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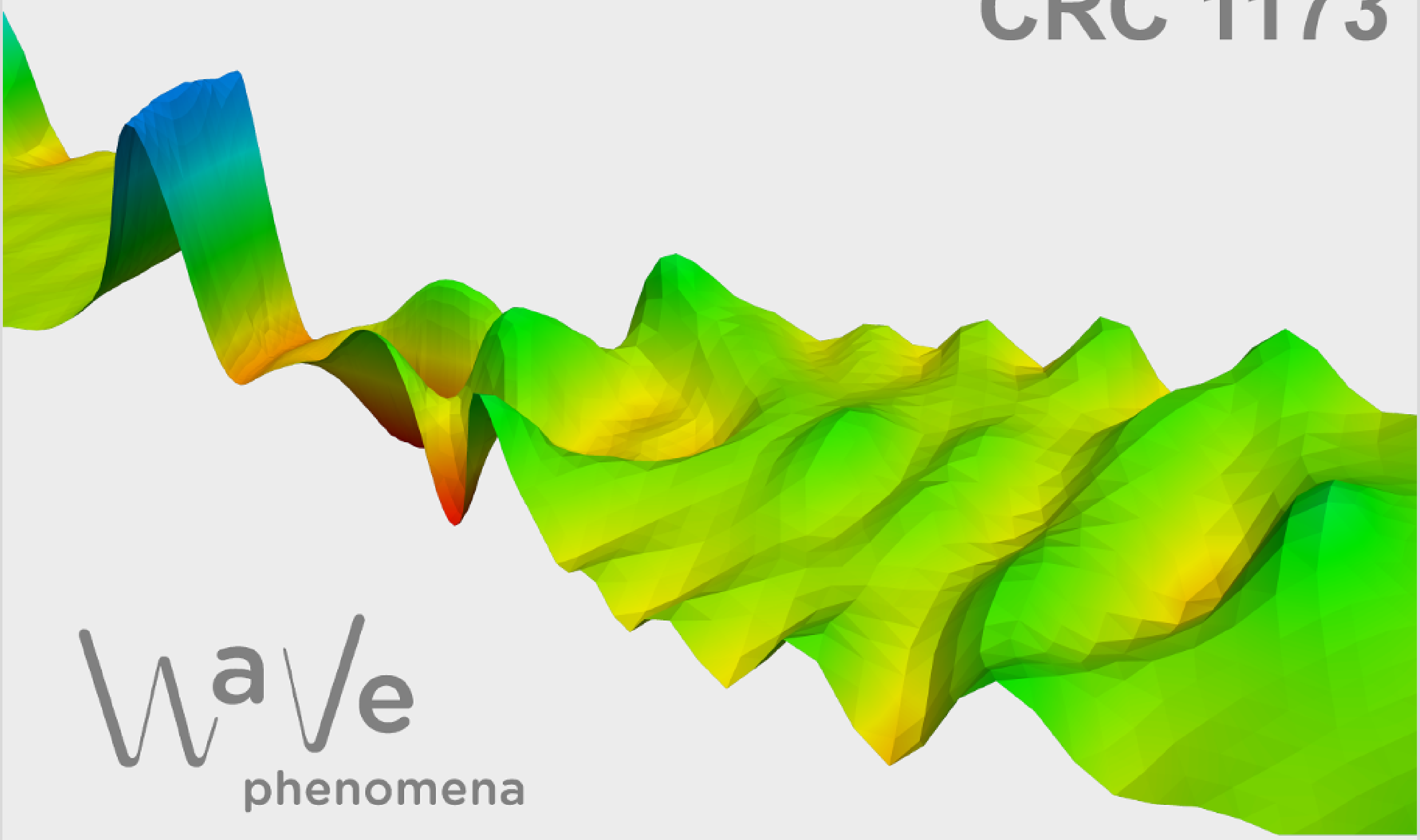

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REAL-VALUED, TIME-PERIODIC WEAK SOLUTIONS FOR A SEMILINEAR WAVE EQUATION WITH PERIODIC δ-POTENTIAL

ANDREAS HIRSCH AND WOLFGANG REICHEL

Abstract. We consider the semilinear wave equation $V(x)u_{tt} - u_{xx} = \pm|u|^{p-1}u$ with $p \in (1, \frac{5}{3})$ and a periodically extended delta potential $V(x) = \alpha + \beta\delta_{\mathrm{per}}(x)$. Both the "+" and the "-" case can be treated. We prove the existence of time-periodic real-valued solutions that are localized in the space direction. Our result builds upon a Fourier-Floquet-Bloch expansion of the solution and a detailed analysis of the spectrum of the wave operator. In fact, it turns out that by a careful choice of the parameters α, β and the spatial and temporal periods, the spectrum of the wave operator $V(x)\partial_t^2 - \partial_x^2$ (considered on suitable space of time-periodic functions) is bounded away from 0. This allows to find weak solutions as critical points of a functional on a suitable Hilbert space and to apply tools for indefinite variational problems.

1. Introduction and results

We study the $1 + 1$ dimensional semilinear wave equation

$$V(x)u_{tt} - u_{xx} = \pm|u|^{p-1}u \text{ in } \mathbb{R}\times\mathbb{R} \tag{1.1$_\pm$}$$

both for the plus and the minus case. Here $V > 0$ is a periodically distributed potential and $1 < p < \frac{5}{3}$. We are looking for real-valued, time-periodic and spatially localized solutions of $(1.1)_\pm$ often called breathers. Equation $(1.1)_\pm$ is a prototype semilinear wave equation which, e.g., can be viewed as an approximation of a second-order in time Maxwell equation for the polarized electric field in the presence of nonlinearities, cf. [4]. Our result is motivated by the work Blank, Chirilus-Bruckner, Lescarret, Schneider [4] who considered an equation of the type

$$s(x)u_{tt} - u_{xx} + q(x)u = u^3 \text{ in } \mathbb{R}\times\mathbb{R} \tag{1.2}$$

with periodic $s, q\colon \mathbb{R} \to \mathbb{R}$. For a very specific choice of periodic step-functions s and q they proved the existence of breathers with the help of spatial dynamics, bifurcation theory and center manifold theory.

In the present paper we assume that the potential $V\colon \mathbb{R} \to \mathbb{R}$ is periodic and has the special form

$$V(x) = \alpha + \beta\delta_{\mathrm{per}}(x), \tag{1.3}$$

where δ_{per} denotes a 2π-periodic delta distribution supported w.l.o.g. on the set $\{2n\pi : n \in \mathbb{Z}\}$. This particular choice allows us to have very good control on the spectrum of the wave operator $L_{x,t} = V(x)\partial_t^2 - \partial_x^2$, and in particular how (after Fourier-transform in time) the spectral gap near zero of a sequence of elliptic operators $(L_k)_{k\in2\mathbb{Z}+1}$ grows w.r.t. k, cf. Lemma 2.4. For future work it will be desirable to replace the periodic delta-potential by a bounded periodic potential, e.g. a periodic step potential. The presence of the δ_{per}-distribution also requires a suitable concept of a weak solution given next. Due to the T-periodicity in time we consider a polychromatic solution ansatz

$$u(x,t) = \sum_{k\in2\mathbb{Z}+1} u_k(x)e^{ik\omega t}, \quad u_k(x) = \bar{u}_{-k}(x), \quad \omega = \frac{2\pi}{T}. \tag{1.4}$$

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The function u generated by the Fourier decomposition (1.4) is T-periodic in time and real-valued due to the assumption $u_k(x) = \bar{u}_{-k}(x)$. Since we only consider coefficients with odd indices $k \in 2\mathbb{Z}+1$ the function u is in fact $T/2$-antiperiodic. The space of antiperiodic-in-time functions is important since it prevents the $k=0$-mode and thus keeps 0 out of the spectrum of the wave operator $L_{x,t} = V(x)\partial_t^2 - \partial_x^2$. At the same time the nonlinearity $\pm|u|^{p-1}u$ is consistent with seeking $T/2$-antiperiodic solutions. The space-time domain on which the solutions are determined is denoted by $D := \mathbb{R}\times[0,T)$.

Definition 1.1. *We call u of the form* (1.4) *with $u \in H^{1/2}(0,T;L^2(\mathbb{R})) \cap H^{-3/2}(0,T;H^1(\mathbb{R}))$ and $u \in L^{p+1}(D)$ a weak T-periodic solution of* $(1.1)_\pm$ *if*

$$\int_D u(-\bar{v}_{xx} + \alpha\bar{v}_{tt})d(x,t) + \beta\sum_{n\in\mathbb{Z}}\langle u(2\pi n,\cdot), v_{tt}(2\pi n,\cdot)\rangle_{H^{-\frac{1}{2}}\times H^{\frac{1}{2}}} = \pm\int_D |u|^{p-1}u\bar{v}d(x,t) \tag{1.5}$$

holds for all $v(x,t) = \sum_{k\in\mathbb{Z}} v_k(x)e^{ik\omega t}$ with $v \in H^r(0,T;L^2(\mathbb{R})) \cap H^s(0,T;H^1(\mathbb{R})) \cap L^2(0,T;H^2(\mathbb{R}))$, $r \geq \frac{5}{2}$, $s \geq 5-r$ and $v \in L^{p+1}(D)$.

Remark 1.2. *The above assumptions on u, v imply by Lemma 7.2 that $\sum_{n\in\mathbb{Z}}\|u(2\pi n,\cdot)\|^2_{H^{-1/2}} < \infty$ and $\sum_{n\in\mathbb{Z}}\|v_{tt}(2\pi n,\cdot)\|^2_{H^{1/2}} < \infty$.*

Based on this concept of a weak solution our main result reads as follows.

Theorem 1.3. *Let $p \in (1,\frac{5}{3})$ and let $V\colon \mathbb{R}\to\mathbb{R}$ be given by* (1.3)*, $\alpha > 0$ and $\beta = 16\alpha$. Then* $(1.1)_\pm$ *possesses a non-trivial $8\pi\sqrt{\alpha}$-periodic weak solution in the sense of Definition 1.1.*

Corollary 1.4. *For $p \in (1,\frac{4}{3})$ the solution u from Theorem 1.3 satisfies $u \in H^{1/2}(D)$ and $u \in H^1(0,T;L^2(\mathbb{R})) \cap H^{-1}(0,T;H^1(\mathbb{R}))$. We can therefore weaken the assumptions on the test functions v, i.e., $v \in H^r(0,T;L^2(\mathbb{R})) \cap H^s(0,T;H^1(\mathbb{R})) \cap L^2(0,T;H^2(\mathbb{R}))$ for all $r \geq 2$, $s \geq 4-r$ and $v \in L^{p+1}(D)$.*

Throughout this paper we write $\mathbb{Z}_{\text{odd}} := 2\mathbb{Z}+1$. Next to the Fourier-decomposition in (1.4) we perform as a further step the Floquet-Bloch-decomposition (cf. Section 3 for details)

$$u_k(x) = \sum_{j\in\mathbb{N}_0}\int_{-1/2}^{1/2}\tilde{u}_{j,k}(s)\psi_{j,k}(x,s)ds \text{ in } L^2(\mathbb{R}) \text{ for all } k \in \mathbb{Z}_{\text{odd}}.$$

Here we expand u_k in terms of Bloch waves $\psi_{j,k}(x,s)$ and Bloch variables $\tilde{u}_{j,k}(s)$. This expansion diagonalizes the wave operator $V(x)\partial_t^2 - \partial_x^2$ and enables us to use variational tools in terms of the Bloch variable $(\tilde{u}_{j,k}(s))_{j\in\mathbb{N}_0,k\in\mathbb{Z}_{\text{odd}},s\in[-1/2,1/2)}$ to find weak solutions of $(1.1)_\pm$. The use of variational tools is the main methodical difference to [4].

Breather solutions of nonlinear wave equations are quite rare. After the discovery of the Sine-Gordon breather family, cf. [1]

$$u_{m,\omega}(x,t) = 4\arctan\left(\frac{m}{\omega}\frac{\sin(\omega t)}{\cosh(mx)}\right), m,\omega > 0, m^2+\omega^2 = 1$$

for the Sine-Gordon equation

$$u_{tt} - u_{xx} + \sin u = 0 \text{ in } \mathbb{R}\times\mathbb{R} \tag{1.6}$$

many results on the non-existence of breathers appeared, e.g. [3] and [7]. By these works it became clear that breathers do not persist in homogeneous nonlinear wave equations if the $\sin u$ nonlinearity in (1.6) is perturbed to $f(u)$ with $f(0) = 0, f'(0) > 0$. The situation is different if one introduces inhomogeneities. For example, nonlinear wave equations on discrete lattices can support breather solutions, cf. [16] for a fundamental result and [15] for an overview with many references. Another way to recover breathers is to introduce inhomogeneities via x-dependent coefficients like in [4] for

(1.2). Recently, the authors in [21] gave an existence result for breathers in the 3 + 1-dimensional semilinear curl-curl wave equation

$$s(x)\partial_t^2 U + \nabla \times \nabla \times U + q(x)U \pm V(x)|U|^{p-1}U = 0, \quad p > 1,$$

for radially symmetric, positive and non-constant functions $V, q, s \colon \mathbb{R}^3 \to (0, \infty)$ satisfying further properties not listed here. Another interesting polychromatic approach for finding coherent spatially localized solutions of the 1+1-dimensional (quasilinear) Maxwell modell is given in [20]. Based on a multiple scale ansatz the field profile is expanded into infinitely many modes which are time-periodic both in the fast and slow time variables. Since the periodicities in the fast and slow time-variables differ, the field becomes quasiperiodic in time. The resulting system for these infinitely many coupled modes is to a certain extent treated analytically, with a rigorous existence proof yet missing. The numerical results of [20] indicate that spatially localized solitary waves could exist, although nonexistence has not yet been ruled out.

The paper is structured as follows: In the next section we briefly recall parts of the general theory of second-order stationary Schrödinger operators with delta point interactions (cf. [2]). Moreover, for our specific choice of parameters from Theorem 1.3 we study the spectrum of a family of elliptic operators $(L_k)_{k\in\mathbb{Z}_{\text{odd}}}$ that arise form the linear wave operator via discrete Fourier-transform in time. It turns out that 0 is in a spectral gap. More precisely, for every $k \in \mathbb{Z}_{\text{odd}}$ we define a suitable operator L_k which corresponds to the frequency $ik\omega$ in (1.4) and we guarantee that 0 is in a spectral gap of all these operators $(L_k)_{k\in\mathbb{Z}_{\text{odd}}}$. In Section 3 we define a Hilbert space (expressed in terms of Bloch-variables) in which we look for appropriate solutions. After having established a functional analytic framework we study the consequences of the *uniform* spectral gap in Section 4. An important part is the integrability properties of functions composed via Bloch-variables as described in Theorem 5.1. Because the proof of this theorem is rather long, we have moved it to Section 6. The use of the integrability properties allows to incorporate nonlinearities into the variational setting. In Section 5 we find minimizers of a suitable functional on the so-called generalized Nehari manifold. Finally, the proof of Theorem 1.3 consists in verifying that these minimizers are indeed weak solutions in the sense of Definition 1.1. In order to keep the main sections non-technical, some technical aspects are shifted to the appendix.

2. The delta point interaction in one dimension and the spectrum of a family of operators

We consider the one-dimensional differential expression

$$Lu := -u'' + (\tilde{\alpha} + \tilde{\beta}\delta_{\text{per}}(x))u \text{ on } \mathbb{R}, \tag{2.1}$$

where $\tilde{\alpha} \in \mathbb{R}$ and $\tilde{\beta} \in \mathbb{R} \setminus \{0\}$. We always assume that δ_{per} is supported on $I_\delta := \{2n\pi : n \in \mathbb{Z}\}$, is 2π-periodic and acts as a delta-distribution at each of the points $2n\pi$ for $n \in \mathbb{Z}$. By Theorem 1 in [6] the operator L in (2.1) is self-adjoint on the domain

$$\begin{aligned} D(L) := \Big\{ & u \in L^2(\mathbb{R}) : u \text{ abs. cont. on } \mathbb{R}, u' \text{ abs. cont. on } \mathbb{R} \setminus I_\delta, \\ & u'(x_+) - u'(x_-) = \tilde{\beta}u(x) \text{ for all } x \in I_\delta \text{ and } -u'' + \tilde{\alpha}u \in L^2(\mathbb{R}) \Big\}. \end{aligned} \tag{2.2}$$

In (2.2) the function u is continuous on $\mathbb{R}$ and u', u'' exists pointwise almost everywhere and are L^2-integrable. We rewrite the domain of definition in (2.2) by making use of weak derivatives. In the following u is a continuous L^2-function with an L^2-integrable weak derivative u', whereas u'' is not a function anymore but a distribution. Thus,

$$D(L) = \{u \in L^2(\mathbb{R}) : Lu \in L^2(\mathbb{R})\} = \Big\{u \in H^1(\mathbb{R}), u|_{(2\pi n, 2\pi(n+1))} \in H^2(2\pi n, 2\pi(n+1))$$

$$\text{for all } n \in \mathbb{Z}, \sum_{n\in\mathbb{Z}} \|u''\|^2_{L^2(2\pi n, 2\pi(n+1))} < \infty, u'(x_+) - u'(x_-) = \tilde{\beta} u(x) \text{ for all } x \in I_\delta \Big\}.$$

We now introduce the concept of a weak solution of $Lu = f$.

Definition 2.1. *For $f \in L^2(\mathbb{R})$ we say that $u \in H^1(\mathbb{R})$ is a weak solution of $Lu = f$ with L as in* (2.1) *if*

$$\int_{\mathbb{R}} (u'(x)\varphi'(x) + \tilde{\alpha} u(x)\varphi(x))\, dx + \tilde{\beta} \sum_{n\in\mathbb{Z}} u(2\pi n)\varphi(2\pi n) = \int_{\mathbb{R}} f(x)\varphi(x) dx$$

holds true for all $\varphi \in C_c^\infty(\mathbb{R})$. Furthermore, for $u, v \in H^1(\mathbb{R})$ we define the bilinear form b associated to L by

$$b(u,v) := \int_{-\infty}^{\infty} \Big(u'(x)\overline{v'(x)} + \tilde{\alpha} u(x)\overline{v(x)}\Big)\, dx + \tilde{\beta} \sum_{n\in\mathbb{Z}} u(2\pi n)\overline{v(2\pi n)}. \tag{2.3}$$

The bilinear form b and operator L are related via $b(u,v) = \langle Lu, v\rangle_{L^2(\mathbb{R})}$ for all $u \in D(L), v \in H^1(\mathbb{R})$, see Theorem VIII.15 in [22]. In [5] it is shown that the classical Sturm-Liouville theory can be generalized to include delta-point interactions, see also the appendix of [6]. In particular, we can describe the spectrum of L by using the so-called discriminant D (compare Chapter 1 and § 2.1 in [10]). Here the discriminant is defined as follows: for $\lambda \in \mathbb{R}$ let $v_1, v_2 : \mathbb{R} \to \mathbb{R}$ be solutions of the initial value problems $Lv_i = \lambda v_i$ with jump-conditions $v_i'(x_+) - v_i'(x_-) = \tilde{\beta} v_i(x)$ for all $x \in I_\delta$, $i = 1, 2$ and initial conditions $v_1(x_0) = 1, v_1'(x_0) = 0$ and $v_2(x_0) = 0, v_2'(x_0) = 1$ for some $x_0 \notin I_\delta$. Then v_1, v_2 is a system of fundamental solutions for the equation $Lu = \lambda u$ and the discriminant is defined as

$$D(\lambda) := v_1(x_0 + 2\pi) + v_2'(x_0 + 2\pi).$$

Following Chapter 1 and § 2.1 in [10], we have the following characterization of the spectrum $\sigma(L)$.

Theorem 2.2. $\sigma(L) = \{\lambda \in \mathbb{R} : |D(\lambda)| \le 2\}$.

Next we present the exact form of D associated to (2.1). The proof is a straightforward computation so we omit it.

Lemma 2.3. *The discriminant $D(\cdot)$ associated to* (2.1) *reads*

$$D(\lambda) = \begin{cases} \frac{\tilde{\beta}}{\sqrt{\lambda-\tilde{\alpha}}} \sin(2\pi\sqrt{\lambda - \tilde{\alpha}}) + 2\cos(2\pi\sqrt{\lambda-\tilde{\alpha}}) & \text{for } \lambda - \tilde{\alpha} > 0, \\ 2 + 2\pi\tilde{\beta} & \text{for } \lambda - \tilde{\alpha} = 0, \\ \frac{\tilde{\beta}}{\sqrt{-(\lambda-\tilde{\alpha})}} \sinh(2\pi\sqrt{-(\lambda-\tilde{\alpha})}) + 2\cosh(2\pi\sqrt{-(\lambda-\tilde{\alpha})}) & \text{for } \lambda - \tilde{\alpha} < 0. \end{cases} \tag{2.4}$$

Plugging ansatz (1.4) in the left-hand side of $(1.1)_\pm$ we formally compute

$$L_{x,t} u := -u_{xx} + V(x) u_{tt} = \sum_{k\in\mathbb{Z}_{\text{odd}}} (-u_k'' - \omega^2 k^2 (\alpha + \beta\delta_{per}(x)) u_k) e^{ik\omega t}.$$

For $k \in \mathbb{Z}_{\text{odd}}$ we abbreviate

$$L_k := -\frac{d^2}{dx^2} - \alpha\omega^2 k^2 - \beta\omega^2 k^2 \delta_{\text{per}}(x). \tag{2.5}$$

Note that L_k has the form (2.1). For $f, g \in H^1(\mathbb{R})$ the associated bilinear $b_k \colon H^1(\mathbb{R}) \times H^1(\mathbb{R}) \to \mathbb{C}$ reads as follows

$$b_k(f,g) = \int_{\mathbb{R}} \Big(f'(x)\overline{g'(x)} - \alpha\omega^2 k^2 f(x)\overline{g(x)}\Big)\, dx - \beta\omega^2 k^2 \sum_{n\in\mathbb{Z}} f(2\pi n)\overline{g(2\pi n)}. \tag{2.6}$$

By Lemma 2.3 the discriminant D_k associated to L_k reads

$$D_k(\lambda) = \begin{cases} -\frac{\beta\omega^2k^2}{\sqrt{\lambda+\alpha\omega^2k^2}}\sin(2\pi\sqrt{\lambda+\alpha\omega^2k^2}) + 2\cos(2\pi\sqrt{\lambda+\alpha\omega^2k^2}) & \text{for } \lambda > -\alpha\omega^2k^2, \\ 2 - 2\pi\beta\omega^2k^2 & \text{for } \lambda = -\alpha\omega^2k^2, \\ -\frac{\beta\omega^2k^2}{\sqrt{-\lambda-\alpha\omega^2k^2}}\sinh(2\pi\sqrt{-\lambda-\alpha\omega^2k^2}) + 2\cosh(2\pi\sqrt{-\lambda-\alpha\omega^2k^2}) & \text{for } \lambda < -\alpha\omega^2k^2. \end{cases} \tag{2.7}$$

We compute $\sigma(L_k)$ depending on $k \in \mathbb{Z}_{\text{odd}}$ by making use of Theorem 2.2. Since k appears in L_k only as k^2 we restrict to $k \in \mathbb{N}_{\text{odd}}$. We give conditions on $(\omega, \alpha, \beta) \in \mathbb{R}^3_+$ s.t. zero lies uniformly in a spectral gap of L_k for all $k \in \mathbb{N}_{\text{odd}}$ in the following sense.

Lemma 2.4. *Let $(\omega, \alpha, \beta) \in \mathbb{R}^3_+$ satisfy*

$$\alpha > 0, \omega = \frac{1}{4\sqrt{\alpha}} \text{ and } \beta = 16\alpha. \tag{2.8}$$

Then there is $c > 0$ independent of $k \in \mathbb{N}_{\text{odd}}$ such that $(-c|k|, c|k|) \subset \rho(L_k)$ for all $k \in \mathbb{N}_{\text{odd}}$.

Remark 2.5. *Assumption* (2.8) *is precisely the assumption of Theorem 1.3 since* (2.8) *leads to $T = \frac{2\pi}{\omega} = 8\pi\sqrt{\alpha}$. Moreover, notice that $\left|D_k\left(-\frac{k^4}{4} - \frac{k^2}{16}\right)\right| = 2e^{-\pi k^2} < 2$ and $\left|D_k\left(\frac{k^4}{4} - \frac{k^2}{16}\right)\right| = 2$, i.e., $\pm\frac{k^4}{4} - \frac{k^2}{16} \in \sigma(L_k)$. Hence there exist elements of $\sigma(L_k)$ to the left and to the right of 0, i.e., 0 lies in a true spectral gap of L_k.*

Proof. We divide the proof into two parts. Part 1:

$$(-\frac{k}{100}, \frac{k}{100}) \subset \rho(L_k) \text{ for all } k \in 2\mathbb{N}+1.$$

By Theorem 2.2 we have to show $|D_k(\lambda)| > 2$ for all $\lambda \in (-\frac{k}{100}, \frac{k}{100})$ and all $k \in 2\mathbb{N}+1$. Since $-\frac{k}{100} > -\alpha\omega^2k^2 = -\frac{k^2}{16}$ for all $k \in \mathbb{N}$ we only have to deal with the first case of the case distinction in (2.7). The result follows if we can guarantee that

$$\left|2\cos\left(2\pi\sqrt{\lambda+\frac{k^2}{16}}\right) - \frac{k^2}{\sqrt{\lambda+\frac{k^2}{16}}}\sin\left(2\pi\sqrt{\lambda+\frac{k^2}{16}}\right)\right| > 2 \text{ for } |\lambda| < \frac{k}{100} \text{ and all } k \in 2\mathbb{N}+1. \tag{2.9}$$

Since $\left|2\cos\left(2\pi\sqrt{\lambda+\frac{k^2}{16}}\right)\right| \le 2$ it is sufficient for (2.9) to prove

$$\frac{k^2}{\sqrt{\lambda+\frac{k^2}{16}}}\left|\sin\left(2\pi\sqrt{\lambda+\frac{k^2}{16}}\right)\right| > 4 \text{ for } |\lambda| < \frac{k}{100} \text{ and all } k \in 2\mathbb{N}+1. \tag{2.10}$$

Note the inequality $\frac{\sqrt{29}}{20}k > \sqrt{\lambda+\frac{k^2}{16}}$ for $|\lambda| < \frac{k^2}{100}$ which is in particular valid for $|\lambda| < \frac{k}{100}$. Hence, a sufficient condition for the validity of (2.10) and therefore also of (2.9) is to verify

$$\left|\sin\left(2\pi\sqrt{\lambda+\frac{k^2}{16}}\right)\right| > \frac{\sqrt{29}}{5k} \text{ for } |\lambda| < \frac{k}{100} \text{ and all } k \in 2\mathbb{N}+1. \tag{2.11}$$

To establish (2.11) we investigate the argument of the sine-function in (2.11). We write $k = 2m+1$ with $m \in \mathbb{N}$ and therefore

$$2\sqrt{\lambda+\frac{k^2}{16}} = \sqrt{4\lambda+m^2+m+\frac{1}{4}} \in \left(\sqrt{m^2+m+\frac{1}{4}-\frac{2m+1}{25}}, \sqrt{m^2+m+\frac{1}{4}+\frac{2m+1}{25}}\right)$$

for $|\lambda| < \frac{k}{100}$. Since $m + \frac{1}{6} < \sqrt{m^2 + m + \frac{1}{4} - \frac{2m+1}{25}} < \sqrt{m^2 + m + \frac{1}{4} + \frac{2m+1}{25}} < m + \frac{5}{6}$ we have

$$2\sqrt{\lambda + \frac{k^2}{16}} \in \left(m + \frac{1}{6}, m + \frac{5}{6}\right) \text{ for } |\lambda| < \frac{k}{100}. \tag{2.12}$$

The periodicity and monotonicity of the sine-function together with (2.12) then gives

$$\left|\sin\left(2\pi\sqrt{\lambda + \frac{k^2}{16}}\right)\right| \geq \left|\sin\left(\frac{\pi}{6}\right)\right| = \frac{1}{2} \text{ for } |\lambda| < \frac{k}{100} \text{ and all } k \in 2\mathbb{N}+1.$$

In summary,

$$\left|\sin\left(2\pi\sqrt{\lambda + \frac{k^2}{16}}\right)\right| - \frac{\sqrt{29}}{5k} \geq \frac{1}{2} - \frac{\sqrt{29}}{15} > 0 \text{ for } |\lambda| < \frac{k}{100} \text{ and all } k \in 2\mathbb{N}+1 \tag{2.13}$$

which verifies (2.11) and finishes the proof of Part 1.

Part 2: $0 \notin \sigma(L_1)$. The estimate (2.13) in the preceeding proof is the only reason why we focus on $k \geq 3$ in Part 1. We conclude $0 \in \rho(L_1)$ since $D_1(0) = -4$. This finishes the proof of the Lemma with a constant $c > 0$ possibly smaller than $\frac{1}{100}$. □

3. The functional analytic framework

In this section we first use the Floquet-Bloch decomposition in order to derive a suitable functional analytic framework for our problem. This leads to a Hilbert space in which we seek for solutions.

3.1. **Calculations via Floquet-Bloch decomposition.** In this section we introduce some notation which will later help us to treat the indefinite quadratic part of the energy functional arising from the family of operators $(L_k)_{k\in\mathbb{Z}_{\text{odd}}}$. Let $\mathcal{P} := [-\pi, \pi)$ denote the interval of periodicity and $\mathcal{B} := [-\frac{1}{2}, \frac{1}{2})$ the Brillouin zone. For each $k \in \mathbb{Z}_{\text{odd}}$ the operator L_k has a sequence of Bloch waves $(\psi_{j,k})_{j\in\mathbb{N}_0}$. Since for each $(j,k) \in \mathbb{N}_0 \times \mathbb{Z}_{\text{odd}}$ the Bloch wave $\psi_{j,k}$ depends on the variables $x \in \mathcal{P}$ and $s \in \mathcal{B}$ we write $\psi_{j,k} : \mathcal{P}\times\mathcal{B} \to \mathbb{C}$. For fixed $s \in \mathcal{B}$ the function $\psi_{j,k}(\cdot, s) \in H^1_{\text{loc}}(\mathbb{R})$ satisfies the s-quasiperiodic problem

$$\begin{cases} L_k^{\text{quasi},s}\psi_{j,k}(\cdot, s) &= \lambda_{j,k}(s)\psi_{j,k}(\cdot, s) \text{ in } \mathcal{P}, \\ \psi_{j,k}(x + 2\pi, s) &= e^{2\pi i s}\psi_{j,k}(x, s) \text{ for all } (x, s, j, k) \in \mathcal{P}\times\mathcal{B}\times\mathbb{N}_0\times\mathbb{Z}_{\text{odd}} \end{cases} \tag{3.1}$$

and the family $(\psi_{j,k}(\cdot, s))_{j\in\mathbb{N}_0}$ is a $\langle\cdot,\cdot\rangle_{L^2(\mathcal{P})}$-orthonormal and complete system of eigenfunctions in $L^2(\mathcal{P})$ with associated s-quasiperiodic eigenvalues

$$\lambda_{1,k}(s) \leq \lambda_{2,k}(s) \leq \cdots \leq \lambda_{j,k}(s) \to \infty \text{ as } j \to \infty.$$

Here, for $s \in \mathcal{B}$, the operator $L_k^{\text{quasi},s}$ (given by the same differential expression as L_k) is defined and self-adjoint on

$$D(L_k^{\text{quasi},s}) := \{f \in L^2(\mathcal{P}), f \text{ cont. on } [-\pi, \pi], f' \text{ cont. on } [-\pi, 0) \cup (0, \pi], f'' \in L^2(-\pi, 0), f'' \in L^2(0, \pi),$$
$$f'(0+) - f'(0-) = -k^2 f(0), f(\pi) = e^{2\pi i s} f(-\pi), f'(\pi) = e^{2\pi i s} f'(-\pi)\}.$$

Then $L_k^{\text{quasi},s}$ has pure point spectrum $\sigma(L_k^{\text{quasi},s}) = \bigcup_{j\in\mathbb{N}_0} \lambda_{j,k}(s)$.

To explain the relation between L_k and $L_k^{\text{quasi},s}$ recall the definition of the Bloch transform of a function $f \in L^2(\mathbb{R})$

$$(\mathcal{T} f)(x, s) := \frac{1}{|\mathcal{B}|}\sum_{n\in\mathbb{Z}} f(x - 2\pi n)e^{2\pi i s n} = \sum_{n\in\mathbb{Z}} f(x - 2\pi n)e^{2\pi i s n}, \quad (x, s) \in \mathcal{P}\times\mathcal{B}.$$

The operator $\mathcal{T} : L^2(\mathbb{R}) \to L^2(\mathcal{P} \times \mathcal{B})$ is an isometric isomorphism. The relation between L_k and $L_k^{\text{quasi},s}$ can now be expressed by

$$\langle \mathcal{T} L_k f(\cdot, s), g\rangle_{\mathcal{P}} = \left\langle \mathcal{T} f(\cdot, s), L_k^{\text{quasi},s} g\right\rangle_{\mathcal{P}} \qquad \text{for all } f \in D(L_k), g \in D(L_k^{\text{quasi},s}), \tag{3.2}$$

where $\langle \cdot, \cdot\rangle_{\mathcal{P}}$ denotes the standard inner product on $L^2(\mathcal{P})$. Recall that $\sigma(L_k) = \bigcup_{s\in\mathcal{B}} \sigma(L_k^{\text{quasi},s}) = \bigcup_{j\in\mathbb{N}_0, s\in\mathcal{B}} \lambda_{j,k}(s)$ for all $k \in \mathbb{Z}_{\text{odd}}$. For a function $u_k : \mathbb{R} \to \mathbb{C}$ with $u_k \in L^2(\mathbb{R})$ we use the notation

$$\tilde{u}_{j,k}(s) := \left\langle \mathcal{T} u_k(\cdot, s), \psi_{j,k}(\cdot, s)\right\rangle_{\mathcal{P}} \tag{3.3}$$

The fact that for fixed k the collection of Bloch waves w.r.t. $j \in \mathbb{N}_0$ and $s \in \mathcal{P}$ are complete can be expressed for $u_k \in L^2(\mathbb{R})$ by

$$u_k(x) = \sum_{j\in\mathbb{N}_0} \int_{\mathcal{B}} \tilde{u}_{j,k}(s)\psi_{j,k}(x, s)ds \text{ in } L^2(\mathbb{R}) \text{ for all } k \in \mathbb{Z}_{\text{odd}}. \tag{3.4}$$

For references, cf. Chapter 3 in [9], § 2.3, § 2.4 and Theorem 5.3.2 in [10], and [17].

The following result explains how the operator L_k diagonalizes with respect to the Bloch waves.

Lemma 3.1. *Fix $k \in \mathbb{Z}_{\text{odd}}$ and recall the definition of b_k from* (2.6)*. Then the following identities hold:*

$$\int_{\mathbb{R}} L_k u_k \overline{v_k} dx = \sum_{j\in\mathbb{N}_0} \int_{\mathcal{B}} \lambda_{j,k}(s)\tilde{u}_{j,k}(s)\overline{\tilde{v}_{j,k}(s)}ds \qquad \text{for } u_k \in D(L_k), v_k \in L^2(\mathbb{R}), \tag{3.5}$$

$$b_k(u_k, v_k) = \sum_{j\in\mathbb{N}_0} \int_{\mathcal{B}} \lambda_{j,k}(s)\tilde{u}_{j,k}(s)\overline{\tilde{v}_{j,k}(s)}ds \qquad \text{for } u_k, v_k \in D(b_k) = H^1(\mathbb{R}), \tag{3.6}$$

$$L_k u_k(x) = \sum_{j\in\mathbb{N}_0} \int_{\mathcal{B}} \lambda_{j,k}(s)\tilde{u}_{j,k}(s)\psi_{j,k}(x, s)ds \qquad \text{in } L^2(\mathbb{R}) \text{ for } u_k \in D(L_k). \tag{3.7}$$

Proof. Let us show (3.5). First, note that (3.4) implies

$$\langle u_k, v_k\rangle_{L^2(\mathbb{R})} = \sum_{j\in\mathbb{N}_0} \int_{\mathcal{B}} \tilde{u}_{j,k}(s)\overline{\tilde{v}_{j,k}(s)}ds \text{ for all } u_k, v_k \in L^2(\mathbb{R}). \tag{3.8}$$

Since $u_k \in D(L_k)$ we may use (3.8) for $L_k u_k, v_k \in L^2(\mathbb{R})$ and (3.2) to find

$$\langle L_k u_k, v_k\rangle_{L^2(\mathbb{R})} = \sum_{j\in\mathbb{N}_0} \int_{\mathcal{B}} \left\langle (\mathcal{T} L_k u_k)(\cdot, s), \psi_{j,k}(\cdot, s)\right\rangle_{L^2(\mathcal{P})} \overline{\tilde{v}_{j,k}(s)}ds = \sum_{j\in\mathbb{N}_0} \int_{\mathcal{B}} \lambda_{j,k}(s)\tilde{u}_{j,k}(s)\overline{\tilde{v}_{j,k}(s)}ds.$$

To see (3.6) recall $b_k(u_k, v_k) = \langle L_k u_k, v_k\rangle$ for all $u_k \in D(L_k)$ and all $v_k \in D(b_k)$. Then (3.6) follows from (3.5) and the fact that $D(L_k)$ is dense in $D(b_k) = H^1(\mathbb{R})$, see Chapter IV, Theorem 2.4 (v) in [11].

Finally, let us verify (3.7), see also Theorem XIII.98 (c) in [23]. For $u_k \in D(L_k)$ we have $w := L_k u_k \in L^2(\mathbb{R})$ and therefore

$$w = \sum_{j\in\mathbb{N}_0} \int_{\mathcal{B}} \tilde{w}_{j,k}(s)\psi_{j,k}(x, s)ds \quad \text{in } L^2(\mathbb{R}).$$

By (3.2) we see that $\tilde{w}_{j,k}(s) = \left\langle \mathcal{T} L_k u_k(\cdot, s), \psi_{j,k}(\cdot, s)\right\rangle_{\mathcal{P}} = \lambda_{j,k}(s)\tilde{u}_{j,k}(s)$ and the proof is done. □

3.2. **The right Hilbert space.** We now introduce a Hilbert space in which we find our solutions. Lemma 3.1 suggests to define

$$\mathcal{H} := \Big\{\tilde{u} = (\tilde{u}_{j,k})_{j\in\mathbb{N}_0,k\in\mathbb{Z}_{\text{odd}}} : \tilde{u}_{j,k} \colon \mathcal{B} \to \mathbb{C} \text{ measurable for all } (j,k) \in \mathbb{N}_0 \times \mathbb{Z}_{\text{odd}},$$
$$\overline{\tilde{u}_{j,k}(s)} = \tilde{u}_{j,-k}(-s) \text{ for all } (j,k,s) \in \mathbb{N}_0 \times \mathbb{Z}_{\text{odd}} \times \mathcal{B} \text{ and } \sum_{j\in\mathbb{N}_0,k\in\mathbb{Z}_{\text{odd}}} \int_{\mathcal{B}} |\lambda_{j,k}(s)||\tilde{u}_{j,k}(s)|^2 ds < \infty\Big\},$$

which is a Hilbert space over the field $\mathbb{R}$ equipped with the canonical inner product and norm

$$\langle \tilde{u}, \tilde{v}\rangle_{\mathcal{H}} := \sum_{j\in\mathbb{N}_0,k\in\mathbb{Z}_{\text{odd}}} \int_{\mathcal{B}} |\lambda_{j,k}(s)|\tilde{u}_{j,k}(s)\overline{\tilde{v}_{j,k}(s)} ds \quad \text{and} \quad \|\tilde{u}\|_{\mathcal{H}} := \sqrt{\langle \tilde{u}, \tilde{u}\rangle} \text{ for } \tilde{u}, \tilde{v} \in \mathcal{H}.$$

We next justify the condition $\overline{\tilde{u}_{j,k}(-s)} = \tilde{u}_{j,-k}(s)$ for all $(j,k,s) \in \mathbb{N}_0 \times \mathbb{Z}_{\text{odd}} \times \mathcal{B}$ incorporated in $\mathcal{H}$.

Lemma 3.2. *For $(j,k) \in \mathbb{N}_0 \times \mathbb{Z}_{\text{odd}}$ and $s \in \mathcal{B}$ we have $\lambda_{j,k}(s) = \lambda_{j,-k}(s) = \lambda_{j,k}(-s)$. Moreover, if $\tilde{u} \in \mathcal{H}$ and u_k is given by*

$$u_k(x) = \sum_{j\in\mathbb{N}_0} \int_{\mathcal{B}} \tilde{u}_{j,k}(s)\psi_{j,k}(x,s) ds \tag{3.9}$$

then $\overline{u_k} = u_{-k}$ for all $k \in \mathbb{Z}_{\text{odd}}$.

Proof. Since $\psi_{j,k}$ satisfies (3.1) which only depends on k^2 we have $\lambda_{j,k}(s) = \lambda_{j,-k}(s)$ and may assume

$$\psi_{j,k} = \psi_{j,-k} \text{ on } \mathcal{P} \times \mathcal{B} \text{ for all } (j,k) \in \mathbb{N}_0 \times \mathbb{Z}_{\text{odd}}. \tag{3.10}$$

Taking complex conjugates of (3.1) leads to

$$-\overline{\psi''_{j,k}(x,s)} - k^2 V(x)\overline{\psi_{j,k}(x,s)} = \lambda_{j,k}(s)\overline{\psi_{j,k}(x,s)}, \quad \overline{\psi_{j,k}(x+2\pi,s)} = \overline{\psi_{j,k}(x,s)}e^{-2\pi i s}.$$

This reveals that $\lambda_{j,k}(s) = \lambda_{j,k}(-s)$ and that $\psi_{j,k}(\cdot,s)$ may be chosen such that

$$\overline{\psi_{j,k}(\cdot,s)} = \psi_{j,k}(\cdot,-s) \text{ for all } (j,k,s) \in \mathbb{N}_0 \times \mathbb{Z}_{\text{odd}} \times \mathcal{B}. \tag{3.11}$$

The claim follows if we ensure $\int_{\mathcal{B}} \overline{\tilde{u}_{j,k}(s)\psi_{j,k}(x,s)} ds = \int_{\mathcal{B}} \tilde{u}_{j,-k}(s)\psi_{j,-k}(x,s) ds$. In the following calculation we first exploit (3.11), then use that $\mathcal{B}$ is symmetric about $\{s = 0\}$, profit from (3.10) and finally use $\overline{\tilde{u}_{j,k}(-s)} = \tilde{u}_{j,-k}(s)$. Hence, for $j \in \mathbb{N}_0$ we deduce

$$\int_{\mathcal{B}} \overline{\tilde{u}_{j,k}(s)\psi_{j,k}(x,s)} ds = \int_{\mathcal{B}} \overline{\tilde{u}_{j,k}(s)}\psi_{j,k}(x,-s) ds = \int_{\mathcal{B}} \overline{\tilde{u}_{j,k}(-s)}\psi_{j,k}(x,s) ds = \int_{\mathcal{B}} \tilde{u}_{j,-k}(s)\psi_{j,-k}(x,s) ds$$

which finishes the proof. □

Next, we introduce some further notation which we use later to deal with the indefinite character of the problem. We introduce the projections $\mathcal{P}^+$ and $\mathcal{P}^-$ by

$$\mathcal{H}^+ := \mathcal{P}^+\mathcal{H} := \{\tilde{u} \in \mathcal{H} : \tilde{u}_{j,k} \equiv 0 \text{ whenever } \lambda_{j,k}(s) < 0 \text{ for all } s \in \mathcal{B}\},$$
$$\mathcal{H}^- := \mathcal{P}^-\mathcal{H} := \{\tilde{u} \in \mathcal{H} : \tilde{u}_{j,k} \equiv 0 \text{ whenever } \lambda_{j,k}(s) > 0 \text{ for all } s \in \mathcal{B}\}$$

and set $\tilde{u}^\pm := \mathcal{P}^\pm \tilde{u}$. Moreover, we consider the bilinear form $B \colon \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ defined by

$$B(\tilde{u}, \tilde{v}) = \sum_{j\in\mathbb{N}_0,k\in\mathbb{Z}_{\text{odd}}} \int_{\mathcal{B}} \lambda_{j,k}(s)\tilde{u}_{j,k}(s)\overline{\tilde{v}_{j,k}(s)} ds \text{ for } \tilde{u}, \tilde{v} \in \mathcal{H}.$$

Since by Lemma 2.4 there is no triple $(j,k,s) \in \mathbb{N}_0 \times \mathbb{Z}_{\text{odd}} \times \mathcal{B}$ such that $\lambda_{j,k}(s) = 0$ we obtain the splitting $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ with

$$B(\tilde{u}, \tilde{u}) = \|\tilde{u}^+\|^2_{\mathcal{H}} - \|\tilde{u}^-\|^2_{\mathcal{H}} \text{ for all } \tilde{u} \in \mathcal{H}. \tag{3.12}$$

Hence, $\|\tilde{u}\|^2_{\mathcal{H}} = \|\tilde{u}^+\|^2_{\mathcal{H}} + \|\tilde{u}^-\|^2_{\mathcal{H}}$, and in particular $\|\tilde{u}^+\|_{\mathcal{H}}, \|\tilde{u}^-\|_{\mathcal{H}} \leq \|\tilde{u}\|_{\mathcal{H}}$ for all $\tilde{u} \in \mathcal{H}$.

The domains of L_k and b_k can be characterized by the variables $\tilde{u}_{j,k}(s)$ as follows.

Lemma 3.3. *Fix $k \in \mathbb{Z}_{\text{odd}}$. Then*

$$D(L_k) = \left\{ u = \sum_{j\in\mathbb{N}_0} \int_{\mathcal{B}} \tilde{u}_{j,k}(s)\psi_{j,k}(x,s)ds : \sum_{j\in\mathbb{N}_0} \int_{\mathcal{B}} \lambda^2_{j,k}(s)|\tilde{u}_{j,k}(s)|^2 ds < \infty \right\}$$

$$D(b_k) = \left\{ u = \sum_{j\in\mathbb{N}_0} \int_{\mathcal{B}} \tilde{u}_{j,k}(s)\psi_{j,k}(x,s)ds : \sum_{j\in\mathbb{N}_0} \int_{\mathcal{B}} |\lambda_{j,k}(s)||\tilde{u}_{j,k}(s)|^2 ds < \infty \right\}.$$

Proof. Since $D(L_k) = \{u_k : L_k u_k \in L^2(\mathbb{R})\}$ we can use (3.7) from Lemma 3.1 and obtain

$$\|L_k u_k\|^2_{L^2(\mathbb{R})} = \sum_{j\in\mathbb{N}_0} \int_{\mathcal{B}} |\lambda_{j,k}(s)|^2 |\tilde{u}_{j,k}(s)|^2 ds$$

which proves the claim concerning $D(L_k)$. The second part then follows from (3.6) in Corollary 3.1 and the second representation theorem (Theorem 2.8 and Section IV.4 in [11]). □

Since $D(b_k) = H^1(\mathbb{R})$ the previous Lemma 3.3 shows that elements of $\mathcal{H}$ generate via (3.4) functions u_k belonging to $H^1(\mathbb{R})$.

Corollary 3.4. *Let $\tilde{u} \in \mathcal{H}$. Then u_k from* (3.4) *satisfies $u_k \in H^1(\mathbb{R})$ for all $k \in \mathbb{Z}_{\text{odd}}$.*

4. Fine tuning of prefactors and resulting estimates

We now give two further estimates for elements $v \in D(b_k) = H^1(\mathbb{R})$ which incorporate a k-dependence. We first introduce some notation. Recall

$$\langle L_k u, \varphi\rangle_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \lambda d\,\langle P_\lambda u, \varphi\rangle \text{ for } u \in D(L_k), \varphi \in L^2(\mathbb{R}),$$

where $(P_\lambda)_{\lambda\in\mathbb{R}}$ denotes the projection-valued measure for L_k. We next introduce for $v \in L^2(\mathbb{R})$ the splitting $v = v^+ + v^-$, where $v^\pm := P^\pm v$ with

$$P^+ v := \int_0^\infty 1d\,\langle P_\lambda v, \cdot\rangle, \quad P^- v := \int_{-\infty}^0 1d\,\langle P_\lambda v, \cdot\rangle.$$

Lemma 4.1. *The operators*

$$L_k^\pm : P^\pm D(L_k) \subset P^\pm L^2(\mathbb{R}) \to P^\pm L^2(\mathbb{R}), \quad L_k^\pm u := L_k u \tag{4.1}$$

are self-adjoint operators. Their associated bilinear forms are restrictions of b_k to $D(b_k)^\pm \times D(b_k)^\pm$ with $D(b_k)^\pm := P^\pm D(b_k) = P^\pm H^1(\mathbb{R})$.

Remark 4.2. *Due to the representation in Lemma 3.1, $D(b_k)^+$, $D(L_k^+)$ contain elements with $\tilde{v}_{j,k}(s) = 0$ whenever $\lambda_{j,k}(s) < 0$. Vice versa, $D(b_k)^-$, $D(L_k^-)$ contain elements where $\tilde{v}_{j,k}(s) = 0$ if $\lambda_{j,k}(s) > 0$, cf. Lemma 3.3.*

Proof. We show selfadjointness for L_k^+, the statement for L_k^- follows in the same manner. Due to

$$\langle L_k^+ u, \varphi\rangle = \int_0^\infty \lambda d\,\langle P_\lambda u, \varphi\rangle = \int_0^\infty \lambda d\,\langle u, P_\lambda \varphi\rangle = \langle u, L_k^+ \varphi\rangle$$

we have that L_k^+ is symmetric. Since L_k and the projection-valued measure P_λ commute we also know that L_k and P^+ commute which implies the mapping property of L_k^+ in (4.1). Since $L_k \pm i\,\mathrm{Id}$ has a bounded inverse, also $L_k^+ \pm i\,\mathrm{Id}$ has a bounded inverse and hence L_k^+ is self-adjoint by Theorem VIII.3 in [22]. □

The next two Theorems are based on the spectral information for L_k as stated in Lemma 2.4.

Theorem 4.3. *There is $c > 0$ such that*

$$b_k(v^+, v^+) - b_k(v^-, v^-) \geq c\,|k|\,\|v\|^2_{L^2(\mathbb{R})} \text{ for all } v \in H^1(\mathbb{R}) \text{ and all } k \in \mathbb{Z}_{\text{odd}}. \tag{4.2}$$

Proof. Recall that for a self-adjoint lower semi-bounded operator $A\colon D(A) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})$ we have

$$\inf_{f \in D(A)} \frac{\langle Af, f\rangle_{L^2(\mathbb{R})}}{\|f\|^2_{L^2(\mathbb{R})}} = \inf \sigma(A). \tag{4.3}$$

The idea is now to use the splitting of the indefinite operator L_k into a positive definite and a negative definite operator $L_k^{\pm}$, apply (4.3) and then use the density of $D(L_k)$ in $H^1(\mathbb{R})$. From (4.3) and Lemma 2.4 we conclude that

$$\inf_{u \in P^+D(L_k)} \frac{\left\langle L_k^+ u, u\right\rangle_{L^2(\mathbb{R})}}{\|u\|^2_{L^2(\mathbb{R})}} \geq \tilde{c}|k|, \qquad \inf_{u \in P^-D(L_k)} -\frac{\left\langle L_k^- u, u\right\rangle_{L^2(\mathbb{R})}}{\|u\|^2_{L^2(\mathbb{R})}} \geq \tilde{c}|k| \tag{4.4}$$

for some $\tilde{c} > 0$. By (4.4) one obtains

$$\langle L_k^+ P^+u, P^+u\rangle_{L^2(\mathbb{R})} - \langle L_k^- P^-u, P^-u\rangle \geq \tilde{c}|k|\left(\|P^+u\|^2_{L^2(\mathbb{R})} + \|P^-u\|^2_{L^2(\mathbb{R})}\right) = \tilde{c}|k|\|u\|^2_{L^2(\mathbb{R})}$$

and (4.2) then follows from the density statement above mentioned.

□

The benefit of an estimate like (4.2) lies in the k-dependence. In the following result we construct a similar lower bound with $\|v'\|^2_{L^2(\mathbb{R})}$ instead of $\|v\|^2_{L^2(\mathbb{R})}$ in the right hand side of (4.2).

Theorem 4.4. *There is a constant $c > 0$ such that*

$$b_k(v^+, v^+) - b_k(v^-, v^-) \geq \frac{c}{|k|^3}\|v'\|^2_{L^2(\mathbb{R})} \text{ for all } v \in H^1(\mathbb{R}) \text{ and all } k \in \mathbb{Z}_{\text{odd}}. \tag{4.5}$$

Proof. For $k \in \mathbb{Z}_{\text{odd}}$ and due to the choices of α, β we abbreviate $V_k(x) := -\alpha\omega^2k^2 - \beta\omega^2k^2\delta_{\text{per}}(x) = -\frac{k^2}{16} - k^2\delta_{\text{per}}(x)$. We prove (4.5) by several case distinctions. Let $\lambda \in (0, 1)$ be fixed for the whole proof.

Case 1): Let $v \in D(b_k)^+$. We distinguish two cases.

a): $\int_{\mathbb{R}}\left(|v'|^2 + \frac{V_k}{1-\lambda}|v|^2\right)dx \geq 0$: Then we directly obtain $\int_{\mathbb{R}}\left(|v'|^2 + V_k|v|^2\right)dx \geq \lambda\int_{\mathbb{R}}|v'|^2dx$.

b): $-\int_{\mathbb{R}}\left(|v'|^2 + \frac{V_k}{1-\lambda}|v|^2\right)dx \geq 0$: Recall from (7.1) that for every $\epsilon > 0$

$$\sum_{n\in\mathbb{Z}} |v(2\pi n)|^2 \leq \left(\frac{1}{2\pi} + \frac{1}{2\varepsilon}\right)\|v\|^2_{L^2(\mathbb{R})} + \frac{\varepsilon}{2}\|v'\|^2_{L^2(\mathbb{R})}. \tag{4.6}$$

Therefore,

$$\begin{aligned}\int_{\mathbb{R}}|v'|^2dx \leq -\int_{\mathbb{R}}\frac{V_k}{1-\lambda}|v|^2dx &= \frac{\alpha\omega^2k^2}{1-\lambda}\|v\|^2_{L^2(\mathbb{R})} + \frac{\beta\omega^2k^2}{1-\lambda}\sum_{n\in\mathbb{Z}}|v(2\pi n)|^2 \\ &\leq \frac{\omega^2k^2}{1-\lambda}\left(\alpha + \beta\left(\frac{1}{2\pi} + \frac{1}{2\varepsilon}\right)\right)\|v\|^2_{L^2(\mathbb{R})} + \frac{\beta\omega^2k^2}{1-\lambda}\frac{\varepsilon}{2}\|v'\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

In particular, for $\varepsilon = \varepsilon_k := \frac{1-\lambda}{\beta\omega^2k^2}$ we have $\frac{\beta\omega^2k^2}{1-\lambda}\frac{\varepsilon_k}{2} = \frac{1}{2}$ and thus

$$\|v'\|^2_{L^2(\mathbb{R})} \leq \frac{2\omega^2k^2}{1-\lambda}\left(\alpha + \beta\left(\frac{1}{2\pi} + \frac{1}{2\varepsilon_k}\right)\right)\|v\|^2_{L^2(\mathbb{R})}.$$

In summary, we conclude

$$\frac{\int_{\mathbb{R}}\left(|v'|^2+V_k(x)|v|^2\right)dx}{\int_{\mathbb{R}}|v'|^2dx}=\frac{\int_{\mathbb{R}}\left(|v'|^2+V_k(x)|v|^2\right)dx}{\|v\|^2_{L^2(\mathbb{R})}}\frac{\|v\|^2_{L^2(\mathbb{R})}}{\|v'\|^2_{L^2(\mathbb{R})}}\geq c|k|\frac{1-\lambda}{2\omega^2k^2}\frac{1}{\alpha+\beta\left(\frac{1}{2\pi}+\frac{1}{2\varepsilon_k}\right)}. \tag{4.7}$$

Since ε_k is of order $\frac{1}{k^2}$ we infer that the right hand side in (4.7) is of order $O\left(\frac{1}{|k|^3}\right)$. Therefore, merging case 1a) and (4.7) we deduce $\int_{\mathbb{R}}\left(|v'|^2+V_k(x)|v|^2\right)dx\geq\frac{c}{|k|^3}\int_{\mathbb{R}}|v'|^2dx$ for all $v\in D(b_k)^+$ and $c>0$.

Case 2): Let $v\in D(b_k)^-$, i.e., $\int_{\mathbb{R}}\left(|v'|^2+V_k|v|^2\right)dx\leq -c|k|\int_{\mathbb{R}}|v|^2dx$. By (4.6) with $\varepsilon=\varepsilon_k=\frac{1}{\beta\omega^2k^2}$ we deduce

$$\int_{\mathbb{R}}|v'|^2dx\leq\left(\alpha\omega^2k^2-c|k|\right)\|v\|^2_{L^2(\mathbb{R})}+\beta\omega^2k^2\left(\frac{1}{2\pi}+\frac{1}{2\varepsilon_k}\right)\|v\|^2_{L^2(\mathbb{R})}+\frac{\beta\omega^2k^2\varepsilon_k}{2}\|v'\|^2_{L^2(\mathbb{R})}$$

which entails

$$\|v'\|^2_{L^2(\mathbb{R})}\leq 2\left(\alpha\omega^2k^2-c|k|+\beta\omega^2k^2\left(\frac{1}{2\pi}+\frac{1}{2\varepsilon_k}\right)\right)\|v\|^2_{L^2(\mathbb{R})}. \tag{4.8}$$

In analogy to the first case we now conclude

$$\frac{-\int_{\mathbb{R}}\left(|v'|^2+V_k|v|^2\right)dx}{\|v'\|^2_{L^2(\mathbb{R})}}=\frac{-\int_{\mathbb{R}}\left(|v'|^2+V_k|v|^2\right)dx}{\|v\|^2_{L^2(\mathbb{R})}}\frac{\|v\|^2_{L^2(\mathbb{R})}}{\|v'\|^2_{L^2(\mathbb{R})}}\geq c|k|\frac{\|v\|^2_{L^2(\mathbb{R})}}{\|v'\|^2_{L^2(\mathbb{R})}}$$

and due to (4.8) the fraction $\frac{\|v\|^2_{L^2(\mathbb{R})}}{\|v'\|^2_{L^2(\mathbb{R})}}$ is of order $\frac{1}{|k|^4}$ which establishes our claim in the case $v\in D(b_k)^-$.

Finally, merging the two estimates for $D(b_k)^+$ and $D(b_k)^-$ we end up with

$$b_k(v^+,v^+)-b_k(v^-,v^-)\geq\frac{\tilde{c}}{|k|^3}\int_{\mathbb{R}}\left(\left(|v^{+'}|\right)^2+\left(|v^{-'}|\right)^2\right)dx\geq\frac{\tilde{c}}{2|k|^3}\int_{\mathbb{R}}|v'|^2dx$$

for a constant $\tilde{c}>0$ and the proof is done. □

5. Minimization on the generalized Nehari manifold

In Corollary 3.4 we were able to deduce $H^1(\mathbb{R})$-regularity in space for each member of the sequence $(u_k)_{k\in\mathbb{Z}_{\text{odd}}}$. Now we establish integrability of the composite function $u(x,t)=\sum_{k\in\mathbb{Z}_{\text{odd}}}u_k(x)e^{ik\omega t}$ in space and time as expressed by the following theorem. The proof, which is rather complex, is given in Section 6.

Theorem 5.1. *The linear operator $\mathcal{S}\colon\mathcal{H}\to L^q(D)$,*

$$(\mathcal{S}\tilde{u})(x,t):=\sum_{j\in\mathbb{N}_0,k\in\mathbb{Z}_{\text{odd}}}\int_{\mathcal{B}}\tilde{u}_{j,k}(s)\psi_{j,k}(x,s)ds\, e^{ik\omega t}$$

is bounded for all $q\in[2,\frac{8}{3}]$ where $D=\mathbb{R}\times[0,T)$.

Now we find the time-periodic solution of $(1.1)_\pm$ as a minimizer of a functional J on the so-called generalized Nehari manifold. We are using Theorem 35, Chapter 4 from [24], where an abstract result is given that guarantees the existence of minimizer of an indefinite functional on the generalized Nehari manifold. We first treat the "+"-case in $(1.1)_\pm$. At the end of this section we explain how the "−"-case can be treated. Let $J\colon\mathcal{H}\to\mathbb{R}$ be given by

$$J(\tilde{u}):=J_0(\tilde{u})-J_1(\tilde{u})$$

with

$$J_0(\tilde{u}) := \frac{1}{2}B(\tilde{u},\tilde{u}), \quad J_1(\tilde{u}) := \frac{1}{T(p+1)}\int_D |\mathcal{S}\tilde{u}|^{p+1} d(x,t)$$

and where $\mathcal{S}$ is the operator from Theorem 5.1 which reproduces $u(x,t)$ from the Bloch-variables $\tilde{u} = (\tilde{u}_{j,k}(s))_{j\in\mathbb{N}_0,k\in\mathbb{Z}_{\text{odd}},s\in\mathcal{B}} \in \mathcal{H}$. Due to Theorem 5.1 the functional J is well-defined on $\mathcal{H}$. The generalized Nehari manifold is defined as

$$\mathcal{M} := \{\tilde{u} \in \mathcal{H}\setminus\mathcal{H}^- : J'(\tilde{u})[\tilde{u}] = 0 \text{ and } J'(\tilde{u})[\tilde{v}] = 0 \text{ for all } \tilde{v}\in\mathcal{H}^-\}.$$

Moreover, for $\tilde{u}\in\mathcal{H}$ we set

$$\mathcal{H}(\tilde{u}) := \mathbb{R}^+\tilde{u}\oplus\mathcal{H}^- = \mathbb{R}^+\tilde{u}^+\oplus\mathcal{H}^-,$$

where $\mathbb{R}^+ = [0,\infty)$. Finally, let S denote the unit ball in $\mathcal{H}$ and define $S^+ := S\cap\mathcal{H}^+$.

By standard calculations (compare Proposition 1.12 in [26]) we deduce $J\in C^1(\mathcal{H})$. Using the conjugation-symmetry of $\tilde{u},\tilde{v}\in\mathcal{H}$ we find

$$J'(\tilde{u})[\tilde{v}] = J_0'(\tilde{u})[\tilde{v}] - J_1'(\tilde{u})[\tilde{v}] = \sum_{j\in\mathbb{N}_0,k\in\mathbb{Z}_{\text{odd}}}\int_{\mathcal{B}}\lambda_{j,k}(s)\tilde{u}_{j,k}(s)\overline{\tilde{v}_{j,k}(s)}ds - \frac{1}{T}\int_D |\mathcal{S}\tilde{u}|^{p-1}\mathcal{S}\tilde{u}\overline{\mathcal{S}\tilde{v}}d(x,t).$$

Notice that $\tilde{u},\tilde{v}\in\mathcal{H}$ imply that $\mathcal{S}\tilde{u},\mathcal{S}\tilde{v}$ are read-valued functions and that $J_0'(\tilde{u})[\tilde{v}], J_1'(\tilde{u})[\tilde{v}]\in\mathbb{R}$. The verification of $J'[\tilde{u}] = 0$ for a suitable $\tilde{u}\in\mathcal{H}$ is a key point in this section. We simplify this task by the following lemma. The proof is given in the Appendix.

Lemma 5.2. *For $k\in\mathbb{Z}_{\text{odd}}$ let*

$$\mathcal{H}_{k,\text{mono}} := \left\{\tilde{\phi} = (\tilde{\phi}_{j,k})_{j\in\mathbb{N}_0} : \tilde{\phi}_{j,k} : \mathcal{B}\to\mathbb{C} \text{ measurable s.t.} \sum_{j\in\mathbb{N}_0}\int_{\mathcal{B}}|\lambda_{j,k}(s)||\tilde{\phi}_{j,k}(s)|^2 ds < \infty\right\}.$$

Let $\tilde{u}\in\mathcal{H}$. Then the following are equivalent:

(i) *for all $k\in\mathbb{Z}_{\text{odd}}$ we have $J'(\tilde{u})[\tilde{\phi}] = 0$ for all $\tilde{\phi}\in\mathcal{H}_{k,\text{mono}}$ for a dense subset of $\mathcal{H}_{k,\text{mono}}$*

(ii) $J'(\tilde{u}) = 0$

The set of all $\tilde{\phi}\in\mathcal{H}_{k,\text{mono}}$ such that $\mathcal{S}\tilde{\phi} := \sum_{j\in\mathbb{N}_0}\int_{\mathcal{B}}\tilde{\phi}_{j,k}(x)\psi_{j,k}(x,s)\,ds e^{ik\omega t}$ has compact support in $\overline{D}$ is dense in $\mathcal{H}_{k,\text{mono}}$.

Remark 5.3. *The set $\mathcal{H}_{k,\text{mono}}$ consists of monochromatic Bloch-variables occupying only the frequency $k\omega$ while all other frequencies $l\omega$ with $l\neq k$ are not occupied. Because of the missing conjugation-symmetry $\mathcal{H}_{k,\text{mono}}$ is not a subset of $\mathcal{H}$. Nevertheless, the functionals J, J' as well as the map $\mathcal{S}$ naturally extend as continuous functions to $\mathcal{H}_{k,\text{mono}}$.*

We start verifying the assumption (B_1), (i) and (ii) of Theorem 35 in [24].

Lemma 5.4. *The following statements hold true:*

(a) *J_1 is weakly lower semicontinuous,*

$$J_1(0) = 0 \quad \text{and} \quad \frac{1}{2}J_1'(\tilde{u})[\tilde{u}] > J_1(\tilde{u}) > 0 \text{ for } \tilde{u}\neq 0. \tag{5.1}$$

(b) $\lim_{\tilde{u}\to 0}\frac{J_1'(\tilde{u})}{\|\tilde{u}\|_{\mathcal{H}}} = 0$ *and* $\lim_{\tilde{u}\to 0}\frac{J_1(\tilde{u})}{\|\tilde{u}\|^2_{\mathcal{H}}} = 0$.

(c) *For a weakly compact set $U\subset\mathcal{H}\setminus\{0\}$ we have $\lim_{s\to\infty}\frac{J_1(s\tilde{u})}{s^2} = \infty$ uniformly w.r.t. $\tilde{u}\in U$.*

Proof. (a) Since J_1 is continuous and convex (recall $\mathcal{S}$ is linear) it is in particular weakly lower semi-continuous. Due to $p > 1$ we obtain

$$J_1'(\tilde{u})[\tilde{u}] = (p+1)J_1(\tilde{u}) \geq 2J_1(\tilde{u}) \geq 0.$$

To see that the last two inequalities in (5.1) are strict for $\tilde{u} \neq 0$ it suffices to prove that $\mathcal{S}\colon \mathcal{H} \to L^{p+1}(D)$ is one-to-one. Therefore, let $\tilde{u} \in \mathcal{H}$ be given with $\mathcal{S}\tilde{u} = 0$. In particular, $\mathcal{S}\tilde{u} \in L^2(D)$ and

$$0 = \|\mathcal{S}\tilde{u}\|^2_{L^2(D)} = \sum_{j\in\mathbb{N}_0, k\in\mathbb{Z}_{\text{odd}}} \int_{\mathcal{B}} |\tilde{u}_{j,k}(s)|^2 ds,$$

i.e., $\tilde{u} = 0$ and (5.1) is verified.

(b) This is immediate by the embedding provided by Theorem 5.1.

(c) Let $U \subset \mathcal{H} \setminus \{0\}$ be weakly compact and $\delta := \inf_{\tilde{u}\in U} \|\mathcal{S}\tilde{u}\|_{L^{p+1}(D)}$. We show that $\delta > 0$. There is a sequence $(\tilde{u}_n)_{n\in\mathbb{N}}$ in U with $\|\mathcal{S}\tilde{u}_n\|_{L^{p+1}(D)} \to \delta$ as $n \to \infty$. Since U is weakly compact there is $\tilde{u} \in U$ and a subsequence such that $\tilde{u}_{n_m} \rightharpoonup \tilde{u}$ in $\mathcal{H}$ as $m \to \infty$. In particular, $\mathcal{S}\tilde{u}_{n_m} \to \mathcal{S}\tilde{u}$ in $L^2(D_{\text{loc}})$ as $m \to \infty$ and therefore by a further diagonal argument we can assume w.l.o.g. that $\mathcal{S}\tilde{u}_{n_m} \to \mathcal{S}\tilde{u}$ pointwise almost everywhere in D. In particular, Fatou's lemma gives

$$\delta = \liminf_{m\to\infty} \|\mathcal{S}\tilde{u}_{n_m}\|^{p+1}_{L^{p+1}(D)} \geq \|\mathcal{S}\tilde{u}\|^{p+1}_{L^{p+1}(D)} > 0$$

due to $0 \notin U$. Thus, for an arbitrary sequence $(s_n)_{n\in\mathbb{N}}$ with $s_n \to \infty$ as $n \to \infty$ and $\tilde{u} \in U$ we infer

$$\frac{J_1(s_n\tilde{u})}{s_n^2} = s_n^{p-1} J_1(\tilde{u}) \geq s_n^{p-1}\delta^{p+1} \to \infty \text{ as } n \to \infty$$

uniformly for $\tilde{u} \in U$. □

Assumption (B_2) of Theorem 35 in [24] is guaranteed by the next result.

Lemma 5.5. *The following statements hold true:*

(a) *For each $\tilde{w} \in \mathcal{H} \setminus \mathcal{H}^-$ there exists a unique nontrivial critical point $m_1(\tilde{w})$ of $J|_{\mathcal{H}(\tilde{w})}$. Moreover, $m_1(\tilde{w}) \in \mathcal{M}$ is the unique global maximizer of $J|_{\mathcal{H}(\tilde{w})}$ as well as $J(m_1(\tilde{w})) > 0$.*
(b) *There exists $\delta > 0$ such that $\|m_1(\tilde{w})^+\|_{\mathcal{H}} \geq \delta$ for all $\tilde{w} \in \mathcal{H} \setminus \mathcal{H}^-$.*

Proof. (a) We can directly follow the lines of proof of Proposition 39 in [24].

(b) First, consider $\tilde{v} \in \mathcal{H}^+$. Then we have $\lim_{\tilde{v}\to 0} \frac{J(\tilde{v})}{\|\tilde{v}\|^2_{\mathcal{H}}} = \frac{1}{2}$ due to Lemma 5.4 (b). Thus there is $\rho_0 > 0$ s.t. $J(\tilde{v}) \geq \frac{1}{4}\|\tilde{v}\|^2_{\mathcal{H}}$ for all $\tilde{v} \in \mathcal{H}^+$ with $\|\tilde{v}\|_{\mathcal{H}} \leq \rho_0$. Hence for $\rho \in (0, \rho_0)$ we find $\eta = \frac{\rho^2}{4}$ with $J(\tilde{v}) \geq \eta$ for all $\tilde{v} \in \mathcal{H}^+$ with $\|\tilde{v}\|_{\mathcal{H}} = \rho$. Now, let $\tilde{w} \in \mathcal{H} \setminus \mathcal{H}^-$. Due to the structure of J we infer that

$$\frac{\|m_1(\tilde{w})^+\|^2_{\mathcal{H}}}{2} \geq J(m_1(\tilde{w})). \tag{5.2}$$

Since $m_1(\tilde{w})$ is the maximizer of $J|_{\mathcal{H}(\tilde{w})}$ we conclude

$$J(m_1(\tilde{w})) \geq J\left(\rho \frac{\tilde{w}^+}{\|\tilde{w}^+\|_{\mathcal{H}}}\right) \geq \eta. \tag{5.3}$$

and the combination of (5.2) and (5.3) finishes the proof of part (b). □

Lemma 5.6. *Any Palais-Smale sequence $(\tilde{u}_n)_{n\in\mathbb{N}}$ of $J|_{\mathcal{M}}$ is bounded.*

Proof. We show that there is a constant $C > 0$ such that

$$\|\tilde{u}\|_{\mathcal{H}} \leq CJ(\tilde{u})^{\frac{p}{p+1}} \text{ for all } \tilde{u} \in \mathcal{M}.$$

Due to $\tilde{u} \in \mathcal{M}$ we have $J'(\tilde{u})[\tilde{u} - \tilde{u}^-] = 0$. Thus,

$$\|\tilde{u}^+\|^2_{\mathcal{H}} = \underbrace{J'(\tilde{u})[\tilde{u}^+]}_{=0} + \frac{1}{T}\int_D |\mathcal{S}\tilde{u}|^{p-1}\mathcal{S}\tilde{u}\mathcal{S}\tilde{u}^+ d(x,t) \leq \frac{1}{T}\|\mathcal{S}\tilde{u}\|^p_{L^{p+1}(D)}\|\mathcal{S}\tilde{u}^+\|_{L^{p+1}(D)}. \tag{5.4}$$

Since $\tilde{u} \in \mathcal{M}$ implies $\|\mathcal{S}\tilde{u}\|^{p+1}_{L^{p+1}(D)} = \frac{2T(p+1)}{(p-1)}J(\tilde{u})$ and since $\|\mathcal{S}\tilde{u}^+\|_{L^{p+1}(D)} \leq \bar{C}\|\tilde{u}^+\|_{\mathcal{H}}$ by Theorem 5.1, we derive from (5.4) that $\|\tilde{u}^+\|_{\mathcal{H}} \leq \tilde{C}J(\tilde{u})^{\frac{p}{p+1}}$. Analogously, one shows $\|\tilde{u}^-\|_{\mathcal{H}} \leq \tilde{C}J(\tilde{u})^{\frac{p}{p+1}}$ and the proof is done. □

Finally, we can turn to our overall goal of this section and verify the following statement.

Theorem 5.7. *The functional J admits a ground state, i.e., there exists $\tilde{u} \in \mathcal{M}$ such that $J'(\tilde{u}) = 0$ and $J(\tilde{u}) = \inf_{\tilde{v}\in\mathcal{M}} J(\tilde{v})$.*

The proof requires the following variant of a concentration-compactness Lemma of P. L. Lions, cf. Lemma 1.21 in [26] for a similar result in non-fractional Sobolev-spaces. Its proof is given in the Appendix. Recall that we interpret $\tilde{u} \in \mathcal{H}$ as a function on D which is continued to $\mathbb{R}^2$ periodically w.r.t. the second component. This is needed since in the following lemma the balls $B_r(y)$ which can exceed the set D.

Lemma 5.8. *Let $q \in [2, \frac{8}{3})$ and $r > 0$ be given. Moreover, let $(\tilde{u}_n)_{n\in\mathbb{N}}$ be a bounded sequence in $\mathcal{H}$ and*

$$\sup_{z\in D}\int_{B_r(z)} |\mathcal{S}\tilde{u}_n|^q d(x,t) \to 0 \text{ as } n \to \infty. \tag{5.5}$$

Then $\mathcal{S}\tilde{u}_n \to 0$ in $L^{\tilde{q}}(D)$ as $n \to \infty$ for all $\tilde{q} \in (2, \frac{8}{3})$.

Proof of Theorem 5.7: Conditions (B1), (B2) and (i) and (ii) of Theorem 35 in [24] are fulfilled, and only (iii) does not hold so that J does not satisfy the Palais-Smale condition. As a consequence, Theorem 35 in [24] only provides a minimizing Palais-Smale $(\tilde{u}_n)_{n\in\mathbb{N}}$ in $\mathcal{M}$ with $J'(\tilde{u}_n) \to 0$ as $n \to \infty$. Lemma 5.6 guarantees that $(\tilde{u}_n)_{n\in\mathbb{N}}$ is bounded. Thus, there is $\tilde{u} \in \mathcal{H}$ such that $\tilde{u}_{n_m} \rightharpoonup \tilde{u}$ as $m \to \infty$. We now proceed in three steps:

First claim: $J'(\tilde{u}) = 0$. By Lemma 5.2 it is enough to check $J'(\tilde{u})[\tilde{v}] = 0$ for $\tilde{v} \in \mathcal{H}$ with $\mathcal{S}\tilde{v}$ having compact support in $\overline{D}$. For such $\tilde{v}$ we conclude first by weak convergence that

$$J_0'(\tilde{u}_n)[\tilde{v}] = B(\tilde{u}_n, \tilde{v}) \to B(\tilde{u}, \tilde{v}) = J_0'(\tilde{u})[\tilde{v}] \text{ as } n \to \infty.$$

Next, due to the compact support property of $\tilde{v}$ and the compact embedding $H^{1/4}_{\text{per}}(\mathbb{R}^2) \hookrightarrow L^{p+1}(K)$, $1 < p < \frac{5}{3}$ for any compact subset $K \subset \mathbb{R}^2$, cf. Corollary 7.2 in [8], we obtain

$$J_1'(\tilde{u}_n)[\tilde{v}] = \frac{1}{T}\int_D |\mathcal{S}\tilde{u}_n|^{p-1}\mathcal{S}\tilde{u}_n\mathcal{S}\tilde{v} d(x,t) \to J_1'(\tilde{u})[\tilde{v}] \text{ as } n \to \infty.$$

Combining the two convergence results and using that $\mathcal{V}$ is dense in $\mathcal{H}$ we deduce $J'(\tilde{u}) = 0$. Note that this chain of arguments only uses that $(\tilde{u}_n)_{n\in\mathbb{N}}$ is a Palais-Smale sequence for J and not $\tilde{u}_n \in \mathcal{M}$.

Second claim: We may choose a new Palais-Smale sequence $(\tilde{v}_n)_{n\in\mathbb{N}}$ such that $J(\tilde{v}_n) \to \inf_{\mathcal{M}} J$ and that its weak limit $\tilde{v}$ belongs to $\mathcal{M}$ (we do not claim that $\tilde{v}_n \in \mathcal{M}$). We first show that

$$\liminf_{n\to\infty}\sup_{z\in D}\int_{B_1(z)} |\mathcal{S}\tilde{u}_n|^2 d(x,t) > 0. \tag{5.6}$$

Suppose (5.6) is violated. Then Lemma 5.8 implies $\|\mathcal{S}\tilde{u}_n\|_{L^{p+1}(D)} \to 0$ as $n \to \infty$ along a subsequence which we again denote by $(\tilde{u}_n)_{n\in\mathbb{N}}$. Therefore, we conclude

$$\int_D |\mathcal{S}\tilde{u}_n|^{p-1}\mathcal{S}\tilde{u}_n\mathcal{S}\tilde{u}_n^+ d(x,t) \to 0 \text{ as } n \to \infty.$$

Due to

$$0 = J'(\tilde{u}_n)\tilde{u}_n^+ = \|\tilde{u}_n^+\|_{\mathcal{H}}^2 - \frac{1}{T}\int_D |\mathcal{S}\tilde{u}_n|^{p-1}\mathcal{S}\tilde{u}_n\mathcal{S}\tilde{u}_n^+ d(x,t)$$

we obtain $\|\tilde{u}_n^+\|_{\mathcal{H}}^2 \to 0$ as $n \to \infty$, a contradiction to Lemma 5.5 (b) (notice that $m_1(\tilde{u}_n) = \tilde{u}_n$ since $\tilde{u}_n \in \mathcal{M}$). Therefore, (5.6) is valid and we find $\delta > 0$, a sequence $(y_n)_{n\in\mathbb{N}}$ in D and a subsequence of $(\tilde{u}_n)_{n\in\mathbb{N}}$ (again denoted by $(\tilde{u}_n)_{n\in\mathbb{N}}$) such that

$$\int_{B_1(y_n)} |\mathcal{S}\tilde{u}_n|^2 d(x,t) \geq \delta > 0 \text{ for all } n \in \mathbb{N}. \tag{5.7}$$

Next we shift $\mathcal{S}\tilde{u}_n$ in such a way that we can make use of compact embeddings for the shifted sequence. For this purpose let us first study the effect of the following transformation on elements of $\mathcal{H}$. For $\tilde{u} \in \mathcal{H}$, $m \in \mathbb{Z}$ denote $\tilde{v} := e^{2\pi i m s}\tilde{u}$ meaning $\tilde{v}_{j,k}(s) = e^{2\pi i m s}\tilde{u}_{j,k}(s)$ for all $j \in \mathbb{N}_0, k \in \mathbb{Z}_{\text{odd}}$ and all $s \in \mathcal{B} = [-\frac{1}{2}, \frac{1}{2})$. Then

$$\tilde{v} \in \mathcal{H}, \quad \|\tilde{v}\|_{\mathcal{H}} = \|\tilde{u}\|_{\mathcal{H}}, \quad (\mathcal{S}\tilde{v})(x - \pi m, t) = (\mathcal{S}\tilde{u})(x,t) \text{ for all } (x,t) \in \mathbb{R}^2. \tag{5.8}$$

For the centers $y_n = (x_n, t_n)^T$ of the balls appearing in (5.7) we have $x_n = 2\pi m_n + r_n$ for some $m_n \in \mathbb{Z}, r_n \in [0, 2\pi)$. The shifted centers are denoted by $y_n' := (r_n, t_n)^T \in [0, 2\pi) \times [0, T)$. Let us define new functions $\tilde{v}_n$ by

$$\tilde{v}_n := e^{2\pi i m_n s}\tilde{u}_n.$$

If we set $\tilde{B} := [-1, 2\pi+1] \times [-1, T+1]$ then $B_1(y_n') \subset \tilde{B}$ for all $n \in \mathbb{N}$. Moreover, (5.7) and (5.8) entail

$$\int_{\tilde{B}} |\mathcal{S}\tilde{v}_n|^2 d(x,t) \geq \int_{B_1(y_n')} |\mathcal{S}\tilde{v}_n|^2 d(x,t) = \int_{B_1(y_n)} |\mathcal{S}\tilde{u}_n|^2 d(x,t) \geq \delta \text{ for all } n \in \mathbb{N}.$$

Up to a selection of a subsequence, $\tilde{v}_n \rightharpoonup \tilde{v} \in \mathcal{H}$ as $n \to \infty$. Using the compact embedding to $L^{p+1}(\tilde{B})$ (cf. Corollary 7.2 in [8]) yields $\|\mathcal{S}\tilde{v}\|_{L^{p+1}(D)} \neq 0$.

We now prove some additional properties of $(\tilde{v}_n)_{n\in\mathbb{N}}$ which ensure that $(\tilde{v}_n)_{n\in\mathbb{N}}$ is also a bounded Palais-Smale sequence for J. We have $\|\tilde{v}_n\|_{\mathcal{H}}^2 = \|\tilde{u}_n\|_{\mathcal{H}}^2$ as well as $B(\tilde{u}_n, \tilde{u}_n) = B(\tilde{v}_n, \tilde{v}_n)$ with B from (3.12). This entails

$$\|\tilde{u}_n^+\|_{\mathcal{H}} = \|\tilde{v}_n^+\|_{\mathcal{H}} \text{ and } \|\tilde{u}_n^-\|_{\mathcal{H}} = \|\tilde{v}_n^-\|_{\mathcal{H}} \text{ for all } n \in \mathbb{N}. \tag{5.9}$$

By the shift-property (5.8) we infer that $\int_D |\mathcal{S}\tilde{u}_n|^{p+1} d(x,t) = \int_D |\mathcal{S}\tilde{v}_n|^{p+1} d(x,t)$. This and (5.9) implies $J(\tilde{u}_n) = J(\tilde{v}_n)$. In order to prove that $(\tilde{v}_n)_{n\in\mathbb{N}}$ is a Palais-Smale sequence it remains to show that $\|J'(\tilde{v}_n)\| \to 0$ as $n \to \infty$. Therefore, for $\tilde{w} \in \mathcal{H}$ we calculate $J_0'(\tilde{u}_n)[\tilde{w}] = J_0'(\tilde{v}_n)[\tilde{w}e^{2\pi i m_n s}]$. Similarly, by using the shift-property (5.8) we find $J_1'(\tilde{u}_n)[\tilde{w}] = J_1'(\tilde{v}_n)[\tilde{w}e^{2\pi i m_n s}]$ which in summary yields

$$J'(\tilde{u}_n)[\tilde{w}] = J'(\tilde{v}_n)[\tilde{w}e^{2\pi i m_n s}]. \tag{5.10}$$

Moreover, for $m \in \mathbb{Z}$ the map $\tilde{w} \mapsto \tilde{w}e^{-2\pi i m s}$ is a bijection on $\mathcal{H}$ with inverse $\tilde{w} \mapsto \tilde{w}e^{2\pi i m s}$. Thus, by (5.10) we conclude that $\|J'(\tilde{u}_n)\| = \|J'(\tilde{v}_n)\|$, i.e., $\|J'(\tilde{v}_n)\| \to 0$ as $n \to \infty$. In summary, we have shown that $(\tilde{v}_n)_{n\in\mathbb{N}}$ is a Palais-Smale sequence with nontrivial weak limit $\tilde{v} \neq 0$. The property $J'(\tilde{v}) = 0$ follows from the first claim.

It remains to show $\tilde{v}^+ \neq 0$. Assume by contradiction that $\tilde{v}^+ = 0$, i.e., $\tilde{v} = \tilde{v}^-$. By testing $J'(\tilde{v}) = 0$ with $\tilde{v}$ we infer

$$-\|\tilde{v}^-\|_{\mathcal{H}}^2 = \frac{1}{T}\int_D |\mathcal{S}\tilde{v}|^{p+1} d(x,t),$$

a contradiction since the two expressions have different signs. Thus, $\tilde{v} \in \mathcal{M}$.

Third claim: $\tilde{u}$ minimizes J on $\mathcal{M}$. Since $\tilde{u} \in \mathcal{M}$ we obviously have $J(\tilde{u}) \geq \inf_{\mathcal{M}} J$. The reverse inequality follows from $J|_{\mathcal{M}} = \frac{p-1}{2} J_1$, the fact that $(\tilde{u}_n)_{n\in\mathbb{N}}$ is a minimizing sequence and Fatou's lemma. □

Remark 5.9. *Let us explain how the case of "−" in* $(1.1)_\pm$ *can be treated. In this case one keeps the functional* J_1 *but replaces* J_0 *by* $-J_0$ *and flips the spaces* $\mathcal{H}^+$ *and* $\mathcal{H}^-$. *Since* J_0 *is an indefinite functional this is without relevance for the proof strategy. All proofs of this section can be carried over with no change.*

It remains to give the proof of Theorem 1.3 and Corollary 1.4. We only do the "+"-case.

Proof of Theorem 1.3: Let $\tilde{u}$ be a ground state of J obtained previously in Theorem 5.7. We will verify that $u := \mathcal{S}\tilde{u}$ is a weak solution of $(1.1)_\pm$ in the sense of Definition 1.1. By Theorem 5.1 we have that $u \in L^{p+1}(D)$. The boundedness of the operator $\mathcal{S}_1 : \mathcal{H} \to \hat{H}$ from Theorem 6.5 shows that

$$\|u\|^2_{H^{1/2}(0,T;L^2)} + \|u\|^2_{H^{-3/2}(0,T;H^1)} = \sum_{k\in\mathbb{Z}} \left(|k| \|u_k\|^2_{L^2(\mathbb{R})} + \frac{1}{|k|^3}\|u_k'\|^2_{L^2(\mathbb{R})} \right) \leq C\|\tilde{u}\|^2_{\mathcal{H}}.$$

Therefore all the integrability and regularity assumptions of Definition 1.1 are fulfilled.

In the following we fix a test function $v = \sum_{k\in\mathbb{Z}} v_k(x)e^{ik\omega t}$ with finitely many nonzero coefficient functions $v_k \in C_c^\infty(\mathbb{R})$. Since L^2-inner products in time on $(0,T)$ between u, $|u|^{p-1}u$ and $v_k e^{ik\omega t}$ trivially vanish whenever $k \in \mathbb{Z}$ is even we may assume w.l.o.g. that only (finitely many) odd indices $k \in \mathbb{Z}_{\text{odd}}$ appear with $|k| \leq K$. For $k \in \{-K, \ldots, K\}$, k odd one finds

$$\begin{aligned} b_k(u_k, v_k) &= \int_{\mathbb{R}} -u_k'\bar{v}_k' - \alpha k^2 u_k \bar{v}_k dx - \beta k^2 \sum_{n\in\mathbb{Z}} u_k(2\pi n)\bar{v}_k(2\pi n) \\ &= \int_{\mathbb{R}} u_k \bar{v}_k'' - \alpha k^2 u_k \bar{v}_k dx - \beta k^2 \sum_{n\in\mathbb{Z}} u_k(2\pi n)\bar{v}_k(2\pi n). \end{aligned} \tag{5.11}$$

Let us set

$$\tilde{v}_{k,j}(s) := \langle \mathcal{T} v_k(\cdot, s), \psi_{j,k}(\cdot, s)\rangle_{\mathcal{P}}, \quad \tilde{v} = (\tilde{v}_{j,k}(s))_{j\in\mathbb{N}_0, k\in\mathbb{Z}_{\text{odd}}, \in\mathcal{B}}.$$

Summing (5.11) over k with $|k| \leq K$ we get

$$J_0'(\tilde{u})[\tilde{v}] = \sum_{|k|\leq K} b_k(u_k, v_k) = \frac{1}{T}\int_D u(-\bar{v}_{xx} + \alpha\bar{v}_{tt})\, d(x,t) + \frac{\beta}{T}\sum_{n\in\mathbb{Z}} \langle u(2\pi n, \cdot), v_{tt}(2\pi n, \cdot)\rangle_{H^{-\frac{1}{2}} \times H^{\frac{1}{2}}}. \tag{5.12}$$

Note that $\tilde{v}$ does not necessarily fulfill the conjugation-symmetry, but it is a finite sum of members of $\mathcal{H}_{k,\text{mono}}$ for $|k| \leq K$. By Lemma 5.2 we deduce $J_0'(\tilde{u})[\tilde{v}] = J_1'(\tilde{u})[\tilde{v}] = \frac{1}{T}\int_D |u|^{p-1}u\bar{v}\, d(x,t)$. In view of (5.12) this shows that u is indeed a weak solution in the sense of Definition 1.1.

It remains to show that the assumption of having only finitely many nonzero compactly supported coefficient functions $v_k \in C_c^\infty(\mathbb{R})$ in the definition of $v = \sum_{k\in\mathbb{Z}} v_k(x)e^{ik\omega t}$ may be relaxed in favor of $v \in L^{p+1}(D)$, $v \in H^r(0,T;L^2(\mathbb{R})) \cap H^s(0,T;H^1(\mathbb{R})) \cap L^2(0,T;H^2(\mathbb{R}))$ for $r \geq \frac{5}{2}$, $s \geq 5 - r$. This will follow from the first part of the theorem by letting the summation index in the definition of v tend to infinity and using the following estimates:

$$\left|\int_D |u|^{p-1}u\bar{v}\,d(x,t)\right| \le \|u\|^p_{L^{p+1}(D)}\|v\|_{L^{p+1}(D)},$$

$$\left|\int_D u(-\bar{v}_{xx}+\alpha\bar{v}_{tt})\,d(x,t)\right| \le \|u\|_{L^2(D)}\|-v_{xx}+\alpha v_{tt}\|_{L^2(D)},$$

$$\left|\sum_{n\in\mathbb{Z}}\langle u(2\pi n,\cdot),v_{tt}(2\pi n,\cdot)\rangle_{H^{-\frac12}\times H^{\frac12}}\right| \le \sum_{n\in\mathbb{Z}}\|u(2\pi n,\cdot)\|_{H^{-1/2}(0,T)}\|v_{tt}(2\pi n,\cdot)\|_{H^{1/2}(0,T)}.$$

The first two of the estimates are guaranteed by $v\in L^{p+1}(D)$ and by the fact that $v\in H^2(0,T;L^2(\mathbb{R}))\cap L^2(0,T;H^2(\mathbb{R}))$ follows from the assumption on u. In the third estimate $\sum_{n\in\mathbb{Z}}\|v_{tt}(2\pi n,\cdot)\|^2_{H^{1/2}(0,T)}$ is finite because the additional assumption $v\in H^r(0,T;L^2(\mathbb{R}))\cap H^s(0,T;H^1(\mathbb{R}))$ with $r\ge\frac52$, $s\ge 5-r$ allows to apply Lemma 7.2. This finishes the proof. □

Proof of Corollary 1.4: Let u be the solution from Theorem 1.3. From Corollary 6.7 and from the assumption $p\in(1,\frac43)$ we find that $\|u\|_{L^{2p}(D)}<\infty$. Testing $J'(\tilde{u})=0$ with $(\tilde{u}_{j,k}(s)\delta_{k,k_0})_{j\in\mathbb{N}_0,k\in\mathbb{Z}_{\text{odd}},s\in\mathcal{B}}$ and with $(\tilde{\lambda}_{j,k}(s)\tilde{u}_{j,k}(s)\delta_{k,k_0})_{j\in\mathbb{N}_0,k\in\mathbb{Z}_{\text{odd}},\in\mathcal{B}}$ we get for every $k_0\in\mathbb{Z}_{\text{odd}}$ the two inequalities, respectively,

$$\left|\int_{\mathbb{R}} L_{k_0}u_{k_0}\bar{u}_{k_0}\,dx\right| \le \|(|u|^{p-1}u)_{k_0}\|^2_{L^2(\mathbb{R})}+\|u_{k_0}\|^2_{L^2(\mathbb{R})}, \tag{5.13}$$

$$\sum_{j\in\mathbb{N}_0}\int_{\mathcal{B}}|\lambda_{j,k_0}(s)|^2|\tilde{u}_{j,k_0}(s)|^2\,ds = \|L_{k_0}u_{k_0}\|^2_{L^2(\mathbb{R})} \le \|(|u|^{p-1}u)_{k_0}\|_{L^2(\mathbb{R})}\|L_{k_0}u_{k_0}\|_{L^2(\mathbb{R})}. \tag{5.14}$$

Lemma 2.4 gives $|\lambda_{j,k}(s)|\ge c|k|$ uniformly in $(j,s)\in\mathbb{N}_0\times\mathcal{B}$. Utilizing (5.14) we see $\|L_{k_0}u_{k_0}\|^2_{L^2(\mathbb{R})}\le\|(|u|^{p-1}u)_{k_0}\|^2_{L^2(\mathbb{R})}$. Summing over $k_0\in\mathbb{Z}_{\text{odd}}$ and using the identity $\sum_{k\in\mathbb{Z}_{\text{odd}}}\|(|u|^{p-1}u)_k\|^2_{L^2(\mathbb{R})}=\|u\|^{2p}_{L^{2p}(D)}<\infty$ we get that

$$\sum_{k\in\mathbb{Z}_{\text{odd}}}k^2\|u_k\|^2_{L^2(\mathbb{R})}<\infty. \tag{5.15}$$

Recall from Lemma 7.1 that for every $\delta>0$ we have the inequality $\sum_{n\in\mathbb{Z}}|u_{k_0}(2\pi n)|^2\le C_\delta k_0^2\|u_{k_0}\|^2_{L^2(\mathbb{R})}+\frac{\delta}{k_0^2}\|u'_{k_0}\|^2_{L^2(\mathbb{R})}$ for all $k_0\in\mathbb{Z}_{\text{odd}}$. Together with (5.13) one finds

$$\|u'_{k_0}\|^2_{L^2(\mathbb{R})}\le\|(|u|^{p-1}u)_{k_0}\|^2_{L^2(\mathbb{R})}+\|u_{k_0}\|^2_{L^2(\mathbb{R})}(1+\alpha k_0^2+C_{1/(2\beta)}k_0^4)+\frac12\|u'_{k_0}\|^2_{L^2(\mathbb{R})}.$$

Summing over $k_0\in\mathbb{Z}_{\text{odd}}$ and using (5.15) yields

$$\sum_{k\in\mathbb{Z}_{\text{odd}}}\frac{1}{k^2}\|u_k\|^2_{LH^1(\mathbb{R})}<\infty. \tag{5.16}$$

Applying the same techniques as in the proof of Theorem 6.5 and using (5.15), (5.16) we can deduce $u\in H^{1/2}(D)$. Moreover, (5.15), (5.16) show that $u\in H^1(0,T;L^2(\mathbb{R}))\cap H^{-1}(0,T;H^1(\mathbb{R}))$. Hence Lemma 7.2 applies and shows $\sum_{n\in\mathbb{Z}}\|u(2\pi n,\cdot)\|^2_{L^2(0,T)}<\infty$. Thus, we can estimate

$$\left|\sum_{n\in\mathbb{Z}}\langle u(2\pi n,\cdot),v_{tt}(2\pi n,\cdot)\rangle_{H^{-\frac12}\times H^{\frac12}}\right|\le\sum_{n\in\mathbb{Z}}\|u(2\pi n,\cdot)\|_{L^2(0,T)}\|v_{tt}(2\pi n,\cdot)\|_{L^2(0,T)}.$$

Finally Lemma 7.2 allows to estimate $\|v_{tt}(2\pi n,\cdot)\|^2_{L^2(0,T)}$ by $\|v\|^2_{H^r(0,T;L^2)}+\|v\|^2_{H^s(0,T;H^1)}$ provided $r+s\ge4$ and $r\ge2$. This verifies the claim on weakening the admissible test functions. □

6. Proof of boundedness of $\mathcal{S}$

We split the proof of Theorem 5.1 in several steps. First, we recall two auxiliary lemmata. The first statement is done within the proof of Proposition 4.1 in [8]. The second can be achieved by standard methods which is omitted here.

Lemma 6.1. *Let $v = (v_1, v_2)^T \in \mathbb{R}^2$. Then there is a constant $c_1 > 0$ s.t.* $\int_{\mathbb{R}^2} \frac{1-\cos(v\cdot x)}{|x|^{5/2}} dx = c_1 \sqrt[4]{v_1^2 + v_2^2}$.

Lemma 6.2. *There is a constant $c > 0$ such that* $\int_0^\infty \int_0^R \frac{x^2}{(x^2+y^2)^{\frac{5}{4}}} dx\, dy = cR^{\frac{3}{2}}$ *for all $R > 0$.*

In order to prove Theorem 5.1 we make use of several intermediate spaces. We denote by $\mathcal{F}$ the Fourier transform with respect to the space-variable $x \in \mathbb{R}$. Let

$$\hat{H} := \Big\{(u_k)_{k\in\mathbb{Z}_{\text{odd}}} : u_k \in H^1(\mathbb{R}) \text{ for all } k \in \mathbb{Z}_{\text{odd}} \text{ s.t. } \|(u_k)_{k\in\mathbb{Z}_{\text{odd}}}\|_{\hat{H}} < \infty\Big\},$$
$$\|(u_k)_{k\in\mathbb{Z}_{\text{odd}}}\|_{\hat{H}}^2 := \sum_{k\in\mathbb{Z}_{\text{odd}}} \left(|k|\, \|u_k\|_{L^2(\mathbb{R})}^2 + \frac{1}{|k|^3}\|u_k'\|_{L^2(\mathbb{R})}^2\right).$$

Moreover, for $r > 0$ and $D = \mathbb{R} \times [0, T)$ let

$$\tilde{H}^r(D) := \Big\{u\colon D \to \mathbb{R}; u(x,t) = \sum_{k\in\mathbb{Z}_{\text{odd}}} u_k(x)e^{ik\omega t} \text{ s.t. } \overline{u_k(x)} = u_{-k}(x)\ \forall k \in \mathbb{Z}_{\text{odd}} \text{ and } \|u\|_{\tilde{H}^r(D)} < \infty\Big\},$$
$$\|u\|_{\tilde{H}^r(D)}^2 := \sum_{k\in\mathbb{Z}_{\text{odd}}} \int_{\mathbb{R}} (1 + \xi^2 + k^2)^r |\mathcal{F} u_k(\xi)|^2 d\xi.$$

Notice that $u \in \tilde{H}^r(D)$ is T-periodic in the second component. Additionally, for $r \in (0, 1)$ and an open subset $\Omega \subseteq \mathbb{R}^2$ recall the definition of the fractional Sobolev space (see [8])

$$H^r(\Omega) := \Big\{u \in L^2(\Omega) : \text{ s.t. } \|u\|_{H^r(\Omega)} < \infty\Big\},$$
$$\|u\|_{H^r(\Omega)}^2 := \int_\Omega |u(x,t)|^2 d(x,t) + \int_\Omega\int_\Omega \frac{|u(x,t) - u(y,s)|^2}{|(x,t) - (y,s)|^{2(1+r)}} d(x,t)d(y,s).$$

Finally, we also introduce a periodic fractional Sobolev space. With $D_n := \mathbb{R} \times (-nT, nT)$ for $n \in \mathbb{N}$ we define

$$H^r_{\text{per}}(\mathbb{R}^2) := \{u\colon \mathbb{R}^2 \to \mathbb{R} : u \in H^r(D_n)\ \forall n \in \mathbb{N} \text{ and } u \text{ is } T\text{-periodic in the second component}\}$$
$$\|u\|_{H^r_{\text{per}}(\mathbb{R}^2)} := \|u\|_{H^r(D_1)}.$$

Next we state two more lemmata of auxiliary character.

Lemma 6.3. *Let $n \in \mathbb{N}$ and $r \in (0, 1)$. Then there is a constant $c = c(n, r) > 0$ such that*

$$\|u\|_{H^r(D_n)} \le c(n, r)\|u\|_{H^r(D_1)} \tag{6.1}$$

for all $u \in H^r_{\text{per}}(\mathbb{R}^2)$.

Proof. We only show (6.1) for $n = 2$. The case $n > 2$ can be established by the same techniques.

Since $\|u\|_{L^2(D_2)}^2 = 2\|u\|_{L^2(D_1)}^2$ it remains to bound the expression

$$\int_{D_2}\int_{D_2} \frac{|u(x,t) - u(y,s)|^2}{|(x,t) - (y,s)|^{2(1+r)}} d(x,t)d(y,s)$$

by a constant multiple of $\|u\|_{H^r(D_1)}^2$. The idea is to split the domain of integration $D_2 \times D_2$ in several parts. Due to symmetry of the integrand in the variables t and s it is enough to consider the three cases

1) $t, s \in (-T, T)$ 2) $t \in [T, 2T), s \in (0, 2T)$ 3) $t \in [T, 2T), s \in (-2T, 0)$

which are treated one after another.

1) $t, s \in (-T, T)$: We directly obtain $\int_{\mathbb{R}\times(-T,T)}\int_{\mathbb{R}\times(-T,T)} \frac{|u(x,t)-u(y,s)|^2}{|(x,t)-(y,s)|^{2(1+r)}} d(x,t)d(y,s) \leq \|u\|^2_{H^r(D_1)}$.

2) $t \in [T, 2T)$, $s \in (0, 2T)$: With the substitution $(\tilde{t}, \tilde{s}) = (t - T, s - T)$ we obtain

$$\int_{\mathbb{R}\times[T,2T)}\int_{\mathbb{R}\times[0,2T)} \frac{|u(x,t) - u(y,s)|^2}{|(x,t) - (y,s)|^{2(1+r)}} d(y,s)d(x,t) \leq \|u\|^2_{H^r(D_1)}.$$

3) $t \in [T, 2T)$, $s \in (-2T, 0)$: Using the substitution $t = \tilde{t} + 3T$ and the observation $\tilde{t} - s + 3T \geq T$ for $\tilde{t}, s \in [-2T, 0)$ together with the T-periodicity of u in its second variable we may estimate

$$\begin{aligned}\int_{\mathbb{R}\times[T,2T)}\int_{\mathbb{R}\times(-2T,0)} &\frac{|u(x,t) - u(y,s)|^2}{|(x,t) - (y,s)|^{2(1+r)}} d(y,s)d(x,t)\\ &\leq \int_{\mathbb{R}\times[-2T,0)}\int_{\mathbb{R}\times(-2T,0)} \frac{2u(x,\tilde{t})^2 + 2u(y,s)^2}{|(x,\tilde{t}+3T) - (y,s)|^{2(1+r)}} d(y,s)d(x,\tilde{t})\\ &= 4\|u\|^2_{L^2(D_2)} \int_{\mathbb{R}\times[T,\infty)} \frac{1}{|(z,\delta)|^{2(1+r)}} d(z,\delta) = \|u\|^2_{L^2(D_1)} C(r)\end{aligned}$$

and the proof is done. □

Lemma 6.4. *For $(z, \delta) \in \mathbb{R}^2$ and $u \in L^2(D)$ we have*

$$\int_D |u(x,s) - u(x+z, s+\delta)|^2 d(x,s) = 2T \sum_{k\in\mathbb{Z}_{\text{odd}}} \int_{\mathbb{R}} (1 - \cos(k\omega\delta + \xi z))|(\mathcal{F}u_k)(\xi)|^2 d\xi.$$

Proof. By Plancherel's Theorem we have

$$\int_D |u(x,s) - u(x+z, s+\delta)|^2 d(x,s) = T \sum_{k\in\mathbb{Z}_{\text{odd}}} \|u_k(\cdot) - u_k(\cdot + z)e^{ik\omega\delta}\|^2_{L^2(\mathbb{R})}$$

$$= T \sum_{k\in\mathbb{Z}_{\text{odd}}} \|\mathcal{F}\left(u_k(\cdot) - u_k(\cdot+z)e^{ik\omega\delta}\right)\|^2_{L^2(\mathbb{R})} = T \sum_{k\in\mathbb{Z}_{\text{odd}}} \int_{\mathbb{R}} \underbrace{|1 - e^{ik\omega\delta+\xi z}|^2}_{=2(1-\cos(k\omega\delta+\xi z))} |(\mathcal{F}u_k)(\xi)|^2 d\xi.$$

□

Now we have all ingredients to deduce several embeddings between the spaces introduced previously. The first result demonstrates a connection between $\mathcal{H}, \tilde{H}^{1/4}(D), H^{1/4}(D)$ and $H^{1/4}_{\text{per}}(\mathbb{R}^2)$.

Theorem 6.5. *The following linear operators are bounded:*

$$\begin{aligned} &\mathcal{S}_1\colon \mathcal{H} \to \hat{H}, && (\mathcal{S}_1\tilde{u})_k(x) := \sum_{j\in\mathbb{N}_0} \int_{\mathcal{B}} \tilde{u}_{j,k}(s)\psi_{j,k}(x,s)ds \text{ for } k \in \mathbb{Z}_{\text{odd}},\\ &\mathcal{S}_2\colon \hat{H} \to \tilde{H}^{1/4}(D), && (\mathcal{S}_2(u_k)_{k\in\mathbb{Z}_{\text{odd}}})(x,t) := \sum_{k\in\mathbb{Z}_{\text{odd}}} u_k(x)e^{ik\omega t},\\ &\mathcal{S}_3\colon \tilde{H}^{1/4}(D) \to H^{1/4}_{\text{per}}(\mathbb{R}^2), && \mathcal{S}_3u(x,t) := u(x,s), \text{ where } s = t \pmod{T}.\end{aligned}$$

Proof. We investigate the four operators separately.

1) Boundedness of $\mathcal{S}_1$: Because of (3.6) in Lemma 3.1 we observe that $b_k(v^+, v^+) - b_k(v^-, v^-) = \sum_{j\in\mathbb{N}_0}\int_{\mathcal{B}} |\lambda_{j,k}(s)||\tilde{v}_{j,k}(s)|^2 ds$. Together with Corollary 4.3 and Theorem 4.4 we know that there is $C > 0$ such that

$$|k|\,\|v\|^2_{L^2(\mathbb{R})} + \frac{1}{|k|^3}\|v'\|^2_{L^2(\mathbb{R})} \leq C \sum_{j\in\mathbb{N}_0} \int_{\mathcal{B}} |\lambda_{j,k}(s)||\tilde{v}_{j,k}(s)|^2 ds \tag{6.2}$$

for all $v \in H^1(\mathbb{R})$. Setting $v = u_k$ in (6.2) and summing over $k \in \mathbb{Z}_{\text{odd}}$ gives

$$\| ((\mathcal{S}_1\tilde{u})_k)_{k\in\mathbb{Z}_{\text{odd}}} \|_{\hat{H}}^2 = \sum_{k\in\mathbb{Z}_{\text{odd}}} (|k|\, \|u_k\|_{L^2(\mathbb{R})}^2 + \frac{1}{|k|^3}\|u_k'\|_{L^2(\mathbb{R})}^2) \le C \sum_{j\in\mathbb{N}_0, k\in\mathbb{Z}_{\text{odd}}} \int_{\mathcal{B}} |\lambda_{j,k}(s)||\tilde{u}_{j,k}(s)|^2 ds = C\|\tilde{u}\|_{\mathcal{H}}^2,$$

which proves the boundedness of $\mathcal{S}_1$.

2) Boundedness of $\mathcal{S}_2$: By Plancherel's identity we obtain $\|u_k'\|_{L^2(\mathbb{R})}^2 = \int_{\mathbb{R}} \xi^2 |(\mathcal{F}u_k)(\xi)|^2 d\xi$. Recall Young's inequality $ab \le \frac{a^4}{4} + \frac{3b^{4/3}}{4}$ for $a, b \ge 0$. Thus, we infer

$$\begin{aligned} \|\mathcal{S}_2(u_k)_{k\in\mathbb{Z}_{\text{odd}}}\|_{\tilde{H}^{1/4}(D)} &= \sum_{k\in\mathbb{Z}_{\text{odd}}} \int_{\mathbb{R}} \left(\frac{1+\xi^2+k^2}{|k|^3}\right)^{1/4} |k|^{\frac{3}{4}} |\mathcal{F}u_k(\xi)|^2 d\xi \\ &\le \sum_{k\in\mathbb{Z}_{\text{odd}}} \int_{\mathbb{R}} \left(\frac{1}{4}\frac{1+\xi^2+k^2}{|k|^3} + \frac{3}{4}|k|\right) |\mathcal{F}u_k(\xi)|^2 d\xi \\ &\le \sum_{k\in\mathbb{Z}_{\text{odd}}} \int_{\mathbb{R}} \left(\frac{1}{4}\frac{\xi^2}{|k|^3} + \frac{5}{4}|k|\right) |\mathcal{F}u_k(\xi)|^2 d\xi \le \frac{5}{4}\| (u_k)_{k\in\mathbb{Z}_{\text{odd}}} \|_{\hat{H}}^2 \end{aligned}$$

which shows the boundedness of $\mathcal{S}_2$.

3) Boundedness of $\mathcal{S}_3$: Fix $n \in \mathbb{N}$. Then due to periodicity

$$\|u\|_{L^2(D_n)}^2 = 2n \int_D |u(x,t)|^2 d(x,t) = 2nT \sum_{k\in\mathbb{Z}_{\text{odd}}} \int_{\mathbb{R}} |u_k(x)|^2 dx \le 2nT\|u\|_{\tilde{H}^{1/4}(D)}^2. \tag{6.3}$$

Moreover, with the help of the substitution $(z, \delta) := (y - x, s - t)$, Fubini and the periodicity of u in the second component we obtain

$$\begin{aligned} \int_{D_n}\int_{D_n} \frac{|u(x,t)-u(y,s)|^2}{|(x,t)-(y,s)|^{5/2}} d(x,t)d(y,s) &= \int_{D_n}\int_{\mathbb{R}\times(-nT-t,nT-t)} \frac{|u(x,t)-u(x+z,t+\delta)|^2}{|(z,\delta)|^{5/2}} d(z,\delta)d(x,t) \\ &\le \int_{\mathbb{R}^2} \frac{1}{|(z,\delta)|^{5/2}} \int_{D_n} |u(x,t)-u(x+z,t+\delta)|^2 d(x,t)d(z,\delta) \\ &= 2n \int_{\mathbb{R}^2} \frac{1}{|(z,\delta)|^{5/2}} \int_D |u(x,t)-u(x+z,t+\delta)|^2 d(x,t)d(z,\delta). \end{aligned}$$

Due to Lemma 6.4 and Lemma 6.1 we can continue the previous estimate by

$$\begin{aligned} \int_{D_n}\int_{D_n} \frac{|u(x,t)-u(y,s)|^2}{|(x,t)-(y,s)|^{5/2}} d(x,t)d(y,s) &\le 4nTc_1 \sum_{k\in\mathbb{Z}_{\text{odd}}} \int_{\mathbb{R}} \sqrt[4]{\omega^2k^2+\xi^2} |(\mathcal{F}u_k)(\xi)|^2 d\xi \\ &\le \tilde{c}(n)T\|u\|_{\tilde{H}^{1/4}(D)}^2 \end{aligned}$$

for a constant $\tilde{c}(n) > 0$. Together with (6.3) this implies the boundedness of $\mathcal{S}_3$. □

The next lemma contains a crucial step in our regularity considerations.

Lemma 6.6. *Let $\varphi\colon \mathbb{R} \to \mathbb{R}$ be defined as follows*

$$\varphi(t) := \begin{cases} 1 & , \text{ if } t \in [-T, T], \\ 2 - \frac{1}{T}t & , \text{ if } t \in (T, 2T), \\ 2 + \frac{1}{T}t & , \text{ if } t \in (-2T, -T), \\ 0 & , \text{ if } t \in (-\infty, -2T] \cup [2T, \infty). \end{cases}$$

Then the multiplication operator $\mathcal{S}_4\colon H^{1/4}_{\text{per}}(\mathbb{R}^2) \to H^{1/4}(\mathbb{R}^2), u \mapsto \varphi u$ is bounded.

Proof. Let $u \in H^{1/4}_{\text{per}}(\mathbb{R}^2)$. Notice that φ is Lipschitz-continuous with Lipschitz constant $\frac{1}{T}$. By definition of φ and the periodicity of u in the second component we have $\|\varphi u\|^2_{L^2(\mathbb{R}^2)} \le 2\|u\|^2_{H^{1/4}_{\text{per}}(\mathbb{R}^2)}$. It remains to bound the expression

$$I := \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{|\varphi(t)u(x,t)-\varphi(s)u(y,s)|^2}{|(x,t)-(y,s)|^{5/2}} d(x,t)d(y,s)$$

by constant multiples of $\|\cdot\|_{L^2(D_1)}$ and $\|\cdot\|_{H^{1/4}(D_1)}$. Therefore, we split the domain of integration into nine subdomains, namely,

$$\Omega_1 := \{(x,t,y,s) \in \mathbb{R}^4 : t,s \in (-2T,2T)\}, \quad \Omega_2 := \{(x,t,y,s) \in \mathbb{R}^4 : t,s \in [2T,\infty)\},$$
$$\Omega_3 := \{(x,t,y,s) \in \mathbb{R}^4 : t,s \in (-\infty,-2T]\}, \quad \Omega_4 := \{(x,t,y,s) \in \mathbb{R}^4 : t \in (-2T,2T), s \in [2T,\infty)\},$$
$$\Omega_5 := \{(x,t,y,s) \in \mathbb{R}^4 : s \in (-2T,2T), t \in [2T,\infty)\},$$
$$\Omega_6 := \{(x,t,y,s) \in \mathbb{R}^4 : t \in (-2T,2T), s \in (-\infty,-2T]\},$$
$$\Omega_7 := \{(x,t,y,s) \in \mathbb{R}^4 : s \in (-2T,2T), t \in (-\infty,-2T]\},$$
$$\Omega_8 := \{(x,t,y,s) \in \mathbb{R}^4 : t \in (-\infty,-2T], s \in [2T,\infty)\},$$
$$\Omega_9 := \{(x,t,y,s) \in \mathbb{R}^4 : s \in (-\infty,-2T], t \in [2T,\infty)\}.$$

Let us write $I = \sum_{r=1}^{9} I_r$ with

$$I_r := \int_{\Omega_r} \frac{|\varphi(t)u(x,t)-\varphi(s)u(y,s)|^2}{|(x,t)-(y,s)|^{5/2}} d(x,t,y,s) \quad \text{for} \quad r \in \{1,2,\dots,9\}$$

Since $\varphi \equiv 0$ on $(-\infty,-2T] \cup [2T,\infty)$ we have $I_2 = I_3 = I_8 = I_9 = 0$. Due to symmetry in the variables (x,t) and (y,s) we infer that $I_4 = I_5$ and $I_6 = I_7$. Therefore, it is sufficient to estimate I_1, I_4 and I_6 which will be done in the following.

Estimation of I_1: We have

$$\begin{aligned} I_1 &= \int_{\mathbb{R}\times(-2T,2T)}\int_{\mathbb{R}\times(-2T,2T)} \frac{|\varphi(t)u(x,t)-\varphi(s)u(y,s)|^2}{|(x,t)-(y,s)|^{5/2}} d(x,t)d(y,s) \\ &\le 2\int_{\mathbb{R}\times(-2T,2T)}\int_{\mathbb{R}\times(-2T,2T)} \left(\frac{|\varphi(t)(u(x,t)-u(y,s))|^2}{|(x,t)-(y,s)|^{5/2}} + \frac{|(\varphi(t)-\varphi(s))u(y,s)|^2}{|(x,t)-(y,s)|^{5/2}}\right) d(x,t)d(y,s) \end{aligned}$$

and both summands will be treated separately. Due to $0 \le \varphi \le 1$ and Lemma 6.3 for $n = 2$ we infer

$$\int_{\mathbb{R}\times(-2T,2T)}\int_{\mathbb{R}\times(-2T,2T)} \frac{|\varphi(t)(u(x,t)-u(y,s))|^2}{|(x,t)-(y,s)|^{5/2}} d(x,t)d(y,s) \le \|u\|^2_{H^{1/4}(D_2)} \le c(2,\frac{1}{4})\|u\|^2_{H^{1/4}(D_1)}$$

with the constant $c(2,\frac{1}{4})$ from Lemma 6.3. For the second summand we use the Lipschitz-continuity of φ and the substitution $(z,\delta) = (x-y, t-s)$ in order to estimate

$$\begin{aligned} &\int_{\mathbb{R}\times(-2T,2T)}\int_{\mathbb{R}\times(-2T,2T)} \frac{|(\varphi(t)-\varphi(s))u(y,s)|^2}{|(x,t)-(y,s)|^{5/2}} d(x,t)d(y,s) \\ &\le \frac{1}{T^2}\int_{\mathbb{R}\times(-2T,2T)}\int_{\mathbb{R}\times(-2T,2T)} \frac{|t-s|^2|u(y,s)|^2}{|(x,t)-(y,s)|^{5/2}} d(x,t)d(y,s) \\ &= \frac{1}{T^2}\int_{\mathbb{R}\times(-2T,2T)} |u(y,s)|^2 \int_{\mathbb{R}\times(-2T-s,2T-s)} \frac{\delta^2}{|(z,\delta)|^{5/2}} d(z,\delta)d(y,s) \\ &\le \frac{1}{T^2}\int_{\mathbb{R}\times(-2T,2T)} |u(y,s)|^2 d(y,s) \int_{\mathbb{R}\times(-4T,4T)} \frac{\delta^2}{|(z,\delta)|^{5/2}} d(z,\delta) \end{aligned}$$

$$\leq \quad 2\|u\|^2_{L^2(D_1)} C T^{-1/2}$$

due to the periodicity of u in the second component and Lemma 6.2.

Estimation of I_4, I_6: Since the technique is the same for I_4 as for I_6 we only do I_4. First of all, notice that for $T > 0$ and $t < 2T$ by polar coordinates

$$\int_{\mathbb{R}\times[2T-t,\infty)} \frac{1}{|(z,\delta)|^{5/2}} d(z,\delta) \leq \int_{\mathbb{R}^2\setminus B_{2T-t}(0)} \frac{1}{|(z,\delta)|^{5/2}} d(z,\delta) = \frac{4\pi}{\sqrt{2T-t}}.$$

Thus, the substitution $(z,\delta) := (y-x, s-t)$ and the Lipschitz-continuity of φ imply

$$\begin{aligned} I_4 &= \int_{\mathbb{R}\times(-2T,2T)} \int_{\mathbb{R}\times[2T,\infty)} \frac{|\varphi(t)u(x,t)|^2}{|(x,t)-(y,s)|^{5/2}} d(y,s)d(x,t) \\ &\leq \int_{\mathbb{R}\times(-2T,2T)} |\varphi(2T)-\varphi(t)|^2 |u(x,t)|^2 \frac{4\pi}{\sqrt{2T-t}} d(x,t) \\ &\leq \frac{4\pi}{T^2} \int_{\mathbb{R}\times(-2T,2T)} (2T-t)^{\frac{3}{2}} |u(x,t)|^2 d(x,t) \leq \frac{64\pi}{\sqrt{T}} \|u\|^2_{L^2(D_1)}. \end{aligned}$$

This finishes the proof. □

We now provide the last embedding that is necessary for the proof of Theorem 5.1.

Corollary 6.7. *For any $u \in H^{1/4}_{\text{per}}(\mathbb{R}^2)$ and any $q \in [2, \frac{8}{3}]$ we have $\|u\|_{L^q(D)} \leq c(q)\|u\|_{H^{1/4}_{\text{per}}(\mathbb{R}^2)}$ with constant $c(q) > 0$ not depending on u.*

Proof. Since the cut-off function φ from Lemma 6.6 satisfies $0 \leq \varphi \leq 1$ and $\varphi \equiv 1$ on D we have $\|u\|_{L^q(D)} \leq \|\varphi u\|_{L^q(\mathbb{R}^2)}$. Since $H^{1/4}(\mathbb{R}^2) \hookrightarrow L^q(\mathbb{R}^2)$ for all $q \in [2, \frac{8}{3}]$, see Theorem 6.5 in [8], we have that $\|u\|_{L^q(D)} \leq \tilde{c}(q)\|\varphi u\|_{H^{1/4}(\mathbb{R}^2)}$. The claim of the corollary then follows from the boundedness of the operator $\mathcal{S}_4 \colon H^{1/4}_{\text{per}}(\mathbb{R}^2) \to H^{1/4}(\mathbb{R}^2), u \mapsto \varphi u$ as shown in Lemma 6.6. □

After these preparations the proof of Theorem 5.1 becomes quite simple.

Proof of Theorem 5.1: By Theorem 6.5 we have $\mathcal{S}\tilde{u} = (\mathcal{S}_3 \circ \mathcal{S}_2 \circ \mathcal{S}_1)\tilde{u}$ in $H^{1/4}_{\text{per}}(\mathbb{R}^2)$. Corollary 6.7 and the boundedness of $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ yield the desired result. □

7. Appendix

Lemma 7.1. *Let $f \in H^1(\mathbb{R})$. Then for $\varepsilon > 0$ we have*

$$\sum_{n\in\mathbb{Z}} |f(2\pi n)|^2 \leq \left(\frac{1}{2\pi} + \frac{1}{2\varepsilon}\right) \|f\|^2_{L^2(\mathbb{R})} + \frac{\varepsilon}{2}\|f'\|^2_{L^2(\mathbb{R})}. \tag{7.1}$$

Proof. Let $u_n(x) := f(2\pi n + x)$. We compute

$$|u_n(0)|^2 = \frac{1}{\pi}\int_{-\pi}^{0} \frac{d}{dt}\left[(t+\varepsilon)|u_n(t)|^2\right] dt \leq \frac{1}{\pi}\int_{-\pi}^{0} |u_n(t)|^2 dt + 2\int_{-\pi}^{0} |u_n(t)u_n'(t)| dt. \tag{7.2}$$

In the same manner

$$|u_n(0)|^2 = -\frac{1}{\pi}\int_{0}^{\pi} \frac{d}{dt}\left[(\varepsilon - t)|u_n(t)|^2\right] dt \leq \frac{1}{\pi}\int_{0}^{\pi} |u_n(t)|^2 dt + 2\int_{0}^{\varepsilon} |u_n(t)u_n'(t)| dt. \tag{7.3}$$

By adding (7.2) and (7.3) we conclude

$$|u_n(0)|^2 \leq \frac{1}{2\pi}\|u_n\|^2_{L^2(-\pi,\pi)} + \|u_n\|_{L^2(-\pi,\pi)}\|u_n'\|_{L^2(-\pi,\pi)} \leq \frac{1}{2}\left(\frac{1}{\epsilon} + \frac{1}{\pi}\right)\|u_n\|^2_{L^2(-\pi,\pi)} + \frac{\epsilon}{2}\|u_n'\|^2_{L^2(-\pi,\pi)}$$

and hence

$$|f(2\pi n)|^2 \le \frac{1}{2}\left(\frac{1}{\epsilon}+\frac{1}{\pi}\right)\|f(2\pi n+\cdot)\|^2_{L^2(-\pi,\pi)} + \frac{\epsilon}{2}\|f'(2\pi n+\cdot)\|^2_{L^2(-\pi,\pi)}.$$

The claim follows by a summation over $n \in \mathbb{Z}$. □

Lemma 7.2. *Let $a,b,c \in \mathbb{R}$ with $b \ge a$ and $b+c \ge 2a$. Then there exists a constant $C = C(a,b,c)$ such that the following estimate holds for functions $w(x,t) = \sum_{k\in\mathbb{Z}} w_k(x)e^{ik\omega t}$, $w_k \in H^1(\mathbb{R})$, $w_0 = 0$ in case $a \ne b$ and $w \in H^b(0,T;L^2(\mathbb{R})) \cap H^c(0,T;H^1(\mathbb{R}))$:*

$$\sum_{n\in\mathbb{Z}} \|w(2\pi n,\cdot)\|^2_{H^a(0,T)} \le C\left(\|w\|^2_{H^b(0,T;L^2)} + \|w\|^2_{H^c(0,T;H^1)}\right).$$

Proof. Use (7.1) from Lemma 7.1 with $\epsilon = |k|^{2(a-b)}$ to get

$$\begin{aligned}\sum_{n\in\mathbb{Z}} \|w(2\pi n,\cdot)\|^2_{H^a(0,T)} = \sum_{n\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} |k|^{2a}|w_k(2\pi n)|^2 &\le \tilde{C}\sum_{k\in\mathbb{Z}}\left(k^{2b}\|w_k\|^2_{L^2(\mathbb{R})} + |k|^{2(2a-b)}\|w_k'\|^2_{L^2(\mathbb{R})}\right)\\ &\le C\left(\|w\|^2_{H^b(0,T;L^2)} + \|w\|^2_{H^c(0,T;H^1)}\right).\end{aligned}$$

□

Proof of Lemma 5.2. For the purpose of this proof let us define the space

$$\mathcal{H}_0 := \Big\{\tilde{\phi} = (\tilde{\phi}_{j,k})_{j\in\mathbb{N}_0,k\in\mathbb{Z}_{\text{odd}}} : \tilde{\phi}_{j,k}\colon \mathcal{B}\to\mathbb{C} \text{ measurable}, \sum_{j\in\mathbb{N}_0,k\in\mathbb{Z}_{\text{odd}}}\int_{\mathcal{B}} |\lambda_{j,k}(s)||\tilde{\phi}_{j,k}(s)|^2 ds < \infty\Big\}.$$

It can be seen as a variant of $\mathcal{H}$ but without the additional requirement of conjugation-symmetry $\overline{\tilde{\phi}_{j,k}(s)} = \tilde{\phi}_{j,-k}(-s)$. Clearly, $\mathcal{H}_{k,\text{mono}} \not\subset \mathcal{H}$ but $\mathcal{H}_{k,\text{mono}} \subset \mathcal{H}_0$.

First we check that $J'(\tilde{u}) = 0$ implies (and hence is equivalent to) $J'(\tilde{u})[\tilde{\phi}] = 0$ for all $\tilde{\phi} \in \mathcal{H}_0$, i.e., that we can allow test functions $\tilde{\phi}$ without the extra conjugation-symmetry. For $\tilde{\phi} \in \mathcal{H}_0$ let us define the splitting

$$\tilde{\phi}_{j,k}(s) = \tilde{\phi}^a_{j,k}(s) + \tilde{\phi}^b_{j,k}(s) \quad \text{with} \quad \tilde{\phi}^a_{j,k}(s) := \frac{\tilde{\phi}_{j,k}(s) + \overline{\tilde{\phi}_{j,-k}(-s)}}{2}, \quad \tilde{\phi}^b_{j,k}(s) := \frac{\tilde{\phi}_{j,k}(s) - \overline{\tilde{\phi}_{j,-k}(-s)}}{2}.$$

Then $\tilde{\phi}^a, i\tilde{\phi}^b \in \mathcal{H}$ and hence $J'(\tilde{u})[\tilde{\phi}^a] = 0$ and $0 = J'(\tilde{u})[i\tilde{\phi}^b] = (-i)J'(\tilde{u})[\tilde{\phi}^b]$. Therefore we also have $J'(\tilde{u})[\tilde{\phi}] = J'(\tilde{u})[\tilde{\phi}^a + \tilde{\phi}^b] = 0$ as claimed.

(i) ⇔ (ii): With the help of the first step we know that $J'(\tilde{u})|_{\mathcal{H}} = 0$ implies $J'(\tilde{u})|_{\mathcal{H}_{k,\text{mono}}} = 0$. Now we verify the reverse: $J'(\tilde{u})|_{\mathcal{H}_{k,\text{mono}}} = 0$ for all $k \in \mathbb{Z}_{\text{odd}}$ implies $J'(\tilde{u})|_{\mathcal{H}} = 0$. For this, note that any $\tilde{\phi} \in \mathcal{H}_0$ can be see as $\tilde{\phi} = \lim_{m\to\infty}\tilde{\phi}^m$ (convergence with respect to the $\|\cdot\|_{\mathcal{H}}$-norm) where for $m \in \mathbb{N}$, m odd, we set

$$\tilde{\phi}^m_{j,k}(s) := \begin{cases} \tilde{\phi}_{j,k}(s) & \text{if } k \in \mathbb{Z}_{\text{odd}}, |k| \le m,\\ 0 & \text{if } k \in \mathbb{Z}_{\text{odd}}, |k| > m.\end{cases}$$

Since $\tilde{\phi}^m$ is a finite sum of members of $\mathcal{H}_{k,\text{mono}}$ for $k = -m, -m+2, \ldots, -1, 1, \ldots, m-2, m$ we have $J'(\tilde{u})[\tilde{\phi}^m] = 0$. Then $J'(\tilde{u})[\tilde{\phi}] = 0$ follows since J' is a continuous linear functional on $\mathcal{H}$ and $\|\tilde{\phi}^m - \tilde{\phi}\|_{\mathcal{H}} \to 0$ as $m \to \infty$. The claim $J'(\tilde{u})|_{\mathcal{H}} = 0$ follows by the first step.

Finally, it remains to show that functions $\tilde{\phi} \in \mathcal{H}_{k,\text{mono}}$ such that $\mathcal{S}\tilde{\phi}$ has compact support in $\overline{D}$ are dense in $\mathcal{H}_{k,\text{mono}}$. For this consider the map

$$\Sigma : \mathcal{H}_{k,\text{mono}} \to H^1(\mathbb{R}), \quad \Sigma(\tilde{\phi}) := \sum_{j\in\mathbb{N}_0}\int_{\mathcal{B}} \tilde{\phi}_{j,k}(s)\psi_{j,k}(x,s)\,dx.$$

It is bounded and $\|\tilde{\phi}\|_{\mathcal{H}}$ and $\|\Sigma\tilde{\phi}\|_{H^1}$ are equivalent, cf. Lemma 3.1. Moreover, Σ is onto because for any $\phi \in H^1(\mathbb{R})$ we may set $\tilde{\phi}_{j,k}(s) := \langle(\mathcal{T}\phi)(\cdot, s), \psi_{j,k}(\cdot, s)\rangle_{\mathcal{P}}$ and get $\tilde{\phi} \in \mathcal{H}_{k,\mathrm{mono}}$ with $\Sigma\tilde{\phi} = \phi$. Therefore, Σ has a bounded inverse and the set $\Sigma^{-1}(C_c^\infty(\mathbb{R}))$ is dense in $\mathcal{H}_{k,\mathrm{mono}}$. Thus $\mathcal{S}(\Sigma^{-1}(C_c^\infty(\mathbb{R}))$ consists of functions having compact support in $\overline{D}$. □

Proof of Lemma 5.8.

Lemma 7.3. *Let $0 < r < T$. Then there is a sequence $(y_l)_{l\in\mathbb{N}}$ in D s.t. $D \subset \bigcup_{l\in\mathbb{N}} B_r(y_l)$ and each point $y \in D$ is contained in at most four balls $B_r(y_l)$.*

Proof. The statement follows if we choose $(y_l)_{l\in\mathbb{N}}$ to be an enumeration of $r\mathbb{Z}^2 \cap D$. □

Lemma 7.4. *With the notation of Lemma 7.3 there is a constant $C > 0$ such that*

$$\sum_{l\in\mathbb{N}} \|\mathcal{S}\tilde{u}\|^2_{L^{8/3}(B_r(y_l))} \le C\|\tilde{u}\|^2_{\mathcal{H}} \text{ for all } \tilde{u} \in \mathcal{H}.$$

Proof. Recall from Corollary 7.2 in [8] the embedding $H^{1/4}(B_r(y_l)) \to L^{8/3}(B_r(y_l))$. Due to Lemma 7.3 we can distinguish balls $B_r(y_l), l \in \mathbb{N}$ which are completely in D and others which protrude from D. However, since the functions $\mathcal{S}\tilde{u}$ are periodic in the second variable and hence their norms in $H^{1/4}(B_r(y_l))$ and $L^{8/3}(B_r(y_l))$ are invariant under translations in t-direction, the distinction between these balls it not needed for deducing the existence of a constant $\tilde{c} > 0$ such that

$$\sum_{l\in\mathbb{N}} \|\mathcal{S}\tilde{u}\|^2_{L^{8/3}(B_r(y_l))} \le \tilde{c} \sum_{l\in\mathbb{N}} \|\mathcal{S}\tilde{u}\|^2_{H^{1/4}(B_r(y_l))} \text{ for all } l \in \mathbb{N}. \tag{7.4}$$

We abbreviate $\tilde{D}_r := \bigcup_{l\in\mathbb{N}} B_r(y_l)$. Due to the overlapping property in Lemma 7.3 we calculate

$$\sum_{l\in\mathbb{N}} \|\mathcal{S}\tilde{u}\|^2_{H^{1/4}(B_r(y_l))} \le 4\int_{\tilde{D}_r} |(\mathcal{S}\tilde{u})(x,t)|^2 d(x,t) + 4\int_{\tilde{D}_r}\int_{\tilde{D}_r} \frac{|(\mathcal{S}\tilde{u})(x,t) - (\mathcal{S}\tilde{u})(y,s)|^2}{|(x,t)-(y,s)|^{5/2}} d(x,t)d(y,s). \tag{7.5}$$

Due to $0 < r < T$ and Lemma 6.3 we conclude

$$\begin{aligned}&\int_{\tilde{D}_r} |(\mathcal{S}\tilde{u})(x,t)|^2 d(x,t) + \int_{\tilde{D}_r}\int_{\tilde{D}_r} \frac{|(\mathcal{S}\tilde{u})(x,t) - (\mathcal{S}\tilde{u})(y,s)|^2}{|(x,t)-(y,s)|^{5/2}} d(x,t)d(y,s)\\ &\le \|\mathcal{S}\tilde{u}\|^2_{H^{1/4}(\mathbb{R}\times[-T,2T])} \le \hat{c}\|\mathcal{S}\tilde{u}\|^2_{H^{1/4}(D)}.\end{aligned} \tag{7.6}$$

Finally, Theorem 6.5 (recall $\mathcal{S}\tilde{u} = (\mathcal{S}_3 \circ \mathcal{S}_2 \circ \mathcal{S}_1)\tilde{u}$ in $H^{1/4}(D)$) and (7.4), (7.5) and (7.6) gives

$$\sum_{l\in\mathbb{N}} \|\mathcal{S}\tilde{u}\|^2_{L^{8/3}(B_r(y_l))} \le 4\tilde{c}\hat{c}\|\mathcal{S}\tilde{u}\|^2_{H^{1/4}(D)} \le C\|\tilde{u}\|^2_{\mathcal{H}}$$

which finishes the proof. □

Proof of Lemma 5.8: W.l.o.g. we may assume $r > 0$ so small that $r \in (0, T)$. Fix $\tilde{u} \in \mathcal{H}$ and $y \in D$. By Hölder interpolation for $s \in (q, \frac{8}{3})$ there is $\lambda = \frac{s-q}{\frac{8}{3}-q}\frac{8}{3s}$ such that

$$\|\mathcal{S}\tilde{u}\|_{L^s(B_r(y))} \le \|\mathcal{S}\tilde{u}\|^{1-\lambda}_{L^q(B_r(y))}\|\mathcal{S}\tilde{u}\|^{\lambda}_{L^{8/3}(B_r(y))}.$$

For $s = 2 + \frac{q}{4}$ we have $\lambda = \frac{2}{s}$ and in particular

$$\|\mathcal{S}\tilde{u}\|^s_{L^s(B_r(y))} \le \|\mathcal{S}\tilde{u}\|^{(1-\lambda)s}_{L^q(B_r(y))}\|\mathcal{S}\tilde{u}\|^2_{L^{8/3}(B_r(y))} \le \|\mathcal{S}\tilde{u}\|^2_{L^{8/3}(B_r(y))} \sup_{z\in D}\|\mathcal{S}\tilde{u}\|^{(1-\lambda)s}_{L^q(B_r(z))}. \tag{7.7}$$

We now choose the sequence $(y_l)_{l\in\mathbb{N}}$ from Lemma 7.3, then use (7.7) for $y = y_l$ and perform a summation over $l \in \mathbb{N}$. Due to Lemma 7.3 we obtain

$$\|\mathcal{S}\tilde{u}\|^s_{L^s(D)} \le \sum_{l\in\mathbb{N}} \|\mathcal{S}\tilde{u}\|^s_{L^s(B_r(y_l))} \le \sum_{l\in\mathbb{N}} \|\mathcal{S}\tilde{u}\|^2_{L^{8/3}(B_r(y_l))} \sup_{z\in D}\|\mathcal{S}\tilde{u}\|^{(1-\lambda)s}_{L^q(B_r(z))}.$$

Lemma 7.4 guarantees the existence of $C>0$ s.t. $\sum_{l\in\mathbb{N}}\|\mathcal{S}\tilde{u}\|^2_{L^{8/3}(B_r(y_l))}\le C\|\tilde{u}\|^2_{\mathcal{H}}$. Thus,

$$\|\mathcal{S}\tilde{u}\|^s_{L^s(D)}\le C\|\tilde{u}\|^2_{\mathcal{H}}\sup_{z\in D}\|\mathcal{S}\tilde{u}\|^{(1-\lambda)s}_{L^q(B_r(z))} \tag{7.8}$$

for any $\tilde{u}\in\mathcal{H}$. Plugging $(\tilde{u}_n)_{n\in\mathbb{N}}$ into (7.8), assumption (5.5) entails $\|\mathcal{S}\tilde{u}_n\|_{L^s(D)}\to 0$ as $n\to\infty$. The desired result $\|\mathcal{S}\tilde{u}_n\|_{L^{\tilde{q}}(D)}$ as $n\to\infty$ for all $\tilde{q}\in(2,\frac{8}{3})$ then follows by Hölder interpolation. □

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