Application of bifurcation theory for existence of travelling waves in examples of semilinear and quasilinear wave equations

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1 Introduction and main results

1.1 Derivation of the main equations

Maxwell equations reads as follows[1]

$$\nabla\cdot\overrightarrow{D} = \rho \qquad (\textit{Gauss' law for electric field}), \tag{1}$$
$$\nabla\cdot\overrightarrow{B} = 0 \qquad (\textit{Gauss' law for magnetic field}), \tag{2}$$
$$\nabla\times\overrightarrow{E} = -\frac{\partial}{\partial t}\overrightarrow{B} \qquad (\textit{Faraday's law}), \tag{3}$$
$$\nabla\times\overrightarrow{H} = \overrightarrow{J} + \frac{\partial}{\partial t}\overrightarrow{D} \qquad (\textit{Ampere's law}). \tag{4}$$

In the above equations $\overrightarrow{D}$ stands for the *electric displacement* $\left[\frac{C}{m^2}\right]$, $\overrightarrow{E}$ the *electric field* $\left[\frac{V}{m}\right]$, $\overrightarrow{B}$ the *magnetic field* $\left[\frac{V\cdot s}{m^2}\right]$, $\overrightarrow{H}$ the *magnetic field intensity* $\left[\frac{A}{m}\right]$, ρ the *volume charge density* $\left[\frac{C}{m^3}\right]$, $\overrightarrow{J}$ the *current density* $\left[\frac{A}{m^2}\right]$, ϵ_0 and μ_0 are called *permittivity* and *permeability* of vacuum[2]. We study the situation with no charges ($\rho = 0$) and no currents ($\overrightarrow{J} = 0$).

Moreover we assume the following dependence between the fields

$$\overrightarrow{D} = \epsilon_0\overrightarrow{E} + \overrightarrow{P}, \tag{5}$$
$$\overrightarrow{B} = \mu_0\overrightarrow{H} + \overrightarrow{M}, \tag{6}$$

with

$$\overrightarrow{P} = \epsilon_0\chi_1(\overrightarrow{x})\overrightarrow{E} + \epsilon_0\chi_3(\overrightarrow{x})\left|\overrightarrow{E}\right|^2\overrightarrow{E}, \tag{7}$$
$$\overrightarrow{M} = \mu_0\chi_\nu\overrightarrow{H}. \tag{8}$$

where $\overrightarrow{P}$ is the *polarization* and $\overrightarrow{M}$ is the *magnetization* of the media. The fields $\overrightarrow{P}$ and $\overrightarrow{M}$ describe how the electric and magnetic fields change the electric and magnetic properties of the material under consideration, respectively.

For the derivation of the equation, which we will later consider, we will assume that

$$\chi_1(\overrightarrow{x}) = \chi_1(x_3) \ \text{ and } \chi_3(\overrightarrow{x}) = \chi_3(x_3)\,. \tag{9}$$

The model example for us is the so-called *slab waveguide*, i.e., when

$$\chi_1(\overrightarrow{x}) = \chi_1(x_3) = \begin{cases} \chi_1^i & (|x_3| < a)\,, \\ \chi_1^o & (|x_3| > a)\,, \end{cases}$$
$$\chi_3(\overrightarrow{x}) = \chi_3(x_3) = \begin{cases} \chi_3^i & (|x_3| < a)\,, \\ \chi_3^o & (|x_3| > a)\,. \end{cases}$$

[1] Here equations are presented in MKS units.

[2] $\epsilon_0 \approx 8.8541878176\ldots\cdot 10^{-10}\left[\frac{F}{m}\right]$, $\mu_0 \approx 1.2566370614\ldots\cdot 10^{-6}\left[\frac{V\cdot s}{A\cdot m}\right]$

Under such assumptions we (formally) consider

$$\begin{aligned}
\nabla\times\nabla\times\overrightarrow{E} &\overset{(3)}{=} -\nabla\times\frac{\partial}{\partial t}\overrightarrow{B} \overset{(6)}{=} -\nabla\times\frac{\partial}{\partial t}\Big(\mu_0\overrightarrow{H}+\overrightarrow{M}\Big)\\
&\overset{(8)}{=} -\nabla\times\frac{\partial}{\partial t}\Big(\mu_0(1+\chi_\nu)\,\overrightarrow{H}\Big) = -\mu_0(1+\chi_\nu)\,\frac{\partial}{\partial t}\nabla\times\overrightarrow{H}\\
&\overset{(4)}{=} -\mu_0(1+\chi_\nu)\,\frac{\partial^2}{\partial t^2}\overrightarrow{D} \overset{(5)}{=} -\mu_0(1+\chi_\nu)\,\frac{\partial^2}{\partial t^2}\Big(\epsilon_0\overrightarrow{E}+\overrightarrow{P}\Big)\\
&\overset{(7)}{=} -\mu_0(1+\chi_\nu)\,\frac{\partial^2}{\partial t^2}\left(\epsilon_0(1+\chi_1)\,\overrightarrow{E}+\epsilon_0\chi_3\left|\overrightarrow{E}\right|^2\overrightarrow{E}\right),
\end{aligned}$$

which amounts to the quasilinear wave equation of the form

$$\nabla\times\nabla\times\overrightarrow{E}+\frac{\partial^2}{\partial t^2}\left(V(\overrightarrow{x})\overrightarrow{E}+\Gamma(\overrightarrow{x})\left|\overrightarrow{E}\right|^2\overrightarrow{E}\right)=0, \qquad (10)$$

where

$$V=\mu_0(1+\chi_\nu)\,\varepsilon_0(1+\chi_1)\,,\ \Gamma=\mu_0(1+\chi_\nu)\,\varepsilon_0\chi_3.$$

When we look for polarised waves, i.e., waves of the form

$$\overrightarrow{E}(\overrightarrow{x},t)=\begin{pmatrix}0\\ U(x_1,x_3,t)\\ 0\end{pmatrix}, \qquad (11)$$

then equation (10) becomes a quasilinear scalar wave equation

$$-\triangle U+\frac{\partial^2}{\partial t^2}\big(V(x_3)U+\Gamma(x_3)\,|U|^2\,U\big)=0 \text{ on } \mathbb{R}^2\times\mathbb{R}. \qquad (12)$$

Note that, if $\overrightarrow{E}$ is as in (11), then we have that $\nabla\cdot\overrightarrow{E}=0$, and in consequence, due to (5), (7) and (9) $\nabla\cdot\overrightarrow{D}=0$. Moreover, if U solves (12) and $\overrightarrow{E}$ is of the form (11), by defining $\overrightarrow{B}$ by the time integration of $-\nabla\times\overrightarrow{E}$ we will obtain a solution of the full Maxwell system (1) - (4).

Our goal is to investigate *travelling waves* solutions of the equation (12), i.e., waves having the form $U(\overrightarrow{x},t)=u(x_1-\omega t,x_3)$, where function the u is $2P$-periodic in its first variable. The *profile* has to satisfy the equation

$$-\triangle u+\omega^2\big(V(x_3)\,u_{11}+\Gamma(x_3)\big(3u^2u_{11}+6uu_1^2\big)\big)=0, \qquad (13)$$

on a strip $D=(-P,P)\times\mathbb{R}$.

In section 6.3 we will prove the existence of a solution for the problem (13) with $V(x_3)=\alpha\delta(x_3)+\gamma$, where $\delta(\cdot)$ is a *Dirac delta function* supported along the line $x_3=0$ (cf. Theorem 101). The result is obtained by an application of the theorem of Crandall-Rabinowitz (cf. Theorem 141). These travelling waves are established by bifurcating from the *linear guided modes*, i.e., functions of the form $\overrightarrow{E}(\overrightarrow{x},t) = \begin{pmatrix} 0 \\ \varphi(x_3) \\ 0 \end{pmatrix} e^{\mathrm{i}(\omega t - kx_1)}$ solving the equation

$$\nabla \times \nabla \times \overrightarrow{E} + V(x_3) \frac{\partial^2}{\partial t^2} \overrightarrow{E} = 0 \text{ on } \mathbb{R}^3 \times \mathbb{R}.$$

In section 5 we will study the existence of real valued travelling waves for the semilinear variant of the equation (12), namely the equation

$$-\triangle W + V(x_3)\, W_{tt} + \mu W = g(x_3, W) \;\text{ on } \mathbb{R}^2 \times \mathbb{R}. \tag{14}$$

After taking the travelling wave asatz $W(\overrightarrow{x},t) = w(x_1 - \omega t, x_3)$, where w is $2P$-periodic in its first variable, the equation (14), becomes

$$-\triangle w + \omega^2 V(x_3)\, w_{11} + \mu w = g(x_3, w) \;\text{ on } D = (-P,P) \times \mathbb{R}. \tag{15}$$

The main result of this section is stated in Theorem 28. The equation (14) can be viewed as an approximation of the equation (12) as discussed in [9, Section A].

1.2 Outline of the work

In the section 3 we introduce and study the basic properties of some anisotropic Sobolev spaces $H^{a,b}(D)$, which distinguish between the differentiability properties of the functions in different directions. The need for studying such spaces is motivated by our considerations of the quasilinear wave equation. The quasilinear equation (12) has a linear potential containing a *Dirac delta function* supported along the line $x_3 = 0$, hence obtained solutions will not be twice differentiable in the x_3 direction at $x_1 = 0$. However, in the x_1 direction they are twice differentiable and this property is important. In this section we study the existence of trace operator at the line $x_3 = 0$ with values in $L^2(-P,P)$ (cf. Lemma 5) the embedding properties (e.g. Lemma 9, Corollary 10), and the Fourier series (in the x_1 direction) in defined spaces.

Section 4 is devoted to studying the so-called *Nemytskii operators* induced by the non-linearities appearing in the considered problems (13) and (15). Namely we investigate the differentiability properties of the mappings of the form

$$H^1(D) \ni u \longmapsto g(\cdot, u(\cdot)) \in L^2(D)\,, \tag{16}$$

and

$$H^{2,1}(D) \ni u \longmapsto 3u^2 u_{11} + 6uu_1^2 \in L^2(D)\,. \tag{17}$$

The results obtained for (16) and (17) will be later applied in the proof of the existence of solutions of the equations (15) and (13). The main results of this section, Lemmas 20, 21 and 27, state that the mappings (16) and (17) are sufficiently smooth to apply the Crandall-Rabinowitz theorem for considered problems.

The sections 5 and 6 are the core of this work. In this sections, we show the existence of travelling waves solving the semilinear and quasilinear wave equation. The main results of this sections are Theorems 28 and 101 which state the existence of solutions of the equations (15) and (13) i.e., of real valued travelling waves solving (14) and (12), under suitable assumptions. The strategy of the proof of these two theorems is similar - in both cases, we apply the Crandall-Rabinowitz theorem (quoted here in Theorem 141). Equations (15) and (13) can be written in the form

$$L_\lambda w = G(w)\,, \tag{18}$$

where λ is the *bifurcation parameter* and G is the corresponding non-linearity. In the quasilinear case, we take $\lambda = \omega^2$, where ω is the *frequency* of the travelling wave, and in the semilinear case we take $\lambda = \mu$. First we study the linear operator L_λ and its properties with respect to the parameter λ. The results, which we want to mention at this point are Lemmas 52 and 106. They state, that the operator L_λ has a bounded inverse T_λ on the space $\texttt{ker}\, {L_\lambda}^\perp$ and describe the behaviour of its norm with respect to the parameter λ. We choose λ_1 in such a way, that the operator L_{λ_1} has a one-dimensional kernel. Then we take φ to be a normalised element of the kernel of L_{λ_1}, and we rewrite the problem (18) as

$$\widetilde{L_\lambda} w = L_\lambda w + \mathtt{P}_\varphi w = G(w) + \mathtt{P}_\varphi w, \tag{19}$$

where $\mathtt{P}_\varphi w$ is the L^2 projection of the function w on the kernel of L_{λ_1}. The idea of considering these projections $\mathtt{P}_\varphi$ comes from my colleague Peter Rupp from the Institute for Analysis. We are able to show, that the operator $\widetilde{L_\lambda}$ is invertible for all values of λ in a sufficiently small neighbourhood of λ_1. In the semilinear case we consider the inverse as $\widetilde{T_\lambda} = \widetilde{L_\lambda}^{-1} : L^2_{\text{odd}}(D) \to H^1_{\text{odd}}(D)$. Correspondingly, in the quasilinear, the inverse is defined as $\widetilde{T_\lambda} \colon L^2_{\text{odd}}(D) \to H^{2,1}_{\text{odd}}(D)$. Moreover we prove that the operator $\widetilde{T_\lambda}$ has a norm bounded uniformly for all λ sufficiently close to λ_1 (cf. Corollaries 86 and 132). With this, we can rewrite the equation 19 into the form $F(w, \lambda) = 0$, where

$$F(w, \lambda) = w - \widetilde{T_\lambda}(w + \mathtt{P}_\varphi w)\,.$$

For such a function F, thanks to results from section 4, we are able to apply the Crandall-Rabinowitz theorem. Due to the projection $\mathtt{P}_\varphi$ some of the Fréchet derivatives of the function F have very simple form, namely for λ being sufficiently close to λ_1 we have

$$\mathrm{D}_u F(0, \lambda)\, h = h - \frac{\langle h, \varphi\rangle_{L^2(D)}}{1 + \lambda - \lambda_1}\varphi,$$

$$\mathrm{D}^2_{u\lambda} F(0, \lambda)\, h = \frac{\langle h, \varphi\rangle^2_{L^2(D)}}{(1 + \lambda - \lambda_1)^2}\varphi,$$

which allows to verify assumptions in the Crandall-Rabinowitz theorem.

Section 6 also contains description of some potentials appearing in the equation (13), where one can compute the solution explicitly (cf. Example 96). Moreover, we study existence of complex valued travelling waves solving (12). These waves have the form

$$U(x_1, x_2, x_3, t) = e^{\mathtt{i}\omega\left(x_1 - \sqrt{\lambda}\, t\right)} v(x_3)\,,$$

which leads to the equation of the form

$$-\frac{\partial^2 v}{\partial x_3} + \omega^2(1 - \lambda V(x_3))\, v = \lambda\omega^2\Gamma(x_3)\, v^3 \text{ on } \mathbb{R}, \tag{20}$$

which is a nonlinear, time independent Schrödinger equation in 1-d. Theorem 100 is an existence result for the equation (20).

In section 7 we quote the Crandall Rabinowitz theorem and some remarks concerning its assumptions coming from [5].

Section 8 contains some definitions and technical results, which are used. It also contains some discussion about the necessity of the assumption $\mathbf{S}^4$ (63) in the semilinear case.

2 Spaces and notations

In this section we will introduce some notations used later.

2.1 Notations

Let $D = (-P, P) \times \mathbb{R}$, where $P > 0$. Elements of the set D will be usually denoted as $(x_1, x_3) \in D$. Let $\varepsilon > 0$. The ε neighbourhood and ε surrounding of the point $x \in \mathbb{R}$ will be denoted by $I_x^\varepsilon = (x-\varepsilon, x+\varepsilon)$ and $J_x^\varepsilon = I_x^\varepsilon \setminus \{x\}$, respectively. We will use the following notations for the derivatives. Let $u \colon \mathrm{D} \to \mathbb{R}$ be sufficiently smooth, then

$$\begin{aligned} u_1 &= u_{x_1} = \frac{\partial u}{\partial x_1}, \\ u_{11} &= u_{x_1 x_1} = \frac{\partial^2 u}{\partial x_1^2}, \\ &\vdots \\ u_{13} &= u_{x_1 x_3} = \frac{\partial^2 u}{\partial x_1 \partial x_3}, \\ &\vdots \end{aligned}$$

2.2 Spaces

We will consider the following sets and spaces

- $\mathcal{C}_c^\infty(D)$ is the space of all $\mathcal{C}^\infty(D)$ functions with compact support,
- $\mathcal{C}_b^\infty(D)$ is the space of all $\mathcal{C}^\infty(D)$ functions with bounded support,
- $\mathcal{C}_{\mathrm{per},T}^\infty(\mathbb{R})$ is the space of all $\mathcal{C}^\infty(\mathbb{R})$ functions which are T periodic,
- $\mathcal{C}_{\mathrm{per},T,b}^\infty(\mathbb{R}^2)$ is the space of all $\mathcal{C}^\infty(\mathbb{R}^2)$ functions, which are T-periodic in the x_1 direction, and have bounded support in the x_3 direction, i.e., for every $f \in \mathcal{C}_{\mathrm{per},T,b}^\infty(D)$, there exists a constant $M > 0$ such that $\operatorname{supp} f \subseteq \mathbb{R} \times [-M, M]$,
- $L^2_{\mathrm{odd}}(D)$ is the set of all $L^2(D)$ functions f which are odd in the x_1 direction, i.e., for almost all $(x_1, x_3) \in D$, $f(-x_1, x_3) = -f(x_1, x_3)$. Equivalently, the space $L^2_{\mathrm{odd}}(D)$ is the space of all $L^2(D)$ functions, which can be written in the form

  $$f(x_1, x_3) = \sum_{k \in \mathbb{N}} f_k(x_3) \sin\Big(k \frac{\pi}{P} x_1\Big),$$

  for some sequence $(f_k)_{k\in\mathbb{N}}$ of $L^2(\mathbb{R})$ functions such that $\sum_{k\in\mathbb{N}} \|f_k\|^2_{L^2(D)} < \infty$,

- the spaces $H^{a,b}(D)$ are anisotropic Sobolev spaces defined in Definition 1 in section 3,
- the space $H^{a,b}_{\text{loc}}(\mathbb{R}^2)$ is the space of all functions f such that $f_{|\Omega} \in H^{a,b}(\Omega)$ for every bounded set $\Omega \subseteq \mathbb{R}^2$,
- $H^{a,b}_{\text{per}}(D)$ is the space of all $H^{a,b}(D)$ functions f, which are $2P$ periodic in the x_1 direction, i.e., there exists a function $\widetilde{f} \in H^{a,b}_{\text{loc}}(\mathbb{R}^2)$ such that $\widetilde{f}$ is $2P$ periodic in the direction of x_1 and $\widetilde{f}_{|D} = f$.
- $H^{a,b}_{\text{odd}}(D)$ is the set of all $H^{a,b}_{\text{per}}(D)$ functions, which are odd in the x_1 direction. As proven in Lemma 14 $f \in H^{a,b}_{\text{odd}}(D)$ if and only if

  $$f(x_1, x_3) = \sum_{k\in\mathbb{N}} f_k(x_3) \sin\Big(k\frac{\pi}{P}x_1\Big) \text{ in } L^2(D)\,,$$

  for some sequence $(f_k)_{k\in\mathbb{N}}$ of $H^b(\mathbb{R})$ functions, such that

  $$\sum_{k\in\mathbb{N}} \|f_k\|^2_{H^\beta(D)}\, k^{2\alpha} < \infty,$$

  for all $(\alpha, \beta) \in I_{(a,b)}$, where $I_{a,b}$ is as in Definition 1.
- $\mathcal{L}(X, Y)$ is the space of all linear and continuous operators between two normed spaces X and Y,
- $\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

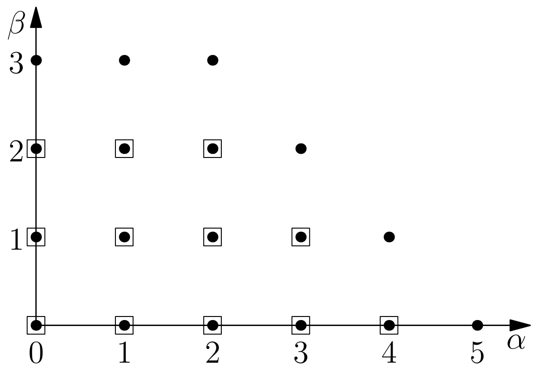


Figure 1: The set $I_{5,3}$ (marked with dots) and the set $I_{4,2}$ (marked with squares) as defined in (21).

3 About the spaces $H^{a,b}$

In this section we introduce some anisotropic Sobolev spaces. We will use them later in the section 6 in the considerations about the quasilinear wave equation (12), with the potential V containing a *Dirac delta function* supported along the line $x_3 = 0$.

3.1 Definition and basic properties

We will begin our considerations with the definition of the function space $H^{a,b}(\Omega)$.

Definition 1. Let $a, b \in \mathbb{N}_0$ and let $\Omega \subseteq \mathbb{R}^2$ be an open set. Define

$$I_{a,b} = \left\{(\alpha,\beta) \in \mathbb{N}_0^2 \colon 0 \leqslant \alpha \leqslant a,\ 0 \leqslant \beta \leqslant b,\ \alpha+\beta \leqslant \max\{a,b\}\right\}. \tag{21}$$

The space $H^{a,b}(\Omega)$ is defined to be

$$H^{a,b}(\Omega) = \left\{ f \in L^2(\Omega) : \frac{\partial^{\alpha+\beta} f}{\partial x^\alpha \partial y^\beta} \in L^2(\Omega) \text{ for all } (\alpha,\beta) \in I_{a,b} \right\} (^3).$$

On the space $H^{a,b}(\Omega)$ we consider the norm $\|\cdot\|_{H^{a,b}(\Omega)}$ defined by

$$\|f\|_{H^{a,b}(\Omega)} = \sqrt{\sum_{(\alpha,\beta)\in I_{a,b}} \left\| \frac{\partial^{\alpha+\beta} f}{\partial x^\alpha \partial y^\beta} \right\|^2_{L^2(\Omega)}}.$$

The set $H^{a,b}_{\mathrm{loc}}(\Omega)$ consists of all functions u such that $u \in H^{a,b}(U)$ for every bounded and open set $U \subseteq \mathbb{R}^2$ such that $\overline{U} \subseteq \Omega$.

[3] the derivative $\frac{\partial^{\alpha+\beta} f}{\partial x^\alpha \partial y^\beta}$ is understood in the distributional sense.

Remark 2. For every $m \in \mathbb{N}$ we have $H^{m,m}(\Omega) = H^m(\Omega)$, where $H^m(\Omega)$ is the usual Sobolev space of functions having $L^2(D)$ integrable derivatives up to order m, cf. [1].

Lemma 3. *The space $\mathcal{C}_b^\infty(D)$ is dense in the space $H^{0,1}(D)$.*

Proof. Let $\varrho\colon \mathbb{R}^2 \to \mathbb{R}$ be a mollifier, for example,

$$\varrho(x) = \begin{cases} ce^{\frac{1}{|x|^2-1}} & \left(|x| = \sqrt{x_1^2 + x_3^2} \leqslant 1\right), \\ 0 & (|x| \geqslant 1)\,, \end{cases}$$

where the constant $c > 0$ is chosen in such a way, that $\int_{\mathbb{R}^2} \varrho \, \mathrm{d}x = 1$. Denote $D' = (-2P, 2P) \times \mathbb{R}$. For $v \in L^1_{\mathrm{loc}}(D')$ and $h > 0$ define

$$v_h(x) = h^{-2} \int_{D'} \varrho\left(\frac{x-y}{h}\right) v(y) \, \mathrm{d}y.$$

The function v_h is a $\mathcal{C}^\infty$ function, for all $h > 0$. Take any $u \in H^{0,1}(D)$. After periodic extension consider the function u as an element of the space $H^{0,1}(D')$.

By [3, Lemma 7.3, p. 150], for sufficiently small $h > 0$, we have that

$$\frac{\partial u_h}{\partial x_3} = \left(\frac{\partial u}{\partial x_3}\right)_h \quad \text{on } D. \tag{22}$$

By [1, Theorem 2.29, p. 36] we have that

$$u_h \xrightarrow[h\to 0]{L^2(D)} u, \; \left(\frac{\partial u}{\partial x_3}\right)_h \xrightarrow[h\to 0]{L^2(D)} \frac{\partial u}{\partial x_3}. \tag{23}$$

Relations (23) and (22) imply, that the set $\mathcal{C}^\infty(D)$ is a dense subset of $H^{0,1}(D)$.

For every $k \in \mathbb{N}$ consider a function $w_k\colon \mathbb{R} \to \mathbb{R}$ such that $w_k \in \mathcal{C}_c^\infty(\mathbb{R})$, $w_k(x) = 1$, for $|x| \leqslant k$, $w_k(x) = 0$, for $|x| \geqslant k+1$ and $\|w_k'\|_{L^\infty(\mathbb{R})} \leqslant 2$. Let $v \in \mathcal{C}^\infty(D) \cap H^{0,1}(D)$. For every $k \in \mathbb{N}$ define

$$v_k(x_1, x_3) = v(x_1, x_3)\, w_k(x_3) \quad ((x_1, x_3) \in D)\,.$$

Note that

$$\|v_k - v\|^2_{L^2(D)} = \|v(1-w_k)\|^2_{L^2(D)} \leqslant \int_{-k}^{k} \int_{-P}^{P} v^2 \, \mathrm{d}x_1 \, \mathrm{d}x_3 \xrightarrow[k\to\infty]{} 0.$$

Denote $R_k = [-k-1, -k] \cup [k, k+1]$. We have

$$\begin{aligned} \left\| \frac{\partial v_k}{\partial x_3} - \frac{\partial v}{\partial x_3} \right\|^2_{L^2(D)} &= \left\| \frac{\partial v}{\partial x_3}(w_k - 1) + v w_k' \right\|^2_{L^2(D)} \\ &\leqslant 2 \left\| \frac{\partial v}{\partial x_3}(w_k - 1) \right\|^2_{L^2(D)} \\ &\quad + 2 \int_{R_k} \int_{-P}^{P} v^2 {w_k'}^2 \, \mathrm{d}x_1 \, \mathrm{d}x_3 \xrightarrow[k\to\infty]{} 0, \end{aligned}$$

Hence $v_k \xrightarrow[k\to\infty]{H^{0,1}(D)} w$, which shows, that that the space $\mathcal{C}_b^\infty(D)$ is dense in $H^{0,1}(D)$. □

Remark 4. By same argumentation as in the proof of Lemma 3, we get that the space $\mathcal{C}_b^\infty(D)$ is dense in $H^{a,b}(D)$, for all $a, b \in \mathbb{N}$.

3.2 Trace operator for the space $H^{0,1}(D)$

Lemma 5. *There exists a continuous trace operator* $\operatorname{tr} : H^{0,1}(D) \to L^2(-P,P)$ *at the line* $x_3 = 0$.

Proof. Let $L = \{0\} \times (-P,P)$, $\xi = (0,1) \in \mathbb{R}^2$. Let $u \in \mathcal{C}_b^\infty(D)$. Since u is continuous, it has a trace $\operatorname{tr} u = u_{|L}$ at $x_3 = 0$. Moreover

$$\begin{aligned}\int_{(-P,P)} (\operatorname{tr} u)^2 \, \mathrm{d}\sigma &= \int_{\partial D^+} u^2 \xi \cdot \nu \, \mathrm{d}\sigma \\ &= \int_{D^+} \nabla \cdot \left(\xi u^2\right) \, \mathrm{d}x = \int_{D^+} 2u \frac{\partial u}{\partial x_3} \, \mathrm{d}x \\ &\leqslant \int_{D^+} u^2 + \left(\frac{\partial u}{\partial x^3}\right)^2 \, \mathrm{d}x = \|u\|^2_{H^{0,1}(D)} .\end{aligned}$$

Note that we can obtain the same estimate using the set D^-. Therefore the operator $\operatorname{tr} : \mathcal{C}_b^\infty(D) \to L^2((-P,P))$ satisfies

$$\|\operatorname{tr} u\|_{L^2(-P,P)} \leqslant \|h\|_{H^{0,1}(D)} ,$$

for all $u \in \mathcal{C}_b^\infty(D)$. By Lemma 3 the space $\mathcal{C}_b^\infty(D)$ is a dense subset of $H^{0,1}(D)$. Hence there is a continuous extension of the trace operator $\operatorname{tr} : \mathcal{C}_b^\infty(D) \to L^2(-P,P)$ to $\operatorname{tr} : H^{0,1}(D) \to L^2(-P,P)$. □

Lemma 6. *There exists a trace operator* $\operatorname{tr} : H^{m,1}(D) \to H^m(-P,P)$ *at* $x_3 = 0$.

Proof. By Lemma 5 for $u \in \mathcal{C}_b^\infty(D)$ and for all $a \leqslant m$, we have that

$$\left\| \frac{\partial^a}{\partial x_1^a} \operatorname{tr} u \right\|_{L^2(-P,P)} = \left\| \operatorname{tr} \frac{\partial^a u}{\partial x_1^a} \right\|_{L^2(-P,P)} \leqslant \left\| \frac{\partial^a u}{\partial x_1^a} \right\|_{H^{0,1}(D)} \leqslant \|u\|_{H^{m,1}(D)} .$$

Therefore the operator $\operatorname{tr} : \mathcal{C}_b^\infty(D) \to L^2((-P,P))$ satisfies

$$\|\operatorname{tr} u\|_{L^2(-P,P)} \leqslant \|h\|_{H^{0,1}(D)} ,$$

for all $u \in \mathcal{C}_b^\infty(D)$. By Remark 4 the space $\mathcal{C}_b^\infty(D)$ is a dense subset of $H^{m,1}(D)$. Thus, there exists a continuous extension of the trace operator $\operatorname{tr} : \mathcal{C}_b^\infty(D) \to H^m(-P,P)$ to $\operatorname{tr} : H^{m,1}(D) \to H^m(-P,P)$.

□

3.3 Embedding properties of the space $H^{2,1}(D)$

Now we will list some embedding theorems for the space $H^{2,1}(D)$.

Lemma 7. *The space $H^{2,1}(D)$ embeds continuously into the space $L^q(D)$ for all $q \in [2, \infty)$.*

Proof. Note that $H^{2,1}(D) \subseteq H^1(D)$ and the space $H^1(D)$ embeds continuously into the space $L^q(D)$ for all $q \in [2, \infty)$ (cf. [1, Theorem 4.12, p. 85]). □

Lemma 8. *The mapping $H^{2,1}(D) \ni u \longmapsto \frac{\partial u}{\partial x_1} \in H^1(D)$ is continuous.*

Proof. Take $u \in H^{2,1}(D)$ and let $f = \frac{\partial u}{\partial x_1}$. By the definition of the space $H^{2,1}(D)$, we have that $f, \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_3} \in L^2(D)$. Therefore $f \in H^1(D)$. □

Lemma 9. *The space $H^{2,1}(D)$ embeds continuously into the space $\mathcal{C}^{0,\beta}\left(\overline{D}\right)$ for all $\beta \in \left(0, \frac{1}{3}\right)$.*

Proof. Take $\beta \in \left(0, \frac{1}{3}\right)$. By Lemma 8 and the fact that the space $H^1(D)$ embeds continuous into the space $L^q(D)$ for all $q \in [2, \infty))$ (cf. [1, Theorem 4.12, p. 85]), we have that

$$u \in L^2(D)\,, \ \frac{\partial u}{\partial x_1} \in L^q(D)\,, \ \frac{\partial u}{\partial x_3} \in L^2(D) \quad \left(u \in H^{2,1}(D)\,, q \in [2, \infty)\right). \tag{24}$$

For each $n \in \mathbb{Z}$ let $D_n = (-P, P) \times (n, n+1)$. Let $q = \frac{2\beta - 1}{3\beta - 1}$. Note that by [6, Theorem 2], there exists a constant $C > 0$ such that for all $n \in \mathbb{N}$ and all $u \in H^{2,1}(D)$

$$\left\|u_{|D_n}\right\|_{\mathcal{C}^{0,\beta}\left(\overline{D_n}\right)} \leqslant C \left(\left\|u_{|D_n}\right\|_{L^2(D_n)} + \left\| \frac{\partial u_{|D_n}}{\partial x_1} \right\|_{L^q(D_n)} + \left\| \frac{\partial u_{|D_n}}{\partial x_3} \right\|_{L^2(D_n)} \right).$$

This, together with (24), gives, that there exists some constant $C > 0$

$$\left\|u_{|D_n}\right\|_{\mathcal{C}^{0,\beta}\left(\overline{D_n}\right)} \leqslant C \left\|u_{|D_n}\right\|_{H^{2,1}(D_n)} \leqslant C \left\|u\right\|_{H^{2,1}(D)} \quad \left(u \in H^{2,1}(D)\,, n \in \mathbb{N}\right).$$

Therefore, for every $\beta \in \left(0, \frac{1}{3}\right)$, there exists a constant $C_\beta > 0$ such that

$$\left\|u\right\|_{\mathcal{C}^{0,\beta}\left(\overline{D}\right)} \leqslant C_\beta \left\|u\right\|_{H^{2,1}(D)} \quad \left(u \in H^{2,1}(D)\right).$$

□

As a consequence of the Lemmas 7 and 9, we get our next result

Corollary 10. *The space $H^{2,1}(D)$ embeds continuously into the space $L^q(D)$ for all $q \in [2, \infty]$.*

3.4 Fourier series in the spaces $H^{a,b}_{\mathrm{odd}}(D)$

In this subsection, we will consider Fourier series in the spaces $H^{a,b}_{\mathrm{odd}}(D)$. The main results of this section are Lemmas 12 and 14 , which are Riesz-Fisher type results.

Lemma 11. *Let $u \in L^2_{\mathrm{odd}}(D)$, i.e., u has the form*

$$u(x_1, x_3) = \sum_{k=1}^{\infty} b_k(x_3) \sin\Big(k\frac{\pi}{P}x_1\Big)\,,\ \textit{for almost all}(x_1, x_3) \in D.$$

Assume that

$$\sum_{k=1}^{\infty} k^2 \left\|b_k\right\|^2_{L^2(\mathbb{R})} < \infty. \tag{25}$$

Then $u \in H^{1,0}_{\mathrm{odd}}(D)$ and

$$\frac{\partial u}{\partial x_1} = \sum_{k=1}^{\infty} k\frac{\pi}{P} b_k(x_1) \cos\Big(k\frac{\pi}{P}x_1\Big)\,,\ \textit{for almost all}\ (x_1, x_3) \in D.$$

Proof. Extend periodically the functions u to the whole $\mathbb{R}^2$. For each $K \in \mathbb{N}$ define

$$u_K(x_1, x_3) = \sum_{k=1}^{K} b_k(x_3) \sin\Big(k\frac{\pi}{P}x_1\Big)\,,\ \text{for almost all}(x_1, x_3) \in \mathbb{R}^2.$$

The functions u_K are weakly differentiable with respect to x_1 and

$$\frac{\partial u_K}{\partial x_1} = \sum_{k=1}^{K} k\frac{\pi}{P} b_k(x_1) \cos\Big(k\frac{\pi}{P}x_1\Big)\,,$$

for almost all $(x_1, x_3) \in \mathbb{R}^2$ and all $K \in \mathbb{N}$. Assumption (25) implies that the sequence $\left(\frac{\partial u_K}{\partial x_1}\right)_{K\in\mathbb{N}}$ is a Cauchy sequence in $L^2((-M, M) \times \mathbb{R})$, for all $M > 0$. Therefore there exists a function $v \in L^2_{\mathrm{loc}}(\mathbb{R}^2)$ such that $\frac{\partial u_K}{\partial x_1} \xrightarrow[K\to\infty]{L^2((-M,M)\times\mathbb{R})} v$, for all $M > 0$. Moreover, $v \in L^2((-M, M) \times \mathbb{R})$, for all $M > 0$. We will show that v is a weak derivative of the function u.

Consider any $\varphi \in \mathcal{C}^\infty_c(\mathbb{R}^2)$. Let $L > P$ be such that $\operatorname{supp}\varphi \subseteq [-L, L] \times$

$[-L, L]$. We have:

$$\begin{aligned}
\int_{\mathbb{R}^2} v(x)\,\varphi(x)\;\mathrm{d}x &= \int_{\mathbb{R}^2} \sum_{k\in\mathbb{N}} k\frac{\pi}{P} b_k(x_1)\cos\Big(k\frac{\pi}{P}x_1\Big)\;\varphi(x_1,x_3)\;\mathrm{d}(x_1,x_3) \\
&= \sum_{k\in\mathbb{N}} \int_{-L}^{L}\int_{-L}^{L} k\frac{\pi}{P} b_k(x_1)\cos\Big(k\frac{\pi}{P}x_1\Big)\;\varphi(x_1,x_3)\;\mathrm{d}x_1\,\mathrm{d}x_3 \\
&= -\sum_{k\in\mathbb{N}} \int_{-L}^{L}\int_{-L}^{L} b_k(x_1)\sin\Big(k\frac{\pi}{P}x_1\Big)\;\frac{\partial\varphi}{\partial x_1}(x_1,x_3)\;\mathrm{d}x_1\,\mathrm{d}x_3 \\
&= -\int_{[-L,L]\times[-L,L]} \sum_{k\in\mathbb{N}} b_k(x_1)\sin\Big(k\frac{\pi}{P}x_1\Big)\;\frac{\partial\varphi}{\partial x_1}(x_1,x_3)\;\mathrm{d}(x_1,x_3) \\
&= -\int_{\mathbb{R}^2} u(x_1,x_3)\;\frac{\partial\varphi}{\partial x_1}(x_1,x_3)\;\mathrm{d}(x_1,x_3)\,.
\end{aligned}$$

Therefore $\frac{\partial u}{\partial x_1} = v$ and $u \in H^{1,0}((-M,M)\times\mathbb{R})$, for all $L > 0$. Moreover this implies, that $u \in H^{1,0}_{\text{odd}}(D)$. □

Lemma 12. *Assume that* $u \in L^2_{\text{odd}}(D)$*, i.e.,*

$$u(x_1,x_3) = \sum_{k=1}^{\infty} u_k(x_3)\sin\Big(k\frac{\pi}{P}x_1\Big) \;\; \textit{in } L^2(D)\,,$$

where $u_k \in H^b(\mathbb{R})$ $(k\in\mathbb{N})$ *and*

$$\sum_{k=1}^{\infty} \|u_k\|^2_{H^\beta(\mathbb{R})}\,k^{2\alpha} < \infty,$$

for all $(\alpha,\beta)\in I_{a,b}$*. Then* $u \in H^{a,b}_{\text{odd}}(D)$ *and for all* $(\alpha,\beta)\in I_{a,b}$

$$\frac{\partial^{\alpha+\beta}u}{\partial x_1^\alpha x_3^\beta} = \sum_{k=1}^{\infty} f_k^{(\beta)}(x_3)(\sin\sqrt{a_k}\,x_1)^{(\alpha)} \;\; \textit{for almost all } (x_1,x_3)\in D.$$

Proof. The proof follows from induction and Lemma 11. □

Lemma 13. *Let* $u \in H^{1,0}_{\text{odd}}(D)$*, i.e.,* u *has the form*

$$u(x_1,x_3) = \sum_{k=1}^{K} u_k(x_3)\sin\Big(k\frac{\pi}{P}x_1\Big) \quad ((x_1,x_3)\in D, K\in\mathbb{N})\,.$$

Then $\sum_{k\in\mathbb{N}} k^2\,\|u_k\|^2_{L^2(\mathbb{R})} < \infty$.

Proof. Since $u \in H^{1,0}(D)$, we have that $\frac{\partial u}{\partial x_1} \in L^2(D)$, therefore, we can write it as

$$\frac{\partial u}{\partial x_1}(x_1, x_3) = \sum_{k=1}^{\infty} c_k(x_3) \cos\Big(k\frac{\pi}{P}x_1\Big) \quad ((x_1, x_3) \in D)\,,$$

for some sequence $(c_k)_{k\in\mathbb{N}}$ of $L^2(\mathbb{R})$ functions, such that $\sum_{k=1}^{\infty} \|c_k\|^2_{L^2(D)} < \infty$. We will show that

$$c_k \overset{L^2(\mathbb{R})}{=} k\frac{\pi}{P}u_k \quad (k \in \mathbb{N}) \tag{26}$$

We have that

$$\int_D u\frac{\partial \varphi}{\partial x_1}\,\mathrm{d}x = -\int_D \frac{\partial u}{\partial x_1}\varphi\,\mathrm{d}x \quad (\varphi \in \mathcal{C}_c^{\infty}(D))\,. \tag{27}$$

Observe that for every $\varphi \in \mathcal{C}_c^{\infty}(D)$

$$\begin{aligned}
&-\sum_{k=1}^{K} k\frac{\pi}{P}u_k(x_3)\cos\Big(k\frac{\pi}{P}x_1\Big)\,\varphi(x_1,x_3)\ \mathrm{d}x_1\,\mathrm{d}x_3 \\
&\quad= \sum_{k=1}^{K}\int_D u_k(x_3)\sin\Big(k\frac{\pi}{P}\Big)\,\frac{\partial\varphi}{\partial x_1}(x_1,x_3)\ \mathrm{d}x_1\,\mathrm{d}x_3 \\
&\quad= \int_D \sum_{k=1}^{K} u_k(x_3)\sin\Big(k\frac{\pi}{P}\Big)\,\frac{\partial\varphi}{\partial x_1}(x_1,x_3)\ \mathrm{d}x_1\,\mathrm{d}x_3 \xrightarrow[K\to\infty]{} \int_D u\frac{\partial\varphi}{\partial x_1}\,\mathrm{d}x,
\end{aligned} \tag{28}$$

and

$$\begin{aligned}
&-\sum_{k=1}^{K}\int_D c_k(x_3)\cos\Big(k\frac{\pi}{P}x_1\Big)\,\varphi(x_1,x_3)\ \mathrm{d}x_1\,\mathrm{d}x_3 \\
&\quad= \int_D -\sum_{k=1}^{K} c_k(x_3)\cos\Big(k\frac{\pi}{P}x_1\Big)\,\varphi(x_1,x_3)\ \mathrm{d}x_1\,\mathrm{d}x_3 \xrightarrow[K\to\infty]{} -\int_D \frac{\partial u}{\partial x_1}\varphi\,\mathrm{d}x.
\end{aligned} \tag{29}$$

Therefore, by (27), (28) and (29) we obtain (26). □

Proof. The proof follows from induction and Lemma 13. □

As a consequence of above statement, we have the following characterization of the space $H^{a,b}_{\text{odd}}(D)$.

Lemma 14. *$u \in H^{a,b}_{\text{odd}}(D)$ if and only if u has the form*

$$u(x_1, x_3) = \sum_{k=1}^{\infty} u_k(x_3)\sin\Big(k\frac{\pi}{P}x_1\Big)\,,\ \textit{for almost all } (x_1, x_3 \in D)\,,$$

for some sequence $(u_k)_{k\in\mathbb{N}}$ of $H^b(\mathbb{R})$ functions such that for all $(\alpha, \beta) \in I_{a,b}$

$$\sum_{k=1}^{n} \|u_k\|^2_{H^{\beta}(\mathbb{R})}\,k^{2\alpha} < \infty.$$

Proof. The proof follows from induction and Lemma 13. □

4 Nemytskii operators

In this section we will study the differentiability properties of certain non-linear maps acting between Banach spaces.

4.1 Results applicable for the semilinear wave equation

In this section we will study the continuity and differentiability of the operators of the form

$$H^1(D) \ni u \longmapsto g(\cdot, u(\cdot)) \in L^2(D)\,.$$

The main results of this section are Lemmas 18, 20 and 21.

We will begin with listing some technical results used later in the proofs.

Remark 15. Because of [1, Theorem 4.12, p. 85] for all $q \in [2,\infty)$, the space $H^1(D)$ embeds continuously into the space $L^q(D)$.

Remark 16. Assume that $\beta \geqslant 1$. Then the function

$$[0,\infty) \ni x \longmapsto x^\beta \in [0,\infty)\,, \tag{30}$$

is superadditive, i.e.,

$$a^\beta + b^\beta \leqslant (a+b)^\beta \quad (a, b \geqslant 0)\,.$$

Remark 17. Let $\beta > 1$ and $p > 1$. Then if $f \in L^p(\Omega)$, then $|f|^\beta \in L^{\frac{p}{\beta}}(\Omega)$. Moreover $\left\| |f|^\beta \right\|_{L^{\frac{p}{\beta}}(\Omega)} = \left(\|f\|_{L^p(\Omega)} \right)^\beta$.

Lemma 18. *Let $r \in [2,\infty)$. Assume that the function $g\colon D \times \mathbb{R} \to \mathbb{R}$ is such that*

- *for all $s \in \mathbb{R}$ the mapping*

  $$D \ni (x_1, x_3) \longmapsto g(x_1, x_3, s) \in \mathbb{R},$$

  is measurable,

- *for almost all $(x_1, x_3) \in D$ the mapping*

  $$\mathbb{R} \ni s \longmapsto g(x_1, x_3, s) \in \mathbb{R}, \tag{31}$$

  is continuous,

- *for some $\alpha \geqslant 1$ and $q \in (r,\infty]$ there exists a function $a \in L^q(D)$ such that*

  $$|g(x_1, x_3, s)| \leqslant a(x_1, x_3)\,|s|^\alpha \quad ((x_1, x_3) \in D, s \in \mathbb{R})\,. \tag{32}$$

Then the mapping $G\colon H^1(D) \to L^r(D)$ *defined as*

$$G(u)(x_1, x_3) = g(x_1, x_3, u(x_1, x_3)) \quad \big(u \in H^1(D)\,, (x_1, x_3) \in D\big)\,, \tag{33}$$

is continuous.

Proof. First we will show that the mapping G introduced in (33) is well-defined. By the Remarks 15 and 17, we get that

$$\||u|^\alpha\|_{L^p(D)} < \infty \quad \left(u \in H^1(D)\,, p \in \left[\frac{2}{\alpha}, \infty\right)\right).$$

There exists $p > 2$ such that $\frac{1}{r} = \frac{1}{q} + \frac{1}{p}$. Using Hölder's inequality we can estimate

$$\|G(u)\|_{L^r(D)} \leqslant \|a \cdot |u|^\alpha\|_{L^r(D)} \leqslant \|a\|_{L^q(D)} \||u|^\alpha\|_{L^p(D)} < \infty.$$

Now we will show the continuity of the mapping G. Let $u \in H^1(D)$ and let $h_n \xrightarrow[n\to\infty]{H^1(D)} 0$ be arbitrary. We will show that every subsequence of $\left(\|G(u + h_n) - G(u)\|_{L^2(D)}\right)_{n\in\mathbb{N}}$ has a subsequence convergent to 0. W.l.o.g (by choosing suitable subsequences) we may assume that $h_n(x_1, x_3) \xrightarrow[n\to\infty]{} 0$ for almost all $(x_1, x_3) \in D$ and that

$$\|h_{n+1} - h_n\|_{H^1(D)} \leqslant \frac{1}{2^n} \quad (n \in \mathbb{N})\,. \tag{34}$$

Note that for every $n \in \mathbb{N}$ we have $h_n = h_1 + \sum_{i=1}^{n-1}(h_{i+1} - h_i)$. By inequality (34) and Remark 15 the function $w = |h_1| + \sum_{i=1}^{\infty} |h_{i+1} - h_i| \in L^s$ for all $s \in [2, \infty)$. Observe that

$$|h_n(x)| \leqslant |h_1(x)| + \sum_{i=1}^{n-1} |h_{i+1}(x) - h_i(x)| \leqslant w(x) \quad (x \in \mathbb{N})\,. \tag{35}$$

For every $n \in \mathbb{N}$ define

$$v_n(x_1, x_3) = g(x_1, x_3, u(x_1, x_3) + h_n(x_1, x_3)) - g(x_1, x_3, u(x_1, x_3))\,.$$

For almost all $x \in D$, we can estimate

$$\begin{aligned} |v_n(x)| &\leqslant |g(x, u(x) + h_n(x))| + |g(x, u(x))| \\ &\overset{(32)}{\leqslant} a(x)(|u(x) + h_n(x)|^\alpha + |u(x)|^\alpha) \\ &\overset{\text{Remark 16}}{\leqslant} a(x)(2\,|u(x)| + |h_n(x)|)^\alpha \\ &\overset{(35)}{\leqslant} a(x)(2\,|u(x)| + w(x))^\alpha\,. \end{aligned}$$

This shows that the functions v_n are bounded by a function from the space $L^r(D)$.

Since the sequence h_n converges to 0 almost everywhere in D and the function g is continuous with respect to the third variable (cf. (31)), we have that the sequence v_n converges to 0 almost everywhere in D. An application of Lebesgue's dominated convergence theorem finishes the proof. □

Lemma 19. *Let $r \in [2, \infty)$. Assume that the function $g\colon D \times \mathbb{R} \to \mathbb{R}$ is such that*

- *for all $s \in \mathbb{R}$ the mapping*
$$D \ni (x_1, x_3) \longmapsto g(x_1, x_3, s) \in \mathbb{R},$$
is measurable,
- *for almost all $(x_1, x_3) \in D$ the mapping*
$$\mathbb{R} \ni s \longmapsto g(x_1, x_3, s) \in \mathbb{R}, \tag{36}$$
is continuous,
- *for some $\alpha \geqslant 1$ and $q \in (r, \infty]$ there exists a function $a \in L^q(D)$ such that*
$$|g(x_1, x_3, s)| \leqslant a(x_1, x_3)\, |s|^\alpha \quad ((x_1, x_3) \in D, s \in \mathbb{R})\,. \tag{37}$$

Then for every sequence $h_n \xrightarrow[n\to\infty]{H^1(D)} 0$ and for every $u \in H^1(D)$ we have
$$\int_0^1 g(x, u(x) + th_n(x)) - g(x, u(x))\ \mathrm{d}t \xrightarrow[n\to\infty]{L^r(D)} 0.$$

Proof. Let $r \geqslant 2$. Denote
$$v_n(x) = \int_0^1 g(x, u(x) + th_n(x)) - g(x, u(x))\ \mathrm{d}t \quad (n \in \mathbb{N}, x \in D)\,.$$
We will show that every subsequence of $\Big(\|v_n\|_{L^r(D)}\Big)_{n\in\mathbb{N}}$ has a subsequence convergent to 0. W.l.o.g (by choosing suitable subsequences) we may assume that $h_n(x_1, x_3) \xrightarrow[n\to\infty]{} 0$ for almost all $(x_1, x_3) \in D$ and that
$$\|h_{n+1} - h_n\|_{H^1(D)} \leqslant \frac{1}{2^n} \quad (n \in \mathbb{N})\,. \tag{38}$$
Note that for every $n \in \mathbb{N}$ we have $h_n = h_1 + \sum_{i=1}^{n-1}(h_{i+1} - h_i)$. By inequality (38) and Remark 15 the function $w = |h_1| + \sum_{i=1}^{\infty} |h_{i+1} - h_i| \in L^s(D)$ for all $s \in [2, \infty)$. Observe that
$$|h_n(x)| \leqslant |h_1(x)| + \sum_{i=1}^{n-1} |h_{i+1}(x) - h_i(x)| \leqslant w(x) \quad (x \in \mathbb{N})\,. \tag{39}$$

For every $n \in \mathbb{N}$ define

$$v_n(x_1, x_3) = \int_0^1 g(x_1, x_3, u(x_1, x_3) + th_n(x_1, x_3))\ \mathrm{d}t - g(x_1, x_3, u(x_1, x_3))\,.$$

For almost all $x \in D$, we can estimate

$$\begin{aligned}
|v_n(x)| &\leqslant \int_0^1 |g(x, u(x) + th_n(x))| + |g(x, u(x))|\ \mathrm{d}t \\
&\overset{(37)}{\leqslant} a(x) \int_0^1 (|u(x) + th_n(x)|^\alpha + |u(x)|^\alpha)\ \mathrm{d}t \\
&\overset{\text{Remark 16}}{\leqslant} a(x) \int_0^1 (2\,|u(x)| + t\,|h_n(x)|)^\alpha\ \mathrm{d}t \\
&\leqslant a(x)(2\,|u(x)| + |h_n(x)|)^\alpha \\
&\overset{(39)}{\leqslant} a(x)(2\,|u(x)| + w(x))^\alpha\,.
\end{aligned}$$

Above estimate shows, that the functions $v_n\,(n \in \mathbb{N})$ are bounded by a function from the space $L^r(D)$.

Since the sequence h_n converges to 0 almost everywhere in D and the function g is continuous with respect to the third variable (cf. (31)), we have that the sequence v_n converges to 0 almost everywhere in D. Application of Lebesgue's dominated convergence theorem finishes the proof. □

Lemma 20. *Assume that the function $g\colon D \times \mathbb{R} \to \mathbb{R}$ is such that:*

- *the for almost all $(x_1, x_3) \in D$ the mapping*

$$\mathbb{R} \ni s \longmapsto g(x_1, x_3, s) \in \mathbb{R},$$

  is differentiable,
- *the mappings g and $\frac{\partial g}{\partial s}$ are Carathéodory functions (cf. Definition 143),*
- *$g(x_1, x_3, 0) = 0$ for almost all $(x_1, x_3) \in D$,*
- *for some $\alpha \geqslant 1$ and $q \in (2, \infty]$ there exit functions $a \in L^q(D)$ such that*

$$\left|\frac{\partial g}{\partial s}(x_1, x_3, s)\right| \leqslant a(x_1, x_3)\,|s|^\alpha \quad ((x_1, x_3) \in D, s \in \mathbb{R})\,, \tag{40}$$

Then the operator $G\colon H^1(D) \to L^2(D)$ defined as

$$G(u)(x_1, x_3) = g(x_1, x_3, u(x_1, x_3)) \quad \big(u \in H^1(D)\,,(x_1, x_3) \in D\big)\,, \tag{41}$$

is of the class $\mathcal{C}^1(H^1(D)\,, L^2(D))$ and

$$\mathrm{D}G(u)\,h = \frac{\partial g}{\partial s}(\cdot, \cdot, u)\,h \quad \big(u, h \in H^1(D)\big)\,. \tag{42}$$

Proof. Observe that, for almost all $(x_1, x_3) \in D$

$$
\begin{aligned}
|g(x_1, x_3, s)| &= \left| \int_0^s \frac{\partial g}{\partial s}(x_1, x_3, \varsigma)\ \mathrm{d}\varsigma \right| \\
&\overset{(40)}{\leqslant} |a(x_1, x_3)| \int_0^{|s|} |\varsigma|^\alpha\ \mathrm{d}\varsigma \\
&= \frac{|a(x_1, x_3)|}{\alpha + 1} |s|^{\alpha+1}.
\end{aligned}
$$

Note that Lemma 18 implies that the operators described in (41) and (42) are well-defined and continuous. We will show that the operator defined in (42) is indeed the Fréchet derivative of the operator G defined in (41). Let $u, h \in H^1(D)$. Consider

$$
\begin{aligned}
G(u + h)(x_1, x_3) - G(u)(x_1, x_3) & \\
&= g(x_1, x_3, u(x_1, x_3) + h(x_1, x_3)) - g(x_1, x_3, u(x_1, x_3)) \\
&= h(x_1, x_3) \int_0^1 \frac{\partial g}{\partial s}(x_1, x_3, u(x_1, x_3) + th(x_1, x_3))\ \mathrm{d}t.
\end{aligned}
$$

Using this we can write

$$
\begin{aligned}
R(u, h)(\cdot, \text{▪}) &= (G(u + h) - G(u) - \mathrm{D}G(u)\, h)(\cdot, \text{▪}) \\
&= h(\cdot, \text{▪}) \left(\int_0^1 \frac{\partial g}{\partial s}(\cdot, \text{▪}, u(\cdot, \text{▪}) + th(\cdot, \text{▪})) - \frac{\partial g}{\partial s}(\cdot, \text{▪}, u(\cdot, \text{▪}))\ \mathrm{d}t \right).
\end{aligned}
$$

This, together with Hölder's inequality, Remark 15 and Lemma 19, shows that

$$
\frac{\|R(u, h)\|_{L^2(D)}}{\|h\|_{H^1(D)}} \xrightarrow[h \xrightarrow{H^1(D)} 0]{} 0.
$$

□

Lemma 21. *Under the assumptions of Lemma 20, assume that* $\alpha > 1$ *in condition* (40). *Then the operator* $G\colon H^1(D) \to L^2(D)$ *defined in* (41) *is twice differentiable at point* 0 *and*

$$
\mathrm{D}^2 G(0)[h, v] = 0 \quad \big(h, v \in H^1(D)\big).
$$

Proof. We have to show that

$$
\sup_{h \in \mathrm{B}_{H^1(D)}(0,1)} \frac{\|\mathrm{D}G(v)\, h - \mathrm{D}G(0)\, h\|_{L^2(D)}}{\|v\|_{H^1(D)}} \xrightarrow[v \xrightarrow[H^1(D)]{} 0]{} 0.
$$

Let $s > 2$ and $p \geqslant \frac{2}{\alpha}$, be such that $\frac{q-2}{2q} = \frac{1}{2} - \frac{1}{q} = \frac{1}{s} + \frac{1}{p}$. Using generalised Hölder's inequality (cf. [14, Theorem 2.1]), we obtain:

$$\begin{aligned}\|\mathrm{D}G(v)\,h - \mathrm{D}G(0)\,h\|_{L^2(D)} &\overset{(42)}{=} \left\|\frac{\partial g}{\partial s}(\cdot,\boldsymbol{\cdot},v)\,h\right\|_{L^2(D)} \leqslant \left\||h|\left|\frac{\partial g}{\partial s}(\cdot,\boldsymbol{\cdot},v)\right|\right\|_{L^2(D)} \\ &\overset{(40)}{\leqslant} \||h|\,a\,|v|^\alpha\|_{L^2(D)} \leqslant \|h\|_{L^s(D)}\,\|a\|_{L^q(D)}\,\left\||v|^2\right\|_{L^p(D)} \\ &\leqslant \|h\|_{L^s(D)}\,\|a\|_{L^q(D)}\,\|v\|^\alpha_{L^{\alpha p}}\,.\end{aligned}$$

Since $\alpha > 1$, above estimate, together with Remark 15, proves the claim. □

Example 22. In Lemma 21 the assumption that $\alpha > 1$ is essential and it can not be replaced by $\alpha \geqslant 1$.

Consider the function $g(x_1, x_3, s) = g(s) = \frac{1}{2}\,|s|\,s$. Then $\frac{\partial g}{\partial s} = |s|$ and the corresponding operator G is not twice differentiable at 0.

4.2 Results applicable for the quasilinear wave equation

In the following lemmas we will discuss the differentiability properties of the mapping (cf. Lemma 27)

$$H^{2,1}(D) \ni u \longmapsto 3u^2 u_{11} + 6uu_1^2 \in L^2(D)\,.$$

Lemma 23. *There exists a constant $C > 0$ such that*

$$\|abc_{11}\|_{L^2(D)} \leqslant C\,\|a\|_{H^{2,1}(D)}\,\|b\|_{H^{2,1}(D)}\,\|c\|_{H^{2,1}(D)} \quad \big(a,b,c \in H^{2,1}(D)\big)\,.$$

Proof. For arbitrary $a, b, c \in H^{2,1}(D)$, by using the Hölder's inequality, we obtain

$$\|abc_{11}\|_{L^2(D)} \leqslant \|a\|_{L^\infty(D)}\,\|b\|_{L^\infty(D)}\,\|c_{11}\|_{L^2(D)}\,.$$

Application of the Corollary 10, gives the claim. □

Lemma 24. *The function $G\colon H^{2,1}(D) \to L^2(D)$ defined as*

$$G(u) = u^2 u_{11} \quad \big(u \in H^{2,1}(D)\big)\,, \tag{43}$$

is of the class $\mathcal{C}^2(H^{2,1}(D)\,, L^2(D))$ and

$$\mathrm{D}G(u)\,h = 2uu_{11}h + u^2 h_{11} \quad \big(u,h \in H^{2,1}(D)\big)\,, \tag{44}$$

$$\mathrm{D}^2G(u)[h,v] = 2uv_{11}h + 2u_{11}vh + 2uvh_{11} \quad \big(u,h,v \in H^{2,1}(D)\big)\,. \tag{45}$$

Proof. Note that by Lemma 23 there exists a constant $C > 0$ such that

$$\|G(u)\|_{L^2(D)} \leqslant C \|u\|^3_{H^{2,1}(D)} \quad \left(u \in H^{2,1}(D)\right),$$

which shows that the function G is well-defined.

Again applying Lemma 23 there exist a constant $C > 0$ such that

$$\|\mathrm{D}G(u)\, h\|_{L^2(D)} \leqslant C \|u\|^2_{H^{2,1}(D)} \|h\|_{H^{2,1}(D)} \quad \left(u, h \in H^{2,1}(D)\right).$$

which shows that the operator $\mathrm{D}G(u) : H^{2,1}(D) \to L^2(D)$ described in the formula (44) is well-defined for each $u \in H^{2,1}(D)$.

Now we will show that the operator $\mathrm{D}G(u)$ defined in (44) is the Fréchet derivative of the function G at the point $u \in H^{2,1}(D)$. Observe that for every functions $u, h \in H^{2,1}(D)$

$$N(u,h) = G(u+h) - G(u) - \mathrm{D}G(u)\, h = u_{11}h^2 + 2uhh_{11} + h^2 h_{11}. \tag{46}$$

By Lemma 23 there is a constant $C > 0$ such that

$$\|N(u,h)\|_{L^2(D)} \leqslant C\Big(\|u\|_{H^{2,1}(D)} \|h\|^2_{H^{2,1}(D)} + \|h\|^3_{H^{2,1}(D)}\Big) \quad \left(u, h \in H^{2,1}(D)\right).$$

Therefore one obtains

$$\frac{\|N(u,h)\|_{L^2(D)}}{\|h\|_{H^{2,1}(D)}} = C\Big(\|u\|_{H^{2,1}(D)} \|h\|_{H^{2,1}(D)} + \|h\|^2_{H^{2,1}(D)}\Big) \xrightarrow[\|h\|_{H^{2,1}(D)} \to 0]{} 0,$$

which, together with (46), shows that the operator $\mathrm{D}G(u)$ defined in (44) is the Fréchet derivative of the function G at the point $u \in H^{2,1}(D)$. For any $u, v, h \in H^{2,1}(D)$ consider the expression

$$R(u,v,h) = \mathrm{D}G(u+v)\, h - \mathrm{D}G(u)\, h = 2uv_{11}h + 2u_{11}vh + 2vv_{11}h + 2uvh_{11} + v^2 h_{11}. \tag{47}$$

By Lemma 23, we obtain that

$$\left\|\mathrm{D}^2 G(u)[h,v]\right\|_{L^2(D)} \leqslant C \|u\|_{H^{2,1}(D)} \|h\|_{H^{2,1}(D)} \|v\|_{H^{2,1}(D)} \quad \left(u, h, v \in H^{2,1}(D)\right), \tag{48}$$

which in particular implies that the operator $\mathrm{D}^2 G(u)$ is well-defined. We will show that this mapping is the Fréchet second derivative of the function G. Note that (cf. formulas (45) and (47))

$$\mathrm{D}G(u+v)\, h - \mathrm{D}G(u) - \mathrm{D}^2 G(u)[h,v] = 2vv_{11}h + v^2 h_{11}.$$

Let B denote the closed unit ball in the space $H^{2,1}(D)$. Applying Lemma 23 for the above relation, we obtain that

$$\sup_{h \in B} \left\|\mathrm{D}G(u+v)\, h - \mathrm{D}G(u) - \mathrm{D}^2 G(u)[h,v]\right\|_{L^2(D)} \leqslant C \|v\|^2_{H^{2,1}(D)}$$

which proves the claim. By inequality (48) the function G is of the class $\mathcal{C}^2(H^{2,1}(D), L^2(D))$. □

Lemma 25. *There exists a constant $C > 0$ such that*

$$\|ab_1c_1\|_{L^2(D)} \leqslant C \|a\|_{H^{2,1}(D)} \|b\|_{H^{2,1}(D)} \|c\|_{H^{2,1}(D)} \quad \big(a, b, c \in H^{2,1}(D)\big) .$$

Proof. For arbitrary $a, b, c \in H^{2,1}(D)$ we have

$$\|ab_1c_1\|_{L^2(D)} \leqslant \|a\|_{L^\infty(D)} \|b_1\|_{L^4(D)} \|c_1\|_{L^4(D)}$$

The claim follows by Lemma 8, Corollary 10 and the fact that the space $H^1(D)$ embeds into the space $L^q(D)$, for all $q \in [2, \infty)$ (cf. [1, Theorem 4.12, p. 85]). ☐

Lemma 26. *The function $G\colon H^{2,1}(D) \to L^2(D)$ defined as*

$$G(u) = uu_1^2 \quad \big(u \in H^{2,1}(D)\big) , \tag{49}$$

is of the class $\mathcal{C}^2(H^{2,1}(D) , L^2(D))$ and

$$\mathrm{D}G(u)\, h = u_1^2 h + 2uu_1h_1 \quad \big(u, h \in H^{2,1}(D)\big) , \tag{50}$$

$$\mathrm{D}^2G(u)[h, v] = 2u_1v_1h + 2uv_1h_1 + 2u_1vh_1 \quad \big(u, h, v \in H^{2,1}(D)\big) . \tag{51}$$

Proof. The functions described in the formulas (49), (50) and (51) are well-defined, indeed by Lemma 25 there exist some constant $C > 0$ such that

$$\begin{aligned} \|G(u)\|_{L^2(D)} &\leqslant C \|u\|^3_{H^{2,1}(D)} \quad \big(u \in H^{2,1}(D)\big) , \\ \|\mathrm{D}G(u)\, h\|_{L^2(D)} &\leqslant C \|u\|^2_{H^{2,1}(D)} \|h\|_{H^{2,1}(D)} \quad \big(u, h \in H^{2,1}(D)\big) , \\ \big\|\mathrm{D}^2G(u)[h, v]\big\|_{L^2(D)} &\leqslant C \|u\|_{H^{2,1}(D)} \|v\|_{H^{2,1}(D)} \|h\|_{H^{2,1}(D)} \ \big(u, v, h \in H^{2,1}(D)\big) . \end{aligned} \tag{52}$$

We will show that the operator defined in (50) is the Fréchet derivative of the function G at the point $u \in H^{2,1}(D)$. Observe that for all functions $u, h \in H^{2,1}(D)$

$$N(u, h) = G(u + h) - G(u) - \mathrm{D}G(u)\, h = hh_1^2 + 2u_1hh_1 + uh_1^2, \tag{53}$$

and by Lemma 25, there exists some constant $C > 0$ such that

$$\|N(u, h)\|_{L^2(D)} \leqslant C\Big(\|h\|^3_{H^{2,1}(D)} + \|u\|_{H^{2,1}(D)} \|h\|^2_{H^{2,1}(D)}\Big) \quad \big(u, h \in H^{2,1}(D)\big) .$$

Thus

$$\frac{\|N(u, h)\|_{L^2(D)}}{\|h\|_{H^{2,1}(D)}} = C\Big(\|u\|_{H^{2,1}(D)} \|h\|_{H^{2,1}(D)} + \|h\|^2_{H^{2,1}(D)}\Big) \xrightarrow[\|h\|_{H^{2,1}(D)} \to 0]{} 0,$$

which, together with (53), shows that the operator $\mathrm{D}G(u)$ defined in (50) is the Fréchet derivative of the function G at the point $u \in H^{2,1}(D)$.

We will show that the operator defined in (51) is the second Fréchet derivative of the function G at the point $u \in H^{2,1}(D)$. For all functions $u, h, v \in H^{2,1}(D)$, we have

$$R(u,h,v) = \mathrm{D}G(u+v)\,h - \mathrm{D}G(u)\,h - \mathrm{D}^2G(u)[h,v] = v_1^2 h + 2vv_1h_1. \tag{54}$$

Application of Lemma 25 yields

$$\|R(u,h,v)\|_{H^{2,1}(D)} \leqslant C\,\|h\|_{H^{2,1}(D)}\,\|v\|^2_{H^{2,1}(D)} \quad \big(u,h,v \in H^{2,1}(D)\big)\,, \tag{55}$$

for some constant $C > 0$ independent of the functions u, h and v. Let B denote the closed unit ball in the space $H^{2,1}(D)$. Therefore, by (54) and (55)

$$\sup_{h\in B} \big\|G(u+v)\,h - G(u) - \mathrm{D}^2G(u)[h,v]\big\|_{L^2(D)} \leqslant C\,\|v\|^2_{H^{2,1}(D)}\,,$$

which proves the claim. Inequality (52) and the fact that the expression $\mathrm{D}^2G(u)[h,v]$ is linear implies the $\mathcal{C}^2(H^{2,1}(D)\,,L^2(D))$ regularity of the mapping G. $\square$

Collecting the result of Lemma 24 and 26 we obtain out final result.

Lemma 27. *The function $G\colon H^{2,1}(D) \to L^2(D)$ defined as*

$$G(u) = 3u^2u_{11} + 6uu_1^2 \quad \big(u \in H^{2,1}(D)\big)\,, \tag{56}$$

is of the class $\mathcal{C}^2(H^{2,1}(D)\,,L^2(D))$ with

$$\begin{aligned}\mathrm{D}G(u)\,h &= 6uu_{11}h + 3u^2h_{11} + 6u_1^2h + 12uu_1h_1,\\ \mathrm{D}^2G(u)[h,v] &= 6uv_{11}h + 6u_{11}vh + 6uvh_{11} + 12u_1v_1h + 12uv_1h_1 + 12u_1vh_1\end{aligned}$$

for all $u, h, v \in H^{2,1}(D)$.

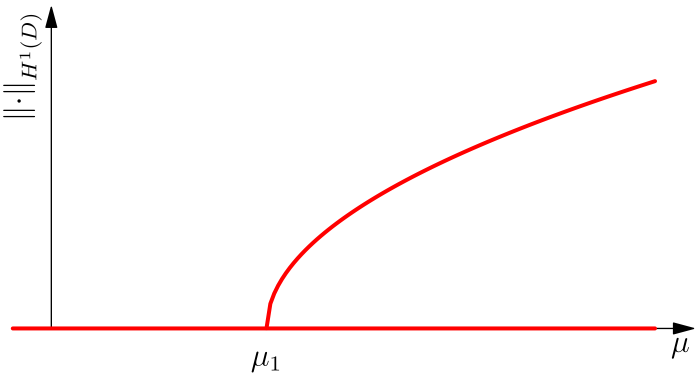


Figure 2: Qualitative sketch of the bifurcation diagram for the equation (58).

5 Semilinear wave equation

In this section we will consider the following semilinear wave equation

$$-\triangle W + V(x_3)\, W_{tt} + \mu W = g(x_3, W) \;\text{ on } \mathbb{R}^2 \times \mathbb{R}, \tag{57}$$

where $V(x_3) = \begin{cases} V_i, & |x_3| < a \\ V_o, & |x_3| > a \end{cases}$. Our goal is to prove the existence of the travelling wave solutions of (57), that is solutions of the form

$$W(x_1, x_3, t) = w(x_1 - \omega t, x_3)\,,$$

where the profile w is a $2P$-periodic function in its first variable and has some decay to 0 in its second variable. The function w and the constant ω have to satisfy the equation

$$-\triangle w + \omega^2 V(x_3)\, w_{11} + \mu w = g(x_3, w) \;\text{ on } D = (-P, P) \times \mathbb{R}. \tag{58}$$

We will work under the assumptions $\mathbf{S}$ and $\mathbf{S}_g$ described in the section 5.1. The main result of this section is the following statement.

Theorem 28. *Assume* $\mathbf{S}$ *and* $\mathbf{S}_g$. *Then there exits a non-trivial, continuously differentiable curve passing through a point* $(0, \mu_1)$

$$\left\{(w(s)\,, \mu(s)) \in H^1_{\text{odd}}(D) \times \mathbb{R} \colon s \in (-\delta, \delta)\,, w(0) = 0, \mu(0) = \mu_1\right\},$$

such that the pair $(w(s)\,, \mu(s))$ *solves* (58) *in a weak sense for all* $s \in (-\delta, \delta)$. *Moreover all solutions of the equation* (58) *in a neighbourhood of the point* $(0, \mu_1)$ *are on the trivial line or on the curve defined above.*

The proof of Theorem 28 can be found in the subsection 5.4.4.

Remark 29. Theorem 28 states the existence of a one parameter family of travelling waves, propagating with the same speed ω in the direction x_1, solving the equation (57). These travelling waves have the profile $w(s)$. Moreover they are *close* (in the sense of $H^1(D)$ norm) to the linear guided modes.

5.1 Assumptions

In this section we will work under the following assumptions on the parameters a, ω, P, V_o, V_i, μ_1 and ε.

Assumption S. We assume that

S^1 The values ω, V_i and V_o are such that

$$1-\omega^2 V_i < 0, \tag{59}$$
$$1-\omega^2 V_o > 0. \tag{60}$$

S^2 The values μ_1 and $\varepsilon > 0$ are such that

$$-\big(1-\omega^2 V_o\big)\,\frac{\pi^2}{P^2} + \varepsilon < \mu_1 < -\big(1-\omega^2 V_i\big)\,\frac{\pi^2}{P^2} - \varepsilon.$$

S^3 Consider equations

$$\frac{\sqrt{(1-\omega^2 V_o)\,\frac{k^2\pi^2}{P^2}+\mu}}{\sqrt{-(1-\omega^2 V_i)\,\frac{k^2\pi^2}{P^2}-\mu}} = \operatorname{tg}\left(\sqrt{-(1-\omega^2 V_i)\,\frac{k^2\pi^2}{P^2}-\mu}\; a\right), \tag{61}$$
$$\frac{\sqrt{(1-\omega^2 V_o)\,\frac{k^2\pi^2}{P^2}+\mu}}{\sqrt{-(1-\omega^2 V_i)\,\frac{k^2\pi^2}{P^2}-\mu}} = -\operatorname{ctg}\left(\sqrt{-(1-\omega^2 V_i)\,\frac{k^2\pi^2}{P^2}-\mu}\; a\right). \tag{62}$$

For all $\mu \in \overline{I^{\varepsilon}_{\mu_1}} = [\mu_1-\varepsilon, \mu_1+\varepsilon]$ and $k \in \mathbb{N}$ one of the equations (61) or (62) is satisfied if and only if $k = k_1$ and $\mu = \mu_1$.

S^4 The values a, ω^2, P, V_i and V_o are such that

$$\frac{\sqrt{-(1-\omega^2 V_i)}\,a}{P} = \frac{j}{n} \in \mathbb{Q}\cap(0,\infty)\,, \text{ where } \mathtt{gcd}(j,n)=1, \tag{63}$$
$$\sqrt{\frac{1-\omega^2 V_o}{-(1-\omega^2 V_i)}} \notin \left\{\operatorname{tg}\frac{jk\pi}{n}, -\operatorname{ctg}\frac{jk\pi}{n} : k = 0,\dots,n\right\}. \tag{64}$$

Moreover we will consider the following non-linearity g.

Assumption S$_g$. Assume that the function $g\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is such that:

S$_g^1$ for almost all $x_3 \in \mathbb{R}$ the mapping

$$\mathbb{R} \ni s \longmapsto g(x_3, s) \in \mathbb{R},$$

is differentiable,

$\mathbf{S}_g^2$ the mappings g and $\frac{\partial g}{\partial s}$ are Carathéodory functions (cf. Definition 143),

$\mathbf{S}_g^3$ for almost all $x_3 \in \mathbb{R}$ the mapping

$$\mathbb{R} \ni s \longmapsto g(x_3, s) \in \mathbb{R},$$

is odd,

$\mathbf{S}_g^4$ for some $\alpha > 1$ and $q \in (2, \infty]$ there exits a function $a \in L^q(\mathbb{R})$ such that

$$\left| \frac{\partial g}{\partial s}(x_3, s) \right| \leqslant a(x_3)\, |s|^\alpha \quad (x_3 \in \mathbb{R}, s \in \mathbb{R})\,.$$

We will use the notation $J_{\mu_1}^\varepsilon = I_{\mu_1}^\varepsilon \setminus \{\mu_1\}$ and we define a set

$$S^\mu = \{k \in \mathbb{N}\colon \text{equations (61) and (62) are not satisfied for } k \text{ and } \mu\}\,. \quad (65)$$

Remark 30. By Lemma 41 the condition

$$\sqrt{\frac{1 - \omega^2 V_o}{-(1 - \omega^2 V_i)}} \in \mathbb{Q} \setminus \{1\}\,,$$

implies the condition (64) in Assumption $\mathbf{S}^4$.

Remark 31. Note that assumptions $\mathbf{S}^1$ and $\mathbf{S}^2$ imply that for all $\mu \in \overline{I_{\mu_1}^\varepsilon}$ and for all $k \in \mathbb{N}$

$$-\big(1 - \omega^2 V_o\big)\, \frac{k^2\pi^2}{P^2} < \mu < -\big(1 - \omega^2 V_i\big)\, \frac{k^2\pi^2}{P^2}, \quad (66)$$

hence the expressions in the equations (61) and (62) are well-defined, for all $k \in \mathbb{N}$ and all $\mu \in \overline{I_\mu^\varepsilon}$.

Remark 32. Assumption $\mathbf{S}^2$ is equivalent to the following. The values μ_1 and $\varepsilon > 0$ are such that

$$\overline{I_{\mu_1}^\varepsilon} \subseteq \left(-\big(1 - \omega^2 V_o\big)\, \frac{\pi^2}{P^2}, -\big(1 - \omega^2 V_i\big)\, \frac{\pi^2}{P^2} \right).$$

Remark 33. Note that assumption $\mathbf{S}^3$ implies that $S^{\mu_1} = \mathbb{N} \setminus \{k_1\}$ and for all $\mu \in \overline{I_{\mu_1}^\varepsilon} \setminus \{\mu_1\}$, we have that $S^\mu = \mathbb{N}$.

Note that the parameters chosen as in Examples 44, 49 and 50 satisfy satisfy $\mathbf{S}$.

5.1.1 Some further remarks about the assumptions S

As we show in Lemma 149 the assumption $\mathbf{S}^4$ (63) in the Lemma 62 is necessary. Therefore, the assumption $\mathbf{S}^4$ (63) is also necessary for presented method of proving the Theorem 28.

This also shows, that the parameter ω^2 can not be used (at least using presented techniques) as a bifurcation parameter in the equation (58), because when ω^2 varies in any non-empty open set, the expression $\frac{\sqrt{-(1-\omega^2 V_i)}\,a}{P}$ takes some irrational values.

Assumption $\mathbf{S}^4$ (64) expresses the fact that the expression $\sqrt{\frac{1-\omega^2 V_o}{-(1-\omega^2 V_i)}}$ is not a limit point of the right hand sides of the equations (61) and (62) in $\mathbf{S}^3$.

5.2 Choice of the parameters

This section is devoted the the choice of the parameters ω, V_i, V_o, P and a in a way suitable for the applications described in the section 5.4.1, i.e. satisfying assumption $\mathbf{S}$. As we will show later, the kernel of the operator $L_\mu = -\triangle + \omega V(x_3)\frac{\partial^2}{\partial x_1^2} + \mu$ (defined later in the formula (80)), which is the linear part of the equation (58), depends on the above parameters. This section contains some examples of the parameters, for which we have a good control of the kernel of the operator L_μ (cf. Examples 44, 49 and 50).

Because the criterion, which describes the eigenvalues of the operator L_μ (cf. Lemma 69 and equation (89) for the definition of the operators $L_{\mu,k}$) contains some trigonometric equations, we will begin our considerations, with listing some results about trigonometric sequences.

5.2.1 Some remarks about the limit points of some trigonometric sequences

In this section we will study the limit points of the sequences of the form $\left(\sin\sqrt{q^2k^2\pi+\mu}\right)_{k\in\mathbb{N}}$, $\left(\cos\sqrt{q^2k^2\pi+\mu}\right)_{k\in\mathbb{N}}$ and $\left(\operatorname{tg}\sqrt{q^2k^2\pi+\mu}\right)_{k\in\mathbb{N}}$, with $q\in\mathbb{Q}$. The main results of this section is Lemma 38 and Corollary 42.

Definition 34. Let $A\subseteq\mathbb{R}^m$ and let $f_n\colon A\to\mathbb{R}$, for every $n\in\mathbb{N}$. We say that the sequence $(f_n)_{n\in\mathbb{N}}$ converges *almost uniformly* to a function $f\colon A\to\mathbb{R}$, if and only if for every compact $I\subseteq A$

$$\sup_{\mu\in I}|f_n(\mu)-f(\mu)| \xrightarrow[n\to\infty]{} 0.$$

Lemma 35. *For every $\mu\in\mathbb{R}$*

$$\lim_{k\to\infty}\sin\sqrt{k^2\pi^2+\mu}=0.$$

Moreover, the convergence is almost uniform in μ.

Proof. Observe that

$$\sqrt{k^2\pi^2+\mu} = k\pi + o\left(\frac{1}{k}\right).$$

□

Definition 36. For a sequence $(a_k)_{k\in\mathbb{N}}$ let $\#_{\mathtt{lp}}(a_k)$ denote the number of the limit points (in $\mathbb{R}\cup\{\pm\infty\}$) of the sequence $(a_k)_{k\in\mathbb{N}}$.

Remark 37. For every $n\in\mathbb{N}$ consider the sequences $(s_k^n)_{k\in\mathbb{N}}$ and $(c_k^n)_{k\in\mathbb{N}}$ defined as follows

$$\begin{aligned} s_k^n &= \sin\frac{k\pi}{n} \quad (k\in\mathbb{N}),\\ c_k^n &= \cos\frac{k\pi}{n} \quad (k\in\mathbb{N}). \end{aligned}$$

Then

$$\#_{\mathtt{lp}}(s_k^n) = \#_{\mathtt{lp}}(c_k^n) = \begin{cases} n & (n\in 2\mathbb{N}+1),\\ n+1 & (n\in 2\mathbb{N}). \end{cases}$$

Moreover the sequences $(s_k^n)_{k\in\mathbb{N}}$ and $(c_k^n)_{k\in\mathbb{N}}$ are $2n$ periodic and their ranges consist of $\#_{\mathtt{lp}}(s_k^n)$ values.

Lemma 38. *Let $n\in\mathbb{N}$ and $\mu\in\mathbb{R}$. Consider the sequences $(\widetilde{s}_k)_{k\in\mathbb{N}}$ and $(\widetilde{c}_k)_{k\in\mathbb{N}}$ defined as follows*

$$\begin{aligned} \widetilde{s}_k &= \sin\sqrt{\frac{k^2\pi^2}{n^2}+\mu} \quad (k\in\mathbb{N}),\\ \widetilde{c}_k &= \cos\sqrt{\frac{k^2\pi^2}{n^2}+\mu} \quad (k\in\mathbb{N}). \end{aligned}$$

Then

$$\#_{\mathtt{lp}}(\widetilde{s}_k) = \#_{\mathtt{lp}}(\widetilde{c}_k) = \begin{cases} n & (n\in 2\mathbb{N}+1),\\ n+1 & (n\in 2\mathbb{N}). \end{cases}$$

Moreover the limit points of the sequences $(\widetilde{s}_k)_{k\in\mathbb{N}}$ and $(\widetilde{c}_k)_{k\in\mathbb{N}}$ do not depend on μ and the convergence of convergent subsequences of them is almost uniform in μ.

Proof. Observe that for every $n\in\mathbb{N}$ and $\mu\in\mathbb{R}$

$$\sqrt{\frac{k^2\pi^2}{n^2}+\mu} = \frac{k\pi}{n} + o\left(\frac{1}{k}\right).$$

The remaining part of the proof follows from the Remark 37. □

Lemma 39. *Let $n, j \in \mathbb{N}$ and $\mu \in \mathbb{R}$. Consider the sequences $(\overline{s}_k)_{k\in\mathbb{N}}$ and $(\overline{c}_k)_{k\in\mathbb{N}}$ defined as follows*

$$\overline{s}_k = \sin\sqrt{\frac{j^2k^2\pi^2}{n^2} + \mu} \quad (k \in \mathbb{N}),$$
$$\overline{c}_k = \cos\sqrt{\frac{j^2k^2\pi^2}{n^2} + \mu} \quad (k \in \mathbb{N}).$$

Then

$$\#_{\mathtt{lp}}(\overline{s}_k) = \#_{\mathtt{lp}}(\overline{c}_k) \leqslant n + 1.$$

Moreover the limit points of the sequences $(\overline{s}_k)_{k\in\mathbb{N}}$ and $(\overline{c}_k)_{k\in\mathbb{N}}$ do not depend on μ and the convergence of convergent subsequences of them is almost uniform in μ.

Proof. Note that the sequence $\left(\sin\frac{jk\pi}{n}\right)_{k\in\mathbb{N}}$ is a subsequence of the sequence $\left(\sin\frac{k\pi}{n}\right)_{k\in\mathbb{N}}$. The claim follows from Lemma 38. In the same way, we treat the sequence $\left(\cos\frac{jk\pi}{n}\right)_{k\in\mathbb{N}}$. □

Lemma 40. *Assume that $c_1 > 0$, $c_2 > 0$ and $\mu_a, \mu_b \in \mathbb{R}$. Then*

$$\lim_{k\to\infty}\frac{\sqrt{c_1k^2 + \mu_a}}{\sqrt{c_2k^2 + \mu_b}} = \sqrt{\frac{c_1}{c_2}},$$

almost uniformly in $\mu = (\mu_a, \mu_b)$. Moreover, the sequence $\left(\frac{\sqrt{c_1k^2+\mu_a}}{\sqrt{c_2k^2+\mu_b}}\right)_{k\in N}$ is increasing [resp. decreasing] if and only if $c_1\mu_b - c_2\mu_a > 0$ [resp. $c_1\mu_b - c_2\mu_a < 0$], where

$$N = \left\{k \in \mathbb{N}\colon c_1k^2 + \mu_a \geqslant 0 \text{ and } c_2k^2 + \mu_b > 0\right\}.$$

Proof. Let $k_0 = \min N$. Consider the function $f\colon (k_0, \infty) \to \mathbb{R}$ defined by

$$f(x) = \frac{c_1x^2 + \mu_a}{c_2x^2 + \mu_b} \quad (x > k_0).$$

We have

$$f'(x) = \frac{2x(c_1\mu_b - c_2\mu_a)}{(c_2x^2 + \mu_b)^2}.$$

Observe that for all $x > k_0$ $f'(x) > 0$ [resp. $f'(x) < 0$] if and only if $c_1\mu_b - c_2\mu_a > 0$ [resp. $c_1\mu_b - c_2\mu_a < 0$]. Above, together with the fact that the function $\mathbb{R} \ni s \longmapsto \sqrt{s} \in \mathbb{R}$ is increasing, proves the claim. □

Lemma 41. *Let $a, b \in \mathbb{N}$. Then $\operatorname{tg}\frac{a\pi}{b} \in (\mathbb{R}\setminus\mathbb{Q}) \cup \{0, \pm 1\}$, unless $\frac{a}{b} \in \frac{1}{2} + \mathbb{Z}$*

Proof. For the proof, we refer to [8, Corollary 1, p. 3]. □

Corollary 42. *Let $n, j \in \mathbb{N}$ and $\mu \in \mathbb{R}$. The sequence* $\left(\operatorname{tg}\sqrt{\frac{j^2k^2\pi^2}{n^2}+\mu}\right)_{k\in\mathbb{N}}$ *has at most $n+1$ limit points and each of these limit points is equal to $\pm\infty$, ± 1, 0 or an irrational number. Moreover, the set of limit points Γ^∞ of this sequence does not depend on μ and*

$$\Gamma^\infty = \left\{\operatorname{tg}\frac{\widetilde{j}k\pi}{\widetilde{n}} : k = 0, \ldots, \widetilde{n}\right\},$$

where $\widetilde{j}, \widetilde{n} \in \mathbb{N}$ are such that $\mathtt{gcd}\left(\widetilde{j}, \widetilde{n}\right) = 1$ and $\frac{\widetilde{j}}{\widetilde{n}} = \frac{j}{n}$. Furthermore the convergence of convergent subsequence of it is almost uniform in μ.

Proof. Denote $\bar{t}_k = \operatorname{tg}\sqrt{\frac{j^2k^2\pi^2}{n^2}+\mu}$ and $t_k^n = \operatorname{tg}\frac{k\pi}{n}$. Since $\sqrt{\frac{j^2k^2\pi^2}{n^2}+\mu} = \frac{jk\pi}{n} + o\left(\frac{1}{k}\right)$ and the sequence $\left(\frac{jk\pi}{n}\right)_{k\in\mathbb{N}}$ is a subsequence of the sequence $\left(\frac{k\pi}{n}\right)_{k\in\mathbb{N}}$, the limit points of the sequence $\left(\bar{t}_k\right)_{k\in\mathbb{N}}$ are one of values of the sequence $\left(t_k^n\right)_{k\in\mathbb{N}}$. By Remark 37, there are at most $n+1$ such limit points and each of these limit points is one of the values of the sequence $\left(t_k^n\right)_{k\in\mathbb{N}}$. The rest follows from Lemma 41. □

Remark 43. Recall than for all $x \in \mathbb{R}$

$$-\operatorname{ctg} x = \operatorname{tg}\left(\frac{\pi}{2} + x\right).$$

5.2.2 Strategy of choosing the parameters in the equation (58)

In this section we will give examples of parameters satisfying assumption **S**. They are described in Examples 44, 49 and 50. Theorem 45 contains a description of a strategy of finding parameters satisfying assumption **S**.

Example 44. Choose

$$a,\ P \in (0,\infty), \tag{67}$$

$$(0,\infty) \ni \omega^2 V_o < 1 \text{ and } 1 - \omega^2 V_i = -\frac{P^2}{a^2}, \tag{68}$$

$$0 < a < \frac{\pi}{4}\frac{\sqrt{13\operatorname{tg}^2\left(\frac{1+\sqrt{13}}{2}\pi\right) - 3}}{\sqrt{1-\omega^2 V_o}\frac{\pi}{P}} = \frac{\sqrt{13\operatorname{tg}^2\left(\frac{1+\sqrt{13}}{2}\pi\right) - 3}}{4}\frac{P}{\sqrt{1-\omega^2 V_o}},$$

where $\frac{\sqrt{13\operatorname{tg}^2\left(\frac{1+\sqrt{13}}{2}\pi\right)-3}}{4} \approx 3.729627607048\ldots$. In order to find values of the constants a, P, ω, V_o and V_i satisfying above requirements one can do the following:

1. take any $\omega \in \mathbb{R}$ and $P \in (0, \infty)$,

2. take $V_o > \frac{1}{\omega^2}$ (which is equivalent to $1 - \omega^2 V_o > 0$),

3. take $a \in \left(0, \frac{\sqrt{13 \text{tg}^2\left(\frac{1+\sqrt{13}}{2}\pi\right) - 3}}{4} \frac{P}{\sqrt{1-\omega^2 V_o}}\right)$,

4. define $V_i = \frac{1}{\omega^2}\left(1 + \frac{P^2}{a^2}\right)$.

Now we will verify the assumptions $\mathbf{S}$ are satisfied. Denote

$$\begin{aligned} h_i &= -\left(1 - \omega^2 V_i\right) \frac{\pi^2}{P^2}, \\ h_o &= \left(1 - \omega^2 V_o\right) \frac{\pi^2}{P^2}. \end{aligned} \tag{69}$$

Note that (68) implies that $\mathbf{S}^1$ and moreover

$$h_o > 0 \text{ and } h_i = \frac{\pi^2}{a^2}. \tag{70}$$

Since the the limit points of the right hand sides of the equations (61) and (62), as $k \to \infty$, are 0 and $\pm\infty$, the assumption $\mathbf{S}^4$ is valid. Define

$$\begin{aligned} L(k, \mu) &= \frac{\sqrt{h_o k^2 + \mu}}{\sqrt{h_i k^2 - \mu}} \overset{(70)}{=} \frac{\sqrt{h_o k^2 + \mu}}{\sqrt{\frac{\pi^2 k^2}{a^2} - \mu}}, \\ R_{\text{tg}}(k, \mu) &= \text{tg}\left(\sqrt{h_i k^2 - \mu}\, a\right) \overset{(70)}{=} \text{tg}\sqrt{\pi^2 k^2 - a^2 \mu}, \\ R_{\text{ctg}}(k, \mu) &= -\text{ctg}\left(\sqrt{h_i k^2 - \mu}\, a\right) = \text{tg}\left(\frac{\pi}{2} + \sqrt{h_i k^2 - \mu}\, a\right) \\ &\overset{(70)}{=} \text{tg}\left(\frac{\pi}{2} + \sqrt{\pi^2 k^2 - a^2 \mu}\right) \end{aligned}$$

We will show that there there exists a unique $\mu_1 \in \left(0, \frac{3}{4}\frac{\pi^2}{a^2}\right)$ such that $L(1, \mu_1) = R_{\text{ctg}}(1, \mu_1)$. Note that

- the functions $L(1, \cdot)$ and $R_{\text{ctg}}(1, \cdot)$ are continuous on the interval $\left(0, \frac{\pi^2}{a^2}\right)$
- $R_{\text{ctg}}(1, \mu) \xrightarrow[\mu \to 0^+]{} +\infty$,
- $R_{\text{ctg}}(1, \mu) \xrightarrow[\mu \to \left(\frac{\pi^2}{a^2}\right)^-]{} -\infty$,
- $R_{\text{ctg}}(1, \mu) = 0$ if and only if $\mu = \frac{3}{4}\frac{\pi^2}{a^2}$,
- the function $\left(0, \frac{3}{4}\frac{\pi^2}{a^2}\right) \ni \mu \longmapsto R_{\text{ctg}}(1, \mu) \in \mathbb{R}$ is decreasing,

- $L(1,0) = \frac{a}{\pi}\sqrt{h_o} > 0$,
- the functions $\left(0, \frac{\pi^2}{a^2}\right) \ni \mu \longmapsto L(1,\mu) \in \mathbb{R}$ is increasing,
- $L(1,\mu) \xrightarrow[\mu \to \frac{\pi^2}{a^2}]{} \infty$,

which proves the claim. Moreover note that $\mu_1 \in \left(0, \frac{3}{4}\frac{\pi^2}{a^2}\right) \overset{(70)}{\subseteq} (-h_o, h_i)$, hence one can choose sufficiently small $\varepsilon > 0$ such that $\mathbf{S}^2$ holds true. In order to verify $\mathbf{S}^3$, we will show that the equations (61) and (62) have solutions different that $k = 1$ and $\mu = \mu_1$, for μ sufficiently close to μ_1. Note that $L(1,\mu_1) \neq R_{\mathrm{tg}}(1,\mu_1)$. Since $\mu_1 \in \left(0, \frac{3}{4}\frac{\pi^2}{a^2}\right)$, then for all μ sufficiently close to μ_1 and all $k \in \mathbb{N}$

$$\frac{\pi}{2} + \pi(k-1) < \frac{\pi}{2} + \pi k\sqrt{1 - \frac{3}{4k^2}} < \frac{\pi}{2} + \pi k\sqrt{1 - \frac{\mu}{h_i k^2}} < \frac{\pi}{2} + \pi k,$$

and the expression $\frac{\pi}{2} + \pi k\sqrt{1 - \frac{\mu}{h_i k^2}}$, gets closer to $\frac{\pi}{2} + \pi k$, for as $k \to \infty$. Hence, for all μ sufficiently close to μ_1

$$R_{\mathrm{ctg}}(k-1,\mu) < R_{\mathrm{ctg}}(k,\mu) \xrightarrow[k\to\infty]{} +\infty. \tag{71}$$

By Lemma 40 the sequence $(L(k,\mu))_{k\in\mathbb{N}}$ is decreasing for all μ in some open neighbourhood of μ_1.

Since the function $\left(0, \frac{\pi^2}{a^2}\right) \ni \mu \longmapsto R_{\mathrm{ctg}}(1,\mu)$ is decreasing, the function $\left(0, \frac{\pi^2}{a^2}\right) \ni \mu \longmapsto L(1,\mu)$ is increasing and (71) there exits $\varepsilon_1 > 0$ such that for all $\mu \in (\mu_1 - \varepsilon_1, \mu_1)$ and all $k \in \mathbb{N}$

$$L(k+1,\mu) < L(k,\mu) \leqslant L(1,\mu) < R_{\mathrm{ctg}}(1,\mu) \leqslant R_{\mathrm{ctg}}(k,\mu) < R_{\mathrm{ctg}}(k+1,\mu).$$

Note that

- the functions $L(2,\cdot)$ and $R_{\mathrm{ctg}}(2,\cdot)$ are continuous on the interval $\left(0, \frac{3}{4}\frac{\pi^2}{a^2}\right)$,
- $R_{\mathrm{ctg}}(2,\mu) \xrightarrow[\mu\to 0^+]{} +\infty$,
- $R_{\mathrm{ctg}}\left(2, \frac{3}{4}\frac{\pi^2}{a^2}\right) = \mathrm{tg}\left(\frac{1+\sqrt{13}}{2}\pi\right) \approx 1.4019283574321805\ldots$,
- the function $\left(0, \frac{3}{4}\frac{\pi^2}{a^2}\right) \ni \mu \longmapsto R_{\mathrm{ctg}}(2,\mu) \in \mathbb{R}$ is decreasing,
- $L(2,0) = \frac{a}{\pi}\sqrt{h_o}$,
- the functions $\left(0, \frac{\pi^2}{a^2}\right) \ni \mu \longmapsto L(2,\mu) \in \mathbb{R}$ is increasing,

- $L\Big(2, \frac{3}{4}\frac{\pi^2}{a^2}\Big) = \frac{\sqrt{13}}{13\pi}\sqrt{3\pi^2 + 16a^2h_o}$.

Therefore, if $a < \frac{\pi}{4}\frac{\sqrt{13R_{\rm ctg}\left(2,\frac{3}{4}\frac{\pi^2}{a^2}\right)^2-3}}{\sqrt{h_o}}$, then for all $\mu \in \Big(0, \frac{3}{4}\frac{\pi^2}{a^2}\Big)$

$$L(2,\mu) < R_{\rm ctg}\,(2,\mu)\,.$$

Therefore, there exists $\varepsilon_2 > 0$ such that for all $\mu \in (\mu_1, \mu_1 + \varepsilon_2)$ and all $k \geqslant 2$

$$L(k+1,\mu) < L(k,\mu) \leqslant L(2,\mu) < R_{\rm ctg}\,(2,\mu) \leqslant R_{\rm ctg}\,(k,\mu) < R_{\rm ctg}\,(k+1,\mu)\,.$$

Hence, by taking $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$, we have that for all $\mu \in (\mu_1 - \varepsilon, \mu_1 + \varepsilon)$ and for all $k \geqslant 2$

$$L(k,\mu) \neq R_{\rm ctg}\,(k,\mu)\,.$$

We have shown that the equation (62) in $\mathbf{S}^3$ is not satisfied for all $k \geqslant 2$ and all values μ in a neighbourhood of μ_1. Moreover, it is satisfied for $k = 1$ if and only if $\mu = \mu_1$, for all μ sufficiently close to μ_1.

It remains to show, that the equation (61) in $\mathbf{S}^3$ is never satisfied for all $k \in \mathbb{N}$ and μ sufficiently close to μ_1. Since $R_{\rm tg}\,(1,\mu_1) \neq R_{\rm ctg}\,(1,\mu_1)$, then that for all μ sufficiently close to μ_1

$$R_{\rm tg}\,(1,\mu) \neq R_{\rm ctg}\,(1,\mu)\,,$$

therefore, for all values μ sufficiently close to μ_1

$$L(1,\mu) \neq R_{\rm ctg}\,(1,\mu)\,. \tag{72}$$

By applying the mean value theorem for the mapping $x \to \sqrt{x}$ on the interval $\Big[1 - \frac{\mu}{h_ik^2}, 1\Big]$, we obtain that

$$\sqrt{h_ik^2 - \mu}\,a = a\sqrt{h_i}\,k\sqrt{1 - \frac{\mu}{h_ik^2}} = a\sqrt{h_i}\,k\left(1 - \frac{1}{2\sqrt{\xi_\mu}}\frac{\mu}{h_ik^2}\right),$$

for some $\xi_\mu \in \Big[1 - \frac{\mu}{h_ik^2}, 1\Big]$. Then

$$\sqrt{h_ik^2 - \mu}\,a = \pi k - \frac{1}{2\sqrt{\xi_\mu}}\frac{a\mu}{\sqrt{h_i}\,k}, \text{ for some } \xi_\mu \in \left[1 - \frac{\mu}{h_ik^2}, 1\right]. \tag{73}$$

For $k \geqslant 2$, we have

$$\begin{aligned} 0 < \frac{1}{2\sqrt{\xi_\mu}}\frac{a\mu_1}{\sqrt{h_i}\,k} &\leqslant \frac{1}{2\sqrt{1 - \frac{\mu_1}{h_ik^2}}}\frac{\frac{3}{4}a\frac{\pi^2}{a^2}}{\frac{\pi}{a}k} \\ &= \frac{3\pi}{8k}\frac{1}{\sqrt{1 - \frac{\mu_1}{h_ik^2}}} \leqslant \frac{\pi}{2k}\frac{1}{\sqrt{1 - \frac{3}{4k^2}}} \\ &\overset{k\geqslant 2}{\leqslant} \frac{\pi}{2}\frac{1}{\sqrt{4 - \frac{3}{4}}} < \frac{\pi}{2} \end{aligned} \tag{74}$$

Relations (73) and (73), yield that for $k \geqslant 2$ and all μ sufficiently close to μ_1

$$R_{\mathrm{tg}}(k,\mu) < 0,$$

therefore, using (72), we have that for all $k \in \mathbb{N}$ and all μ in an open neighbourhood of μ_1

$$L(k,\mu) \neq R_{\mathrm{tg}}(k,\mu),$$

which shows that the equation (61) has no solutions for all $k \in \mathbb{N}$ and all μ sufficiently close to μ_1. Hence the assumption $\mathbf{S}^3$ holds true.

Now we will describe a more general strategy of finding parameters satisfying assumption **S**. Denote $\alpha_o = \frac{1-\omega^2 V_o}{P^2}$ and $\alpha_i = -\frac{1-\omega^2 V_i}{P^2}$. Define

$$\begin{aligned} L(k,\mu) &= \frac{\sqrt{\alpha_o k^2\pi^2 + \mu}}{\sqrt{\alpha_i k^2\pi^2 - \mu}}, \\ R_{\mathrm{tg}}(k,\mu) &= \operatorname{tg}\left(\sqrt{\alpha_i k^2\pi^2 - \mu}\, a\right), \\ R_{\mathrm{ctg}}(k,\mu) &= -\operatorname{ctg}\left(\sqrt{\alpha_i k^2\pi^2 - \mu}\, a\right). \end{aligned}$$

From the point of view of assumption **S** (cf. $\mathbf{S}^3$) it is important to describe for which values of $k \in \mathbb{N}$ the condition

$$L(k,\mu) = R_{\mathrm{tg}}(k,\mu) \text{ or } L(k,\mu) = R_{\mathrm{ctg}}(k,\mu),$$

holds true, where μ varies in a bounded interval, which does not depend on k. We will present a strategy of reducing above problem to finitely many values of k.

Theorem 45. *Assume that*

$$\sqrt{\alpha_i}\, a \in \mathbb{Q}, \tag{75}$$

$$\sqrt{\frac{\alpha_o}{\alpha_i}} \notin \left\{ \operatorname{tg}\frac{jk\pi}{n}, -\operatorname{ctg}\frac{jk\pi}{n} : k = 0,\dots,n \right\}, \tag{76}$$

where $j,n \in \mathbb{N}$ are such that $\texttt{gcd}(j,n) = 1$ and $\frac{j}{k} = \sqrt{\alpha_i}\, a$, and let $I \subseteq \mathbb{R}$ be a bounded interval. There exists a number $K \in \mathbb{N}$, which can be computed explicitly, such that for all $k \geqslant K$ and all $\mu \in I$

$$L(k,\mu) \neq R_{\mathrm{tg}}(k,\mu) \text{ and } L(k,\mu) \neq R_{\mathrm{ctg}}(k,\mu).$$

Proof. Let $\Gamma_{\mathrm{tg}} \subseteq \mathbb{R}\cup\{\infty\}$ and $\Gamma_{\mathrm{ctg}} \subseteq \mathbb{R}\cup\{\infty\}$ denote the sets of limit points of the sequences $\left(R_{\mathrm{tg}}(k,\mu)\right)_{k\in\mathbb{N}}$ and $\left(R_{\mathrm{ctg}}(k,\mu)\right)_{k\in\mathbb{N}}$ respectively. By Corollary 42 and assumption (75) we get that the sets Γ_{tg} and Γ_{ctg} are finite, μ independent and

$$\begin{aligned}\Gamma_{\text{tg}} &= \left\{ \operatorname{tg} \frac{jk\pi}{n} : k = 0, \dots, n \right\}, \\ \Gamma_{\text{ctg}} &= \left\{ -\operatorname{ctg} \frac{jk\pi}{n} : k = 0, \dots, n \right\}.\end{aligned}$$

Lemma 40 implies that the sequence $L(k,\mu) \xrightarrow[k\to\infty]{} \sqrt{\frac{\alpha_o}{\alpha_i}}$. Using assumption (76), we have that $\sqrt{\frac{\alpha_o}{\alpha_i}} \notin \Gamma_{\text{tg}} \cup \Gamma_{\text{ctg}}$. Let $\delta > 0$ be such that $\sqrt{\frac{\alpha_o}{\alpha_i}} \notin (\Gamma_{\text{tg}} \cup \Gamma_{\text{ctg}})_\delta$, where $X_\delta = \bigcup_{x\in X}(x-\delta, x+\delta)$. Since $I \subseteq \mathbb{R}$ is a bounded interval then, there exists $K \in \mathbb{N}$ such that for all $k \geqslant K$ and for all $\mu \in I$

$$L(k,\mu) \neq R_{\text{tg}}(k,\mu) \text{ and } L(k,\mu) \neq R_{\text{ctg}}(k,\mu).$$

Now we will show, that the value K can be computed explicitly. Without loss of generality, we may assume that $I \subseteq (-\infty, 0]$ or $I \subseteq [0,\infty)$ (Otherwise consider $I = I^+ \cup I^-$, where $I^\pm \subseteq \mathbb{R}^\pm \cup \{0\}$.) Let $I = (\mu_a, \mu_b)$, where

$$\mu_a \mu_b \geqslant 0, \tag{77}$$

Note that[4], for every $k \in \mathbb{N}$ the function $\mathbb{R} \ni \mu \longmapsto L(k,\mu) \in \mathbb{R}$ is increasing, therefore for all $k \in \mathbb{N}$ and for all $\mu \in I$

$$L(k,\mu_a) < L(k,\mu) < L(k,\mu_b).$$

By Lemma 40 and by assumption (77) the sequence $(L(k,\mu))_{k\in\mathbb{N}}$ is monotone for all $\mu \in I$. Therefore, the sequence $\left(\left|L(k,\mu) - \sqrt{\frac{\alpha_o}{\alpha_i}}\right|\right)_{k\in\mathbb{N}}$ is decreasing (to 0) for all $\mu \in I$. In case, when the sequence $(L(k,\mu))_{k\in\mathbb{N}}$ is increasing, we have that for all $\mu \in I$ and all $k \in \mathbb{N}$

$$0 < \sqrt{\frac{\alpha_o}{\alpha_i}} - L(k,\mu) \leqslant \sqrt{\frac{\alpha_o}{\alpha_i}} - L(k,\mu_a),$$

on the other hand, if the sequence $(L(k,\mu))_{k\in\mathbb{N}}$ is decreasing, we have that for all $\mu \in I$ and all $k \in \mathbb{N}$

$$0 < L(k,\mu) - \sqrt{\frac{\alpha_o}{\alpha_i}} \leqslant L(k,\mu_b) - \sqrt{\frac{\alpha_o}{\alpha_i}}.$$

Therefore, for every $\varepsilon > 0$ there exists $K_1 \in \mathbb{N}$, which can be computed explicitly, such that for all $\mu \in I$ and for all $k \geqslant K_1$

$$\left|\sqrt{\frac{\alpha_o}{\alpha_i}} - L(k,\mu)\right| < \varepsilon. \tag{78}$$

[4] $\frac{\mathrm{d}}{\mathrm{d}\mu}\left(\frac{\alpha_o k^2\pi^2+\mu}{\alpha_i k^2\pi^2-\mu}\right) = \frac{k^2\pi^2(\alpha_i+\alpha_o)}{(\alpha_i k^2\pi^2-\mu)^2}$.

If $I \subseteq (0,\infty)$, then we have that for all $k \in \mathbb{N}$ and all $\mu \in I$

$$0 < a\sqrt{\alpha_i}\,k\pi - \sqrt{\pi^2 k^2 \alpha_i - \mu}\,a < a\sqrt{\alpha_i}\,k\pi - \sqrt{\pi^2 k^2 \alpha_i - \mu_a}\,a \xrightarrow[k\to\infty]{} 0,$$

in a monotone way. On the other hand, if $I \subseteq (-\infty, 0)$, then we have that for all $k \in \mathbb{N}$ and all $\mu \in I$

$$0 < \sqrt{\pi^2 k^2 \alpha_i - \mu}\,a - a\sqrt{\alpha_i}\,k\pi < \sqrt{\pi^2 k^2 \alpha_i - \mu_b}\,a - a\sqrt{\alpha_i}\,k\pi \xrightarrow[k\to\infty]{} 0,$$

Hence for every $\varepsilon > 0$, there exists $K_2 \in \mathbb{N}$, which can be computed explicitly, such that for all $k \geqslant K$ and all $\mu \in I$

$$\left|\sqrt{\alpha_i k^2 \pi^2 - \mu}\,a - a\sqrt{\alpha_i}\,k\pi\right| < \varepsilon. \tag{79}$$

Let $d_{\mathrm{tg}} < \operatorname{dist}\left(\sqrt{\frac{\alpha_o}{\alpha_i}}, \Gamma_{\mathrm{tg}}\right)$. Take $\varepsilon = \frac{d_{\mathrm{tg}}}{2}$. Let $K_1 \in \mathbb{N}$ be such as (78). Consider $M_{\mathrm{tg}} = \frac{d_{\mathrm{tg}}}{2} + \max\left\{\max\{|x| \in \mathbb{R}\colon x \in \Gamma_{\mathrm{tg}} \setminus \{\pm\infty\}\}, \sqrt{\frac{\alpha_o}{\alpha_i}}\right\}$ and let $\widetilde{\mathrm{tg}} : \mathbb{R} \to \mathbb{R}$ be defined as follows

$$\widetilde{\mathrm{tg}}\, x = \begin{cases} \mathrm{tg}\, x, & \text{if } \mathrm{tg}\, x \in (-M_{\mathrm{tg}}, M_{\mathrm{tg}}), \\ M_{\mathrm{tg}}, & \text{if } \mathrm{tg}\, x \geqslant M_{\mathrm{tg}}, \\ -M_{\mathrm{tg}}, & \text{if } \mathrm{tg}\, x \leqslant -M_{\mathrm{tg}}. \end{cases}$$

Note that the function $\widetilde{\mathrm{tg}}$ is Lipschitz continuous on each of the intervals $\left(-\frac{\pi}{2}+\nu, \frac{\pi}{2}+\nu\right)$, $\nu \in \mathbb{Z}$, with the constant $L_{\mathrm{tg}} = 1 + M_{\mathrm{tg}}^2$. Observe that for every $k \in \mathbb{N}$, there exists $g_k \in \Gamma_{\mathrm{tg}}$ such that $g_k = \mathrm{tg}\left(a\sqrt{\alpha_i}\,k\pi\right)$. Let $K_2 \in \mathbb{N}$ be as in (79) (with $\varepsilon = \frac{d}{2L_{\mathrm{tg}}}$). Let $K_{\mathrm{tg}} = \max(K_1, K_2)$. For all $\mu \in I$ and for all $k \geqslant K_{\mathrm{tg}}$ we have

$$\begin{aligned}\left|\widetilde{\mathrm{tg}}\left(\sqrt{\alpha_i k^2\pi^2 - \mu}\,a\right) - \widetilde{\mathrm{tg}}\,(a\sqrt{\alpha_i}\,k\pi)\right| &\leqslant L_{\mathrm{tg}}\left|\sqrt{\alpha_i k^2 \pi^2 - \mu}\,a - a\sqrt{\alpha_i}\,k\pi\right| \\ &\overset{(79)}{<} \frac{d_{\mathrm{tg}}}{2}.\end{aligned}$$

Therefore, for all $k \geqslant K_{\mathrm{tg}}$ and $\mu \in I$

$$\operatorname{dist}\left(R_{\mathrm{tg}}(k,\mu), \Gamma_{\mathrm{tg}} \cup (-\infty, -M) \cup (M, \infty)\right) < \frac{d_{\mathrm{tg}}}{2},$$

which together with (78), shows that for all $k \geqslant K_{\mathrm{tg}}$ and all $\mu \in I$

$$L(k,\mu) \neq R_{\mathrm{tg}}(k,\mu).$$

Analogously we treat the R_{ctg} case. □

Remark 46. By Lemma 41 the condition

$$\sqrt{\frac{\alpha_o}{\alpha_i}} \in \mathbb{Q} \setminus \{0, \pm 1\},$$

implies assumption (76) in the Theorem 45.

In the following Example 47 and 48 we will illustrate the strategy described in the Theorem 45.

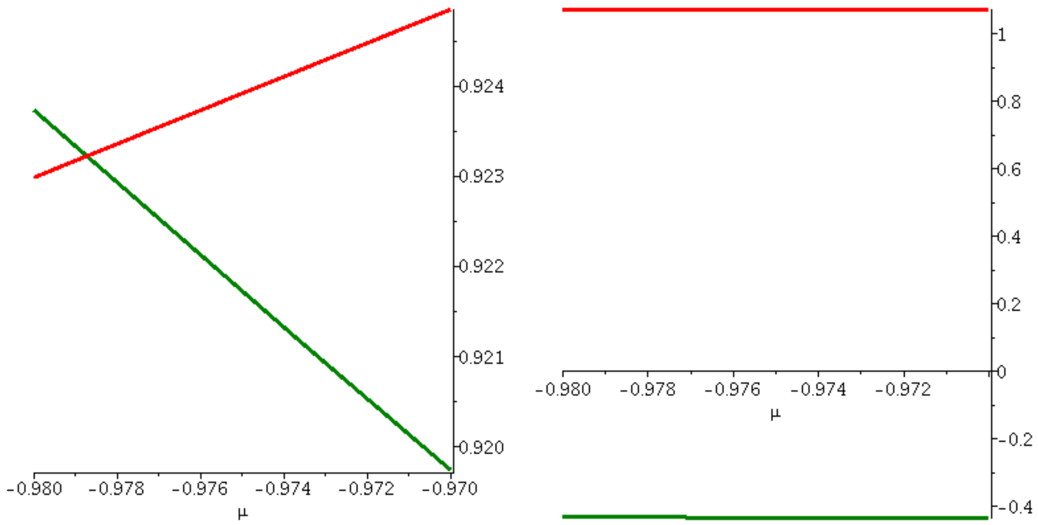
Example 47. Using the notations from the Theorem 45, let $a = 1$, $\alpha_i = \frac{4}{9}$, $\alpha_o = \frac{9}{16}$. Then we have

- $\sqrt{\alpha_i}\, a = \frac{2}{3}$,
- $\lim_{k\to\infty} L(k,\mu) = \frac{9}{8}$,
- $\Gamma_{\mathrm{tg}} = \{-1.7320508\ldots, 0, 1.7320508\ldots\}$,
- $\Gamma_{\mathrm{ctg}} = \{-0.5773502\ldots, 0.5773502\ldots, \pm\infty\}$,
- $\mathrm{dist}\left(\sqrt{\frac{\alpha_o}{\alpha_i}}, \Gamma_{\mathrm{tg}}\right) \approx 0.6070508076\ldots$,
- $\mathrm{dist}\left(\sqrt{\frac{\alpha_o}{\alpha_i}}, \Gamma_{\mathrm{ctg}}\right) \approx 0.5476497308\ldots$,
- define μ_1 such that $L(1,\mu_1) = R_{\mathrm{ctg}}(1,\mu_1)$, for example consider $\mu_1 \approx -0.9787174\ldots$,
- take $\mu_a = -0.98$, $\mu_b = -0.97$,
- note that $(\mu_a,\mu_b) \subseteq (-\alpha_o\pi^2, \alpha_i\pi^2) = (-5.551652\ldots, 4.386490\ldots)$
- take $d_{\mathrm{tg}} = 0.6$ and $d_{\mathrm{ctg}} = 0.54$,
- $M_{\mathrm{tg}} = 2.032050808\ldots$, $M_{\mathrm{ctg}} = 1.395$,
- $K_1^{\mathrm{tg}} = 1$, $K_1^{\mathrm{ctg}} = 1$,
- $L_{\mathrm{tg}} \approx 5.129230485\ldots$, $L_{\mathrm{ctg}} = 2.946025$,
- $K_2^{\mathrm{tg}} = 4$, $K_2^{\mathrm{ctg}} = 3$,
- for all $\mu \in (\mu_a,\mu_b)$, $R_{\mathrm{tg}}(1,\mu) < 0$.

By Theorem 45

- the equation $L(k,\mu) = R_{\mathrm{ctg}}(k,\mu)$ has no solutions for $k \geqslant 3$ and $\mu \in (\mu_a,\mu_b)$,
- the equation $L(k,\mu) = R_{\mathrm{tg}}(k,\mu)$ has no solutions for $k \geqslant 4$ and $\mu \in (\mu_a,\mu_b)$.

Fig. 3 and 4 show the plots of the functions $L(\mu,k)$, $\mathbb{R}_{\mathrm{tg}}(\mu,k)$ and $\mathbb{R}_{\mathrm{ctg}}(\mu,k)$ for *small* values of k and $\mu \in (\mu_a,\mu_b)$. Observe, that the equation $L(k,\mu) = R_{\mathrm{tg}}(k,\mu)$ has no solutions for all $k \in \mathbb{N}$ and $\mu \in (\mu_a,\mu_b)$ and that $L(k,\mu) = R_{\mathrm{ctg}}(k,\mu)$ if and only of $k = 1$ and $\mu = \mu_1$.

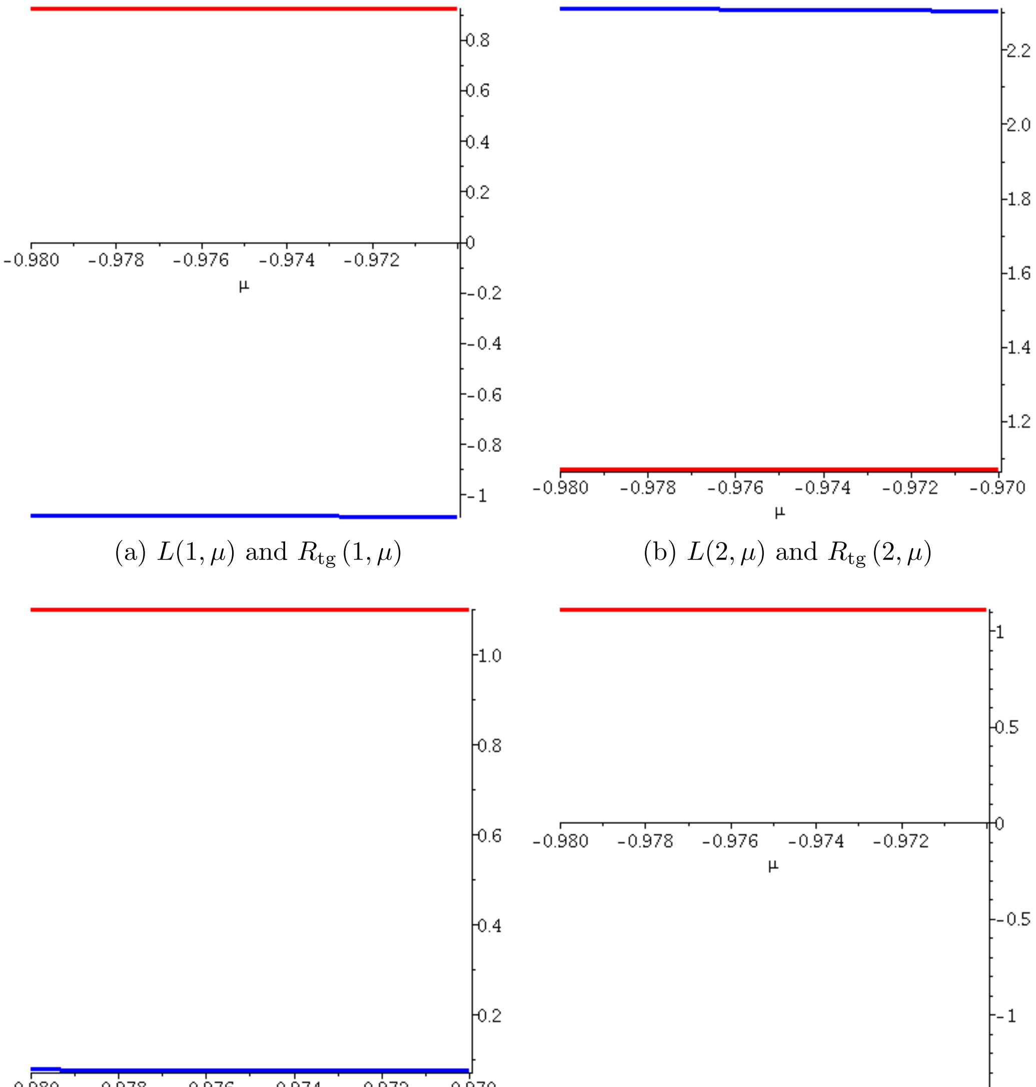


(a) $L(1,\mu)$ and $R_{\text{ctg}}(1,\mu)$ (b) $L(2,\mu)$ and $R_{\text{ctg}}(2,\mu)$

Figure 3: Plots of the functions $L(k,\mu)$, $R_{\text{ctg}}(k,\mu)$, for $k \in \{1,2\}$ and $\mu \in (\mu_a,\mu_b)$, with parameters as described in Example 47. The function L is marked in red and R_{ctg} in green.

Example 48. Using the notations from the Theorem 45, let $a = 1$, $\alpha_i = \frac{9}{25}$, $\alpha_o = \sqrt{2}$. Then we have

- $\sqrt{\alpha_i}\, a = \frac{4}{5}$,
- $\lim_{k\to\infty} L(k,\mu) = \frac{5}{3} \cdot 2^{\frac{1}{4}} \approx 1.982011858\ldots$,
- $\Gamma_{\text{tg}} = \{0, \pm 3.077683537\ldots, \pm 0.726542528\ldots\}$,
- $\Gamma_{\text{ctg}} = \{\pm 0.3249196962\ldots, \pm 1.37638192\ldots, \infty\}$,
- $\text{dist}\left(\sqrt{\frac{\alpha_o}{\alpha_i}}, \Gamma_{\text{tg}}\right) \approx 1.095671679\ldots$,
- $\text{dist}\left(\sqrt{\frac{\alpha_o}{\alpha_i}}, \Gamma_{\text{ctg}}\right) \approx 0.6056299379\ldots$,
- define μ_1 such that $L(1,\mu_1) = R_{\text{tg}}(1,\mu_1)$, for example consider $\mu_1 \approx 1.95547737271124\ldots$,
- take $\mu_a = 1.955$, $\mu_b = 1.956$,
- note that $(\mu_a,\mu_b) \subseteq (-\alpha_o\pi^2, \alpha_i\pi^2) = (-3.5530\ldots, 13.9577\ldots)$,
- take $d_{\text{tg}} = 1.09$ and $d_{\text{ctg}} = 0.605$,
- $K_1^{\text{tg}} = 2$, $K_1^{\text{ctg}} = 2$,
- $L_{\text{tg}} \approx 14.12383601\ldots$, $L_{\text{ctg}} = 6.218994431\ldots$,



(a) $L(1, \mu)$ and $R_{\text{tg}}(1, \mu)$

(b) $L(2, \mu)$ and $R_{\text{tg}}(2, \mu)$

(c) $L(3, \mu)$ and $R_{\text{tg}}(3, \mu)$

(d) $L(4, \mu)$ and $R_{\text{tg}}(4, \mu)$

Figure 4: Plots of the functions $L(k, \mu)$, $R_{\text{tg}}(k, \mu)$, for $k \in \{1, 2, 3, 4\}$ and $\mu \in (\mu_a, \mu_b)$, with parameters as described in Example 47. The function L is marked in red and R_{tg} in blue.

- $K_2^{\mathrm{tg}} = 14$, $K_2^{\mathrm{ctg}} = 11$.

By Theorem 45

- the equation $L(k, \mu) = R_{\mathrm{ctg}}(k, \mu)$ has no solutions for $k \geqslant 11$ and $\mu \in (\mu_a, \mu_b)$,
- the equation $L(k, \mu) = R_{\mathrm{tg}}(k, \mu)$ has no solutions for $k \geqslant 14$ and $\mu \in (\mu_a, \mu_b)$.

Fig. 5 presents the graphs of the functions $L(1, \mu)$ and $R_{\mathrm{tg}}(1, \mu)$ for $\mu \in (\mu_a, \mu_b)$. In order to exclude other intersection points for small values of k we compute that

- the range of the function $L(k, (\mu_a, \mu_b))$, for $k \in \{1, 2, \ldots, 13\}$:

$$\begin{aligned}
L(1, \mu) &\in [3.155558\ldots, 3.156645\ldots],\\
L(2, \mu) &\in [2.171275\ldots, 2.171382\ldots],\\
L(3, \mu) &\in [2.061382\ldots, 2.061424\ldots],\\
L(4, \mu) &\in [2.025806\ldots, 2.025829\ldots],\\
L(5, \mu) &\in [2.009796\ldots, 2.009811\ldots],\\
L(6, \mu) &\in [2.001216\ldots, 2.001226\ldots],\\
L(7, \mu) &\in [1.996081\ldots, 1.996089\ldots],\\
L(8, \mu) &\in [1.992764\ldots, 1.992770\ldots],\\
L(9, \mu) &\in [1.990497\ldots, 1.990501\ldots],\\
L(10, \mu) &\in [1.988879\ldots, 1.988882\ldots],\\
L(11, \mu) &\in [1.987683\ldots, 1.987686\ldots],\\
L(12, \mu) &\in [1.986775\ldots, 1.986777\ldots],\\
L(13, \mu) &\in [1.986069\ldots, 1.986071\ldots],
\end{aligned}$$

- the range of the function $R_{\mathrm{tg}}(k,(\mu_a,\mu_b))$, for $k \in \{1,2,\dots,13\}$:

$$\begin{aligned}
R_{\mathrm{tg}}(1,\mu) &\in [3.153812\dots,3.158147\dots],\\
R_{\mathrm{tg}}(2,\mu) &\in [0.375601\dots,0.375764\dots],\\
R_{\mathrm{tg}}(3,\mu) &\in [-1.037904\dots,-1.037715\dots],\\
R_{\mathrm{tg}}(4,\mu) &\in [2.096830\dots,2.097194\dots],\\
R_{\mathrm{tg}}(5,\mu) &\in [-0.104727\dots,-0.104673\dots],\\
R_{\mathrm{tg}}(6,\mu) &\in [-4.322421\dots,-4.321544\dots],\\
R_{\mathrm{tg}}(7,\mu) &\in [0.618606\dots,0.618659\dots],\\
R_{\mathrm{tg}}(8,\mu) &\in [-0.830923\dots,-0.830867\dots],\\
R_{\mathrm{tg}}(9,\mu) &\in [2.563717\dots,2.563941\dots],\\
R_{\mathrm{tg}}(10,\mu) &\in [-0.052003\dots,-0.051976\dots],\\
R_{\mathrm{tg}}(11,\mu) &\in [-3.656787\dots,-3.656440\dots],\\
R_{\mathrm{tg}}(12,\mu) &\in [0.662396\dots,0.662428\dots],\\
R_{\mathrm{tg}}(13,\mu) &\in [-0.789430\dots,-0.789396\dots],
\end{aligned}$$

- the range of the function $R_{\mathrm{ctg}}(k,(\mu_a,\mu_b))$, for $k \in \{1,2,\dots,10\}$:

$$\begin{aligned}
R_{\mathrm{ctg}}(1,\mu) &\in [-0.317077\dots,-0.316641\dots],\\
R_{\mathrm{ctg}}(2,\mu) &\in [-2.662401\dots,-2.661247\dots],\\
R_{\mathrm{ctg}}(3,\mu) &\in [0.963480\dots,0.963656\dots],\\
R_{\mathrm{ctg}}(4,\mu) &\in [-0.476910\dots,-0.476828\dots],\\
R_{\mathrm{ctg}}(5,\mu) &\in [9.548633\dots,9.553580\dots],\\
R_{\mathrm{ctg}}(6,\mu) &\in [0.231352\dots,0.231399\dots],\\
R_{\mathrm{ctg}}(7,\mu) &\in [-1.616537\dots,-1.616399\dots],\\
R_{\mathrm{ctg}}(8,\mu) &\in [1.203481\dots,1.203562\dots],\\
R_{\mathrm{ctg}}(9,\mu) &\in [-0.390059\dots,-0.390025\dots],\\
R_{\mathrm{ctg}}(10,\mu) &\in [19.229690\dots,19.239558\dots].
\end{aligned}$$

  Observe, that the equation $L(k,\mu) = R_{\mathrm{ctg}}(k,\mu)$ has no solutions for all $k \in \mathbb{N}$ and $\mu \in (\mu_a,\mu_b)$ and that $L(k,\mu) = R_{\mathrm{tg}}(k,\mu)$ if and only of $k = 1$ and $\mu = \mu_1$.

Now, using above Examples 47 and 48 we will present another suitable set of parameters.

Example 49. Let $a = 1$, and ω, P, V_o and V_i be such that

$$\frac{1-\omega^2 V_o}{P^2} = \alpha_o = \frac{9}{16} \text{ and } -\frac{1-\omega^2 V_i}{P^2} = \alpha_i = \frac{4}{9}.$$

As shown in the Example 47 such constants fulfil all of the requirements described in the Assumption **S**.

Figure 5: Plots of the functions $L(1,\mu)$ and $R_{\rm tg}(1,\mu)$ for $\mu \in (\mu_a, \mu_b)$, with parameters as described in Example 48. The function L is marked in red, $R_{\rm tg}$.

Example 50. Let $a = 1$, and ω, P, V_o and V_i be such that

$$\frac{1-\omega^2 V_o}{P^2} = \alpha_o = \sqrt{2} \text{ and } -\frac{1-\omega^2 V_i}{P^2} = \alpha_i = \frac{9}{25}.$$

As shown in the Example 48 such constants fulfil all of the requirements described in the Assumption **S**.

5.3 About the linear part

5.3.1 About the equation $L_\mu w = f$

We formally introduce a family of operators L_μ defined by the formula

$$L_\mu w = -w_{33} - \big(1-\omega^2 V(x)\big)\, w_{11} + \mu w \quad (\mu \in \mathbb{R})\,. \tag{80}$$

Note that the expression $L_\mu w$ is the left hand side of the equation (58). Here we will treat the linear version of the equation (58), namely we will study the solvability in the space $H^1_{\rm odd}(D)$ of the equation

$$L_\mu w = f \text{ on D}, \tag{81}$$

where $f \in L^2_{\rm odd}(D)$ is a given function. Note that the equation (81) is a linear variant of the equation (58). The spaces $H^1_{\rm odd}(D)$ and $L^2_{\rm odd}(D)$ are defined in section 2.2. We will look for *weak solutions* defined as follows

Definition 51. Let $f \in L^2_{\rm odd}(D)$. We say that a function $w \in H^1_{\rm odd}(D)$ is a *weak solution* of the equation (81) if and only if

$$\int_D w_3\psi_3 + \big(1-\omega^2 V(x)\big)\, w_1\psi_1 + \mu w\psi \,\mathrm{d}x = \int_D f\psi \,\mathrm{d}x \quad \big(\psi \in \mathcal{C}^\infty_{{\rm per},2P,b}(D)\big)\,.$$

Our goal is to prove the following statement.

Lemma 52. *Assume* **S**. *Let*

- $L^2_\mu(D)$ *be a set of all functions* $f \in L^2_{\mathrm{odd}}(D)$, *which can be represented in a form*
$$f(x_1,x_3) = \sum_{k\in S^\mu} f_k(x_3)\sin\Big(k\frac{\pi}{P}x_1\Big)\,,$$
- $H^1_\mu(D)$ *be a set of all functions* $w \in H^1_{\mathrm{odd}}(D)$ *which can be represented in a form*
$$w(x_1,x_3) = \sum_{k\in S^\mu} w_k(x_3)\sin\Big(k\frac{\pi}{P}x_1\Big)\,,$$

where S^μ *is defined in* (65). *Then the following statements are true:*

(i) *For every* $f \in L^2_\mu(D)$, *there exits a unique* $w \in H^1_\mu(D)$ *solving the equation* (81) *in the sense of the Definition 51. In other words, there exists a solution operator* $T_\mu \in \mathcal{L}\big(L^2_\mu(D)\,, H^1_\mu(D)\big)$ *for the equation* (81).

(ii) *There exists a constant* $M > 0$ *such that for all* $\mu \in \overline{I^\varepsilon_{\mu_1}}$
$$\|w\|_{H^1(D)} \leqslant M\,\|f\|_{L^2(D)} \quad \big(f \in L^2_{\mu_1}(D)\big)\,,$$
where $w = T_\mu(f)$.

Proof. Above statement follows directly from Lemma 67 and 68. □

Remark 53. Recall Remark 33. For all $\mu \in \overline{I^\varepsilon_{\mu_1}}\backslash\{\mu_1\}$ we have $L^2_\mu(D) = L^2_{\mathrm{odd}}(D)$. As we will see later $L^2_{\mu_1}(D) = \{\varphi\}^{\perp_{L^2_{\mathrm{odd}}(D)}}$, where φ is an eigenfunction of the operator L_{μ_1} corresponding to the zero eigenvalue. In other words $\varphi \in \mathtt{ker}\,L_{\mu_1}$.

Remark 54. Note that for all $\mu \in J^\varepsilon_{\mu_1}$ we have $L^2_\mu(D) = L^2_{\mathrm{odd}}(D)$ and $H^1_\mu(D) = H^1_{\mathrm{odd}}(D)$. Moreover, for all $\mu \in J^\varepsilon_{\mu_1}$ the operator L_μ has a bounded inverse $T_\mu = L^{-1}_\mu \colon L^2_{\mathrm{odd}}(D) \to H^1_{\mathrm{odd}}(D)$. Let M_μ denote the norm of the operator T_μ. Observe that $M_\mu \xrightarrow[\mu\to\mu_1]{} \infty$.

Section 5.3.1.3 contains a description of the eigenvalues of the operators L_μ.

5.3.1.1 About the equation $L_{\mu,k}w = f$

We write
$$\begin{aligned} w(x_1,x_3) &= \sum_{k\in S^\mu} w_k(x_3)\sin\Big(k\frac{\pi}{P}x_1\Big)\,,\\ f(x_1,x_3) &= \sum_{k\in S^\mu} f_k(x_3)\sin\Big(k\frac{\pi}{P}x_1\Big)\,. \end{aligned} \tag{82}$$

Define

$$\begin{aligned} b_o = b_o(k,\mu) &= \big(1-\omega^2 V_o\big)\,\frac{k^2\pi^2}{P^2} + \mu \quad (k \in S^\mu)\,,\\ b_i = b_i(k,\mu) &= \big(1-\omega^2 V_i\big)\,\frac{k^2\pi^2}{P^2} + \mu \quad (k \in S^\mu)\,. \end{aligned} \tag{83}$$

As we observed in Remark 31, assumptions $\mathbf{S}^1$ and $\mathbf{S}^2$ imply that for all $k \in \mathbb{N}$ and for all $\mu \in \overline{I^{\varepsilon}_{\mu_1}}$.

$$b_o(k,\mu) > 0 > b_i(k,\mu)\,. \tag{84}$$

Let $\lambda_o = \lambda_o(\mu,k)$ and $\lambda_i = \lambda_i(\mu,k)$ $\big(\mu \in \overline{I^{\varepsilon}_{\mu_1}}, k \in S^\mu\big)$ be defined by

$$\lambda_o(\mu,k) = \sqrt{b_o(\mu,k)}\,,\ \lambda_i(\mu,k) = \sqrt{-b_i(\mu,k)}\,. \tag{85}$$

Observe that

$$\lambda_o = \Theta(k)\,,\ \ \lambda_i = \Theta(k)\,, \tag{86}$$

i.e.

$$\begin{aligned} &\limsup_{k\to\infty} \frac{\lambda_i(\mu,k)}{k} < \infty,\ \liminf_{k\to\infty} \frac{\lambda_i(\mu,k)}{k} > 0,\\ &\limsup_{k\to\infty} \frac{\lambda_o(\mu,k)}{k} < \infty,\ \liminf_{k\to\infty} \frac{\lambda_o(\mu,k)}{k} > 0. \end{aligned}$$

Define the function $b\colon \mathbb{R} \to \mathbb{R}$ as

$$b(x) = b_{\mu,k}(x) = \begin{cases} b_i(\mu,k) & (|x| < a)\,,\\ b_o(\mu,k) & (|x| > a)\,. \end{cases} \tag{87}$$

For every $\mu \in \mathbb{R}$ and $k \in \mathbb{N}$ consider an operator $L_{\mu,k}$ defines as

$$L_{\mu,k} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + b_{\mu,k}(x)\,, \tag{88}$$

where the function $b = b_{\mu,k}$ was defined in (87).

After taking the ansatz described in (82) the coefficients $(w_k)_{k\in S^\mu}$ and $(f_k)_{k\in S^\mu}$ have to satisfy the equation

$$L_{\mu,k} w_k = -w_k'' + b(x)\, w_k = f_k \text{ on } \mathbb{R}. \tag{89}$$

The main result of this section is the following.

Lemma 55. *Assume* $\mathbf{S}$*. Then the following statements are true:*

(i) If $\mu \in \overline{I^{\varepsilon}_{\mu_1}}$ and $k \in \mathbb{N}$ are such that $(\mu,k) \neq (\mu_1,k_1)$, then for every $f \in L^2(\mathbb{R})$, there exists a unique function $w \in H^1(\mathbb{R})$ solving equation (89). In other words, there exists a solution operator $T_{\mu,k} \in \mathcal{L}(L^2(\mathbb{R})\,, H^1(\mathbb{R}))$ for the equation (89).

(ii) There exits a constant $M > 0$ such that for all $\mu \in \overline{I^{\varepsilon}_{\mu_1}}$, all $k \in S^{\mu_1} = \mathbb{N} \setminus \{k_1\}$ and all $f \in L^2(\mathbb{R})$

$$\begin{aligned} \|w\|_{L^2(\mathbb{R})} &\leqslant \frac{M}{k}\,\|f\|_{L^2(\mathbb{R})}\,, \\ \|w'\|_{L^2(\mathbb{R})} &\leqslant M\,\|f\|_{L^2(\mathbb{R})}\,. \end{aligned} \tag{90}$$

Proof. This is a consequence of Lemmas 65 and 66. □

Remark 56. For every $\mu \in \overline{I^{\varepsilon}_{\mu_1}}$ let M_μ denote the norm of the operator $T_{\mu,k_1} = L^{-1}_{\mu,k_1} : L^2(\mathbb{R}) \to H^1(\mathbb{R})$. Observe, that $M_\mu \xrightarrow[\mu\to\mu_1]{} \infty$

5.3.1.1.1 Solving the equation $L_{\mu,k}w = f$ using the variation of constants Now, using variation of constants, we will derive a formula of the function w_k solving equation (89). Consider two problems, which are the homogeneous variants of the equation of the form (89)

$$\begin{cases} -u'' + b(x)\,u &= 0 \text{ on } [0,+\infty)\,, \\ u'(0) &= 0, \end{cases} \tag{91}$$

and

$$\begin{cases} -u'' + b(x)\,u &= 0 \text{ on } [0,+\infty)\,, \\ u(0) &= 0. \end{cases} \tag{92}$$

Consider the functions

$$\begin{aligned} \varphi_1^N(x) &= \begin{cases} \cos \lambda_i x & (0 < x < a)\,, \\ C e^{\lambda_o x} + D e^{-\lambda_o x} & (x > a)\,, \end{cases} \\ \varphi_1^D(x) &= \begin{cases} \sin \lambda_i x & (0 < x < a)\,, \\ \widetilde{C} e^{\lambda_o x} + \widetilde{D} e^{-\lambda_o x} & (x > a)\,, \end{cases} \\ \varphi_2(x) &= \begin{cases} A \cos \lambda_i x + B \sin \lambda_i x & (0 < x < a)\,, \\ e^{-\lambda_o x} & (x > a)\,, \end{cases} \end{aligned} \tag{93}$$

where the appearing constants are chosen in such a way, that the functions φ_1^N, φ_1^D and φ_2 are of the class $\mathcal{C}^1$. The choice of this constants is described by the following statement.

Lemma 57. *The functions $\varphi_1^N, \varphi_1^D, \varphi_2$ are of the class $\mathcal{C}^1$ if and only if the* *constants* $A, B, C, \widetilde{C}, D, \widetilde{D}$ *are set to be*

$$\begin{aligned} A &= e^{-\lambda_o a} \frac{\lambda_i \cos \lambda_i a + \lambda_o \sin \lambda_i a}{\lambda_i}, \\ B &= -e^{-\lambda_o a} \frac{\lambda_o \cos \lambda_i a - \lambda_i \sin \lambda_i a}{\lambda_i}, \\ C &= e^{-\lambda_o a} \frac{\lambda_o \cos \lambda_i a - \lambda_i \sin \lambda_i a}{2\lambda_o}, \\ D &= e^{\lambda_o a} \frac{\lambda_o \cos \lambda_i a + \lambda_i \sin \lambda_i a}{2\lambda_o}, \\ \widetilde{C} &= e^{-\lambda_o a} \frac{\lambda_i \cos \lambda_i a + \lambda_o \sin \lambda_i a}{2\lambda_o}, \\ \widetilde{D} &= -e^{\lambda_o a} \frac{\lambda_i \cos \lambda_i a - \lambda_o \sin \lambda_i a}{2\lambda_o}. \end{aligned} \tag{94}$$

Let the constants $A, B, C, \widetilde{C}, D, \widetilde{D}$ be chosen as in the Lemma 57 (cf. formulas (94)), then the pairs $\left(\varphi_1^D, \varphi_2\right)$ $\left(\varphi_1^N, \varphi_2\right)$ are fundamental solutions of the equations (91) and (92) respectively. We will introduce the Wrońskians of the sets of fundamental solutions, namely

$$\begin{aligned} W_N = W_N(\mu, k) &= \varphi_1^N \varphi_2' - \left(\varphi_1^N\right)' \varphi_2, \\ W_D = W_D(\mu, k) &= \varphi_1^D \varphi_2' - \left(\varphi_1^D\right)' \varphi_2. \end{aligned} \tag{95}$$

Lemma 59 contains another formula for W_N and W_D.

Remark 58. The expressions W_D and W_N defined in (95) do not depend on x.

Proof. Consider the function $W_N(x)$, which is continuous on $\mathbb{R}$ (cf. Lemma 57) and continuously differentiable on the set $\mathbb{R} \setminus \{\pm a\}$. Note that, for all $x \in \mathbb{R}$

$$W_N'(x) = \varphi_1^N(x)\, \varphi_2''(x) - \left(\varphi_1^N\right)''(x)\, \varphi_2(x) \overset{(91)}{=} 0.$$

Therefore, the function $W_N(x)$ is constant. The same applies for W_D. □

Lemma 57 implies the following statement.

Lemma 59. *For every* $\mu \in I_{\mu_1}^{\varepsilon}$ *and* $k \in S^{\mu}$

$$\begin{aligned} W_N(\mu, k) &= -(\lambda_o \cos \lambda_i a - \lambda_i \sin \lambda_i a)\, e^{-\lambda_o a}, \\ W_D(\mu, k) &= -(\lambda_i \cos \lambda_i a + \lambda_o \sin \lambda_i a)\, e^{-\lambda_o a}, \end{aligned}$$

where the λ_o *and* λ_i *are defined in* (85).

Remark 60. By equations (94) and Lemma 59 we get

$$A = -\frac{W_D}{\lambda_i},\ B = \frac{W_N}{\lambda_i},\ C = -\frac{W_N}{2\lambda_o}, \widetilde{C} = -\frac{W_D}{2\lambda_o}.$$

Remark 61. Observe that as a consequence of Remark 60 we have

$$\frac{C}{W_N} - \frac{\widetilde{C}}{W_D} = 0, \tag{96}$$

$$\frac{C}{W_N} + \frac{\widetilde{C}}{W_D} = -\frac{1}{\lambda_o}. \tag{97}$$

Now, using variation of constants, we will derive a formula for the function $w_k\colon \mathbb{R} \to \mathbb{R}$ solving equation (89). To simply notation, we will write v instead of w_k. Take any $f_k = f \in L^2(\mathbb{R})$ and split it into an odd and even part, i.e. $f = f_o + f_e$, where

$$f_o(x) = \frac{1}{2}(f(x) - f(-x)) \quad (x \in \mathbb{R}), f_e(x) = \frac{1}{2}(f(x) + f(-x)) \quad (x \in \mathbb{R}).$$

Observe, that f_o and f_e are odd and even respectively. Moreover, we can consider the functions f_o and f_e as elements of the space $L^2(0,\infty)$. Solving (89) is equivalent to solving

$$L_{\mu,k} v_o = f_o \text{ on } \mathbb{R}, \tag{98}$$

$$L_{\mu,k} v_e = f_e \text{ on } \mathbb{R}, \tag{99}$$

when $v = v_o + v_e$. Now we will derive a solution formula for the function v_o. Formula for v_e can be obtained by the same way. We write (98) as a first order system

$$\begin{cases} v_o' &= z, \\ z' &= b(x) - f_o. \end{cases} \tag{100}$$

Every solution of the system (100) has a form

$$\begin{pmatrix} v_o \\ z \end{pmatrix} = \xi(x) \begin{pmatrix} \varphi_1^D \\ \varphi_1^{D\prime} \end{pmatrix} + \zeta(x) \begin{pmatrix} \varphi_2 \\ \varphi_2' \end{pmatrix}. \tag{101}$$

By differentiating (101) and using (100), we get

$$\xi' \begin{pmatrix} \varphi_1^D \\ \varphi_1^{D\prime} \end{pmatrix} + \zeta' \begin{pmatrix} \varphi_2 \\ \varphi_2' \end{pmatrix} = \begin{pmatrix} 0 \\ -f_o \end{pmatrix},$$

and therefore

$$\begin{pmatrix} \xi' \\ \zeta' \end{pmatrix} = M_{\text{odd}}^{-1} \begin{pmatrix} 0 \\ -f_o \end{pmatrix}, \tag{102}$$

where

$$\begin{aligned} M_{\text{odd}} &= M_{\text{odd}}(\mu,k) = \begin{bmatrix} \varphi_1^D & \varphi_2 \\ \varphi_1^{D\prime} & \varphi_2' \end{bmatrix}, \\ M_{\text{odd}}^{-1} &= M_{\text{odd}}(\mu,k)^{-1} = \frac{1}{\det M_{\text{odd}}} \begin{bmatrix} \varphi_2' & -\varphi_2 \\ -\varphi_1^{D\prime} & \varphi_1^D \end{bmatrix}, \\ \det M_{\text{odd}} &= W_D(\mu,k). \end{aligned} \tag{103}$$

By integrating (102), we obtain

$$\begin{aligned} \xi(x) &= -\frac{1}{W_D}\int_0^x \varphi_2(s) f_o(s)\,\mathrm{d}s \quad (x \geqslant 0)\,, \\ \zeta(x) &= -\frac{1}{W_D}\int_0^x \varphi_1^D(s) f_o(s)\,\mathrm{d}s \quad (x \geqslant 0)\,. \end{aligned} \tag{104}$$

This, together with (101), gives, for all $x \geqslant 0$

$$v_o(x) = -\frac{1}{W_D}\left(\varphi_1^D(x)\int_x^\infty \varphi_2(s)\, f_o(s)\ \mathrm{d}s + \varphi_2(x)\int_0^x \varphi_1^D(s)\, f_o(s)\ \mathrm{d}s\right). \tag{105}$$

By the same reasoning applied for the equation (99), we obtain for all $x \geqslant 0$

$$v_e(x) = -\frac{1}{W_N}\left(\varphi_1^N(x)\int_x^\infty \varphi_2(s)\, f_e(s)\ \mathrm{d}s + \varphi_2(x)\int_0^x \varphi_1^N(s)\, f_e(s)\ \mathrm{d}s\right). \tag{106}$$

Recall that $w_k = v_o + v_e$ and $f_k = f_o + f_e$, where f_o and f_e are odd and even part of f respectively.

By applying Corollaries 145 and 146 for formulas (105) and (106) we obtain

$$\begin{aligned} v_o'(x) &= -\frac{1}{W_D}\left(\varphi_1^{D\prime}(x)\int_x^\infty \varphi_2(s)\, f_o(s)\ \mathrm{d}s + \varphi_2'(x)\int_0^x \varphi_1^D(s)\, f_o(s)\ \mathrm{d}s\right), \\ v_e'(x) &= -\frac{1}{W_N}\left(\varphi_1^{N\prime}(x)\int_x^\infty \varphi_2(s)\, f_e(s)\ \mathrm{d}s + \varphi_2'(x)\int_0^x \varphi_1^N(s)\, f_e(s)\ \mathrm{d}s\right), \end{aligned} \tag{107}$$

for all $x \geqslant 0$. Recall that Lemma 59 describes the values of W_D and W_N.

5.3.1.1.2 Derivation of the Green's function for $L_{\mu,k}$ Now we will find the representation of the function $w_k = v$ in the terms of the Green's function. We can rewrite formulas (105) and (106) as

$$v_o(x) = \begin{cases} \int_0^\infty G_D(x,y)\, f_o(y)\ \mathrm{d}y & (x > 0)\,, \\ -v_o(-x) = -\int_0^\infty G_D(-x,y)\, f_o(y)\ \mathrm{d}y & (x < 0)\,, \end{cases}$$

$$v_e(x) = \begin{cases} \int_0^\infty G_N(x,y)\, f_e(y)\ \mathrm{d}y & (x > 0)\,, \\ v_e(-x) = \int_0^\infty G_N(-x,y)\, f_e(y)\ \mathrm{d}y & (x < 0)\,, \end{cases}$$

where G_N and G_D are the Green's functions for the corresponding problems (91) and (92), i.e.

$$G_N(x,y) = \begin{cases} -\frac{1}{W_N}\varphi_1^N(x)\,\varphi_2(y) & (0 < x < y)\,, \\ -\frac{1}{W_N}\varphi_1^N(y)\,\varphi_2(x) & (0 < y < x)\,, \end{cases}$$

$$G_D(x,y) = \begin{cases} -\frac{1}{W_D}\varphi_1^D(x)\,\varphi_2(y) & (0 < x < y)\,, \\ -\frac{1}{W_D}\varphi_1^D(y)\,\varphi_2(x) & (0 < y < x)\,, \end{cases}$$

where Lemma 59 contains the values of W_N and W_D, the functions φ_1^N, φ_1^D and φ_2 were defined in the formula (93), and Lemma 57 described the values of the constants appearing there. Define the function $G\colon \mathbb{R}^2 \to \mathbb{R}$ as follows

$$G(x,y) = \begin{cases} \frac{1}{2}(G_N(x,y) + G_D(x,y)) & (x \geqslant 0, y \geqslant 0) \\ \frac{1}{2}(G_N(x,-y) - G_D(x,-y)) & (x \geqslant 0, y \leqslant 0) \\ \frac{1}{2}(G_N(-x,y) - G_D(-x,y)) & (x \leqslant 0, y \geqslant 0) \\ \frac{1}{2}(G_N(-x,-y) + G_D(-x,-y)) & (x \leqslant 0, y \leqslant 0) \end{cases} \tag{108}$$

We have that

$$v(x) = \int_{\mathbb{R}} G(x,y)\, f(y)\, \mathrm{d}y \quad (x \in \mathbb{R})\,. \tag{109}$$

Notice that

$$G(x,y) = G(-x,-y) \quad \big((x,y) \in \mathbb{R}^2\big)\,. \tag{110}$$

Consider the sets illustrated on fig. 6

$$\Omega_1 = \big\{(x,y) \in \mathbb{R}^2 \colon x < -a, y > a, y < -x\big\}\,,$$

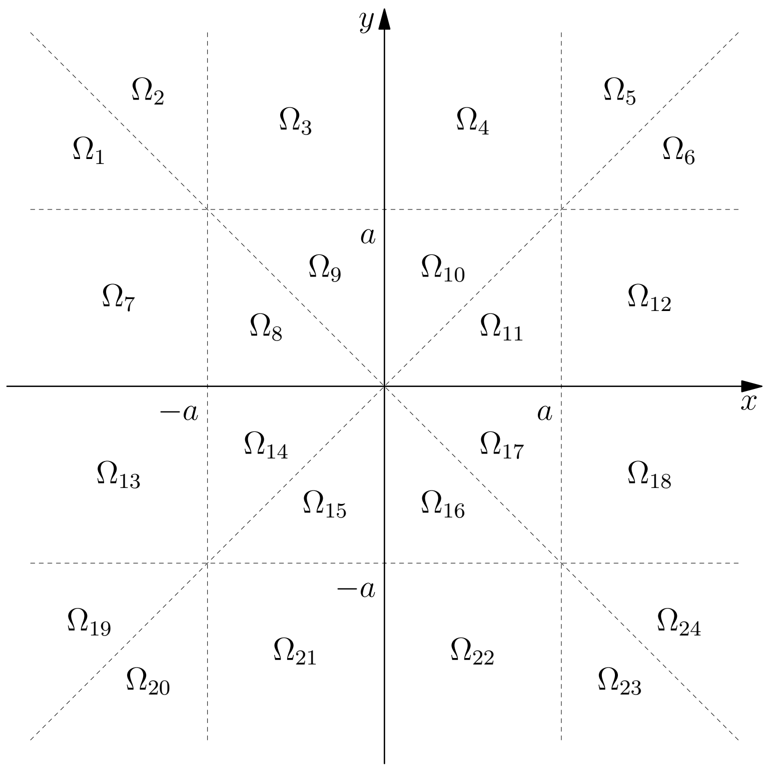


Figure 6: The sets $\Omega_1, \ldots, \Omega_{24}$.

$$\Omega_2 = \big\{(x,y) \in \mathbb{R}^2 \colon x < -a, y > a, y > -x\big\}\,,$$

$$
\begin{aligned}
\Omega_3 &= \Big\{(x,y) \in \mathbb{R}^2 \colon -a < x < 0, y > a\Big\}\,,\\
\Omega_4 &= \Big\{(x,y) \in \mathbb{R}^2 \colon 0 < x < a, y > a\Big\}\,,\\
\Omega_5 &= \Big\{(x,y) \in \mathbb{R}^2 \colon x > a, y > a, y > x\Big\}\,,\\
\Omega_6 &= \Big\{(x,y) \in \mathbb{R}^2 \colon x > a, y > a, y < x\Big\}\,,\\
\Omega_7 &= \Big\{(x,y) \in \mathbb{R}^2 \colon x < -a, 0 < y < a\Big\}\,,\\
\Omega_8 &= \Big\{(x,t) \in \mathbb{R}^2 \colon -a < x < 0, 0 < y < a, y < -x\Big\}\,,\\
\Omega_9 &= \Big\{(x,t) \in \mathbb{R}^2 \colon -a < x < 0, 0 < y < a, y > -x\Big\}\,,\\
\Omega_{10} &= \Big\{(x,t) \in \mathbb{R}^2 \colon 0 < x < a, 0 < y < a, y > x\Big\}\,,\\
\Omega_{11} &= \Big\{(x,t) \in \mathbb{R}^2 \colon 0 < x < a, 0 < y < a, y < x\Big\}\,,\\
\Omega_{12} &= \Big\{(x,y) \in \mathbb{R}^2 \colon x > a, 0 < y < a\Big\}\,,\\
\Omega_{13} &= \Big\{(x,y) \in \mathbb{R}^2 \colon x < -a, -a < y < 0\Big\}\,,\\
\Omega_{14} &= \Big\{(x,t) \in \mathbb{R}^2 \colon -a < x < 0, -a < y < 0, y > x\Big\}\,,\\
\Omega_{15} &= \Big\{(x,t) \in \mathbb{R}^2 \colon -a < x < 0, -a < y < 0, y < x\Big\}\,,\\
\Omega_{16} &= \Big\{(x,t) \in \mathbb{R}^2 \colon 0 < x < a, -a < y < 0, y < -x\Big\}\,,\\
\Omega_{17} &= \Big\{(x,t) \in \mathbb{R}^2 \colon 0 < x < a, -a < y < 0, y > -x\Big\}\,,\\
\Omega_{18} &= \Big\{(x,y) \in \mathbb{R}^2 \colon x > a, -a < y < 0\Big\}\,,\\
\Omega_{19} &= \Big\{(x,y) \in \mathbb{R}^2 \colon x < -a, y < -a, y > x\Big\}\,,\\
\Omega_{20} &= \Big\{(x,y) \in \mathbb{R}^2 \colon x < -a, y < -a, y < x\Big\}\,,\\
\Omega_{21} &= \Big\{(x,y) \in \mathbb{R}^2 \colon -a < x < 0, y < -a\Big\}\,,\\
\Omega_{22} &= \Big\{(x,y) \in \mathbb{R}^2 \colon 0 < x < a, y < -a\Big\}\,,\\
\Omega_{23} &= \Big\{(x,y) \in \mathbb{R}^2 \colon a < x, y < -a, y < -x\Big\}\,,\\
\Omega_{24} &= \Big\{(x,y) \in \mathbb{R}^2 \colon a < x, y < -a, y > -x\Big\}\,.
\end{aligned}
$$

Inserting the formulas (93) into (108), we obtain

$$
G(x,y) =
$$

$$
\begin{cases}
-\left(\frac{\sin\lambda_i x}{2W_D} + \frac{\cos\lambda_i x}{2W_N}\right) e^{-\lambda_o y} & ((x,y) \in \Omega_4)\,,\\
-\left(\frac{C}{2W_N} + \frac{\widetilde{C}}{2W_D}\right) e^{\lambda_o(x-y)} - \left(\frac{D}{2W_N} + \frac{\widetilde{D}}{2W_D}\right) e^{\lambda_o(-x-y)} & ((x,y) \in \Omega_5)\,,\\
-\left(\frac{C}{2W_N} + \frac{\widetilde{C}}{2W_D}\right) e^{\lambda_o(y-x)} - \left(\frac{D}{2W_N} + \frac{\widetilde{D}}{2W_D}\right) e^{\lambda_o(-y-x)} & ((x,y) \in \Omega_6)\,,\\
-\frac{A\sin\lambda_i x\cos\lambda_i y}{2W_D} - \frac{B\sin\lambda_i x\sin\lambda_i y}{2W_D} - \frac{A\cos\lambda_i x\cos\lambda_i y}{2W_N} - \frac{B\cos\lambda_i x\sin\lambda_i y}{2W_N} & ((x,y) \in \Omega_{10})\,,\\
-\frac{A\cos\lambda_i x\sin\lambda_i y}{2W_D} - \frac{B\sin\lambda_i x\sin\lambda_i y}{2W_D} - \frac{A\cos\lambda_i x\cos\lambda_i y}{2W_N} - \frac{B\sin\lambda_i x\cos\lambda_i y}{2W_N} & ((x,y) \in \Omega_{11}\cup\Omega_{17})\,,\\
-e^{-\lambda_o x}\left(\frac{\sin\lambda_i y}{2W_D} + \frac{\cos\lambda_i y}{2W_N}\right) & ((x,y) \in \Omega_{12}\cup\Omega_{18})\,,\\
-\frac{A\cos\lambda_i x\cos\lambda_i y}{2W_N} + \frac{B\cos\lambda_i x\sin\lambda_i y}{2W_N} + \frac{A\sin\lambda_i x\cos\lambda_i y}{2W_D} - \frac{B\sin\lambda_i x\sin\lambda_i y}{2W_D} & ((x,y) \in \Omega_{16})\,,\\
-\left(\frac{\cos\lambda_i x}{2W_N} - \frac{\sin\lambda_i x}{2W_D}\right) e^{\lambda_o y} & ((x,y) \in \Omega_{22})\,,\\
-\underbrace{\left(\frac{C}{2W_N} - \frac{\widetilde{C}}{2W_D}\right)}_{\overset{(96)}{=}0} e^{\lambda_o(x+y)} + \left(\frac{D}{2W_N} + \frac{\widetilde{D}}{2W_D}\right) e^{\lambda_o(-x+y)} & ((x,y) \in \Omega_{23}\cup\Omega_{24})\,.
\end{cases}
$$

Using the relations from Remark 60 one can rewrite above formulas as

$$G(x,y)=\begin{cases} -\left(\frac{\sin\lambda_i x}{2W_D}+\frac{\cos\lambda_i x}{2W_N}\right)e^{-\lambda_o y} & \big((x,y)\in\Omega_4\big)\,,\\ -\left(\frac{C}{2W_N}+\frac{\widetilde{C}}{2W_D}\right)e^{\lambda_o(x-y)}-\left(\frac{D}{2W_N}+\frac{\widetilde{D}}{2W_D}\right)e^{\lambda_o(-x-y)} & \big((x,y)\in\Omega_5\big)\,,\\ -\left(\frac{C}{2W_N}+\frac{\widetilde{C}}{2W_D}\right)e^{\lambda_o(y-x)}-\left(\frac{D}{2W_N}+\frac{\widetilde{D}}{2W_D}\right)e^{\lambda_o(-y-x)} & \big((x,y)\in\Omega_6\big)\,,\\ -\frac{1}{2\lambda_i}\sin(\lambda_i(x-y))-\frac{B\sin\lambda_i x\sin\lambda_i y}{2W_D}-\frac{A\cos\lambda_i x\cos\lambda_i y}{2W_N} & \big((x,y)\in\Omega_{10}\big)\,,\\ -\frac{1}{2\lambda_i}\sin(\lambda_i(x-y))-\frac{B\sin\lambda_i x\sin\lambda_i y}{2W_D}-\frac{A\cos\lambda_i x\cos\lambda_i y}{2W_N} & \big((x,y)\in\Omega_{11}\cup\Omega_{17}\big)\,,\\ -e^{-\lambda_o x}\left(\frac{\sin\lambda_i y}{2W_D}+\frac{\cos\lambda_i y}{2W_N}\right) & \big((x,y)\in\Omega_{12}\cup\Omega_{18}\big)\,,\\ -\frac{A\cos\lambda_i x\cos\lambda_i y}{2W_N}+\frac{1}{2\lambda_i}\sin(\lambda_i(x-y))-\frac{B\sin\lambda_i x\sin\lambda_i y}{2W_D} & \big((x,y)\in\Omega_{16}\big)\,,\\ -\left(\frac{\cos\lambda_i x}{2W_N}-\frac{\sin\lambda_i x}{2W_D}\right)e^{\lambda_o y} & \big((x,y)\in\Omega_{22}\big)\,,\\ -\left(\frac{D}{2W_N}-\frac{\widetilde{D}}{2W_D}\right)e^{\lambda_o(-x+y)} & \big((x,y)\in\Omega_{23}\cup\Omega_{24}\big)\,. \end{cases}$$

Let $r_1\colon\mathbb{R}\to\mathbb{R}$ be defined as

$$r_1(s)=\left(\frac{C}{2W_N}+\frac{\widetilde{C}}{2W_D}\right)e^{-\lambda_o|s|}\quad (s\in\mathbb{R})\,.$$

Define function $r_2\colon\mathbb{R}^2\to\mathbb{R}$ as

$$r_2(x,y)=G(x,y)-r_1(x-y)\quad \big((x,y)\in\mathbb{R}^2\big)\,.$$

Observe, that by (110), we have that

$$r_2(x,y)=r_2(-x,-y)\quad \big((x,y)\in\mathbb{R}^2\big)\,.$$

We can write the solution of (89) as

$$v(x)=(r_1*f)(x)+\int_{\mathbb{R}} r_2(x,y)\,f(y)\,\mathrm{d}y\quad (x\in\mathbb{R})\,. \tag{111}$$

Application of Young's inequality and Hölder's inequality in (111) yields

$$\|v\|_{L^2(\mathbb{R})}\leqslant\Big(\|r_1\|_{L^1(\mathbb{R})}+\|r_2\|_{L^2(\mathbb{R}^2)}\Big)\,\|f\|_{L^2(\mathbb{R})}\,. \tag{112}$$

Now we will derive some estimates of the norms $\|r_1\|_{L^1(\mathbb{R})}$, $\|r_2\|_{L^2(\mathbb{R}^2)}$. Observe that

$$\|r_1\|_{L^1(\mathbb{R})}=\frac{1}{\lambda_o}\left|\frac{C}{W_N}+\frac{\widetilde{C}}{W_D}\right|, \tag{113}$$

and

$$\begin{aligned} \|r_2\|_{L^2(\Omega_4\cup\Omega_{22})}&\leqslant\sqrt{a}\left(\frac{1}{|W_D|}+\frac{1}{|W_N|}\right)\frac{e^{-\lambda_o a}}{2\sqrt{\lambda_o}}+\left|\frac{C}{W_N}+\frac{\widetilde{C}}{W_D}\right|\frac{\sqrt{1-e^{-2\lambda_o a}}}{2\lambda_o},\\ \|r_2\|_{L^2(\Omega_5\cup\Omega_6)}&\leqslant\frac{1}{4}\left|\frac{\widetilde{D}}{W_D}+\frac{D}{W_N}\right|\frac{e^{-2\lambda_o a}}{\lambda_o},\\ \|r_2\|_{L^2(\Omega_{23}\cup\Omega_{24})}&\leqslant\frac{1}{4}\left|\frac{D}{W_N}-\frac{\widetilde{D}}{W_D}-\frac{C}{W_N}-\frac{\widetilde{C}}{W_D}\right|\frac{e^{-2\lambda_o a}}{\lambda_o},\\ \|r_2\|_{L^2(\Omega_{10}\cup\Omega_{11}\cup\Omega_{16}\cup\Omega_{17})}&\leqslant a\left(\frac{|A|+|B|}{|W_D|}+\frac{|A|+|B|}{|W_N|}\right)+\frac{1}{2}\left|\frac{C}{W_N}+\frac{\widetilde{C}}{W_D}\right|\frac{\sqrt{4\lambda_o a-1+e^{-4\lambda_o a}}}{2\lambda_o},\\ \|r_2\|_{L^2(\Omega_{12}\cup\Omega_{18})}&\leqslant\sqrt{a}\left(\frac{1}{|W_D|}+\frac{1}{|W_N|}\right)\frac{e^{-\lambda_o a}}{2\sqrt{\lambda_o}}+\left|\frac{C}{W_N}+\frac{\widetilde{C}}{W_D}\right|\frac{\sqrt{1-e^{-2\lambda_o a}}}{2\lambda_o}. \end{aligned} \tag{114}$$

Now we will show, that the values $\|r_1\|_{L^1(\mathbb{R})}$ and $\|r_2\|_{L^2(\mathbb{R}^2)}$ have a bound of order $\frac{1}{k^{\frac{3}{2}}}$, which is uniform in $\mu \in I^{\varepsilon}_{\mu_1}$. This, together with inequality (112), will lead to the proof of the estimates (90) in Lemma 55 part (ii).

5.3.1.1.3 Estimates of the constants In this section we will study the behaviour of the constants appearing in the estimates (113) and (114), as $k \to \infty$.

In order to control the behaviour of W_N and W_D as $k \to \infty$, we will use the following lemma. It will be used later in the proof of Lemma 65 which is a part of the proof of Lemma 55.

Lemma 62. *Assume* **S***. There exist constants $C_1, C_2 > 0$ such that for all $\mu \in \overline{I^{\varepsilon}_{\mu_1}}$ and all $k \in S^{\mu_1} = \mathbb{N} \setminus \{k_1\}$(*[5]*)*

$$C_1 k \leqslant |\lambda_o \cos \lambda_i a - \lambda_i \sin \lambda_i a| \leqslant C_2 k,$$
$$C_1 k \leqslant |\lambda_i \cos \lambda_i a + \lambda_o \sin \lambda_i a| \leqslant C_2 k.$$

Proof. Because of (86), the upper bounds are clear. Note that

$$\frac{\lambda_i}{k} \xrightarrow[k\to\infty]{} \sqrt{-(1-\omega^2 V_i)}\,\frac{\pi}{P}, \quad \frac{\lambda_o}{k} \xrightarrow[k\to\infty]{} \sqrt{1-\omega^2 V_o}\,\frac{\pi}{P}. \tag{115}$$

Consider the sequences $(a^{\mu}_k)_{k\in\mathbb{N}}, (b^{\mu}_k)_{k\in\mathbb{N}}$ defined by

$$a^{\mu}_k = \frac{\lambda_o}{k} \cos \lambda_i a - \frac{\lambda_i}{k} \sin \lambda_i a,$$
$$b^{\mu}_k = \frac{\lambda_i}{k} \cos \lambda_i a + \frac{\lambda_o}{k} \sin \lambda_i a,$$

Recall that by (85) $\lambda_i = \sqrt{-(1-\omega^2 V_i)\,\frac{k^2\pi^2}{P^2} - \mu}$. By condition (63) in the assumption $\mathbf{S}^4$ and by Lemma 39 and relation (115) the sequences defined above have finitely many limit points, which do not depend on the value of $\mu \in \overline{I^{\varepsilon}_{\mu_1}}$. Let $\mathfrak{A}^{\infty}$, $\mathfrak{B}^{\infty}$ denote the sets of limit points of the sequences $(a^{\mu}_k)_{k\in\mathbb{N}}$, $(b^{\mu}_k)_{k\in\mathbb{N}}$ respectively. We will show that the sets $\mathfrak{A}^{\infty}$ and $\mathfrak{B}^{\infty}$ do not contain 0. Suppose that $0 \in \mathfrak{A}^{\infty}$. Then, we have that

$$\sqrt{1-\omega^2 V_o}\,\frac{\pi}{P}\kappa - \sqrt{-(1-\omega^2 V_i)}\,\frac{\pi}{P}\varsigma = 0, \tag{116}$$

where κ and ς are one of the limit points of the sequences $(\cos \lambda_i a)_{k\in\mathbb{N}}$ and $(\sin \lambda_i a)_{k\in\mathbb{N}}$ (note, that the sets of the limits points of these sequences do not

[5] Recall that the values λ_i, λ_o depend on k and μ, cf. formula (85) for definitions of λ_i and λ_o.

depend on $\mu \in \overline{I^{\varepsilon}_{\mu_1}}$). Moreover, there exists a subsequence $(k_\nu)_{\nu\in\mathbb{N}}$ of natural numbers, such that

$$\sin \lambda_i(k_\nu)\, a \xrightarrow[\nu\to\infty]{\mathbb{R}} \varsigma \text{ and } \cos \lambda_i(k_\nu)\, a \xrightarrow[\nu\to\infty]{\mathbb{R}} \kappa. \tag{117}$$

Let $\mathfrak{T}^\infty$ denote the set of limit points of the sequence $(\operatorname{tg} \lambda_i a)_{k\in\mathbb{N}}$. Relation (116) is equivalent to

$$\frac{\varsigma}{\kappa} = \sqrt{\frac{1-\omega^2 V_o}{-(1-\omega^2 V_i)}} \overset{(64)}{\notin} \mathfrak{T}^\infty,$$

which gives a contradiction, hence $0 \notin \mathfrak{A}^\infty$. By similar arguments , we prove that $0 \notin \mathfrak{B}^\infty$.

The sets $\mathfrak{A}^\infty$ and $\mathfrak{B}^\infty$ consist of finitely many points, therefore, there exists $\varepsilon > 0$ such that the sets $\overline{\mathfrak{A}^\infty_\varepsilon}$ and $\overline{\mathfrak{B}^\infty_\varepsilon}$, do not contain 0. where we use the notation $X_\varepsilon = \bigcup_{x\in X}(x-\varepsilon, x+\varepsilon)$. Using the fact that the sequences $(a^\mu_k)_{k\in\mathbb{N}}$ and $(b^\mu_k)_{k\in\mathbb{N}}$ converge uniformly, with respect to μ, on the set $\overline{I^{\varepsilon}_{\mu_1}}$, there exists $K \in \mathbb{N}$ such that for all $\mu \in \overline{I^{\varepsilon}_{\mu_1}}$ and for all $k \geqslant K$ we have $a^\mu_k \in \overline{\mathfrak{A}^\infty_\varepsilon}$ and $b^\mu_k \in \overline{\mathfrak{B}^\infty_\varepsilon}$. Let $C_1 = \min\big\{|x| : x \in \overline{\mathfrak{A}^\infty_\varepsilon} \cup \overline{\mathfrak{B}^\infty_\varepsilon}\big\}$. Then we have that, for all $\mu \in \overline{I^{\varepsilon}_{\mu_1}}$ and $k \geqslant K$

$$C_1 k \leqslant |\lambda_o \cos \lambda_i a - \lambda_i \sin \lambda_i a| \text{ and } C_1 k \leqslant |\lambda_i \cos \lambda_i a + \lambda_o \sin \lambda_i a|\,.$$

By condition $\mathbf{S}^3$ in Assumption **S**, we have that the right hand sides of the expressions above can be equal to 0 only if $k = k_1$ and $\mu = \mu_1$. This ends the proof. ☐

In Lemma 149 in section 8.3 we show, that the assumption $\mathbf{S}^4$ (63) is necessary in Lemma 62.

Remark 63. Now we will estimate all of the constants appearing in the inequalities (114):

- note that all of the constants A, B, C, D, $\widetilde{C}$ and $\widetilde{D}$ depend on μ in a locally continuous way,
- there exits a constant $M_1 > 0$ such that for all $\mu \in \overline{I^{\varepsilon}_{\mu_1}}$ and all $k \in S^{\mu_1}$

$$\frac{1}{\lambda_o}\left|\frac{C}{W_N} + \frac{\widetilde{C}}{W_D}\right| \overset{(97)}{=} \frac{1}{\lambda_o^2} \leqslant \frac{M_1}{k^2},$$

- by (86) and Lemmas 59 and 62, there exists a constant $M_2 > 0$ such that for all $\mu \in \overline{I^{\varepsilon}_{\mu_1}}$ and all $k \in S^{\mu_1}$

$$\left(\frac{1}{|W_D|} + \frac{1}{|W_N|}\right)\frac{e^{-\lambda_o a}}{\sqrt{\lambda_o}} \leqslant \frac{M_2}{k^{\frac{3}{2}}},$$

- there exists a constant $M_3 > 0$ such that for all $\mu \in \overline{I^{\varepsilon}_{\mu_1}}$ and all $k \in S^{\mu_1}$

$$\left|\frac{C}{W_N} + \frac{\widetilde{C}}{W_D}\right| \frac{\sqrt{1 - e^{-2\lambda_o a}}}{2\lambda_o} \leqslant \frac{M_3}{k^2},$$

- by Lemmas 57, 59 and 62 there exists a constant $M_4 > 0$ such that for all $\mu \in \overline{I^{\varepsilon}_{\mu_1}}$ and all $k \in S^{\mu_1}$

$$\left|\frac{\widetilde{D}}{W_D} + \frac{D}{W_N}\right| \frac{e^{-2\lambda_o a}}{\lambda_o} \leqslant \frac{M_4}{k^2},$$

- by Lemmas 57, 59 and 62 and formula (97) there exists a constant $M_5 > 0$ such that for all $\mu \in \overline{I^{\varepsilon}_{\mu_1}}$ and all $k \in S^{\mu_1}$

$$\left|\frac{D}{W_N} - \frac{\widetilde{D}}{W_D} - \frac{C}{W_N} - \frac{\widetilde{C}}{W_D}\right| \frac{e^{-2\lambda_o a}}{\lambda_o} \leqslant \frac{M_5}{k^2}$$

- by Lemmas 57, 59 and 62 there exists a constant $M_6 > 0$ such that for all $\mu \in \overline{I^{\varepsilon}_{\mu_1}}$ and all $k \in S^{\mu_1}$

$$\left(\frac{|A| + |B|}{|W_D|} + \frac{|A| + |B|}{|W_N|}\right) \leqslant \frac{M_6}{k}$$

- by formula (97) there exists a constant $M_7 > 0$ such that for all $\mu \in \overline{I^{\varepsilon}_{\mu_1}}$ and all $k \in S^{\mu_1}$

$$\left|\frac{C}{W_N} + \frac{\widetilde{C}}{W_D}\right| \frac{\sqrt{4\lambda_o a - 1 + e^{-4\lambda_o a}}}{2\lambda_o} \leqslant \frac{M_7}{k^{\frac{3}{2}}}.$$

Remark 64. As a consequence of Remark 63 and inequalities (113) and (114) we obtain that there exists a constant $M > 0$ such that for all $k \in S^{\mu_1}$ and all $\mu \in \overline{I^{\varepsilon}_{\mu_1}}$

$$\|r_1\|_{L^1(\mathbb{R})} \leqslant \frac{M}{k^2},$$
$$\|r_2\|_{L^2(\mathbb{R}^2)} \leqslant \frac{M}{k}.$$

Thus, using inequality (111), we get that

$$\|v\|_{L^2(\mathbb{R})} = \|w_k\|_{L^2(\mathbb{R})} \leqslant \frac{M}{k} \|f_k\|_{L^2(\mathbb{R})}.$$

Moreover, it seems that $\|r_2\|_{L^2(\Omega_{10} \cup \Omega_{11} \cup \Omega_{16} \cup \Omega_{17})}$ are of order $\frac{1}{k}$. Note that the remaining norms can be bounded by terms of the order at least $\frac{1}{k^{\frac{3}{2}}}$.

5.3.1.1.4 $L^2(\mathbb{R})$ estimates for the functions w_k and w_k' Now we are ready to prove the following statement.

Lemma 65. *There exists a constant $M > 0$ such that for all $\mu \in \overline{I^{\varepsilon}_{\mu_1}}$, all $k \in S^{\mu_1} = \mathbb{N} \setminus \{k_1\}$ and all $f \in L^2(\mathbb{R})$*

$$\|w_k\|_{L^2(\mathbb{R})} \leqslant \frac{M}{k} \|f\|_{L^2(\mathbb{R})}, \tag{118}$$

$$\|w_k'\|_{L^2(\mathbb{R})} \leqslant M \|f\|_{L^2(\mathbb{R})}, \tag{119}$$

where $w_k = v$ is as in the formula (109).

Proof. Remark 64 gives (118). Recall (107). By similar reasoning, we get (119). □

5.3.1.1.5 Soundness of the functions w_k Now we will show, that the function $w_k = v$ defined in (109) solves the equation (89).

Lemma 66. *Let $\mu \in \overline{I^{\varepsilon}_{\mu_1}}$ and $k \in S^{\mu}$ be such that $(\mu, k) \neq (\mu_1, k_1)$. For every $f \in L^2(D)$, the function v defined in* (109) *satisfies the equation* (89) *almost everywhere in $\mathbb{R}$.*

Proof. Recall that $f = f_o + f_e$, where f_o is the odd part of f and f_e is the even part of f, and that $v = v_o + v_e$, where v_o and v_e are as in (105) and (106) respectively. By applying Corollaries 145 and 146, we can differentiate formulas (107), and obtain that for almost all $x \geqslant 0$

$$\begin{aligned} v_o''(x) = &- \frac{1}{W_D}\Bigg({\varphi_1^D}''(x) \int_x^\infty \varphi_2(s)\, f_o(s)\, \mathrm{d}s - {\varphi_1^D}'(x)\, \varphi_2(x)\, f_o(x) \\ &+ \varphi_2''(x) \int_0^x \varphi_1^D(s)\, f_o(s)\, \mathrm{d}s + \varphi_2'(x)\, \varphi_1^D(x)\, f_o(x) \Bigg) \\ &\overset{(95)}{=} - f_o - \frac{1}{W_D}\left({\varphi_1^D}''(x) \int_x^\infty \varphi_2(s)\, f_o(s)\, \mathrm{d}s + \varphi_2''(x) \int_0^x \varphi_1^D(s)\, f_o(s)\, \mathrm{d}s \right). \end{aligned}$$

Since the functions φ_1^D and φ_2 solve the homogeneous equation (92), we have that for almost all $x \geqslant 0$

$$L_{\mu,k} v_o \overset{(88)}{=} -v_o'' + b_{\mu,k} v_o = f_o.$$

By same argument, we obtain, that for almost all $x \geqslant 0$

$$L_{\mu,k} v_e \overset{(88)}{=} -v_e'' + b_{\mu,k} v_e = f_e.$$

This finishes the proof. □

5.3.1.2 Proof of Lemma 52

Recall the ansatz (82). We define the function $w\colon D \to \mathbb{R}$ as

$$w(x_1, x_3) = \sum_{k\in S^\mu} w_k(x_3) \sin\Big(x\frac{\pi}{P}x_1\Big)\,, \tag{120}$$

where the functions $(w_k)_{k\in S^\mu}$ are as in Lemma 55.

Lemma 67. *Assume* **S**. *Then for every $f \in L^2_\mu(D)$, the function w defined in* (120) *is an element of the space $H^1_\mu(D)$. Moreover, the part (ii) of the Lemma 52 holds true.*

Proof. By applying Lemma 12 and the inequalities (90) from Lemma 55 part (ii) we obtain

$$\begin{aligned}
\left\|\frac{\partial w}{\partial x_1}\right\|^2_{L^2(D)} &\leqslant \sum_{k\in S^{\mu_1}} k^2\frac{M}{k^2}\left\|f_k\right\|^2_{L^2(\mathbb{R})} \leqslant M\left\|f\right\|^2_{L^2(D)}\,,\\
\left\|\frac{\partial w}{\partial x_3}\right\|^2_{L^2(D)} &\leqslant \sum_{k\in S^{\mu_1}} M\left\|f_k\right\|^2_{L^2(\mathbb{R})} \leqslant M\left\|f\right\|^2_{L^2(D)}\,,
\end{aligned}$$

where the constant M does not depend on the choice of $\mu \in \overline{I^\varepsilon_\mu}$ and $k \in S^{\mu_1} = \mathbb{N}\setminus\{k_1\}$. □

Lemma 68. *For every $f \in L^2_\mu(D)$ the function $w \in H^1_\mu(D)$ defined* (120) *solves the equation* (81) *in the sense of the Definition 51.*

Proof. By Lemma 55, we have that for all $\varphi \in \mathcal{C}^\infty_c(\mathbb{R})$ and all $k \in S^\mu$

$$\int_{\mathbb{R}} w_k'\varphi' + b_{\mu,k}(x)\, w_k\varphi \,\mathrm{d}x = \int_{\mathbb{R}} f_k\varphi\,\mathrm{d}x, \tag{121}$$

where $b_{\mu,k}$ is defined in (87). Take any $\psi \in \mathcal{C}^\infty_{\mathrm{per},2P,b}(D)$.

$$\begin{aligned}
\int_D f\psi\,\mathrm{d}x &= \sum_{k\in S^\mu}\int_{-P}^{P}\int_{\mathbb{R}} f_k(x_3)\sin\Big(k\frac{\pi}{P}x_1\Big)\,\psi(x_1,x_3)\;\mathrm{d}x_3\,\mathrm{d}x_1\\
&\overset{(121)}{=} \sum_{k\in S^\mu}\int_{-P}^{P}\int_{\mathbb{R}} w_k'(x_3)\sin\Big(k\frac{\pi}{P}x_1\Big)\,\psi(x_1,x_3)\\
&\qquad + b_{\mu,k}(x_3)\,w_k(x_3)\sin\Big(k\frac{\pi}{P}x_1\Big)\,\psi(x_1,x_3)\;\mathrm{d}x_3\,\mathrm{d}x_1\\
&= \int_{\mathrm{D}} w_3\psi_3 + \big(1-\omega^2 V(x_3)\big)\,w_1\psi_1 + \mu w\psi\,\mathrm{d}x,
\end{aligned}$$

hence the function w is a weak solution of the equation (81). □

5.3.1.3 Some remarks about the eigenvalues of the operators L_μ

In this section we will identify the eigenvalues of the operator $L_{\mu,k}$ defined in (89) considered on the space $H^1(\mathbb{R})$. Moreover we will give a characterization of the property the operator L_μ defined by the formula (80) has 0 as a eigenvalue. It is stated in the Lemma 73.

We will begin with discussion of the spectral properties of the operator $L_{\mu,k}v = -v'' + b(x)\,v$, with $b(x) = \begin{cases} b_i, & |x| < a \\ b_o, & |x| > a \end{cases}$, namely we will study the eigenvalue problem

$$-v'' + b(x)\,v = \lambda v \text{ on } \mathbb{R},$$

where $v \in L^2(\mathbb{R})$.

Lemma 69. *Assume that $b_o > \lambda > b_i$ are such that*[6]

$$\begin{aligned} \sqrt{-\lambda + b_o} &= \sqrt{-b_i + \lambda}\,\mathrm{tg}\,\sqrt{-b_i + \lambda}\,a \text{ or} \\ \sqrt{-\lambda + b_o} &= -\sqrt{-b_i + \lambda}\,\mathrm{ctg}\,\sqrt{-b_i + \lambda}\,a, \end{aligned} \tag{122}$$

then the equation

$$-v'' + b(x)\,v = \lambda v \text{ on } \mathbb{R}, \tag{123}$$

has one dimensional set of solutions in the space $H^1(\mathbb{R})$. Moreover, if $v \in H^1(\mathbb{R}) \setminus \{0\}$ solves 123, then λ satisfies condition (122)*.*

Proof. Function of the form

$$v(x) = \begin{cases} Ae^{\mathtt{i}\sqrt{-b_i+\lambda}\,x} + Be^{-\mathtt{i}\sqrt{-b_i+\lambda}\,x} & (|x| < a)\,, \\ Ce^{-\sqrt{-\lambda+b_o}(x-a)} & (x > a)\,, \\ De^{\sqrt{-\lambda+b_o}(x+a)} & (x < -a)\,, \end{cases}$$

solves the equation $-v'' + b(x)\,v = \lambda v$ for $x \in \mathbb{R} \setminus \{\pm a\}$. The constants A, B, C, D are chosen in such a way, that the function v is of the class $\mathcal{C}^1$. This leads to the following system, with unknowns A, B, C, D:

$$\begin{cases} Ae^{\mathtt{i}\sqrt{-b_i+\lambda}\,a} + Be^{-\mathtt{i}\sqrt{-b_i+\lambda}\,a} = C \\ Ae^{-\mathtt{i}\sqrt{-b_i+\lambda}\,a} + Be^{\mathtt{i}\sqrt{-b_i+\lambda}\,a} = D \\ \mathtt{i}A\sqrt{-b_i+\lambda}\,e^{\mathtt{i}\sqrt{-b_i+\lambda}\,a} - \mathtt{i}B\sqrt{-b_i+\lambda}\,e^{-\mathtt{i}\sqrt{-b_i+\lambda}\,a} = -C\sqrt{-\lambda+b_o} \\ \mathtt{i}A\sqrt{-b_i+\lambda}\,e^{-\mathtt{i}\sqrt{-b_i+\lambda}\,a} - \mathtt{i}B\sqrt{-b_i+\lambda}\,e^{\mathtt{i}\sqrt{-b_i+\lambda}\,a} = D\sqrt{-\lambda+b_o} \end{cases}.$$

The above system has a non-trivial solution if and only if the determinant of

[6] cf. Remark 71.

its matrix is equal to 0. Computing the determinant we find

$$\det\begin{pmatrix} e^{\mathtt{i}\sqrt{-b_i+\lambda}\,a} & e^{-\mathtt{i}\sqrt{-b_i+\lambda}\,a} & -1 & 0 \\ e^{-\mathtt{i}\sqrt{-b_i+\lambda}\,a} & e^{\mathtt{i}\sqrt{-b_i+\lambda}\,a} & 0 & -1 \\ \mathtt{i}\sqrt{-b_i+\lambda}\,e^{\mathtt{i}\sqrt{-b_i+\lambda}\,a} & -\mathtt{i}\sqrt{-b_i+\lambda}\,e^{-\mathtt{i}\sqrt{-b_i+\lambda}\,a} & \sqrt{-\lambda+b_o} & 0 \\ \mathtt{i}\sqrt{-b_i+\lambda}\,e^{-\mathtt{i}\sqrt{-b_i+\lambda}\,a} & -\mathtt{i}\sqrt{-b_i+\lambda}\,e^{\mathtt{i}\sqrt{-b_i+\lambda}\,a} & 0 & -\sqrt{-\lambda+b_o} \end{pmatrix}$$

$$= \det\begin{pmatrix} \left(\mathtt{i}\sqrt{-b_i+\lambda}+\sqrt{-\lambda+b_o}\right)e^{\mathtt{i}\sqrt{-b_i+\lambda}\,a} & \left(\sqrt{-\lambda+b_o}-\mathtt{i}\sqrt{-b_i+\lambda}\right)e^{-\mathtt{i}\sqrt{-b_i+\lambda}\,a} \\ \left(\mathtt{i}\sqrt{-b_i+\lambda}-\sqrt{-\lambda+b_o}\right)e^{-\mathtt{i}\sqrt{-b_i+\lambda}\,a} & -\left(\sqrt{-\lambda+b_o}+\mathtt{i}\sqrt{-b_i+\lambda}\right)e^{\mathtt{i}\sqrt{-b_i+\lambda}\,a} \end{pmatrix}$$

$$= -\left(\mathtt{i}\sqrt{-b_i+\lambda}+\sqrt{-\lambda+b_o}\right)^2 e^{2\mathtt{i}\sqrt{-b_i+\lambda}\,a} + \left(\mathtt{i}\sqrt{-b_i+\lambda}-\sqrt{-\lambda+b_o}\right)^2 e^{-2\mathtt{i}\sqrt{-b_i+\lambda}\,a}$$

The determinant is equal to 0 if

$$\left(\mathtt{i}\sqrt{-b_i+\lambda}+\sqrt{-\lambda+b_o}\right)e^{\mathtt{i}\sqrt{-b_i+\lambda}\,a} = \pm\left(\mathtt{i}\sqrt{-b_i+\lambda}-\sqrt{-\lambda+b_o}\right)e^{-\mathtt{i}\sqrt{-b_i+\lambda}\,a},$$

which is equivalent to the condition (122).

In order to finish the proof, we have to show that λ is a simple eigenvalue. Let $\varphi, \psi\colon \mathbb{R} \to \mathbb{R}$ be eigenfunctions corresponding to λ. We will show that the functions φ and ψ are linearly dependent. Consider

$$W(x) = \det\begin{bmatrix} \varphi(x) & \psi(x) \\ \varphi'(x) & \psi'(x) \end{bmatrix} = \varphi(x)\,\psi'(x) - \varphi'(x)\,\psi(x)\,.$$

Since both of the functions φ and ψ solve the equation (123), we have that $W'(x) = 0$, and hence $W(x) = \text{const.} = c$. Since

$$\lim_{x\to\pm\infty}\varphi(x) = \lim_{x\to\pm\infty}\varphi'(x)\lim_{x\to\pm\infty} = \psi(x) = \lim_{x\to\pm\infty}\psi'(x) = 0,$$

we have that $c = 0$. Let $x_0 \in \mathbb{R}$ be such that $\varphi'(x_0) = 0$. By the uniqueness of the solution of the initial value problem, we have that $\varphi(x_0) \neq 0$. Since $c = 0$, we have that $\psi'(x_0) = 0$. Then, again by the uniqueness of the solution of the initial value problem, we have that $\psi(x_0) \neq 0$. Hence, there exists $a \in \mathbb{R}$ such that

$$a\varphi(x_0) = \psi(x_0)\,,$$
$$a\varphi'(x_0) = \psi'(x_0) = 0.$$

Finally, by the uniqueness of the solution of the initial value problem, we have that $a\varphi = \psi$. □

Remark 70. Assume that $\lambda \in \mathbb{R}$ is an eigenvalue of the operator $L_{\mu,k}$ defined in (89). Let $\varphi_k \in H^1(\mathbb{R})\setminus\{0\}$ be such that $L_{\mu,k}\varphi_k = \lambda\varphi_k$. Consider the function $\Phi\colon D \to \mathbb{R}$ defined as

$$\Phi(x_1, x_3) = \varphi_k(x_3)\sin\left(k\frac{\pi}{P}x_1\right).$$

Note that $\Phi \in H^1_{\text{odd}}(D)$ and $L_\mu\Phi = \lambda\Phi$, hence λ is an eigenvalue of the operator L_μ.

Remark 71. Recall (59). Note that

- $W_N = W_N(\mu, k) = 0$ if and only if

$$\sqrt{b_o(\mu,k)} = \sqrt{-b_i(\mu,k)}\,\mathrm{tg}\,\sqrt{-b_i(\mu,k)}\,a,$$

- $W_D = W_D(\mu, k) = 0$ if and only if

$$\sqrt{b_o(\mu,k)} = -\sqrt{-b_i(\mu,k)}\,\mathrm{ctg}\,\sqrt{-b_i(\mu,k)}\,a.$$

The proof follows from formula (85) and Lemma 59. The condition $\mathbf{S}^3$ expresses the fact that that for all $k \in \mathbb{N}$ and for all $\mu \in \overline{I^{\varepsilon}_{\mu_1}}$.

$$W_N(\mu, k) = 0 \text{ or } W_D(\mu, k) = 0$$

if and only of $k = k_1$ and $\mu = \mu_1$.

Remark 72. Assume $\mathbf{S}$. Let $\mu \in \overline{I^{\varepsilon}_{\mu_1}}$ and $k \in \mathbb{N}$. The operator $L_{\mu,k}$ (89) has 0 eigenvalue if and only if $(\mu, k) = (\mu_1, k_1)$.

Lemma 73. *Assume* $\mathbf{S}$ *and let* $\mu \in \overline{I^{\varepsilon}_{\mu_1}}$. *The operator* $L_\mu \colon H^1_{\mathrm{odd}}(D) \to L^2_{\mathrm{odd}}(D)$ *defined in 80 has zero eigenvalue if and only if* $\mu = \mu_1$. *Moreover* 0 *is a simple eigenvalue of the operator* L_{μ_1}, *i.e.* $\dim \mathtt{ker}\, L_{\mu_1} = 1$.

Proof. Assume that $\Psi \in H^1_{\mathrm{odd}}(D) \setminus \{0\}$ is such that $L_\mu \Psi = 0$. The function Ψ has a form

$$\Psi(x_1, x_3) = \sum_{\mu \in \mathbb{N}} \psi_k \sin\Big(k \frac{\pi}{P} x_1\Big) \text{ for almost all } (x_1, x_3) \in D.$$

Then, for every $k \in \mathbb{N}$, we have that $L_{\mu,k}\psi_k = 0$. By Remark 72, we have that $\psi_k \neq 0$ if and only if $k = k_1$ and $\mu = \mu_1$.

It remains to show, that the eigenvalue is simple. Suppose that $\Psi, \Phi \in H^1(D) \setminus \{0\}$ are such that $L_{\mu_1}\Psi = L_{\mu_1}\Phi = 0$. Write

$$\Psi(x_1, x_3) = \sum_{\mu \in \mathbb{N}} \psi_k \sin\Big(k \frac{\pi}{P} x_1\Big) \text{ for almost all } (x_1, x_3) \in D,$$

$$\Phi(x_1, x_3) = \sum_{\mu \in \mathbb{N}} \phi_k \sin\Big(k \frac{\pi}{P} x_1\Big) \text{ for almost all } (x_1, x_3) \in D.$$

We have that for all $k \in \mathbb{N}$, that $L_{\mu_1,k} = \psi_k = L_{\mu_1,k} = \phi_k = 0$. By Remark 72, we obtain, that $\psi_k \neq 0$ if and only if $k = k_1$. We have the same for the functions $(\phi_k)_{k \in \mathbb{N}}$ i.e., $\phi_k \neq 0$ if and only if $k = k_1$. Since the operator L_{μ_1,k_1} has zero as a simple eigenvalue, we have, that for some $a \in \mathbb{R}$ $\phi_{k_1} = a\psi_{k_1}$, and hence $\Phi = a\Psi$. □

As a consequence of Lemma 52 we get

Remark 74. Assume **S**. The operator L_μ has an inverse $T_\mu = L^\mu \colon L^2_{\text{odd}}(D) \to H^1_{\text{odd}}(D)$ if and only if $\mu \neq \mu_1$. Moreover $\texttt{ker}\, L_{\mu_1} = \operatorname{lin}\{\varphi\}$, where $\varphi(x_1, x_3) = \widetilde{\varphi}(x_3) \sin\bigl(k_1 \frac{\pi}{P} x_1\bigr)$ with $\widetilde{\varphi} \in \texttt{ker}\, L_{\mu_1,k_1}$.

Remark 75. In the Examples 44, 49 and 50 we present a choice of the constants a, ω, P, V_o and V_i satisfying assumption **S**. in such a way,

We will finish this section with determining the essential spectrum of the operator $L_{\mu,k}$ defined in (88).

Lemma 76. *Let* $V, W \in L^\infty(\mathbb{R})$ *and assume that the set* $\operatorname{supp} W$ *compact. Consider the operators* $A, B \colon H^2(\mathbb{R}) \to L^2(\mathbb{R})$

$$
\begin{aligned}
A &= -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V, \\
B &= A + W.
\end{aligned}
$$

Then $\sigma_{\text{ess}}(A) = \sigma_{\text{ess}}(B)$.

Proof. Define an operator $T \colon L^2(\mathbb{R}) \to H^2(\mathbb{R})$ as

$$
T = (A - \mathtt{i} \cdot \mathrm{Id})^{-1} - (B - \mathtt{i} \cdot \mathrm{Id})^{-1} .
$$

We will show, that the operator T is compact. Note that

$$
T = -(B - \mathtt{i} \cdot \mathrm{Id})^{-1} (A - B)(A - \mathtt{i} \cdot \mathrm{Id})^{-1} .
$$

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of $L^2(\mathbb{R})$ functions, which is bounded. Since the operator $(A - \mathtt{i} \cdot \mathrm{Id})^{-1}$ is continuous, we have that, there exists a constant $M > 0$ such that

$$
\left\| (A - \mathtt{i} \cdot \mathrm{Id})^{-1} (f_n) \right\|_{H^2(\mathbb{R})} \leqslant M \left\| f_n \right\|_{L^2(\mathbb{R})} \quad (n \in \mathbb{N}) .
$$

Observe, that $(A - B) = W \cdot$. Therefore there exist $L > 0$ such that

$$
\operatorname{supp} (A - B)(A - \mathtt{i} \cdot \mathrm{Id})^{-1} (f_n) \subseteq [-L, L] \quad (n \in \mathbb{N}) .
$$

By [1, Theorem 6.3, p. 168], the space $H^1(-L, L)$ embeds compactly into the space $L^2(-L, L)$, hence the sequence $\bigl((A - B)(A - \mathtt{i} \cdot \mathrm{Id})^{-1}(f_n)\bigr)_{n\in\mathbb{N}}$ has a convergent subsequence. Using the fact that the operator $(B - \mathtt{i} \cdot \mathrm{Id})^{-1}$ is continuous, we have that the sequence $(T f_n)_{n\in\mathbb{N}}$ has a convergent subsequence, which shows that the operator T is compact. In order to obtain the claim, we apply [16, Theorem 5.35, p. 244]. □

As a consequence of Lemma 76, we get the following.

Lemma 77. *Let* $L_{\mu,k}$ *be as in* (88). *Then* $\sigma_{\text{ess}}(L_{\mu,k}) = [b_o(k, \mu), +\infty)$, *for all* $\mu \in \overline{I^\varepsilon_{\mu_1}}$ *and* $k \in \mathbb{N}$, *where* $b_o(k, \mu)$ *is as in* (83).

Proof. Consider two operators

$$A = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + b_o(k,\mu)\,,$$
$$B = A + W,$$

where $W = \begin{cases} b_i(\mu,k) - b_o(\mu,k) & |x| < a, \\ 0 & |x| > a, \end{cases}$. Note that $\operatorname{supp} W = [-a,a]$ and that $B = L_{\mu,k}$. By Lemma 76, we have that $\sigma_{\mathrm{ess}}(A) = \sigma_{\mathrm{ess}}(B) = \sigma_{\mathrm{ess}}(L_{\mu,k})$. Moreover, we have that $\sigma_{\mathrm{ess}}(A) = [b_o(k,\mu)\,,\infty)$. □

5.3.2 About the equation $\widetilde{L_\mu} w = f$

Recall that $\mu_1 \in \mathbb{R}$ is such that $S^{\mu_1} = \mathbb{N} \setminus \{k_1\}$, and for all $\mu \in J^{\varepsilon}_{\mu_1}$, $S^{\mu} = \mathbb{N}$, where the set S^μ was defined in (65). Zero is a simple eigenvalue of the operator L_{μ_1} (cf. (80)). We formally introduce an operator $\widetilde{L_\mu}$ by the formula

$$\widetilde{L_\mu} = L_\mu + \mathtt{P}_\varphi, \tag{124}$$

where the operator L_μ is as in (80) and $\mathtt{P}_\varphi w = \langle \varphi, w\rangle_{L^2(D)}\,\varphi$, with $\varphi \in \mathtt{ker}\, L_{\mu_1}$ such that $\|\varphi\|_{L^2(D)} = 1$.

In this section we will consider a linear equation of the form

$$\widetilde{L_\mu} w \overset{(124)}{=} L_\mu w + \mathtt{P}_\varphi w = f \text{ on } D, \tag{125}$$

where $f \in L^2_{\mathrm{odd}}(\mathrm{D})$ is a given function. We seek for $w \in H^1_{\mathrm{odd}}(D)$ being a weak solution. The existence of a solution operator for this problem is stated in Corollary 86.

For $\mu \in \mathbb{R}$ and $k \in \mathbb{N}$ consider a family of operators $\widetilde{L_{\mu,k}}$ defined as

$$\widetilde{L_{\mu,k}} u = \begin{cases} L_{\mu,k} u, & k \neq k_1, \\ L_{\mu,1} u, + \mathtt{P}_{\widetilde{\varphi}} u, & k = k_1, \end{cases} \tag{126}$$

where $\mathtt{P}_{\widetilde{\varphi}} h = \langle h, \widetilde{\varphi}\rangle_{L^2(\mathbb{R})}\,\widetilde{\varphi}$, for all $h \in L^2(\mathbb{R})$ with $\widetilde{\varphi}$ being the k_1-th Fourier coefficient of φ, i.e.

$$\varphi(x_1, x_3) = \widetilde{\varphi}(x_3) \sin\Big(k_1 \frac{\pi}{P} x_1\Big)\,,$$

and where the operator $L_{\mu,k}$ is defined in (89).

We write the functions w and f as

$$\begin{aligned} w(x_1,x_3) &= \sum_{k\in\mathbb{N}} w_k(x_3) \sin\Big(k\frac{\pi}{P}x_1\Big)\,, \\ f(x_1,x_3) &= \sum_{k\in\mathbb{N}} f_k(x_3) \sin\Big(k\frac{\pi}{P}x_1\Big)\,. \end{aligned} \tag{127}$$

Then the equation (125) becomes

$$\widetilde{L_{\mu,k}} w_k = f_k. \tag{128}$$

The existence result (in case of $k = k_1$) is stated in Lemma 85. The equation (128) for $k \neq k_1$ was studied in section 5.3.1.1 (cf. Lemma 55).

In the following we will consider the problem of existence of the inverse of the operator $\widetilde{L_{\mu,k_1}}$.

5.3.2.1 Self-adjointness of the operator $\widetilde{L_{\mu,k_1}}$

Recall the following fact.

Lemma 78. *Let $v \in L^2(\mathbb{R})$. Then $\int_{\mathbb{R}} (-1)^k \varphi^{(k)} v \,\mathrm{d}x = \int_{\mathbb{R}} w\varphi \,\mathrm{d}x$ for all $\varphi \in \mathcal{C}_c^\infty(\mathbb{R})$ if and only if $\mathcal{F}(w) = (\mathtt{i}\xi)^k \mathcal{F}(v)$, where $\mathcal{F}$ denotes the Fourier transform.*

Lemma 79. *$v \in H^2(\mathbb{R})$ if and only if $v \in L^2(\mathbb{R})$ and there exists $w \in L^2(\mathbb{R})$ such that*

$$\int_{\mathbb{R}} \varphi'' v \,\mathrm{d}x = \int_{\mathbb{R}} w\varphi \,\mathrm{d}x \quad \big(\varphi \in H^2(\mathbb{R})\big).$$

Proof. It is sufficient to show the "⇒" part. Define

$$h = \mathcal{F}^{-1}\left(\frac{\mathtt{i}\xi}{1+\xi^2}\mathcal{F}(-w+v)\right). \tag{129}$$

We will show that $h \in L^2(\mathbb{R})$ and $h = v'$ (i.e. h is a weak derivative of v). Observe that

$$\left|\frac{\mathtt{i}\xi}{1+\xi^2}\right| \leqslant \frac{1}{2} \quad (\xi \in \mathbb{R}). \tag{130}$$

We have that

$$\begin{aligned}\|h\|_{L^2(\mathbb{R})} &\leqslant \left\|\frac{-\mathtt{i}\xi}{1+\xi^2}\mathcal{F}(-w+v)\right\|_{L^2(\mathbb{R})} \\ &\overset{(130)}{\leqslant} \frac{1}{2}\|\mathcal{F}(-w+v)\|_{L^2(\mathbb{R})} = \frac{1}{2}\|-w+v\|_{L^2(\mathbb{R})},\end{aligned}$$

therefore $h \in L^2(\mathbb{R})$. By Lemma 78, we have that

$$\mathcal{F}(-w) = \xi^2 \mathcal{F}(v). \tag{131}$$

Write

$$\mathcal{F}(h) \overset{(129)}{=} \frac{\mathtt{i}\xi}{1+\xi^2}\mathcal{F}(-w+v) \overset{(131)}{=} \mathtt{i}\xi\mathcal{F}(v).$$

Using the Lemma 78 again, we obtain that $v' = h$.

□

Lemma 80. *Let $\widetilde{V} \in L^\infty(\mathbb{R})$. Define the operator $A\colon H^2(\mathbb{R}) \subseteq L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by the formula*

$$Av = -v'' + \widetilde{V}(x)\, v \quad \big(v \in H^2(\mathbb{R})\big)\,.$$

Let $h \in H^2(\mathbb{R}) \setminus \{0\}$ be such that $Ah = \lambda h$ for some $\lambda \in \mathbb{R}$. Define $\mathfrak{L} = L^2(\mathbb{R}) \cap \{h\}^{\perp_{L^2(\mathbb{R})}}$ and $\mathfrak{H} = H^2(\mathbb{R}) \cap \{h\}^{\perp_{L^2(\mathbb{R})}}$. Then, the operator $A\colon \mathfrak{H} \subseteq \mathfrak{L} \to \mathfrak{L}$ is self-adjoint.

Proof. The adjoint $A^\star$ of A is defined as follows. $\mathrm{Dom}(A^\star)$ is the set of all $v \in \mathfrak{L}$ such that, there exists $w \in \mathfrak{L}$ with

$$\langle Au, v\rangle_{L^2(\mathbb{R})} = \langle u, w\rangle_{L^2(\mathbb{R})} \quad (u \in \mathrm{Dom}(A) = \mathfrak{H})\,. \tag{132}$$

We put $A^\star v = w$. Note that

$$\langle Ah, v\rangle_{L^2(\mathbb{R})} = \langle \lambda h, v\rangle_{L^2(\mathbb{R})} = 0 = \langle h, w\rangle_{L^2(\mathbb{R})}\,.$$

Therefore, the formula (132) holds true for all $u \in H^2(\mathbb{R})$ i.e.

$$\langle Au, v\rangle_{L^2(\mathbb{R})} = \langle u, w\rangle_{L^2(\mathbb{R})} \quad \big(u \in H^2(\mathbb{R})\big)\,. \tag{133}$$

The formula (133) is equivalent to

$$\int_{\mathbb{R}} -u''v \,\mathrm{d}x = \int_{\mathbb{R}} \Big(w - \widetilde{V}(x)\, v\Big)\, u \,\mathrm{d}x \quad \big(u \in H^2(\mathbb{R})\big)\,,$$

which means that v has a second weak derivative given by $-w + \widetilde{V}(x)\, v$ (cf. Lemma 79). We have that

$$-v'' + \widetilde{V}(x)\, v = w,$$

hence $A^\star v = -v'' + \widetilde{V}(x)\, v$, so that $\mathrm{Dom}(A) = \mathrm{Dom}(A^\star) = \mathfrak{H}$ and $A = A^\star$. □

Lemma 81. *For every $\mu \in \mathbb{R}$ the operator $\widetilde{L_{\mu,k_1}}\colon H^2(\mathbb{R}) \subseteq L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is self-adjoint.*

Proof. Note that the operator $\widetilde{L_{\mu,k_1}}$ is symmetric. By [12, Theorem VIII.3, p. 256] it is enough to show, that

$$\mathtt{im}\left(\widetilde{L_{\mu,k_1}} \pm \mathtt{i}\right) = L^2(\mathbb{R})\,,$$

i.e., for every $f \in L^2(\mathbb{R})$, there exists $v \in H^2(\mathbb{R})$ such that

$$\left(\widetilde{L_{\mu,k_1}} \pm \mathtt{i}\right) v \overset{(126)}{=} (L_{\mu,k_1} + \mathtt{P}_{\widetilde{\varphi}} \pm \mathtt{i})\, v = f. \tag{134}$$

Denote $\mathfrak{L} = L^2(\mathbb{R}) \cap \{h\}^{\perp_{L^2(\mathbb{R})}}$ and $\mathfrak{H} = H^2(\mathbb{R}) \cap \{h\}^{\perp_{L^2(\mathbb{R})}}$. We have

$$L^2(\mathbb{R}) = \mathrm{lin}\,\{\widetilde{\varphi}\} \oplus \mathfrak{L},$$
$$H^2(\mathbb{R}) = \mathrm{lin}\,\{\widetilde{\varphi}\} \oplus \mathfrak{H}.$$

Write $f = s\widetilde{\varphi} + g$ with $s \in \mathbb{R}$, $g \in \mathfrak{L}$.

Note that by Lemma 80 the operator $\widetilde{L_{\mu,k_1}}_{|\mathfrak{H}}$ is self-adjoint, hence, again by [12, Theorem VIII.3, p. 256] for every $g \in \mathfrak{L}$, there exists $w \in \mathfrak{H}$ such that $\left(\widetilde{L_{\mu,k_1}} \pm \mathtt{i}\right) w = g$. Define $t = \frac{s}{\mu-\mu_1+1\pm\mathtt{i}}$. Observe that the function $v = t\widetilde{\varphi} + g$ is an element of $H^2(\mathbb{R})$. Then

$$\left(\widetilde{L_{\mu,k_1}} \pm \mathtt{i}\right) v = t(\mu - \mu_1 + 1 \pm \mathtt{i})\, \widetilde{\varphi} + \left(\widetilde{L_{\mu,k_1}} \pm \mathtt{i}\right) w = s\widetilde{\varphi} + g = f,$$

hence (134) holds true. □

5.3.2.2 Properties of the spectrum the operator $\widetilde{L_{\mu,k_1}}$

Lemma 82. *Assume* **S**, *then* 0 *is not an eigenvalue of the operator* $\widetilde{L_{\mu,k_1}}$ *for all* $\mu \in \overline{I^{\varepsilon}_{\mu_1}}$.

Proof. Suppose that there there exists a function $h \in H^2(\mathbb{R}) \setminus \{0\}$ such that

$$\widetilde{L_{\mu,k_1}} h \overset{(126)}{=} L_{\mu,k_1} h + \langle \widetilde{\varphi}, h \rangle\, \widetilde{\varphi} = 0, \tag{135}$$

for some $\mu \in \overline{I^{\varepsilon}_{\mu_1}}$. Since $L_{\mu_1,k_1}\widetilde{\varphi} = 0$, we have that

$$L_{\mu,k_1}\widetilde{\varphi} = (\mu - \mu_1)\, \widetilde{\varphi}. \tag{136}$$

After testing (135) with $\widetilde{\varphi}$ we obtain

$$0 = \langle L_{\mu,k_1} h, \widetilde{\varphi} \rangle + \langle \widetilde{\varphi}, h \rangle = \langle L_{\mu,k_1} \widetilde{\varphi}, h \rangle + \langle \widetilde{\varphi}, h \rangle \overset{(136)}{=} (\mu - \mu_1 + 1)\, \langle h, \widetilde{\varphi} \rangle\,.$$

If μ is in a sufficiently small neighbourhood of μ_1, we obtain that $\langle h, \widetilde{\varphi} \rangle = 0$, and therefore $L_{\mu,k_1} h = 0$ (cf. equation (135)), i.e., 0 is an eigenvalue of the operator L_{μ,k_1}. By Remark 72 we have to have that $\mu = \mu_1$, and therefore $h = \widetilde{\varphi}$. This, together with $\langle h, \widetilde{\varphi} \rangle = 0$ gives a contradiction. □

Lemma 83. *Assume* **S**. *For all* $\mu \in \overline{I^{\varepsilon}_{\mu_1}}$ *the value* 0 *is isolated from the spectrum* $\sigma\left(\widetilde{L_{\mu,k_1}}\right)$. *In other words, there exists* $\delta > 0$ *such that* $\inf \left|\sigma\left(\widetilde{L_{\mu,k_1}}\right)\right| \geqslant \delta$, *for all* $\mu \in \overline{I^{\varepsilon}_{\mu_1}}$.

Proof. Note that for all $\mu \in \mathbb{R}$ the operator $\widetilde{L_{\mu,k_1}} - L_{\mu,k_1} = \mathtt{P}_{\widetilde{\varphi}}$ is a compact operator. Therefore, by [16, Theorem 5.35, p. 244], $\sigma_{\text{ess}}\left(\widetilde{L_{\mu,k_1}}\right) = \sigma_{\text{ess}}(L_{\mu,k_1})$, which implies that 0 is isolated from $\sigma_{\text{ess}}\left(\widetilde{L_{\mu,k_1}}\right)$. Moreover by Lemma 82, we have that 0 is not an eigenvalue of $\widetilde{L_{\mu,k_1}}$ for all μ being in a neighbourhood of μ_1. By Lemma 77, we have that $\sigma_{\text{ess}}(L_{\mu,k_1}) = [b_o(k_1, \mu)\,, +\infty)$, where $b_o(k_1, \mu)$ is as in the formula (83). Let $\delta = \min_{\mu \in \overline{I^{\varepsilon}_{\mu_1}}} b_o(k_1, \mu)$. By (84) the value δ is positive. □

5.3.2.3 Existence of the inverse of the operator $\widetilde{L_{\mu,k_1}}$

Lemma 84. *There exist constants $B > 0$ and $C > 0$ such that for all $\mu \in \overline{I^{\varepsilon}_{\mu_1}}$ and all $g \in H^2(\mathbb{R})$*

$$\|g\|_{H^2(\mathbb{R})} \leqslant B\sqrt{2\left\|\widetilde{L_{\mu,k_1}}g\right\|^2_{L^2(\mathbb{R})} + C\|g\|^2_{L^2(\mathbb{R})}}.$$

Proof. Note that there exists a constant $B > 0$ such that for all $g \in H^2(\mathbb{R})$

$$\|g\|^2_{H^2(\mathbb{R})} \leqslant B^2\Big(\|-g''\|^2_{L^2(\mathbb{R})} + \|g\|^2_{L^2(\mathbb{R})}\Big) \quad \big(g \in H^2(\mathbb{R})\big). \tag{137}$$

Let $V_{\max} = \max\Big\{\big(1-\omega^2 V_o\big)^2, \big(1-\omega^2 V_i\big)^2\Big\}\frac{k_1^4\pi^4}{P^4}$. Observe that, there exists a constant $C > 0$ (which does not depend on the value of $\mu \in (\mu_1-\varepsilon, \mu_1+\varepsilon)$), such that for all $g \in H^2(\mathbb{R})$

$$\begin{aligned}\|-g''\|^2_{L^2(\mathbb{R})} + \|g\|^2_{L^2(\mathbb{R})} &= \left\|\widetilde{L_{\mu,k_1}}g - b(\cdot)\,g - \mathtt{P}_{\widetilde{\varphi}}g\right\|^2_{L^2(\mathbb{R})} + \|g\|^2_{L^2(\mathbb{R})}\\ &\leqslant 2\left\|\widetilde{L_{\mu,k_1}}g\right\|^2_{L^2(\mathbb{R})} + 2\,\|\mathtt{P}_{\widetilde{\varphi}}g\|^2_{L^2(\mathbb{R})} + 2\big(V_{\max}+\mu^2+1\big)\,\|g\|^2_{L^2(\mathbb{R})}\\ &\leqslant 2\left\|\widetilde{L_{\mu,k_1}}g\right\|^2_{L^2(\mathbb{R})} + C\,\|g\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

Above estimate, together with inequality (137) gives the claim. □

Lemma 85. *Assume* **S**. *For every $\mu \in \overline{I^{\varepsilon}_{\mu_1}}$ and for every $f \in L^2(\mathbb{R})$, there exists a unique $w \in H^2(\mathbb{R})$ such that*

$$\widetilde{L_{\mu,k_1}}w = L_{\mu,k_1}w + \mathtt{P}_{\widetilde{\varphi}}w = f.$$

Moreover, there exists a constant $M > 0$ such that

$$\|w\|_{H^2(\mathbb{R})} \leqslant M\,\|f\|_{L^2(\mathbb{R})} \quad \big(f \in L^2(\mathbb{R}), \mu \in \overline{I^{\varepsilon}_{\mu_1}}\big).$$

Proof. By Lemma 83 the inverse of the operator $\widetilde{L_{\mu,k_1}}\colon H^2(\mathbb{R}) \to L^2(\mathbb{R})$ exists, for all μ in some neighbourhood of μ_1. It remains to show that the norm of the operator $\left(\widetilde{L_{\mu,k_1}}\right)^{-1} : L^2(\mathbb{R}) \to H^2(\mathbb{R})$ can be bounded by a constant, which does not depend on $\mu \in \overline{I^{\varepsilon}_{\mu_1}}$. In order to simplify the notation denote $A = \widetilde{L_{\mu}}$, with $\mathrm{Dom}(A) = H^2(\mathbb{R})$ and $H = L^2(\mathbb{R})$.

Let $(\mathcal{P}_\nu)_{\nu\in\mathbb{R}}$ be the family of spectral projections of the operator A. Define

$$\begin{aligned}P^+ &= \int_0^\infty 1\,\mathrm{d}\mathcal{P}_\nu,\\ P^- &= \int_{-\infty}^0 1\,\mathrm{d}\mathcal{P}_\nu.\end{aligned}$$

Denote $H^{\pm} = P^{\pm}(H)$, then $H = H^{+} \oplus H^{-}$. The operators $A^{\pm} = P^{\pm}A = AP^{\pm}\colon \mathrm{Dom}(A^{\pm}) = \mathrm{Dom}(A) \cap H^{\pm} \to H^{\pm}$ are self-adjoint. Moreover, the operators A^{+}, A^{-} are positively, negatively definite, respectively. Since $A = A^{+} + A^{-}$, we have that $A^{-1} = (A^{+})^{-1} P^{+} + (A^{-})^{-1} P^{-}$, and in a consequence

$$\left\| A^{-1} \right\| \leqslant \left\| A^{+-1} \right\| + \left\| (A^{-})^{-1} \right\|. \tag{138}$$

$$\left\| (A^{+})^{-1} \right\| = \sup_{f^{+} \in H^{+}} \frac{\left\| (A^{+})^{-1} f^{+} \right\|_{H^2(\mathbb{R})}}{\| f^{+} \|_{L^2(\mathbb{R})}} = \sup_{u \in \mathrm{Dom}(A^{+})} \frac{\| u \|_{H^2(\mathbb{R})}}{\| A^{+} u \|_{L^2(\mathbb{R})}}.$$

Using Lemma 84, we have that there exist constants $B > 0$ and $C > 0$ such that

$$\left\| (A^{+})^{-1} \right\| \leqslant B \sup_{u \in \mathrm{Dom}(A^{+})} \frac{\sqrt{2 \| A^{+} u \|^2_{L^2(\mathbb{R})} + C \| u \|^2_{L^2(\mathbb{R})}}}{\| A^{+} u \|_{L^2(\mathbb{R})}}. \tag{139}$$

Applying Bunyakovsky-Cauchy–Schwarz inequality for all $u \in \mathrm{Dom}(A^{+})$ we have $|\langle A^{+} u, u \rangle| \leqslant \| u \|_{L^2(\mathbb{R})} \| A^{+} u \|_{L^2(\mathbb{R})}$, hence

$$\frac{\| u \|^2_{L^2(\mathbb{R})}}{\| A^{+} u \|^2_{L^2(\mathbb{R})}} \leqslant \frac{\| u \|^4_{L^2(\mathbb{R})}}{\langle A^{+} u, u \rangle^2} \quad \left(u \in \mathrm{Dom}\left(A^{+} \right) \right). \tag{140}$$

By [13, Theorem 4.3.1 p. 78], we have that

$$\inf_{u \in \mathrm{Dom}(A^{+})} \frac{\langle A^{+} u, u \rangle}{\| u \|^2_{L^2(\mathbb{R})}} = \inf \sigma\left(A^{+} \right) \geqslant \delta. \tag{141}$$

Using inequalities (140) and (141), we conclude that $\frac{\| u \|^2_{L^2(\mathbb{R})}}{\| A^{+} u \|^2_{L^2(\mathbb{R})}} \leqslant \frac{1}{\delta^2}$ for all $u \in \mathrm{Dom}(A^{+})$ and in a consequence, together with (139) gives

$$\left\| (A^{+})^{-1} \right\| \leqslant B \sqrt{2 + \frac{C}{\delta^2}}. \tag{142}$$

The same argument can be performed for the operator $-A^{-}$. This, together with (138) finishes the proof. □

5.3.2.4 Existence of the inverse of the operator $\widetilde{L_{\mu}}$

We will summarise this section with a statement about the existence of a solution of the equation (125). As a consequence of Lemmas 55 and 85 we get the following:

Corollary 86. *Assume* **S**. *Then for all* $\mu \in \overline{I^{\varepsilon}_{\mu_1}}$ *and all* $f \in L^2_{\mathrm{odd}}(D)$ *there exists a unique* $w \in H^1_{\mathrm{odd}}(D)$ *solving weakly equation* (125). *Denote* $\widetilde{T_{\mu}}(f) = w$.

Moreover, there exits a constant $M > 0$ such that for all $\mu \in \overline{I^{\varepsilon}_{\mu_1}}$ and for all $f \in L^2_{\text{odd}}(D)$

$$\left\| \widetilde{T_\mu}(f) \right\|_{H^1(D)} \leqslant M \left\| f \right\|_{L^2(D)} .$$

Proof. Recall ansatz described in (127) and (126) for the definition of the operators $\widetilde{L_{\mu,k}}$. As a consequence of Lemma 55, we have that:

(a) For all $k \in S^{\mu_1} = \mathbb{N} \setminus \{k_1\}$, all $\mu \in \overline{I^{\varepsilon}_{\mu_1}}$ and for every $f_k \in L^2(\mathbb{R})$ the function $w_k \in H^1(\mathbb{R})$ is a unique solution the equation (128).

(b) There exists a constant $M > 0$ such that for all $k \in S^{\mu_1} = \mathbb{N} \setminus \{k_1\}$, all $\mu \in \overline{I^{\varepsilon}_{\mu_1}}$ and for every $f_k \in L^2(\mathbb{R})$

$$\begin{aligned} \left\| w_k \right\|_{L^2(\mathbb{R})} &\leqslant \frac{M}{k} \left\| f_k \right\|_{L^2(\mathbb{R})} , \\ \left\| w_k' \right\|_{L^2(\mathbb{R})} &\leqslant M \left\| f_k \right\|_{L^2(\mathbb{R})} . \end{aligned}$$

Lemma 85 implies that

(c) For all $\mu \in \overline{I^{\varepsilon}_{\mu_1}}$ and all $f_{k_1} \in L^2(\mathbb{R})$, the function $w_{k_1} \in H^1(\mathbb{R})$ is a unique solution of the equation (128) (with $k = k_1$).

(d) There exists a constant $\widetilde{M} > 0$ such that for all $\mu \in \overline{I^{\varepsilon}_{\mu_1}}$ and all $f_{k_1} \in L^2(\mathbb{R})$

$$\left\| w_{k_1} \right\|_{H^1(\mathbb{R})} \leqslant \widetilde{M} \left\| f_{k_1} \right\|_{L^2(\mathbb{R})} .$$

Having (a), (b), (c) and (d) we proceed as in the proof of Lemma 52. □

Remark 87. A possible choice of the parameters a, ω, P, V_o, V_i and μ_1 such that the assumptions of the Corollary 86 are satisfied was described in the Examples 44, 49 and 50.

Remark 88. For all μ sufficiently close to μ_1

$$\widetilde{L_\mu}(\varphi) = (1 + \mu - \mu_1)\, \varphi, \tag{143}$$

$$\widetilde{T_\mu}(\varphi) = \frac{1}{1 + \mu - \mu_1} \varphi, \tag{144}$$

$$\widetilde{T_\mu}(\mathtt{P}_\varphi h) = \frac{\langle h, \varphi \rangle_{L^2(D)}}{1 + \mu - \mu_1} \varphi \quad \left(h \in H^1_{\text{odd}}(D) \right) . \tag{145}$$

Proof. Calculate

$$\widetilde{L_\mu}\varphi \overset{(124)}{=} L_\mu \varphi + \mathtt{P}_\varphi \varphi = \underbrace{L_{\mu_1}\varphi}_{=0} + (\mu - \mu_1)\, \varphi + \varphi = (1 + \mu - \mu_1)\, \varphi.$$

Since $\widetilde{T_\mu} = \left(\widetilde{L_\mu}\right)^{-1}$ and the operator $\widetilde{T_\mu}$ is linear, we have the claims (143) and (144). In order to prove (145), for every $h \in H^1_{\text{odd}}(D)$ calculate

$$\widetilde{T_\mu}(\mathtt{P}_\varphi h) = \widetilde{T_\mu}(\langle h, \varphi\rangle \varphi) \overset{(144)}{=} \frac{\langle h, \varphi\rangle_{L^2(D)}}{1-\mu-\mu_1}\varphi.$$

$\square$

5.4 Application of the Crandall-Rabinowitz theorem for the semilinear equation

In this section we will apply the Crandall-Rabinowitz theorem, in order to prove Theorem 28. Recall assumptions $\mathbf{S}$ and $\mathbf{S}_g$.

5.4.1 Reformulation of the problem

In this section we will reformulate the problem (58) in a way suitable for the Crandall-Rabinowitz theorem.

As assumed in $\mathbf{S}$ let $\mu_1 \in \mathbb{R}$ be such that 0 is a simple eigenvalue of the operator L_{μ_1} (cf. Lemma 73). Rewrite (58) as[7]

$$L_\mu w + \mathtt{P}_\varphi w = g(x_3, w) + \mathtt{P}_\varphi w \text{ on } D, \tag{146}$$

where $\mathtt{P}_\varphi w = \langle \varphi, w\rangle_{L^2(D)} \varphi$, with $\varphi \in \mathtt{ker}\, L_{\mu_1}$ such that $\|\varphi\|_{L^2(D)} = 1$. By Corollary 86 there exists $\varepsilon > 0$ such that for all $\mu \in (\mu_1 - \varepsilon, \mu_1 + \varepsilon)$, the operator $\widetilde{L_\mu} \overset{(124)}{=} L_\mu + \mathtt{P}_\varphi$ is invertible and $\widetilde{T_\mu} = (L_\mu + \mathtt{P}_\varphi)^{-1} : L^2_{\text{odd}}(D) \to H^1_{\text{odd}}(D)$ is bounded and continuous with respect to $\mu \in (\mu_1 - \varepsilon, \mu_1 + \varepsilon)$.

Define $F \colon H^1_{\text{odd}}(D) \times (\mu_1 - \varepsilon, \mu_1 + \varepsilon) \to H^1_{\text{odd}}(D)$ as

$$F(w, \mu) = w - \widetilde{T_\mu}(g(\cdot, w) + \mathtt{P}_\varphi w) \quad \left(w \in H^1_{\text{odd}}(D)\,, \mu \in I^\varepsilon_{\mu_1}\right). \tag{147}$$

Note that finding a pair $(w, \mu) \in H^1_{\text{odd}}(D) \times \mathbb{R}$ such that $F(w, \mu) = 0$ is equivalent to solving the problem described in (146) and, in a consequence, equivalent to solving the problem (58).

5.4.2 Statement about the regularity of the function F

In this section we will derive some statement about the differentiability of the function F defined in (147). Our goal is to show, that the Remark 142 is applicable for function F. Lemma 90 is the main result of this section. For writing the derivatives with respect to real arguments, we will use the convention described in the Remark 140.

[7] At this point I want to mention, that idea of introducing the projection $\mathtt{P}_\varphi$ comes from Peter Rupp.

5.4.2.1 Differentiability of the mapping $\mu \longmapsto \widetilde{T_\mu}$

Now we will discuss the differentiability of the mapping

$$\mathbb{R} \ni \mu \longmapsto T_\mu \in \mathcal{L}\big(L^2(D)\,, H^1_{\text{odd}}(D)\big)\,,$$

where the operator $T_\mu = \left(\widetilde{L_\mu}\right)^{-1}$ (cf. formula (124) for the definition of the operator $\widetilde{L_\mu}$ and Corollary 86 for the existence of the operator $\widetilde{T_\mu}$).

Lemma 89. *The mapping $\mathbb{R} \ni \mu \longmapsto \widetilde{T_\mu} \in \mathcal{L}(L^2(D)\,, H^1_{\text{odd}}(D))$ is of the class $\mathcal{C}^\infty$ in an open neighbourhood of μ_1. Moreover*

$$\frac{\mathrm{d}\widetilde{T_\mu}}{\mathrm{d}\mu} = -\widetilde{T_\mu} \circ \widetilde{T_\mu}, \tag{148}$$

for all μ sufficiently close to μ_1.

Proof. First we will show that for all μ sufficiently close to μ_1

$$\widetilde{T_{\mu+h}} \xrightarrow[h\to 0]{\mathcal{L}\left(L^2(D),H^1(D)\right)} \widetilde{T_\mu}. \tag{149}$$

Indeed, note that by Corollary 86 there exists $\varepsilon > 0$ and $M > 0$ such that for all $\mu \in (\mu_1 - \varepsilon, \mu_1 + \varepsilon)$ and for all $f \in L^2(D)$ such that

$$\left\|\widetilde{T_\mu}(f)\right\|_{H^1(D)} \leqslant C\,\|f\|_{L^2(D)}\,. \tag{150}$$

Denote

$$\begin{aligned} v &= \widetilde{T_{\mu+h}}(f)\,,\\ w &= \widetilde{T_\mu}(f)\,. \end{aligned} \tag{151}$$

Note that (151) is equivalent to

$$\begin{aligned} \widetilde{L_{\mu+h}}v &= f,\\ \widetilde{L_\mu}w &= f. \end{aligned} \tag{152}$$

By (152) we get

$$\widetilde{L_\mu}(v-w) + hv = 0,$$

which is equivalent to

$$v - w = -h\widetilde{T_\mu}(v)\,.$$

With the aid of (151), above relation can be rewritten as

$$\widetilde{T_{\mu+h}}(f) - \widetilde{T_\mu}(f) = -h\widetilde{T_\mu}\Big(\widetilde{T_{\mu+h}}(f)\Big)\,. \tag{153}$$

Denote $B = \overline{\mathrm{B}_{L^2(D)}\,(0,\,1)}$. Now, for μ sufficiently close to μ_1 and h sufficient small, we calculate

$$\begin{aligned}\sup_{f\in B}\left\|\widetilde{T_{\mu+h}}(f)-\widetilde{T_\mu}(f)\right\|_{H^1(D)} &\overset{(153)}{=} \sup_{f\in B}|h|\left\|\widetilde{T_\mu}\Big(\widetilde{T_{\mu+h}}(f)\Big)\right\|_{H^1(D)}\\ &\overset{(150)}{\leqslant} |h|\,C^2,\end{aligned}$$

which proves (149).

For μ and $\mu+h$ sufficiently close to μ_1 consider the quotient

$$\begin{aligned}&\sup_{f\in B}\left\|\frac{\widetilde{T_{\mu+h}}(f)-\widetilde{T_\mu}(f)}{h}+\widetilde{T_\mu}\Big(\widetilde{T_\mu}(f)\Big)\right\|_{H^1(D)}\\ &\qquad\overset{(153)}{=} \sup_{f\in B}\left\|-\widetilde{T_\mu}\Big(\widetilde{T_{\mu+h}}(f)\Big)+\widetilde{T_\mu}\Big(\widetilde{T_\mu}(f)\Big)\right\|_{H^1(D)}\xrightarrow[h\to 0]{} 0,\end{aligned}$$

by relation (149), which proves the fact that the mapping $\mathbb{R}\ni\mu\longmapsto\widetilde{T_\mu}\in\mathcal{L}(L^2(D)\,,H^2_{\mathrm{odd}}(D))$ is differentiable in an open neighbourhood of μ_1 and justifies formula (148). Note that by relation (149) the operator $\frac{\mathrm{d}\widetilde{T_\mu}}{\mathrm{d}\mu}$ is continuous in an open neighbourhood of μ_1. The claim follows by induction. □

5.4.2.2 Differentiability of the function F

We will apply Lemma 21 and Lemma 89 for the function F defined in (147).

Lemma 90. *Denote $I^\varepsilon_{\mu_1}=(\mu_1-\varepsilon,\mu_1+\varepsilon)$. There exists $\varepsilon>0$ such that the function $F\colon H^1_{\mathrm{odd}}(D)\times I^\varepsilon_{\mu_1}\to H^1_{\mathrm{odd}}(D)$ defined in* (147) *is of the class $\mathcal{C}^1\big(H^1_{\mathrm{odd}}(D)\times I^\varepsilon_{\mu_1},H^1_{\mathrm{odd}}(D)\big)$ and the differential $\mathrm{D}^2_{w\mu}F(w,\mu)$ exists and is continuous for all $w\in H^1(D)$ and $\mu\in I^\varepsilon_{\mu_1}$. Moreover for al $w,h\in H^1_{\mathrm{odd}}(D)$ and all $\mu\in I^\varepsilon_{\mu_1}$*

$$\mathrm{D}_\mu F(w,\mu)=\widetilde{T_\mu}\Big(\widetilde{T_\mu}(g(\cdot,w)+\mathtt{P}_{\widetilde{\varphi}}w)\Big)\,, \tag{154}$$

$$\mathrm{D}_w F(w,\mu)\,h=h-\widetilde{T_\mu}\left(\frac{\partial g}{\partial s}(\cdot,w)\,h+\mathtt{P}_{\widetilde{\varphi}}h\right), \tag{155}$$

$$\mathrm{D}^2_{w\mu}F(w,\mu)\,h=\widetilde{T_\mu}\left(\widetilde{T_\mu}\left(\frac{\partial g}{\partial s}(\cdot,w)\,h+\mathtt{P}_{\widetilde{\varphi}}h\right)\right),$$

in particular, for all $\mu\in I^\varepsilon_{\mu_1}$

$$\mathrm{D}_w F(0,\mu)=\mathrm{Id}_{H^1_{\mathrm{odd}}(D)}-\widetilde{T_\mu}\circ\mathtt{P}_\varphi, \tag{156}$$

$$\mathrm{D}^2_{w\mu}F(0,\mu)=\widetilde{T_\mu}\circ\widetilde{T_\mu}\circ\mathtt{P}_\varphi. \tag{157}$$

Proof. The proof follows from Lemmas 21 and 89 and the fact that the operator $\widetilde{T_\mu}$ is linear and continuous. □

Remark 91. As a consequences of the relation (145) in the Remark 88 we can rewrite relations (156) and (157) as

$$\begin{aligned} \mathrm{D}_w F(0,\mu)\,h &= h - \frac{\langle h,\varphi\rangle_{L^2(D)}}{1+\mu-\mu_1}\varphi, \\ \mathrm{D}^2_{w\mu} F(0,\mu)\,h &= \frac{\langle h,\varphi\rangle^2_{L^2(D)}}{(1+\mu-\mu_1)^2}\varphi, \end{aligned}$$

for all $h \in H^1_{\mathrm{odd}}(D)$.

In order to apply Remark 142, we need the following statement

Lemma 92. *Let B denote a closed unit ball in $H^1(D)$, then*

$$\begin{aligned} \sup_{\mu\in I^\varepsilon_{\mu_1},h\in B} \left\|\mathrm{D}_w F(w,\mu)\,h - \mathrm{D}_w F(0,\mu)\,h - \mathrm{D}^2_{ww} F(0,\mu)[w,h]\right\|_{H^1(D)} \\ = o\Big(\|w\|_{H^1(D)}\Big)\,, \end{aligned} \tag{158}$$

and

$$\sup_{\mu\in I^\varepsilon_{\mu_1}} \left\|\mathrm{D}_\mu F(w,\mu) - \mathrm{D}^2_{\mu w} F(0,\mu)\,w\right\|_{H^1(D)} = o\Big(\|w\|_{H^1(D)}\Big)\,. \tag{159}$$

Proof. We will verify condition (158). By Lemma 21

$$\mathrm{D}^2_{ww} F(0,\mu)[w,h] = 0 \quad \big(w,h \in H^1_{\mathrm{odd}}(D)\big)\,.$$

Using Corollary 86, we get that there exists a constant $M > 0$ such that for all $\mu \in I^\varepsilon_\mu$ and all $f \in L^2(D)$

$$\left\|\widetilde{T_\mu}(f)\right\|_{H^1(D)} \leqslant M\,\|f\|_{L^2(D)}\,. \tag{160}$$

By (155) from Lemma 90 we get for every $\mu \in I^\varepsilon_{\mu_1}$ and $h \in B$ that

$$\begin{aligned} \|\mathrm{D}_w F(w,\mu)\,h - \mathrm{D}_w F(0,\mu)\,h\|_{H^1(D)} &= \left\|\widetilde{T_\mu}\left(\frac{\partial g}{\partial s}(\cdot,w)\,h\right)\right\|_{H^1(D)} \\ &\overset{(160)}{\leqslant} M\left\|\frac{\partial g}{\partial s}(\cdot,w)\,h\right\|_{L^2(D)} \end{aligned}$$

Applying Lemma 21 we conclude that condition (158) holds true.

Now we will proof (159). By the assumption $\mathbf{S}^3_g$, we have that for almost all $x_3 \in \mathbb{R}$

$$g(x_3,0) = 0. \tag{161}$$

Observe, that by $\mathbf{S}^4_g$, we have

$$\Bigg|g(x_3,t) - \underbrace{g(x_3,0)}_{\overset{(161)}{=}0}\Bigg| = \left|\int_0^t \frac{\partial g}{\partial s}(x_3,s)\;\mathrm{d}s\right| \overset{\mathbf{S}^4_g}{\leqslant} a(x_3)\,\frac{t^{\alpha+1}}{\alpha+1}. \tag{162}$$

By formulas (154) and (157) from Lemma 90 we get for all $w \in H^1(D)$ and all $\mu \in I^{\varepsilon}_{\mu_1}$

$$\left\| \mathrm{D}_\mu F(w,\mu) - \mathrm{D}^2_{\mu w}(0,\mu)\, w \right\|_{H^1(D)} = \left\| \widetilde{T_\mu} \circ \widetilde{T_\mu}(g(\cdot,w)) \right\|_{H^1(D)} \overset{160}{\leqslant} M^2 \left\| g(\cdot,w) \right\|_{L^2(D)} .$$

Hence, the claim (159) follows from Lemma 20 and (162). □

5.4.3 Algebraic properties of the function F

This section is devoted for deriving the statements, which will be used, to verify assumptions (261) and (262) in Theorem 141 applied for function F defined in (147).

Lemma 93. *Assume* $\mathbf{S}$ *and* $\mathbf{S}_g$*, then*

$$\mathtt{ker}\, \mathrm{D}_w F(0,\mu_1) = \mathtt{ker}\, L_{\mu_1}.$$

Moreover $\dim \mathtt{ker}\, \mathrm{D}_w F(0,\mu_1) = 1$.

Proof. If $h \in \mathtt{ker}\, D_w F(0,\mu_1)$ (cf. formula (156)), then

$$h - \widetilde{T_{\mu_1}} \mathtt{P}_\varphi h = 0,$$

which is equivalent to

$$L_{\mu_1} h + \mathtt{P}_\varphi h - \mathtt{P}_\varphi h = 0,$$

which means that $h \in \mathtt{ker}\, L_{\mu_1}$. The rest follows from Remark 74. □

Lemma 94. *Assume* $\mathbf{S}$ *and* $\mathbf{S}_g$*, then*

$$\dim \mathtt{ker}\, \mathrm{D}_w F(0,\mu_1) = \operatorname{codim} \mathtt{im}\, \mathrm{D}_w F(0,\mu_1) = 1.$$

Proof. We will show that

$$H^1_{\mathrm{odd}}(D) = \mathtt{im}\, \mathrm{D}_w F(0,\mu_1) \oplus \mathtt{ker}\, \mathrm{D}_w F(0,\mu_1)\,,$$

which together with the Lemma 93 will prove the claim.

Note that $h \in \mathtt{im}\, \mathrm{D}_w F(0,\mu_1)$, if and only if, there exists $z \in H^1_{\mathrm{odd}}(D)$ such that $h = z - \langle z,\varphi\rangle_{L^2(D)}\, \varphi$. Moreover $h \in \mathtt{ker}\, \mathrm{D}_w F(0,\mu_1)$ if and only of $h = \xi\varphi$, for some $\xi \in \mathbb{R}$. Observe, that for all $h \in H^1_{\mathrm{odd}}(D)$

$$h = \underbrace{h - \langle h,\varphi\rangle_{L^2(D)}\, \varphi}_{\in \mathtt{im}\, \mathrm{D}_w F(0,\mu_1)} + \underbrace{\langle h,\varphi\rangle_{L^2(D)}\, \varphi}_{\in \mathtt{ker}\, \mathrm{D}_w F(0,\mu_1)},$$

hence $H^1_{\mathrm{odd}}(D) = \mathtt{im}\, \mathrm{D}_w F(0,\mu_1) + \mathtt{ker}\, \mathrm{D}_w F(0,\mu_1)$. Suppose that $h \in H^1_{\mathrm{odd}}(D)$ is such that $h \in \mathtt{im}\, \mathrm{D}_w F(0,\mu_1) \cap \mathtt{ker}\, \mathrm{D}_w F(0,\mu_1)$. Then, for some $\xi \in \mathbb{R}$ and $z \in H^1(D)$, we have that $z - \langle z,\varphi\rangle_{L^2(D)} = \xi\varphi$. After multiplying this relation by φ, we get that $0 = \xi$, hence $h = 0$, which means that $\mathtt{im}\, \mathrm{D}_w F(0,\mu_1) \cap \mathtt{ker}\, \mathrm{D}_w F(0,\mu_1) = \{0\}\,.$ □

Lemma 95. *Assume* $\mathbf{S}$ *and* $\mathbf{S}_g$, *then*

$$\mathrm{D}^2_{w\mu}F(0,\mu_1)\,\varphi \notin \mathtt{im}\,\mathrm{D}_w F(0,\mu_1)\,.$$

Proof. Suppose that $\mathrm{D}^2_{w\mu}F(0,\mu_1)\,\varphi \in \mathtt{im}\,\mathrm{D}_w F(0,\mu_1)$, i.e., there exists some $\psi \in H^1_{\mathrm{odd}}(D)$ such that

$$\mathrm{D}^2_{w\mu}F(0,\mu_1)\,\varphi = \mathrm{D}_w F(0,\mu_1)\,\psi,$$

which is equivalent to (cf. formulas (156) and (157), for the definitions of the corresponding operators)

$$\widetilde{T_{\mu_1}}\left(\widetilde{T_{\mu_1}}(\varphi)\right) = \psi - \widetilde{T_{\mu_1}}(\mathtt{P}_\varphi \psi)\,.$$

By Remark 88, we can rewrite above as

$$\varphi = \psi - \langle \varphi, \psi \rangle_{L^2(D)}\,\varphi.$$

By testing above with φ, we get

$$1 = \langle \varphi, \varphi \rangle_{L^2(D)} = \langle \varphi, \psi \rangle_{L^2(D)} - \langle \varphi, \psi \rangle_{L^2(D)} = 0.$$

Contradiction. □

5.4.4 Proof of the main result for the semilinear wave equation

Proof of Theorem 28. In order to obtain the statement, we will apply Remark 142 for the function $F\colon H^1_{\mathrm{odd}}(D) \times I^\varepsilon_\mu \to H^1_{\mathrm{odd}}(D)$ defined in (147). Note that by Lemmas 90 and 92 the function F satisfies the regularity conditions. The assumption (261) is fulfilled by Lemma 94. The assumption (262) is satisfied due to Lemma 95. □

6 Quasilinear wave equation

In this section we will consider the quasilinear wave equation (12) derived in section 1.1.

We look for polarized travelling wave solutions of the equation (12). i.e. solutions having the form

$$\overrightarrow{E}(x,t) = \begin{pmatrix} 0 \\ u(x_1 - \omega t, x_3) \\ 0 \end{pmatrix},$$

where the function u is $2P$-periodic in its first variable and has some decay in its second variable. The profile $u = u(x_1, x_3)$ has to satisfy the equation

$$- \triangle u + \omega^2 \big(V(x_3)\, u_{11} + \Gamma(x_3)\big(3u^2 u_{11} + 6uu_1^2\big)\big) = 0, \tag{163}$$

on a strip $D = (-P, P) \times \mathbb{R}$.

6.1 Preliminary remarks and examples concerning the quasilinear wave equation

In the following example we will explicitly construct some coefficients and a solution of the equation (163), which can be written as

$$- u_{11} - u_{33} + \omega^2 V(x_3)\, u_{11} + \omega^2 \Gamma(x_3) \big(|u|^2\, u\big)_{11} = 0 \text{ on } D = (-P, P) \times \mathbb{R}, \tag{164}$$

where we look for a function $u = u(x_1, x_3)$ being $2P$-periodic in its first variable and having some decay in its second variable.

Example 96. Now we will construct examples of coefficients V, Γ, for which we can solve (164). If we write $u = \frac{\partial z}{\partial x_1}$ for some function z, then the function z has to satisfy

$$-z_{111} - z_{331} + \omega^2 V z_{111} + \Gamma \omega^2 \partial_{x_1^2} \left(|z_1|^2\, z_1\right) = 0 \text{ on } D,$$

and by integrating with respect to the x_1 variable, we will get

$$- z_{33} + \big(\omega^2 V - 1\big)\, z_{11} + \Gamma \omega^2 \partial_{x_1} \left(|z_1|^2\, z_1\right) = 0 \text{ on } D. \tag{165}$$

After separating the variables

$$z(x_1, x_3) = a(x_1)\, b(x_3) \quad (x_1 \in (-P, P)\,, x_3 \in \mathbb{R})\,,$$

the equation (165) can be rewritten into the form

$$-ab'' + \big(\omega^2 V(x_3) - 1\big)\, a''b + \omega^2 \Gamma(x_3) \Big(a'^3\Big)'\, b^3 = 0 \text{ on } D.$$

After multiplying both sides of the above relation by factor $\frac{1}{a''b''}$, we obtain

$$-\frac{a}{a''} + \underbrace{\left(\omega^2 V(x_3) - 1\right)\frac{b}{b''}}_{=-\gamma} + \underbrace{\omega^2 \Gamma(x_3)\frac{b^3}{b''}}_{=-\delta} \cdot \frac{\left(a'^3\right)'}{a''} = 0 \text{ on } D.$$

We look for b such that

$$\left(\omega^2 V(x_3) - 1\right)\frac{b}{b''} = \text{const.} = -\gamma,$$
$$\omega^2 \Gamma(x_3)\frac{b^3}{b''} = \text{const.} = -\delta,$$

We want to find functions V, Γ and b and the constants γ, δ, P and ω such that b decays at $\pm\infty$ and simultaneously satisfies

$$-b'' = \frac{1}{\gamma}\left(\omega^2 V(x_3) - 1\right) b \qquad (x_3 \in \mathbb{R}), \tag{166}$$

$$-b'' = \frac{1}{\delta}\omega^2 \Gamma(x_3)\, b^3 \qquad (x_3 \in \mathbb{R}), \tag{167}$$

and a function a, which can be extended to a periodic function, satisfying

$$-\gamma a'' - \delta\left(a'^3\right)' = a \qquad (x_1 \in (-P, P)), \tag{168}$$

Note that by solving the above problem, we will obtain an example of a solution of the equation (165). By defining $u = \frac{\partial z}{\partial x_1} = a'b$, we will obtain a solution of the equation (164).

We will obtain a solution of the equation (168) by a minimization procedure. We consider the functional $I \colon W_0^{1,4}(0,1) \to \mathbb{R}$ given by

$$I(h) = \int_{-1}^{1} \frac{(h')^2}{2} + \frac{(h')^4}{4}\,\mathrm{d}x \quad \left(h \in W_0^{1,4}(0,1)\right). \tag{169}$$

We want to minimize the functional I under the constrain

$$\int_0^1 h^2\,\mathrm{d}x = 1. \tag{170}$$

Note that the minimizer of above problem satisfies (in a weak sense)

$$-h'' - \left(h'^3\right)' = \lambda h \text{ on } (0,1),$$

with a Lagrange multiplier $\lambda \in \mathbb{R}$, i.e.,

$$\int_0^1 h'\psi' + h'^3\psi'\,\mathrm{d}x = \lambda \int_0^1 h\psi\,\mathrm{d}x \quad \left(\psi \in W_0^{1,4}(0,1)\right). \tag{171}$$

Note that in fact $\lambda > 0$. Indeed if $h \in W_0^{1,4}(0,1)$ is a minimizer of the above problem, then by multiplying the equality (96) by h and integrating by parts, we get

$$\lambda \overset{(170)}{=} \lambda \int_0^1 h^2(x)\,\mathrm{d}x \overset{(171)}{=} \int_0^1 (h')^2(x)\,\mathrm{d}x + \int_0^1 (h')^4(x)\,\mathrm{d}x > 0.$$

We will show now, that the above problem has a solution. Observe that

- the functional I, defined in (169), is *coercive*. Indeed note that, by Poincaré inequality, the norms $\|\cdot\|_{W^{1,4}(0,1)}$ and $\|\cdot\|_*$ are equivalent, where $\|u\|_* = \|u'\|_{L^4(0,1)}$. Therefore, there exist constants $C_1 > 0, C_2 > 0$ such that for all $w \in W_0^{1,4}(0,1)$

$$\|w\|^2_{W_0^{1,4}(0,1)} \leqslant C_1 \int_0^1 w'^4\,\mathrm{d}x \leqslant C_1 \int_0^1 w'^2 + w'^4\,\mathrm{d}x \leqslant C_2 I(w)\,,$$

  hence, if for some sequence $(w_n)_{n\in\mathbb{N}}$ of $W_o^{1,4}(0,1)$ functions the sequence $(I(w_n))_{n\in\mathbb{N}}$ is bounded, then the sequence $(w_n)_{n\in\mathbb{N}}$ is bounded in the $W^{1,4}(0,1)$ norm, which proves, the claim.

- the mapping $\mathbb{R} \ni p \longmapsto \frac{p^2}{2} + \frac{p^4}{4} \in \mathbb{R}$ is convex. Then, by [2, Theorem 1, p. 446], the functional I is sequentially weakly lower semicontinuous on $W_0^{1,4}(0,1)$.

- the set $X = \left\{ w \in W_0^{1,4}(0,1) : \int_0^1 w^2\,\mathrm{d}x = 1 \right\}$ is (sequentially) weakly closed. This is a consequence of the fact that the space $W_0^{1,4}(0,1)$ is compactly embedded into the space $L^2(\mathbb{R})$ (cf. [1, Theorem 6.3, p. 168]) and that the constrain is given by the $L^2(\mathbb{R})$ norm.

By [10, Theorem 1.2, p. 4], there exist a solution of the above minimization problem.

Let h be a solution of the above problem and define a function $g\colon (-1,1) \to \mathbb{R}$ as an odd reflection of h, i.e.,

$$g(x) = \begin{cases} h(x) & (x \geqslant 0)\,, \\ -h(-x) & (x \leqslant 0)\,. \end{cases} \tag{172}$$

Since $h \in W_0^{1,4}(0,1)$, we have that $g \in W_0^{1,4}(-1,1)$. Note that the function g can be extended to a 2-periodic function. Next we will show that it solves the equation

$$-g'' - \left(g'^3\right)' = \lambda g \text{ on } (-1,1)\,,$$

in a weak sense, i.e.,

$$\int_{-1}^1 g'\psi' + g'^3\psi'\,\mathrm{d}x = \lambda \int_{-1}^1 g\psi\,\mathrm{d}x \quad \left(\psi \in \mathcal{C}^\infty_{\mathrm{per},2}(\mathbb{R})\right), \tag{173}$$

where $\mathcal{C}^\infty_{\text{per},T}(\mathbb{R})$ denotes the set of all $\mathcal{C}^\infty$ functions defined on $\mathbb{R}$, which are T periodic. Indeed, take an arbitrary $\psi \in \mathcal{C}^\infty_{\text{per},2}(\mathbb{R})$, and let $\widetilde{\psi}(x) = \psi(x) - \psi(-1)$. Since the function ψ is 2 periodic, we obtain $\widetilde{\psi}(-1) = \widetilde{\psi}(1) = 0$. Write $\widetilde{\psi} = \widetilde{\psi_o} + \widetilde{\psi_e}$, where $\widetilde{\psi_o}$ and $\widetilde{\psi_e}$ are odd and even part of $\widetilde{\psi}$ respectively, i.e.

$$\widetilde{\psi_o}(x) = \frac{\widetilde{\psi}(x) - \widetilde{\psi}(-x)}{2}, \ \widetilde{\psi_e}(x) = \frac{\widetilde{\psi}(x) + \widetilde{\psi}(-x)}{2}.$$

Finally observe, that

$$\psi(x) = \underbrace{\widetilde{\psi_e}(x) + \psi(-1)}_{\text{even}} + \underbrace{\widetilde{\psi_o}(x)}_{\text{odd}} \quad (x \in \mathbb{R}),$$

and define $\psi_e, \psi_o \colon \mathbb{R} \to \mathbb{R}$ as follows

$$\psi_e(x) = \widetilde{\psi_e}(x) + \psi(-1), \ \psi_o(x) = \widetilde{\psi_o}(x).$$

Moreover note that $\psi_o = \widetilde{\psi_o} \in W^{1,4}_0(-1,1)$. Since the function g defined in (172) is odd, we have

$$\int_{-1}^1 g'\psi_e' \,\mathrm{d}x = 0, \tag{174}$$

$$\int_{-1}^1 {g'}^3\psi_e' \,\mathrm{d}x = 0, \tag{175}$$

$$\int_{-1}^1 g\psi_e \,\mathrm{d}x = 0. \tag{176}$$

In order to verify (173) calculate

$$\begin{aligned}\int_{-1}^1 g'\psi' + {g'}^3\psi' \,\mathrm{d}x &= \int_{-1}^1 g'\psi_o' \,\mathrm{d}x + \underbrace{\int_{-1}^1 g'\psi_e' \,\mathrm{d}x}_{\overset{(174)}{=}0} + \int_{-1}^1 {g'}^3\psi_o' \,\mathrm{d}x + \underbrace{\int_{-1}^1 {g'}^3\psi_e' \,\mathrm{d}x}_{\overset{(175)}{=}0} \\ &\overset{(172)}{=} 2\int_0^1 h'\psi_o' \,\mathrm{d}x + 2\int_0^1 {h'}^3\psi_o' \,\mathrm{d}x \overset{(171)}{=} 2\lambda \int_0^1 h\psi_o \,\mathrm{d}x \\ &\overset{(172)}{=} \lambda\int_{-1}^1 g\psi_o \,\mathrm{d}x + \lambda \underbrace{\int_{-1}^1 g\psi_e \,\mathrm{d}x}_{\overset{(176)}{=}0} = \lambda\int_{-1}^1 g\psi \,\mathrm{d}x.\end{aligned}$$

Define a function $a \colon \left[-\frac{1}{s}, \frac{1}{s}\right] \to \mathbb{R}$ by

$$a(x) = tg(sx) \quad \left(x \in \left[-\frac{1}{s}, \frac{1}{s}\right]\right), \tag{177}$$

where

$$s = \frac{1}{\sqrt{\gamma\lambda}}, \ t = \gamma\sqrt{\frac{\lambda}{\delta}}, \ P = \frac{1}{s} = \sqrt{\gamma\lambda}.$$

Note that then the equation (168) is valid, indeed for all $x \in \left[-\frac{1}{s}, \frac{1}{s}\right]$

$$\begin{aligned}
&-\gamma a''(x) - \delta\Big(a'^3(x)\Big)' - a(x) \\
&\qquad = -\frac{\gamma\sqrt{\frac{\lambda}{\delta}}}{\lambda}\left(-g''\Big(\frac{x}{\sqrt{\gamma\lambda}}\Big) - \Big(g'^3\Big)'\Big(\frac{x}{\sqrt{\gamma\lambda}}\Big) + \lambda g\Big(\frac{x}{\sqrt{\gamma\lambda}}\Big)\right) \overset{(96)}{=} 0
\end{aligned}$$

Moreover the function a can be extended to a periodic function on $\mathbb{R}$. Therefore, we can solve the equation (168) for all values $\gamma > 0$ and $\delta > 0$.

To simplify further considerations take $\gamma = \delta = 1$. The idea is to choose some function b and define the functions V and Γ in such a way, that the equations (166) and (167) will be satisfied. There are two ways to do it:

- solve the equation
$$-b'' = \big(\omega^2 V(x_3) - 1\big)\, b \text{ on } \mathbb{R},$$
and define the function Γ using the formula
$$\omega^2\Gamma(x_3) = \frac{-b''}{b^3} = \frac{\omega^2 V(x_3) - 1}{b^2}.$$
- solve the equation
$$-b'' = \omega^2\Gamma(x_3)\, b^3 \text{ on } \mathbb{R},$$
and define function V using the formula
$$\omega^2 V(x_3) - 1 = \frac{-b''}{b} = \omega^2\Gamma(x_3)\, b^2.$$

In the first case let $V = \begin{cases} V_i, & |x| < a \\ V_o, & |x| > a \end{cases}$, where a, ω, V_i and V_o satisfy

$$\begin{aligned}
\sqrt{1-\omega^2 V_o} &= \sqrt{\omega^2 V_i - 1}\,\mathrm{tg}\,\sqrt{\omega^2 V_i - 1}\,a \text{ or} \\
\sqrt{1-\omega^2 V_o} &= \sqrt{\omega^2 V_i - 1}\,\mathrm{ctg}\,\sqrt{\omega^2 V_i - 1}\,a,
\end{aligned} \tag{178}$$

and let b be a guided mode in a wave-guide described by V as in Lemma 69 in section 5.3.1.3). Since the function $\left(0, \frac{\pi^2}{4}\right)\, \xi \longmapsto \sqrt{\xi}\,\mathrm{tg}\,\sqrt{\xi} \in (0, \infty)$ is onto, for every values ω V_o and $a > 0$ such that $1 - \omega^2 V_o > 0$, we can find V_i in such a way, that the condition (178) is satisfied. Observe, that for $|x| > a$

$$\omega^2 V(x_3) - 1 = \omega^2 V_o - 1 < 0, \tag{179}$$

hence, due to the exponential decay of the function b, the function Γ is unbounded, namely we have that $\Gamma(x_3) \xrightarrow[|x_3|\to\infty]{} -\infty$. Observe, that for $|x|\, a < a$

$$\omega^2 V(x_3) - 1 = \omega^2 V_i - 1 > 0, \tag{180}$$

hence considered equation is hyperbolic.

In the second case let, e.g.,

$$b(x) = \left(1 + x^2\right)^{-\frac{1}{2}} \quad (x \in \mathbb{R})\,.$$

Then, for each $x \in \mathbb{R}$

$$\begin{aligned} b'(x) &= -x\left(1+x^2\right)^{-\frac{3}{2}}, \\ b''(x) &= -\left(1+x^2\right)^{-\frac{3}{2}} + 3x^2\left(1+x^2\right)^{-\frac{5}{2}} = \frac{2x^2-1}{(1+x^2)^{\frac{5}{2}}}. \end{aligned}$$

For each $x \in \mathbb{R}$ set

$$\begin{aligned} \omega^2\Gamma(x) &= \frac{-b''(x)}{b(x)^3} = \frac{1-2x^2}{1+x^2}, \\ \omega^2 V(x) &= 1 + \omega^2\Gamma(x)b(x)^2 = \frac{x^4+2}{\left(x^2+1\right)^2}. \end{aligned} \tag{181}$$

Note that $\Gamma(x) \xrightarrow[|x|\to\infty]{} -\frac{2}{\omega^2}$ and $V(x) \xrightarrow[|x|\to\infty]{} \frac{1}{\omega^2}$ hence the functions Γ and V are bounded. Moreover V is positive (cf. Fig. 7), which shows the physical relevance of constructed example. Furthermore observe that there exists a constant $L > 0$ such that

$$\omega^2 V(x_3) - 1 > 0 \quad (|x_3| < L)\,, \tag{182}$$

hence again, the considered equation has a hyperbolic nature.

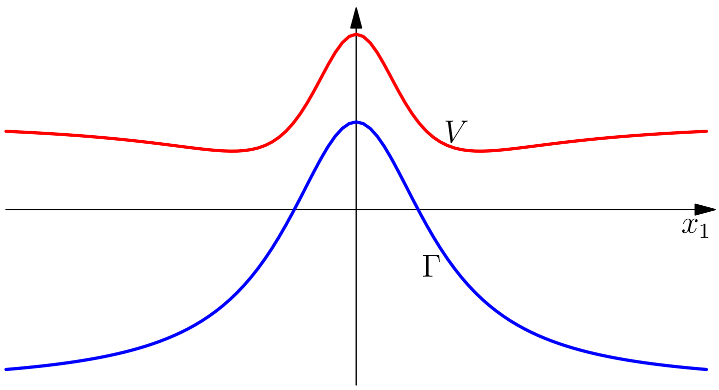


Figure 7: The sketch of the graphs of the functions V and Γ defined in (181), for $\omega = 1$.

Remark 97. In the above Example 96, the constructed potential V always leads to the situation, when the equation (164) is hyperbolic (at some points, near ∞). Indeed note that by equations (180) and (182) there exists a constant $L > 0$ such that the expression $\omega^2 V(x_3) - 1$ is negative, if $|x_3| > L$. The following theorem demonstrates, that this is a necessary condition, to obtain non-trivial solutions.

Note that the equation (165) is elliptic if and only if $\omega^2 \|V\|_{L^\infty(\mathbb{R})} < 1$.

Theorem 98. *If* $\omega^2 \|V\|_{L^\infty(D)} < 1$ *then the equation*

$$-\left(1-\omega^2 V\right) z_{11} - z_{33} + 3\omega^2 \Gamma z_1 z_1 z_{11} = 0 \text{ on } D, \tag{183}$$

has no non-zero solution $z \in \mathcal{C}^2(D) \cap \mathcal{C}\left(\overline{D}\right)$ *satisfying*

$$z(x_1, x_3) \xrightarrow[|x_3|\to\infty]{} 0 \text{ uniformly in } x_1, \tag{184}$$

which is P*-periodic in the* x_1 *direction.*

Proof. Suppose that $z \in \mathcal{C}^2(D) \cap \mathcal{C}\left(\overline{D}\right)$ is a solution of the problem (183) satisfying (184). Moreover, after periodic extension consider z as a function defined on $\mathbb{R}^2$. Consider the operator defined by

$$L = -\left(1-\omega^2 V\right)\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_3^2} + 3\omega^2\Gamma z_1 z_{11}\frac{\partial}{\partial x_1}.$$

Clearly $L(z) = 0$. Note that since $\omega^2 \|V\|_{L^\infty(D)} < 1$ the operator L is elliptic. Because of the assumption (184) and periodicity, z attain its maximum or minimum somewhere on the set $\mathbb{R}^2$. Therefore, by the strong maximum principle (cf. [3, Theorem 3.5]), $z = \text{const.} \overset{(184)}{=} 0$. □

Remark 99. Observe that by Theorem 98, if $\omega^2 \|V\|_{L^\infty(\mathbb{R})} < 1$, the equation (164) has no non-trivial solutions, which decay to 0 as $|x_3| \to 0$ (cf. Example 96).

6.2 Complex valued travelling waves

In this section we will prove the existence of complex valued travelling waves for the quasilinear wave equation (12). We look for solutions of (12) of the form

$$U(x_1, x_2, x_3, t) = e^{\mathrm{i}\omega\left(x_1 - \sqrt{\lambda}t\right)} v(x_3) \quad (x_1, x_2, x_3, t \in \mathbb{R})$$

The above ansatz leads to the equation of the form

$$\omega^2 v - \frac{\partial^2 v}{\partial x_3} - \omega^2 \lambda V(x_3)\, v - \lambda\omega^2\Gamma(x_3)\, v^3 = 0 \text{ on } \mathbb{R},$$

which we will rewrite into the form

$$-\frac{\partial^2 v}{\partial x_3} + \omega^2(1-\lambda V(x_3))\, v = \lambda\omega^2\Gamma(x_3)\, v^3 \text{ on } \mathbb{R}, \tag{185}$$

which is a nonlinear, time independent Schrödinger equation in 1-d.

Denote

$$\widetilde{V}(x_3) = \omega^2(1-\lambda V(x_3)) \quad (x_3 \in \mathbb{R}),$$
$$\widetilde{\Gamma}(x_3) = \lambda\omega^2\Gamma(x_3) \quad (x_3 \in \mathbb{R}),$$

The equation (185) is equivalent to

$$-\frac{\partial^2 v}{\partial x_3} + \widetilde{V}(x_3)\, v = \widetilde{\Gamma}(x_3)\, v^3 \text{ on } \mathbb{R}, \tag{186}$$

Consider the following functional $J\colon H^1(\mathbb{R}) \to \mathbb{R}$ defined by the formula

$$J[u] = \int_{\mathbb{R}} |u'|^2 + \widetilde{V}(x)\, u^2 \,\mathrm{d}s \quad \big(u \in H^1(\mathbb{R})\big)\,.$$

We want to minimize the functional J over the set

$$S = \left\{ u \in H^1(\mathbb{R}) : \int_{\mathbb{R}} \widetilde{\Gamma}(s)\, |u|^4 \,\mathrm{d}s = 1 \right\}.$$

Assume that $v^* \in S$ is a minimizer i.e., $J[v^*] = \inf_{v\in S} J[v]$. Then v^* satisfies

$$-(v^*)'' + \widetilde{V}(x)\, v^* = c\Gamma(x)(v^*)^3 \text{ on } \mathbb{R},$$

where c is a Lagrange multiplier. One can show, that $c = J[v^*]$. Assume that $c \geqslant 0$([8]). Since v^* is a minimizer, we even have, that $c > 0$. When we define $\widetilde{v} = c^{\frac{1}{2}} v^*$, then one can check that the function $\widetilde{v}$ is a solution of the equation (186). Such a solution is called a *ground state* for the equation (186).

Theorem 100. *Assume that the functions $\widetilde{V}, \widetilde{\Gamma} \in L^\infty(D)$ are such that*

- *the expression $\left(\int_{\mathbb{R}} (u')^2 + \widetilde{V}(s)\, u^2 \,\mathrm{d}s\right)^{\frac{1}{2}}$ defines a norm, which is equivalent to the $H^1(\mathbb{R})$ norm,*
- *moreover*

$$\lim_{|x_3|\to\infty} \widetilde{V}(x_3) = \sup_{x_3\in\mathbb{R}} \widetilde{V}(x_3) > 0,$$
$$\lim_{|x_3|\to\infty} \widetilde{\Gamma}(x_3) = \inf_{x_3\in\mathbb{R}} \widetilde{\Gamma}(x_3) > 0,$$

then the equation (186) *has a ground state solution.*

Proof. The proof follows from [11, Theorem 2.5, p. 17] and [10, Theorem 4.2, p. 32]. □

6.3 Real valued travelling waves

In this section we will look for real valued solutions of the equation (163), where the potential V has the form $V(x_3) = \alpha\delta(x_3) + \gamma$, where $\delta(\cdot)$ is a *Dirac delta function* supported along the line $x_3 = 0$ and $\alpha \in \mathbb{R}$, $\gamma \in \mathbb{R}$.

We work under the assumptions $\mathbf{Q}$ and $\mathbf{Q}_\Gamma$ described in section 6.3.1. The main result of this section if the following statement

[8]actually this is implied by the assumption that $\widetilde{V}, \widetilde{\Gamma} \in L^\infty(\mathbb{R})$ in Theorem 100

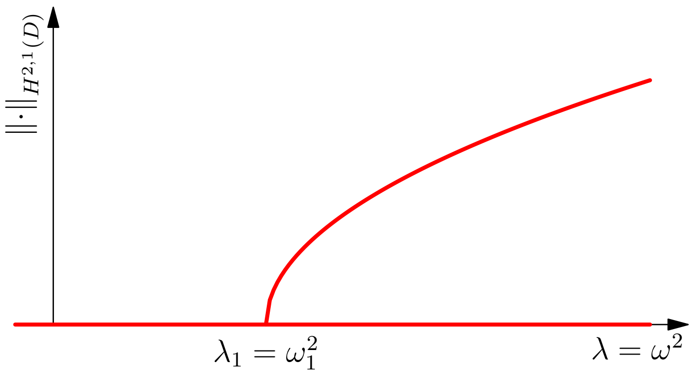


Figure 8: Qualitative sketch of the bifurcation diagram for the equation (163).

Theorem 101. *Assume* **Q** *and* $\mathbf{Q}_\Gamma$*. Then there exists a non-trivial, continuously differentiable curve passing through a point* $(0, \omega_1^2)$

$$\left\{(u(s), \lambda(s)) \in H^{2,1}_{\mathrm{odd}}(D) \times \mathbb{R} \colon s \in (-\delta, \delta), u(0) = 0, \lambda(0) = \omega_1^2\right\},$$

such that the pair $(u(s), \lambda(s))$ *solves* (163) *for all* $s \in (-\delta, \delta)$*. Moreover all solutions of the equation* (163) *in a neighbourhood of point* $(0, \omega_1^2)$ *are on the trivial line or on the curve defined above.*

The proof of the above theorem can be found in section 6.3.4.4.

6.3.1 Assumptions

In this section we will work under the following assumptions on the parameters α, γ, ω, P.

Assumption Q. We will assume that

Q^1 There exists $\varepsilon > 0$ such that for all $\omega^2 \in \overline{I^{\varepsilon}_{\omega_1^2}} = [\omega_1^2 - \varepsilon, \omega_1^2 + \varepsilon]$ and for all natural numbers $k \in \mathbb{N}$

$$k^2 = \frac{4(1 - \omega^2\gamma)\, P^2}{\alpha\omega^2\pi^2} \text{ if and only if } \omega^2 = \omega_1^2 \text{ and } k = k_1.$$

Q^2 For all $\omega^2 \in \overline{I^{\varepsilon}_{\omega_1^2}}$

$$\omega_1^2 + \varepsilon < \frac{1}{\gamma}.$$

Assumption $\mathbf{Q}_\Gamma$**.** The function $\Gamma \colon \mathbb{R} \to \mathbb{R}$ is an element of the space $L^\infty(\mathbb{R})$.

We will also use the notation $\lambda_1 = \omega_1^2$, $\lambda = \omega^2$ and $J^{\varepsilon}_{\omega_1^2} = I^{\varepsilon}_{\omega_1^2} \setminus \{\omega_1^2\}$. Define

$$S^{\omega^2} = \left\{ k \in \mathbb{N} \colon k^2 \neq \frac{4(1 - \omega^2\gamma)\, P^2}{\alpha\omega^2\pi^2} \right\}. \tag{187}$$

Note that assumption $\mathbf{Q}^1$ implies that $S^{\omega_1^2} = \mathbb{N}\backslash\{k_1\}$ and that for all $\omega^2 \in J^{\varepsilon}_{\omega^2}$, we have that $S^{\omega^2} = \mathbb{N}$.

Remark 102. Assumption $\mathbf{Q}^2$ is equivalent to the following: For every $\omega^2 \in \overline{I^{\varepsilon}_{\omega_1^2}}$

$$1 - \omega^2\gamma > 0.$$

6.3.2 Notion of the solution

In this section we will discuss the notion of the weak solution of the equation (cf. Definition 103).

$$-w_{33} - \left(1 - \omega^2 V(x_3)\right) w_{11} = f \quad (x_1 \in (-P, P)\,, x_3 \in \mathbb{R})\,. \tag{188}$$

Definition 103. We say that a function $u \in H^{2,1}_{\text{per}}(D)$ solves the equation (188), where $f \in L^2_{\text{per}}(D)$ is given, in the *weak sense* if for all functions $\psi \in H^{2,1}_{\text{per}}(D)$

$$\int_D u_3\psi_3 \,\mathrm{d}x + \left(1 - \omega^2\gamma\right) \int_D u_1\psi_1 \,\mathrm{d}x - \alpha\omega^2 \int_{-P}^{P} u_1(x_1, 0)\, \psi_1(x_1, 0)\ \mathrm{d}x_1 = \int_D f\psi \,\mathrm{d}x. \tag{189}$$

Since $u \in H^{2,1}_{\text{per}}(D)$, then $u_1 \in H^1(D) \subseteq H^{0,1}(D)$. By Lemma 5, the function u_1 has a trace in the space $L^2(-P, P)$, hence the integral $\int_{-P}^{P} u_1(x_1, 0)\, \psi_1(x_1, 0)\ \mathrm{d}x_1$ in the formula (189) is well-defined.

The remaining considerations are devoted to the equation, which one obtains by taking the Fourier transform of the equation (188) with respect to the x_1 variable.

Definition 104. Let $\widehat{f} \in L^2_{\text{loc}}(\mathbb{R})$. Let $\widehat{V} = \alpha\widehat{\delta} + \gamma$, where $\widehat{\delta}$ the *Dirac delta function* supported at the point 0 and $\alpha, \gamma \in \mathbb{R}$. Denote $a_k = \frac{k^2\pi^2}{P^2}$. We say that a function $h \in H^1_{\text{loc}}(\mathbb{R})$ solves the equation

$$-h'' + \left(1 - \omega^2\widehat{V}\right) a_k h = \widehat{f} \text{ in } \mathbb{R}, \tag{190}$$

in the *weak sense* if for all functions $\varphi \in \mathcal{C}^\infty_c(\mathbb{R})$

$$\int_{\mathbb{R}} h'\varphi' \,\mathrm{d}x \;+\; a_k\left(1 - \omega^2\gamma\right) \int_{\mathbb{R}} h\varphi \,\mathrm{d}x \;-\; \alpha a_k\omega^2 h(0)\, \varphi(0) \;=\; \int_{\mathbb{R}} \widehat{f}\varphi \,\mathrm{d}x. \tag{191}$$

Lemma 105. *Let $\widehat{f} \in L^2(\mathbb{R})$ and that $h \in H^1(\mathbb{R})$. The following conditions are equivalent*

- *the function $h \in H^2((-\infty, 0]) \cap H^2([0, \infty))$ solves the equation*

$$-h'' + a_k\left(1 - \omega^2\gamma\right) h = \widehat{f}, \tag{192}$$

pointwise almost everywhere on $\mathbb{R}$ and satisfies

$$h'\left(0^-\right) - h'\left(0^+\right) = a_k\alpha\omega^2 h(0) \tag{193}$$

- *the function h solves the equation* (190) *in the sense of Definition 104.*

Proof. Assume that $h \in H^2((-\infty,0]) \cap H^2([0,\infty))$ solves the equation (192) pointwise almost everywhere on $(-\infty,0]$ and on $[0,\infty)$ and satisfies condition (193). Let $\varphi \in \mathcal{C}_c^\infty(\mathbb{R})$. Integration by parts yields

$$\int_{-\infty}^{0} -h''\varphi \,\mathrm{d}x = -[h'\varphi]_{-\infty}^{0} + \int_{-\infty}^{0} h'\varphi' \,\mathrm{d}x = -h'\big(0^-\big)\,\varphi(0) + \int_{-\infty}^{0} h'\varphi' \,\mathrm{d}x,$$
$$\int_{0}^{\infty} -h''\varphi \,\mathrm{d}x = -[h'\varphi]_{0}^{\infty} + \int_{0}^{\infty} h'\varphi' \,\mathrm{d}x = h'\big(0^+\big)\,\varphi(0) + \int_{0}^{\infty} h'\varphi' \,\mathrm{d}x,$$

therefore

$$\begin{aligned}\int_{\mathbb{R}} -h''\varphi \,\mathrm{d}x &= \varphi(0)\big(h'\big(0^+\big) - h'\big(0^-\big)\big) + \int_{\mathbb{R}} h'\varphi' \,\mathrm{d}x \\ &\overset{(193)}{=} -a_k\alpha\omega^2 h(0)\,\varphi(0) + \int_{\mathbb{R}} h'\varphi' \,\mathrm{d}x.\end{aligned} \tag{194}$$

Since h solves pointwise the equation (192), we get

$$\begin{aligned}\int_{\mathbb{R}} \widehat{f}\varphi \,\mathrm{d}x &= \int_{\mathbb{R}} -h''\varphi \,\mathrm{d}x + a_k\big(1-\omega^2\gamma\big)\int_{\mathbb{R}} h\varphi \,\mathrm{d}x \\ &\overset{(194)}{=} \int_{\mathbb{R}} h'\varphi \,\mathrm{d}x + a_k\big(1-\omega^2\gamma\big)\int_{\mathbb{R}} h\varphi \,\mathrm{d}x - \alpha a_k\omega^2 h(0)\,\varphi(0)\,.\end{aligned}$$

Therefore, h solves the equation (190) in the sense of definition 104.

Assume that the h solves the equation (190) in the sense of Definition 104. By [3, Theorem 8.8, p. 173], we have that h solves (192) pointwise almost everywhere on $(-\infty,0)$ and on $(0,\infty)$. Moreover, one can show, that $h \in H^2((-\infty,0]) \cap H^2([0,\infty))$. Since the space $\mathcal{C}_c^\infty(\mathbb{R})$ is dense in $H^1(\mathbb{R})$, we have that the condition (191) hold for all $\varphi \in H^1(\mathbb{R})$. For $\varepsilon > 0$ define a function $\varphi_\varepsilon \colon \mathbb{R} \to \mathbb{R}$ by the formula

$$\varphi_\varepsilon(x) = \begin{cases} 1, & |x| \leqslant \varepsilon, \\ 2 - \frac{|x|}{\varepsilon} & |x| \in (\varepsilon, 2\varepsilon)\,, \\ 0, & |x| \leqslant 2\varepsilon. \end{cases}$$

Observe that $\|\varphi_\varepsilon\|_{L^2(\mathbb{R})} = \sqrt{\frac{8}{3}\varepsilon} \xrightarrow[\varepsilon\to 0]{} 0$ and

$$\begin{aligned}\int_{\varepsilon}^{2\varepsilon} \frac{h'}{\varepsilon}\,\mathrm{d}x &= \frac{h(2\varepsilon) - h(\varepsilon)}{\varepsilon} \xrightarrow[\varepsilon\to 0]{} h'\big(0^+\big)\,, \\ \int_{-2\varepsilon}^{-\varepsilon} \frac{h'}{\varepsilon}\,\mathrm{d}x &= \frac{h(-\varepsilon) - h(-2\varepsilon)}{\varepsilon} \xrightarrow[\varepsilon\to 0]{} h'\big(0^-\big)\,.\end{aligned} \tag{195}$$

By inserting φ_ε into (191), we get

$$\int_{-2\varepsilon}^{-\varepsilon} \frac{h'}{\varepsilon}\,\mathrm{d}x - \int_{\varepsilon}^{2\varepsilon} \frac{h'}{\varepsilon}\,\mathrm{d}x + \int_{-2\varepsilon}^{2\varepsilon} \Big(a_k\big(1-\omega^2\gamma\big)\,h - \widehat{f}\Big)\,\varphi_\varepsilon \,\mathrm{d}x = \alpha a_k\omega^2 h(0)\,. \tag{196}$$

Note that

$$\left|\int_{-2\varepsilon}^{2\varepsilon} \left(a_k\left(1-\omega^2\gamma\right)h - f\right)\varphi_\varepsilon \,\mathrm{d}x\right|$$
$$\leqslant \left(a_k\left(1-\omega^2\gamma\right)\|h\|_{L^2(\mathbb{R})} + \|f\|_{L^2(\mathbb{R})}\right)\|\varphi_\varepsilon\|_{L^2(\mathbb{R})}$$
$$= \sqrt{\frac{8}{3}\varepsilon}\left(a_k\left(1-\omega^2\gamma\right)\|h\|_{L^2(\mathbb{R})} + \|f\|_{L^2(\mathbb{R})}\right) \xrightarrow[\varepsilon\to 0]{} 0.$$

This, together with (195) and (196) gives that

$$h'\left(0^-\right) - h'\left(0^+\right) = \alpha a_k \omega^2 h(0)\,,$$

which finishes the proof. □

6.3.3 About the linear part

6.3.3.1 About the equation $L_{\omega^2}w = f$

We formally introduce a family of operators defined by the formula

$$L_{\omega^2} = -\frac{\partial^2}{\partial x_3^2} - \left(1-\omega^2 V\right)\frac{\partial^2}{\partial x_1^2} \quad (\omega\in\mathbb{R})\,. \tag{197}$$

In this section we will consider the solvability of the linear version of the equation (163), namely

$$L_{\omega^2}w = f \text{ on } D, \tag{198}$$

where $f \in L^2_{\mathrm{odd}}(D)$ is given function. Observe that constants are element of the kernel of the operator L_{ω^2}. Hence, as before in our considerations about semilinear wave equation, we restrict our considerations to the functions, which are odd in the x_1 direction, i.e. we will work on the spaces $L^2_{\mathrm{odd}}(D)$ and $H^{2,1}_{\mathrm{odd}}(D)$. Our goal is to prove the following statement.

Lemma 106. *Assume* **Q**. *Let*

- $L^2_{\omega^2}(D)$ *be a set of all functions* $f \in L^2_{\mathrm{odd}}(D)$, *which can be represented in a form*
$$f(x_1,x_3) = \sum_{k\in S^{\omega^2}} f_k(x_3)\sin\left(k\frac{\pi}{P}x_1\right),$$
- $H^{2,1}_{\omega^2}(D)$ *be a set of all functions* $w \in H^{2,1}_{\mathrm{odd}}(D)$ *which can be represented in a form*
$$w(x_1,x_3) = \sum_{k\in S^{\omega^2}} w_k(x_3)\sin\left(k\frac{\pi}{P}x_1\right),$$

where S^{ω^2} *is defined in* (187). *Then the following statements hold true:*

(i) For every $f \in L^2_{\omega^2}(D)$*, there exits a unique* $w \in H^{2,1}_{\omega^2}(D)$ *solving the equation* (198) *in the sense of the Definition 103. In other words, there exists a solution operator* $T_{\omega^2} \in \mathcal{L}\big(L^2_{\omega^2}(D)\,, H^{2,1}_{\omega^2}(D)\big)$ *for the equation* (198)*.*

(ii) Moreover, there exists a constant $M > 0$ *such that for all* $\omega^2 \in \overline{I^{\varepsilon}_{\omega_1^2}}$

$$\|w\|_{H^{2,1}(D)} \leqslant M\,\|f\|_{L^2(D)} \quad \Big(f \in L^2_{\omega_1^2}(D)\Big)\,, \tag{199}$$

where $w = T_\mu(f)$*.*

Proof. Above statement is a consequence of Lemmas 114 and 116 proved in the section 6.3.3.1.2. □

Remark 107. Observe that for all $\omega^2 \in J^{\varepsilon}_{\omega_1^2}$ we have $L^2_{\omega^2}(D) = L^2_{\mathrm{odd}}(D)$ and $H^{2,1}_{\omega^2}(D) = H^{2,1}_{\mathrm{odd}}(D)$, where the spaces $L^2_{\omega^2}(D)$ and $H^{2,1}_{\omega^2}(D)$ are as in Lemma 106. Moreover, for all $\omega^2 \in J^{\varepsilon}_{\omega_1^2}$ the operator L_{ω^2} has a bounded inverse $T_{\omega^2} = L^{-1}_{\omega^2}\colon L^2_{\mathrm{odd}}(D) \to H^{2,1}_{\mathrm{odd}}(D)$. Let M_{ω^2} denote the norm of the operator T_{ω^2}. Observe that $M_{\omega^2} \xrightarrow[\omega^2 \to \omega_1^2]{} \infty$. As we will see later $L^2_{\omega_1^2}(D) = \{\varphi\}^{\perp_{L^2_{\mathrm{odd}}(D)}}$, where $0 \neq \varphi \in \mathtt{ker}\, L_{\omega_1^2}$.

6.3.3.1.1 About the equation $L_{\omega^2,k} = f$

For each $\omega \in \mathbb{R}$ and $k \in \mathbb{N}$, the operator $L_{\omega^2,k}$ is set to be

$$L_{\omega^2,k} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \Big(1 - \omega^2 \widehat{V}(x)\Big)\, a_k, \tag{200}$$

where $\widehat{V}(x) = \alpha\widehat{\delta} + \gamma$, with $\widehat{\delta}$ being the *Dirac delta function* supported at the point 0. We will also use the notation

$$a_k = \frac{k^2\pi^2}{P^2} \quad (k \in \mathbb{N})\,. \tag{201}$$

We take the following ansatz

$$w(x_1, x_3) = \sum_{k \in S^{\omega^2}} w_k(x_3)\sin\Big(k\frac{\pi}{P}x_1\Big) \qquad (x_1 \in (-P, P)\,, x_3 \in \mathbb{R})\,, \tag{202}$$

$$f(x_1, x_3) = \sum_{k \in S^{\omega^2}} f_k(x_3)\sin\Big(k\frac{\pi}{P}x_1\Big) \qquad (x_1 \in (-P, P)\,, x_3 \in \mathbb{R})\,, \tag{203}$$

where S^{ω^2} is defined in (187). The Fourier coefficients $(w_k)_{k \in S^{\omega^2}}, (f_k)_{k \in S^{\omega^2}}$ by (198) have to satisfy

$$L_{\omega^2,k} w_k \overset{(200)}{=} -w_k'' + \Big(1 - \omega^2\widehat{V}\Big)\, a_k w_k = f_k(x_3) \quad (x_3 \in \mathbb{R}, k \in S)\,, \tag{204}$$

where the sequence $(a_k)_{k \in S}$ is as in (201).

We will prove the following statement.

Lemma 108. *Assume* **Q**. *The following statements are true:*

1. *If* $\omega^2 \in \overline{I^{\varepsilon}_{\omega_1^2}}$ *and* $k \in \mathbb{N}$ *are such that* $(\omega^2, k) \neq (\omega_1^2, k_1)$ *then for every* $f \in L^2(\mathbb{R})$, *there exists a unique function* $w \in H^1(\mathbb{R})$ *solving equation* (204) *in the sense of the Definition 104. In other words, there exists a solution operator* $T_{\omega^2,k} \in \mathcal{L}(L^2(\mathbb{R}), H^1(\mathbb{R}))$ *for the equation* (204).

2. *There exits a constant* $M > 0$ *such that for all* $\omega^2 \in \overline{I^{\varepsilon}_{\omega_1^2}}$, $k \in S^{\omega_1^2} = \mathbb{N} \setminus \{k_1\}$ *and all* $f \in L^2(\mathbb{R})$

$$\|w\|_{L^2(\mathbb{R})} \leqslant \frac{M}{k^2} \|f\|_{L^2(\mathbb{R})},$$
$$\|w'\|_{L^2(\mathbb{R})} \leqslant \frac{M}{k} \|f\|_{L^2(\mathbb{R})},$$

where $w = T_{\omega^2,k}(f)$.

Proof. This is a consequence of Lemma 109 and 113. □

Derivation of the Green's function for $L_{\omega^2,k}$ Now we will derive an explicit formula for the function w_k solving the equation (204), namely we will find the representation of w_k in the terms of Green's function.

Consider the homogeneous version of the equation (204), i.e.

$$-h'' + \left(1 - \omega^2 \widehat{V}\right) a_k h = 0, \tag{205}$$

that is, we are looking for a function $h \in \mathcal{C}^2(\mathbb{R} \setminus \{0\}) \cap \mathcal{C}(\mathbb{R})$ satisfying for all test functions $\varphi \in \mathcal{C}_c^\infty(\mathbb{R})$

$$\int_{\mathbb{R}} -h(x_3)\varphi''(x_3)\,\mathrm{d}x_3 + a_k\big(1 - \omega^2\gamma\big) \int_{\mathbb{R}} h(x_3)\varphi(x_3)\,\mathrm{d}x_3 - \alpha a_k \omega^2 h(0)\varphi(0) = 0. \tag{206}$$

By Lemma 105 we will look for solutions of the form

$$h(x) = \begin{cases} Ae^{-\beta x} + (1-A)\, e^{\beta x} & (x \leqslant 0) \\ e^{-\beta x} & (x \geqslant 0). \end{cases} \tag{207}$$

By integrating by parts, we have

$$\int_{\mathbb{R}} -h(x)\varphi''(x)\,\mathrm{d}x = -\beta^2 \int_{\mathbb{R}} h(x)\varphi(x)\,\mathrm{d}x + 2\beta(1-A)\,\varphi(0). \tag{208}$$

(206) and (208) yield that if

$$\begin{aligned} \beta &= \beta\big(\omega^2, k\big) = \sqrt{a_k(1-\omega^2\gamma)}, \\ A &= A\big(\omega^2, k\big) = 1 - \frac{\alpha a_k \omega^2}{2\sqrt{a_k(1-\omega^2\gamma)}}, \end{aligned} \tag{209}$$

h having the above form solves problem (206). Note that by the assumption $\mathbf{Q}^2$ the numbers A and β are well defined for $k \in S^{\omega^2}$. The equation (205) is equivalent to the first order system

$$\begin{cases} h' &= g, \\ g' &= \left(1 - \omega^2 \widehat{V}\right) a_k h. \end{cases} \tag{210}$$

There are two linearly independent solutions $\begin{pmatrix} h_1 \\ h_1' \end{pmatrix}$, $\begin{pmatrix} h_2 \\ h_2' \end{pmatrix}$ of the system (210). The functions h_1, h_2 have the form

$$\begin{aligned} h_1(x) &= h(x) \quad (x \in \mathbb{R}), \\ h_2(x) &= h_1(-x) \quad (x \in \mathbb{R}), \end{aligned} \tag{211}$$

Note that $h_1(x) \xrightarrow[x\to\infty]{} 0$ and $h_2(x) \xrightarrow[x\to-\infty]{} 0$. Each solution of the system (210) has the form $\xi \begin{pmatrix} h_1 \\ h_1' \end{pmatrix} + \zeta \begin{pmatrix} h_2 \\ h_2' \end{pmatrix}$ for some constants $\xi, \zeta \in \mathbb{R}$.

Write the equation (204) as a first order system

$$\begin{cases} w_k' &= z_k, \\ z_k' &= \left(1 - \omega^2 \widehat{V}\right) a_k w_k - f_k, \end{cases} \tag{212}$$

where $f_k \in L^2(\mathbb{R})$. We will solve it using the variation of constants. Each solution of (212) has the form

$$\begin{pmatrix} w_k \\ z_k \end{pmatrix} = \xi_k(x) \begin{pmatrix} h_1 \\ h_1' \end{pmatrix} + \zeta_k(x) \begin{pmatrix} h_2 \\ h_2' \end{pmatrix}. \tag{213}$$

By differentiating (213) and using (212), we get

$$\xi_k' \begin{pmatrix} h_1 \\ h_1' \end{pmatrix} + \zeta_k' \begin{pmatrix} h_2 \\ h_2' \end{pmatrix} = \begin{pmatrix} 0 \\ -f_k \end{pmatrix},$$

and therefore

$$\begin{pmatrix} \xi_k' \\ \zeta_k' \end{pmatrix} = M^{-1} \begin{pmatrix} 0 \\ -f_k \end{pmatrix}, \tag{214}$$

where

$$\begin{aligned} M = M\left(\omega^2, k\right) &= \begin{bmatrix} h_1 & h_2 \\ h_1' & h_2' \end{bmatrix}, \\ M^{-1} = M\left(\omega^2, k\right)^{-1} &= \frac{1}{\det M} \begin{bmatrix} h_2' & -h_2 \\ -h_1' & h_1 \end{bmatrix}, \\ \det M = \det M\left(\omega^2, k\right) &= 2A\beta, \end{aligned} \tag{215}$$

where the values A and β were defined in (209). Note the condition $\mathbf{Q}^1$ is equivalent to saying that for every $\omega^2 \in \overline{I^{\varepsilon}_{\omega_1^2}}$ and $k \in \mathbb{N}$ the determinant $\det M(\omega^2, k) = 0$ if and only if $\omega^2 = \omega_1^2$ and $k = k_1$.

By integrating (214), we obtain

$$\begin{aligned}\xi_k(x) &= \frac{1}{\det M}\int_{-\infty}^{x} h_2(s) f_k(s)\,\mathrm{d}s \quad (x\in\mathbb{R})\,,\\ \zeta_k(x) &= \frac{1}{\det M}\int_{x}^{\infty} h_1(s) f_k(s)\,\mathrm{d}s \quad (x\in\mathbb{R})\,.\end{aligned} \tag{216}$$

Note that, since $f_k \in L^2(\mathbb{R})$, by Corollary 147, both expressions in (216) are well-defined. Substituting (216) into (213) and using (212), we have for each $x \in \mathbb{R}$

$$w_k(x) = \frac{1}{\det M}\left(h_1(x)\int_{-\infty}^{x} h_2(s) f_k(s)\,\mathrm{d}s + h_2(x)\int_{x}^{\infty} h_1(s) f_k(s)\,\mathrm{d}s\right). \tag{217}$$

Moreover, again by Corollary 147, the function w_k described in (217) is almost everywhere differentiable and its derivative, is indeed described by the formula (218). Hence, for almost all $x \in \mathbb{R}$

$$w_k'(x) = \frac{1}{\det M}\left(h_1'(x)\int_{-\infty}^{x} h_2(s) f_k(s)\,\mathrm{d}s + h_2'(x)\int_{x}^{\infty} h_1(s) f_k(s)\,\mathrm{d}s\right). \tag{218}$$

Later, in the proof of Lemma 108, we will show, that since $f_k \in L^2(\mathbb{R})$, then $w_k \in H^1(\mathbb{R})$, for all $k \in S^{\omega^2}$ and that the function w_k is indeed a solution of the equation (204) in the sense of the Definition 104.

Now we will rewrite the formulas (217) and (218) into terms of Greens functions. Note that if

$$G(x,y) = \begin{cases} \frac{h_1(x)h_2(y)}{\det M} & (y \leqslant x)\,,\\ \frac{h_2(x)h_1(y)}{\det M} & (x \leqslant y)\,,\end{cases}$$

then the formula (217) can be written as

$$w_k(x) = \int_{\mathbb{R}} G(x,y) f_k(y)\,\mathrm{d}y \quad (x\in\mathbb{R}). \tag{219}$$

Let

$$\begin{aligned} r_1(s) &= A e^{-\beta|s|} \quad (s\in\mathbb{R})\,,\\ r_2(x,y) &= (1-A)\, e^{-\beta(|x|+|y|)} \quad (x,y\in\mathbb{R})\,,\end{aligned} \tag{220}$$

Then one can write (219) as

$$w_k(x) = \frac{1}{\det M}(r_1 * f_k)(x) + \frac{1}{\det M}\int_{\mathbb{R}} r_2(x,y) f_k(y)\,\mathrm{d}y \quad (x\in\mathbb{R})\,. \tag{221}$$

$L^2(\mathbb{R})$ **estimates for** w_k **and** w_k' Now we will derive $L^2(\mathbb{R})$ estimates for the functions w_k and w_k' in the terms of $\|f_k\|_{L^2(\mathbb{R})}$, where w_k is as in (217).

Lemma 109. *Assume* **Q**. *There exits a constant* $M > 0$ *such that for all* $\omega^2 \in \overline{I^{\varepsilon}_{\omega_1^2}}$, $k \in S^{\omega_1^2} = \mathbb{N} \setminus \{k_1\}$ *and all* $f_k \in L^2(\mathbb{R})$

$$\begin{aligned} \|w_k\|_{L^2(\mathbb{R})} &\leqslant \frac{M}{k^2} \|f\|_{L^2(\mathbb{R})} , \\ \|w_k'\|_{L^2(\mathbb{R})} &\leqslant \frac{M}{k} \|f\|_{L^2(\mathbb{R})} , \end{aligned}$$

where the function w_k *is as in the formula* (217).

Proof. Applying Young's inequality and Hölder's inequality in formula (221), we get the following estimate

$$\|w_k(x)\|_{L^2(\mathbb{R})} \leqslant \frac{1}{|\det M|} \Big(\|r_1\|_{L^1(\mathbb{R})} + \|r_2\|_{L^2(\mathbb{R}^2)} \Big) \|f_k\|_{L^2(\mathbb{R})} . \tag{222}$$

We find that

$$\begin{aligned} \|r_1\|_{L^1(\mathbb{R})} &= \frac{2\,|A|}{\beta}, \\ \|r_2\|_{L^2(\mathbb{R}^2)} &= \frac{|A-1|}{\beta}. \end{aligned} \tag{223}$$

Formulas (215), (222) and (223) yield

$$\|w_k(x)\|_{L^2(\mathbb{R})} \leqslant \frac{2\,|A| + |A-1|}{2\,|A|\,\beta^2} \|f_k\|_{L^2(\mathbb{R})} = \frac{1}{\beta^2}\left(1 + \frac{|A-1|}{2\,|A|} \right) \|f_k\|_{L^2(\mathbb{R})} .$$

By (209) we have that

$$\frac{|A-1|}{2\,|A|} = \frac{\alpha k \pi \omega^2}{\alpha \pi k \omega^2 - 2P\sqrt{1-\gamma\omega^2}}.$$

By assumption $\mathbf{Q}^1$, we have that $\alpha\pi k\omega^2 - 2P\sqrt{1-\gamma\omega^2} = 0$ if and only if $\omega^2 = \omega_1^2$ and $k = k_1$. Therefore, the expression $\frac{|A-1|}{2|A|}$ is bounded for all $\omega^2 \in \overline{I^{\varepsilon}_{\omega_1^2}}$ and all $k \in S^{\omega_1^2}$.

Recall (209) and observe, that there exists a constant $C_1 > 0$ such that for all $\omega^2 \in \overline{I^{\varepsilon}_{\omega_1^2}}$ and all $k \in S^{\omega_1^2}$

$$\|w_k(x)\|_{L^2(\mathbb{R})} \leqslant \frac{C_1}{k^2} \|f_k\|_{L^2(\mathbb{R})} \quad \big(f_k \in L^2(\mathbb{R})\big) .$$

Recall formula (218). By similar reasoning, we get that there exists a constant $C_2 > 0$ such that for all $\omega^2 \in \overline{I^{\varepsilon}_{\omega_1^2}}$ and all $k \in S^{\omega_1^2}$

$$\|w_k'(x)\|_{L^2(\mathbb{R})} \leqslant \frac{C_2}{k} \|f_k\|_{L^2(\mathbb{R})} \quad \big(f_k \in L^2(\mathbb{R})\big) .$$

□

Soundness of w_k Now we will show, that the function w_k defined in (217) solves equation (204) in the sense of the Definition 104. This is stated in the Lemma 113.

Lemma 110. *The function $h\colon \mathbb{R} \to \mathbb{R}$ defined in* (207) *with coefficients β and A as in the formula* (209) *solves the equation* (205) *in the sense of the Definition 104.*

Proof. Note that

$$h'(x) = \begin{cases} -A\beta e^{-\beta x} + (1-A)\beta e^{\beta} x & (x<0) \\ -\beta e^{-\beta x} & (x>0)\,. \end{cases} \tag{224}$$

and

$$h''(x) = \beta^2 h(x) \quad (x \in \mathbb{R}\setminus\{0\})\,. \tag{225}$$

Equation (224) yields

$$h'\big(0^-\big) - h'\big(0^+\big) = 2\beta(1-A)\,. \tag{226}$$

By integrating by parts and applying formulas (225) and (226), for every $\varphi \in \mathcal{C}_c^\infty(\mathbb{R})$ we obtain

$$\begin{aligned}\int_{\mathbb{R}} h'\varphi' \,\mathrm{d}x &= \int_{-\infty}^{0} h'\varphi' \,\mathrm{d}x + \int_{0}^{\infty} h'\varphi' \,\mathrm{d}x \\ &= [h'\varphi]_{-\infty}^{0} + [h'\varphi]_{0}^{\infty} - \int_{-\infty}^{0} h''\varphi \,\mathrm{d}x - \int_{0}^{\infty} h''\varphi \,\mathrm{d}x \\ &\overset{(225),\,(226)}{=} 2\beta(1-A)\,\varphi(0) - \beta^2 \int_{\mathbb{R}} h\varphi \,\mathrm{d}x.\end{aligned}$$

Observe that $h(0) = 1$ and that the constants A and β (cf. formula (209)) are chosen in such a way that the formula (191) (with $f = 0$) holds true for all $\varphi \in \mathcal{C}_c^\infty(\mathbb{R})$. □

We will show that the function w_k defined in (204) satisfy the assumptions of the Lemma 105.

Lemma 111. *Let $k \in S^{\omega^2}$ and let the function w_k be as in formula* (217)*. Then*

$$w_k'\big(0^-\big) - w_k'\big(0^+\big) = a_k \alpha \omega^2 w_k(0)\,.$$

Proof. Recall (211) for the definition for the functions h_1, h_2 and (207) for the formula for the function h (the constants A, β are defined in (209)). Note that

$$h_1'(x) = h'(x) = \begin{cases} -A\beta e^{-\beta x} + (1-A)\,\beta e^{\beta x} & (x<0) \\ -\beta e^{-\beta x} & (x>0)\,, \end{cases}$$

and

$$h_2'(x) = -h'(-x) = \begin{cases} \beta e^{\beta x} & (x < 0) \\ A\beta e^{\beta x} - (1-A)\,\beta e^{-\beta x} & (x > 0)\,. \end{cases}$$

Observe

$$h_1'(0^-) - h_1'(0^+) = h_2'(0^-) - h_1'(0^+) = 2\beta(1-A) \overset{(209)}{=} \alpha a_k \omega^2. \tag{227}$$

By (218), we have

$$\begin{aligned} w_k'(0^-) - w_k'(0^+) =& \frac{1}{\det M}\big(h_1'(0^-) - h_1'(0^-)\big) \int_{-\infty}^{0} h_2(s)\, f_k(s)\; \mathrm{d}s \\ &+ \frac{1}{\det M}\big(h_2'(0^-) - h_2'(0^-)\big) \int_0^{\infty} h_1(s)\, f_k(s)\; \mathrm{d}s \\ \overset{(227)}{=}& \alpha a_k \omega^2 w_k(0)\,. \end{aligned}$$

□

Lemma 112. *Let $k \in S^{\omega^2}$. The function w_k defined in formula* (217)*) is twice differentiable almost everywhere in $\mathbb{R}$ and $w_k \in H^2((-\infty,0])\cap H^2([0,\infty))$ Moreover w_k solves the equation*

$$-w_k'' + a_k\big(1-\omega^2\gamma\big)\, w_k = f_k,$$

pointwise almost everywhere on $(-\infty, 0)$ *and on* $(0, \infty)$.

Proof. Observe that for all $x \in \mathbb{R}\setminus\{0\}$

$$h_1'(x)\, h_2(x) - h_1(x)\, h_2'(x) = -2A\beta \overset{(215)}{=} -\det M \tag{228}$$

By differentiating formula (218), at point $x \neq 0$ (cf. formula (211) for the definition of functions h_1, h_2 and (207), (209) for definition for the function h) we get that for almost all $x \in \mathbb{R}\setminus\{0\}$ (cf. Corollary 147)

$$\begin{aligned} w_k''(x) =& \frac{1}{\det M}\bigg(h_1''(x) \int_{-\infty}^{x} h_2(s) f_k(s)\,\mathrm{d}s + h_1'(x)\, h_2(x)\, f_k(x) + \\ & h_2''(x) \int_x^{\infty} h_1(s) f_k(s)\,\mathrm{d}s - h_1(x)\, h_2'(x)\, f_k(x)\bigg) \\ \overset{(228)}{=}& \frac{1}{\det M}\bigg(h_1''(x) \int_{-\infty}^{x} h_2(s) f_k(s)\,\mathrm{d}s + h_2''(x) \int_x^{\infty} h_1(s) f_k(s)\,\mathrm{d}s\bigg) - f_k(x)\,. \end{aligned}$$

The rest of the claim follows from the fact that the functions h_1, h_2 solve the homogeneous problem. □

As a consequence of Lemma 105 and Lemmas 111 and 112, we have the following statement:

Lemma 113. *Let $k \in S^{\omega^2}$. The function w_k defined in formula* (217) *solves equation* (204) *in the sense of the Definition 104.*

6.3.3.1.2 Proof of Lemma 106 As indicated in (202) consider the function w of the form

$$w(x_1,x_3)=\sum_{k\in S^{\omega^2}} w_k(x_3)\sin\Big(k\frac{\pi}{P}x_1\Big)\,, \tag{229}$$

where the functions w_k are as in the formula (217).

Lemma 114. *Assume* **Q**. *For every* $f\in L^2_{\omega_1^2}(D)$, *the function* w *defined in* (229) *is an element of the space* $H^{2,1}_{\omega_1^2}(D)$. *Moreover, the estimate* (199) *in part (ii) of the Lemma 106 holds true.*

Proof. Application of Lemma 12 and estimates from Lemma 108 give

$$\begin{aligned}
\left\|\frac{\partial w}{\partial x_1}\right\|^2_{L^2(D)} &\leqslant \sum_{k\in S^{\omega_1^2}} \frac{C^2}{k^4}\,\|f_k\|^2_{L^2(\mathbb{R})}\,k^2 \leqslant M\,\|f\|^2_{L^2(D)}\,,\\
\left\|\frac{\partial w}{\partial x_3}\right\|^2_{L^2(D)} &\leqslant \sum_{k\in S^{\omega_1^2}} \|w_k'\|^2_{L^2(\mathbb{R})} \leqslant M\,\|f\|^2_{L^2(D)}\,,\\
\left\|\frac{\partial^2 w}{\partial x_1\partial x_3}\right\|^2_{L^2(D)} &\leqslant \sum_{k\in S^{\omega_1^2}} k^2\,\|w_k'\|^2_{L^2(\mathbb{R})} \leqslant M\,\|f\|^2_{L^2(D)}\,,\\
\left\|\frac{\partial^2 w}{\partial x_1^2}\right\|^2_{L^2(D)} &\leqslant \sum_{k\in S^{\omega_1^2}} \frac{C^2}{k^4}\,\|f_k\|^2_{L^2(\mathbb{R})}\,k^4 \leqslant M\,\|f\|^2_{L^2(D)}\,,
\end{aligned}$$

for some constant $M>0$ independent of $f\in L^2_{\omega_1^2}(D)$, $\omega^2\in\overline{I_{\omega_1^2}}^{\varepsilon}$ and $k\in S^{\omega_1^2}=\mathbb{N}\setminus\{k_1\}$, hence $w\in H^{2,1}_{\omega_1^2}(D)$. □

Remark 115. Let $D^+=(-P,P)\times(0,\infty)$ and $D^-=(-P,P)\times(-\infty,0)$. If $f\in L^2_{\omega_1^2}(D)$ then $w\in W^{2,2}(D^\pm)$. Moreover here exits a constant $M>0$ such that for all $\omega\neq\omega_1$ sufficiently close to ω and for all $f\in L^2_{\omega_1^2}(D)$

$$\|w\|_{W^{2,2}(D^\pm)}\leqslant M\,\|f\|_{L^2(D)}\,.$$

Proof. By Lemma 105, we have that

$$-w_k''=a_k\big(\omega^2\gamma-1\big)\,w_k+f_k \text{ on } (-\infty,0) \text{ and on } (0,\infty)\,.$$

Hence, by Lemma 108 we obtain

$$\left\|\frac{\partial^2 w}{\partial x_3^2}\right\|^2_{L^2(D^\pm)} \leqslant \sum_{k\in S^{\omega_1^2}} C\Big(\|f_k\|^2_{L^2(\mathbb{R})}\Big)^2 \leqslant M\,\|f\|^2_{L^2(D)}\,.$$

The rest of the proof follows from Lemma 114. □

It remains to show that the function w solves equation (198) in the sense of the Definition 103.

Lemma 116. *Assume* $\mathbf{Q}$*. For every* $f \in L^2_{\omega^2}(D)$*, the function* $w \in H^{2,1}_{\omega^2}(D)$ *defined in* (229) *solves the equation* (198) *in the sense of the Definition 103.*

Proof. We need to verify that the formula (189) holds true. Consider arbitrary $\psi \in \mathcal{C}^\infty_{\mathrm{per},2P,b}(D)$. By Lemma 108 the functions $(w_k)_{k\in S^{\omega^2}}$ and $(f_k)_{k\in S^{\omega^2}}$ satisfy the equation (204) in the weak sense of the Definition 104, i.e., for all $k \in S^{\omega^2}$ and for all $\varphi \in \mathcal{C}^\infty_c(\mathbb{R})$

$$\int_{\mathbb{R}} w_k' \varphi' \,\mathrm{d}x_3 + a_k\big(1-\omega^2\gamma\big)\int_{\mathbb{R}} w_k\varphi \,\mathrm{d}x_3 - \alpha a_k \omega^2 w_k(0)\,\varphi(0) = \int_{\mathbb{R}} f_k\varphi\,\mathrm{d}x_3. \tag{230}$$

Using integraton by parts, we obtain that for all $k \in S^{\omega^2}$

$$\int_{-P}^{P} -\alpha\omega^2 a_k w_k(0)\sin\sqrt{a_k}\,x_1 \psi(x_1,0)\;\mathrm{d}x_1 = \int_{-P}^{P} -\alpha\omega^2 w_k\sqrt{a_k}\,\cos\sqrt{a_k}\,x_1\psi_1(x_1,0)\;\mathrm{d}x_1. \tag{231}$$

As mentioned in the Definition 103, by Lemma 5 the expression $u_1(\cdot,0)$ defines a $L^2(-P,P)$ function. Calculate

$$\begin{aligned}
\int_D f\psi\,\mathrm{d}x &= \sum_{k\in S^{\omega^2}}\int_{-P}^{P}\left(\int_{\mathbb{R}} f_k(x_3)\sin\sqrt{a_k}\,x_1\cdot\psi(x_1,x_3)\;\mathrm{d}x_3\right)\mathrm{d}x_1\\
&\overset{(230)}{=} \sum_{k\in S^{\omega^2}}\int_{-P}^{P}\Bigg(\int_{\mathbb{R}} w_k'(x_3)\sin\sqrt{a_k}\,x_1\cdot\psi_3(x_1,x_3)\;\mathrm{d}x_3\\
&\quad +\big(1-\omega^2\gamma\big)\int_{\mathbb{R}} a_k w_k(x_3)\sin\sqrt{a_k}\,x_1\cdot\psi(x_1,x_3)\;\mathrm{d}x_3\\
&\quad -\alpha\omega^2 a_k w_k(0)\sin\sqrt{a_k}\,x_1\cdot\psi(x_1,0)\Bigg)\,\mathrm{d}x_1\\
&\overset{(231)}{=} \int_{\mathrm{D}} u_3\psi_3\,\mathrm{d}x - \big(1-\omega^2\gamma\big)\int_{\mathrm{D}} w_{11}\psi\,\mathrm{d}x\\
&\quad -\alpha\omega^2\int_{-P}^{P} u_1(x_1,0)\,\psi_1(x_1,0)\;\mathrm{d}x_1,
\end{aligned}$$

because

$$u_1(x_1,x_3) = \sum_{k\in N} u_k(x_3)\,\sqrt{a_k}\,\cos\sqrt{a_k}\,x_1.$$

□

Proof of Lemma 106. Statement is a direct consequence of the Lemmas 114 and 116. □

6.3.3.2 Some remarks about the spectrum of the operator $L_{\omega^2,k}$

We want to find the eigenvalues of the operator $L_{\omega^2,k}$ defined in (200), i.e., we want to find $f \in H^1(\mathbb{R})$ and $\mu \in \mathbb{R}$ solving (in a distributional sense) the equation

$$L_{\omega^2,k} f = \mu f \text{ in } \mathbb{R}.$$

Lemma 117. *Assume that*

$$\mu = -a_k\left(\frac{\alpha^2 a_k \omega^4}{4} - \left(1 - \omega^2\gamma\right)\right), \tag{232}$$

then, the equation

$$L_{\mu,k} f = \mu f \text{ in } \mathbb{R}, \tag{233}$$

has one dimensional set of solutions in the space $H^1(\mathbb{R})$. Furthermore, if $f \in H^1(\mathbb{R}) \setminus \{0\}$ solves (233)*, then μ has to satisfy condition* (232)*.*

Proof. Consider a function $f\colon \mathbb{R} \to \mathbb{R}$ defined as

$$f(x) = \begin{cases} e^{\beta x} & (x \leqslant 0) \\ e^{-\beta x} & (x \geqslant 0) \end{cases},$$

where $\beta > 0$. Note that $h'(0^-) - h'(0^+) = 2\beta$. Moreover, for all $x \in \mathbb{R} \setminus 0$, we have $h''(x) = \beta^2 h(x)$. Hence, by Lemma 105 function h solves equation (233) if and only if

$$\mu = -a_k\left(\frac{\alpha^2 a_k \omega^4}{4} - \left(1 - \omega^2\gamma\right)\right) \text{ and } \beta = \frac{\alpha a_k \omega^2}{2}.$$

By the same argument as in the proof of Lemma 69 we show, that the eigenvalues are simple. □

Remark 118. As a consequence of Lemma 117, we have that the operator $L_{\omega^2,k}$ has a zero as a simple eigenvalue, if and only if

$$\alpha^2 a_k \omega^2 = 4\left(1 - \omega^4\gamma\right). \tag{234}$$

Remark 119. Assume **Q**. Among all of the operators $\left\{L_{\omega^2,k}\colon \omega^2 \in \overline{I^{\varepsilon}_{\omega_1^2}}, k \in \mathbb{N}\right\}$ only the operator $L_{\omega_1^2,k_1}$ has zero eigenvalue. Moreover it is simple.

Remark 120. Assume that $\mu \in \mathbb{R}$ is an eigenvalue of the operator $L_{\omega^2,k}$. Let $\varphi \in H^1(\mathbb{R}) \setminus \{0\}$ be such that $L_{\omega^2,k}\varphi = \mu\varphi$. Consider the function $\Psi\colon \mathrm{D} \to \mathbb{R}$, defined as

$$\Psi(x_1, x_3) = \varphi(x_3) \sin\left(k\frac{\pi}{P}k\right).$$

Note that $\Psi \in H^{2,1}_{\text{odd}}(D)$ and $L_{\omega^2}\Psi = \mu\Psi$, hence μ is an eigenvalue of the operator L_{ω^2}.

Lemma 121. *Assume* $\mathbf{Q}$ *and let* $\omega^2 \in \overline{I^{\varepsilon}_{\omega_1^2}}$*. The operator* $L_{\omega^2}\colon H^{2,1}_{\mathrm{odd}}(D) \to L^2_{\mathrm{odd}}(D)$ *has* 0 *eigenvalue if and only if* $\omega^2 = \omega_1^2$*. Moreover* 0 *is a simple eigenvalue of the operator* $L_{\omega_1^2}$*, i.e.,* $\dim \mathtt{ker}\, L_{\omega_1^2} = 1$.

Proof. We proceed as in the proof of Lemma 73. □

Now we will investigate the essential spectrum of the operator $L_{\omega^2,k}$ defined in (200).

Lemma 122. *For every* $\omega^2 \in \overline{I^{\varepsilon}_{\omega_1^2}}$ *and* $k \in \mathbb{N}$*, we have*

$$\sigma_{\mathrm{ess}}(L_{\omega^2,k}) = \big[a_k\big(1 - \omega^2\gamma\big)\,, \infty\big)\,,$$

where $a_k = \frac{k^2\pi^2}{P^2}$ *is as in* (201).

Proof. By [17, Theorem 3.1.4 p. 78], we have that for all $a \in \mathbb{R}$ the essential spectrum of the operator $-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + a\delta$ is the set $[0,\infty)$. □

6.3.3.3 About the equation $\widetilde{L_\lambda} w = f$

This section is analogous to 5.3.2 presented for the semilinear problem. For clarity, we repeat the proofs. We formally introduce a family of the operators $\widetilde{L_\lambda}$ defined as follows:

$$\widetilde{L_\lambda} = L_\lambda + \mathtt{P}_\varphi, \tag{235}$$

where the operator L_λ was defined in (197) (with $\lambda = \omega^2$) and $\varphi \in H^{2,1}_{\mathrm{odd}}(D)$ is such that $L_{\lambda_1}\varphi = 0$ and $\|\varphi\|_{L^2(D)} = 1$. We want to study the linear equation of the form

$$\widetilde{L_\lambda} w \overset{(235)}{=} L_\lambda w + \mathtt{P}_\varphi w = f \text{ on } D, \tag{236}$$

where $f \in L^2_{\mathrm{odd}}(D)$ is a given function. Our goal is to find $w \in H^1_{\mathrm{odd}}(D)$ being a weak solution of (236). The existence of a solution operator for this problem (236) is stated in Corollary 132.

For $k \in \mathbb{N}$ and $\lambda \in \mathbb{R}$ define an operator $\widetilde{L_{\lambda,k}}$ by formula

$$\widetilde{L_{\lambda,k}} = \begin{cases} L_{\lambda,k}u, & k \neq k_1, \\ L_{\lambda,k_1}u + \mathtt{P}_{\widetilde{\varphi}}u, & k = k_1, \end{cases} \tag{237}$$

where $\widetilde{\varphi}$ is the k_1-th Fourier coefficient of φ, i.e.

$$\varphi(x_1,x_3) = \widetilde{\varphi}(x_3)\sin\Big(k_1\frac{\pi}{P}x_1\Big)\,,$$

and where the operator $L_{\lambda,k}$ is defined in (200) (with $\omega^2 = \lambda$). As before, we write the functions w and f as

$$\begin{aligned} w(x_1,x_3) &= \sum_{k\in S} w_k(x_3)\sin\Big(k\frac{\pi}{P}x_1\Big)\,, \\ f(x_1,x_3) &= \sum_{k\in S} f_k(x_3)\sin\Big(k\frac{\pi}{P}x_1\Big)\,. \end{aligned} \tag{238}$$

Then the equation (236) becomes

$$\widetilde{L_{\lambda,k}} w_k = f_k. \tag{239}$$

The existence result (in case of $k = k_1$) for the problem (239) is stated in Lemma 130. Note that the remaining cases $k \neq k_1$ are treated by Lemma 108.

6.3.3.3.1 Domain of the operators L_{λ,k_1} and $\widetilde{L_{\lambda,k_1}}$ In the following, we will consider the domain of the operators $L_{\lambda,k}$ and $\widetilde{L_\lambda, k}$ and its properties.

Definition 123. Consider the set $\mathcal{H}$, being the set of all $u \in L^2(\mathbb{R})$ such that

- $u' \in L^2(\mathbb{R})$,
- $u'' \in L^2((-\infty, 0))$, $u'' \in L^2((0, \infty))$,
- $u'(0^-) - u'(0^+) = a_k \alpha \omega^2 u(0)$.

Here u' and u'' are understood in the weak sense. Note that the expressions $u(0)$ is well defined, because $u \in H^1(\mathbb{R})$, and therefore, by [1, Theorem 4.12, p. 85], $u \in \mathcal{C}(\mathbb{R})$. Since $u \in H^2((-\infty, 0))$ and $u \in H^2((0, \infty))$, again by [1, Theorem 4.12, p. 85], we have that $u \in \mathcal{C}^1((-\infty, 0])$ and $u \in \mathcal{C}^1([0, \infty))$, and therefore it also makes sense to consider the limits $u'(0^\pm) = \lim_{\varepsilon \to 0^\pm} u'(\varepsilon)$. We will consider two norms on the space $\mathcal{H}$, namely

$$\begin{aligned}
\|u\|_{\mathcal{H}_{L_{\lambda,k}}} &= \|u\|_{L^2(\mathbb{R})} + \|L_{\lambda,k} u\|_{L^2(\mathbb{R})}, \\
\|u\|_{\mathcal{H}_{\widetilde{L_{\lambda,k}}}} &= \|u\|_{L^2(\mathbb{R})} + \left\|\widetilde{L_{\lambda,k}} u\right\|_{L^2(\mathbb{R})}.
\end{aligned}$$

Observe that for every $u \in \mathcal{H}$ (cf. formula (237))

$$\begin{aligned}
\|u\|_{\mathcal{H}_{\widetilde{L_{\lambda,k}}}} &\leqslant \|u\|_{L^2(\mathbb{R})} + \|L_{\lambda,k} u\|_{L^2(\mathbb{R})} + \|\mathtt{P}_{\widetilde{\varphi}} u\|_{L^2(\mathbb{R})} \\
&\leqslant 2\|u\|_{L^2(\mathbb{R})} + \|L_{\lambda,k} u\|_{L^2(\mathbb{R})} \leqslant 2\|u\|_{\mathcal{H}_{L_{\lambda,k}}},
\end{aligned}$$

and on the other hand for every $u \in \mathcal{H}$

$$\begin{aligned}
\|u\|_{\mathcal{H}_{L_{\lambda,k}}} &= \|u\|_{L^2(\mathbb{R})} + \|L_{\lambda,k} u + \mathtt{P}_{\widetilde{\varphi}} u - \mathtt{P}_{\widetilde{\varphi}} u\|_{L^2(\mathbb{R})} \\
&\leqslant \|u\|_{L^2(\mathbb{R})} + \left\|\widetilde{L_{\lambda,k}} u\right\|_{L^2(\mathbb{R})} + \|\mathtt{P}_{\varphi} u\|_{L^2(\mathbb{R})} \\
&\leqslant 2\|u\|_{L^2(\mathbb{R})} + \left\|\widetilde{L_{\lambda,k}} u\right\|_{L^2(\mathbb{R})} \leqslant 2\|u\|_{\mathcal{H}_{\widetilde{L_{\lambda,k}}}}.
\end{aligned}$$

Therefore the norms $\|\cdot\|_{\mathcal{H}_{L_{\lambda,k}}}$ and $\|\cdot\|_{\mathcal{H}_{\widetilde{L_{\lambda,k}}}}$ are equivalent. In the situations, when it will not lead to confusion, we will use the notation $\|\cdot\|_{\mathcal{H}}$ do denote one of these two norms.

Lemma 124. *For every $\varepsilon > 0$ and every $u \in H^1(\mathbb{R})$*

$$u(0)^2 \leqslant \frac{1}{\varepsilon} \|u\|_{L^2(\mathbb{R})}^2 + \varepsilon \|u'\|_{L^2(\mathbb{R})}^2 .$$

Proof. Let $u \in \mathcal{C}_c^\infty(\mathbb{R})$. Then

$$u(0)^2 = \int_{-\infty}^0 \frac{\mathrm{d}}{\mathrm{d}t} u^2 \,\mathrm{d}t \leqslant 2 \int_{-\infty}^0 |uu'| \,\mathrm{d}t,$$
$$u(0)^2 = \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}t} u^2 \,\mathrm{d}t \leqslant 2 \int_0^\infty |uu'| \,\mathrm{d}t.$$

Hence, using Hölder's inequality, we obtain.

$$u(0)^2 \leqslant 2 \int_{\mathbb{R}} |uu'| \,\mathrm{d}t \leqslant 2 \|u\|_{L^2(\mathbb{R})} \|u'\|_{L^2(\mathbb{R})} .$$

By applying Young's inequality in above relation we obtain, that for every $\varepsilon > 0$

$$u(0)^2 \leqslant \frac{\|u\|_{L^2(\mathbb{R})}^2}{\varepsilon} + \varepsilon \|u'\|_{L^2(\mathbb{R})}^2 .$$

Since the space $\mathcal{C}_c^\infty(\mathbb{R})$ is a dense subset of $H^2(\mathbb{R})$, we have the claim. ☐

Lemma 125. *Let $I \subseteq (0, \infty)$([9]) be a bounded interval. Let $\mathcal{H}$ be as in the Definition 123. There exists a constant $C > 0$ such that, for all $u \in \mathcal{H}$ and for all $\lambda \in I$*

$$\|u\|_{H^1(\mathbb{R})} \leqslant C \|u\|_{\mathcal{H}_{L_{\lambda,k}}} .$$

Proof. Let $u \in \mathcal{H}$ and let $f = L_{\lambda,k} u$. After testing this equality with u, we get (cf. formula (237) for the definition of the operator $L_{\lambda,k}$)

$$\begin{aligned}\int_{\mathbb{R}} u'^2 \,\mathrm{d}x + (1 - \lambda\gamma a_k)\, u^2 \,\mathrm{d}x - \lambda\alpha a_k u(0)^2 &= \int_{\mathbb{R}} f u \,\mathrm{d}x \\ &\leqslant \|L_{\lambda,k} u\|_{L^2(\mathbb{R})} \|u\|_{L^2(\mathbb{R})} ,\end{aligned}$$

hence, using Lemma 124, for all $\varepsilon > 0$, for all $\lambda \in I$ and for all $u \in H^1(\mathbb{R})$

$$\begin{aligned}\|u'\|_{L^2(\mathbb{R})}^2 &\leqslant \|L_{\lambda,k} u\|_{L^2(\mathbb{R})} \|u\|_{L^2(\mathbb{R})} + |1 - \lambda\gamma a_k| \|u\|_{L^2(\mathbb{R})}^2 + \lambda\alpha a_k u(0)^2 \\ &\leqslant \|L_{\lambda,k} u\|_{L^2(\mathbb{R})} \|u\|_{L^2(\mathbb{R})} + \left(|1 - \lambda\gamma a_k| + \frac{\lambda\alpha a_k}{\varepsilon} \right) \|u\|_{L^2(\mathbb{R})}^2 \\ &\quad + \lambda\alpha a_k \varepsilon \|u'\|_{L^2(\mathbb{R})}^2 .\end{aligned}$$

Choose $\varepsilon > 0$ such that $1 - \lambda\alpha a_k \varepsilon > 0$ for all $\lambda \in I$, then for some constants $C_1 > 0$ and $C_2 > 0$ for all $\lambda \in I$ and for all $u \in H^1(\mathbb{R})$

$$\|u'\|_{L^2(\mathbb{R})}^2 \leqslant C_1 \|L_{\lambda,k} u\|_{L^2(\mathbb{R})} \|u\|_{L^2(\mathbb{R})} + C_2 \|u\|_{L^2(\mathbb{R})}^2 .$$

[9] Formally there are no obstacles to consider $I \subseteq \mathbb{R}$, however we apply this lemma for $\lambda = \omega^2 > 0$.

Therefore, there exits a constant $C_3 > 0$ such that, for all $\lambda \in I$ and for all $u \in \mathcal{H}$

$$\|u'\|_{L^2(\mathbb{R})} \leqslant C_3 \|u\|_{\mathcal{H}_{L_{\lambda,k}}}.$$

This suffices to prove the claim. □

6.3.3.3.2 Self-adjointness of the operator $\widetilde{L_{\lambda,k_1}}$

Lemma 126. *Define $\mathfrak{L} = L^2(\mathbb{R}) \cap \{\widetilde{\varphi}\}^{\perp_{L^2(\mathbb{R})}}$ and $\mathfrak{H} = \mathcal{H} \cap \{\widetilde{\varphi}\}^{\perp_{L^2(\mathbb{R})}}$. Then, for all $\lambda \in \overline{I^{\varepsilon}_{\lambda_1}}$ the operator $L_{\lambda,k_1} \colon \mathfrak{H} \subseteq \mathfrak{L} \to \mathfrak{H}$ is self-adjoint.*

Proof. To avoid confusion denote

$$\begin{aligned} L_0 &= L_{\lambda,k_1} \colon \mathcal{H} \subseteq L^2(\mathbb{R}) \to L^2(\mathbb{R}), \\ L_1 &= L_{\lambda,k_1} \colon \mathfrak{H} \subseteq \mathfrak{L} \to \mathfrak{L}. \end{aligned}$$

By [18, Theorem 1] the operator L_0 is self-adjoint. We will show that the operator L_1 is self-adjoint. The adjoint $L_1^\star$ of L_1 is defined as follows. $\mathrm{Dom}(L_1^\star)$ is the set of all $v \in \mathfrak{L}$ such that, there exists $w \in \mathfrak{L}$ such that

$$\langle L_1 u, v\rangle_{L^2(\mathbb{R})} = \langle u, w\rangle_{L^2(\mathbb{R})} \quad (u \in \mathrm{Dom}(L_1) = \mathfrak{H}). \tag{240}$$

We put $L_1^\star v = w$. Note that

$$\langle L_1 \widetilde{\varphi}, v\rangle_{L^2(\mathbb{R})} = \langle\!\langle (\lambda - \lambda_1)\, \widetilde{\varphi}, v\rangle_{L^2(\mathbb{R})} = 0 = \langle \widetilde{\varphi}, w\rangle_{L^2(\mathbb{R})}.$$

Therefore, by (240) and because $\mathfrak{H} \subseteq \mathcal{H}$ we have that

$$\langle L_0 u, v\rangle_{L^2(\mathbb{R})} = \langle u, w\rangle_{L^2(\mathbb{R})} \quad (u \in \mathcal{H}). \tag{241}$$

Since L_0 is self-andjoint, we have that $v \in \mathrm{Dom}(L_0^\star) = \mathrm{Dom}(L_0) = \mathcal{H}$ and $L_0^\star v = L_0 v = w$. Recall that $v \in \mathfrak{L}$. Hence

$$v \in \mathfrak{L} \cap \mathcal{H} = \mathfrak{H}. \tag{242}$$

Therefore $L_1^* v = w = L_0 v \overset{(242)}{=} L_1 v$ and $\mathrm{Dom}(L_1^\star) = \mathfrak{H}$. □

Lemma 127. *For every $\lambda \in \mathbb{R}$ the operator $\widetilde{L_{\lambda,k_1}} \colon \mathcal{H} \subseteq L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is self-adjoint.*

Proof. Observe that the operator $\widetilde{\lambda, k_1}$ is symmetric. By [12, Theorem VIII.3, p. 256] it is enough to show, that

$$\mathtt{im}\left(\widetilde{L_{\lambda,k_1}} \pm \mathtt{i}\right) = L^2(\mathbb{R}),$$

i.e., for every $f \in L^2(\mathbb{R})$, there exists $v \in \mathcal{H}$ such that

$$\left(\widetilde{L_{\lambda,k_1}} \pm \mathtt{i}\right) v \overset{(237)}{=} (L_{\lambda,k_1} + \mathtt{P}_{\widetilde{\varphi}} \pm \mathtt{i})\, v = f. \tag{243}$$

Denote $\mathfrak{L} = L^2(\mathbb{R}) \cap \{\widetilde{\varphi}\}^{\perp_{L^2(\mathbb{R})}}$ and $\mathfrak{H} = \mathcal{H} \cap \{\widetilde{\varphi}\}^{\perp_{L^2(\mathbb{R})}}$. We have

$$\begin{aligned} L^2(\mathbb{R}) &= \operatorname{lin}\{\widetilde{\varphi}\} \oplus \mathfrak{L}, \\ \mathcal{H} &= \operatorname{lin}\{\widetilde{\varphi}\} \oplus \mathfrak{H}. \end{aligned}$$

Write $f = s\widetilde{\varphi} + g$ with $s \in \mathbb{R}$, $g \in \mathfrak{L}$.

Note that by Lemma 126 the operator $\widetilde{L_{\lambda,k_1|\mathfrak{H}}}$ is self-adjoint, hence, again by [12, Theorem VIII.3, p. 256] for every $g \in \mathfrak{L}$, there exists $w \in \mathfrak{H}$ such that $\left(\widetilde{L_{\lambda,k_1}} \pm \mathtt{i}\right) w = g$. Define $t = \frac{s}{\lambda - \lambda_1 + 1 \pm \mathtt{i}}$. Observe, that $v = t\widetilde{\varphi} + g \in \mathcal{H}$. Then

$$\left(\widetilde{L_{\lambda,k_1}} \pm \mathtt{i}\right) v = t(\lambda - \lambda_1 + 1 \pm \mathtt{i})\,\widetilde{\varphi} + \left(\widetilde{L_{\lambda,k_1}} \pm \mathtt{i}\right) w = s\widetilde{\varphi} + g = f,$$

hence (243) holds true. □

6.3.3.3.3 Properties of the spectrum the operator $\widetilde{L_{\lambda,k_1}}$ Now we are ready to study the existence of the inverse of the operator $\widetilde{L_{\lambda,k_1}}$.

Lemma 128. *0 is not an eigenvalue of the operator $\widetilde{L_{\lambda,k_1}}$ for all λ in some open neighbourhood of λ_1.*

Proof. Suppose that there exists a function $h \in \mathcal{H} \setminus \{0\}$ such that

$$\widetilde{L_{\lambda,k_1}} h \overset{(237)}{=} L_{\lambda,k_1} h + \langle \widetilde{\varphi}, h \rangle\, \widetilde{\varphi} = 0. \tag{244}$$

Since $L_{\lambda_1,k_1}\widetilde{\varphi} = 0$, we have that

$$L_{\lambda,k_1}\widetilde{\varphi} = (\lambda - \lambda_1)\,\widetilde{\varphi}. \tag{245}$$

After testing (244) with $\widetilde{\varphi}$ we obtain

$$0 = \langle L_{\lambda,k_1} h, \widetilde{\varphi} \rangle + \langle \widetilde{\varphi}, h \rangle = \langle L_{\lambda,k_1} \widetilde{\varphi}, h \rangle + \langle \widetilde{\varphi}, h \rangle \overset{(245)}{=} (\lambda - \lambda_1 + 1)\,\langle h, \widetilde{\varphi} \rangle\,.$$

If λ is in a sufficiently small neighbourhood of λ_1, we obtain that $\langle h, \widetilde{\varphi} \rangle = 0$, and therefore $L_{\lambda,k_1} h = 0$ (cf. equation (244)), i.e., the operator L_{λ,k_1} has zero as an eigenvalue. Since, for $\lambda \neq \lambda_1$ in an open neighbourhood of λ_1 zero is not an eigenvalue of the operator L_{λ,k_1}, we have to have that $\lambda = \lambda_1$, and therefore $h = \widetilde{\varphi}$. This, together with $\langle h, \widetilde{\varphi} \rangle = 0$ gives a contradiction. □

Lemma 129. *Assume* **Q**. *For all $\lambda \in \overline{I^{\varepsilon}_{\lambda_1}}$ the value 0 is isolated from the spectrum $\sigma\left(\widetilde{L_{\lambda,k_1}}\right)$. In other words, there exists $\delta > 0$ such that* $\inf \left|\sigma\left(\widetilde{L_{\lambda,k_1}}\right)\right| \geqslant \delta$, *for all $\lambda \in \overline{I^{\varepsilon}_{\lambda_1}}$.*

Proof. Note that for all $\lambda \in \mathbb{R}$ the operator $\widetilde{L_{\lambda,k_1}} - L_{\lambda,k_1} = \mathtt{P}_{\widetilde{\varphi}}$ is compact. Hence, applying [16, Theorem 5.35, p. 244], we obtain $\sigma_{\text{ess}}\left(\widetilde{L_{\lambda,k_1}}\right) = \sigma_{\text{ess}}(L_{\lambda,k_1})$, which implies that 0 is isolated from $\sigma_{\text{ess}}\left(\widetilde{L_{\lambda,k_1}}\right)$. Moreover by Lemma 128, we have that 0 is not an eigenvalue of $\widetilde{L_{\lambda,k_1}}$ for all λ being in a neighbourhood of λ_1. By Lemma 122, we have that

$$\sigma_{\text{ess}}(L_{\lambda,k_1}) = \left[a_k\left(1-\omega^2\gamma\right), +\infty\right),$$

where $\lambda = \omega^2$ and $a_{k_1} = \frac{k_1^2\pi^2}{P^2}$. Let $\delta = \frac{k_1^2\pi^2}{P^2} \min_{\omega^2 \in \overline{I_{\omega_1^2}^{\varepsilon}}}(1-\omega^2\gamma)$. By assumption $\mathbf{Q}^2$, we have that $\delta > 0$. □

6.3.3.3.4 Existence of the inverse of the operator $\widetilde{L_{\lambda,k_1}}$

Lemma 130. *Assume* $\mathbf{Q}$. *For every* $\lambda \in \overline{I_{\lambda_1}^{\varepsilon}}$ *and for every* $f \in L^2(\mathbb{R})$, *there exists an unique* $w \in \mathcal{H}$ *such that*

$$\widetilde{L_{\lambda,k_1}} w \overset{(237)}{=} L_{\lambda,k_1} w + \mathtt{P}_{\widetilde{\varphi}} w = f. \tag{246}$$

Moreover, there exists a constant $M > 0$ *such that*

$$\|w\|_{\mathcal{H}_{\widetilde{L_{\lambda,k_1}}}} \leqslant M \|f\|_{L^2(\mathbb{R})} \quad \left(f \in L^2(\mathbb{R}), \lambda \in \overline{I_{\lambda_1}^{\varepsilon}}\right).$$

Proof. By Lemma 129 the inverse of the operator $\widetilde{L_{\lambda,1}} \colon \mathcal{H} \to L^2(\mathbb{R})$ exists, for all λ in some neighbourhood of λ_1. It remains to show that the norm of the operator $\left(\widetilde{L_{\lambda,k_1}}\right)^{-1} : L^2(\mathbb{R}) \to \mathcal{H}$ can be bounded with a constant, which does not depend on λ. In order to simplify the notation denote $A = \widetilde{L_\lambda}$, with $\text{Dom}(A) = \mathcal{H}$ and $H = L^2(\mathbb{R})$.

Let $(\mathcal{P}_\nu)_{\nu\in\mathbb{R}}$ be the family of spectral projections of the operator A. Define

$$P^+ = \int_0^\infty 1 \,\mathrm{d}\mathcal{P}_\nu,$$
$$P^- = \int_{-\infty}^0 1 \,\mathrm{d}\mathcal{P}_\nu.$$

Denote $H^\pm = P^\pm(H)$, then $H = H^+ \oplus H^-$. The operators $A^\pm = P^\pm A = AP^\pm \colon \text{Dom}(A^\pm) = \text{Dom}(A) \cap H^\pm \to H^\pm$ are self-adjoint. Moreover, the operators A^+, A^- are positively, negatively definite, respectively. Since $A = A^+ + A^-$, we have that $A^{-1} = (A^+)^{-1} P^+ + (A^-)^{-1} P^-$, and in a consequence

$$\|A^{-1}\| \leqslant \left\|A^{+^{-1}}\right\| + \left\|(A^-)^{-1}\right\|. \tag{247}$$

$$
\begin{aligned}
\left\|\left(A^+\right)^{-1}\right\| &= \sup_{f^+\in H^+} \frac{\left\|\left(A^+\right)^{-1} f^+\right\|_{\mathcal{H}}}{\|f^+\|_{L^2(\mathbb{R})}} = \sup_{u\in \mathrm{Dom}(A^+)} \frac{\|u\|_{\mathcal{H}}}{\|A^+u\|_{L^2(\mathbb{R})}} \\
&= 1 + \sup_{u\in \mathrm{Dom}(A^+)} \frac{\|u\|_{L^2(\mathbb{R})}}{\|A^+u\|_{L^2(\mathbb{R})}}.
\end{aligned} \tag{248}
$$

Using Bunyakovsky-Cauchy–Schwarz inequality we have that for all $u \in \mathrm{Dom}(A^+)$, we have $|\langle A^+u, u\rangle| \leqslant \|u\|_{L^2(\mathbb{R})} \|A^+u\|_{L^2(\mathbb{R})}$, hence

$$
\frac{\|u\|_{L^2(\mathbb{R})}}{\|A^+u\|_{L^2(\mathbb{R})}} \leqslant \frac{\|u\|^2_{L^2(\mathbb{R})}}{\langle A^+u, u\rangle} \leqslant \frac{\|u\|^2_{\mathcal{H}}}{\langle A^+u, u\rangle} \quad \left(u \in \mathrm{Dom}\left(A^+\right)\right). \tag{249}
$$

By [13, Theorem 4.3.1 p. 78], we have that

$$
\inf_{u\in \mathrm{Dom}(A^+)} \frac{\langle A^+u, u\rangle}{\|u\|^2_{\mathcal{H}}} = \inf \sigma\left(A^+\right) \geqslant \delta, \tag{250}
$$

where $\delta > 0$ because of Lemma 129. In the virtue of inequalities (249) and (250), we conclude that $\frac{\|u\|_{L^2(\mathbb{R})}}{\|A^+u\|_{L^2(\mathbb{R})}} \leqslant \frac{1}{\delta}$ for all $u \in \mathrm{Dom}(A^+)$ and in a consequence, together with (248) gives

$$
\left\|\left(A^+\right)^{-1}\right\| \leqslant 1 + \frac{1}{\delta}. \tag{251}
$$

Same argument can be performed for the operator $-A^-$. This, together with (247) finishes the proof. □

Remark 131. As a consequence of Lemma 125 and inequality (130), we get that, there exists $\varepsilon > 0$ and a constant $M > 0$ such that

$$
\|w_{k_1}\|_{H^1(\mathbb{R})} \leqslant M \|f_{k_1}\|_{L^2(\mathbb{R})} \quad \left(f_{k_1} \in L^2(\mathbb{R}), \lambda \in \overline{I^{\varepsilon}_{\lambda_1}}\right),
$$

where w_{k_1} solves the equation (246).

6.3.3.3.5 Existence of the inverse of the operator $\widetilde{L_\lambda}$ We will finish our considerations in this section with a statement about the existence of a solution of the equation (236). As a consequence of Lemmas 108 and 130 we get the following:

Corollary 132. *Assuming* **Q** *For all $f \in L^2_{\mathrm{odd}}(D)$ there exists a unique $w \in H^{2,1}_{\mathrm{odd}}(D)$ solving weakly equation (236). Denote $\widetilde{T_\lambda}(f) = w$. Moreover, there exits a constant $M > 0$ such that for all $\lambda \in \overline{I^{\varepsilon}_{\lambda_1}}$ and for all $f \in L^2_{\mathrm{odd}}(D)$*

$$
\left\|\widetilde{T_\lambda}(f)\right\|_{H^{2,1}(D)} \leqslant M \|f\|_{L^2(D)}.
$$

In other words, there exists an operator $T_\lambda = \left(\widetilde{L_\lambda}\right)^{-1} : L^2_{\mathrm{odd}}(\mathbb{R}) \to H^{2,1}_{\mathrm{odd}}(D)$, which is a solution operator for the equation (236), which has a norm bounded uniformly in $\lambda \in \overline{I^{\varepsilon}_{\lambda_1}}$.

Proof. Recall ansatz described in (238) and (237) for the definition of the operators $\widetilde{L_{\lambda,k}}$. As a consequence of Lemma 108, we have that:

(a) For all $k \in S^{\lambda_1} = \mathbb{N} \setminus \{k_1\}$, all $\lambda \in \overline{I^{\varepsilon}_{\lambda_1}}$ and for every $f_k \in L^2(\mathbb{R})$ the function $w_k \in H^1(\mathbb{R})$ is a unique solution the equation (239).

(b) There exists a constant $M > 0$ such that for all $k \in S^{\lambda_1} = \mathbb{N} \setminus \{k_1\}$, all $\lambda \in \overline{I^{\varepsilon}_{\lambda_1}}$ and for every $f_k \in L^2(\mathbb{R})$

$$\|w_k\|_{L^2(\mathbb{R})} \leqslant \frac{M}{k_k} \|f_k\|_{L^2(\mathbb{R})},$$
$$\|w_k'\|_{L^2(\mathbb{R})} \leqslant M \|f_k\|_{L^2(\mathbb{R})}.$$

Remark 131 implies that

(c) For all $\lambda \in \overline{I^{\varepsilon}_{\lambda_1}}$ and all $f_{k_1} \in L^2(\mathbb{R})$, the function $w_{k_1} \in H^1(\mathbb{R})$ is a unique solution of the equation (239) (with $k = k_1$).

(d) There exists a constant $\widetilde{M} > 0$ such that for all $\lambda \in \overline{I^{\varepsilon}_{\lambda_1}}$ and all $f_{k_1} \in L^2(\mathbb{R})$

$$\|w_{k_1}\|_{H^1(\mathbb{R})} \leqslant \widetilde{M} \|f_{k_1}\|_{L^2(\mathbb{R})}.$$

Having (a), (b), (c) and (d) we proceed as in the proof of Lemma 106. □

Remark 133. For all λ sufficiently close to λ_1

$$\widetilde{L_\lambda}(\varphi) = (1 + \lambda - \lambda_1)\,\varphi,$$
$$\widetilde{T_\lambda}(\varphi) = \frac{1}{1 + \lambda - \lambda_1}\varphi,$$
$$\widetilde{T_\lambda}(\mathtt{P}_\varphi h) = \frac{\langle h, \varphi\rangle_{L^2(D)}}{1 + \lambda - \lambda_1}\varphi \quad \big(h \in H^1_{\text{odd}}(D)\big). \tag{252}$$

Proof. Proof of above statements is analogous to the proof of Lemma 88. □

6.3.4 Application of the Crandall-Rabinowitz for the quasilinear equation

6.3.4.1 Reformulation of the problem

In this section we will reformulate the problem (163) in a way suitable for the Crandall-Rabinowitz theorem.

As assumed in **Q** let $\omega_1 \in \mathbb{R}$ be such that the operator $L_{\omega_1^2}$ has a zero as a simple eigenvalue. Take $\lambda_1 = \omega_1^2$ and $\lambda = \omega^2$ and rewrite (163) as[10]

$$L_\lambda u + \mathtt{P}_\varphi u = -\lambda \Gamma(x_3)\, G(u) + \mathtt{P}_\varphi u \text{ on } D, \tag{253}$$

[10] At this point I want to mention, that idea of introducing the projection $\mathtt{P}_\varphi$ comes from Peter Rupp.

where the function $G\colon H^{2,1}_{\mathrm{odd}}(D) \to L^2_{\mathrm{odd}}(D)$ is defined as

$$G(u) = 3u^2 u_{11} + 6uu_1^2 \quad \big(u \in H^{2,1}_{\mathrm{odd}}(D)\big)\,, \tag{254}$$

and $\mathtt{P}_\varphi u = \langle \varphi, u\rangle_{L^2(D)}\, \varphi$, with $\varphi \in \mathtt{ker}\, L_{\lambda_1}$ such that $\|\varphi\|_{L^2(D)} = 1$. By Corollary 132 there exists $\varepsilon > 0$ such that for all $\lambda \in I^\varepsilon_{\lambda_1} = (\lambda_1 - \varepsilon, \lambda_1 + \varepsilon)$, the operator $\widetilde{L_\lambda} \overset{(235)}{=} L_\lambda + \mathtt{P}_\varphi$ is invertible and $\widetilde{T_\lambda} = (L_\lambda + \mathtt{P}_\varphi)^{-1} : L^2_{\mathrm{odd}}(D) \to H^{2,1}_{\mathrm{odd}}(D)$ is bounded with respect to $\lambda \in I^\varepsilon_{\lambda_1}$. The non-linearity G maps the x_1-odd functions to x_1-odd functions. Moreover, by Lemma 27, we have that $G\colon H^{2,1}_{\mathrm{odd}} \to L^2_{\mathrm{odd}}(D)$ is well-defined.

Define $F\colon H^{2,1}_{\mathrm{odd}}(D) \times I^\varepsilon_{\lambda_1} \to H^{2,1}_{\mathrm{odd}}(D)$ as

$$F(u, \lambda) = u - \widetilde{T_\lambda}(-\lambda\Gamma(x_3)\, G(u) + \mathtt{P}_\varphi u) \quad \big(u \in H^{2,1}_{\mathrm{odd}}(D)\,, \lambda \in I^\varepsilon_{\lambda_1}\big)\,. \tag{255}$$

Note that the function F is well-defined, since $\Gamma \in L^\infty(D)$ (cf. assumption $\mathbf{Q}_\Gamma$).

Note that finding a pair $(u, \lambda) \in H^1_{\mathrm{odd}}(D) \times \mathbb{R}$ such that $F(u, \lambda) = 0$ is equivalent to solving the problem described in (253) and, in a consequence, equivalent to solving the problem (163).

6.3.4.2 Statement about the regularity of the function F

In this section we will prove that the function F defined in (255) is of the class $\mathcal{C}^2$. This is stated in Lemma 135. For writing the derivatives with respect to real arguments, we will use the convention described in the Remark 140.

6.3.4.2.1 Differentiability of the mapping $\lambda \longmapsto \widetilde{T_\lambda}$ Now we will discuss the differentiability of the mapping

$$\mathbb{R} \ni \lambda \longmapsto T_\lambda \in \mathcal{L}\big(L^2(D)\,, H^{2,1}_{\mathrm{odd}}(D)\big)\,,$$

where the operator $T_\lambda = \Big(\widetilde{L_\lambda}\Big)^{-1}$ (the operator $\widetilde{L_\lambda}$ was defined in (235), Corollary 132 states the existence of the operator $\widetilde{T_\lambda}$).

Lemma 134. *The mapping $\mathbb{R} \ni \lambda \longmapsto \widetilde{T_\lambda} \in \mathcal{L}\big(L^2(D)\,, H^{2,1}_{\mathrm{odd}}(D)\big)$ is of the class $\mathcal{C}^\infty$ in an open neighbourhood of λ_1. Moreover*

$$\frac{\mathrm{d}\widetilde{T_\lambda}}{\mathrm{d}\lambda} = -\widetilde{T_\lambda} \circ \widetilde{T_\lambda}, \tag{256}$$

for all λ sufficiently close to λ_1.

Proof. By Corollary 132 there exists $\varepsilon > 0$ and $M > 0$ such that for all $\lambda \in (\lambda_1 - \varepsilon, \lambda_1 + \varepsilon)$ and for all $f \in L^2(D)$ such that

$$\Big\|\widetilde{T_\lambda}(f)\Big\|_{H^{2,1}(D)} \leqslant C\, \|f\|_{L^2(D)}\,. \tag{257}$$

Having inequality (257), we proceed as in the proof of Lemma 89. □

6.3.4.2.2 Differentiability of the function F This section we will apply Lemmas 134 and 27 in order to show that the function F defined in (255) is of the class $\mathcal{C}^2$.

Lemma 135. *Denote $I^{\varepsilon}_{\lambda_1} = (\lambda_1 - \varepsilon, \lambda_1 + \varepsilon)$ There exists $\varepsilon > 0$ such that the function $F\colon H^{2,1}_{\text{odd}}(D) \times I^{\varepsilon}_{\lambda_1} \to H^{2,1}_{\text{odd}}(D)$ defined in (255) is of the class $\mathcal{C}^2\big(H^{2,1}_{\text{odd}}(D) \times I^{\varepsilon}_{\lambda_1}, H^{2,1}_{\text{odd}}(D)\big)$ and for all all $u, h \in H^{2,1}_{\text{odd}}(D)$ and all $\lambda \in I^{\varepsilon}_{\lambda_1}$ we have*

$$\begin{aligned}
\mathrm{D}_\lambda F(u,\lambda) &= \widetilde{T_\lambda}\Big(\widetilde{T_\lambda}(-\lambda\Gamma(x_3)\, G(u) + \mathtt{P}_\varphi u)\Big) + \widetilde{T_\lambda}(\Gamma(x_3)\, G(u))\,,\\
\mathrm{D}_u F(u,\lambda)\, h &= h - \widetilde{T_\lambda}(-\lambda\Gamma(x_3)\, \mathrm{D}G(u)\, h + \mathtt{P}_\varphi h)\,,\\
\mathrm{D}^2_{u\lambda} F(u,\lambda)\, h &= \widetilde{T_\lambda}\Big(\widetilde{T_\lambda}(-\lambda\Gamma(x_3)\, \mathrm{D}G(u)\, h + \mathtt{P}_\varphi h)\Big) + \widetilde{T_\lambda}(\Gamma(x_3)\, \mathrm{D}G(u)\, h)\,,
\end{aligned}$$

where the non-linearity G was defined in (254) and

$$\mathrm{D}G(u)\, h = 6uu_{11}h + 3u^2 h_{11} + 6u_1^2 h + 12uu_1h_1,$$

In particular for all $\lambda \in I^{\varepsilon}_{\lambda_1}$

$$\mathrm{D}_u F(0,\lambda) = \mathrm{Id}_{H^{2,1}(D)} - \widetilde{T_\lambda} \circ \mathtt{P}_\varphi, \tag{258}$$

$$\mathrm{D}^2_{u\lambda} F(0,\lambda) = \widetilde{T_\lambda} \circ \widetilde{T_\lambda} \circ \mathtt{P}_\varphi. \tag{259}$$

Proof. The proof follows the differentiability of G as described in Lemma 27, Lemma 134 and the fact that the operator $\widetilde{T_\lambda}$ is linear and continuous. □

Remark 136. As a consequences of the relation (252) in the Remark 133 we can rewrite relations (258) and (259) as

$$\begin{aligned}
\mathrm{D}_w F(0,\lambda)\, h &= h - \frac{\langle h, \varphi\rangle_{L^2(D)}}{1 + \lambda - \lambda_1}\varphi,\\
\mathrm{D}^2_{w\lambda} F(0,\lambda)\, h &= \frac{\langle h, \varphi\rangle^2_{L^2(D)}}{(1 + \lambda - \lambda_1)^2}\varphi,
\end{aligned}$$

for all $h \in H^{2,1}_{\text{odd}}(D)$.

6.3.4.3 Algebraic properties of the function F

Now we will prove some statements, which we will use, to verify assumptions (261) and (262) in Theorem 141 applied for function F defined in (255). Results of this section are analogous to the ones presented in the section 5.4.3 devoted to semilinear problem.

Lemma 137. *Assume* $\mathbf{Q}$ *and* $\mathbf{Q}_\Gamma$*, then*

$$\mathtt{ker}\, \mathrm{D}_u F(0,\lambda_1) = \mathtt{ker}\, L_{\lambda_1},$$

Moreover $\dim \mathtt{ker}\, \mathrm{D}_u F(0,\lambda_1) = 1$.

Proof. If $h \in \mathtt{ker}\, D_u F(0, \lambda_1)$ (cf. formula (258)), then

$$h - \widetilde{T_{\lambda_1}} \mathtt{P}_\varphi h = 0,$$

which is equivalent to

$$L_{\lambda_1} h + \mathtt{P}_\varphi h - \mathtt{P}_\varphi h = 0,$$

which means that $h \in \mathtt{ker}\, L_{\lambda_1}$. The rest follows from Lemma 121. ☐

Lemma 138. *Assume* $\mathbf{Q}$ *and* $\mathbf{Q}_\Gamma$*, then*

$$\dim \mathtt{ker}\, \mathrm{D}_u F(0, \lambda_1) = \operatorname{codim} \mathtt{im}\, \mathrm{D}_u F(0, \lambda_1) = 1.$$

Proof. We will show that

$$H^{2,1}_{\mathrm{odd}}(D) = \mathtt{im}\, \mathrm{D}_u F(0, \lambda_1) \oplus \mathtt{ker}\, \mathrm{D}_u F(0, \lambda_1)\,,$$

which together with the Lemma 137 will prove the claim.

Note that $h \in \mathtt{im}\, \mathrm{D}_u F(0, \lambda_1)$, if and only if, there exists $z \in H^{2,1}_{\mathrm{odd}}(D)$ such that $h = z - \langle z, \varphi \rangle_{L^2(D)} \varphi$. Moreover $h \in \mathtt{ker}\, \mathrm{D}_u F(0, \lambda_1)$ if and only of $h = \xi \varphi$, for some $\xi \in \mathbb{R}$. Observe, that for all $h \in H^{2,1}_{\mathrm{odd}}(D)$

$$h = \underbrace{h - \langle h, \varphi \rangle_{L^2(D)} \varphi}_{\in \mathtt{im}\, \mathrm{D}_u F(0,\lambda_1)} + \underbrace{\langle h, \varphi \rangle_{L^2(D)} \varphi}_{\in \mathtt{ker}\, \mathrm{D}_u F(0,\lambda_1)},$$

hence $H^{2,1}_{\mathrm{odd}}(D) = \mathtt{im}\, \mathrm{D}_u F(0, \lambda_1) + \mathtt{ker}\, \mathrm{D}_u F(0, \lambda_1)$. Suppose that $h \in H^{2,1}_{\mathrm{odd}}(D)$ is such that $h \in \mathtt{im}\, \mathrm{D}_u F(0, \lambda_1) \cap \mathtt{ker}\, \mathrm{D}_u F(0, \lambda_1)$. Then, for some $\xi \in \mathbb{R}$ and $z \in H^{2,1}_{\mathrm{odd}}(D)$, we have that $z - \langle z, \varphi \rangle_{L^2(D)} = \xi \varphi$. After multiplying this relation by φ, we get that $0 = \xi$, hence $h = 0$, which means that $\mathtt{im}\, \mathrm{D}_u F(0, \lambda_1) \cap \mathtt{ker}\, \mathrm{D}_u F(0, \lambda_1) = \{0\}$. ☐

Lemma 139. *Assume* $\mathbf{Q}$ *and* $\mathbf{Q}_\Gamma$*, then*

$$\mathrm{D}^2_{u\lambda} F(0, \lambda_1)\, \varphi \notin \mathtt{im}\, \mathrm{D}_u F(0, \lambda_1)\,.$$

Proof. Suppose that $\mathrm{D}^2_{u\lambda} F(0, \lambda_1)\, \varphi \in \mathtt{im}\, \mathrm{D}_u F(0, \lambda_1)$, i.e., there exists some $\psi \in H^{2,1}_{\mathrm{odd}}(D)$ such that

$$\mathrm{D}^2_{u\lambda} F(0, \lambda_1)\, \varphi = \mathrm{D}_u F(0, \lambda_1)\, \psi,$$

which is equivalent to (cf. formulas (258) and (259), for the definitions of the corresponding operators)

$$\widetilde{T_{\lambda_1}}\Big(\widetilde{T_{\lambda_1}}(\varphi)\Big) = \psi - \widetilde{T_{\lambda_1}}(\mathtt{P}_\varphi \psi)\,.$$

By Remark 133, we can rewrite this as

$$\varphi = \psi - \langle \varphi, \psi \rangle_{L^2(D)} \varphi.$$

By testing with φ, we get

$$1 = \langle \varphi, \varphi \rangle_{L^2(D)} = \langle \varphi, \psi \rangle_{L^2(D)} - \langle \varphi, \psi \rangle_{L^2(D)} = 0.$$

Contradiction. ☐

6.3.4.4 Proof of the main result for the quasilinear wave equation

Proof of Theorem 101. As mentioned earlier, we will apply Theorem 141 for the function $F\colon H^{2,1}_{\mathrm{odd}}(D) \times I^{\varepsilon}_{\lambda_1} \to H^{2,1}_{\mathrm{odd}}(D)$ defined in (255). Note that by Lemma 135 the function is of the class $\mathcal{C}^2$. The assumption (261) is fulfilled by Lemma 138 and the assumption (262) is satisfied due to Lemma 139. $\square$

7 Crandall-Rabinowitz theorem

In this section we present the Crandall-Rabinowitz theorem, cf. [4, Theorem I.5.1, p. 15].

Let X, Y, Z be Banach spaces, $U \subseteq X$, $V \subseteq Y$ open sets and $F\colon U \times V \to Z$ be a function. The theorem is about the structure of set of solutions of the equation

$$F(x, y) = 0. \tag{260}$$

Remark 140. Let $f\colon \mathbb{R} \to Z$, where Z is some normed space, and let $\mathrm{D}f(x_0) \in \mathcal{L}(\mathbb{R}, Z)$ be a Fréchet derivative of f at some point $x_0 \in \mathbb{R}$. Notice that in this case we can identify $\mathrm{D}f(x_0)$ with element $\mathrm{D}f(x_0)1 \in Z$.

Theorem 141 (Crandall-Rabinowitz)**.** *Assume that X, Z are Banach spaces, $U \times V \subseteq X \times \mathbb{R}$ is an open set, $F \in \mathcal{C}^2(U \times V, Z)$, $F(0, \lambda) = 0$ $(\lambda \in V)$ and that there is $\lambda_0 \in V$ such that*

$$\dim \mathtt{ker}\, \mathrm{D}_x F(0, \lambda_0) = \operatorname{codim} \mathtt{im}\, \mathrm{D}_x F(0, \lambda_0) = 1. \tag{261}$$

Furthermore assume that there is some $\widehat{v}_0 \in X$, $\|\widehat{v}_0\| = 1$ such that

$$\mathtt{ker}\, \mathrm{D}_x F(0, \lambda_0) = \operatorname{lin}\{\widehat{v}_0\},$$

and

$$\mathrm{D}^2_{x\lambda} F(0, \lambda_0)\widehat{v}_0 \notin \mathtt{im}\, \mathrm{D}_x F(0, \lambda_0) \quad ({}^{11}). \tag{262}$$

Then there is a non-trivial continuously differentiable curve through point $(0, \lambda_0)$,

$$\{(x(s), \lambda(s)) \in X \times \mathbb{R}\colon s \in (-\delta, \delta), x(0) = 0, \lambda(0) = \lambda_0\}, \tag{263}$$

such that

$$F(x(s), \lambda(s)) = 0 \quad (s \in (-\delta, \delta)),$$

and all solutions of equation (260) *in a neighbourhood of point* $(0, \lambda_0)$ *are on the trivial line or on the non-trivial curve* (263)*.*

It is important to notice that the Theorem 141 can be formulated more generally:

Remark 142 (Theorem A.7, p. 121 in [5]). The Crandall-Rabinowitz Theorem 141 holds true under weakening the regularity assumptions for the function F. It is enough to assume that function $F \in \mathcal{C}^1(U \times V, Z)$ and for each non-empty compact set $I_0 \subseteq V$ the following holds true

$$\begin{aligned} \sup_{\lambda \in I_0} \left\| \mathrm{D}_x F(x, \lambda)[\cdot] - \mathrm{D}_x F(0, \lambda)[\cdot] - \mathrm{D}^2_{xx}(0, \lambda)[x, \cdot] \right\| &= o(\|x\|), \\ \sup_{\lambda \in I_0} \left\| \mathrm{D}_\lambda F(x, \lambda) - \mathrm{D}^2_{\lambda, x}(0, \lambda)[x] \right\| &= o(\|x\|). \end{aligned}$$

[11] cf. Remark 140

According to [5, p. 120] the function F of the form $F(x, \lambda) = \sum_{\mu=1}^{k} F_\mu(x)\Lambda_\mu(\lambda)$ fulfils the above condition, when the functions $F_1, \ldots, F_k$ and $\Lambda_1, \ldots, \Lambda_k$ are continuously differentiable and the functions $F_1, \ldots, F_k$ are twice differentiable at the point 0.

8 Some definitions and technical results

In this section we collect some definitions and technical statements, which were used before.

8.1 Carathéodory functions

In the section 4.1 and in the assumptions $\mathbf{S}_g$ in the section 5.1 we use the notion of *Carathéodory function*. It is defined as follows.

Definition 143. let $G \subseteq \mathbb{R}^m$ be a measurable set. A function $f\colon G \times \mathbb{R}^n \to \mathbb{R}$ is called a *Carathéodory function* if

- for all $s \in \mathbb{R}^n$, the mapping
$$G \ni x \longmapsto f(x, s)\,,$$
is measurable,
- for almost all $x \in G$ the mapping
$$\mathbb{R}^n \ni s \longmapsto f(x, s) \in \mathbb{R},$$
is continuous.

8.2 Lebesgue differentiation theorem

In order to differentiate formulas (105), (106) and (217) we need the following statements.

Theorem 144 (Lebesgue differentiation theorem in 1-d)**.** *Assume that $g \in L^1(\mathbb{R})$, and define the function $F\colon \mathbb{R} \to \mathbb{R}$ by a formula*

$$G(x) = \int_{-\infty}^{x} g(s)\ \mathrm{d}x \quad (x \in \mathbb{R})\,.$$

Then the function G is continuous, almost everywhere differentiable (with respect to the Lebesgue measure) and

$$G'(x) = g(x) \ \text{ for almost all } x \in \mathbb{R}.$$

Proof. For the proof of above theorem we refer to [7, Theorem 1.6.11, p. 136] □

While deriving a solution formula for the equation (89), the following corollaries from Theorem 144 are used. As mentioned earlier, they are used while differentiating formulas (105), (106).

Corollary 145. *Let $f \in L^2(0,\infty)$ and let $h\colon (0,\infty) \to \mathbb{R}$ be a function having the form*

$$h(x) = \begin{cases} B(x) & (x \in (0,a))\,, \\ Ae^{-\beta x} & (x > a)\,, \end{cases}$$

where $A \in \mathbb{R}$ and $\beta \in (0,\infty)$([12]) and $B \in L^2(0,a)$, for some $a > 0$. Then, the function $v\colon (0,\infty) \to \mathbb{R}$ defined by

$$v(x) = \int_x^\infty h(s)\, f(s)\ \mathrm{d}s \quad (x \in (0,\infty))\,,$$

is continuous, almost everywhere differentiable (with respect to the Lebesgue measure) and

$$v'(x) = -h(x)\, f(x) \ \textit{ for almost all } x \in (0,\infty)\,.$$

Proof. Note that the function $hf \in L^1(0,1)$. Applying Theorem 144 gives the claim. □

Corollary 146. *Let $f \in L^2(0,\infty)$ and let $h\colon (0,\infty) \to \mathbb{R}$ be a function having the form*

$$h(x) = \begin{cases} B(x) & (x \in (0,a))\,, \\ A_1 e^{-\beta_1 x} + A_2 e^{\beta_2 x} & (x > a)\,, \end{cases}$$

where $A_1, A_2 \in \mathbb{R}$ and $\beta_1, \beta_2 \in (0,\infty)$([13]) and $B \in L^2(0,a)$, for some $a > 0$. Then, the function $v\colon (0,\infty) \to \mathbb{R}$ defined by

$$v(x) = \int_0^x h(s)\, f(s)\ \mathrm{d}s \quad (x \in (0,\infty))\,,$$

is continuous, almost everywhere differentiable (with respect to the Lebesgue measure) and

$$v'(x) = h(x)\, f(x)\,, \ \textit{ for almost all } x \in (0,\infty)\,.$$

Proof. For every $N \in \mathbb{N}$ consider a function $v_N\colon (0,\infty) \to \mathbb{R}$ defined by

$$v_N(x) = \int_0^x h(s)\, f(x)\, \chi_{(0,N)}(s)\ \mathrm{d}s \quad (x \in (0,\infty))\,.$$

Since $h \in L^2(0,N)$ and $f \in L^2(\mathbb{R})$, we have that the function $hf\chi_{(0,N)} \in L^1(0,\infty)$, which implies that the function v_N is well-defined for all $N \in \mathbb{N}$. Moreover, we have that

$$v(x) = v_N(x) \quad (x \in (0,N)\,, N \in \mathbb{N})\,.$$

[12] Observe, that $\lim_{x\to\infty} h(x) = 0$.
[13] Observe, that $\lim_{x\to\infty} h(x) = \infty$.

By Theorem 144, the function v_N is continuous, almost everywhere differentiable and

$$v_N'(x) = h(x)\, f(x)\, \chi_{(0,N)}(x)\,, \text{ for almost all } x \in (x)\,, \text{ and all } N \in \mathbb{N}.$$

From above, we conclude the claim. □

Similarly, we need the following while solving the equation (204). The following statement is used to differentiate equation (217).

Corollary 147. *Let $f \in L^2(\mathbb{R})$ and let $h\colon \mathbb{R} \to \mathbb{R}$ be a function having the form*

$$h(x) = \begin{cases} A_1 e^{\beta_1 x} & (x \leqslant 0)\,, \\ A_2 e^{-\beta_2 x} + A_3 e^{\beta_3 x} & (x > 0)\,, \end{cases}$$

where $A_1, A_2, A_3 \in \mathbb{R}$ and $\beta_1, \beta_2, \beta_3 \in (0,\infty)$ ([14]). Then, the function $v\colon \mathbb{R} \to \mathbb{R}$ defined by

$$v(x) = \int_{-\infty}^{x} h(s)\, f(s)\; \mathrm{d}s \quad (x \in \mathbb{R})\,,$$

is continuous, almost everywhere differentiable (with respect to the Lebesgue measure) and

$$v'(x) = h(x)\, f(x)\,, \textit{ for almost all } x \in \mathbb{R}.$$

Proof. For every $N \in \mathbb{N}$ define a function $v_N\colon \mathbb{R} \to \mathbb{R}$ by a formula

$$v_N(x) = \int_{-\infty}^{x} h(s)\, f(s)\, \chi_{(-\infty,N)}(s)\; \mathrm{d}s \quad (x \in \mathbb{R})\,.$$

Since $h \in L^2(-\infty, N)$ and $f \in L^2(\mathbb{R})$, we have that the function $hf\chi_{(-\infty,N)} \in L^1(\mathbb{R})$, which implies that the function v_N is well-defined, for all $N \in \mathbb{N}$. Moreover

$$v(x) = v_N(x) \quad (x \in (-\infty, N)\,, N \in \mathbb{N})\,.$$

By Theorem 144, the function v_N is continuous, almost everywhere differentiable (with respect to the Lebesgue measure) and

$$v_N'(x) = h(x)\, f(x)\, \chi_{(-\infty,N)}(x)\,, \text{ for almost all } x \in \mathbb{R}, \text{ and all } N \in \mathbb{N}.$$

From the above, we conclude that the function v has the desired properties. □

[14] Observe, that $\lim_{x\to-\infty} h(x) = 0$ and $\lim_{x\to\infty} |h(x)| = \infty$.

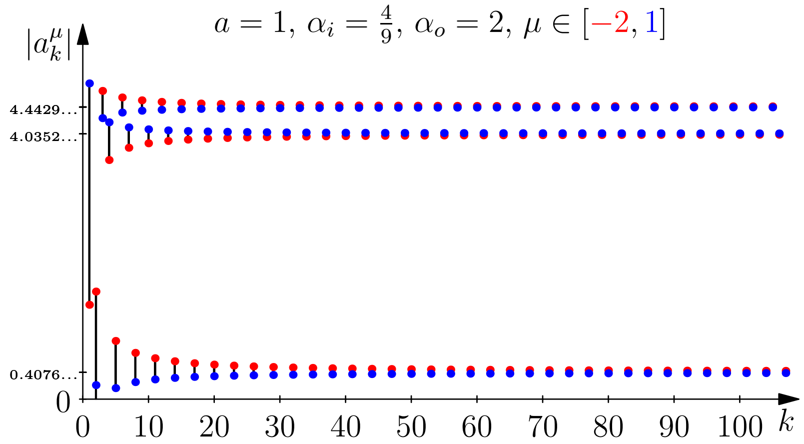


Figure 9: Behaviour of the sequence $\big(|a_k^\mu|_{k\in\mathbb{N}}\big)$. On the graph the ranges of the functions $[-2,1] \ni \mu \longmapsto |a_k^\mu| \in \mathbb{R}$ are marked.

8.3 Some remarks about Lemma 62

In the following we will present some comments about the necessity of the Assumption $\mathbf{S}^4$ (63) in the Lemma 62. We will study the limit points (as $k \to \infty$) of the sequences

$$\begin{aligned} a_k^\mu =& \frac{\sqrt{\alpha_o\pi^2k^2+\mu}}{k} \cos\sqrt{\alpha_i\pi^2k^2+\mu}\,a \\ &- \frac{\sqrt{\alpha_i\pi^2k^2+\mu}}{k} \sin\sqrt{\alpha_i\pi^2k^2+\mu}\,a, \\ b_k^\mu =& \frac{\sqrt{\alpha_i\pi^2k^2+\mu}}{k} \cos\sqrt{\alpha_i\pi^2k^2+\mu}\,a \\ &+ \frac{\sqrt{\alpha_o\pi^2k^2+\mu}}{k} \sin\sqrt{\alpha_i\pi^2k^2+\mu}\,a. \end{aligned} \tag{264}$$

On the Figures 9 and 10 the behaviour of the sequence $(|a_k^\mu|)_{k\in\mathbb{N}}$, for some rational values of $\sqrt{\alpha_i}\,a \in \mathbb{Q}$ was presented. The figures 11, 12 and 13 show the behaviour of the sequence $(|a_k^\mu|)_{k\in\mathbb{N}}$, when $\sqrt{\alpha_i}\,a \notin \mathbb{Q}$. All above figures contains the ranges of the functions $[\mu_a, \mu_b] \ni\longmapsto a_k^\mu \in \mathbb{R}$, for corresponding values of $k \in \mathbb{N}$.

Lemma 148. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function with a period $T \in \mathbb{R}\setminus\mathbb{Q}$. Then, the set $f(\mathbb{N})$ is a dense subset of $f(\mathbb{R})$.*

Proof. Consider a function $\lfloor\cdot\rceil : \mathbb{R} \to [0,1)$ defined by the formula

$$\lfloor x\rceil = x - \lfloor x\rfloor \quad (x \in \mathbb{R}),$$

where $\lfloor x\rfloor = \max\{z \in \mathbb{Z}\colon z \leqslant x\}$. By [15, Theorem 2], if $\alpha \in \mathbb{R}\setminus\mathbb{Q}$, the sequence $(\lfloor n\rceil)_{n\in\mathbb{N}}$ is a dense subset of the interval $[0,1]$. Let $y \in f(\mathbb{R})$ and let

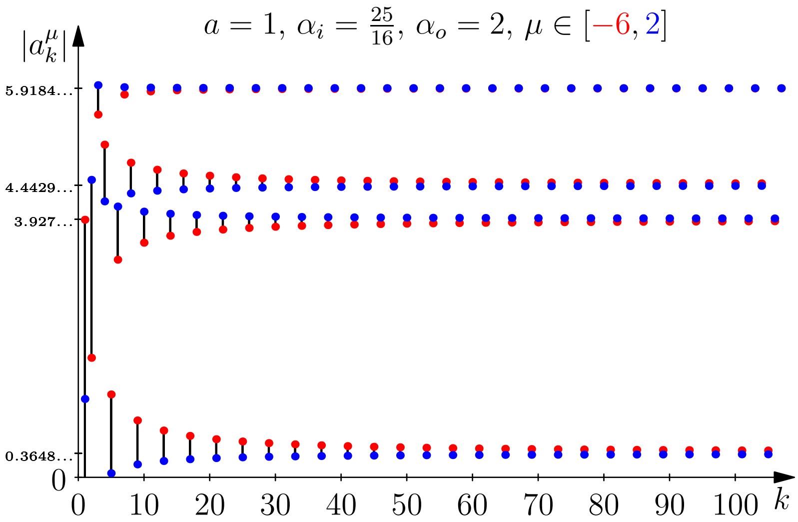


Figure 10: Behaviour of the sequence $\left(|a_k^\mu|_{k\in\mathbb{N}}\right)$.

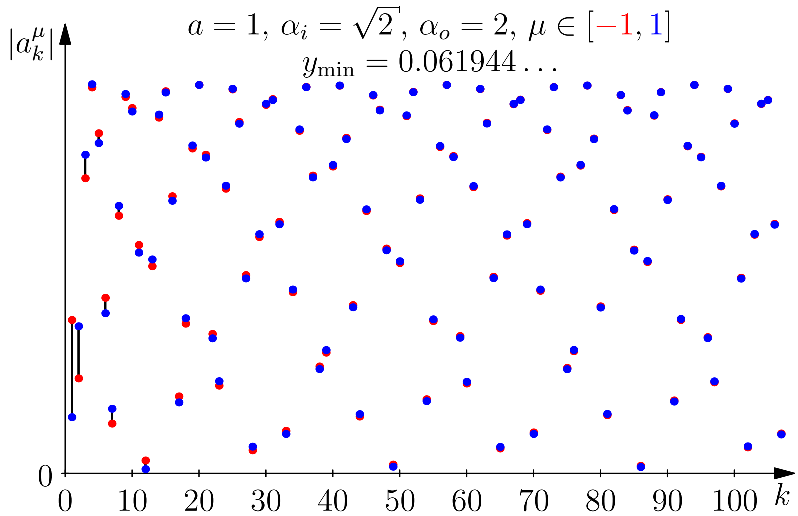


Figure 11: Behaviour of the sequence $\left(|a_k^\mu|_{k\in\mathbb{N}}\right)$. The value y_{min} represents the smallest value marked on the graph.

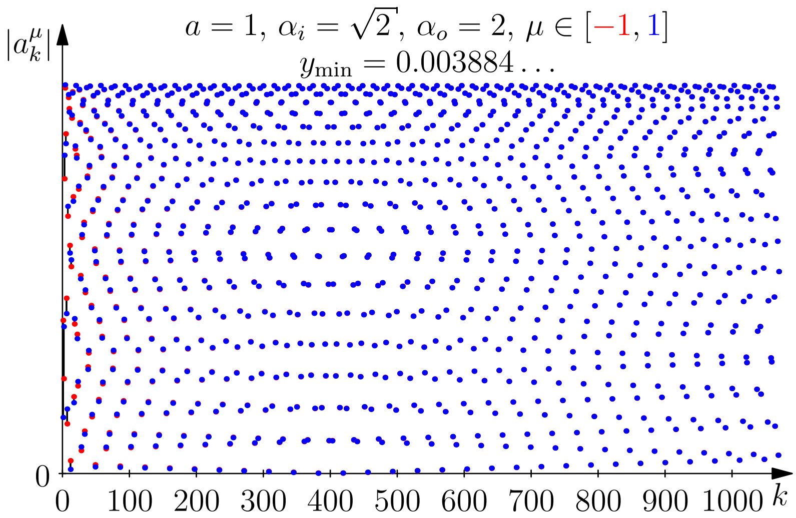


Figure 12: Behaviour of the sequence $\left(|a_k^\mu|_{k\in\mathbb{N}}\right)$.

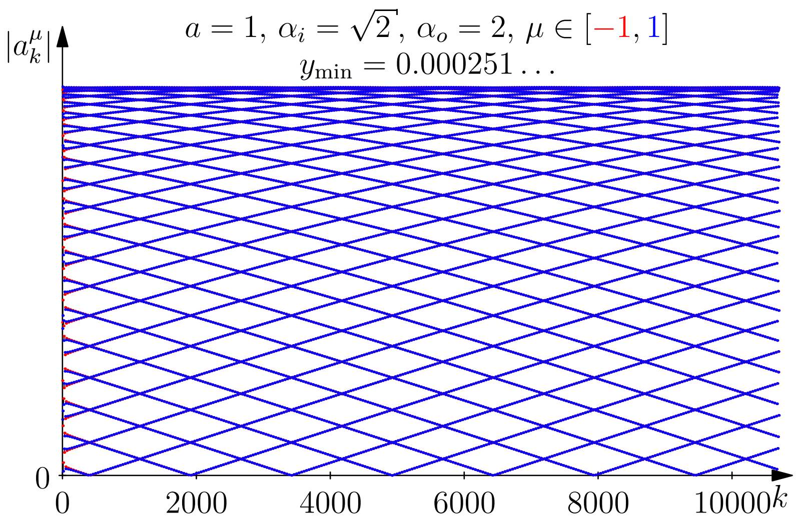


Figure 13: Behaviour of the sequence $\left(|a_k^\mu|_{k\in\mathbb{N}}\right)$.

$\varepsilon > 0$ be arbitrary. There exist $x_0 \in [0, T]$ such that $f(x_0) = y$. Let $\delta > 0$ be such that, if $|x - x_0| < \delta$, then

$$|f(x) - y| < \frac{\varepsilon}{2}. \tag{265}$$

Since T is an irrational number, then $\frac{1}{T}$ is also irrational. The sequence $\left(\left\lfloor \frac{n}{T} \right\rceil\right)_{n\in\mathbb{N}}$ is a dense subset of $[0, 1]$. Let $n \in \mathbb{N}$ be such that $\left|\frac{x_0}{T} - \left\lfloor \frac{n}{T} \right\rceil\right| \leqslant \frac{\delta}{T}$, hence

$$\left|x_0 - T \left\lfloor \frac{n}{T} \right\rceil\right| \leqslant \delta.$$

By (265), we get

$$\left|f\left(T \left\lfloor \frac{n}{T} \right\rceil\right) - y\right| < \varepsilon.$$

Note that

$$T \left\lfloor \frac{n}{T} \right\rceil = T\left(\frac{n}{T} - \left\lfloor \frac{n}{T} \right\rfloor\right) = n - T \underbrace{\left\lfloor \frac{n}{T} \right\rfloor}_{\in\mathbb{N}}.$$

Hence $f(n) = f\left(T \left\lfloor \frac{n}{T} \right\rceil\right)$. □

Lemma 149. *Let $\alpha_i, \alpha_o, a > 0$, $\sqrt{\alpha_i}\,a \in \mathbb{R} \setminus \mathbb{Q}$ and let $\mu \in \mathbb{R}$. Then the sequences $(a_k^\mu)_{k\in\mathbb{N}}$ and $(b_k^\mu)_{k\in\mathbb{N}}$ defined in* (264) *have* 0 *as a limit point.*

Proof. Consider a function $f\colon \mathbb{R} \to \mathbb{R}$ defined as

$$f(x) = \sqrt{\alpha_o}\,\pi \cos \sqrt{\alpha_i}\,a\pi x - \sqrt{\alpha_i}\,\pi \sin \sqrt{\alpha_i}\,a\pi x \quad (x \in \mathbb{R}). \tag{266}$$

Note that the function f is periodic, with the period $\frac{2}{\sqrt{\alpha_i}\,a} \in \mathbb{R} \setminus \mathbb{Q}$. Observe that $f(0) > 0$ and $f\left(\frac{1}{\sqrt{\alpha_i}\,a}\right) < 0$, hence $0 \in f(\mathbb{R})$. Therefore, by Lemma 148, there exists a sequence $(k_n)_{n\in\mathbb{N}}$ of natural numbers, such that $f(k_n) \xrightarrow[n\to\infty]{} 0$. Observe

$$\frac{\sqrt{\alpha_o \pi^2 k^2 + \mu}}{k} \xrightarrow[k\to\infty]{} \sqrt{\alpha_o}\,\pi,$$
$$\frac{\sqrt{\alpha_i \pi^2 k^2 + \mu}}{k} \xrightarrow[k\to\infty]{} \sqrt{\alpha_i}\,\pi,$$
$$\sqrt{\alpha_i \pi^2 k^2 + \mu}\,a - \sqrt{\alpha_i}\,a\pi k \xrightarrow[k\to\infty]{} 0.$$

This implies that $a_{k_n}^\mu - f(k_n) \xrightarrow[n\to\infty]{} 0$, which shows that $a_{k_n} \xrightarrow[n\to\infty]{} 0$. By same reasoning the get the claim for the sequence $(b_k)_{k\in\mathbb{N}}$. □

Remark 150. Above Lemma 149, shows, that the assumption $\mathbf{S}^4$ (63) in the Lemma 62 is necessary.

We will conclude this section with we following summary

- if assumption $\mathbf{S}^4$ (63) holds true then, both of the sequences $(|a_k^\mu|)_{k\in\mathbb{N}}$ and $(|b_k^\mu|)_{k\in\mathbb{N}}$ have finitely many limit points,

- if assumption $\mathbf{S}^4$ (64) holds true, then 0 is not a limit point of the sequences $(|a_k^\mu|)_{k\in\mathbb{N}}$ and $(|b_k^\mu|)_{k\in\mathbb{N}}$,
- if assumption $\mathbf{S}^4$ (64) is not satisfied, then 0 is a limit point of one of the sequences $(|a_k^\mu|)_{k\in\mathbb{N}}$ or $(|b_k^\mu|)_{k\in\mathbb{N}}$,

- if assumption $\mathbf{S}^4$ (63) is not satisfied then, both of the sequences $(|a_k^\mu|)_{k\in\mathbb{N}}$ and $(|b_k^\mu|)_{k\in\mathbb{N}}$ have infinitely many limit points and 0 is one of them. More precisely, for every value $\mu \in \mathbb{N}$, the set $\{a_k^\mu \in \mathbb{R} \colon k \in \mathbb{N}\}$ is a dense subset of the set $f((0,\infty))$, where f is in (266). Similar holds for the sequence $(b_k^\mu)_{k\in\mathbb{N}}$.

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Karlsruhe, 12.09.2017,

Declaration

I confirm that this is my own work and the use of all material from other sources has been properly and fully acknowledged.

Karlsruhe, 23.08.2017 Piotr Idzik