{t}{\varepsilon}\right)\right)\,, \quad m = -\infty, \ldots, \infty\,, \tag{10}$$

with the abbreviation

$$\omega_{[jklm]} = j^2 - k^2 + l^2 - m^2.$$

Since we have for arbitrary $\varepsilon \neq 0$

$$\left|\exp\left(-\mathrm{i}\omega_{[jklm]}\widehat{\phi}\left(\tfrac{t}{\varepsilon}\right)\right)\right| = 1\,,$$

the right-hand side of equation the tDMNLS is bounded for $\varepsilon \to 0$ if the sequence $\big(y_m(t)\big)_{m\in\mathbb{Z}}$ decays sufficiently fast. This is our motivation to study the tDMNLS instead of the DMNLS throughout the paper. However, it is clear that both formulations are equivalent. For convenience we abbreviate

$$Y_{jkl}(t) = y_j(t)\overline{y}_k(t)y_l(t)\,, \quad \widehat{Y}_{jklm}(t) = Y_{jkl}(t)\exp(-\mathrm{i}\omega_{[jklm]}t\alpha) \tag{11}$$

and write the tDMNLS subsequently as

$$y_m'(t) = \mathrm{i}\sum_{I_m} \widehat{Y}_{jklm}(t)\exp\left(-\mathrm{i}\omega_{[jklm]}\phi\left(\tfrac{t}{\varepsilon}\right)\right)\,, \qquad m = -\infty, \ldots, \infty\,. \tag{12}$$

3 The Limit System

The right-hand side of the tDMNLS (12) still contains rapidly oscillating phases. In order to approximate the solution analytically, the periodic exponential function in (12) can be replaced by its averaged value over one period, i.e.

$$\exp\left(-\mathrm{i}\omega_{[jklm]}\phi\left(\tfrac{t}{\varepsilon}\right)\right) \approx \frac{1}{2\varepsilon}\int_0^{2\varepsilon} \exp\left(-\mathrm{i}\omega_{[jklm]}\phi\left(\tfrac{s}{\varepsilon}\right)\right)\mathrm{d}s\,. \tag{13}$$

Thanks to the periodicity and symmetry of ϕ we have for $p \in \mathbb{N}$

$$\int_{p\varepsilon}^{(p+1)\varepsilon} \exp\left(-\mathrm{i}\omega_{[jklm]}\phi\left(\tfrac{s}{\varepsilon}\right)\right) \mathrm{d}s = \varepsilon \int_0^1 \exp(\mathrm{i}\omega_{[jklm]}\delta\xi)\,\mathrm{d}\xi$$
$$= \begin{cases} \varepsilon \frac{\exp(\mathrm{i}\omega_{[jklm]}\delta)-1}{\mathrm{i}\omega_{[jklm]}\delta} & \text{if} \quad \omega_{[jklm]} \neq 0, \\ \varepsilon & \text{if} \quad \omega_{[jklm]} = 0\,. \end{cases} \tag{14}$$

Substituting (13) into (12) yields the system

$$v_m'(t) = \mathrm{i} \sum_{I_m} \widehat{V}_{jklm}(t) \int_0^1 \exp(\mathrm{i}\omega_{[jklm]}\delta\xi)\,\mathrm{d}\xi\,, \quad m = -\infty, \dots, \infty\,, \tag{15}$$

with

$$V_{jkl}(t) = v_j(t)\overline{v}_k(t)v_l(t) \quad \text{and} \quad \widehat{V}_{jklm}(t) = V_{jkl}(t)\exp(-\mathrm{i}\omega_{[jklm]}t\alpha)\,. \tag{16}$$

In fact, we will show in Theorem 1 that solutions of the tDMNLS converge to solutions of (15) in the limit $\varepsilon \to 0$; see also [33]. The advantage of the limit system (15) is that the right-hand side is independent of ε and no longer contains the discontinuous coefficient function γ.

This approach is closely related to the GTE, which can be derived as follows. First, the phase $\widehat{\phi}(t/\varepsilon) = \phi(t/\varepsilon) + \alpha t$ is replaced by $\phi(t/\varepsilon)$ in the transformation (8). Then, the oscillating phase is replaced by its mean as in (13). This yields the discrete counterpart of the GTE

$$w_m'(t) = -\mathrm{i}m^2\alpha w_m(t) + \mathrm{i}\sum_{I_m} w_j(t)\overline{w}_k(t)w_l(t) \int_0^1 \exp(\mathrm{i}\omega_{[jklm]}\delta\xi)\,\mathrm{d}\xi\,, \quad m \in \mathbb{Z}\,. \tag{17}$$

Each of these equations still contains a linear term, whereas the linear part is completely eliminated in (15). A sequence $\big(w_m(t)\big)_{m\in\mathbb{Z}}$ is a solution of (17) if and only if

$$v_k(t) = \exp(\mathrm{i}k^2\alpha t)w_k(t)\,, \quad k \in \mathbb{Z}$$

is a solution of (15). Hence, (17) and (15) are equivalent, but neither of the two equations is equivalent to (12).

The original GTE proposed in [14,15] is obtained analogously if the DMNLS is considered on $\mathbb{R}$ instead of $\mathbb{T}$, the Fourier series (4) is replaced by the Fourier transform, and the double sum in the nonlinear term is exchanged for a double integral.

4 The Adiabatic Midpoint Rule

4.1 Construction

In this section we construct a novel numerical method for the tDMNLS (12) by extending ideas from [25, 26]. The aim is to obtain a method which attains the desired accuracy with a significantly larger step-size than traditional methods.

We choose $N \in \mathbb{N}$ and let $t_n = n\tau$ with step-size $\tau = T/N$. Integrating the tDMNLS from t_{n-1} to t_{n+1} yields for every $m \in \mathbb{Z}$

$$y_m(t_{n+1}) = y_m(t_{n-1}) + \mathrm{i} \sum_{I_m} \int_{t_{n-1}}^{t_{n+1}} \widehat{Y}_{jklm}(s) \exp\left(-\mathrm{i}\omega_{[jklm]} \phi\left(\tfrac{s}{\varepsilon}\right)\right) \mathrm{d}s\,. \tag{18}$$

We employ the abbreviations

$$Y_{jkl}^{(n)} = y_j^{(n)} \overline{y}_k^{(n)} y_l^{(n)} \quad \text{and} \quad \widehat{Y}_{jklm}^{(n)} = Y_{jkl}^{(n)} \exp(-\mathrm{i}\omega_{[jklm]} t_n \alpha)$$

as in (11) and obtain a two-step method by approximating $y_m(t_n) \approx y_m^{(n)}$ via

$$y_m^{(n+1)} = y_m^{(n-1)} + \mathrm{i} \sum_{I_m} \widehat{Y}_{jklm}^{(n)} \int_{t_{n-1}}^{t_{n+1}} \exp\left(-\mathrm{i}\omega_{[jklm]} \phi\left(\tfrac{s}{\varepsilon}\right)\right) \mathrm{d}s\,. \tag{19}$$

Because the integral in (19) is highly oscillatory it is essential not to approximate it ingenuously by a quadrature formula. However, as ϕ is a piecewise linear function the remaining integral in (19) can be computed exactly and efficiently by a partition of the interval $[t_{n-1}, t_{n+1}]$ at multiples of ε and using the relation (14).

As starting step of method (19) we propose the corresponding one-step method

$$y_m^{(n+1)} = y_m^{(n)} + \mathrm{i} \sum_{I_m} \widehat{Y}_{jklm}^{(n)} \int_{t_n}^{t_{n+1}} \exp\left(-\mathrm{i}\omega_{[jklm]} \phi\left(\tfrac{s}{\varepsilon}\right)\right) \mathrm{d}s\,. \tag{20}$$

The numerical method (19) will be referred to as adiabatic midpoint rule and we call method (20) the adiabatic Euler method, respectively. Both names are derived from [24] where the construction idea originated from.

For space discretization the infinite sum (4) may be replaced by a finite sum via the spectral collocation method: the solution $u(t,x)$ is approximated by a trigonometric polynomial satisfying (1) only in $L \in \mathbb{N}$ equidistant collocation points $x_k = -\pi + 2\pi k/L$, $k = 0, \ldots, L-1$; see [11] for details. However, in this paper we focus on the semi-discretization in time with the adiabatic midpoint rule.

4.2 Relation to the Limit System

Since for $\varepsilon \to 0$ the tDMNLS converges to the limit system (15) (see Theorem 1 below), we also investigate the behavior of the numerical method in the limit $\varepsilon \to 0$. For arbitrary $a, b \in \mathbb{R}$ the partition $a = L_1\varepsilon - r_1^*$ and $b = L_2\varepsilon + r_2^*$ with $L_1, L_2 \in \mathbb{N}$ and $r_1^*, r_2^* \in [0, \varepsilon)$ leads for $\omega = \omega_{[jklm]}$ to the decomposition

$$\int_a^b \exp\left(-\mathrm{i}\omega\phi\left(\tfrac{s}{\varepsilon}\right)\right) \mathrm{d}s = I_1 + I_2 + I_3$$

with

$$I_1 = \int_{L_1\varepsilon - r_1^*}^{L_1\varepsilon} \exp\left(-\mathrm{i}\omega\phi\left(\tfrac{s}{\varepsilon}\right)\right) \mathrm{d}s\,, \qquad I_2 = \int_{L_1\varepsilon}^{L_2\varepsilon} \exp\left(-\mathrm{i}\omega\phi\left(\tfrac{s}{\varepsilon}\right)\right) \mathrm{d}s\,,$$

$$I_3 = \int_{L_2\varepsilon}^{L_2\varepsilon + r_2^*} \exp\left(-\mathrm{i}\omega\phi\left(\tfrac{s}{\varepsilon}\right)\right) \mathrm{d}s\,.$$

As $|I_1| \le \varepsilon$ and $|I_3| \le \varepsilon$ we have $\lim_{\varepsilon\to 0} I_1 = 0$ and $\lim_{\varepsilon\to 0} I_3 = 0$, respectively. From (14) it follows that

$$\lim_{\varepsilon\to 0} I_2 = (b-a)\int_0^1 \exp\left(\mathrm{i}\omega\delta\xi\right)\mathrm{d}\xi\,.$$

Consequently, for fixed τ the adiabatic midpoint rule (19) for the tDMNLS reduces in the limit $\varepsilon \to 0$ to the classical explicit midpoint rule

$$v_m^{(n+1)} = v_m^{(n-1)} + 2\tau\mathrm{i}\sum_{I_m}\widehat{V}_{jklm}^{(n)}\int_0^1 \exp\left(\mathrm{i}\omega_{[jklm]}\delta\xi\right)\mathrm{d}\xi \tag{21}$$

for the limit system (15).

Furthermore, there is an additional relation to the limit system (15) for *fixed* ε. If we choose step-sizes $\tau = k\varepsilon$ for some $k \in \mathbb{N}$, the integral in (19) simplifies to

$$\int_{t_{n-1}}^{t_{n+1}} \exp\left(-\mathrm{i}\omega_{[jklm]}\phi\left(\tfrac{s}{\varepsilon}\right)\right)\mathrm{d}s = 2\tau\int_0^1 \exp\left(\mathrm{i}\omega_{[jklm]}\delta\xi\right)\mathrm{d}\xi\,,$$

cf. (14). In this case the adiabatic midpoint rule (19) is equivalent to

$$y_m^{(n+1)} = y_m^{(n-1)} + 2\tau\mathrm{i}\sum_{I_m}\widehat{Y}_{jklm}^{(n)}\int_0^1 \exp\left(\mathrm{i}\omega_{[jklm]}\delta\xi\right)\mathrm{d}\xi\,, \tag{22}$$

which is again the classical explicit midpoint rule applied to the limit equation (15), i.e. $y_m^{(n)} \approx v_m(t_n)$. This results in an advantageous error behavior for these specific step-sizes; cf. Theorem 4 below. In order to understand this behavior we analyze the accuracy of the limit system (15) as an approximation for the tDMNLS in section 6.1.

5 The Analytic Setting

In this section we provide a suitable analytic setting to study the DMNLS (5) and the adiabatic midpoint rule (19). The setting is adopted from [11, VII.2.]. We define $|m|_+ := \max\{1, m\}$ and consider for $z = \left(z_m\right)_{m\in\mathbb{Z}}$ with $z_m \in \mathbb{C}$ the norm

$$\|z\|_{\ell_s^1} = \sum_{m\in\mathbb{Z}} |m|_+^s\,|z_m|\,, \quad s \ge 0$$

and the corresponding Banach space

$$\ell_s^1 := \left\{ z \subset \mathbb{C} \mid \|z\|_{\ell_s^1} < \infty \right\}.$$

In addition, we introduce the norm

$$\|z\|_{\ell_s^2} = \left(\sum_{m\in\mathbb{Z}} |m|_+^{2s}\,|z_m|^2\right)^{\frac{1}{2}}, \quad s \ge 0$$

and define the space

$$\ell_s^2 := \left\{ z \subset \mathbb{C} \mid \|z\|_{\ell_s^2} < \infty \right\} \cong H^s(\mathbb{T})\,,$$

where $H^s(\mathbb{T})$ is the classical Sobolev space on the one-dimensional torus. We identify $H^0(\mathbb{T}) = L_2(\mathbb{T})$. The connection between the sequence space and the function space is provided by the Fourier transform which allows to associate a complex sequence $\big(z_m\big)_{m\in\mathbb{Z}}$ with a complex function u and vice versa.

Observe that the Fourier transform (4) as well as transformation (9) are isometries in $L_2(\mathbb{T})$. Therefore, the natural choice to investigate the tDMNLS and the adiabatic midpoint rule (19) is the ℓ_0^2 norm. However, to cope with the polynomial nonlinearity the space ℓ_0^1 is much more convenient because it is a Banach algebra, see [38, IX.1]. This occurs in the following principle which is frequently used in Sections 7 and 8: if $a, b, c \in \ell_0^1$ and $d = \big(d_m\big)_{m\in\mathbb{Z}}$ is given by $d_m = \sum_{I_m} a_j b_k c_l$, then $d \in \ell_0^1$ and

$$
\begin{aligned}
\|d\|_{\ell_0^1} &= \sum_{m\in\mathbb{Z}} \Big| \sum_{I_m} a_j b_k c_l \Big| \\
&\le \Big(\sum_{j\in\mathbb{Z}} |a_j| \Big) \Big(\sum_{k\in\mathbb{Z}} |b_k| \Big) \Big(\sum_{l\in\mathbb{Z}} |c_l| \Big) = \|a\|_{\ell_0^1} \|b\|_{\ell_0^1} \|c\|_{\ell_0^1} \, .
\end{aligned}
$$

Since we have for s and r with $r - s \ge 1/2$ the embedding

$$
\ell_r^2 \hookrightarrow \ell_s^1 \hookrightarrow \ell_s^2 \,, \quad \text{i.e.} \quad \|z\|_{\ell_s^2} \le \|z\|_{\ell_s^1} \le C \, \|z\|_{\ell_r^2} \tag{23}
$$

for some constant C, we may prove error bounds in ℓ_0^1 to obtain error bounds in ℓ_0^2.

To the best of our knowledge there are no analytical results concerning existence, uniqueness and regularity of solutions for the DMNLS, we thus make the following assumption.

Assumption 1. *We suppose that there exists a unique global strong solution u of the DMNLS with either*

$$
a) \quad u \in H^1(\mathbb{T}) \qquad\qquad b) \quad u \in H^3(\mathbb{T}) \qquad\qquad c) \quad u \in H^5(\mathbb{T})
$$

for $t \in [0, T]$.

Remark. Assumption 1 implies via the embedding (23) that there exists a unique global strong solution y of the tDMNLS with either

$$
a) \quad y \in \ell_1^2 \subset \ell_0^1 \qquad\qquad b) \quad y \in \ell_3^2 \subset \ell_2^1 \qquad\qquad c) \quad y \in \ell_5^2 \subset \ell_4^1
$$

for $t \in [0, T]$.

In addition, we make the following assumption regarding solutions of the limit system (15)

Assumption 2. *We suppose that there exists a unique global strong solution v of the limit system* (15) *with either*

$$
a) \quad v \in \ell_0^1 \qquad\qquad b) \quad v \in \ell_2^1 \qquad\qquad c) \quad v \in \ell_4^1
$$

for $t \in [0, T]$.

Henceforth, we will write ℓ^1 instead of ℓ_0^1 for simplicity. Throughout the paper, $C > 0$ and $C(\cdot) > 0$ denote universal constants, possibly taking different values at various appearances. The notation $C(\cdot)$ means that the constant depends only on the values specified in the brackets.

6 Main Results

6.1 Convergence of the tDMNLS to the Limit System

We observed in section 4 that for step-sizes which are multiples of ε the accuracy of the adiabatic midpoint rule (19) depends on error bounds between solutions of the tDMNLS (12) and the limit system (15). Such bounds are stated in the following theorem; see also [33, 41]. We employ the abbreviations

$$M_s^y := \max_{t\in[0,T]} \|y(t)\|_{\ell_s^1}\,, \qquad M_s^v := \max_{t\in[0,T]} \|v(t)\|_{\ell_s^1} \tag{24}$$

and

$$M_s := \max\{M_s^y, M_s^v\}\,.$$

Theorem 1. *Let y and v be solutions of the tDMNLS* (12) *and the limit system* (15)*, respectively.*

(i) Suppose that assumption 1a) and 2a) hold, and that $y(0) = v(0) \in \ell^1$. Then, we have for $t \in [0,T]$

$$\|y(t) - v(t)\|_{\ell^1} \le \varepsilon C(t,\alpha,\delta,M_0)e^{tC(M_0)}\,.$$

(ii) Suppose that assumption 1b) and 2b) hold, and $y(0) = v(0) \in \ell_2^1$. Then, we have for $t_k = \varepsilon k$ with $k \in \mathbb{N}$ and $t_k \in [0,T]$

$$\|y(t_k) - v(t_k)\|_{\ell^1} \le \frac{\varepsilon^2}{\delta}\big(C(t_k,\alpha,M_0) + \alpha C(t_k,\delta,M_2)\big)e^{t_kC(M_0)}\,.$$

The proof of Theorem 1 is postponed to section 7.

Remarks.

(a) If $\alpha = 0$, then it is sufficient to assume 1a) and 2a) instead of 1b) and 2b) in part (ii) of Theorem 1.

(b) Corresponding results for the GTE on $\mathbb{R}$ have been obtained in [33] under slightly higher regularity requirements.

In the following, we illustrate Theorem 1 with a numerical example. We consider the DMNLS with $\alpha = 0.1$, $\delta = 0.1$ and $T = 1$ with initial value[1] $u_0(x) = e^{-3x^2}e^{3ix}$ and 128 grid points in the interval $[-\pi,\pi]$ for $\varepsilon = 0.1, 0.05, 0.01$. Figure 1 shows the evolution in time of the real and imaginary part of the coefficient $y_k(t)$ for $k = -5$. For decreasing values of ε the speed of the small scale oscillations increases but their amplitude decreases. In fact, we observe the convergence for $\varepsilon \to 0$ to the corresponding coefficient of the limit system (15) started with the same parameters and initial value.

Moreover, we observe intersections with the limit equation close to multiples of ε. Figure 2 shows the evolution over time of the real part and imaginary part of the difference $y_k(t) - v_k(t)$ for fixed $\varepsilon = 0.1$ and $k = 0,\dots,-15$. We see that the difference is much smaller for all coefficients at multiples of ε, in accordance with the improved error bound of part (ii) of Theorem 1.

[1] The initial value is only approximately periodic, but this can be neglected.

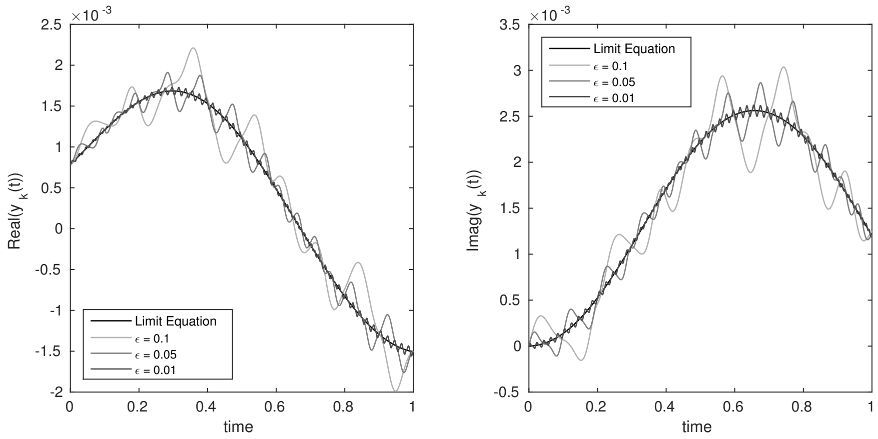

Figure 1: Evolution over time of the real part (left) and imaginary part (right) of one coefficient of a solution for the limit system (15) and the corresponding coefficient of a solution for the tDMNLS computed with $\varepsilon = 0.1,\ 0.05,\ 0.01$.

6.2 Accuracy of the Adiabatic Midpoint Rule

This section is devoted to the results of the error analysis for the adiabatic midpoint rule (19). We make the following assumption.

Assumption 3. *Let $y^{(n)}$ be the approximation to the tDMNLS by the adiabatic midpoint rule* (19)*. We suppose that $y^{(n)}$ is bounded in ℓ^1 for all $\tau n \leq T$ with a constant independent of τ.*

Now, we abbreviate

$$M_0^\star := \{M_0^y, \max_{\tau n \leq T} \|y_n\|_{\ell^1}\} \quad \text{and} \quad M_0^\bullet := \{M_0^v, \max_{\tau n \leq T} \|y_n\|_{\ell^1}\}$$

with M_0^y and M_0^v defined in (24).

Theorem 2. *Under assumption 1b) and 3 we have for $\tau n \leq T$*

$$\left\| y(t_n) - y^{(n)} \right\|_{\ell^1} \leq \tau\big(C(T, M_0^\star) + \alpha C(T, M_0^\star, M_2^y)\big) e^{TC(\alpha, M_0^\star)} .$$

In contrast to the "classical" error behavior of the explicit midpoint rule applied to a non-oscillatory problem, we do not obtain second order convergence for arbitrary step-sizes. Instead, Theorem 2 yields first-order convergence, but with a constant independently of ε. Such estimates can usually not be obtained for "classical" methods applied to the DMNLS, because here the constant typically contains the factor $1/\varepsilon$, and reasonable accuracy can only be expected for $\tau \ll \varepsilon$.

The next theorem states that the accuracy improves by a factor of ε for specific choices of step-sizes τ.

Theorem 3. *Suppose that assumption 1c) and 3 hold. If we choose the step-size $\tau = \varepsilon/k$ for some $k \in \mathbb{N}$ for the adiabatic midpoint rule* (19)*, then we have for $\tau n \leq T$*

$$\left\| y(t_n) - y^{(n)} \right\|_{\ell^1} \leq \varepsilon\tau\Big(C(T, M_0^\star, M_0^y) + \alpha C(T, M_0^\star, M_4^y)\Big) e^{TC(\alpha, M_0^\star)} .$$

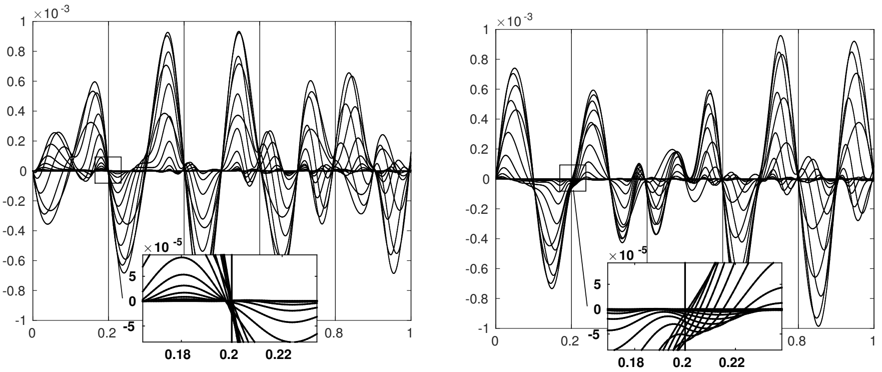

Figure 2: Evolution over time of the real part (left) and imaginary part (right) of the difference between various coefficients of a solution for the tDMNLS with $\varepsilon = 0.1$ and the corresponding coefficients of the solution for the limit system (15) computed with the same initial value.

The last theorem is based on the fact that for step-sizes $\tau = k\varepsilon$ the adiabatic midpoint rule (19) reduces to the explicit midpoint rule applied to the limit equation (15); cf. (22). Again, this results in an improved error behavior compared to Theorem 2.

Theorem 4. *Suppose that assumption 1b), 2c) and 3 hold. If we choose the step-size $\tau = \varepsilon k$ for some $k \in \mathbb{N}$ for the adiabatic midpoint rule* (19)*, then we have for $\tau n \leq T$*

$$\left\| y(t_n) - y^{(n)} \right\|_{\ell^1} \leq \left(\tfrac{\varepsilon^2}{\delta} + \tau^2\right)\left(C(T, \alpha, M_0^\bullet, M_2^v) + \alpha C(T, M_0^\bullet, M_2, M_4^v)\right) e^{TC(\alpha, M_0^\bullet)} .$$

Theorems 2-4 are proved in section 8.

Remark. If $\alpha = 0$, then the regularity requirements for Theorems 2-4 lower significantly.

In the following we demonstrate the assertions of Theorems 2-4 by a numerical example. We consider the tDMNLS with $\alpha = 0.1$, $\varepsilon \in \{0.005, 0.01\}$, $T = 1$, $\delta = 1$ and the initial value $u_0(x) = e^{-3x^2} e^{3ix}$ with 64 equidistant grid points in the interval $[-\pi, \pi]$. To this setting we apply the Strang splitting method, the adiabatic Euler method (20) and the adiabatic midpoint rule (19). The reference solution is computed by the Strang splitting method with a large number of steps ($\approx 10^6$).

The left panels of Figure 3 show the accuracy of the three methods for roughly logarithmically spread step-sizes τ and $\varepsilon = 0.01$ (top) and $\varepsilon = 0.005$ (bottom). The behavior of the Strang splitting and the adiabatic midpoint rule appears to be somewhat erratic in the sense that small changes of the step-size may change the error by a factor of 10 to 100. Although the adiabatic Euler is only a first-order method it yields a significantly higher accuracy than Strang splitting with the same step-size. The highest accuracy is obtained with the adiabatic midpoint rule. Even though this method is "better than order one for many step-sizes", several outliers stipulate first order convergence as claimed in Theorem 2. The right panels of Figure 3 display solely the error of the adiabatic

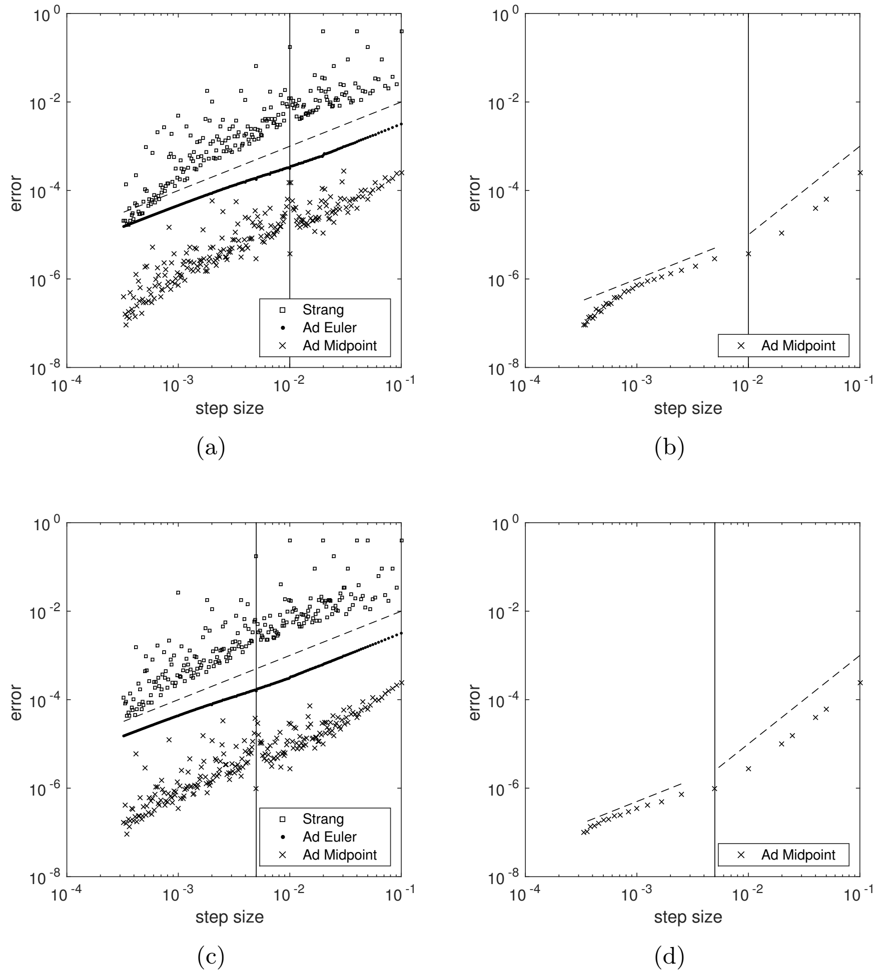

Figure 3: Maximal ℓ_0^2-error over time of the adiabatic midpoint rule (19) for $\varepsilon = 0.01$ (top), $\varepsilon = 0.005$ (bottom). In 3a and 3c we included the error of the adiabatic Euler (20) and the Strang splitting as reference and chose logarithmically spread step-sizes τ. In 3b and 3d the step-sizes are chosen according to Theorem 3 and Theorem 4.

midpoint rule for step-sizes chosen according to Theorem 3 and Theorem 4, i.e. integer multiples and fractions of ε, again for $\varepsilon = 0.01$ (top) and $\varepsilon = 0.005$ (bottom). We observe second order convergence for $\tau > \varepsilon$ and convergence in $O(\tau\varepsilon)$ for $\tau < \varepsilon$ as stated in Theorem 3 and Theorem 4.

Conclusion. The previous theorems and observations show that in applications where a moderate accuracy of $O(\tau^2)$ with $\tau \geq \varepsilon$ is sufficient, one can solve the tDMNLS with the adiabatic midpoint rule and step-sizes $\tau = k\varepsilon$. This yields exactly the same approximation as the classical explicit midpoint rule applied to the limit equation (15).

In principle, one could use a higher-order method to solve the limit system (15). For a method of order p this would improve the accuracy from $O(\varepsilon^2+\tau^2) = O(\tau^2)$ to $O(\varepsilon^2 + \tau^p) = O(\max\{\varepsilon^2, \tau^p\})$. For $p > 2$, however, more than one evaluation of the right-hand side of (15) is necessary, and a very high order will not pay because the total error cannot be smaller than $O(\varepsilon^2)$ due to the error of the limit equation. If a better accuracy than $O(\varepsilon^2)$ is desired, (12) must be solved with the adiabatic midpoint rule and the special step-size $\tau = \varepsilon/k$ for some $k \in \mathbb{N}$. This gives an accuracy of $O(\tau\varepsilon) = O(\varepsilon^2/k)$, whereas an arbitrary step-size $\tau \leq \varepsilon$ only ensures an accuracy of $O(\tau)$.

After solving the tDMNLS numerically in either case, approximations to the solution of the DMNLS are obtained via the back-transformation of (8). Since the step-size is an integer multiple or a fraction of ε, this is also true for all t_n, however, approximations at other times $t \in (t_n, t_{n+1})$ can easily be obtained, e.g., by using the Strang splitting with a tiny step-size, but now only on the small interval $[t_n, t]$ of length $O(\tau)$. The same principle is used in numerical stroboscopic averaging, cf. [6].

7 Proof of Theorem 1

Before we start with the proof of Theorem 1, we make a few preparations. First, we define for $\omega \neq 0$

$$g_\omega(\sigma) = \exp\big(-\mathrm{i}\omega\phi(\sigma)\big) - \frac{\exp(\mathrm{i}\omega\delta) - 1}{\mathrm{i}\omega\delta}\,. \tag{25}$$

Second, we prove the following lemma which contains estimates for integrals involving the product of $g_\omega(\sigma)$ with a sufficiently smooth function.

Lemma 1. *For $\varepsilon > 0$, $\omega \neq 0$ and $f \in C^2(\mathbb{R})$ we have*

$$(i)\quad \left|\int_0^1 f(\varepsilon\sigma)g_\omega(\sigma)\,\mathrm{d}\sigma\right| \leq \frac{\varepsilon}{\delta} C \max_{\sigma\in[0,1]} \left|\omega^{-1} f'(\varepsilon\sigma)\right|\,,$$

$$(ii)\quad \left|\int_1^2 f(\varepsilon\sigma)g_\omega(\sigma)\,\mathrm{d}\sigma\right| \leq \frac{\varepsilon}{\delta} C \max_{\sigma\in[1,2]} \left|\omega^{-1} f'(\varepsilon\sigma)\right|\,,$$

$$(iii)\quad \left|\int_0^2 f(\varepsilon\sigma)g_\omega(\sigma)\,\mathrm{d}\sigma\right| \leq \frac{\varepsilon^2}{\delta} C \max_{\sigma\in[0,2]} \left|\omega^{-1} f''(\varepsilon\sigma)\right|\,.$$

Remark. Part (i) and (ii) of Lemma 1 will lead to the $O(\varepsilon)$ bound in Theorem 1, whereas part (iii) is the backbone of the improved error bound of $O(\varepsilon^2)$.

Proof. Equation (2), (3) and (7) yield

$$g_\omega(\sigma) = \begin{cases} g_{\omega,1}(\sigma) & \text{if} \quad \sigma \in [0,1)\,, \\ g_{\omega,2}(\sigma) & \text{if} \quad \sigma \in [1,2) \end{cases}$$

with

$$g_{\omega,1}(\sigma) = \exp(\mathrm{i}\omega\delta\sigma) - \frac{\exp(\mathrm{i}\omega\delta) - 1}{\mathrm{i}\omega\delta}$$

and

$$g_{\omega,2}(\sigma) = \exp(\mathrm{i}\omega\delta(2-\sigma)) - \frac{\exp(\mathrm{i}\omega\delta) - 1}{\mathrm{i}\omega\delta}\,.$$

In order to prove (i) we define the function

$$G_{\omega,1}(\sigma) = \frac{\exp(\mathrm{i}\omega\delta\sigma) - 1}{\mathrm{i}\omega\delta} - \sigma\frac{\exp(\mathrm{i}\omega\delta) - 1}{\mathrm{i}\omega\delta}\,.$$

It is easily seen that $G'_{\omega,1}(\sigma) = g_{\omega,1}(\sigma)$ and $G_{\omega,1}(0) = G_{\omega,1}(1) = 0$. Since

$$\max_{\sigma\in[0,1]} |\omega G_{\omega,1}(\sigma)| \le \frac{4}{\delta}\,,$$

integration by parts yields inequality (i). The second inequality follows analogously with

$$G_{\omega,2}(\sigma) = \frac{\exp(\mathrm{i}\omega\delta(2-\sigma)) - \exp(\mathrm{i}\omega\delta)}{-\mathrm{i}\omega\delta} - (\sigma - 1)\frac{\exp(\mathrm{i}\omega\delta) - 1}{\mathrm{i}\omega\delta}\,,$$

using that $G'_{\omega,2}(\sigma) = g_{\omega,2}(\sigma)$, $G_{\omega,2}(1) = G_{\omega,2}(2) = 0$, and

$$\max_{\sigma\in[1,2]} |\omega G_{\omega,2}(\sigma)| \le \frac{4}{\delta}\,.$$

For inequality (iii), we set

$$\widetilde{G}_{\omega,1}(\sigma) = \frac{1}{\mathrm{i}\omega\delta}\left(\frac{\exp(\mathrm{i}\omega\delta\sigma) - 1}{\mathrm{i}\omega\delta} - \sigma - \frac{\sigma^2}{2}\big(\exp(\mathrm{i}\omega\delta) - 1\big)\right)$$

and

$$\widetilde{G}_{\omega,2}(\sigma) = \frac{1}{-\mathrm{i}\omega\delta}\left(\frac{\exp(\mathrm{i}\omega\delta(2-\sigma)) - 1}{-\mathrm{i}\omega\delta} \right. \\ \left. - (\sigma - 2)\exp(\mathrm{i}\omega\delta) + \frac{\sigma^2 - 2\sigma}{2}\big(\exp(\mathrm{i}\omega\delta) - 1\big)\right)\,.$$

It is easy to check that

$$\widetilde{G}'_{\omega,1}(\sigma) = G_{\omega,1}(\sigma)\,, \quad \widetilde{G}'_{\omega,2}(\sigma) = G_{\omega,2}(\sigma)$$

and

$$\widetilde{G}_{\omega,1}(0) = \widetilde{G}_{\omega,2}(2) = 0\,, \quad \widetilde{G}_{\omega,1}(1) = \widetilde{G}_{\omega,2}(1)\,.$$

Since

$$\left|\frac{\exp(\mathrm{i}\omega\delta\theta) - 1}{\mathrm{i}\omega\delta}\right| = \left|\int_0^\theta \exp(\mathrm{i}\omega\delta\xi)\,\mathrm{d}\xi\right| \le |\theta|\,, \tag{26}$$

we obtain the estimates

$$\max_{\sigma\in[0,1]} \left|\omega \widetilde{G}_{\omega,1}(\sigma)\right| \le \frac{3}{\delta} \quad \text{and} \quad \max_{\sigma\in[1,2]} \left|\omega \widetilde{G}_{\omega,2}(\sigma)\right| \le \frac{3}{\delta}\,.$$

Hence, applying integration by parts two times yields

$$\begin{aligned}\left|\int_0^2 f(\varepsilon\sigma) g_\omega(\sigma)\,\mathrm{d}\sigma\right| &= \left|\int_0^1 f(\varepsilon\sigma) g_{\omega,1}(\sigma)\,\mathrm{d}\sigma + \int_1^2 f(\varepsilon\sigma) g_{\omega,2}(\sigma)\,\mathrm{d}\sigma\right| \\ &\le \frac{\varepsilon^2}{\delta} C \max_{\sigma\in[0,2]} \left|\omega^{-1} f''(\varepsilon\sigma)\right|\,.\end{aligned}$$

□

We are now in a position to prove Theorem 1. It is convenient to split the index set I_m into

$$R_m = \left\{(j,k,l)\in I_m : \omega_{[jklm]} = 0\right\} \quad \text{and} \quad N_m = \left\{(j,k,l)\in I_m : \omega_{[jklm]} \neq 0\right\}$$

due to (14). Then, integrating (15) gives

$$v_m(t) = v_m(0) + \mathrm{i}\sum_{R_m}\int_0^t V_{jkl}(s)\,\mathrm{d}s + \mathrm{i}\sum_{N_m} \tfrac{\exp(\mathrm{i}\omega_{[jklm]}\delta)-1}{\mathrm{i}\omega_{[jklm]}\delta} \int_0^t \widehat{V}_{jklm}(s)\,\mathrm{d}s\,. \quad (27)$$

Likewise, we obtain with (12)

$$y_m(t) = y_m(0) + \mathrm{i}\sum_{R_m}\int_0^t Y_{jkl}(s)\,\mathrm{d}s + \mathrm{i}\sum_{N_m}\int_0^t \widehat{Y}_{jklm}(s)\exp\left(-\mathrm{i}\omega_{[jklm]}\phi\left(\tfrac{s}{\varepsilon}\right)\right)\mathrm{d}s\,. \quad (28)$$

If we now subtract (27) from (28), we arrive at

$$\begin{aligned}\|y(t)-v(t)\|_{\ell^1} \le &\left\|\sum_{R_m}\int_0^t \Big(Y_{jkl}(s) - V_{jkl}(s)\Big)\,\mathrm{d}s\right\|_{\ell^1} \\ &+ \left\|\sum_{N_m}\int_0^t \Big(\widehat{Y}_{jklm}(s) - \widehat{V}_{jklm}(s)\Big)\exp\left(-\mathrm{i}\omega_{[jklm]}\phi\left(\tfrac{s}{\varepsilon}\right)\right)\mathrm{d}s\right\|_{\ell^1} \\ &+ \left\|\sum_{N_m}\int_0^t \widehat{V}_{jklm}(s) g_{\omega_{[jklm]}}\left(\tfrac{s}{\varepsilon}\right)\mathrm{d}s\right\|_{\ell^1}\end{aligned}$$

with $g_{\omega_{[jklm]}}$ defined in (25). Since

$$\begin{aligned}\left|\widehat{Y}_{jklm}(s) - \widehat{V}_{jklm}(s)\right| &= |Y_{jkl}(s) - V_{jkl}(s)| \\ &\le |y_j(s)-v_j(s)|\cdot|\bar{y}_k(s)|\cdot|y_l(s)| \\ &\quad + |v_j(s)|\cdot|\bar{y}_k(s)-\bar{v}_k(s)|\cdot|y_l(s)| \\ &\quad + |v_j(s)|\cdot|\bar{v}_k(s)|\cdot|y_l(s)-v_l(s)|\,,\end{aligned}$$

we obtain

$$\begin{aligned}\|y(t)-v(t)\|_{\ell^1} \le C(M_0)&\int_0^t \|y(s)-v(s)\|_{\ell^1}\,\mathrm{d}s \\ &+ \left\|\sum_{N_m}\int_0^t \widehat{V}_{jklm}(s) g_{\omega_{[jklm]}}\left(\tfrac{s}{\varepsilon}\right)\mathrm{d}s\right\|_{\ell^1}\,. \quad (29)\end{aligned}$$

To complete the proof of Theorem 1, we deduce an estimate for the second term in (29) under the assumptions (i) and (ii), respectively. Then, the two assertions follow from Gronwall's lemma.

First, we fix $t \in [0,T]$ and use the partition $t = (L+\theta)\varepsilon + t_\varepsilon$ with $L \in \mathbb{N}$ even, $\theta \in \{0,1\}$ and $t_\varepsilon \in [0,\varepsilon)$ to obtain the decomposition

$$\int_0^t \widehat{V}_{jklm}(s) g_{\omega_{[jklm]}}\left(\tfrac{s}{\varepsilon}\right) \mathrm{d}s = T^{(1)}_{jklm} + T^{(2)}_{jklm} + T^{(3)}_{jklm}$$

with

$$T^{(1)}_{jklm} = \sum_{p=0}^{\frac{L}{2}-1} \int_{2\varepsilon p}^{2\varepsilon(p+1)} \widehat{V}_{jklm}(s) g_{\omega_{[jklm]}}\left(\tfrac{s}{\varepsilon}\right) \mathrm{d}s\,, \tag{30}$$

$$T^{(2)}_{jklm} = \int_{\varepsilon L}^{\varepsilon(L+\theta)} \widehat{V}_{jklm}(s) g_{\omega_{[jklm]}}\left(\tfrac{s}{\varepsilon}\right) \mathrm{d}s \tag{31}$$

and

$$T^{(3)}_{jklm} = \int_{\varepsilon(L+\theta)}^{\varepsilon(L+\theta)+t_\varepsilon} \widehat{V}_{jklm}(s) g_{\omega_{[jklm]}}\left(\tfrac{s}{\varepsilon}\right) \mathrm{d}s\,.$$

We conclude from (26) that $\left|g_{\omega_{[jklm]}}\left(\tfrac{s}{\varepsilon}\right)\right| \le 2$ and since $\left|\widehat{V}_{jklm}(s)\right| = |V_{jkl}(s)|$, we immediately obtain

$$\left\|\sum_{N_m} T^{(2)}_{jklm}\right\|_{\ell^1} \le \varepsilon C(M_0) \quad \text{and} \quad \left\|\sum_{N_m} T^{(3)}_{jklm}\right\|_{\ell^1} \le \varepsilon C(M_0)\,.$$

If we substitute $\sigma = s/\varepsilon$ in each summand of (30) and apply part (i) and (ii) of Lemma 1, we get the estimate

$$\left\|\sum_{N_m} T^{(1)}_{jklm}\right\|_{\ell^1} \le \frac{\varepsilon}{\delta} C t \max_{\xi\in[0,t]} \sum_{m\in\mathbb{Z}} \sum_{N_m} \left|\omega^{-1}_{[jklm]} \widehat{V}'_{jklm}(\xi)\right|\,.$$

Since $0 \le |\omega^{-1}_{[jklm]}| \le 1$ for $m \in \mathbb{Z}$ and $(j,k,l) \in N_m$, we obtain

$$\left|\omega^{-1}_{[jklm]} \widehat{V}'_{jklm}(s)\right| \le \left|V'_{jkl}(s)\right| + \alpha \left|V_{jkl}(s)\right|\,. \tag{32}$$

Applying Lemma 3 (see Appendix) gives

$$\left\|\sum_{N_m} \int_0^t \widehat{V}_{jklm}(s) g_{\omega_{[jklm]}}\left(\tfrac{s}{\varepsilon}\right) \mathrm{d}s\right\|_{\ell^1} \le \varepsilon C(t,\delta,\alpha,M_0)\,. \tag{33}$$

Now, combining (29) with (33) results in

$$\|y(t) - v(t)\|_{\ell^1} \le C(M_0) \int_0^t \|y(s) - v(s)\|_{\ell^1} \mathrm{d}s + \varepsilon C(t,\delta,\alpha,M_0)$$

and Gronwall's lemma yields part (i) of Theorem 1.

We attain part (ii) by improving the estimate (33). Since now t_k is a multiple of ε, we have $t_k = (L+\theta)\varepsilon$ with $L \in \mathbb{N}$ even, $\theta \in \{0,1\}$, and hence

$$\int_0^{t_k} \widehat{V}_{jklm}(s) g_{\omega_{[jklm]}}\left(\tfrac{s}{\varepsilon}\right) \mathrm{d}s = T^{(1)}_{jklm} + T^{(2)}_{jklm}$$

with $T^{(1)}_{jklm}$ and $T^{(2)}_{jklm}$ given in (30) and (31). After substituting $\sigma = s/\varepsilon$ Lemma 1 (iii) yields the estimate

$$\left\| \sum_{N_m} T^{(1)}_{jklm} \right\|_{\ell^1} \le \frac{\varepsilon^2}{\delta} C t_k \max_{s\in[0,t]} \sum_{m\in\mathbb{Z}} \sum_{N_m} \left| \omega^{-1}_{[jklm]} \widehat{V}''_{jklm}(s) \right|$$

and Lemma 1 (i) yields

$$\left\| \sum_{N_m} T^{(2)}_{jklm} \right\|_{\ell^1} \le \frac{\varepsilon^2}{\delta} C t_k \max_{s\in[0,t]} \sum_{m\in\mathbb{Z}} \sum_{N_m} \left| \omega^{-1}_{[jklm]} \widehat{V}'_{jklm}(s) \right| .$$

Combining (32) and

$$\left| \omega^{-1}_{[jklm]} \widehat{V}''_{jklm}(s) \right| \le \left| V''_{jkl}(s) \right| + 2\alpha \left| V'_{jkl}(s) \right| + \alpha^2 \left| \omega_{[jklm]} V_{jkl}(s) \right|$$

with Lemma 3 (see Appendix) leads to the improved estimate

$$\left\| \sum_{N_m} \int_0^{t_k} \widehat{V}_{jklm}(s) g_{\omega_{[jklm]}}\left(\tfrac{s}{\varepsilon}\right) \mathrm{d}s \right\|_{\ell^1} \le \frac{\varepsilon^2}{\delta} \big(C(t_k, M_0) + \alpha C(t_k, M_2) \big) . \tag{34}$$

Now part (ii) of Theorem 1 follows by inserting (34) into (29) and applying Gronwall's lemma. □

8 Error Analysis: Proof of Theorems 2-4

The foundation for the proofs of Theorems 2-4 is an adaptation of the error recursion formula for the explicit midpoint rule in [24]. For this purpose, we reformulate the two-step method (19) as a one-step method. We define for $\mu = (\mu_m)_{m\in\mathbb{Z}}$ and $\nu = (\nu_m)_{m\in\mathbb{Z}}$

$$\left(A\left(\mu, t, \tfrac{s}{\varepsilon}\right) \nu \right)_m = \mathrm{i} \sum_{I_m} \mu_j \bar{\mu}_k \nu_l \exp\left(-\mathrm{i}\omega_{[jklm]} (\alpha t + \phi\left(\tfrac{s}{\varepsilon}\right)) \right) . \tag{35}$$

Then, the two-step method (19) reads

$$y^{(n+1)} = y^{(n-1)} + \int_{t_{n-1}}^{t_{n+1}} A\left(y^{(n)}, t_n, \tfrac{s}{\varepsilon} \right) y^{(n)} \, \mathrm{d}s .$$

With the abbreviations

$$z_{n+1} = \begin{pmatrix} y^{(n+1)} \\ y^{(n)} \end{pmatrix}, \; \mathcal{J} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \; \mathcal{M}_n = \begin{pmatrix} \int_{t_{n-1}}^{t_{n+1}} A\left(y^{(n)}, t_n, \frac{s}{\varepsilon}\right) \mathrm{d}s & 0 \\ 0 & 0 \end{pmatrix} \tag{36}$$

we obtain the one-step formulation

$$z_{n+1} = \left(\mathcal{J} + \mathcal{M}_n \right) z_n \tag{37}$$

of the adiabatic midpoint rule. Defining the error terms

$$d_{n+1} = \big(\mathcal{J} + \mathcal{M}_n \big) z(t_n) - z(t_{n+1}) \quad \text{with} \quad z(t_{n+1}) = \begin{pmatrix} y(t_{n+1}) \\ y(t_n) \end{pmatrix} \tag{38}$$

allows to express the global error $e_N = z_N - z(t_N)$ by the recursion formula

$$e_{N+1} = \mathcal{J}^N e_1 + \sum_{n=1}^{N} \mathcal{J}^{N-n} \mathcal{M}_n e_n + \sum_{n=1}^{N} \mathcal{J}^{N-n} d_{n+1}\,, \qquad N \geq 1\,. \tag{39}$$

The formula can be shown by induction; cf. [24]. Henceforth, we will establish estimates for the ℓ_1-norms for each part of the recursion formula (39) and then apply the discrete Gronwall lemma.

Recall that the starting step is conducted by the adiabatic Euler method (20). For arbitrary $y(0) = y_0$ it can be shown by straightforward computation that

$$\left\| \mathcal{J}^N e_1 \right\|_{\ell^1} \leq \| e_1 \|_{\ell^1} \leq \tau^2 C(\alpha, M_0^\star)\,. \tag{40}$$

Moreover, let $[\mathcal{M}_n e_n]_m$ denote the m-th entry of $\mathcal{M}_n e_n$. By (36) all non-zero entries of $\mathcal{M}_n e_n$ are of the form

$$[\mathcal{M}_n e_n]_m = \mathrm{i} \sum_{m \in \mathbb{Z}} y_j^{(n)} \overline{y}_k^{(n)} \left(y_l^{(n)} - y_l(t_n) \right) \int_{t_{n-1}}^{t_{n+1}} \exp\left(-\mathrm{i}\omega_{[jklm]}(\alpha t_n + \phi\left(\tfrac{s}{\varepsilon}\right)) \right) \mathrm{d}s\,.$$

Hence, we have

$$\left\| \sum_{n=1}^{N} \mathcal{J}^{N-n} \mathcal{M}_n e_n \right\|_{\ell^1} \leq \tau C(M_0^\star) \sum_{n=1}^{N} \| e_n \|_{\ell^1}\ . \tag{41}$$

Estimates (40) and (41) will be used in the proofs of Theorem 2-4. Ultimately, the term

$$\left\| \sum_{n=1}^{N} \mathcal{J}^{N-n} d_{n+1} \right\|_{\ell^1} \tag{42}$$

of the recursion formula (39) has to be estimated. In the following we deduce a bound for (42) in each of the settings of Theorem 2-4. Then, the discrete Gronwall lemma yields the desired error estimates.

8.1 Proof of Theorem 2

We prove the linear convergence of the adiabatic midpoint rule with a constant which does not depend on ε. Let $[d_{n+1}]_m$ denote the m-th entry of d_{n+1}. By (38) all non-zero entries of d_{n+1} are of the form

$$\begin{aligned}[d_{n+1}]_m = \mathrm{i} \sum_{I_m} \int_{t_{n-1}}^{t_{n+1}} & y_j^{(n)} \overline{y}_k^{(n)} y_l(t_n) \exp\left(-\mathrm{i}\omega_{[jklm]}(\alpha t_n + \phi\left(\tfrac{s}{\varepsilon}\right)) \right) \mathrm{d}s \\ & + y_m(t_{n-1}) - y_m(t_{n+1})\,. \end{aligned} \tag{43}$$

If we substitute

$$\begin{aligned} y_m(t_{n+1}) = y_m(t_{n-1}) &+ \mathrm{i} \sum_{I_m} \int_{t_{n-1}}^{t_{n+1}} \widehat{Y}_{jklm}(t_n) \exp\left(-\mathrm{i}\omega_{[jklm]} \phi\left(\tfrac{s}{\varepsilon}\right) \right) \mathrm{d}s \\ &+ \mathrm{i} \sum_{I_m} \int_{t_{n-1}}^{t_{n+1}} \int_{t_n}^{s} \widehat{Y}'_{jklm}(\sigma)\, \mathrm{d}\sigma \exp\left(-\mathrm{i}\omega_{[jklm]} \phi\left(\tfrac{s}{\varepsilon}\right) \right) \mathrm{d}s\,, \end{aligned}$$

we obtain the partition

$$[d_{n+1}]_m = [d^{(1)}_{n+1}]_m - [d^{(2)}_{n+1}]_m \tag{44}$$

with

$$[d^{(1)}_{n+1}]_m = \mathrm{i}\sum_{I_m}\int_{t_{n-1}}^{t_{n+1}} \big(y^{(n)}_j\overline{y}^{(n)}_k - y_j(t_n)\overline{y}_k(t_n)\big)y_l(t_n) \exp\big(-\mathrm{i}\omega_{[jklm]}(\alpha t_n + \phi\left(\tfrac{s}{\varepsilon}\right))\big)\,\mathrm{d}s$$

and

$$[d^{(2)}_{n+1}]_m = \mathrm{i}\sum_{I_m}\int_{t_{n-1}}^{t_{n+1}}\int_{t_n}^{s} \widehat{Y}'_{jklm}(\sigma)\,\mathrm{d}\sigma \exp\big(-\mathrm{i}\omega_{[jklm]}\phi\left(\tfrac{s}{\varepsilon}\right)\big)\,\mathrm{d}s\,. \tag{45}$$

Since

$$\Big|\big(y^{(n)}_j\overline{y}^{(n)}_k - y_j(t_n)\overline{y}_k(t_n)\big)y_l(t_n)\exp\big(-\mathrm{i}\omega_{[jklm]}(\alpha t_n + \phi\left(\tfrac{s}{\varepsilon}\right))\big)\Big| \\ \le \big(|y^{(n)}_j - y_j(t_n)||\overline{y}^{(n)}_k| + |\overline{y}^{(n)}_k - \overline{y}_k(t_n)|\,|y_j(t_n)|\,\big)\,|y_l(t_n)|\ ,$$

we obtain the estimate

$$\Big\|[d^{(1)}_{n+1}]_m\Big\|_{\ell^1} \le \tau C(M_0^\star)\,\|e_n\|_{\ell^1}\ . \tag{46}$$

Furthermore, differentiating (11) yields

$$\Big|\widehat{Y}'_{jklm}(\sigma)\Big| \le \big|Y'_{jkl}(\sigma)\big| + \alpha\,\big|\omega_{[jklm]}Y_{jkl}(\sigma)\big|\ . \tag{47}$$

Hence, applying Lemma 4 (see Appendix) results in

$$\Big\|[d^{(2)}_{n+1}]_m\Big\|_{\ell^1} \le \tau^2\big(C(M_0^\star) + \alpha C(M_0^\star, M_2^y)\big) \tag{48}$$

and we arrive at

$$\Bigg\|\sum_{n=1}^{N}\mathcal{J}^{N-n}d_{n+1}\Bigg\|_{\ell^1} \le \tau C(M_0^\star)\sum_{n=1}^{N}\|e_n\|_{\ell^1} + \tau\big(C(T,M_0^\star) + \alpha C(T,M_0^\star,M_2^y)\big)\,. \tag{49}$$

Combining (40), (41) and (49) with the recursion formula (39) gives

$$\|e_{N+1}\|_{\ell^1} \le \tau C(\alpha, M_0^\star)\sum_{n=1}^{N}\|e_n\|_{\ell^1} + \tau\big(C(T,M_0^\star) + \alpha C(T,M_0^\star,M_2^y)\big)\,.$$

Now, the discrete Gronwall lemma yields

$$\|e_{N+1}\|_{\ell^1} \le \tau\big(C(T,M_0^\star) + \alpha C(T,M_0^\star,M_2^y)\big)e^{TC(\alpha,M_0^\star)}$$

which completes the proof of Theorem 2. □

8.2 Proof of Theorem 3

In the setting of Theorem 3, i.e. $\tau = \varepsilon/k$ for some $k \in \mathbb{N}$, the estimate (49) can be improved. The starting point is the partition (44). Whereas we may reuse the estimate (46) for $d^{(1)}_{n+1}$, we enhance the estimate for $d^{(2)}_{n+1}$. Decomposing $N = 2kL + n^*$ with $L \in \mathbb{N}$ and $n^* \in \{0, \dots, 2k-1\}$ yields

$$\left\| \sum_{n=1}^{N} \mathcal{J}^{N-n} d^{(2)}_{n+1} \right\|_{\ell^1} \leq \left\| \sum_{n=1}^{2kL-1} \mathcal{J}^{N-n} d^{(2)}_{n+1} \right\|_{\ell^1} + \left\| \sum_{n=2kL}^{2kL+n^*} \mathcal{J}^{N-n} d^{(2)}_{n+1} \right\|_{\ell^1} . \tag{50}$$

Since $n^*\tau^2 \leq 2k\varepsilon\tau^2 \leq 2\tau\varepsilon$, equation (45) shows that

$$\left\| \sum_{n=2kL}^{2kL+n^*} \mathcal{J}^{N-n} d^{(2)}_{n+1} \right\|_{\ell^1} \leq \varepsilon\tau \big(C(M_0^\star) + \alpha C(M_0^\star, M_2^y) \big) , \tag{51}$$

by (47) and Lemma 4 (see Appendix). The crucial step to refine (49) is avoiding the triangle inequality to estimate the remaining sum in (50). Our aim is to employ instead the partition

$$\left\| \sum_{n=1}^{2kL-1} \mathcal{J}^{N-n} d^{(2)}_{n+1} \right\|_{\ell^1} \leq \left\| \sum_{\substack{n=1 \\ n \text{ odd}}}^{2kL-1} d^{(2)}_{n+1} \right\|_{\ell^1} + \left\| \sum_{\substack{n=1 \\ n \text{ even}}}^{2kL-1} d^{(2)}_{n+1} \right\|_{\ell^1}$$

and then estimate the sums over odd and even n separately by exploiting cancellation effects of double integrals of the form

$$\mathcal{I}_n = \int_{t_{n-1}}^{t_{n+1}} \int_{t_n}^{s} \exp\left(-\mathrm{i}\omega\phi\left(\tfrac{\sigma}{\varepsilon}\right)\right) \mathrm{d}\sigma \exp\left(-\mathrm{i}\widehat{\omega}\phi\left(\tfrac{s}{\varepsilon}\right)\right) \mathrm{d}s \,. \tag{52}$$

These cancellation effects are specified in the following lemma.

Lemma 2. *Let $k, L \in \mathbb{N}$ and suppose that $\tau = \varepsilon/k$. Then we have for a sequence $(a_n)_{n\in N}$ and $\mathcal{I}_n$ given in* (52)

$$(i) \quad \left| \sum_{\substack{n=1 \\ n \text{ even}}}^{2kL-1} a_n \mathcal{I}_n \right| \leq 2\varepsilon\tau \sum_{n=1}^{kL-2} \left| a_{2(n+1)} - a_{2n} \right|$$

and

$$(ii) \quad \left| \sum_{\substack{n=1 \\ n \text{ odd}}}^{2kL-1} a_n \mathcal{I}_n \right| \leq 2\varepsilon\tau \sum_{n=1}^{kL-2} \left| a_{2n+1} - a_{2n-1} \right| .$$

Proof. Applying the summation by parts formula gives

$$\begin{aligned} \sum_{\substack{n=1 \\ n \text{ even}}}^{2kL-1} a_n \mathcal{I}_n &= \sum_{n=1}^{kL-1} a_{2n} \mathcal{I}_{2n} \\ &= \left(\sum_{n=1}^{kL-1} \mathcal{I}_{2n} \right) a_{2(kL-1)} - \sum_{n=1}^{kL-2} \left(\sum_{j=1}^{n} \mathcal{I}_{2j} \right) \left(a_{2(n+1)} - a_{2n} \right) . \end{aligned} \tag{53}$$

We have $n = (kl-1) + n^*$ with $l \in \mathbb{N}$ and $n^* \in \{0, \ldots, k\}$ and thus

$$\sum_{j=1}^{n} \mathcal{I}_{2j} = \sum_{j=1}^{lk-1} \mathcal{I}_{2j} + \sum_{j=lk}^{lk+n^*} \mathcal{I}_{2j} \, .$$

Hence, if we prove that

$$\sum_{n=1}^{lk-1} \mathcal{I}_{2n} = 0 \quad \text{for} \quad l \in \mathbb{N} \, , \tag{54}$$

we obtain with

$$\left| \sum_{j=lk}^{lk+n^*} \mathcal{I}_{2j} \right| \le 2\tau^2 n^*$$

the estimate (i) by

$$\left| \sum_{\substack{n=1 \\ n \text{ even}}}^{2kL-1} a_n \mathcal{I}_n \right| \le \sum_{n=1}^{kL-2} 2\tau^2 n^* \left| a_{2(n+1)} - a_{2n} \right| \le 2\varepsilon\tau \sum_{n=1}^{kL-2} \left| a_{2(n+1)} - a_{2n} \right| \, .$$

It remains to prove (54). By definition (7), ϕ is symmetric and periodic, i.e.

$$\phi(1+s) = \phi(1-s) \, , \qquad \phi(2+s) = \phi(2-s) \tag{55}$$

and

$$\phi(s) = \phi(2+s) \, . \tag{56}$$

Since $t_{2k} = 2\varepsilon$ and $t_1 = \varepsilon/k$, it follows with (56) and (55) that

$$\begin{aligned} &\int_{t_{2k}}^{t_{2k+1}} \int_{t_{2k}}^{s} \exp\left(-\mathrm{i}\omega\phi\left(\tfrac{\sigma}{\varepsilon}\right)\right) \mathrm{d}\sigma \exp\left(-\mathrm{i}\widehat{\omega}\phi\left(\tfrac{s}{\varepsilon}\right)\right) \mathrm{d}s \\ &\qquad = -\int_{t_{2k-1}}^{t_{2k}} \int_{t_{2k}}^{s} \exp\left(-\mathrm{i}\omega\phi\left(\tfrac{\sigma}{\varepsilon}\right)\right) \mathrm{d}\sigma \exp\left(-\mathrm{i}\widehat{\omega}\phi\left(\tfrac{s}{\varepsilon}\right)\right) \mathrm{d}s \end{aligned}$$

and therefore

$$\mathcal{I}_{2k} = 0 \, . \tag{57}$$

Thanks to (56) we have in addition

$$\mathcal{I}_n = \mathcal{I}_{n+2k} \, . \tag{58}$$

In the following we assume that k is even; the case k is odd follows with minor modifications. Using (57) and (58) yields

$$\sum_{n=1}^{kL-1} \mathcal{I}_{2n} = \sum_{l=1}^{L} \sum_{n=(l-1)k+1}^{kl-1} \mathcal{I}_{2n} = L \sum_{n=1}^{k-1} \mathcal{I}_{2n} \, .$$

Rearranging the summands symmetrically results in

$$\sum_{n=1}^{kL-1} \mathcal{I}_{2n} = L \left(\mathcal{I}_k + \sum_{n=1}^{k/2-1} \left(\mathcal{I}_{2n} + \mathcal{I}_{2(k-n)} \right) \right) \, .$$

With (56) and straightforward computation we obtain

$$\mathcal{I}_k = 0, \quad \mathcal{I}_{2n} + \mathcal{I}_{2(k-n)} = 0\,, \qquad\qquad \text{for} \quad n = 1, \dots, k/2 - 1$$

which completes the proof of (54) and thus of estimate (i).

Estimate (ii) follows analogously with

$$\sum_{n=1}^{kL} \mathcal{I}_{2n-1} = L \sum_{n=1}^{k/2} \big(\mathcal{I}_{2n-1} + \mathcal{I}_{2(k-n)+1}\big)$$

and

$$\mathcal{I}_{2n-1} + \mathcal{I}_{2(k-n)+1} = 0 \quad \text{for} \quad n = 1, \dots, k/2\,.$$

□

We may now complete the improvement of estimate (49) and thus prove Theorem 3. By definition (11) we have

$$\widehat{Y}'_{jklm}(\sigma) = \big(Y'_{jkl}(\sigma) - \mathrm{i}\omega_{[jklm]}\alpha Y_{jkl}(\sigma)\big) \exp\big(-\mathrm{i}\omega_{[jklm]}\alpha\sigma\big)\ .$$

Hence, we get the partition

$$[d^{(2)}_{n+1}]_m = [S^{(1)}_{n+1}]_m + [S^{(2)}_{n+1}]_m$$

with

$$[S^{(1)}_{n+1}]_m = \mathrm{i} \sum_{I_m} \int_{t_{n-1}}^{t_{n+1}} \int_{t_n}^{s} Y'_{jkl}(\sigma) \exp\big(-\mathrm{i}\omega_{[jklm]}(\alpha\sigma + \phi\left(\tfrac{s}{\varepsilon}\right))\big)\, \mathrm{d}\sigma\, \mathrm{d}s\,,$$

and

$$[S^{(2)}_{n+1}]_m = \alpha \sum_{I_m} \int_{t_{n-1}}^{t_{n+1}} \int_{t_n}^{s} \omega_{[jklm]} \widehat{Y}_{jklm}(\sigma) \exp\big(-\mathrm{i}\omega_{[jklm]}\phi\left(\tfrac{s}{\varepsilon}\right)\big)\, \mathrm{d}\sigma\, \mathrm{d}s\,.$$

The aim for the next two steps is to split off suitable parts of $S^{(1)}_{n+1}$ and $S^{(2)}_{n+1}$, respectively, for which we then employ Lemma 2, before we finally apply Gronwall's lemma.

Step 1. By definition (11) we have $[S^{(1)}_{n+1}]_m = [T^{(1)}_{n+1}]_m + [T^{(2)}_{n+1}]_m + [T^{(3)}_{n+1}]_m$ with

$$[T^{(1)}_{n+1}]_m = \mathrm{i} \sum_{I_m} \int_{t_{n-1}}^{t_{n+1}} \int_{t_n}^{s} y'_j(\sigma)\overline{y}_k(\sigma)y_l(\sigma) \exp\big(-\mathrm{i}\omega_{[jklm]}(\alpha\sigma + \phi\left(\tfrac{s}{\varepsilon}\right))\big)\, \mathrm{d}\sigma\, \mathrm{d}s\,, \tag{59}$$

$$[T^{(2)}_{n+1}]_m = \mathrm{i} \sum_{I_m} \int_{t_{n-1}}^{t_{n+1}} \int_{t_n}^{s} y_j(\sigma)\overline{y}'_k(\sigma)y_l(\sigma) \exp\big(-\mathrm{i}\omega_{[jklm]}(\alpha\sigma + \phi\left(\tfrac{s}{\varepsilon}\right))\big)\, \mathrm{d}\sigma\, \mathrm{d}s\,, \tag{60}$$

$$[T^{(3)}_{n+1}]_m = \mathrm{i} \sum_{I_m} \int_{t_{n-1}}^{t_{n+1}} \int_{t_n}^{s} y_j(\sigma)\overline{y}_k(\sigma)y'_l(\sigma) \exp\big(-\mathrm{i}\omega_{[jklm]}(\alpha\sigma + \phi\left(\tfrac{s}{\varepsilon}\right))\big)\, \mathrm{d}\sigma\, \mathrm{d}s\,. \tag{61}$$

Since (59)-(61) are structured similarly it is sufficient to consider the term (59). The estimates for (60) and (61) follow analogously. Replacing $y_j'(\sigma)$ by the differential equation (12) yields

$$[T_{n+1}^{(1)}]_m = -\sum_{I_m}\sum_{I_j}\int_{t_{n-1}}^{t_{n+1}}\int_{t_n}^{s} Y_{pqrkl}(\sigma)\exp\left(-\mathrm{i}(\omega_{[pqrj]}+\omega_{[jklm]})\alpha\sigma\right)$$
$$\exp\left(-\mathrm{i}\omega_{[pqrj]}\phi\left(\tfrac{\sigma}{\varepsilon}\right)\right)\mathrm{d}\sigma\exp\left(-\mathrm{i}\omega_{[jklm]}\phi\left(\tfrac{s}{\varepsilon}\right)\right)\mathrm{d}s \tag{62}$$

where we abbreviate

$$Y_{pqrkl}(\sigma) = y_p(\sigma)\overline{y}_q(\sigma)y_r(\sigma)\overline{y}_k(\sigma)y_l(\sigma)$$

in the spirit of (11) and (16). For simplicity, we fix $m \in \mathbb{Z}$, $(j,k,l) \in I_m$, $(p,q,r) \in I_j$, and write for short $\widehat{\omega} = \omega_{[jklm]}$, $\omega = \omega_{[pqrj]}$ and in particular

$$F(\sigma) = Y_{pqrkl}(\sigma)\exp\left(-\mathrm{i}(\omega_{[pqrj]}+\omega_{[jklm]})\alpha\sigma\right)\,.$$

Hence, we obtain for any summand of (62)

$$\int_{t_{n-1}}^{t_{n+1}}\int_{t_n}^{s} F(\sigma)\exp\left(-\mathrm{i}\omega\phi\left(\tfrac{\sigma}{\varepsilon}\right)\right)\mathrm{d}\sigma\exp\left(-\mathrm{i}\widehat{\omega}\phi\left(\tfrac{s}{\varepsilon}\right)\right)\mathrm{d}s = R_{n+1}^{(1)} + R_{n+1}^{(2)}$$

with

$$R_{n+1}^{(1)} = F(t_n)\mathcal{I}_n\,,$$
$$R_{n+1}^{(2)} = \int_{t_{n-1}}^{t_{n+1}}\int_{t_n}^{s}\int_{t_n}^{\sigma_1} F'(\sigma_2)\,\mathrm{d}\sigma_2\exp\left(-\mathrm{i}\omega\phi\left(\tfrac{\sigma_1}{\varepsilon}\right)\right)\mathrm{d}\sigma_1\exp\left(-\mathrm{i}\widehat{\omega}\phi\left(\tfrac{s}{\varepsilon}\right)\right)\mathrm{d}s$$

and $\mathcal{I}_n$ from (52). It is clear that

$$\left|\sum_{n=1}^{2kL-1} R_{n+1}^{(2)}\right| \le \tau^2 C(T)\max_{\sigma\in[0,T]}|F'(\sigma)|\,. \tag{63}$$

Moreover, Lemma 2 provides the estimates

$$\left|\sum_{\substack{n=1\\ n \text{ even}}}^{2kL-1} R_{n+1}^{(1)}\right| \le \varepsilon\tau C(T)\max_{\sigma\in[0,T]}|F'(\sigma)| \tag{64}$$

and

$$\left|\sum_{\substack{n=1\\ n \text{ odd}}}^{2kL-1} R_{n+1}^{(1)}\right| \le \varepsilon\tau C(T)\max_{\sigma\in[0,T]}|F'(\sigma)|\,. \tag{65}$$

Since

$$|F'(\sigma)| \le \left|Y_{pqrkl}'(\sigma)\right| + \alpha\left|(\omega_{[jklm]}+\omega_{[pqrj]})Y_{pqrkl}(\sigma)\right|\,,$$

combining (63)-(65) with Lemma 4 (see Appendix) results in

$$\left\|\sum_{n=1}^{2kL-1}\mathcal{J}^{N-n}S_{n+1}^{(1)}\right\|_{\ell^1} \le \varepsilon\tau\big(C(T,M_0^y)+\alpha C(T,M_2^y)\big)\,. \tag{66}$$

Step 2. We fix $m \in \mathbb{Z}$, $(j,k,l) \in I_m$ and write for short $\widehat{\omega} = \omega_{[jklm]}$ and

$$\widehat{F}(\sigma) = \omega_{[jklm]} \widehat{Y}_{jklm}(\sigma)\,.$$

For any summand of $[S^{(2)}_{n+1}]_m$ we have the partition

$$\alpha \int_{t_{n-1}}^{t_{n+1}} \int_{t_n}^{s} \widehat{F}(\sigma) \exp\left(-\mathrm{i}\widehat{\omega}\phi\left(\tfrac{s}{\varepsilon}\right)\right) \mathrm{d}\sigma\, \mathrm{d}s = \alpha\big(R^{(3)}_{n+1} + R^{(4)}_{n+1}\big)$$

with

$$\begin{aligned} R^{(3)}_{n+1} &= \widehat{F}(t_n)\mathcal{I}_n\,, \\ R^{(4)}_{n+1} &= \int_{t_{n-1}}^{t_{n+1}} \int_{t_n}^{s} \int_{t_n}^{\sigma_1} \widehat{F}'(\sigma_2)\, \mathrm{d}\sigma_2 \exp\left(-\mathrm{i}\widehat{\omega}\phi\left(\tfrac{s}{\varepsilon}\right)\right) \mathrm{d}\sigma_1\, \mathrm{d}s\,. \end{aligned}$$

and $\mathcal{I}_n$ from (52) with $\omega = 0$. It is clear that

$$\left|\sum_{n=1}^{2kL-1} R^{(4)}_{n+1}\right| \leq \tau^2 C(T) \max_{\sigma\in[0,T]} \left|\widehat{F}'(\sigma)\right|\,. \tag{67}$$

As before, Lemma 2 with $\omega = 0$ yields

$$\left|\sum_{\substack{n=1\\ n \text{ even}}}^{2kL-1} R^{(3)}_{n+1}\right| \leq \varepsilon\tau C(T) \max_{\sigma\in[0,T]} \left|\widehat{F}'(\sigma)\right|\,, \tag{68}$$

$$\left|\sum_{\substack{n=1\\ n \text{ odd}}}^{2kL-1} R^{(3)}_{n+1}\right| \leq \varepsilon\tau C(T) \max_{\sigma\in[0,T]} \left|\widehat{F}'(\sigma)\right| \tag{69}$$

Since

$$\left|\widehat{F}'(\sigma)\right| \leq \left|\omega_{[jklm]} Y'_{jkl}(\sigma)\right| + \alpha \left|\omega^2_{[jklm]} Y_{jkl}(\sigma)\right|\,,$$

combining (67)-(69) with Lemma 4 (see Appendix) results in

$$\left\|\sum_{n=1}^{2kL-1} \mathcal{J}^{N-n} S^{(2)}_{n+1}\right\|_{\ell^1} \leq \varepsilon\tau\Big(C(T, M^y_2) + \alpha C(T, M^y_4)\Big)\,. \tag{70}$$

Step 3. Finally, we arrive with (46), (51), (66) and (70) at

$$\left\|\sum_{n=1}^{N} \mathcal{J}^{N-n} d_{n+1}\right\|_{\ell^1} \leq \tau C(M^\star_0) \sum_{n=1}^{N} \|e_n\|_{\ell^1} + \varepsilon\tau\Big(C(T, M^y_0) + \alpha C(T, M^y_4)\Big)\,. \tag{71}$$

Combining (40), (41) and (71) with the recursion formula (39) gives

$$\|e_{N+1}\|_{\ell^1} \leq \tau C(\alpha, M^\star_0) \sum_{n=1}^{N} \|e_n\|_{\ell^1} + \varepsilon\tau\Big(C(T, M^\star_0, M^y_0) + \alpha C(T, M^\star_0, M^y_4)\Big)$$

and applying the discrete Gronwall lemma completes the proof. □

8.3 Proof of Theorem 4

We recall that for step-sizes $\tau = k\varepsilon$ the adiabatic midpoint rule (19) applied to the tDMNLS coincides with the classical explicit midpoint rule applied to the limit system (15), cf. section 4. Thus, we obtain with Theorem 1 the estimate

$$\begin{aligned}\left\| y(t_n) - y^{(n)} \right\|_{\ell^1} &\le \| y(t_n) - v(t_n)\|_{\ell^1} + \left\| v(t_n) - y^{(n)} \right\|_{\ell^1} \\ &\le \frac{\varepsilon^2}{\delta}\big(C(t_n, \alpha, M_0) + \alpha C(t_n, \delta, M_2)\big) e^{t_n C(M_0)} \\ &\quad + \left\| v(t_n) - y^{(n)} \right\|_{\ell^1}\end{aligned}$$

and it only remains to show that the explicit midpoint rule applied to the limit system (15) is of order two.

Since the limit system and therefore in particular $\widehat{V}_{jklm}(s)$ is independent of ε, this can be done with a suitable adaptation of the error recursion formula (39) and Taylor expansion. Hence, we omit the details of this proof. The occurring second order derivative

$$\left| \widehat{V}''_{jklm}(s) \right| \le \left| V''_{jkl}(s) \right| + 2\alpha \left| \omega_{[jklm]} V'_{jkl}(s) \right| + \alpha^2 \left| \omega^2_{[jklm]} V_{jkl}(s) \right| ,$$

can be estimated by Lemma 3 (see Appendix) and finally the discrete Gronwall lemma yields

$$\left\| v(t_n) - y^{(n)} \right\|_{\ell^1} \le \tau^2 \Big(C(T, M_0^{\bullet}, M_2^v) + \alpha C(T, M_0^{\bullet}, M_4^v) \Big) e^{TC(\alpha, M_0^{\star})}$$

which concludes the proof of Theorem 4.

□

Appendix

In this section we state and prove two lemmas containing rather technical estimates for various quantities arising in the above computations. These lemmas are used frequently throughout the paper.

The first lemma concerns various estimates for the quantity $V_{jkl}(t)$.

Lemma 3. *Under assumption 2 we have the estimates*

$$\begin{aligned}
&(i) && \sum_{m\in\mathbb{Z}} \sum_{I_m} \left| V'_{jkl}(t) \right| \le C(M_0^v) && \text{for all} \quad t \in [0,T] ,\\
&(ii) && \sum_{m\in\mathbb{Z}} \sum_{I_m} \left| \omega_{[jklm]} V_{jkl}(t) \right| \le C(M_2^v) && \text{for all} \quad t \in [0,T] ,\\
&(iii) && \sum_{m\in\mathbb{Z}} \sum_{I_m} \left| \omega_{[jklm]} V'_{jkl}(t) \right| \le C(M_2^v) && \text{for all} \quad t \in [0,T] ,\\
&(iv) && \sum_{m\in\mathbb{Z}} \sum_{I_m} \left| \omega^2_{[jklm]} V_{jkl}(t) \right| \le C(M_4^v) && \text{for all} \quad t \in [0,T] ,\\
&(v) && \sum_{m\in\mathbb{Z}} \sum_{I_m} \left| V''_{jkl}(t) \right| \le C(M_0^v) + \alpha C(M_2^v) && \text{for all} \quad t \in [0,T] .
\end{aligned}$$

Proof. (i) We differentiate (16) and obtain

$$V'_{jkl}(t) = v'_j(t)\overline{v}_k(t)v_l(t) + v_j(t)\overline{v}'_k(t)v_l(t) + v_j(t)\overline{v}_k(t)v'_l(t)\,. \tag{72}$$

Furthermore, (15) yields

$$\|v'(t)\|_{\ell^1} = \sum_m |v'_m(t)| = \sum_{m\in\mathbb{Z}}\sum_{I_m} |v_j(t)|\,|\overline{v}_k(t)|\,|v_l(t)| \le C(M_0^v)\,. \tag{73}$$

and we acquire

$$\sum_{m\in\mathbb{Z}}\sum_{I_m} \left|V'_{jkl}(t)\right| \le C(M_0^v)\,. \tag{74}$$

(ii) Since we have for $(j,k,l)\in I_m$ by definition

$$\omega_{[jklm]} = (j^2 - k^2 + l^2 - m^2) = -2(k^2 + jk - jl + kl)\,, \tag{75}$$

we obtain

$$\begin{aligned}\sum_{m\in\mathbb{Z}}\sum_{I_m} \left|\omega_{[jklm]}V_{jkl}(t)\right| &= 2\sum_{m\in\mathbb{Z}}\sum_{I_m} \left|(k^2 + jk - jl + kl)V_{jkl}(t)\right| \\ &\le 2\big(\,\|v(t)\|^2_{\ell^1_0}\,\|v(t)\|_{\ell^1_2} + 3\,\|v(t)\|_{\ell^1_0}\,\|v(t)\|^2_{\ell^1_1}\,\big) \\ &\le C(M_2^v)\,. \end{aligned} \tag{76}$$

(iii) Combining (75) with (72) and (73) yields

$$\sum_{I_m} \left|\omega_{[jklm]}V'_{jkl}(t)\right| \le C(M_2^v)\,.$$

(iv) A short computation starting from (75) leads to

$$\begin{aligned}\omega^2_{[jklm]} &= 4(k^2 + jk - jl + kl)^2 \\ &= 4\big((k^2+jk)^2 - 2(k^2+jk)(jl+kl) + (jl+kl)^2\big) \\ &= 4\big(k^4 + 2k^3j + j^2k^2 - 2(k^3l + j^2kl + 2k^2jl) + j^2l^2 + 2jkl^2 + k^2l^2\big)\,.\end{aligned}$$

Hence, we obtain

$$\sum_{m\in\mathbb{Z}}\sum_{I_m} \left|\omega^2_{[jklm]}V_{jkl}(t)\right| \le C(M_4^v)\,.$$

(v) By definition (16) and chain rule we obtain

$$\begin{aligned}V''_{jkl}(t) &= v''_j(t)\overline{v}_k(t)v_l(t) + v'_j(t)\overline{v}'_k(t)v_l(t) + v'_j(t)\overline{v}_k(t)v'_l(t) \\ &\quad + v'_j(t)\overline{v}'_k(t)v_l(t) + v_j(t)\overline{v}''_k(t)v_l(t) + v_j(t)\overline{v}'_k(t)v'_l(t) \\ &\quad + v'_j(t)\overline{v}_k(t)v'_l(t) + v_j(t)\overline{v}'_k(t)v'_l(t) + v_j(t)\overline{v}_k(t)v''_l(t)\,.\end{aligned} \tag{77}$$

If we differentiate (15) we get

$$\|v''(t)\|_{\ell^1} \le \sum_{m\in\mathbb{Z}}\sum_{I_m} \left|V'_{jkl}(t) - \mathrm{i}\omega_{[jklm]}\alpha V_{jkl}(t)\right|\,. \tag{78}$$

Substituting (74) and (76) into (78) yields

$$\|v''(t)\|_{\ell^1} \le C(M_0^v) + \alpha C(M_2^v)\,. \tag{79}$$

Then, combining (73) and (79) results in

$$\sum_{m\in\mathbb{Z}}\sum_{I_m}\left|V''_{jkl}(t)\right| \le C(M_0^v) + \alpha C(M_2^v)\,. \tag{80}$$

□

The second lemma concerns various estimates for the quantities $Y_{jkl}(t)$ and $Y_{pqrkl}(t)$.

Lemma 4. *Under assumption 1 we have the estimates*

$$\begin{aligned}
&(i) && \sum_{m\in\mathbb{Z}}\sum_{I_m}\left|Y'_{jkl}(t)\right| \le C(M_0^y) && \text{for all} \quad t\in[0,T]\,,\\
&(ii) && \sum_{m\in\mathbb{Z}}\sum_{I_m}\left|\omega_{[jklm]}Y_{jkl}(t)\right| \le C(M_2^y) && \text{for all} \quad t\in[0,T]\,,\\
&(iii) && \sum_{m\in\mathbb{Z}}\sum_{I_m}\left|\omega_{[jklm]}Y'_{jkl}(t)\right| \le C(M_2^y) && \text{for all} \quad t\in[0,T]\,,\\
&(iv) && \sum_{m\in\mathbb{Z}}\sum_{I_m}\left|\omega^2_{[jklm]}Y_{jkl}(t)\right| \le C(M_4^y) && \text{for all} \quad t\in[0,T]\,,\\
&(v) && \sum_{m\in\mathbb{Z}}\sum_{I_m}\left|Y'_{pqrkl}(t)\right| \le C(M_0^y) && \text{for all} \quad t\in[0,T]\,,\\
&(vi) && \sum_{m\in\mathbb{Z}}\sum_{I_m}\left|(\omega_{[jklm]}+\omega_{[pqrj]})Y_{pqrkl}(t)\right| \le C(M_2^y) && \text{for all} \quad t\in[0,T]\,.
\end{aligned}$$

Proof. (i)-(iv) can be shown analogously to (i)-(iv) in Lemma 3 with obvious modifications. Then (v) and (vi) follow immediately from (i) and (iii). □

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